RESEARCH
IN
APPLIED GEOPHYSICS

NO. 21 OCTOBER, 1981

THE INDUCTIVE RESPONSE OF THIN PLATES IN A STRATIFIED SPACE

by
PETER WALKER

GEOPHYSICS LABORATORY
DEPARTMENT OF PHYSICS
UNIVERSITY OF TORONTO
THE INDUCTIVE RESPONSE OF THIN PLATES
IN A STRATIFIED SPACE

by

Peter Walker

A thesis submitted in conformity with
the requirements for the degree of
Master of Science in the
University of Toronto

© 1981 Peter Walker
ABSTRACT

The equivalent source method, the method whereby conductors are replaced by induced current distributions, can be a valuable tool for solving EM problems. The method can be used to reduce problems with complex analytical solutions into ones which have a simple numerical formulation. This is accomplished by replacing conductors with predetermined source currents of unknown amplitude. The Green's functions of complex spaces can therefore be replaced with the free space Green's function or other simpler Green's functions. The choice of which conductors to replace with induced currents, and consequently which Green's function to use is merely a matter of convenience.

This approach is applied to solve for the EM response of two or more plates in a stratified space by invoking the Galerkin method. The plates were replaced with predetermined current distributions and their interactions were computed by using the Green's functions for a stratified space.

A program was then written to demonstrate the feasibility of the method. Results of the program were compared with the results of existing programs and with scale model experiments. Computational problems were encountered when calculating the responses of plates in a layered Earth, due to the excessive time required to calculate the
Green's functions. However, the results for plates in free space agreed well enough with model experiments to confirm that the method works.
Acknowledgements

Of the many people who have contributed to this thesis, I would particularly like to thank my supervisor, Dr. G.F. West for his support and encouragement. His insight into the problems that cropped up during the course of this thesis has proved to have been invaluable.

I would also like to thank the staff and students for their many helpful suggestions, and for making work here most pleasant. I would particularly like to thank Scott Holladay, Marc Vallee and Jim MacNae for the discussions that I had with them about the problems that I encountered during this work. Mark Bloore was often invaluable when computing problems occurred.

The model experiments were conducted by Andrew Platzer, and the modelling programs that were used to verify some of the results of this work were written by Alf Dyck and Tom Eadie. I would also like to thank Khader Khan and Raul Cunha for drafting the diagrams and Drs. Bailey and Peltier for reading the final drafts of this thesis.

Finally, I would like to thank my wife, Joyce, for her support and assistance during the course of this thesis, and for proof reading much of the manuscript.
Financial support for this work was provided through an Ontario Geoscience grant to Dr. West.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter One</th>
<th>INTRODUCTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>The Problem</td>
</tr>
<tr>
<td>1.2</td>
<td>Previous Work</td>
</tr>
<tr>
<td>1.3</td>
<td>Thesis Outline</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter Two</th>
<th>DERIVATION OF THE FREDHOLM INTEGRAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Overview</td>
</tr>
<tr>
<td>2.2</td>
<td>The Fredholm Integral for a Homogenous Body</td>
</tr>
<tr>
<td>2.3</td>
<td>The Equation for a Thin Sheet</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter Three</th>
<th>THE GALERKIN METHOD</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>3.2</td>
<td>Derivation of the Galerkin Method</td>
</tr>
<tr>
<td>3.3</td>
<td>Application of the Galerkin Method to the Induction Equation</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter Four</th>
<th>THE SOLUTION OF THE INDUCTION EQUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Inversion of the Induction Matrix</td>
</tr>
<tr>
<td>4.2</td>
<td>Implementation of the Solution</td>
</tr>
</tbody>
</table>

| Chapter Five | THE EFFECT OF A NONUNIFORM EXTERNAL MEDIUM |
5.1 Introduction ................................................................. 31
5.2 The Effect of a Non-Uniform Medium .............................. 33
5.3 The Effect of Other Plates ............................................. 36
5.4 Solutions for a Inhomogenous Space ............................... 38

Chapter Six GREEN'S FUNCTIONS IN A STRATIFIED EARTH

6.1 Overview ................................................................. 45
6.2 Notation ..................................................................... 46
6.3 The Hertz Potentials ..................................................... 47
6.4 Calculation of Potential Ratios ....................................... 56
6.5 Calculation of the Total Potential at the
Source Level .................................................................... 62
6.6 Calculation of the Electric and Magnetic
Fields from the Source Currents ....................................... 68
6.7 Transformation of the Fields into the Space
Domain ........................................................................... 74
6.8 Summary ..................................................................... 79

Chapter Seven THE PROGRAMMED SOLUTION: ORGANIZATION AND
RESULTS

7.1 Organization of the Program ........................................... 81
7.2 Test Results ............................................................... 83

Chapter Eight CONCLUSIONS ............................................. 95

Appendix A VECTOR RELATIONS ......................................... 98
<table>
<thead>
<tr>
<th>Appendix</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appendix B</td>
<td>CALCULATION OF THE FIELD IN FREE SPACE</td>
<td>99</td>
</tr>
<tr>
<td>Appendix C</td>
<td>INTEGRATION OF THE REACTANCE MATRIX</td>
<td>104</td>
</tr>
<tr>
<td>Appendix D</td>
<td>SYMMETRY IN PLATE INTERACTION</td>
<td>113</td>
</tr>
<tr>
<td>Appendix E</td>
<td>THE SOLUTION TO THE HELMHOLTZ EQUATION</td>
<td>116</td>
</tr>
<tr>
<td>Appendix F</td>
<td>THE PRIMARY FIELDS AT THE SOURCE LEVEL DUE TO DIPOLE CURRENTS</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>IN A LAYERED EARTH</td>
<td></td>
</tr>
<tr>
<td>Table No.</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>3-1</td>
<td>Elements of the Galerkin Matrices</td>
<td></td>
</tr>
<tr>
<td>5-1</td>
<td>The Distribution of Current over the Plate</td>
<td></td>
</tr>
<tr>
<td>5-2</td>
<td>The Solution for Many Plates in a Non-Uniform Space</td>
<td></td>
</tr>
<tr>
<td>6-1</td>
<td>Continuation Between the Top and Bottom of a Layer</td>
<td></td>
</tr>
<tr>
<td>6-2</td>
<td>Propagation Matrices for Interfaces</td>
<td></td>
</tr>
<tr>
<td>6-3</td>
<td>The Primary Hertz Potentials at the Source Level</td>
<td></td>
</tr>
<tr>
<td>6-4</td>
<td>Recursion Relations for the Reflection Coefficients</td>
<td></td>
</tr>
<tr>
<td>6-5</td>
<td>Total Potentials at the Source Point</td>
<td></td>
</tr>
<tr>
<td>6-6</td>
<td>Propagation Matrices</td>
<td></td>
</tr>
<tr>
<td>6-7</td>
<td>Definition of $\alpha, \beta, \gamma, \delta, \epsilon, \phi$</td>
<td></td>
</tr>
<tr>
<td>6-8</td>
<td>The Vectors $\mathbf{f}$ and $\mathbf{g}$</td>
<td></td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>6–1</td>
<td>The Geometry of the Layered Earth Problem</td>
<td></td>
</tr>
<tr>
<td>6–2</td>
<td>Calculation of the Total Hertz Potentials at the Source Level</td>
<td></td>
</tr>
<tr>
<td>6–3</td>
<td>Calculation of the Potential Ratios</td>
<td></td>
</tr>
<tr>
<td>7–1</td>
<td>The Program Structure</td>
<td></td>
</tr>
<tr>
<td>7–2, 7–3, 7–4</td>
<td>Comparison of the Results of the Program with Those Produced by Plate</td>
<td></td>
</tr>
<tr>
<td>7–5</td>
<td>Comparison of the Results for Two Plates with Scale Model Experiments</td>
<td></td>
</tr>
<tr>
<td>7–6</td>
<td>Comparison of HLEM Responses produced by the Program with Those of Eadie</td>
<td></td>
</tr>
<tr>
<td>7–7</td>
<td>The Response of a Plate Below an Overburden Layer</td>
<td></td>
</tr>
<tr>
<td>C–1</td>
<td>Integration of the Reactance Matrix</td>
<td></td>
</tr>
</tbody>
</table>
INDEX OF SYMBOLS

E — Electric Field
H — Magnetic Field
J — Current Density
M — Magnetic Current Density
K — Surface Current Density
L — Magnetic Surface Current Density
\( \mathcal{M} \) — Magnetisation
A — Magnetic Vector Potential
F — Electric Vector Potential
V — Electric Scalar Potential
\( \mathcal{F} \) — Magnetic Scalar Potential
\( \Pi \) — Electric Hertz Potential
\( \mathcal{\Pi} \) — Magnetic Hertz Potential

SYMBOLS USED IN THE SOLUTION FOR THE PLATE

\( \phi \) — Trial Function
\( T_n \) — Chebychev Polynomial of Order \( n \)
\( a_1 \) — \( x \) half dimension of the plate
\( a_2 \) — \( y \) half dimension of the plate
\( \chi \) — \( x \) coordinate normalized by \( a \)
\( \xi \) — \( y \) coordinate normalized by \( a \)

SYMBOLS USED IN THE LAYERED EARTH SOLUTION

\( p,q \) — Fourier Wavenumbers in \( x \) and \( y \)
\( \lambda,\beta \) — Radial Wavenumbers in \( x \) and \( y \)
\( \hat{F} \) — Fourier Transform of \( F \)
\( \tilde{F} \) – Hankel Transform of \( F \)

\( \eta \) – z wavenumber

RFP – Potential Ratio on the Positive Side of the Layer

RFN – Potential Ratio on the Negative Side of the Layer
CHAPTER ONE

INTRODUCTION

1.1 The Problem

Electromagnetic (EM) prospecting has come to play an increasingly important role since its inception in the late nineteen forties. Its major application has been in mineral exploration, but it has also been used successfully in geological mapping and in engineering. In this technique a low frequency radio signal is used to probe the Earth for regions exhibiting high conductivity. These regions may either be caused by or associated with the presence of ore, or they may be indicative of underlying geological structure. When the signals impinge on a conductive region, eddy currents are excited which cause a secondary field to be produced. By detecting this field, characteristics of the conductor such as its depth, attitude and conductivity can be deduced.

In favourable terrain, where the conductor is isolated from other conductive bodies, electromagnetic prospecting has been very profitable. Interpretation of the secondary field is simple, and accurate model responses have been produced by assuming that the conductor resides alone in free space. However, as more and more of the
Earth is explored, the amount of unexplored favourable terrain is being diminished. Exploration activity is gradually being forced into less favourable areas.

In these areas, the conductor is not isolated from the effects of other conductive zones, many of which may be of little or no interest on their own. These zones distort the secondary field of the conductor in two ways: They alter the field impinging on the conductor and they produce a secondary field of their own. Interpretation of the secondary field therefore becomes quite difficult.

In this thesis, a computational model for calculating the responses of multiple planar conductors in a stratified medium is presented. This model is useful for simulating the responses of two-dimensional geological structures (such as veins and faults) underlying a layer of overburden.

1.2 Previous Work

The response of conductors to electromagnetic systems has been examined with the use of analytical solutions, scale modelling, case histories and numerical simulation. One of the first analytical solutions was derived by Sommerfeld in 1897: that for the potential field of a pole source in a space containing a half plane of infinite
conductivity. This was later expanded by West (1960) to include the potential of a dipole source. Similar problems involving dipole sources – that of the infinitely conductive strip and disk – were solved by Martin (1960) and Douloff (1961).

Previous studies of the responses of multiple conductors have been made by Ranasinghe (1962) and Lowrie and West (1965). Ranasinghe studied the analytical frequency domain response of two loops to an external field while Lowrie and West examined the effects of overburden on the response of a buried conductor with scale model experiments. Villegas–Garcia (1980) extended this study by examining the scale model response of ridges and valleys concealed by conductive overburden.

Analytical solutions have been obtained for a stratified earth by Freischknect (1967), Wait (1970) and Weaver (1971). These solutions have been used to compute the response of a plate in a conducting host medium under a more conductive overburden layer by (La-jole and West (1975) and Hanneson (1981)). Lamontagne and West (1971) and Annan (1974) produced solutions for a plate in a uniform and infinitely resistive space. Other analyses have been done by Geyer (1972), who studied the response of a dipping contact, and Parry and Ward (1971), who studied the response of two-dimensional buried structures. Gaur et al (1971) studied the enhancement of EM anomalies due to overburden. To this date, however, no numerical solutions have been produced for multiple conductors that reside either
in free space or beneath a conductive overburden. That is the subject of the present work.

The model examined in this thesis consists of thin rectangular plates of constant conductivity imbedded in a stratified earth. The plates are restricted to strata of infinite resistivity because the present analysis is only capable of evaluating inductive responses. In spite of this restriction, the model nevertheless has geological applicability since many conductive two-dimensional geological structures such as faults, shears, lenses and veins reside in highly resistive host rocks.

1.3 Thesis Outline

The method for determining the EM response of many plates in a stratified space is developed in chapters two through five. In chapter two, the induction equation for a plate residing in free space is derived. This is converted into matrix form in chapter three using the Galerkin method. In chapter four, the solution to the induction equation is developed. This solution is then modified in chapter five to include the effects produced by the interaction of the plates with each other and with a stratified halfspace.

In chapter six, a method for computing the Green's functions in a stratified space is developed. The practicalities involved in
computing the solution are stressed, and intermediate results are tabulated as a reference.

Finally, in chapter seven the structure of the computer programme based on the results of chapters two through six is outlined. Some results produced by this program are then presented and compared with those of previous work.
CHAPTER 2

DERIVATION OF THE FREDHOLM INTEGRAL EQUATION

2.1 Overview

The computer modelling of electromagnetic data is difficult and for finite bodies has been confined to computing solutions to the forward problem. As such, the use of a modelling routine to simulate field results often requires that several models be run before an acceptable fit between the field data and computed results is obtained. It is therefore imperative that computer routines use as much data from previously computed models as possible, so that costly duplication of results is minimized. This last criterion is an important factor in determining the nature of the solution for a modelling programme.

Several approaches can be used for deriving the solution to the problem. It is conceptually possible, for example, to determine the Green's functions of any space however complex, and consequently the fields. Such an approach to the problem however, is likely to be difficult. Moreover, it is unlikely that results from one case could be retained for use in subsequent models. Such a direct method of solution is therefore not likely to produce viable algorithms for a
modelling routine.

An alternative is found by modelling the response of a single plate alone, and then combining the responses of the plate with other plates and with the stratified earth. The currents on each plate would then be considered to be unknown source distributions. The responses of plates could then be kept separate from effects external to each plate. This approach would then allow intermediate results to be preserved for future reference.

2.2 The Fredholm Equation for a Homogeneous Body

Consider a conductive body with uniform permeability and admittivity (conductivity and permittivity) residing in free space. Now assume that currents are flowing in the body due to the presence of an externally generated source current. The forces driving the source currents do not concern us. The total electric and magnetic fields, \( \mathbf{E} \) and \( \mathbf{H} \), that are energized by the source currents \( \mathbf{J}^* \) are given by Maxwell's equations:

\[
\nabla \times \mathbf{H} = \varepsilon \frac{\partial}{\partial t} \mathbf{E} + \mathbf{J}^* + \mathbf{J} \tag{2-1a}
\]

\[
-\nabla \times \mathbf{E} = \mu \frac{\partial \mathbf{H}}{\partial t} \tag{2-1b}
\]
Using phasor notation, these equations reduce to

\[ \nabla \times \mathbf{H} = y \mathbf{E} + \mathbf{J}^* \quad 2-1c \]

\[ \nabla \times \mathbf{E} = z \mathbf{H} \quad 2-1d \]

and \[ y = \sigma + i \omega \epsilon \quad z = -i \omega \mu \quad \mathbf{J} = \sigma \mathbf{E} \]

where \( y \) and \( z \) have different values inside and outside the conductor. Inside the conductor, properties are subscripted with a 'c' whereas outside it an 's' subscript is used.

The conductor is now replaced with an induced current distribution \( \mathbf{J}' \) which is used to represent its electromagnetic response. These currents only exist within the volume of the conductor and behave as though they are in a medium with properties \( y_s \) and \( z_s \). \( \mathbf{J}' \) is defined by

\[ \mathbf{J}' = \sigma_p \mathbf{E} \]

since the plate has free space permeability and permittivity and resides in a space of zero conductivity. The subscript 'p' refers to the conductor, which in this thesis is a plate.

Maxwell's equations now become:

\[ \nabla \times \mathbf{H} = y_s \mathbf{E} + \mathbf{J}' + \mathbf{J}^* \quad 2-1e \]
\[ \nabla \times E = Z_s H \] 2–1f

The curl of the second Maxwell equation and identity A–4 combine to produce:

\[ \nabla \times \nabla \times E = y_s Z_s E + z \mathcal{J} + z \mathcal{J}^* \] 2–2

Now, identity A–6 and equations 2–1e and 2–2 imply:

\[ -\nabla^2 E + \nabla \left( \nabla \cdot E \right) = y_s Z_s E + z \mathcal{J} + z \mathcal{J}^* \] 2–3

or

\[ \nabla^2 E + y_s Z_s E = \nabla \left( \nabla \cdot E \right) - z \mathcal{J} - z \mathcal{J}^* \] 2–4

The continuity equation for the total current,

\[ \nabla \cdot (y_s E + \mathcal{J} + \mathcal{J}^*) = 0 \] 2–5

is now used to evaluate the divergence of the electric field.

If there are no source currents in the body,

\[ \nabla \cdot (y_s E + \sigma_p E) = 0 \] 2–6
\[ u_s \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla y_s + \sigma_r \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \sigma_p = 0 \]

Since the space and the plate are homogenous, \( \nabla y_s = 0 \) and \( \nabla \sigma_p = 0 \). Therefore,

\[ \nabla \cdot \mathbf{E} = 0 \quad 2-7 \]

Equation 2-7 is an expression for the divergence of the electric field inside a homogenous body. Substituting 2-7 into equation 2-3 yields the Helmholtz equation for a homogenous body,

\[ \nabla^2 \mathbf{E} + u_s \mathbf{Z}_s \mathbf{E} = -\mathbf{Z}_s \left( \mathbf{J}' + \mathbf{J}^* \right) \quad 2-8 \]

This is integrated to produce the following solution for the electric field in free space,

\[ \mathbf{E}(\mathbf{R}') = -\mathbf{Z}_s \int_V G(\mathbf{R}, \mathbf{R}') \mathbf{J}'(\mathbf{R}) \, dV - \mathbf{Z}_s \int_V G(\mathbf{R}, \mathbf{R}') \mathbf{J}^*(\mathbf{R}) \, dV \quad 2-9 \]

where \( G \) is the Green's function for equation 2-8 with delta function current at \( \mathbf{R} \):

\[ \frac{1}{4\pi |\mathbf{R} - \mathbf{R}'|} \exp \left( \frac{iy_s \mathbf{Z}_s}{4\pi |\mathbf{R} - \mathbf{R}'|} \right) \]
and \[ R = (x, y, z), \quad R' = (x', y', z') \]

The primed coordinates represent the field point, and the unprimed coordinates the source point.

The term on the right represents the electric field due to the externally generated currents. Replacing this term with \( \mathbf{E}_o \) and recognizing that the lefthand side can be replaced by

\[ \mathbf{E}(R') = \frac{\mathbf{J}'(R')}{\sigma_p} \]

we get

\[ \frac{\mathbf{J}'(R')}{\sigma_p} + z_s \int_V G(R,R') \mathbf{J}'(R) \, dV = \mathbf{E}_o(R') \quad 2\text{-}10 \]

Equation 2–10 is a Fredholm integral equation for a body of constant conductivity, permeability and free space permittivity residing in a space of zero conductivity.

2.3 The Equation for a Thin Sheet

The general result of equation 2–10 must now be reduced to an equation representing the response of a thin sheet. To do this, equation 2–10 is rewritten for the case of a parallelepiped. This
equation is then reduced into one which is descriptive of a thin sheet by letting one of the dimensions of the parallelopiped approach a length of zero. For a parallelopiped of dimensions $2a_1$, $2a_2$, and $2a_3$, equation 2–10 is:

$$\int_{V'} \mathbf{J}(\mathbf{r}') \, dV + Z_s \int_{-a_3}^{a_3} \int_{-a_2}^{a_2} \int_{-a_1}^{a_1} \mathbf{G}(\mathbf{r},\mathbf{r}') \, \mathbf{J}(\mathbf{r}) \, dV = \mathbf{E}_o(\mathbf{r}')$$

2–10

If the sheet is sufficiently thin, the current density $\mathbf{J}$ becomes a surface current, $\mathbf{K}$. In the limit as $a_3$ goes to zero, then:

$$K_x = 2a_3J_x' \quad K_y = 2a_3J_y'$$

2–11

where $a_3$ must be much smaller than the skin depth $\delta$, since the current density is assumed to be constant across the thickness of the plate. i.e:

$$a_3 \ll \delta = \frac{l}{\omega \mu \sigma}$$

Inserting equations 2–11 into 2–10, and letting $a_3$ approach zero:

$$\frac{\mathbf{K}(\mathbf{r}')}{2a_3\sigma_o} + Z \int_{V'} \mathbf{G}(\mathbf{r},\mathbf{r}') \mathbf{K}(\mathbf{r}) \, dV = \mathbf{E}_o(\mathbf{r}')$$

2–12
Since $K$ is constant through the thickness of the plate, the integral in 2–12 becomes a surface integral, and the Green's function becomes:

$$g(r, r') = \frac{1}{4\pi ||r - r'||} \exp \left[-\sqrt{||z|| - ||r - r'||}\right], \quad C = (x, y).$$

To prevent the first term in equation 2–12 from approaching infinity during the collapse of the parallelepiped, the conductivity–thickness product of the plate, $s = 2a_3\sigma_o$, must remain finite as $a_3$ approaches zero. Equation 2–12 becomes

$$\frac{K(r')}{5} + z \int_A g(r, r') K(r) dA = E_o(r') \quad 2–13$$

Taking the curl of equation 2–13, we get

$$\nabla' \times K(r') + z \int_A \nabla' \times g(r, r') K(r) dA = \nabla' \times E_o(r') \quad 2–14$$

Because the current in the plate has zero divergence, the Helmholtz theorem implies that the current can be derived from a vector potential, $U$ in the following way,

$$K(r') = \nabla' \times U(r') = -\hat{a}_3 \times \nabla' U(r') \quad 2–15$$

where $U(r') = U(r)|\hat{a}_3$. 
since the current is confined to the surface of the plate.

Also:
\[ \nabla' \times K(r') g(r,r') = \nabla' g(r,r') \times K(r) + g(r,r') \nabla' \times K(r) \]

and
\[ \nabla' \times K(r) g(r,r') = -\nabla g(r,r') \times K(r) \quad 2-16 \]

since \( K \) is a function of \( \mathbf{r} \) alone, and:
\[ \nabla' g(r,r') = -\nabla g(r,r') \quad 2-17 \]

Substituting 2-15 into 2-16 we get
\[ \nabla' \times K(r) g(r,r') = -\nabla g(r,r') \times K(r) \quad 2-18 \]
\[ = (\nabla g \cdot \nabla u) \hat{a}_3 \]

An expression for the curl of the electric field can also be obtained:
\[ \hat{a}_3 \cdot (\nabla' \times \mathbf{E}(r')) = -j\omega \mu_0 \mathbf{H}(r') \cdot \hat{a}_3 \]

Using these results, equation 2-14 is transformed into an integral equation relating the current potential to the magnetic field (Annan, 1974):
\[ \nabla^2 u(r') + \mathbf{z} \int_A \nabla q \cdot \nabla u(r) \, dA = -z \mathbf{H}(r') \quad 2-19 \]

where \( \mathbf{H} \) is the magnetic field of the source over the plate.

Equation 2-19 is the induction equation, a Fredholm integral equation for the response of a conductive thin sheet to an externally generated magnetic field. In the following chapter, an outline of the method used to solve for the current distribution in the plate will be presented.
CHAPTER THREE

THE GALERKIN METHOD

3.1 Introduction

No analytical solutions have been found for the surface current in the induction equation 2-19. A numerical approach is therefore necessary. Many types of numerical solutions can be used to solve such equations. The most commonly used of these are the grid method, the finite element method and variational method. In all of these methods, the current must be determined from a set of predetermined basis functions.

Numerical methods can be classified according to whether the basis functions are local or global, and whether the governing equation is solved at each point or a functional is maximized or minimized. In the grid method the basis functions are defined and solved locally whereas the finite element method uses the variational approach on locally defined functions. Both of these methods have been used successfully to solve electromagnetic problems. An example of the application of the grid method to a plate in free space is given by Lamontagne (1970). Sylvester and Kisak (1975) described the application of the finite element method to magnetotelluric model-
In this thesis, the problem is solved by using a variational approach in which each trial function is defined over the surface of the plate. These functions must form an independent (although not necessarily orthogonal) basis. In addition, the trial functions should conform to the boundary conditions of the problem to ensure that the solution along the boundary is correct. Annan (1974) has successfully applied variational techniques to a number of electromagnetic models. One of these techniques, the Galerkin method, was applied to a conducting plate in free space and is also used in the present development. (A good reference is Mikhlin and Smolentsky, 1967)

3.2 Derivation of the Galerkin Method

The problem being tackled in this thesis is a boundary value problem, and as such it can be expressed in the form:

\[ \mathbf{A}u = f \]  \hspace{1cm} 3-1

where \( u \) and \( f \) are functions, and \( \mathbf{A} \) is an operator which maps \( u \) onto \( f \). In the Galerkin method, both \( u \) and \( f \) are in the same space. "\( u \)" is unknown, and in the present problem represents the surface current through \( \mathbf{K}^{*} \nabla \times \mathbf{U} \). The boundary condition on \( u \) is that \( \mathbf{K} \cdot \mathbf{n} \) vanishes on the edge of the plate, (ie. no current flows
out of the plate). This is equivalent to setting $u$ to a constant value along the edges.

The current potential is now expressed in terms of the trial functions:

$$V = \sum_{i=1}^{n} C_i \phi_i$$  

where $V$ is the approximate form of $u$ and converges to $u$ as the number of trial functions approach infinity. The set, $\{\phi_i\}$, is the set of trial functions, and $\{C_i\}$ is the set of unknown coefficients. (Note that trial function are the approximate forms of the current potential. The curl of the trial functions, or eigencurrents, are the approximate forms of the surface current.)

The following abbreviation is now used for the inner product between the elements in the set of trial functions:

$$\left(\phi_i, \phi_i\right) = \int_s \phi_i \phi_i d\mathbf{s}$$  

In the Galerkin method, the inner product of the difference between the function, $f$, and its approximation, $\hat{f}$, and each trial function is set to zero. Thus:

$$\left(\phi_i, \hat{f} - f\right) = 0$$  

3-4
This results in the set of linear equations:

\[
\left( \sum_{i=1}^{N} A C_i \phi_i, \phi_i \right) = \left( f, \phi_i \right) \tag{3-5}
\]

Condition 3–4 ensures that the error in the numerical solution,

\[ \epsilon = A V - f \]

is forced to be orthogonal to the trial functions, \( \phi \). Consequently for sufficiently many trial functions, the error in the solution can effectively be made as small as desired.

3.3 Application of the Galerkin Method to the Induction Equation

When 3–5 is applied to equation 2–19, the following results:

\[
-Z \int_{A} \phi_i(r) H_3(r) \, dA' = -\sum_{i=1}^{N} \frac{1}{2} \int_{A} C_i \phi_i(r) \nabla \phi_i(r) \, dA' + \sum_{i=1}^{N} \int_{A} \int_{A} \nabla q(r,r') \cdot \nabla C_i \phi_i(r) \phi_i(r') \, dA \, dA' \tag{3-6}
\]

Using equation 2–17, and integrating by parts changes the induction equation into:
\[-Z \int_{A} \phi(r') H_s(r') dA' = - \sum_{i=1}^{N} \frac{1}{2} \int_{A} C_i \nabla' \phi_s(r') \cdot \nabla' \phi_i(r') dA' \]
\[+ \sum_{m=1}^{N} C_i z \int_{A} g(r,r') \nabla' \phi_i(r') \cdot \nabla \phi_i(r) dA dA' \]

(Here the boundary condition, \( \phi(a) = 0 \), has been used). Equation 3-7 can now be written explicitly in matrix form,

\[
\left( \frac{1}{2} F - Z L \right) C = Z H \tag{3-8a}
\]

or

\[
Z C = Z H \tag{3-8b}
\]

where the matrices \( F, L \), and \( H \) are identified in Table 3-1, and the induction matrix, \( Z \), is defined by

\[
Z = \frac{1}{2} F - Z L
\]

The Galerkin method ensures that both the resistance matrix, \( F \), and the inductance matrix, \( L \), are positive definite symmetric matrices. The vector \( C \) is an unknown coefficient vector, with each coefficient corresponding to the amplitude of a trial function. \( H \) is a vector describing the interaction of the primary magnetic field with each trial function.
In the following chapter, a method for obtaining a solution to the matrix form of the induction equation will be outlined.
\[ F_{ij} = \int_{A'} \frac{\partial}{\partial x'} \phi_i(x', y') \frac{\partial}{\partial x} \phi_j(x', y') + \frac{\partial}{\partial y'} \phi_i(x', y') \frac{\partial}{\partial y} \phi_j(x', y') \ dx' \ dy' \]

\[ L_{ij} = \int_{A'} \int_{A} g(r, r') \left\{ \frac{\partial}{\partial x'} \phi_i(x, y) \frac{\partial}{\partial x} \phi_j(x, y) + \frac{\partial}{\partial y'} \phi_i(x, y) \frac{\partial}{\partial y} \phi_j(x, y) \right\} \ dx \ dy \ dx' \ dy' \]

\[ H_j = \int_{A'} \phi_j(x, y') H_3(x, y') \ dx' \ dy' \]
CHAPTER 4

THE SOLUTION OF THE INDUCTION EQUATION

4.1 Inversion of the Induction Matrix

A solution to the induction equation may be obtained as soon as the induction matrix is inverted. This can be done by using standard inversion techniques, and does not present a problem. However, a more useful inverse can be obtained by using the weighted eigenvector method described below (Annan, 1974). With this method, an inverse can be easily generated once certain eigenvectors and eigenvalues have been obtained. These data are only a function of the dimensions of the plate, and as such they are independent of its conductivity and permeability, and the frequency of the impressed field. By saving the eigenvectors and eigenvalues, a substantial amount of computation can be saved the next time a plate of the same dimensions is modelled. This method of inversion thus allows many problems to be solved both cheaply and quickly.

The induction equation, (3-8), is

\[ \frac{Z}{C} = z H \]  \hspace{1cm} 4-1
where
\[
\mathbf{Z} = \frac{1}{S} \mathbf{E} - \mathbf{Z}
\]

Examination of table 3–1 reveals that both \( \mathbf{E} \) and \( \mathbf{Z} \) are symmetric, and therefore that \( \mathbf{Z} \) is also symmetric. Moreover, the Galerkin method ensures that \( \mathbf{E} \) is positive definite (Mikhlin and Smolentsky, 1967), and thus permits \( \mathbf{Z} \) to be inverted by using the weighted eigenvector method. In order to use this method, the equation 4–1 is transformed so that the resistance matrix, \( \mathbf{E} \), becomes the identity matrix. The transformation is accomplished by first diagonalizing with the matrix of the eigenvectors of \( \mathbf{E} \), and then normalizing the elements of the diagonalized matrix with the associated eigenvalues. If \( \mathbf{Y}_i \) is an eigenvector of \( \mathbf{E} \), ie:

\[
\mathbf{E} \mathbf{Y}_i = f_i \mathbf{Y}_i
\]

then

\[
\mathbf{Y}^T \mathbf{E} \mathbf{Y} = \mathbf{D}(f_i)
\]

where \( \mathbf{Y} \) is the matrix which has the eigenvectors of \( \mathbf{E} \) as columns and \( \mathbf{D}(f_i) \) is a diagonal matrix composed of the eigenvalues. Therefore

\[
\mathbf{E}' = \mathbf{D}^{\frac{1}{2}}(f_i) \mathbf{Y}^T \mathbf{E} \mathbf{Y} \mathbf{D}^{\frac{1}{2}}(f_i) = \mathbf{I}
\]

The prime indicates that the matrix is written in the new
coordinate system. For $D^x(f_i)$ to be real, the eigenvalues of $\mathbf{P}$ must be positive, and $\mathbf{P}$ is thus required to be positive definite.

The same transformation is now applied to the reactance matrix, $L$, to yield $L'$:

$$L' = D^x(f_i)\nu^T L \nu D^x(f_i)$$  \hspace{1cm} 4-3

Therefore the transformed induction matrix, $Z'$, becomes

$$Z' = \frac{1}{5} I + ZL'$$  \hspace{1cm} 4-4

Now if vectors $C_n$ are defined in the following fashion, they can be used to find $Z''$

$$Z C_n = \lambda_n C_n$$  \hspace{1cm} 4-5

Equation 4–5 is the weighted eigenvector equation. In the primed coordinate system, equation 4–5 becomes:

$$\left(\frac{1}{5} I + ZL'\right)C'_n = \lambda_n I C'_n$$  \hspace{1cm} 4-6

where

$$C'_n = \nu D^x(f_i) C_n$$
has been used. The transformed weighted equation reduces to

\[ Z \mathcal{L} \mathcal{C}_n' = (\lambda_n - \frac{1}{S}) \mathcal{C}_n' \]  

alternately

\[ \mathcal{L} \mathcal{C}_n' = -\frac{1}{Z} \left( \frac{1}{S} - \lambda_n \right) \mathcal{C}_n' \]  

From this equation, it is clear that the \( \{c_n'\} \) are the eigen-vectors of \( \mathcal{L}_n \), and that the eigenvalues \( \ell_n' \) of \( \mathcal{L}_n \) are

\[ \ell_n' = -\frac{1}{Z} \left( \frac{1}{S} - \lambda_n \right) \]  

so that

\[ \lambda_n = \frac{1}{S} + Z \ell_n' \]  

If we use

\[ \mathcal{C}_n' = D^{\frac{1}{2}}(f) \mathcal{V}^T \mathcal{C}_n \]

equation 4–7b can be transformed into the unprimed (original) coordinate system:

\[ \mathcal{L} \mathcal{C}_n = \ell_n' \mathcal{V} D^{\frac{1}{2}}(f) D^{\frac{1}{2}}(f) \mathcal{V}^T \mathcal{C}_n \]  

4–9
Multiplying 4–9 by $C^\mathbf{m}$ produces:

$$C^\mathbf{m} L C^\mathbf{n} = \ell' \mathbf{v} \mathbf{D}^\mathbf{y}(f^I) \mathbf{D}^\mathbf{y}(f^J) \mathbf{v}^T C^\mathbf{n} \quad 4–10a$$

or

$$C^\mathbf{m} \mathbf{L} C^\mathbf{n} = \ell' \mathbf{v} C^\mathbf{m} C^\mathbf{n} = \ell' \delta_{MN} \quad 4–10b$$

therefore $C^\mathbf{m} \mathbf{L} C^\mathbf{n}$ is a diagonal matrix with eigenvalues of $\ell'$ as elements. Similarly:

$$C^\mathbf{m} \mathbf{F} C^\mathbf{n} = \delta_{MN} \quad 4–11$$

so that

$$C^\mathbf{m} \mathbf{Z} C^\mathbf{n} = \lambda \delta_{MN} \quad 4–12a$$

Therefore

$$C^\mathbf{m} \mathbf{Z} C^\mathbf{n} = \mathbf{D}(\lambda \mathbf{n}) \quad 4–12b$$

and $\mathbf{Z}$ can be written as:

$$\mathbf{Z} = (C^\mathbf{n})^T \mathbf{D}(\lambda \mathbf{n}) C^\mathbf{n} \quad 4–13$$
The inverse of 4–13 yields the inverse of $Z$, and consequently the solution of the induction equation is obtained.

$$Z^{-1} = C_n D \left( \frac{1}{\lambda_n} \right) C_n^T$$  \hspace{1cm} 4–14

and

$$C = Z C_n D \left( \frac{1}{\lambda_n} \right) C_n^T H$$  \hspace{1cm} 4–15

Equation 4–15 provides a solution for the eigencurrent excitation coefficients, $C$, due to the action of any magnetic field.

The solution to the induction equation, 4–15, is much more appealing than that resulting from a simple inversion of the matrix. This is because the electrical and geometrical properties of the plate are now decoupled. Changes in the plate not involving a change of scale now do not require a completely new inversion of $Z$, thus saving an expensive and time consuming procedure. Moreover, by storing the vectors $C_n$ and the eigenvalues $\lambda_n$, the solution to the induction equation for specific plate dimensions can be stored compactly. In effect then, only one inversion needs to be done for a plate of a given size and number of trial functions. This result is completely independent of the primary field.

4.2 Implementation of the Solution
In principle, any set of independent functions satisfying the boundary conditions for the current can be used. Practically, however, it is best to choose functions which are easily computed, and which converge to the solution fairly quickly. The functions used by Annan (1974) were chosen because they represent the current along the edges of the plate well. They were also used in the present work as they have been extensively well tested. The trial functions are:

\[ y_i = (1 - \chi^2)(1 - \xi^2) \Gamma_n(\chi) \Gamma_m(\xi) \]  

where: \( \Gamma_p \) is the Chebychev polynomial of order \( p \), \( a_i \) is half of the dimension of the plate in the \( \hat{x}_i \) direction and

\[ \chi = \frac{x}{a_1}, \quad \xi = \frac{y}{a_2} \]

\( y_i \) is the trial function corresponding to the Chebychev polynomial of order \( n \) in \( x \) and \( m \) in \( y \).

There is no reason why these functions cannot be replaced with an alternate set of trial functions. These alternate functions could be chosen so that they approximate the current in other parts of the plate better than the functions given above do. Such functions would be used at the expense of representing the current well along all of the edges of the plate. The benefits of replacing the
trial functions in 4–16 would depend on the problem being solved. For example, when a large suboutcropping steeply dipping conductor is close to a transmitter, the current distribution along the upper edge of the conductor is likely to be rapidly varying, whereas the current along the lower edge is likely to be well behaved. The trial functions, 4–16, do not approximate this situation well in that they model current along all edges of the plate equally well. A better set of trial functions for this problem would be ones which would represent the current flow along one edge of the plate at the expense of representing the current elsewhere. Such a set of functions would ensure faster convergence for problems in which current flow is stimulated along one side of the plate, and hence would produce a cheaper solution.
CHAPTER 5

THE EFFECT OF A NON–UNIFORM EXTERNAL MEDIUM

5.1 Introduction

In the previous chapters, a method has been outlined for determining the EM response of a plate in free space. In this chapter, this solution will be extended into one for a plate in non-uniform media. Specifically, for a plate in residing in a stratified space and in the presence of other plates.

The effect of non-uniform media can be accounted for in two ways: The Green's function in the reactance matrix can be altered to reflect the properties of the new space, or the primary field, \( H_3 \), can be changed to account in some manner for the change in the self-interaction of the eigencurrents i.e. a magnetic field component may be introduced due to reflection of the field of the eigencurrents resulting from inhomogeneities in the medium. Both methods are equivalent, however the latter method is more convenient from a computational point of view. This is particularly true for the Green's functions for a stratified space as they cannot be computed with an accuracy much greater than one part in ten thousand and it is helpful to divide them into direct and scattered parts. This method also al-
lows the trial functions in multiple plate systems to be located in each plate separately. This has an advantage in that the currents represented by high order polynomials can be confined to physically important regions, rather than being disseminated amongst several plates.

We therefore choose to modify the field exciting the eigen-currents and hence the vector $\vec{H}$. From table 3-1,

$$H_j = \int_S \phi_j (\vec{H} \cdot \hat{a}_j) \, ds$$

$$= \int_S \phi_j H_j \, ds$$

where $H_j$ is the total field on the plate. Now, $H_j$ can be divided into a component produced by the source field, $H_j^s$, a component due to the self-interaction of a medium different from free space, $H_j^a$, and a component due to the presence of other plates, $H_j^p$. Therefore

$$H_j = H_j^s + H_j^a + H_j^p$$

The component due to the source field, $H_j^s$, is similar to the formulation in table 3-1

$$H_j^s = \int_S \phi_j H_j^{\text{Source}} \, ds$$

The derivation of the other components of $H_j$ will be outlined in the following sections. It should be noted that in the fol-
lowing sections, coordinate subscripts are in the frame of the plate being examined.

5.2 The Effect of a Non-Uniform Medium

The component of the source vector due to the self interaction of a plate in a non-uniform medium, \( H_j^s \), is

\[
H_j^s = \int_S \phi_j H^s_{\phi} \, ds
\]

where \( H^s_{\phi} \) is the difference between the magnetic field normal to the plate when the plate is not in free space and when it is in free space. The component of \( H_j^s \) due to eigencurrent \( i \) is

\[
H_{3i}^s = c_i \int_{S'} \left[ J_{x_i} \tilde{G}_{3i}^s + J_{z_i} \tilde{G}_{3z}^s \right] \, ds'
\]

where \( \tilde{G}_{3i}^s \) is the difference in Green's functions produced by a point electric current density of unit amplitude in the \( \hat{x}_k \) direction between currents situated in free space and in a layered earth, \( J_{x_i} \) is the \( k \) component of the \( i \) electric eigencurrent (of unit strength), and \( c_i \) is the excitation coefficient of eigencurrent \( i \). Now

\[
\bar{J}_i = \hat{e}_\alpha \times \left( u_{x_i}, u_{y_i}, 0 \right), \quad u_k = \frac{\partial}{\partial x_k}
\]
so that

\[ \mathbf{J}_i = (u_x, -u_y, 0) \]

Expressions for \( U_x \) and \( U_y \) are listed in Table 5-1. Now, \( H_{3i}^\alpha \) is expressed as

\[ H_{3i}^\alpha = c_i \int_{s'} u_{yi} G_{3i}^s - u_{xi} G_{3x}^s \, ds' \quad 5-3 \]

or

\[ H_{3i}^\alpha = \mathbf{c} \cdot \mathbf{A} \quad 5-4 \]

where

\[ A_i(x,y) = \int_{s'} u_{yi} G_{3i}^s - u_{xi} G_{3x}^s \, ds' \quad 5-5 \]

(Note the \( \mathbf{A} \) in 5-4 is not the same as the operator \( \mathbf{A} \) in 3-1.) Therefore

\[ \mathbf{H}_i = \int_{S} \phi_i(x,y) \mathbf{c} \cdot \mathbf{A}(x,y) \, dx \, dy \]

and

\[ \mathbf{H} = \sum_{i} \mathbf{S} \mathbf{C} \quad \text{where} \quad S_{ij} = \int_{S} \phi_i A_i \, ds \quad 5-6 \]

A non-uniform medium is accounted for by constructing a matrix which corrects for the difference between the Green's functions.
TABLE 5-1

THE DISTRIBUTION OF CURRENT OVER THE PLATE

\[ U_x = \frac{1}{a_x} \left( 1 - \frac{x^2}{a_x^2} \right) T_n \left( \frac{x}{a_x} \right) \left\{ \left( 1 - \frac{y^2}{a_y^2} \right) T'_m \left( \frac{y}{a_y} \right) - 2 \frac{y}{a_y} T_m \left( \frac{y}{a_y} \right) \right\} \]

\[ U_y = \frac{1}{a_y} \left( 1 - \frac{y^2}{a_y^2} \right) T_m \left( \frac{y}{a_y} \right) \left\{ \left( 1 - \frac{x^2}{a_x^2} \right) T'_n \left( \frac{x}{a_x} \right) - 2 \frac{x}{a_x} T_n \left( \frac{x}{a_x} \right) \right\} \]

Where, \( i = \frac{(n+m+1)(n+m+2)}{2} \)

\[ a_x \text{ and } a_y \text{ are the half dimensions of the plate in the x and y directions} \]

\( T_k(z) \) is the Chebychev polynomial of order \( k \) evaluated at \( z \), and

\( T'_k(z/b) \) is the derivative with respect to \( z \) of the Chebychev polynomial of order \( k \) evaluated at \( z/b \).
of a uniform and non-uniform space. This matrix acts on the coefficients of the eigencurrents and thus modifies the current distribution within the plate.

5.3 The Effect of Other Plates

In this section, the effect of currents flowing in one plate on the current distribution in another will be considered. However, the solution obtained by this method can be generalized any set of multiple plates. This generalization will be dealt with in the following section.

The component of $\mathbf{H}$ considered here is $H^p_j$, and is defined by

$$H^p_j = \int \phi_i H^\text{PLATE}_3 \, dS \quad 5-7$$

This integral is defined over the plate upon which the magnetic field is acting. This plate is labelled the receiver plate, and is referred to by the superscript $R$. The other plate, the one that is transmitting the magnetic field, is denoted by the superscript $T$. In this notation, equation 5–7 becomes

$$\mathbf{H}^p = \int_{S_T} \phi_i \mathbf{H}_3 \, dS \quad 5-8$$
Now, the explicit expression for $^T \mathbf{H}_3$ is

\[ ^T \mathbf{H}_3 = \sum_{i=1}^{N} \tau C_i \int_{S} \left\{ ^T \mathbf{J}_i \mathbf{G}_{x_i} + ^T \mathbf{J}_2i \mathbf{G}_{y_2i} \right\} d^R S \tag{5-9} \]

where $^T S$ is the surface of the transmitter plate, $^T \mathbf{J}_k$ is the $k^{th}$ component of the eigencurrent on the transmitter plate of order $i$ with an excitation coefficient of unity, and $^T \mathbf{G}_{jk}$ is the vertical component of the magnetic field on the receiver plate due to a unit current dipole in the direction $x$ on the transmitter plate. $^T \mathbf{G}$ is a Green's function, and not a specific magnetic field.

Equation 5–9 is now rewritten using the notation of the previous section

\[ ^T \mathbf{H}_3 = \int_S \tau u_y \mathbf{G}_{y_2i} - \tau u_x \mathbf{G}_{x_i} d^R S \tag{5-10} \]

or

\[ ^T \mathbf{H}_3 = C_i \times \mathbf{A}_i \tag{5-11} \]

where

\[ \mathbf{A}_i = \int_S \tau u_y \mathbf{G}_{y_2i} - \tau u_x \mathbf{G}_{x_i} d^R S \tag{5-12} \]

Therefore, the $^j$th component of the vector $^T \mathbf{H}$ is
\[ r^{\mathbf{H}}_{ij} = \int_{S}^{k} \phi_{i}^{\mathbf{r}} C \cdot A_{i} d^{k}S \]  

5-13

which in matrix notation becomes

\[ r^{\mathbf{H}} = W^{\mathbf{r}} C \]  

5-14

where

\[ r^{\mathbf{W}}_{ij} = \int_{S}^{k} \phi_{i}^{\mathbf{r}} A_{i} d^{k}S \]

As in the case of a non-uniform space, the current distribution in a plate is modified by the presence of other plates. Moreover, the effect of an eigencurrent in a plate on an eigencurrent in a different plate, is the same as the effect of the latter eigencurrent on the former one. This symmetry is proven in Appendix D, and allows the solution of the multiple plate problem to be determined by using the weighted eigenvector method.

In the following section, the induction equation for one or more plates in a uniform or non-uniform space is derived.

5.4 Solutions for an Inhomogenous Space

5.4.1 The Single Plate in a Non-Uniform Space:
From equation 4–15, we have

\[ C = \sum C_n \hat{D}(\lambda) C_n^T H \]  \hspace{1cm} 5–15

which is abbreviated to

\[ C = MH \]  \hspace{1cm} 5–16

where

\[ M = \sum C_n \hat{D}(\lambda) C_n^T \]

Using the fact that

\[ H = H^\Lambda + H^\Sigma \]  \hspace{1cm} 5–17

where, from equation 5–6

\[ H^\Lambda = SC \]  \hspace{1cm} 5–6

equation 5–15 is now written as

\[ IC = MH^\Lambda + MSC \]  \hspace{1cm} 5–18

and the following solution for the excitation coefficients is obtained

\[ C = (I - MS)^T MH^\Sigma \]  \hspace{1cm} 5–19
Note that \((I - M_Z)^n\) represents the change in the inverse of \(Z\) caused by using the Green's function for an inhomogeneous space instead of the free space Green's function.

5.4.2 The Solution for Two Plates in Free Space:

In this section, the plates are labelled \(P\) and \(Q\). Using equation 5–16, the solution for the excitation coefficients of plate \(P\) is:

\[
C^p = M^p \Sigma^p + M^p H^{qp}
\]

And for plate \(Q\) is:

\[
C^q = M^q \Sigma^q + M^q H^{pq}
\]

Substituting from equation 5–14:

\[
H^{qp} = W^{qp} C^q
\]

\[
H^{pq} = W^{pq} C^p
\]

So that the excitation coefficients for plates \(P\) and \(Q\) become:
\[ C^p = M^p H^{\Sigma p} + M^p W^{q^p} C^q \quad 5-23a \]
\[ C^q = M^q H^{\Sigma q} + M^q W^{p^q} C^p \quad 5-23b \]

Rewriting equations 5-23

\[ M^p H^{\Sigma p} = C^p - M^p W^{q^p} C^q \quad 5-24a \]
\[ M^q H^{\Sigma q} = -M^q W^{p^q} C^p + C^q \quad 5-24b \]

Equations 5-24 can be combined to produce the following matrix equation which can be subsequently solved for the excitation coefficients:

\[
\begin{bmatrix}
M^p & 0 \\
0 & M^q
\end{bmatrix}
\begin{bmatrix}
H^{\Sigma p} \\
H^{\Sigma q}
\end{bmatrix} =
\begin{bmatrix}
I & -M^p W^{q^p} \\
-M^q W^{p^q} & I
\end{bmatrix}
\begin{bmatrix}
C^p \\
C^q
\end{bmatrix} \quad 5-25
\]

5.4.3 The Solution of the Multiple Plate Problem in a Non-uniform Space

If the equations 5-20 and 5-25 are combined, the solution for two plates in a non-uniform space is obtained:
\[
\begin{bmatrix}
M^p Q^p & H^p \\
O & M^q H^q
\end{bmatrix} = \begin{bmatrix}
I - M^p S^p & -M^p W^q \\
M^q W^p & I - M^q S^q
\end{bmatrix} C^p C^q
\]

The matrix on the right side of the equation is composed of blocks constructed of mutual and self interaction matrices for each of the plates. The size of these blocks is determined by the number of eigencurrents used on each plate. This allows great flexibility in choosing how accurately the current system on each plate is to be represented, and in turn, allows one to include only the currents critical to the problem at hand. This flexibility would not be present if a simple set of eigencurrents were distributed across more than one plate. The generalized form of equation 5–26 for many plates in a non-uniform space is outlined in table 5–2.

By inverting the matrix on the right side of 5–26 and multiplying it by the matrix on the left side, a matrix is obtained which is descriptive of the response of a distribution of plates in a stratified space. Once this "system response matrix" is determined, the current distribution on the plates can be found by generating the concatenated vector, \( H \), in equation 5–26 and multiplying it by the system response matrix. The response of the system of plates can then be found by evaluating the fields resulting from the current distribution on the plates.
The Solution for Many Plates in a Non-Uniform Medium

<table>
<thead>
<tr>
<th>...</th>
<th>...</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

TABLE 5-2
CHAPTER 6

GREEN'S FUNCTIONS IN A STRATIFIED EARTH

6.1 Overview

The method used here to derive the Green's functions in a layered Earth is based on a method outlined by West (1980). It produces results for electric and magnetic dipoles located at any position within the layered Earth. Previous solutions for the response of a stratified medium were either confined to a few layers or to the response of dipoles located in the half space overlying the Earth (Wait, 1971; Lajoie, Alfonso-Roche and West, 1975).

The geometry of this problem is depicted in figure 6-1. A right handed coordinate system is used, with the positive z axis perpendicular to the strata and pointing upwards. Layers are labelled from the top down, with the first layer being the upper half space and the n being the lower half space. The index, i, is used to count the layers. There are n-1 interfaces in the problem, with each interface having the index of the layer above it and lying at a depth of \( d_i \). Depths are positive, i.e., the interfaces are located at \( z = -d \). The thickness of each layer, \( t_i \), is therefore \( d_i - d_{i+1} \).

The solution involves the use of Hertz potentials in the Fourier transform domain. The Hertz potentials are used in this
Figure 6-1: Exploded View of the Stratified Earth
problem because their use reduces the the number of variables to two unknown scalar quantities– the z components of the electric and magnetic Hertz potentials. These potentials are examined in further detail in sections 6.3, 6.4 and 6.5.

In order to solve the problem, the Fourier transformed free space potentials at the source level are found in terms of the electric and magnetic current dipoles following Lajoie(1973) and Weaver(1970). The secondary field potentials are then calculated by using boundary conditions imposed by the strata on the potentials. Using the solution of the Helmholtz equation and boundary conditions on the potentials, the source potentials are transformed into the potentials at the field point. The Fourier transformed electric and magnetic fields are then computed from the Hertz potentials. The solution is then converted into the radial wave number domain and then into the space domain using the fast Hankel transform method of Anderson (1979).

6.2 Notation

Several notational conventions are used here to solve for the electric and magnetic fields in a layered Earth. The solution of the problem is facilitated by separating the potentials into upwardly and downwardly propagating components because these components are attenuated by propagation in different directions (refer to Appendix E ).
To denote this, field symbols are superscripted with a '+' and '-' to indicate upward and downward propagating fields respectively. Unsuperscripted field variables are the sum of both of the superscripted components. The same superscripts are used on the symbol for depth, d. In this case they imply the following limits:

\[ d_+^i = \lim_{\epsilon \to 0} (d_i + \epsilon) \]
\[ d_-^i = \lim_{\epsilon \to 0} (d_i - \epsilon) \]

Subscripts are also used. Those appearing on the left side of a symbol specify the directional component of a vector, with unsubscripted vectors being in the vertical direction. Subscripts on the right side indicate partial differentiation in the direction indicated by the subscript, except for unit vectors where right side subscripts indicate direction. Letter subscripts 'f' and 's' refer to the field and source points respectively, and 'i' to any layer.

The symbols \( \sim \) and \( \triangledown \) above a variable refer to the Cartesian wave number and radial wave number domains respectively. An absence of either symbol indicates an untransformed value.

6.3 The Hertz Potentials

The Hertz potentials are a pair of vector potentials that are convenient to use in the stratified earth problem. The first poten-
tial, $\Pi$, is associated solely with electrical sources and the second, $\vec{J}$, is produced by magnetic sources (equivalently, divergence-free electrical sources). These potentials are easily derived from Maxwell's equations (Ward, 1967), and are related to the electric and magnetic fields by

$$\vec{E} = \nabla \times \nabla \times \Pi - \nabla \times \vec{J} \quad 6-1a$$

$$\vec{H} = \nabla \times \nabla \times \vec{J} + \nabla \times \Pi \quad 6-1b$$

Because the stratified earth has symmetry, all three components of the Hertz potentials are not required in the present problem. It is sufficient to represent the fields either in terms of the radial component or the vertical component of the potentials. In this development, the latter components are used. Equations 6–1 become

$$\vec{E} = \left( \begin{array}{c} \Pi_{13} - Z \Pi_{22} \\ \Pi_{23} + Z \Pi_{12} \\ -\Pi_{11} \end{array} \right) \quad 6-2a$$

$$\vec{H} = \left( \begin{array}{c} \Pi_{13} + Y \Pi_{22} \\ \Pi_{23} - Y \Pi_{12} \\ -\Pi_{11} \end{array} \right) \quad 6-2b$$

The Hertz potentials are also defined in terms of sources through the Helmholtz equation,
\[ \nabla^2 \mathbb{\Pi} - K^2 \mathbb{\Pi} = -\mathbb{J} / \gamma \]  
\[ \nabla^2 \mathbb{\Gamma} - K^2 \mathbb{\Gamma} = \mathbb{M} / \mathbb{Z} = \mathbb{\mathcal{M}} \]

where \( \mathbb{\mathcal{M}} \) is the magnetization. The Helmholtz equation can be solved by transforming it into the Fourier wavenumber domain (refer to Appendix E). This results in solutions of the form

\[ \hat{F}(p, q, z) = \hat{F}(0, 0, 0) e^{-ipx} e^{-iqy} e^{-\gamma Dz} \]

where \( p \) and \( q \) are Fourier wavenumbers in \( x \) and \( y \),
\[ \gamma = (\lambda^2 + K^2)^{1/2} \]
\[ \lambda^2 = p^2 + q^2 \]
\( \hat{F} \) is either \( \hat{\Pi} \) or \( \hat{\Gamma} \),

and \( D = +1 \) for waves travelling in the positive \( z \) direction and \( -1 \) for waves travelling in the negative \( z \) direction.

The \( z \) derivatives of \( F \) are therefore given by

\[ \frac{\partial}{\partial z} \hat{F} = -\gamma D \hat{F} \]

The Hertz potentials are now split into upward and downward propagating components, \( \hat{F}^+ \) and \( \hat{F}^- \). Transforming equations 6–2 into the Fourier domain, the following equations for \( \hat{\Pi} \) and \( \hat{\Gamma} \) in terms of \( \hat{\Pi} \) and \( \hat{\Gamma} \) are obtained.
\[ \hat{E} = \{ i \eta \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right) + z q \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right), \]
\[ i q \eta \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right) - z i \rho \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right), \]
\[ \lambda^2 \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right) \} \]

\[ \hat{\mathbf{A}} = \{ i \eta \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right) - y i q \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right), \]
\[ i q \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right) + y i \rho \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right), \]
\[ \lambda^2 \left( \hat{\tau}_{+3} \hat{\tau}_{-3} \right) \} \]

Using the results of equation 6-4, it is possible to calculate the potentials at any point in a homogeneous medium provided the potentials at some other point in the medium are known. This is done with the following matrix operation

\[
\begin{bmatrix}
\hat{\mathbf{E}}^+ \\
\hat{\mathbf{F}}^+ (\rho, q)
\end{bmatrix} =
\begin{bmatrix}
\mathbf{e}^{-q(z_s-z_t)} & 0 \\
0 & \mathbf{e}^{-q(z_s-z_t)}
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{E}}^- \\
\hat{\mathbf{F}}^- (Z_s)
\end{bmatrix}
\]

where \( z_s \) is the \( z \) coordinate where the potentials are known, and \( z_t \) is the \( z \) coordinate where the potentials are to be found. Using these relationships, it is also possible to find the potentials at one side of a layer if the potentials on its other side are known. Results of this process are given in table 6-1.

To find the potentials in a medium different from the one in
TABLE 6-1

CONTINUATION BETWEEN THE TOP AND BOTTOM OF A LAYER

UPWARD PROPAGATION:

\[
\begin{bmatrix}
\hat{F}^+(z = -d_{i-1}) \\
\hat{F}^-(z = -d_i)
\end{bmatrix} =
\begin{bmatrix}
e^{-niti} & 0 \\
0 & e^{niti}
\end{bmatrix}
\begin{bmatrix}
\hat{F}^+(z = -d_i) \\
\hat{F}^-(z = -d_i)
\end{bmatrix}
\]

DOWNWARD PROPAGATION:

\[
\begin{bmatrix}
\hat{F}^+(z = -d_i) \\
\hat{F}^-(z = -d_i)
\end{bmatrix} =
\begin{bmatrix}
e^{niti} & 0 \\
0 & e^{-niti}
\end{bmatrix}
\begin{bmatrix}
\hat{F}^+(z = -d_i) \\
\hat{F}^-(z = -d_i)
\end{bmatrix}
\]
which they are known, the boundary conditions on and must be used. These conditions are derivable from the following boundary conditions on the electric and magnetic fields

\[ J _ { d _ i } = J _ { d _ i } \quad H _ { d _ i } = H _ { d _ i } \quad H _ { d _ i } = H _ { d _ i } \]

and

\[ B _ { d _ i } = B _ { d _ i } \quad E _ { d _ i } = E _ { d _ i } \quad E _ { d _ i } = E _ { d _ i } \]

The boundary conditions on the Fourier transformed Hertz potentials can then be derived by using equations 6–5:

\[ y _ i ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) = y _ i ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) = Z _ { i n } ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) = Z _ { i n } ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) \]

\[ j \hat{F} _ { d _ i } = j \hat{F} _ { d _ i } \quad j \hat{F} _ { d _ i } = j \hat{F} _ { d _ i } \]

Using \( \frac{\partial}{\partial z} = - \eta \partial \), the last two boundary conditions become:

\[ \eta ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) = \eta ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) \quad \eta ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) = \eta ( \hat{F} _ { d _ i } \hat{F} _ { d _ i } ) \]

These equations can be used to solve for the potentials on one side of an interface in terms of the potentials on the other side of it. (Refer to table 6–2 for results.) When these results are combined with the results of table 6–1, the potentials anywhere in the layered earth can be found provided that the potentials at some other
TABLE 6-2

PROPAGATION MATRICES FOR INTERFACES

of the form: \( \hat{\mathbf{F}}_d^a = S^a \hat{\mathbf{F}}_d^c \), \( \hat{\mathbf{F}}_d^a = S^a \hat{\mathbf{F}}_d^c \)

\[
\begin{bmatrix}
\hat{F}_d^a^+ \\
\hat{F}_d^a^-
\end{bmatrix} = \begin{bmatrix}
(Y_i^+ N_i^+)/2 & (Y_i^- N_i^-)/2 \\
(Y_i^+ N_i^-)/2 & (Y_i^- N_i^+)/2
\end{bmatrix}
\begin{bmatrix}
\hat{F}_d^a^- \\
\hat{F}_d^a^+
\end{bmatrix}
\]

\( Y_i^+ = y_{in}/y_i \)
\( Y_i^- = y_i/y_{in} \)
\( N_i^+ = \eta_i/\eta_{in} \)
\( N_i^- = \eta_{in}/\eta_i \)

of the form: \( \hat{\mathbf{F}}_d^a = Q^a \hat{\mathbf{F}}_d^c \), \( \hat{\mathbf{F}}_d^a = Q^a \hat{\mathbf{F}}_d^c \)

\[
\begin{bmatrix}
\hat{F}_d^a^+ \\
\hat{F}_d^a^-
\end{bmatrix} = \begin{bmatrix}
(Z_i^+ N_i^+)/2 & (Z_i^- N_i^-)/2 \\
(Z_i^- N_i^-)/2 & (Z_i^+ N_i^+)/2
\end{bmatrix}
\begin{bmatrix}
\hat{F}_d^a^- \\
\hat{F}_d^a^+
\end{bmatrix}
\]

\( Z_i^+ = z_{in}/z_i \)
\( Z_i^- = z_i/z_{in} \)

\[
\begin{bmatrix}
\hat{F}_d^a^+ \\
\hat{F}_d^a^-
\end{bmatrix} = \begin{bmatrix}
(Z_i^+ N_i^+)/2 & (Z_i^- N_i^-)/2 \\
(Z_i^- N_i^-)/2 & (Z_i^+ N_i^+)/2
\end{bmatrix}
\begin{bmatrix}
\hat{F}_d^a^- \\
\hat{F}_d^a^+
\end{bmatrix}
\]
point are known.

From the Helmholtz equation, it is also possible to derive the potentials of the primary fields at the source level due to a dipole current. Writing equation 6-3 in generalized notation,

\[ \nabla^2 \mathcal{F} - \kappa^2 \mathcal{F} = S \]

where \( S = -\mathcal{I}_0 \) if \( \mathcal{F} = \mathcal{I} \) and \( S = -\mathcal{M}/2 \) if \( \mathcal{F} = \mathcal{M} \). In the Fourier domain, the solution for \( \mathcal{F} \) at the source point is as follows

\[ \kappa \hat{\mathcal{F}} = \frac{1}{\lambda \eta} \kappa \hat{S} \]  

6-7

A cusp is therefore present in the primary potential at the source level. This prevents the potential from being continued from one side of the source level to the other. The steps leading to equation 6-7 are detailed in appendix F.

Equation 6-6 is now used in conjunction with 6-7 to obtain an expression for the \( \hat{\vec{F}} \) in terms of all of the components of the source dipole. Since the \( z \) components of \( \mathcal{E} \) and \( \mathcal{H} \) must be the same irrespective of the potentials from which they are derived,

\[ \hat{\vec{\mathcal{E}}} (\hat{\vec{n}}, \hat{\vec{r}}) = \hat{\vec{\mathcal{E}}} (\hat{\vec{n}}, \hat{\vec{r}}, \hat{\vec{n}}, \hat{\vec{r}}, \hat{\vec{n}}, \hat{\vec{r}}) \]  

6-8a
THE PRIMARY HERTZ POTENTIALS AT THE SOURCE LEVEL

\[
3 \hat{F}_p^+ = \frac{-i \rho \cdot \jmath}{2 \gamma \lambda^3} - \frac{i q \cdot \jmath}{2 \gamma \lambda^3} - \frac{3 \jmath}{2 \gamma \eta} - \frac{i \rho \cdot \jmath}{2 \gamma \eta} \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta}
\]

\[
3 \hat{F}_p^- = \frac{i \rho \cdot \jmath}{2 \gamma \lambda^3} + \frac{i q \cdot \jmath}{2 \gamma \lambda^3} - \frac{3 \jmath}{2 \gamma \eta} - \frac{i \rho \cdot \jmath}{2 \gamma \eta} \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta}
\]

\[
3 \hat{F}_p^+ = \frac{-i \rho \cdot \jmath}{2 \gamma \lambda^3} - \frac{i q \cdot \jmath}{2 \gamma \lambda^3} - \frac{3 \jmath}{2 \gamma \eta} + \frac{i \rho \cdot \jmath}{2 \gamma \eta} \frac{i q \cdot \jmath}{2 \gamma \eta} - \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta}
\]

\[
3 \hat{F}_p^- = \frac{i \rho \cdot \jmath}{2 \gamma \lambda^3} + \frac{i q \cdot \jmath}{2 \gamma \lambda^3} - \frac{3 \jmath}{2 \gamma \eta} + \frac{i \rho \cdot \jmath}{2 \gamma \eta} \frac{i q \cdot \jmath}{2 \gamma \eta} - \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta} + \frac{i q \cdot \jmath}{2 \gamma \eta}
\]

TABLE 6-3b

THE PRIMARY POTENTIALS IN MATRIX NOTATION

\[
\begin{bmatrix}
\hat{F}_p^+ \\
\hat{F}_p^- \\
\end{bmatrix} = 
\begin{bmatrix}
A_1 & A_2 & A_3 & A_4 & A_5 \\
-A_1 & -A_2 & A_3 & A_4 & A_5 \\
\end{bmatrix} 
\begin{bmatrix}
J & J & J & M & M \\
J & J & J & M & M \\
\end{bmatrix}^T
\]

\[
\begin{bmatrix}
\hat{F}_p^+ \\
\hat{F}_p^- \\
\end{bmatrix} = 
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 \\
-B_1 & -B_2 & B_3 & B_4 & B_5 \\
\end{bmatrix} 
\begin{bmatrix}
J & J & J & M & M \\
J & J & J & M & M \\
\end{bmatrix}^T
\]

\begin{align*}
A_1 &= A_2 = -i/2 \gamma \lambda^3 \\
A_3 &= -1/2 \gamma \eta \\
A_4 &= -i/2 \gamma \lambda^3 \\
A_5 &= i/2 \gamma \lambda^3 \\
B_1 &= B_2 = -i/2 \lambda^3 \\
B_3 &= -1/2 \gamma \eta \\
B_4 &= +i/2 \gamma \lambda^3 \\
B_5 &= -i/2 \gamma \lambda^3
\end{align*}
\[
3 \hat{H}_{3} = 3 \hat{H}(3, \hat{n}, \hat{r}, \hat{r})
\]

Therefore

\[
-3 \hat{\Gamma}_{n} - 3 \hat{\Gamma}_{z} = I \hat{\Gamma}_{n} + z I \hat{\Gamma}_{z} + y(2 \hat{\Gamma}_{n} - I \hat{\Gamma}_{z})
\]

\[
-3 \hat{\Pi}_{n} - 3 \hat{\Pi}_{z} = I \hat{\Pi}_{n} + z I \hat{\Pi}_{z} - z(2 \hat{\Pi}_{n} - I \hat{\Pi}_{z})
\]

Solving for \(3 \hat{\Pi}\) and \(3 \hat{r}\) we obtain expressions which relate the vertical component of the potentials to the horizontal components. The result, when converted into the Fourier wavenumber domain is,

\[
3 \hat{r} = \frac{i \rho \hat{D}}{\lambda^2} \hat{r} + \frac{i \rho \hat{D}_{2}}{\lambda^2} \hat{r} - \frac{y \rho \hat{L}}{\lambda^2} \hat{r} + \frac{y \rho \hat{L}_{2}}{\lambda^2} \hat{r}
\]

\[
3 \hat{\Pi} = \frac{i \rho \hat{D}}{\lambda^2} \hat{\Pi} + \frac{i \rho \hat{D}_{2}}{\lambda^2} \hat{\Pi} + \frac{z \rho \hat{L}}{\lambda^2} \hat{\Pi} - \frac{z \rho \hat{L}_{2}}{\lambda^2} \hat{\Pi}
\]

Substituting 6–7 into equations 6–10 yields the expressions for the vertical Hertz potential in terms of the source dipoles. These expressions are listed in tables 6–3.

6.4 Calculation of Potential Ratios
In section 6.3, the free space Hertz potentials were found in terms of the source dipoles. When these dipoles are placed in a stratified Earth, the free space potentials are scattered and produce a secondary potential. It is therefore necessary to determine this secondary field so that the total field at the source level can be found. Once this is done, continuation can be used to find the potentials, and therefore the fields, at any level in the layered Earth. This is done following the method of Lajoie, 1973.

In order to find the scattered field at the source level, it is first necessary to determine the ratios of the upwardly propagating potentials to the downwardly propagating potentials there. These ratios can be calculated at any point in the stratified Earth if they are known at any other point merely by continuing the potentials from one point to the other. This assumes, however, that the structure of the stack and that the position of the source are known.

Consider that the source is in the $i^{th}$ layer of the Earth (refer to figure 6-2). If an arbitrary boundary is placed at the source level (despite the fact that there is no property contrast there) then the source will have interface $i-1$ above it and $i+1$ below it. At interface $i-1$, the $\Pi$ potential, for example, in layer $i-1$ is:

$$\hat{\Pi}_3(d_{i-1}) = -\eta_i \hat{\Pi}(d_{i-1}) \ast \eta_i \hat{\Pi}(d_{i-1})$$  \hspace{1cm} 6-11
Figure 6-2: Calculation of the Potential Ratios
and its vertical derivative is

\[ \hat{\Pi}_3 (d_i^*) = \eta_{i-1} \hat{\Pi} (d_i^*) + \eta_{i+1} \hat{\Pi} (d_i^*) \]  

In layer 1 at the same interface:

\[ \hat{\Pi} (d_i) = \hat{\Pi}' (d_i) + \hat{\Pi}^{-} (d_i) \]  

\[ \hat{\Pi}_2 (d_i) = \eta_i \hat{\Pi} (d_i) + \eta_i \hat{\Pi} (d_i) \]  

The potentials at the top and bottom interfaces in the same layer are related by:

\[ \hat{\Pi}' (d_i) = \hat{\Pi}' (d_i) e^{-n_i b_i} \]  

\[ \hat{\Pi}^{-} (d_i) = \hat{\Pi}^{-} (d_i) e^{n_i b_i} \]  

The ratio of the downgoing to the upgoing potential on the positive side of interface \( i \), \( \text{RFP}(i) \), is defined as:

\[ \text{RFP}(i) = \frac{\hat{\Pi} (d_i^*)}{\hat{\Pi}^{-} (d_i^*)} \]  

Equations 6–15 and 6–16 therefore imply that the ratios of the fields at adjacent interfaces are
\[
\frac{\hat{\Pi}^+(d_{i-})}{\hat{\Pi}^-(d_{i-})} = \frac{e^{-2\eta_l^i}}{RF_{P(i)}}
\]

6-17

Substituting 6-17 into 6-13 and 6-14, and setting

\[
\alpha = \frac{e^{-2\eta_l^i}}{RF_{P(i)}}
\]

we get

\[
\hat{\Pi}^+(d_{i-}) = \hat{\Pi}^-(d_{i-})\{1+\alpha\}
\]

6-18

\[
\hat{\Pi}^-(d_{i-}) = \hat{\Pi}^-(d_{i-})\{1-\alpha\}
\]

6-19

Applying the boundary conditions on \(\Pi\) at interface \(i-1\) to equations 6-11 to 6-14, and then equating 6-11 to 6-13 and 6-12 to 6-14,

\[
y_{i-}\{\hat{\Pi}^+(d_{i-}) + \hat{\Pi}^-(d_{i-})\} = y_i \hat{\Pi}^-(d_{i-})\{1+\alpha\}
\]

6-20

\[
-\eta_{i-}\{\hat{\Pi}^+(d_{i-}) + \hat{\Pi}^-(d_{i-})\} = \eta_i \hat{\Pi}^+(d_{i-})\{1-\alpha\}
\]

6-21

Eliminating \(\hat{\Pi}^-(d_{i-})\), and setting \(y_{i-}/y_i\), \(N/\eta_{i-}/\eta_i\) equations

6-20 and 6-21 become

\[
\gamma (\hat{\Pi}^+ + \hat{\Pi}^-)/(1-\alpha) = \chi (\hat{\Pi}^+ - \hat{\Pi}^-)/(1-\alpha)
\]

6-22
\[ \hat{\Pi}^{-} \left( \frac{Y}{1 + \alpha} - \frac{N}{1 - \alpha} \right) = \hat{\Pi}^{+} \left( \frac{Y}{1 + \alpha} + \frac{N}{1 - \alpha} \right) \] 6–23

Now if \( \hat{\Pi}^{-} \) is divided by \( \hat{\Pi}^{+} \), then an expression for RFP\((i-1)\) in terms of RFP\((i)\) can be obtained:

\[ RFP\((i-1)\) = -Y(1-\alpha) + N(1+\alpha) \] 6–24

\[ RFP\((i-1)\) = \frac{(N+Y) e^{2\eta t} RFP\((i)\) + (N-Y)}{(Y-N) e^{2\eta t} RFP\((i)\) - (N+Y)} \] 6–25

RFP\((i)\) can also be expressed in terms of RFP\((i-1)\):

\[ RFP\((i)\) = \frac{RFP\((i-1)\) (N+Y) - (N-Y)}{(N+Y) RFP\((i-1)\) (Y-N)} \] 6–26

This expression can be used iteratively to compute the potential ratios for the \( \Pi \) potential if the potential ratio at any point in the layered earth above the source is known.

The potential ratio on the first interface of the layered Earth is now derived. Once this is done, the problem is solved for the electric Hertz potential above the source layer. When the source is below the first interface, there is no downward propagating potential above that interface. This is because there is nothing present
to reflect upward propagating waves. Hence, RFP(1)=0. Also, if the source is above the first interface, the potential ratio on the positive side of the source layer is also zero, because there is no secondary potential present. From a similar argument, it is deduced that RFN(n)=0.

In this section a method has been outlined for expressing the potential ratios on the upper side of any level in a stratified space in terms of the ratios at the interface between layers immediately overlying that level. The method for deriving the potential ratios just below the source layer is virtually the same as the method outlined here. The results for the magnetic Hertz potential can be easily obtained by substituting the appropriate boundary conditions and are recorded in table 6-4.

6.5 Calculation of the Total Potential at the Source Level

Expressions for the primary field potentials were obtained in section 6.4. It is now necessary to find the total potential at the source level and to propagate it to the field level.

The notation is now generalized, with F defined as being either \( \Pi \) or \( \Gamma \). The superscripts \( p \) and \( s \) denote the primary and secondary fields respectively. Additionally, RFN, the potential ratio on the negative side of the source layer is defined as being the
TABLE 6-4

RECURSION RELATIONS FOR THE REFLECTION COEFFICIENTS

ELECTRIC HERTZ POTENTIAL, \( \pi \)

\[
RFP(i) = \left\{ \frac{RFP(i-1) \cdot (N' - Y') + (Y' - N')}{RFP(i-1) \cdot (Y' - N') + (N' + Y')} \right\} e^{-2\eta_{it} t_i} \\
RFN(i) = \left\{ \frac{(Y' - N') + (N' + Y') \cdot RFN(i+1)}{(N' + Y') \cdot (Y' - N') \cdot RFN(i+1)} \right\} e^{-2\eta_{rin} t_{in}}
\]

\[Y' = y_{Ki} / y_i\]
\[Y' = y_{i-2} / y_{in}\]
\[N' = \eta_{i-2} / \eta_i\]
\[N' = \eta_{rin} / \eta_{in}\]

MAGNETIC HERTZ POTENTIAL, \( \Gamma \)

\[
RFP(i) = \left\{ \frac{RFP(i-1) \cdot (Z' + N') + (Z' - N')}{RFP(i-1) \cdot (Z' - N') + (Z' + N')} \right\} e^{-2\eta_{it} t_i} \\
RFN(i) = \left\{ \frac{(Z' - N') + (N' + Z') \cdot RFN(i+1)}{(N' + Z') \cdot (Z' - N') \cdot RFN(i+1)} \right\} e^{-2\eta_{rin} t_{in}}
\]

\[Z' = z_{Ki} / z_i\]
\[Z' = z_{i-2} / z_{in}\]

THE FIRST TERMS OF THE RECURSION FOR BOTH POTENTIALS

\[RFP(1) = 0 \quad RFN(N-1) = 0\]
ratio of the transformed total upward propagating potential, \( \hat{F}^+ \), to the transformed total downward propagating potential, \( \hat{F}^- \), just below the source layer. The potential ratio on the positive side of the layer, RFP, is defined in a similar manner. Both RFN and RFP were derived in the previous section. Therefore, from figure 6–3

\[
RFN = \frac{\hat{F}^+}{\hat{F}^-} \text{ at } d_i \quad 6–27a
\]

\[
RFP = \frac{\hat{F}^-}{\hat{F}^+} \text{ at } d_i \quad 6–27b
\]

where \( d_i \) is the depth of the source point. Expressing the total potentials in terms of primary and secondary components:

\[
RFN = \frac{\hat{F}_s^+}{(\hat{F}_p^- + \hat{F}_s^-)} \quad 6–28a
\]

\[
RFP = \frac{\hat{F}_s^-}{(\hat{F}_p^+ + \hat{F}_s^+)} \quad 6–28b
\]

Solving 6–28 for the secondary potentials:

\[
\hat{F}_s^+ = \frac{RFN \cdot RFP \hat{F}_p^+ + RFN \hat{F}_p^-}{1 - RFN \cdot RFP} \quad 6–29a
\]

\[
\hat{F}_s^- = \frac{RFP \hat{F}_p^+ RFN \cdot RFP \hat{F}_p^-}{1 - RFN \cdot RFP} \quad 6–29b
\]
Figure 6-3: The Total Field at the Source Level
Equations 6–29 can be written in matrix notation:

\[
\begin{bmatrix}
F_s^+ \\
F_s^-
\end{bmatrix} = \begin{bmatrix}
\text{RFP \cdot RFN} & \text{RFN} \\
\text{1-RFP \cdot RFN} & \text{1-RFP \cdot RFN}
\end{bmatrix} \begin{bmatrix}
\text{RFN} \\
\text{RFP \cdot RFN}
\end{bmatrix} \begin{bmatrix}
\hat{F}_p^+ \\
\hat{F}_p^-
\end{bmatrix}
\]

The secondary potentials have now been determined at the source level. To obtain the total potential at the source level, the appropriate primary potential must be added to the secondary potential. If the field point is above the source point, the total Hertz potential is the sum of the upwardly propagating free space potential and the upwardly propagating secondary potential. Similarly, by adding the downwardly propagating free space potential to the secondary potential, the total potential just below the source point can be obtained. The total potentials can thus be propagated to the field point without traversing the cusp in the potential at the source level. Details of the results of this procedure are tabulated in table 6–5.

It is now necessary to propagate the total fields to the field level. Using the results of tables 6–1 and 6–2, a propagation matrix for the total Hertz potentials can be derived. When multiplied by the matrix in equation 6–30, a transformation which takes the primary potentials at the source level into the total potentials at the field level is obtained. If these transformations are la-
TABLE 6-5
TOTAL POTENTIALS AT THE SOURCE POINT

UPWARD PROPAGATION

\[
\begin{bmatrix}
\hat{F}_T^+ \\
\hat{F}_T^-
\end{bmatrix} =
\begin{bmatrix}
1/(1-RFN \cdot RFP) & RFN/(1-RFN \cdot RFP) \\
RFP/(1-RFN \cdot RFP) & RFN \cdot RFP/(1-RFN \cdot RFP)
\end{bmatrix}
\begin{bmatrix}
\hat{F}_P^+ \\
\hat{F}_P^-
\end{bmatrix}
\]

DOWNWARD PROPAGATION

\[
\begin{bmatrix}
\hat{F}_T^+ \\
\hat{F}_T^-
\end{bmatrix} =
\begin{bmatrix}
RFN \cdot RFP/(1-RFN \cdot RFP) & RFN/(1-RFN \cdot RFP) \\
RFP/(1-RFN \cdot RFP) & 1/(1-RFN \cdot RFP)
\end{bmatrix}
\begin{bmatrix}
\hat{F}_P^+ \\
\hat{F}_P^-
\end{bmatrix}
\]

SIMPLIFIED VERSIONS AT THE TOP AND BOTTOM OF THE STACK

UPWARD PROPAGATION ABOVE THE STACK

\[
\begin{bmatrix}
\hat{F}_T^+ \\
\hat{F}_T^-
\end{bmatrix} =
\begin{bmatrix}
1 & RFN \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{F}_P^+ \\
\hat{F}_P^-
\end{bmatrix}
\]

DOWNWARD PROPAGATION BELOW THE STACK

\[
\begin{bmatrix}
\hat{F}_T^+ \\
\hat{F}_T^-
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
RFP & 1
\end{bmatrix}
\begin{bmatrix}
\hat{F}_P^+ \\
\hat{F}_P^-
\end{bmatrix}
\]
belled T for the \( \Pi \) potential and R for the \( \Gamma \) potential, then

\[
\begin{bmatrix}
\hat{\Pi}^+_f \\
\hat{\Pi}^-_f
\end{bmatrix} = T \begin{bmatrix}
\hat{\Pi}^+_p \\
\hat{\Pi}^-_p
\end{bmatrix}
\quad \begin{bmatrix}
\hat{\Gamma}^+_f \\
\hat{\Gamma}^-_f
\end{bmatrix} = R \begin{bmatrix}
\hat{\Gamma}^+_p \\
\hat{\Gamma}^-_p
\end{bmatrix}
\]

The structure of these transformations is outlined in Table 6–6.

6.6 Calculation of the Electric and Magnetic Fields From the Source Currents

In the preceding sections of this chapter, a method for determining the transformed Hertz potentials of the total field of a dipole source in a layered Earth was derived. It is now necessary to convert these potentials into the electric and magnetic fields and then to derive an expression for the fields in terms of the source currents. Once this is done, the fields can be converted into the space domain. Several methods are available for doing this conversion. The one that is used here is the Hankel transform because the functions being transformed have arbitrary radial wave number dependence and simple angular wavenumber dependence. Thus instead of a fully two dimensional transform, a few one dimensional transforms can be done.

The Hankel transform is analogous to the Fourier transform,
TABLE 6-6

PROPAGATION MATRICES

Propagation matrices relating the Hertz potentials at the field point level to the Hertz potentials of the primary field at the source level in the same or another layer.

\[
T = \begin{bmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{bmatrix}, \quad R = \begin{bmatrix}
R_1 & R_2 \\
R_3 & R_4
\end{bmatrix}
\]

\(T\) and \(R\) are constructed by successive multiplication of submatrices as follows

\[
G(\text{field point to interface}) \times Q(\text{through interface})
\]

\[
\times G(\text{one side of layer to the other side}) \ldots \ldots
\]

\[
\ldots \ldots \times G(\text{one side of layer to the other side})
\]

\[
\times Q(\text{through interface}) \times G(\text{interface to source point}) \times X
\]

where

\(G\) is a continuation matrix (refer to table 6-1)

\(Q\) is a matrix for propagation across interfaces (refer to table 6-2)

\(X\) is the matrix which takes the primary field at the source point into the total field there (refer to table 6-5)
except that the transformed functions are functions of radial and angular wavenumbers instead of Cartesian wavenumbers. In order to use the Hankel transform, the angular dependence of the fields, must therefore be separated from the radial dependence. Once this is done, the fields can be converted into the space domain with the inverse Hankel transform. The present section is devoted to setting up the fields so that they can be conveniently represented in the Hankel transform domain. The following section actually deals with the conversion of the fields into radial coordinates and the transformation of the fields into the space domain.

From table 6-3b, the primary potentials in matrix notation are

\[
\begin{align*}
\begin{bmatrix}
\hat{\Pi}^+ \\hat{\Pi}^- \n \end{bmatrix} &= \begin{bmatrix}
A_1 p & A_2 q & A_3 & A_4 p & A_5 q \\
A_1 p & A_2 q & A_3 & A_4 p & A_5 q 
\end{bmatrix} \begin{bmatrix}
1 j \\
2 j \\
3 j \\
2 m \\
1 m 
\end{bmatrix} \quad 6-31a
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\hat{\Gamma}^+ \\hat{\Gamma}^- \n \end{bmatrix} &= \begin{bmatrix}
B_1 p & B_2 q & B_3 & B_4 p & B_5 q \\
B_1 p & B_2 q & B_3 & B_4 p & B_5 q 
\end{bmatrix} \begin{bmatrix}
1 m \\
2 m \\
3 m \\
2 j \\
1 j 
\end{bmatrix} \quad 6-31b
\end{align*}
\]
Multiplying by $T$ and $R$, an expression for the Hertz potentials at the field point level as a function of the source currents is obtained:

$$\vec{\Pi}_f^+ = p\alpha_1 + q\beta_1 + \gamma_1$$
$$\vec{\Pi}_f^- = p\alpha_2 + q\beta_2 + \gamma_2$$

$$\vec{F}_f^+ = p\delta_1 + q\varepsilon_1 + \phi_1$$
$$\vec{F}_f^- = p\delta_2 + q\varepsilon_2 + \phi_2$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ and $\phi$ are defined in table 6–7.

Now by combining equations 6–5 with 6–32, expressions for the total electric field at the field point level in terms of the source currents are obtained.

$$\vec{E}^+ = i p\eta_f (p\alpha_1 + q\beta_1 + \gamma_1 - p\alpha_2 - q\beta_2 - \gamma_2) +$$
$$i z_i q (p\delta_1 + q\varepsilon_1 + \phi_1 + p\delta_2 + q\varepsilon_2 + \phi_2)$$

$$\vec{E}^- = i q\eta_f (p\alpha_1 + q\beta_1 + \gamma_1 - p\alpha_2 - q\beta_2 - \gamma_2) -$$
$$i z_i p (p\delta_1 + q\varepsilon_1 + \phi_1 + p\delta_2 + q\varepsilon_2 + \phi_2)$$

$$\vec{E} = \lambda^l (p\alpha_1 + q\beta_1 + \gamma_1 + p\alpha_2 + q\beta_2 + \gamma_2)$$

6–33
$\alpha_1 = A_{1,1} j \left( T_1 - T_2 \right) + A_{4,2} m \left( T_1 + T_2 \right)$
$\alpha_2 = A_{1,1} j \left( T_3 - T_4 \right) + A_{4,2} m \left( T_3 + T_4 \right)$

$\beta_1 = A_{2,2} j \left( T_1 - T_2 \right) + A_{5,1} m \left( T_1 + T_2 \right)$
$\beta_2 = A_{2,2} j \left( T_3 - T_4 \right) + A_{5,1} m \left( T_3 + T_4 \right)$

$\gamma_1 = A_{3,1} j \left( T_1 + T_2 \right)$
$\gamma_2 = A_{3,1} j \left( T_3 + T_4 \right)$

$\delta_1 = B_{1,1} m \left( R_1 - R_2 \right) + B_{4,2} j \left( R_1 + R_2 \right)$
$\delta_2 = B_{1,1} m \left( R_3 - R_4 \right) + B_{4,2} j \left( R_3 + R_4 \right)$

$\epsilon_1 = B_{2,2} m \left( R_1 - R_2 \right) + B_{5,1} j \left( R_1 + R_2 \right)$
$\epsilon_2 = B_{2,2} m \left( R_3 - R_4 \right) + B_{5,1} j \left( R_3 + R_4 \right)$

$\phi_1 = B_{3,1} m \left( R_1 + R_2 \right)$
$\phi_2 = B_{3,1} m \left( R_3 + R_4 \right)$
which reduces to, for example

\[ \hat{E} = p^t i\eta_t (\alpha_t - \alpha_t) + q^t i\eta_t (\beta_t + \beta_t) + l \eta_l (\delta_l + \delta_l) + m^t i\eta_l (\gamma_l - \gamma_l) + n^t i\eta_l (\phi_t + \phi_t) \]

6–34

Analogous expressions for the magnetic field can also be derived.

Equation 6–34 is the kernel for the electric field in the two-dimensional Fourier transform domain. In order to effect an efficient transformation into the space domain, fast Hankel transform routines must be used. Since these routines are available only for Bessel functions of orders 0 and 1, it is necessary to rewrite our two-dimensional Fourier transform kernels in a form compatible with the \( J_0 \) and \( J_1 \) Hankel transforms (i.e., as functions of \( \lambda \) only or \( \lambda \) and \( \exp(i \arctan(p/q)) \)). This is accomplished by first separating all parts of the expressions for the transformed fields that are functions of the radial wave number, \( \lambda \), from the specific \( p \) and \( q \) dependent parts. Thus it is convenient to write

\[ \hat{\omega} = (p^t, q^t, pq, p, q, l) \] 6–35

and then introduce a vector \( \hat{\omega}(\lambda) \) such that the electric
field can be expressed as

$$ k \hat{E} = \hat{\omega} \cdot k \vec{S}(\lambda) \quad \text{(6-35a)} $$

Similarly, using the vector $k \vec{H}$ for the magnetic field, we get

$$ k \hat{H} = \hat{\omega} \cdot k \vec{Z}(\lambda) \quad \text{(6-36b)} $$

where the vectors $k \vec{S}$ and $k \vec{Z}$ are tabulated in Table 6-8. The elements of $k \vec{S}$ and $k \vec{Z}$ are functions of $\lambda$ only, and as such the fields in the Hankel domain may now be determined by converting $\hat{\omega}$ into radial coordinates. A description of this conversion is given in the following section.

6.7 Transformation of the Fields into the Space Domain

Expressions were derived in the last section which relate the Fourier transformations of the electric and magnetic fields to the source dipoles and the properties of the layered Earth. It is now necessary to transform these fields into the space domain. This is accomplished by first converting these fields into the radial wavenumber (Hankel transform) domain, and then transforming the fields.
TABLE 6-8

THE VECTORS $\mathbf{\xi}$ AND $\mathbf{\chi}$

\begin{align*}
1 \mathbf{\xi}_1 &= i \eta_1 (\alpha_i - \alpha_k) \\
1 \mathbf{\chi}_1 &= i \eta_1 (\epsilon_i + \epsilon_k) \\
1 \mathbf{\xi}_2 &= i \eta_2 (\beta_i + \beta_k) \\
1 \mathbf{\chi}_2 &= i \eta_2 (\phi_i + \phi_k)
\end{align*}
\begin{align*}
1 \mathbf{\xi}_3 &= 0 \\
1 \mathbf{\chi}_3 &= 0 \\
1 \mathbf{\xi}_4 &= \lambda^i (\alpha_i + \alpha_k) \\
1 \mathbf{\chi}_4 &= \lambda^i (\beta_i + \beta_k)
\end{align*}
\begin{align*}
2 \mathbf{\xi}_1 &= -i \eta_1 (\delta_i - \delta_k) \\
2 \mathbf{\chi}_1 &= -i \eta_1 (\sigma_i - \sigma_k) \\
2 \mathbf{\xi}_2 &= i \eta_2 (\epsilon_i - \epsilon_k) \\
2 \mathbf{\chi}_2 &= i \eta_2 (\phi_i - \phi_k)
\end{align*}
\begin{align*}
2 \mathbf{\xi}_3 &= 0 \\
2 \mathbf{\chi}_3 &= 0 \\
2 \mathbf{\xi}_4 &= \lambda^i (\delta_i - \delta_k) \\
2 \mathbf{\chi}_4 &= \lambda^i (\sigma_i - \sigma_k)
\end{align*}
\begin{align*}
3 \mathbf{\xi}_1 &= 0 \\
3 \mathbf{\chi}_1 &= 0 \\
3 \mathbf{\xi}_2 &= \lambda^i (\epsilon_i + \epsilon_k) \\
3 \mathbf{\chi}_2 &= \lambda^i (\phi_i + \phi_k)
\end{align*}

NOTATION: SUBSCRIPT 'f' DENOTES THE PROPERTY AT THE FIELD POINT
THE NUMBER SUBSCRIPTS ON THE RIGHT SIDE OF $\mathbf{\chi}$ AND $\mathbf{\xi}$ INDICATE VECTOR COMPONENTS
back into the space domain with an inverse Hankel transform.

In order to do the transformation into the space domain, functions of Fourier the wavenumbers \( p \) and \( q \) must be converted into functions of \( \lambda \) and \( \beta \). This is accomplished with the following identity,

\[
\hat{f}_n(\lambda) = \frac{1}{2\pi} \int_0^{\pi} e^{i\eta(\beta - \pi \lambda)} \hat{f}(\lambda \cos \beta, \lambda \sin \beta) \, d\beta \quad 6-37
\]

where \( p = \lambda \cos \beta \) and \( q = \lambda \sin \beta \) and \( n \) is an integer which both counts the periodic variation in the angular coordinate and is the order of the Bessel function. Applying this identity to the first element of \( \hat{\omega} \), we get \( \hat{\omega}_1 \), the first element of \( \hat{\omega} \) in radial coordinates;

\[
\hat{\omega}_1(\eta) = \frac{1}{2\pi} \int_0^{\pi} e^{i\eta(\beta - \pi \lambda)} \lambda \cos \beta \, d\beta \quad 6-38
\]

where \( \hat{f}(\lambda \cos \beta, \lambda \sin \beta) \) in 6-37 is \( \lambda \cos \beta \) since the first coordinate of \( \hat{\omega} \) is \( \rho \).

Similarly,

\[
\hat{\omega}_1(\eta) = \frac{1}{2\pi} \int_0^{\pi} e^{-i\eta(\beta \pi \lambda)} \lambda \sin \beta \, d\beta \quad 6-39a
\]
\[ \hat{\omega}(n) = \frac{1}{2\pi} \int_{\pi}^{0} e^{-i\alpha \cdot \beta} \cos \beta \sin \beta \, d\beta \]  

6-39b

Similar expressions can be obtained for \( \hat{\omega}_4, \hat{\omega}_5 \) and \( \hat{\omega}_6 \).

Since \( n \) is discrete, the set \( \{\hat{\omega}\} \) takes the form of a two-dimensional matrix \( \Omega \), with the rows of \( \Omega \) corresponding to the various values of \( n \) and the columns to the corresponding elements of \( \hat{\omega} \).

Performing the integration in \( \beta \), \( \Omega \) becomes

\[ \Omega(n = 2) = \left( \begin{array}{cccc} -\frac{\lambda}{4} & \frac{\lambda}{4} & i\frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ -\frac{\lambda}{4} & \frac{\lambda}{4} & -i\frac{\lambda}{4} & 0 & 0 & 0 \end{array} \right) \]  

6-40a

\[ \Omega(n = 1) = \left( \begin{array}{cccc} -\frac{\lambda}{4} & \frac{\lambda}{4} & i\frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ -\frac{\lambda}{4} & \frac{\lambda}{4} & -i\frac{\lambda}{4} & 0 & 0 & 0 \end{array} \right) \]  

6-40b

\[ \Omega(n = 0) = \left( \begin{array}{cccc} -\frac{\lambda}{4} & \frac{\lambda}{4} & i\frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ -\frac{\lambda}{4} & \frac{\lambda}{4} & -i\frac{\lambda}{4} & 0 & 0 & 0 \end{array} \right) \]  

6-40c

\[ \Omega(n = 1) = \left( \begin{array}{cccc} -\frac{\lambda}{4} & \frac{\lambda}{4} & i\frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ -\frac{\lambda}{4} & \frac{\lambda}{4} & -i\frac{\lambda}{4} & 0 & 0 & 0 \end{array} \right) \]  

6-40d

\[ \Omega(n = 2) = \left( \begin{array}{cccc} -\frac{\lambda}{4} & \frac{\lambda}{4} & i\frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ -\frac{\lambda}{4} & \frac{\lambda}{4} & -i\frac{\lambda}{4} & 0 & 0 & 0 \end{array} \right) \]  

6-40e

which can be written as a matrix,

\[
\Omega = \begin{bmatrix}
-\frac{\lambda}{4} & \frac{\lambda}{4} & i\frac{\lambda}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\
\frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -i\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\
-\frac{\lambda}{4} & \frac{\lambda}{4} & -i\frac{\lambda}{4} & 0 & 0 & 0
\end{bmatrix}
\]  

6-41
The electric and magnetic fields in the angular wavenumber domain now become

\[ i \vec{H} = \Omega \chi(\lambda) \]  
\[ \text{6-42a} \]

and

\[ i \vec{E} = \Omega \vec{f}(\lambda) \]  
\[ \text{6-42b} \]

where:

\[ i \vec{H}^T = \left( \vec{H}(n=2), \vec{H}(n=1), \vec{H}(n=0), \vec{H}(n=1), \vec{H}(n=0), \vec{H}(n=1) \right) \]

Since we can only transform functions of orders zero and one alone, and because the angular wavenumber, \( n \), corresponds to the order of the Bessel function in the transform into the space domain, the form of \( \Omega \) must be changed, i.e. \( \Omega \) must be made a function of \( n=0 \) and \( n=1 \) alone. Now the transformation is done by using the following relationship between functions in the radial wave number and space domains,

\[ \bar{F}(r, \theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \bar{F}_n(\lambda) e^{-in\theta} J_n(\lambda r) \lambda d\lambda \]  
\[ \text{6-43} \]

The following properties of the Bessel function allow us to write this transformation in terms of \( n=0 \) and \( n=1 \) alone:
\[ J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x), \quad J_{-n}(x) = (-1)^n J_n(x) \]

Equation 6.43 can be rewritten in the following compressed form when it contains no elements for \(-2 < n < 2\):

\[
F(r, \theta) = \frac{1}{2\pi} \left\{ \int_0^\infty \left[ \left( \frac{F_1(\lambda)}{\lambda} \right) e^{-i\theta} - \frac{F_2(\lambda)}{\lambda} e^{i\theta} \right] \lambda J_1(\lambda r) d\lambda \right\} + \frac{1}{2\pi} \left\{ \int_0^\infty \left[ \frac{F_2(\lambda)}{\lambda} e^{-2i\theta} + \frac{F_3(\lambda)}{\lambda} e^{2i\theta} \right] \lambda J_0(\lambda r) d\lambda \right\}
\]

Substitution of \( E \) or \( H \) for \( F \) yields the solution for the component of the electric and magnetic fields due to a dipole in a stratified Earth.

6.8 Summary

A programme has been written to compute the fields in a stratified earth using the results of the chapter above. The fields calculated by the program were found to be in concordance with those produced by Eadie (ref. Frischknecht (1967)) for the horizontal loop EM system. The program was also tested against the reciprocity theorem, and against the boundary conditions for the electric and magnetic fields at layer interfaces. A further check was made by setting
the properties of all media to constant values, and comparing results with those expected from a homogenous space. All test results were in accordance with the expected theoretical results.
CHAPTER 7

THE PROGRAMMED SOLUTION: ORGANIZATION AND RESULTS

7.1 Organization of the Program

A program has been written to evaluate the feasibility of the results outlined in chapters two through six. It is presently capable of computing the horizontal loop EM response of two plates in a stratified medium. However, the program can be easily extended to compute the results of many plates to any EM system.

The structure of the program is outlined in figure 7-1. It consists of two major parts: In the first part, the system response matrix is evaluated, and in the second part the response of the system due to external source currents is generated. The system response matrix can therefore be used to compute the response of a particular set of plates to any source-receiver combination.

In order to compute the system response matrix, the matrices M, W, and S are either generated or read from a file. Once these matrices have been found, they are deposited in a larger matrix to form the matrix on the right side of equation 5-26. This matrix is then inverted and multiplied by the matrix on the left side of 5-26, thus
FIGURE 7-1

THE PROGRAM STRUCTURE:

Compute the matrices $F$ and $L$,
Invert to form $M$

Compute the Interaction Matrix, $W$
between all pairs of plates:

Compute the Self Interaction, $S$
between all plates and the
stratified Earth

Synthesize the System Response
Matrix from $M$, $W$ and $S$

Compute the Interaction, $H$, between
the plates and the source field

Multiply $H$ by the System Response to
obtain the Excitation Coefficients, $C$

Compute the Secondary Field from the
plate by multiplying the Fields due to
unit Eigencurrents by $C$

Plot the Secondary Field as a Ratio of
the Primary Field at the Receiver
generating the system response matrix.

In the second part of the program the vector, $H$, which describes the interaction between the trial functions and the source field, is generated every time the source field is changed. This vector is then multiplied by the system response matrix to generate the excitation coefficients for each eigencurrent. The fields due to these eigencurrents are then generated by integrating over the current distributions in the plates.

7.2 Test Results

The results produced by the program for a single plate in free space are presented in figures 7-2a, 7-3a and 7-4a. Fifteen eigencurrents were used. The results of the program developed for this thesis are consistent with those produced by the program, PLATE (ref. Dyck), which calculates the EM response of a single plate in free space.

Small discrepancies between the results of the program and Plate occur in the peak amplitudes of the primary to secondary field ratios. These discrepancies may have been caused by the numerical integration schemes which were used to calculate the excitation coefficients and the secondary fields. It was found that the secondary field predicted by the program written for this thesis was sensitive
Figure 7-2a: A Comparison of HLEM Responses Generated by "Plate" and the Present Routine for the Geometry Depicted in Figure 7-2b.

Figure 7-2b: The Geometry Used for the Comparison of HLEM Responses in figure 7-2a
Figure 7–3a: A Comparison of HLEM Responses Computed with "Plate" and the Present Routine for the Geometry Depicted in figure 7–3b.

Figure 7–3b: The Geometry Used for the Comparison of HLEM Responses in figure 7–3a.
Figure 7-4a: A Comparison of HLEM Responses Computed by "Plate" and the Program for the Geometry Depicted in figure 7-4b.

Figure 7-4b: The Geometry Used for the Comparison of HLEM Responses in figure 7-4a
to the order of quadrature. This was particularly true where the effect of high order eigencurrents was strong, i.e. when the receiver or transmitter was close to the plate. However, as the order of quadrature was made larger, variation in the secondary field was less extreme. The discrepancy between the programs, therefore, could be due to the choice of the order of quadrature.

In order to test the program for the case of multiple plates, the results of the program were compared with data from a scale model experiment. This comparison is detailed in figure 7–5. The model consisted of two rectangular pieces of foil separated by five centimeters (see figure 7–5c). The program was run against this data using three, six, ten, and fifteen eigencurrents in each plate. Agreement between the scale model experiment and the computed results was reasonable, except near the peak of the anomaly, where the computed results scattered about the experimental ones. In this region of the quadrature anomaly, additional eigencurrents reduced the size of the anomaly, and seemed to overcompensate for the excessive current in the plate that was generated when only three eigencurrents were used. The computed in phase anomaly also diverged at the peak of the anomaly from the model results as the number of eigencurrents were increased. In all other parts of the profile, the agreement between the two sets of data was quite good. The difference between the two sets of data could again have been the result of low order numerical quadrature, which was likely to have been more severe when high order
Figure 7-5a: A Comparison of the In Phase Anomaly for Two Plates Produced by the Program and from Scale Model Experiments

Figure 7-5b: A Comparison of the Quadrature Anomaly for Two Plates Produced by the Program and from Scale Model Experiments
Figure 7-5c: HLEM Configuration used to Compare the Results of the Program with Scale Model Experiments for Multiple Conductors
eigencurrents are used. However, the difference could also have been caused by a poor choice of trial functions. Recall that the Galerkin method attempted to reduce the error in the current potential only. In doing so, the method may have added current which actually made the secondary field worse in some places before it converged to the correct solution. A remedy for this problem could be found by altering the trial functions.

In figure 7-8, the results of the stratified Earth routine are compared with those published by Eadie (1979). Agreement between the two routines is very good. The response of a plate under overburden is presented in figure 7-7a and is compared with the response of the same plate in free space in figure 7-7b. The location of the plate with respect to the overburden is depicted in figure 7-7c.

In computing the response of the plate under overburden, only the distortion of the primary and secondary fields due to the overburden was taken into account. No correction was made for the distortion of the current distribution within the plate itself: The response was calculated by assuming that the self interaction matrix of the plate due to the secondary field, S, was zero. This may not have been a valid assumption, particularly if the interaction between the plate and the overburden was strong. With the current version of the program, however, S could not be calculated to the required accuracy without consuming prohibitively large computing resources. This is because the Green's function routines are very slow. In the fu-
Figure 7-6: Argand Diagram Used to Compare Helix Responses for a Stratified Earth
Figure 7-7a: The Approximate Response of a Dipping Plate Under a Conductive Overburden

Figure 7-7b: The Response of a Dipping Plate in Free Space
Figure 7-7c: The Model Used in the Calculations for figures 7-7a and 7-7b.
future, with the Green's function routines modified for an array processor, it should be possible to effect both accurate and economical evaluations of this matrix.
CHAPTER 8

CONCLUSIONS

In this thesis, a method was developed for computing the EM response of more than one body by using the equivalent source method. The electromagnetic response of two conductors residing in free space was calculated and was found to agree with scale model results. These calculations were made by treating the current system in the conductors as a set of sources with unknown amplitudes. The amplitudes of these sources were then determined by solving a set of linear equations derived by invoking the Galerkin method.

The method of treating currents as sources of unknown amplitude was also used to compute the response of a plate in a stratified earth. In this scenario, the plate interacted with itself rather than with another plate to produce a modified current distribution. However, the solution to this problem could not be properly calculated because of the time required to compute the Green's functions for a source in a stratified earth. Although the method outlined in this thesis for computing the EM response of more than one conductor has been shown to work, the manner in which the solution for a plate in a stratified earth was implemented was found to be impractical.

Further improvement of the program could be attained by in-
creasing the order and type of the trial functions. The efficiency of the Green's function routines could be increased by replacing the electric current distribution on the plates with a magnetic current distribution, by invoking the reciprocity theorem, by tabulating intermediate results, and by using an array processor.

It is also possible to monitor the accuracy of the results produced by the program. One way to do this would be by comparing the total magnetic field on the surface of the plates with that produced by the induced current distribution. This would indicate how well the current distribution on the plate was modelled. A simpler method would be to examine the behavior of the excitation coefficients as a function of the order of the trial functions. If the coefficients did not converge to zero with increasing order, then it would be likely (but not certain) that more trial functions were needed to ensure an accurate solution.

The procedures outlined in this thesis could also be used to model the responses of three dimensional conductors by allowing the trial functions to occupy a third dimension. It may also be possible to use the Galerkin method to model the response of a conductor in contact with a layer of overburden. This would require an auxiliary set of trial functions in addition to the ones already in use so that the current entering and leaving the edge of the plate could be modelled. This current would then produce a distribution of sources along the edge of the plate in contact with the overburden. The am-
plitudes of these sources would be determined once the current distribution in the plate was known.

This method could then be extended for the case of three-dimensional body in a conductive space by evaluating the source distributions along the bounding planes of the body. The solution for an inhomogenous body in a conducting space could then be determined by creating sources within the body itself. Considerable fineness would be required, however, to ensure that the trial functions met the boundary conditions imposed by such a problem.
APPENDIX A

VECTOR RELATIONS

\[ \nabla \cdot \nabla \times A = 0 \quad \text{A-1} \]

\[ \nabla \times \nabla \phi = 0 \quad \text{A-2} \]

\[ \nabla \cdot \phi A = A \cdot \nabla \phi + \phi \nabla \cdot A \quad \text{A-3} \]

\[ \nabla \times \phi A = \nabla \phi \times A + \phi \nabla \times A \quad \text{A-4} \]

\[ \nabla \cdot A \times B = B \cdot \nabla \times A - A \cdot \nabla \times B \quad \text{A-5} \]

\[ \nabla \times \nabla \times A = \nabla (\nabla \cdot A) - \nabla^2 A \quad \text{A-6} \]

GREEN'S THEOREM

\[ \oint_S (f \nabla g - g \nabla f) \cdot n \, ds = \int_V (f \nabla g - g \nabla^2 f) \, dv \quad \text{A-7} \]

GREEN'S FIRST IDENTITY

\[ \oint_S g \nabla f \cdot n \, ds = \int_V (\nabla g \cdot \nabla f + g \nabla^2 f) \, dv \quad \text{A-8} \]
APENDIX B

CALCULATION OF THE FIELD IN FREE SPACE

The electric and magnetic fields can be expressed in terms of their vector potentials, $\mathbf{A}$ and $\mathbf{F}$:

$$
\mathbf{E} = \frac{i}{\mu y} \nabla \times \nabla A - \frac{1}{\varepsilon} \nabla \times \mathbf{F} \tag{B-1}
$$

$$
\mathbf{H} = \frac{1}{\mu z} \nabla \times \nabla F + \frac{1}{\mu} \nabla \times \mathbf{A} \tag{B-2}
$$

If the source currents are dipoles only, then the vector potentials can be expressed in terms of the Green's function:

$$
A = \mu g \mathcal{J}, \quad g = \frac{i}{4\pi R} e^{-ikR} \tag{B-3}
$$

$$
F = \varepsilon g \mathcal{M} \tag{B-4}
$$
We put
\[ G = \phi \ell \]

where \( \phi = \frac{e^{-jkr}}{4\pi R} \) and \( \ell = (l_1, l_2, l_3) \)

Now if the electric and magnetic dipoles are replaced with \( \ell \), then the fields will be specified once curl \( G \) and curl curl \( G \) are known. Putting the source point at the origin, \( R = (x^2 + y^2 + z^2)^{1/2} \). Curl \( G \) can be written:

\[
\nabla \times G = \nabla \times \phi \ell = \nabla \phi \times \ell + \phi \nabla \times \ell = \nabla \phi \times \ell \quad B-5
\]

which, with a bit of algebra reduces to:

\[
\nabla \times G = \frac{-1}{4\pi} \left( \frac{1}{R} + jk \right) \frac{\exp(-jkR)}{R^2} \left[ y_2 z_l_2 - z_2 y_l_2, z_2 x_l_2 - x_2 z_l_2, x_2 y_l_2 - y_2 x_l_2 \right] B-6
\]

Curl curl \( G \) is calculated is a similar fashion, except that the algebra is a bit messier. We set
\[ \nabla \times \nabla \times \mathbf{G} = \nabla \times \psi \mathbf{\hat{r}} \]

where
\[ \psi = -\left( \frac{1}{R} + jk \right) \frac{\exp(-jkr)}{4\pi R^2} \]

and \[ \mathbf{\hat{r}} = (y \hat{j} - z \hat{k}, z \hat{i} - x \hat{k}, x \hat{i} - y \hat{j}) \]

Now,
\[ \nabla \times \psi \mathbf{\hat{r}} = \nabla \psi \times \mathbf{\hat{r}} + \psi \nabla \times \mathbf{\hat{r}} \]

\[ \nabla \psi \] works out to be
\[ \frac{1}{4\pi} \frac{(x, y, z)}{R^3} \frac{\exp(-jkr)}{R} \left\{ -k^2 + \frac{3jk}{R} + \frac{3}{R^2} \right\} \]

Taking the cross product of \[ \nabla \psi \] and \[ \mathbf{\hat{r}} \], the first term on the right hand side of equation B–8 is obtained.
\[ \nabla \psi \times \vec{S} = \frac{\exp(-jkR)}{4\pi R^3} \left( -k^2 + \frac{3jk}{R} + \frac{3}{R^2} \right). \]  

\[ \left\{ x_L(y_Lz + l_x y) - \text{Int}_L(y^2 z + z^2), \quad y_L(z_x z + l_z x) - \text{Int}_L(x^2 z^2), \quad z \left( l_x z + l_y y \right) - \text{Int}_L(y^2 + x^2) \right\} \]

The second term of B-8, \( \nabla \psi \times \vec{S} \), becomes,

\[ \psi \nabla \times \vec{S} = \left( \frac{1}{R} + jk \right) \frac{\exp(-jkR)}{R^2} \cdot \mathbf{2} (l_x, l_y, l_z) \]

Therefore, curl curl \( \vec{G} \) is

\[ \nabla \times \nabla \times \vec{G} = \frac{\exp(-jkR)}{R^3} \left( -k^2 + \frac{3jk}{R} + \frac{3}{R^2} \right). \]

\[ \left\{ x_L(y_Lz + l_x y) - \text{Int}_L(y^2 z + z^2), \quad y_L(z_x z + l_z x) - \text{Int}_L(x^2 z^2), \quad z \left( l_x z + l_y y \right) - \text{Int}_L(y^2 + x^2) \right\} + \]

\[ 2 \left( \frac{1}{R} + jk \right) \frac{\exp(-jkR)}{R^2} \cdot \mathbf{2} (l_x, l_y, l_z) \]

and curl \( \vec{G} \) is
\[ \nabla \times \mathbf{G} = -\left(\frac{1}{R} + jk\right) \frac{\exp(-jkR)}{R^2} \left[ (y_1, -z_2, 0), (z_1 - x_2, x_1 - y_1, 0) \right] \]

By substituting equations B–12 and B–13 into equations B–1 and B–2, the fields in free space due to current dipoles can be obtained.
APPENDIX C

INTEGRATION OF THE REACTANCE MATRIX

From table 3–1 the following expression for the reactance matrix is obtained:

\[
\begin{aligned}
\mathbf{M} &= \int_{S_x} \int_{S_y} g \left\{ \frac{\partial \overline{\phi}(x,y)}{\partial x} \frac{\partial \overline{\psi}(u,v)}{\partial u} + \\
&\quad \frac{\partial \overline{\phi}(x,y)}{\partial y} \frac{\partial \overline{\psi}(u,v)}{\partial v} \right\} dx \, dy \, du \, dv \\
\end{aligned}
\]

where \( S_x \) and \( S_y \) are the surface of the plate in the coordinates \( x,y \) and \( u,v \) respectively and \( g \) is the Green's function in free space. When the distance between the source point and the field point is much less than one wavelength, and both points lay on the same plane, the Green's function is

\[
g = \frac{1}{4 \pi \rho}, \quad \rho^2 = (x-u)^2 + (y-v)^2
\]

where the source point lies at \((x,y)\) and the field point at
(u,v). Now examine the integral "I" over the source points alone.

\[ I = \frac{\partial \Phi}{\partial u} \int_{s_{x'}} g \frac{\partial \Phi^T}{\partial x} \, dx \, dy + \frac{\partial \Phi}{\partial u} \int_{s_{y'}} \frac{\partial \Phi^T}{\partial y} \, dx \, dy \quad C-2 \]

for the component in row i and column j in l, C-2 becomes:

\[ I_{ij} = \frac{\partial \Phi}{\partial u} \int_{s_{x'}} g \frac{\partial \Phi}{\partial x} \, dx \, dy + \frac{\partial \Phi}{\partial u} \int_{s_{y'}} \frac{\partial \Phi}{\partial y} \, dx \, dy \]

We now note that \( \Phi \) is composed of Tchebychev polynomials.

If the plate has half lengths \( z \) and \( w \) in \( x \) and \( y \), then \( \Phi \) can be written as

\[ \phi_i = P_{\text{odd}} \left( \frac{X}{Z} \right) P_{\text{odd}} \left( \frac{Y}{W} \right) = \left( 1 - \frac{X^2}{Z^2} \right) \left( 1 - \frac{Y^2}{W^2} \right) T_x \left( \frac{X}{Z} \right) T_y \left( \frac{Y}{W} \right) \]

\[ \phi_j = P_{\text{odd}} \left( \frac{U}{Z} \right) P_{\text{odd}} \left( \frac{V}{W} \right) = \left( 1 - \frac{U^2}{Z^2} \right) \left( 1 - \frac{V^2}{W^2} \right) T_x \left( \frac{U}{Z} \right) T_y \left( \frac{V}{W} \right) \]
where $P_i(x/z)$ is a polynomial of order $i$ evaluated at $x/z$.

Let $Q_{ij}(x/z)$ be the $x$ derivative of $P_i(x/z)$. Writing the $i$th term of the polynomial $Q$ as $Q(i-1,j)$, then $Q(i-1,j)=P(i,j+1)\times(j+1)/z$.

Expressing $I_{ij}$ in this notation, we get

$$I_{ij} = Q_{y^1} \left( \frac{y}{z} \right) P_{x^2} \left( \frac{u}{w} \right) \int_{s^{xy}} g Q_{\alpha\mu} \left( \frac{x}{z} \right) P_{\beta\nu} \left( \frac{y}{w} \right) ds^{xy}$$

$$+ Q_{\delta^1} \left( \frac{y}{w} \right) P_{x^2} \left( \frac{u}{z} \right) \int_{s^{xy}} g Q_{\beta\mu} \left( \frac{y}{w} \right) P_{\alpha\nu} \left( \frac{x}{z} \right) ds^{xy}$$

Now consider the integral in the first term of equation C-3.

$$\int_{s^{xy}} g Q_{\alpha\mu} \left( \frac{x}{z} \right) P_{\beta\nu} \left( \frac{y}{w} \right) ds^{xy}$$

Writing D-4 explicitly, we get

$$\frac{1}{4\pi} \int_{-\omega}^{\omega} \int_{-2}^{2} \frac{1}{\rho} Q_{\alpha\mu} \left( \frac{x}{z} \right) P_{\beta\nu} \left( \frac{y}{w} \right) d\rho dy$$
Notice that the integral is a surface integral. The singularity is of order \(1/\rho\) and will disappear if the integration is done in polar coordinates. (A similar trick can be used to get rid of \(1/\rho^2\) singularities in volume integrals.) Now, putting \(x-u=r\) and \(y-v=s\), equation C-5 becomes

$$\frac{1}{4\pi} \int \int \frac{1}{\rho} Q_{\alpha \beta} \left( \frac{r+u}{z} \right) P_{\beta \gamma} \left( \frac{s+v}{\omega} \right) \, dr \, ds$$

C-6

where

\[ dr = dx \quad ds = dy \]

Putting \(r^2 + s^2 = \rho^2\), \(\Theta = \arctan(s/r)\), \(r = \rho \cos \Theta\), \(s = \rho \sin \Theta\), the integral is expressable in polar coordinates having a pole at \((u,v)\) on the plate (refer to figure C-1),

\[
\int_0^{\Theta_1} \int_0^{\rho_1} Q_{\alpha \beta} \left( \frac{\rho \cos \Theta + u}{z} \right) P_{\beta \gamma} \left( \frac{\rho \sin \Theta + s}{\omega} \right) \, dp \, d\Theta \\
+ \int_0^{\Theta_1} \int_0^{\rho_1} Q_{\alpha \beta} \left( \frac{\rho \cos \Theta + u}{z} \right) P_{\beta \gamma} \left( \frac{\rho \sin \Theta + s}{\omega} \right) \, dp \, d\Theta \\
+ \int_0^{\Theta_1} \int_0^{\rho_1} Q_{\alpha \beta} \left( \frac{\rho \cos \Theta + u}{z} \right) P_{\beta \gamma} \left( \frac{\rho \sin \Theta + s}{\omega} \right) \, dp \, d\Theta \\
+ \int_0^{\Theta_1} \int_0^{\rho_1} Q_{\alpha \beta} \left( \frac{\rho \cos \Theta + u}{z} \right) P_{\beta \gamma} \left( \frac{\rho \sin \Theta + s}{\omega} \right) \, dp \, d\Theta
\] C-7
where the variables in equation C–7 are depicted in figure C–1. Since $P_{\beta \alpha}$ and $Q_{\alpha \eta}$ are polynomials, they can be expressed as

$$P_{\beta \alpha}(x) = \sum_{k=0}^{\beta+2} p(\beta+2,k) x^k$$

$$Q_{\alpha \eta}(y) = \sum_{I=0}^{\alpha+1} q(\alpha+1,I) y^I$$

so that

$$P_{\beta \alpha} \left( \frac{\rho \sin \theta + u}{\omega} \right) = \sum_{k=0}^{\beta+2} \sum_{L=0}^{K} p(\beta+2,k) \binom{K}{L} \rho^L \sin^L \theta \frac{u^L}{\omega^k}$$

$$Q_{\alpha \eta} \left( \frac{\rho \cos \theta + u}{Z} \right) = \sum_{I=0}^{\alpha+1} \sum_{J=0}^{I} q(\alpha+1,I) \binom{I}{J} \rho^J \cos^J \frac{u^J}{Z^I}$$
Figure C-1: Integration of the Reactance Matrix
The first term in equation C–7 now becomes

\[ \int_{\theta_{1}}^{\theta_{2}} \int_{0}^{\rho} \sum_{I=0}^{w_{I}} \sum_{K=0}^{\beta_{2}} \sum_{J=0}^{x_{J}} \sum_{L=0}^{k_{L}} p(\beta_{2}, k) q(\alpha_{H}, I) (I | J | K) \rho \frac{\tilde{u}^{j}}{Z^{i} \omega^{k}} \cos^{j} \theta \sin^{i} \theta \, d\theta \, d\rho \]

Integrating with respect to \( \rho \) and for convenience putting

\[ \gamma = \sum_{I=0}^{d_{I}} \sum_{K=0}^{\beta_{K}} \sum_{J=0}^{x_{J}} \sum_{L=0}^{k_{L}} p(\beta_{2}, k) q(\alpha_{H}, I) (I | J | K) \frac{\tilde{u}^{i} \tilde{v}^{k}}{Z^{i} \omega^{k}} \]

the first term in equation C–3 becomes:

\[ \gamma \left\{ \frac{1}{J+L+1} \left( (w-u)^{J+L+1} \int_{\theta_{1}}^{\theta_{2}} \frac{\cos^{j} \theta}{\sin^{j} \theta} d\theta + \right) \right. \]

\[ \left. \left( -z-u \right)^{J+L+1} \int_{\theta_{1}}^{\theta_{2}} \frac{\sin^{i} \theta}{\cos^{i} \theta} d\theta + \right) \]

\[ \left. \left( -w-u \right)^{J+L+1} \int_{\theta_{1}}^{\theta_{2}} \frac{\cos^{j} \theta}{\sin^{j} \theta} d\theta + \right) \]

\[ \left. \left( z-u \right)^{J+L+1} \int_{\theta_{1}}^{\theta_{2}} \frac{\sin^{i} \theta}{\cos^{i} \theta} d\theta \right\} \]
Similarly, the second term in C-3 is:

\[
\sum_{x\leq 2} \sum_{x=0}^{b+1} \sum_{j=0}^{s} \sum_{l=0}^{t} \left( \begin{array}{c} I \\ J \\ L \end{array} \right) \left( \begin{array}{c} K \\ L \end{array} \right) q^{(\beta, \delta, \kappa)} p^{(\alpha, \zeta, \eta)} \frac{u^{k \ell}}{\omega^k z^\ell}.
\]

\[
\left\{ (w-u)^{j \ell} \int_{\theta_2}^{\theta_3} \cos^2 \theta \right\}
\]

\[
\left. \sin^2 \theta \right\} d\theta
\]

\[
+ (-z-u)^{j \ell} \int_{\theta_2}^{\theta_3} \sin \theta \cos \theta d\theta
\]

\[
+ (-w-u)^{j \ell} \int_{\theta_3}^{\theta_4} \cos \theta \sin \theta d\theta
\]

\[
+ (z-u)^{j \ell} \int_{\theta_4}^{\theta} \frac{\sin \theta}{\cos \theta} d\theta \right\}
\]

The integrals with respect to \( \Theta \) still remain to be evaluated. By using the chain rule

\[
\frac{d}{dx} \sin^n x \cos^n x = \frac{\sin^m x}{\cos^n x} - \frac{m-1}{l-n} \frac{\sin^{m-2} x}{\cos^{n-2} x}
\]

Therefore
\[ \int \frac{\sin^m x}{\cos^n x} \, dx = \frac{1}{-n} \frac{\sin^m x}{\cos^n x} + \frac{m-1}{-n} \int \frac{\sin^{m^2} x}{\cos^n x} \, dx \quad \text{C-12} \]

Similarly,

\[ \frac{1}{n-1} \frac{d}{dx} \sin^m x \cos^n x = \frac{1-n}{m-n} \frac{\cos^m x}{\sin^n x} - \frac{m-1}{m-n} \frac{\cos^{m^2} x}{\sin^{n^2} x} \quad \text{C-13} \]

so that

\[ \int \frac{\cos^m x}{\sin^n x} \, dx = \frac{1}{1-n} \frac{\cos^{m-1} x}{\sin^{n-1} x} + \frac{m-1}{1-n} \int \frac{\cos^{m^2} x}{\sin^{n^2} x} \, dx \quad \text{C-14} \]

Equations C-12 and C-14 can now be solved iteratively for the integrals in theta from the integrals of the secant and cosecant functions. The induction matrix is then generated by integrating over the field point "(u,v)" numerically.
APPENDIX D

SYMMETRY IN PLATE INTERACTION

Using the notation of chapter five, the interaction matrix is defined as follows:

$$\mathbf{H} = \mathbf{U}^T \mathbf{C}$$

Now, $\mathbf{U}_{ij}$ is defined from equations 5-11 and 5-14 to be:

$$\mathbf{U}_{ij} = \int_{S^5} \phi_i(x_i, y_i) \tilde{A}_i(x_i, y_i) \ d^4s$$

$$= \int_{S^5} \phi_i \int_{S^5} \mathbf{J}_i(x, y) \mathbf{H}_i(x, y, x_i, y_i) \ d^4s \ d^4v \ d^4s \ d^4v$$

The electric current is expressable in terms of magnetic current, if the electric current is divergence free:

$$\frac{1}{2} \nabla \times \mathbf{M} = \mathbf{J}_i$$
But,

\[ \nabla \times \phi_i \hat{e}_3 = \mathbf{J}_i, \quad \mathbf{M}_i = z \mathbf{M}_i. \]

Therefore

\[ \mathbf{M}_i = \phi_i \hat{e}_z. \]

Equation D-3 now becomes

\[ \mathbf{G}^{T} \mathbf{W}_{ij} = \int_{S} \int_{S} \phi_i \phi_j \mathbf{H}_{m} d^r S \ d^s S \quad \text{D-4} \]

where \( \mathbf{H}_{m} \) is the magnetic field normal to the receiver plate produced by a unit magnetic dipole current normal to the transmitter plate.

Therefore, from equation 6-1b

\[ \mathbf{H}_{m} = \nabla \times \nabla \times \mathbf{F} \]

since the \( \mathbf{F} \) potential is produced by electric current sources.

The magnetic field is now written in terms of its vector potential,

\[ \mathbf{H}_{m} = \frac{1}{\varepsilon} \nabla \times \nabla \times \mathbf{F} \quad \text{D-5} \]

Since the magnetic current is due to a dipole, there is no
need to integrate over the source distribution. Therefore,

$$F = \varepsilon g M$$

where $g$ is a Green's function. The magnetic field is now written as a Green's function:

$$H_m = \frac{1}{Z} \varepsilon \nabla_x \nabla_x \hat{E}_z g$$

$$= \frac{-1}{Z} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) g$$

$$= \frac{-1}{Z} \varepsilon \nabla_{x}^2 g \quad \text{D-6}$$

Substituting D-6 into D-3 and using the chain rule to move the gradient symbol, the expression for $\varepsilon^\tau \omega_{ij}$ becomes

$$\varepsilon^\tau \omega_{ij} = \iint _{S} \int _{S} \left( \varepsilon \nabla_j \varepsilon \phi_i \cdot \nabla_j \varepsilon \phi_i \right) g \, dS \, dS \quad \text{D-7}$$

From equation D-7 we see that $\varepsilon^\tau \omega_{ij} = \tau^\varepsilon \omega_{ji}$. The effect of an eigencurrent $i$ in the transmitter plate on eigencurrent $j$ in the receiver plate is therefore the same as the effect of eigencurrent $j$ in the receiver plate on eigencurrent $i$ in the transmitter plate. This implies that the composite matrix in equation 5-25 is symmetric and can be inverted by using the weighted eigenvector method.
APPENDIX E

THE SOLUTION TO THE HELMHOLTZ EQUATION

Let $F$ be a function which satisfies the Helmholtz equation.

Then,

$$\nabla^2 F - k^2 F = 0 \tag{E-1}$$

Assume that $F$ has an $x$, $y$ and $z$ dependence of:

$$F(x, y, z) = X(x) \ Y(y) \ Z(z)$$

then

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} - k^2 = 0 \tag{E-2}$$

where "indicates a double derivative. Setting

$$\frac{\ddot{X}}{X} = -p^2, \quad \frac{\ddot{Y}}{Y} = -q^2$$
the following solutions for $X(x)$ and $Y(y)$ are obtained;

$$X(x) = X(0) e^{z^{ipx}} \quad Y(y) = Y(0) e^{z^{iqy}}$$

Therefore

$$\frac{\partial^2}{\partial z^2} \mathcal{Z} = k^2 + p^2 + q^2$$

Putting $p^2 + q^2 = k^2$, $Z(z)$ becomes

$$Z(z) = Z(0) e^{\frac{z^{ipx}}{\sqrt{k^2 + p^2 + q^2}}}$$

so that $F(x,y,z)$ has the form of

$$F(x,y,z) = F(0,0,0) e^{z^{ipx}} e^{z^{iqy}} e^{z^{iDx}}$$

where $p^2 = k^2 + p^2$ and $D = +1$ for positive propagation and $-1$ for propagation in the negative direction.
APPENDIX F

THE PRIMARY FIELDS AT THE SOURCE LEVEL DUE TO DIPOLE CURRENTS IN A LAYERED EARTH

In section 6.3 it was shown that the Hertz potentials could be expressed in terms of electric and magnetic current sources. In this section, an explicit representation for the Hertz potentials at the source level in terms of $M$ and $J$ will be developed.

The equivalence theorem (Jordan and Balmain, 1968) states that electric and magnetic fields can be replaced by distributions of magnetic and electric surface currents, $\mathbf{k}$ and $\mathbf{l}$:

\[ \mathbf{n} \times (\mathbf{H}^+ - \mathbf{H}^-) = \mathbf{k} \quad \text{F-1} \]

\[ -\mathbf{n} \times (\mathbf{E}^+ - \mathbf{E}^-) = \mathbf{l} \quad \text{F-2} \]

where $\mathbf{n}$ is a unit vector normal to the surface current, and the superscripts '+' and '-' designate positively and negatively propagating fields respectively. Equations F-1 and F-2 become:
\[-2 \mathbb{H}^+ + 2 \mathbb{H}^- = \mathbb{K}\]  
\[\mathbb{I} \mathbb{H}^+ - \mathbb{H}^- = \mathbb{K}\]  
\[2 \mathbb{E}^+ - 2 \mathbb{E}^- = \mathbb{L}\]  
\[\mathbb{I} \mathbb{E}^+ + \mathbb{E}^- = \mathbb{L}\]

As we have seen in section 6.3, \(E\) and \(H\) can be expressed in terms of \(\Pi\) and \(\Gamma\). Equations F-3 now become:

\[y \left( \Pi^+_{11} - \Pi^-_{11} \right) - \left( \Gamma^+_{22} - \Gamma^-_{22} \right) = \mathbb{K}\]  
\[y \left( \Pi^+_{22} - \Pi^-_{22} \right) + \left( \Gamma^+_{33} - \Gamma^-_{33} \right) = \mathbb{K}\]  
\[Z \left( \Gamma^+_{22} - \Gamma^-_{22} \right) + \left( \Pi^+_{22} - \Pi^-_{22} \right) = \mathbb{L}\]  
\[Z \left( \Gamma^+_{33} - \Gamma^-_{33} \right) + \left( \Pi^+_{33} - \Pi^-_{33} \right) = \mathbb{L}\]

where \(\Pi_{ij} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} \Pi\)
and the subscript designating the \( x_3 \) component has been discarded as the potentials are parallel to the vertical axis of the medium. We now put:

\[
\gamma = \Gamma_3^+ - \Gamma_3^-
\]

F-5a

\[
\chi = \gamma (P_1^+ - P_1^-)
\]

F-5b

\[
\Omega = \Omega_3^+ - \Omega_3^-
\]

F-5c

\[
\Phi = \chi (\Gamma^+ - \Gamma^-)
\]

F-5d

and then substitute equations F-5 into F-4. Therefore:

\[
\chi_1 - \gamma_2 = \chi K
\]

F-6a

\[
\chi_2 + \gamma_1 = \chi K
\]

F-6b

\[
\Phi_1 + \Omega_2 = \chi L
\]

F-6c

\[
\Phi_2 - \Omega_1 = \chi L
\]

F-6d
Differentiating equations F–6,

\[ \chi_{11} - \gamma_{21} = _1 K_1 \]  \hspace{1cm} F–7a

\[ \chi_{22} + \gamma_{12} = _2 K_2 \]  \hspace{1cm} F–7b

\[ \Phi_{11} + \bigcap_{12} = _1 L_1 \]  \hspace{1cm} F–7c

\[ \Phi_{22} - \bigcap_{12} = _2 L_2 \]  \hspace{1cm} F–7d

Rearranging equations F–7, and setting \( D^x = \nabla \cdot K \) and \( D^L = \nabla \cdot L \), we get

\[ \chi_{11} + \chi_{22} = \nabla \cdot K = D^x \]  \hspace{1cm} F–8a

\[ \Phi_{11} + \Phi_{22} = \nabla \cdot L = D^L \]  \hspace{1cm} F–8b

Again differentiating equations F–6:
\[ \chi_{l_2} - \gamma_{22} = K_l \]  
F-9a

\[ \chi_{x_1} - \gamma_{11} = K_l \]  
F-9b

\[ \Phi_{i_2} + \Omega_{22} = L_2 \]  
F-9c

\[ \Phi_{x_1} - \Omega_{11} = L_l \]  
F-9d

Setting \( C^k = V \times K \cdot n_3 \) and \( C^l = V \times L \cdot n_3 \), equations F-9 become

\[ (\gamma_{11} + \gamma_{22}) = C^k \]  
F-10a

\[ (\Omega_{11} + \Omega_{22}) = C^l \]  
F-10b

Now equations F-8 and F-10 are transformed into the Fourier wave number domain. Since

\[ \frac{\partial}{\partial x_i} \rightarrow -i \rho \quad \frac{\partial}{\partial x_2} \rightarrow -i q \]
and \[ \lambda^2 = \rho^2 + q^2 \] , we get

\[
\begin{align*}
\hat{\chi} &= -\frac{\hat{D}^x}{\lambda} \\
\hat{\phi} &= -\frac{\hat{D}^l}{\lambda} \\
\hat{\gamma} &= -\frac{\hat{C}^x}{\lambda} \\
\hat{\Omega} &= -\frac{\hat{C}^l}{\lambda}
\end{align*}
\]  \hspace{1cm} F–11

where \( \hat{\cdot} \) indicates a Fourier transformed function. Equations F–5 and F–11 now imply:

\[
\begin{align*}
\hat{\Gamma}^+ - \hat{\Gamma}^- &= -\frac{1}{\eta \lambda} \hat{D}^x \\
\hat{\Gamma}^* - \hat{\Gamma}^- &= -\frac{1}{\eta \lambda} \hat{D}^l \\
\hat{\Gamma}_x^* - \hat{\Gamma}_x^- &= -\frac{1}{\lambda^2} \hat{C}^x \\
\hat{\Pi}^* - \hat{\Pi}^- &= -\frac{1}{\lambda^2} \hat{C}^l
\end{align*}
\]  \hspace{1cm} F–12

Replacing the \( z \) derivative in the transform domain with \( -\eta \hat{D} \) (see Appendix E) equation F–13 becomes:

\[
\begin{align*}
\hat{\Gamma}^+ + \hat{\Gamma}^- &= \frac{1}{\eta \lambda^3} \hat{C}^x, \\
\hat{\Pi}^+ + \hat{\Pi}^- &= \frac{1}{\eta \lambda} \hat{C}^l
\end{align*}
\]  \hspace{1cm} F–14
The Hertz potentials have now been related to the curl and divergence of current at the source level. It is now necessary to find an expression for the curl and divergence of the current in terms of the current itself. For current dipoles, \( J = (j_1, j_2, j_3) \) and \( M = (m_1, m_2, m_3) \), and

\[
\mathbf{D}^k = \nabla \cdot j \quad \rightarrow \quad \mathbf{\tilde{D}}^k = -ip_j - iq_j
\]

F-15a

Similarly:

\[
\mathbf{\tilde{D}}^t = -ip_m - iq_m
\]

F-15b

\[
\mathbf{C}^k = -ip_j + iq_j
\]

F-15c

\[
\mathbf{C}^t = -ip_m + iq_m
\]

F-15d

Equations F-12 and F-15 imply:

\[
\begin{aligned}
\mathbf{\hat{D}}^+ \quad &\mathbf{\hat{D}}^- = \frac{1}{2 \lambda^t} (ip_m + iq_m) \\
\end{aligned}
\]

F-16
The $i^{th}$ component of $\Gamma$ only results from the $i^{th}$ component of magnetic current as stated previously. This is in apparent contradiction to equation F-16. However, because of the symmetry of the layered earth problem, the Hertz potentials are degenerate. Using the results of section 6.3 to equate the $\chi_3$ component of $\Gamma$ to its $\chi$ and $\lambda$ components, equation F-16 now becomes:

$$\hat{\Gamma}_3^- - \hat{\Gamma}_3^+ = \frac{\eta i q}{\lambda} \hat{\Gamma}_3^+ + \frac{\eta i q}{\lambda} \hat{\Gamma}_3^-$$  \hspace{1cm} F-17

or:

$$\hat{\Gamma}_1^+ + \hat{\Gamma}_1^- = \frac{1}{\eta Z} \hat{\Gamma}_1^- \hat{\Gamma}^- \hspace{1cm} F-18a$$

$$\hat{\Gamma}_2^+ + \hat{\Gamma}_2^- = \frac{1}{\eta Z} \hat{\Gamma}_2^- \hat{\Gamma}^- \hspace{1cm} F-18b$$

By symmetry, $\hat{\Gamma}_1^+ = \hat{\Gamma}_1^-$ and $\hat{\Gamma}_2^+ = \hat{\Gamma}_2^-$ at the source point since $\Gamma$ is the free space field there. Therefore

$$\kappa \hat{\Gamma} = \frac{\kappa m}{2 \eta Z}$$  \hspace{1cm} F-19
Similarly,

\[ \tilde{\Pi} = \frac{\kappa j}{2\eta y} \]

F–20

Again using the results of section 6.3 the \( \chi_3 \) component of the free space Hertz potentials due to all components of the source dipole can be obtained by using the results of equations F–19 and F–20. These results are expressed in table 6–3.
REFERENCES


Dyck, A.V., Bloore, M., and Vallee, M.A., 1980, User Manual for Programs Plate and Sphere, Research in Applied Geophysics, 14, University of Toronto

Eadie, T., 1979, Stratified Earth Interpretation using Standard Horizontal Loop Electromagnetic Data, Research in Applied Geophysics, 9, University of Toronto


Kisak, E. and Sylvester, P., 1975, A Finite Element Package for Magnetotelluric Modelling, Computer Physics Communications, 10

Lajoie, J.J. and West, G.F., 1976, Electromagnetic Response of a Conductive Inhomogeneity in a Layered Earth, Geophysics, vol. 41, no. 6A


Lamontagne, Y. and West, G.F. 1971, EM Response of a Rectangular Thin Plate, Geophysics, vol 36, no. 6


Martin, L., Field Outside a Conducting Strip in the Presence of a Magnetic Dipole, Masters Thesis, University of Toronto


Ranasinghe, V.V.C., 1962, Inductive Interaction Between Nearby Conducting Bodies in a Time Varying Magnetic Field, Masters Thesis, University of Toronto


West, G.F., 1960, Quantitative Interpretation of Electromagnetic Prospecting Measurements, PhD Thesis, University of Toronto