Convergence in shape of Steiner symmetrized line segments

by

Arthur Korneychuk

A thesis submitted in conformity with the requirements
for the degree of Master of Science
Graduate Department of Mathematics
University of Toronto

© Copyright 2015 by Arthur Korneychuk
Abstract

Convergence in shape of Steiner symmetrized line segments

Arthur Korneychuk
Master of Science
Graduate Department of Mathematics
University of Toronto
2015

Building on the recent work of Burchard, Bianchi, Gronchi and Volcic, it is shown that for every finite union of line segments, every sequence of Steiner symmetrals will converge in shape. It is also shown via counterexample that the rotation scheme used in the aforementioned paper to establish the square summable case will not work generally.
## Contents

Introduction  

1 Preliminaries  
   1.1 Steiner Symmetrization  
   1.2 Rotations  
   1.3 Summability and convergence  

2 Convergence of Symmetrals  
   2.1 Symmetrals do not always converge  
   2.2 Rotation scheme for convergence in shape  
   2.3 Limitation of rotation scheme  

3 Main Result  
   3.1 Convergence in shape of line segments  
   3.2 Concluding remarks  

Bibliography
Introduction

Given a compact set $K$ and a sequence of symmetrizations $S_{u_n}$, we can consider the convergence properties of the sequence of symmetrals

$$K_n = S_{u_n} ... S_{u_1} K$$

Such sequences enjoy a host of monotonicity properties useful in optimization problems, but there are open questions regarding their convergence. In a recent paper [1], it is shown that for any sequence of Steiner symmetrals whose successive angles are square summable and any compact set $K$, every sequence of symmetrals $K_n$ converges up to isometry. It leaves as an open question what happens when the assumption of square summability is dropped. This thesis gives a partial answer to this question. In particular, two results are established. First, the construction used in [1] does not work when the assumption of square summability is dropped. This is the subject of chapter two, where we sketch out the construction of a counterexample.

The second result is that any finite union of line segments converge in shape under any sequence of symmetrizations. This is the subject of chapter three. Thus for this restricted class of sets, the open question is resolved affirmatively. The proof relies on the fact that symmetrization acts very similarly to projection when acting on line segments.
Chapter 1

Preliminaries

1.1 Steiner Symmetrization

Following [2], we define Steiner symmetrization as follows: let $K$ be a compact set in $\mathbb{R}^d$, and $u$ a direction, by which we mean that $u$ has unit norm, i.e. $u \in S^{d-1}$. For each point $z \perp u$, we compute the one-dimensional measure of $K$ with the line through $z$ in the direction of $u$, and replace it with the closed interval of the same length centered on $u\perp$. If the intersection is empty, so is the interval; if it’s nonempty and of measure zero, then it is a single point. The resulting set is $S_uK$. Formally:

Given a compact set $K \subset \mathbb{R}^d$ and a direction $u \in S^{d-1}$, we define the **Steiner symmetral of $K$ in the direction** $u$ as

$$ S_uK = \{ z + tu \mid z \in \pi_uK \text{ and } |t| \leq \frac{1}{2}m_1(l_z \cap K) \} $$

$\pi_u$ is projection onto $u\perp$. $m_1$ is one-dimensional Lebesgue measure. For a more comprehensive discussion of Steiner symmetrization, see [3]. A few remarks about this definition are in order. Since our focus is on line segments, note that the symmetral of a centralized line segment (one endpoint is the negative of the other) is almost always equivalent to projection in the same direction, i.e. $S_u l = \pi_u l$. The two exceptions to this are when $u$ and $l$ are colinear, i.e. when $u = \pm \frac{l}{||l||}$. This breaks the linearity of the operation in a non-trivial way, as a sequence of directions may overlap with a line segment infinitely many times without resulting in any major degeneracy.

Another remark is that the symmetral of a finite union of line segments is almost always the union of the symmetrals of the individual line segments, i.e.

$$ S_u(l_1 \cup \ldots \cup l_k) = S_{u_1}l_1 \cup \ldots \cup S_{u_k}l_k $$
The exception is when there are two colinear, non-overlapping line segments and we symmetrize in their direction. This will not be a serious obstacle since it can only happen finitely many times in a given sequence, given that the two segments will become one. Given that we’re concerned only about convergence, and hence the tail end of a sequence, we can consider a new sequence starting from the point in the old sequence where we no longer have line segments merging. The two sequences will have the same limit, if they converge. We will use this idea several times in the paper to assume without loss of generality that certain special cases do not occur anywhere in a sequence of symmetrals.

Throughout the paper, we use the Hausdorff metric. We denote the ball around the origin of radius $\delta$ as $B_\delta$. By $K + H$ for sets $K$ and $H$ we mean the Minkowski sum $\{k + h \mid k \in K, h \in H\}$. Given compact sets $K$ and $G$, the Hausdorff distance between $K$ and $G$ is defined as

$$d_H(K, G) = \inf\{\delta > 0 \mid K \subset G + B_\delta \text{ and } G \subset K + B_\delta\}$$

The results concern the convergence of line segments, which can be characterized by their endpoints. We denote the line segment between points $a$ and $b$ by $ab$. The following proof is little more than validation that the Hausdorff metric provides a reasonable notion of convergence of line segments:

**Proposition 1.** Let $l_n = \overline{a_nb_n}$ be a sequence of line segments. If $a_n \to a$, and $b_n \to b$, then $l_n$ converges to $l = \overline{ab}$ in the Hausdorff metric.

**Proof.** Suppose $a_n \to a$ and $b_n \to b$. Let $\epsilon > 0$ and choose $N$ sufficiently large so that for all $n \geq N$,

$$|a_n - a| < \epsilon/2 \text{ and } |b_n - b| < \epsilon/2$$

Then the convexity of norm gives us the result. For all $n \geq N$, $t \in (0, 1)$,

$$|ta_n + (1 - t)b_n - (tb_n + (1 - t)b)| = |t(a_n - a) + (1 - t)(b_n - b)|$$

$$\leq t|a_n - a| + (1 - t)|b_n - b|$$

$$< \epsilon$$

So in particular, for each point on one segment (represented as a convex combination of endpoints) the corresponding convex combination of the other segment’s endpoints is within $\epsilon$. Thus

$$l_n \subset l + B_\epsilon \text{ and } l \subset l_n + B_\epsilon$$
As a corollary, we get that for centrally symmetric line segments, it suffices to show convergence for a single endpoint. So rather than dealing with entire sets of points, we will look at just individual endpoints for which the analysis is much simpler. Unfortunately, we don’t have \( S_u a(−a) = (S_u a)(S_u(−a)) \) in the case where \( a \) lies in the direction \( u \). So we define a new operator \( S'_u \) which gives us this relation. In particular, define \( S'_u \) as a function on \( R^m \) that acts as \( S_u \) (in particular, as projection) on points that don’t lie in the direction \( u \), and as identity on points that do. Thus

\[
S_u a(−a) = (S'_u a)(S'_u(−a))
\]

The upshot of all of this is that the convergence of the symmetrals of a centralized line segment has been reduced to the convergence of a single sequence of points, where each point is related to the previous by either a projection or by identity. We derive the convergence of arbitrary line segments as a corollary of the centralized case.

To get from the centralized case to the arbitrary case, the following is a useful characterization of line segments. We can represent a line segment as a parallel, centralized line segment of the same length, translated by the midpoint of the original segment.

**Proposition 2.** Let \( l = ab, \ c = \frac{b - a}{2}, \ L = (l - \frac{b - a}{2}) \).

Then \( l = c + L \) and \( S_u l = S_u c + S_u L \)

**Proof.** Immediate from definitions.

In particular, convergence of both the midpoint \( c \) and the centralized segment \( L \) give us convergence of \( l \).

### 1.2 Rotations

Simple examples exist of sequences of symmetrals failing to converge even under strong conditions on the sequence of directions. This is explored in more detail in chapter two. In all known cases, however, they converge in shape. Formally, a sequence of compact sets \( \{K_n\}_{n\in N} \) **converges in shape** if there exists a sequence of isometries \( \{R_n\}_{n\in N} \) such that \( R_n K_n \) converges.

Given a pair of orthonormal vectors \( v, w \) we can define a rotation that rotates the \( vw \) plane by \( \theta \) while fixing all vectors orthogonal to this plane. Treating \( v, w \) as column
vectors, it can be checked that this rotation is given by the matrix

\[ R_{v,w} = I_d + \sin \theta (vw^T - wv^T) - (1 - \cos \theta)(vv^T + ww^T) \]  

(1.1)

This type of rotation is of special interest to us as it rotates one vector to another in a minimal fashion, in the sense that the entire space orthogonal to the span of those two vectors is fixed. This allows us to derive the existence of a rotation needed for the main result:

**Proposition 3.** Given a point \( x = (x_1, ..., x_d) \) in \( \mathbb{R}^d \), and \( n < d \), there exists a rotation \( R \) such that \( Rx = (x_1, ..., x_{n-1}, x'_n, 0, ..., 0) \) for some \( x'_n \in \mathbb{R} \), and \( R \) fixes the span of \( e_1, ..., e_{n-1} \).

**Proof.** We apply the rotation \( R = R_{e_n, e_{n+1}} R_{e_{n+1}, e_{n+2}} ... R_{e_{d-1}, e_d} \) where the angle in each rotation \( R_{e_i, e_{i+1}} \) is chosen such that \( (x_1, ..., x_i, x'_{i+1}, 0, ..., 0) \) is sent to \( (x_1, ..., x'_i, 0, ..., 0) \) for some \( x'_i \). Each rotation fixes every other component so we end up with the desired \( Rx \), and since each rotation fixes at least the span of \( e_1, ..., e_{n-1} \), the composite rotation \( R \) will do so as well.

### 1.3 Summability and convergence

Here we record two propositions relating summability and convergence. The first is a condition for convergence from basic analysis that arises frequently in this context. We omit the simple proof:

**Proposition 4.** If \( \sum_{i=1}^{\infty} |a_n - a_{n-1}| = M < \infty \), for \( a_n \in \mathbb{R} \), then \( a_n \) converges.

In both of the following chapters, we are interested in whether a sequence of numbers is summable or merely square summable.

**Proposition 5.** Suppose \( \{u_n\}_{n \in \mathbb{N}} \) is a sequence of directions and \( \{p_n\}_{n \in \mathbb{N}} \) a sequence of points in \( \mathbb{R}^d \) such that \( p_n = S_{u_n} p_{n-1} \) for \( n \geq 1 \). Then

\[ \sum_{i=1}^{\infty} ||p_i - p_{i-1}||^2 < \infty \]

**Proof.** The proof rests on showing the following equality:

\[ ||p_0||^2 = ||p_n||^2 + \sum_{i=1}^{n} ||p_i - p_{i-1}||^2 \]
If this is true for every \( n \), then

\[
\sum_{i=1}^{n} ||p_i - p_{i-1}||^2 \leq ||p_0||^2
\]

This immediately gives us

\[
\sum_{i=1}^{\infty} ||p_i - p_{i-1}||^2 \leq ||p_0||^2 < \infty
\]

as desired.

We prove the equality by induction. Consider the point \( p_{n-1} - p_n = p_{n-1} - S'_{u_n} p_{n-1} \). If \( S' \) acts as identity, this point is the origin. Otherwise, if \( S' \) acts as projection, then \( p_{n-1} - \pi_{u_n} p_{n-1} \) is orthogonal to \( p_{n-1} \). This observation allows us to use the Pythagorean Theorem as follows:

\[
||p_0||^2 = ||p_1 + (p_0 - p_1)||^2 = ||p_1||^2 + ||p_0 - p_1||^2
\]

This proves the base case. Assuming it holds for \( n - 1 \),

\[
||p_0||^2 = ||p_{n-1}||^2 + \sum_{i=1}^{n-1} ||p_i - p_{i-1}||^2
\]

\[
= ||p_n + (p_{n-1} - p_n)||^2 + \sum_{i=1}^{n-1} ||p_i - p_{i-1}||^2
\]

\[
= ||p_0||^2 + \sum_{i=1}^{n} ||p_i - p_{i-1}||^2
\]

This proves the claim, and hence also the proposition. \( \square \)
Chapter 2

Convergence of Symmetrals

In this chapter we take a closer look at convergence in shape of symmetrals, explain the main result of [1] for the special class of sequences of directions whose angles are square summable, and show how this result does not generalize beyond that class. Nothing in this chapter is used in the third chapter, and may be skipped to get to the main result.

2.1 Symmetrals do not always converge

It was mentioned in the first chapter that there exist simple counterexamples to the convergence of symmetrals. One such example involves taking a line segment in two dimensions and sequence of directions \( u_n \) such that the angle between successive rotations, defined by \( \cos \theta_n = \langle u_n, u_{n-1} \rangle \), is square summable but not summable, and always rotated the same way (e.g. clockwise). Specifically, we take the line segment \( l \) from the origin to \( e_2 \), and take \( e_1 \) as the first direction \( u_1 \). We define \( u_n \) as the direction that forms an angle of \( 1/n \) radians with respect to \( u_{n-1} \), in the clockwise direction. Each symmetral then shrinks the segment by a factor of \( \cos(1/n) \), and rotates it by \( 1/n \) radians. Square summability implies that the partial products \( \prod \cos(1/n) \) converge to a positive number, so the length converges. Lack of summability implies that the partial sums \( \sum 1/n \), which quantify the spinning of the segment, do not converge. Thus the line segment does not converge.

We’re shrinking the norm by a monotone factor, so the length of the segments clearly converges. But we’re also introducing a rotation by projecting onto a sequence of moving hyperplanes \( u_n \perp \), which is responsible for the divergence.
2.2 Rotation scheme for convergence in shape

One idea of how to fix the unwanted rotational component is to rotate the hyperplane to a fixed place at each iteration. Specifically, we apply the following rotation scheme:

Let $R'_1$ send $u_1$ to $e_1$, fixing $(\text{span}\{e_1, u_1\})^\perp$.
Let $R'_n$ send $R'_{n-1} \ldots R'_1 u_n$ to $e_1$, fixing $(\text{span}\{e_1, R'_{n-1} \ldots R'_1 u_n\})^\perp$.
Let $R_n = R'_n \ldots R'_1$.

This gives us a sequence of rotated symmetrals $K_n = R_n S_{u_n} \ldots S_{u_1} K$ for arbitrary compact sets $K$. Using the unproven commutation relation $R S_u H = S_{R u} R H$, we can rewrite the expression for $K_n$ as

$$K_n = S_{e_1} R'_n S_{e_1} R'_{n-1} \ldots S_{e_1} R'_1 K$$

In particular, this gives us the relation

$$K_n = S_{e_1} R'_n K_{n-1}$$

So effectively we’re implementing the simplest rotations that force the symmetrizations to be in the same direction, so that the hyperplane being projected against is always $e_1^\perp$. It’s easy to see that this gives us convergence in shape of the two-dimensional line segment, as each iteration yields a line segment from the origin to a shrinking point along $e_2$. What’s remarkable is that if the sequence of directions has square summable angles, this exact rotation scheme works in every dimension for every compact set $K$. This is the main result of [1]. This constructively proves that with the square summable assumption, every sequence of symmetrals converges in shape.

An open question is to what extent we can relax the assumption of square summability. If we permit orthogonal directions, this same rotation scheme clearly won’t work, as can be seen by taking the line segment from $-e_1$ to $e_1$ and the sequence of directions $u_{2n-1} = e_1, e_{2n} = e_2$. Every symmetrization fixes the set, but the rotations rotate the line segment by $\pi/2$ each time. It’s clear where we go wrong: symmetrizing in the direction of a line segment produces a dramatically different effect from symmetrizing in an arbitrarily close direction, and the rotation scheme does not reflect this. But even if we restrict the angles to not being orthogonal, a counterexample can still be produced (albeit far less trivially).
2.3 Limitation of rotation scheme

Now we sketch the construction of a counterexample in three dimensions. We start with a line segment in the $yz$ plane (i.e. $e_1$). By (2.1) the symmetral at each iteration can be computed by performing a rotation on such a segment, and then deleting the $x$-component, since symmetrization reduces to projection. We want to see what happens to a line segment (or more precisely, its endpoint) in the $yz$ plane, $l = (0, b, c)$, when we apply a rotation of the form specified, i.e. a rotation from $u = (u_x, u_y, u_z)$ to $e_1$. Without loss of generality, we suppose that $\langle u, e_1 \rangle$ is between $0$ and $\pi/2$. This is because $u$ and $-u$ define the same symmetrals, we can replace a direction with its negative without affecting convergence. In order to use (1.1), we need a pair of orthonormal vectors. In the notation of (1.1) we can take $w = e_1$ and

$$v = \frac{u - \langle u, e_1 \rangle e_1}{\|u - \langle u, e_1 \rangle e_1\|} = \frac{1}{\sin \theta} (0, u_y, u_z)$$

where $\theta$ is the angle between $u$ and $e_1$.

Computing the corresponding rotation matrix $R$ and applying it to the segment $l$ yields

$$Rl = (-\langle u, l \rangle, b - \frac{u_y}{1 + \cos \theta} \langle u, l \rangle, c - \frac{u_z}{1 + \cos \theta} \langle u, l \rangle)$$

Thus

$$RS_u l = S_{e_1} R l = (0, b - \frac{u_y}{1 + \cos \theta} \langle u, l \rangle, c - \frac{u_z}{1 + \cos \theta} \langle u, l \rangle)$$

In order to determine convergence, we are particularly interested in the angle $\alpha$ between successive symmetrals. Of special interest is therefore the following inner product:

$$\|RS_u l\| \|l\| \cos \alpha = \langle RS_u l, l \rangle = \|l\|^2 - \frac{\langle u, l \rangle^2}{1 + \cos \theta}$$

The left equality is the definition of $\alpha$, the right equality is the computation of the dot product. Rearranging, we get

$$\cos \alpha = \frac{\|l\|^2}{\|l\| \|RS_u l\|} - \frac{\langle u, l \rangle^2}{(1 + \cos \theta) \|l\| \|RS_u l\|}$$

Now consider a sequence $\{u_n\}_{n \in \mathbb{N}}$ in $S^{d-1}$, with $\theta_n \in (0, \pi/2)$, for $\cos \theta_n = \langle u_n, u_{n-1} \rangle$ and a line segment identified by its endpoint $l = (0, b, c)$, with $l_n$ being the $n$th rotated
symmetrical. If the norm converges to a positive number $A$, then

$$\cos \alpha_n \approx 1 - \frac{\langle u_n, l_{n-1} \rangle^2}{(1 + \cos \theta_n)A^2}$$

This tells us that

$$|\alpha_n| \approx |A_n \langle u_n, l_{n-1} \rangle|$$

The sequence of numbers $A_n = \frac{1}{\sqrt{(1 + \cos \theta_n)A}}$ is bounded by $1/A$ and $1/\sqrt{2}A$. Thus the upshot of this computation is that the summability of $\alpha_n$, which tells us whether the net rotation converges or not, is equivalent to the summability of the inner products $\langle u_n, l_{n-1} \rangle$. From this computation, we see that a counterexample would need to satisfy two conditions: the norm must converge to a positive number and the sequence of inner products $\langle u_n, l_{n-1} \rangle$ must not be summable.

Satisfying the above conditions is not difficult. Start with some line segment $(0, 1, 0)$, then pick a sequence of directions $u_n = (e^{-n}, y_n, z_n)$ such that $\langle u_n, l_{n-1} \rangle = \frac{C}{n}$ for a constant $C$ chosen to ensure the norm converges to a positive number. There are only two such directions satisfying this equality, and we can pick the one that makes the line segment spin in the same direction each time to avoid cancellation of the net angle.
Chapter 3

Main Result

This chapter is structured as follows. Lemma 1 shows that inner products converge under the operation $S'$. This gives us convergence of angles between line segments that don’t collapse to the origin. Lemma 2 shows that a maximal linearly independent set is preserved under $S'$. Theorem 4 establishes convergence in shape for finite unions of centralized line segments. The proof of the theorem looks at a set of points consisting of one endpoint from each line segment. From this it selects a maximal linearly independent set, which acts as a ‘frame’ for the shape of the set. In particular, by fixing the orientation of this frame, we fix the rest of the linearly dependent points. By Lemma 2 this frame is preserved at every iteration. By Lemma 1, the points in this frame converge, and hence the linearly dependent ones do as well. This proves convergence of the union.

Finally, we use Theorem 4 to derive the general case, and also to show that the result holds if we use projections instead of symmetrizations.

3.1 Convergence in shape of line segments

Lemma 1. Let $a,b \in \mathbb{R}^n$.
Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of directions.
Let $a_n = S'_{u_n} \ldots S'_{u_1} a$, $b_n = S'_{u_n} \ldots S'_{u_1} b$.
Then $P_n = \langle a_n, b_n \rangle$ converges.

Proof. If $a_n, b_n$ ever fall on the same line through the origin, the result becomes trivial as the inner product reduces to the product of norms, each of which converges by monotonicity.

So suppose this never happens. We first prove the claim under the assumption that the $S'$ operator acts as projection at all but finitely many iterations. Without loss of
Chapter 3. Main Result

generality, assume it acts as projection for every iteration. We show that the quantity
\[ D_n = P_n - P_{n-1} = \langle a_n, b_n \rangle - \langle a_{n-1}, b_{n-1} \rangle \]
is absolutely summable, which proves the lemma by Proposition 4. First, consider the following two quantities:

\[ \langle a_n, b_n - b_{n-1} \rangle \quad \text{and} \quad \langle b_n, a_n - a_{n-1} \rangle \]
a_\text{n is being projected onto the subspace } u_n^\perp, \text{ while } b_n - b_{n-1} \text{ lies entirely on } u_n. \text{ Thus the first inner product is 0, and by the same reasoning, so is the second. Thus}

\[ \langle a_n, b_n \rangle = \langle a_n + (a_n - a_{n-1}), b_n + (b_n - b_{n-1}) \rangle \]
\[ = \langle a_{n-1}, b_{n-1} \rangle + \langle a_n - a_{n-1}, b_n - b_{n-1} \rangle + \langle a_n - a_{n-1}, b_n - b_{n-1} \rangle \]
\[ = \langle a_{n-1}, b_{n-1} \rangle + \langle a_n - a_{n-1}, b_n - b_{n-1} \rangle \]

Rearranging, we end up with

\[ D_n = \langle a_n - a_{n-1}, b_n - b_{n-1} \rangle \]

Taking absolute values,

\[ |D_n| \leq ||a_n - a_{n-1}|| \cdot ||b_n - b_{n-1}|| \leq ||a_n - a_{n-1}||^2 + ||b_n - b_{n-1}||^2 \]

Thus by Proposition 5, \(|D_n|\) is summable.

Next we prove the lemma under the assumption that \(S'\) acts as identity infinitely many times. We show the inner product must converge to 0. Consider an index \(n\) where \(S'_{u_n}\) acts as identity on one of the points (and only one, by assumption). One point is fixed on the direction \(u_n\), while the other gets projected into \(u_n^\perp\), so they’re orthogonal. In particular, \(P_n = 0\) for every index where \(S'_{u_n}\) acts as identity.

Consider an index for which \(S'\) does not act as identity on either point. Denote this index by \(k+n\), where \(k\) is the biggest previous index such that \(S'\) last operated as identity. In particular, \(P_n = 0\) for \(k \leq i \leq k+n\). We know that \(P_k = 0\).

Thus it follows that

\[ P_{k+n} = P_k + \sum_{i=k+1}^{k+n} D_i = \sum_{i=k+1}^{k+n} D_i \]

Taking absolute values and using the computation for \(D_i\) from above, we get

\[ |P_{k+n}| \leq \sum_{i=k+1}^{k+n} |D_i| \leq \sum_{i=k+1}^{\infty} (||a_i - a_{i-1}||^2 + ||b_i - b_{i-1}||^2) \]
This is the tail end of a convergent series, so it goes to zero as $k$ grows large, which happens as $k+n$ is taken large.

Remark: it may amuse the reader to note that Lemma 1 can be proved immediately for projections using the polarization identity and the fact that projections are linear operators with operator norm not greater than one. Nonetheless the proof used above was chosen since we need the computation for $|D_n|$.

**Lemma 2.** If $A = \{b_0, ..., b_k\}$ is a linearly dependent set in $\mathbb{R}^d$, and $u$ is a direction, then $A' = \{S'_u b_0, ..., S'_u b_k\}$ is linearly dependent.

**Proof.** It is clear that projection preserves linear dependence in this way, as projection is linear. $S'_u$ reduces to projection except when it acts on a point in the direction $u$, so if $A$ contains no points in this direction, then $A'$ is dependent. Suppose without loss of generality that $b_0$ is in the direction $u$. If another point $b_i$ is in this direction, then $b_0$ and $b_i$ are dependent (being equal up to a scalar multiple), and since $S'$ fixes both, $A'$ remains dependent. So suppose $b_0$ is the only point in this direction. If $b_1, ..., b_k$ are dependent, then since none of them lie in the direction $u$, $S'$ acts as projection on them and they remain dependent. This leaves the case where $b_0$ is a linear combination of $b_1, ..., b_k$. Thus

$$b_0 = c_1 b_1 + ... + c_k b_k$$

where $c_i$ are not all zero. Now if we project along $u$ and use linearity we get

$$0 = \pi_u b_0 = c_1 \pi_u b_1 + ... + c_k \pi_u b_k$$

This is a non-trivial linear combination of points in $A'$ equal to 0, showing that $A'$ is dependent.

**Corollary 3.** Given any finite set of points $a_1, ..., a_k \in \mathbb{R}^d$ and any sequence of directions $u_n$, there exists an $m \in \mathbb{N}$ such that

For all $n > m$, \{$(a^{(n)}_{k_1}, ..., a^{(n)}_{k_p})$\} is linearly independent iff \{$(a^{(n+1)}_{k_1}, ..., a^{(n+1)}_{k_p})$\} is.

**Proof.** By Lemma 2, $S'$ cannot create a new linearly independent set. Thus the number of linearly independent sets, which is finite, is monotonically decreasing and hence is eventually fixed after some index. That index is the $m$ for the claim.

**Theorem 4.** Let $l_1, ..., l_k$ be centralized line segments, $\{u_n\}_{n \in \mathbb{N}}$ a sequence of directions. Then $L_n = S_{u_n} ... S_{u_1} (l_1 \cup ... \cup l_k)$ converges in shape.
Chapter 3. Main Result

Proof. Let $a_i$ be an endpoint of the $l_k$ which does not collapse to the origin. To show that the union converges under a suitable sequence of isometries, we show that each line segment converges, using the fact that the symmetral of a union of centralized line segments is the union of the symmetrals. To show that each line segment converges, it suffices to show that a single endpoint from each segment converges, by the corollary to Proposition 1.

By Corollary 3, linear dependence is eventually preserved so we assume without loss of generality that it’s preserved across the entire sequence for notational simplicity.

Now we construct the isometries. Let $b_1, \ldots, b_m \in \{a_1, \ldots, a_k\}$ denote a maximal linearly independent set, chosen arbitrarily. By the previous paragraph, it remains a maximal linearly independent set after every finite iteration. For a given $n$, we construct a sequence of basic isometries $R_1^n, \ldots, R_k^n$ as follows. Let $R_1^n$ be the isometry sending $b_1^{(n)}$ to a positive multiple of $e_1$. In general, define $R_i^n$ as the isometry embedding $b_i^{(n)}$ into span\{${e_1, \ldots, e_i}$\} such that $\langle b_i^{(n)}, e_i \rangle > 0$. We do this by using Proposition 3 and then reflecting across $e_i$ if necessary. Define $R^n = R_k^n \ldots R_1^n$. This is the sequence of isometries used to determine convergence in shape. From this point on, when we refer to a point, we mean its image under the corresponding $R^n$. Since rotations are isometries, this will not affect any inner products or linear dependencies.

To see that this process determines a unique position for each point $b_i^{(n)}$, note that $b_i^{(n)}$ is fixed given that it has a particular norm, and is on the positive $e_1$ axis. For $b_i^{(n)}$, it is contained in the $i$-dimensional space span\{${e_1, \ldots, e_i}$\} and has a fixed inner product with each $b_j^{(n)}$ for $j < i$. Since $b_j^{(n)}$ form a spanning set for \{${e_1, \ldots, e_{i-1}}$\}, the first $i - 1$ coordinates of $b_i^{(n)}$ are fixed. Fixing the norm determines two solutions for the $i$th coordinate equal up to sign change, and $\langle b_i^{(n)}, e_i \rangle > 0$ fixes the sign.

A similar inductive argument yields convergence: it is easy but tedious to show that each coordinate of each point is a continuous function of converging inner products and norms. All that’s left to show is convergence of the linearly dependent $a_i$ not among the $b$’s. But note that since $a_i^{(n)}$ are always linearly dependent, that means $S'$ never acts as identity on them, nor on the elements of their linearly independent spanning sets among the $b$’s. In particular, $S'$ acts as projection, a continuous function. Thus the convergence of the $b_i^{(n)}$ immediately imply convergence of the $a_i$. Thus every one of the endpoints converges. Thus each rotated line converges, proving the result.

Corollary 5. The result holds if symmetrization operations are replaced with projections in the same directions.

Proof. Symmetrization and projection on line segments (points) differ only when the operation occurs in the direction of the segment (point) in question. Projecting in this
direction simply kills the segment (point) by collapsing it to the origin. Since there are only finitely many segments (points) in question, this can only happen finitely many times, after which we can replace the projections with symmetrizations and apply Theorem 4 to get the result.

**Corollary 6.** Theorem 4 holds without the assumption of centrality.

**Proof.** As we’ve seen by Proposition 2, we can view a line segment as a centralized segment translated by the midpoint of the original segment. Let $L_1, ..., L_k$ denote the not necessarily centralized segments. Let $a_1, ..., a_k$ denote their midpoints, and $l_1, ..., l_k$ the centralized segments, so that $L_i = a_i + l_i$. If we symmetrize in the direction of an $a_i$, it collapses to the origin and the $L_i$ becomes centralized. This can happen only finitely many times. So without loss of generality, we assume it never happens. But then $S_{u_n}$ applied to $a_i$ is equivalent to $S_{u_n}'$, so we can treat $a_i$ like the endpoint of a centralized line segment $a_i'$, and then apply Theorem 4 to $a_1', ..., a_k', l_1, ..., l_k$ to get convergence. □

### 3.2 Concluding remarks

For the case of square summable angles, it was enough to get a single rotation scheme that worked for every line segment and depended only on the choice of directions. This same rotation scheme works for every compact set. It is thus a potentially serious defect in our construction that the choice of isometries depends on the line segments. However, it is interesting to note that the proof for Theorem 4 could easily be adapted for projections in a way that does not depend on line segments. The essential idea behind the proof of Theorem 4 is to take a frame on which the points in our set are linearly dependent, and rotate the points in the frame so that they converge. For projections, there is an obvious frame to pick: any basis for $\mathbb{R}^d$. Thus for projections, we have a rotation scheme depending only on the choice of directions.

For symmetrizations, the difficulty is dealing with the fact that symmetrizing in the direction of a line segment will make it linearly independent from the symmetral of any basis which does not contain that direction. But this defines a narrow class of exceptions, and it seems likely that a simple argument will get around the issue. Doing so will give us a sequence of isometries that depends only on the directions, and from this we may have the necessary sequence of isometries to solve the general problem for arbitrary compact sets.
Bibliography

