Adaptive Fault Tolerant Control For Nonlinear Systems With Constraints

by

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Abstract

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In this thesis, fault tolerant control (FTC) design for nonlinear systems with input and output/state constraints are studied. To handle the input constraints, auxiliary systems are integrated with the controller. To deal with output constraints, Barrier Lyapunov Functions (BLFs) are used. When state constraints are in force, command filters are used in conjunction with BLFs. A novel adaptive fault tolerant cooperative tracking control (AFTCTC) scheme is developed for a class of input and output constrained nonlinear multiagent systems. Exponential convergence of the cooperative output tracking error into a small set around zero is guaranteed, while the constraints on the output will not be violated. The results are then extended to the design of a novel adaptive fault tolerant control (AFTC) scheme for a class of input and state constrained multi-input-multi-output (MIMO) nonlinear systems, where the constraints on the system state will not be violated.
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List of abbreviations

- **AFTC**: Adaptive fault tolerant control
- **AFTCTC**: Adaptive fault tolerant cooperative tracking control
- **BLF**: Barrier Lyapunov function
- **CR**: Control reconfiguration
- **FA**: Fault accommodation
- **FDI**: Fault detection and isolation
- **FTC**: Fault tolerant control
- **MIMO**: Multi-input multi-output
- **RHS**: Right hand side
- **SISO**: Single-input single-output
- **UAV**: Unmanned aerial vehicles
Chapter 1

Introduction

Reliability has always been an important consideration in designing and operating complex engineering systems. As nowadays the systems are depending heavily on the integration of sensors, actuators and other components, faults that occur in a local part can potentially affect the whole system in an undesirable way. Serious problems like degradation of performance or even safety hazards can arise if no proper action is considered. This is particularly true for large complex life-critical systems like aircrafts [1], [2].

In industry, hardware redundancy has been used to handle failures for some mission-critical components. Hardware redundancy refers to the design methodology of installing more than one component that performs the same function. In the event of any component failure, the faulty component will be replaced by the redundant healthy parts. For example, the first made-in-Singapore earth observation satellite XSAT [3], launched in 2011, has no single point of failure through hardware redundancy for all modules. Single point failure means any failure of a single component that could result in the failure of the whole system. In fact, XSAT has 20 redundant processing nodes, 4 redundant network elements, dual redundant command bus link, dual redundant LVDS data bus link, dual redundant power supply module, and 20
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distributed memory banks. In computing, triple modular redundancy, sometimes called triple-
mode redundancy (TMR) is a fault-tolerant form of N-modular redundancy. In TMR, three
systems perform a process and the result is processed by a majority-voting system to produce
a single output. If any one of the three systems fails, the other two systems can correct and
mask the fault. This arrangement has also been used in some of the computing units and some
of the satellites.

However, it is not always cost effective to install redundant hardware for every single compo-
nents. It may dramatically increase the cost of the whole system, or simply there is not enough
space to install redundant components. In such situations, it is desirable to design control algo-
rithms to maintain satisfactory performance, in the presence of potential system faults. This is
the design of fault-tolerant control (FTC) [4]-[7], and it is becoming an increasingly important
research topic. FTC generally can be categorized into two types, passive FTC and active FTC.
In passive FTC frameworks, which are usually based on robust control techniques, the faults
that have occurred are treated as special kinds of disturbance or uncertainties. The controller
structure remains the same for both normal and faulty situations. The passive FTC framework
can only be effective if it can handle faults as the worst case scenario. Active FTC, on the other
hand, determines the information about system faults and use such information in the controller
design. Active FTC schemes usually consist of two parts, the fault diagnosis/detection and iso-
lation (FDI) module and the fault accommodation (FA) or control reconfiguration module. The
objective of FDI module is to detect and estimate faults, whereas the FA module is to provide
compensation in the presence of faults, based on the information received from the FDI module.
Control reconfiguration modifies the control law so that a reduced set of control objectives can
still be achieved.

In addition, all engineering systems in practice are subject to constraints in one way or the
other. Failure to consider such constraints may result in degradation of system performance, or even hazards and system failure in the worst case. This is an important consideration when the system is subject to faults, which may perturb the system in an undesirable way that could lead to the violation of constraint requirements. Therefore, the effective handling of constraints has also received much attention from the research community. In the literature, usually two types of system constraints are discussed, but often separately. One is the control input constraint, which means that due to hardware limitations, the actuator can only supply a limited range of signals to the system. The other is the constraint on the system state and/or output, which requires that the system state/output to remain in some compact sets during system operation. Such requirements may come not only from system specifications, but also from safety considerations. For example, the velocity and acceleration of a bus on the road should be limited, where the velocity constraint is most commonly due to safety considerations, and the acceleration constraint may be due to hardware limitations and the comfort of drivers and passengers. In a robotic team, if the position of a certain agent is deviating too much from the position of the leader, it may lose contact with the leader, as some commonly used communication modules like XBEE can work effectively only within a certain range. As a result, such an agent may be lost in operation. When a team of unmanned aerial vehicles (UAVs) is passing between buildings with narrow gaps or pathways, if the position of a particular UAV deviates too much from the desired trajectory generated by the leader, it may hit the wall or other obstacles and become damaged. How to ensure the desired system performance under the effects of system faults and input constraints, while guaranteeing that the system state/output constraint requirements are not violated during operation, is an interesting and important research topic that deserves attention. To the best of our knowledge, this has not been addressed properly yet in the literature. A review of the state of the art will be given in
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the next Chapter.

1.1 Problems Description

First, we consider a class of multiagent systems, which could be the model of ground robots, autonomous surface vessels, etc. For such systems under input and output constraints, we wish to develop a FTC control scheme to effectively achieve the cooperative tracking objective despite the presence of actuator faults and system uncertainties. The objective is such that each follower agent in the multiagent network should track the leader’s trajectory. It is worth noticing that unlike the problem with single systems, which may have direct access to or be able to estimate the system state, here we assume that only the neighborhood synchronization error is available for control law designs, where the neighborhood synchronization error is defined in terms of the relative state information between the agent and its neighbours.

Next, for a general form of multi-input multi-output (MIMO) nonlinear systems with both parametric and nonparametric uncertainties, where the control input is constrained and subject to actuator faults, we wish to develop an effective FTC scheme to guarantee that the system output tracks the desired output trajectory, while ensuring that the predefined constraint requirements on the system state vectors are not violated during operation.

1.2 Contributions

To the best of our knowledge, there is no previous work in the networked cooperative tracking control literature that considers both the effects of control input constraints and constraint on agents’ output with potential actuator faults. Also, to the best of our knowledge, we are not aware of any previous work in the FTC literature that considers both the effects of control input
constraints and system state constraints for MIMO nonlinear systems under actuator faults. The main contributions of this thesis are summarized in the following points:

1. The issue of input and output constraints under actuator faults for multiagent systems is studied for the first time using our proposed FTC methods.

2. For a general class of MIMO nonlinear systems, state and input constraints can be effectively dealt with by the proposed novel FTC scheme. Both the state and input constraints can be asymmetric, and the state constraint requirements can be time-varying.

3. By the novel design of Lyapunov functions and controller structures, the proposed FTC structure can be regarded as a generalized scheme which can deal with systems with or without state/output constraints. In this regard, the systems without state/output constraints can be viewed as being constrained by an infinitely large bound.

4. The issue of “explosion of complexity” [46] in calculating the stabilizing functions in the backstepping analysis for nonlinear systems, which will be discussed in detail in the next chapter, is avoided by incorporating command filters into the FTC scheme.

5. Both parametric and nonparametric system uncertainties can be effectively dealt with, where the unknown parametric uncertainties can be time-varying, whereas the existing literature only considers constant unknown parameters.

1.3 Thesis Structure

This thesis is organized as follows. An introduction to some background and literature review related to our work is presented in Chapter 2. Limitations of some existing literature will be discussed. In Chapter 3, we discuss FTC schemes for a class of multi-agent systems for cooperative tracking control, where the actuators are faulty and the agents are subject to
input and output constraints. In Chapter 4, for a general MIMO nonlinear systems with both parametric and nonparametric uncertainties, a novel FTC control structure is proposed to deal with system state and input constraints under actuator faults. Two simulation studies further demonstrate the effectiveness of the proposed method. Finally, Chapter 5 presents the concluding remarks and future research directions.
Chapter 2

Background and Literature Review

To facilitate the discussion, some common mathematical notations used in this chapter will first be introduced. Then, we present an overview of fault tolerant control. Since actuator faults are the main focus in this thesis, common formulation about the actuator faults will be presented as well. Next, we review different schemes adopted in the literature for constrained systems, namely system state constraints and system input constraints, respectively. Benefits or drawbacks about some of these schemes will be discussed. It should be pointed out that when particular works from the literature are discussed, the notations and variables will be consistent with the works under discussion, that is, the same notations used in these works will be presented. The literature on multiagent systems will then be reviewed, with emphasis on how the issues of fault tolerance and system constraints are dealt with in the literature. Last but not least, as the backstepping control design approach will be used extensively in the next two chapters, we will provide a brief introduction about the this method and give a short illustrative example.
2.1 Mathematical Notations

The mathematical notations used in this chapter are fairly standard. In particular, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^{m \times n} \) denotes the set of real-valued matrices with the dimension \( m \times n \). If \( A \) is a \( m \times n \) real matrix, we will say \( A \in \mathbb{R}^{m \times n} \). The identity matrix is denoted as \( I \). If we want to emphasize the identity matrix is of the dimension \( m \times m \), we would write \( I_{m \times m} \). The notation \( |x| \) denotes the absolute value of the real value \( x \), while \( ||x|| \) denotes the Euclidean norm of the real-valued vector \( x \). The variable \( t \) is used to represent time, which is usually omitted for notational convenience. However, the notation \( x(0) \) represents the value of the variable \( x \) at the initial time \( t = 0 \).

For notational convenience and to avoid ambiguity, for the subsequent technical chapters, the notations used there will be introduced in each of the chapters in a self-contained manner.

2.2 Overview of Fault Tolerant Control

In this section, we introduce some background about fault tolerant control (FTC), which is one of the main themes of this thesis.

FTC can be generally categorized into passive FTC and active FTC, depending on the controller design. In passive FTC design, the controller design is based on robust control techniques, and the faults occurred are treated as special kinds of disturbance or uncertainties. It does not depend on any fault information, therefore it does not require a fault detection or estimation module. The control law is unchanged for normal and faulty situations. The advantage of passive FTC is that the control design is simple, and it does not suffer from time-delay effects, which may occur if fault detection or estimation models are incorporated. However, since passive FTC does not use any fault information, it has to work for all possible
faults. This can be a strong requirement. First, the information about all possible faults may not be readily available. In some situations, even if the system is normal, that is, with no faults occurring, the passive FTC still considers the system as having the worst case scenario. This can provide the system with unnecessary robustness, which can result in very conservative designs in many applications.

The so-called active FTC determines or estimates the faults that have occurred, and make use of the fault information in the controller design. An active FTC design usually consists of two major components, namely the fault detection and isolation module (FDI), and the control reconfiguration (CR) or fault accommodation (FA) module. Typically, the FDI module requires the knowledge of the system input and state/output information to detect and isolate a fault that has occurred. The term “fault detection” refers to the process of using the input and state/output information to determine whether the system behavior is normal. If the system state is not directly observable, often an observer-based controller design is incorporated to estimate the system state. Fault detection can be performed via different techniques like residual generation or fault estimation. A residual is a signal which is normally zero when there are no system faults, but becomes nonzero in the presence of faults. The reader is referred to [8] for more information about residual generation. However, just knowing a fault has occurred may not be sufficient for the CR/FA module to deal with the faults. “Fault isolation” refers to the process of identifying which faults may have occurred. One way is to design a residual signal which has a different response to different kinds of faults. Another way is to use estimators to estimate the faults. The processes of detecting and isolating the faults are together referred as FDI. Then, using the knowledge of the type of faults that have occurred, the objective now becomes to design the CR/FA module to handle the system faults.

Among the different kinds of faults, sensor and actuator faults are most commonly encoun-
tered for the industrial systems. Sensor faults refer to the situations where the sensors can no longer provide reliable measurement of the system state or other feedback signals. Therefore, the controller designed has to take the possibility of faulty measurements into consideration.

Sensor faults have been discussed in many works in the literature. For example, in [9], by correcting the faulty measurement with an estimate of the fault obtained from the sliding mode FDI scheme, good closed-loop performance is still maintained. In [10], multiple model adaptive estimation is used to detect and identify sensor failures in a mobile robot. In [11], the problem of sensor fault diagnosis in the class of nonlinear Lipschitz systems is considered. Reference [12] presents a robust fault isolation scheme for a class of nonlinear systems with sensor bias type of faults. A method of Bayesian belief network (BBN)-based sensor fault detection and identification is presented in [13], which is applicable to processes operating in transient or at steady-state. More works about sensor faults can be found in the survey paper [14]. Actuator faults are considered as serious threats to the system performance and safety of operators, since actuator faults mean that operators cannot change the dynamics of the system effectively via control input signals. This is also the main focus of this thesis. The issue of actuator faults has been studied in different works. Reference [15] proposes the design of a reconfigurable FTC and trajectory planning scheme with emphasis on online decision making using differential flatness. In the fault case, the proposed active FTC system consists of synthesizing a reconfigurable feedback control along with modified reference trajectories once an actuator fault has been diagnosed. In [16], using radial basis function neural networks to approximate the unknown nonlinear functions, an adaptive fault tolerant control scheme is designed. Both loss of effectiveness and lock-in-place have been considered for actuator faults. In [17], by manipulating the steady-state values of system states with the detection weighting matrix, a residual is then generated, through which actuator stuck faults including actuator outages can be detected ef-
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 EFFECTIVELY. Reference [18] considers finite-time attitude control for an over-activated spacecraft subject to actuator faults. A finite-time attitude compensation control scheme is developed for an over-activated rigid spacecraft subject to actuator faults in [19].

Since actuator faults will be the main focus in this thesis, we will examine the common formulations about actuator faults in detail. Regarding a faulty actuator control signal $u^F(t) = [u^F_1(t), \ldots, u^F_m(t)]^T \in \mathcal{R}^m$, where $m > 1$ is a positive integer to denote the number of actuator components, there are three common faults discussed in the literature [20, 21, 22, 23] in the following two scenarios:

1. **Stuck actuator faults:**

   $$u^F_i(t) = \bar{u}_i(t),$$  \hspace{1cm} (2.2.1)

   where $i \in [i_1, \ldots, i_{m-k}] \subset [1, \ldots, m]$, $\bar{u}_i(t)$ is the stuck actuator value for the $i$-th component of $u^F(t)$ that cannot be controlled.

2. **Multiplicative and additive actuator faults:**

   $$u^F_j(t) = \rho_j(t)u_j(t) + \phi_j(t),$$  \hspace{1cm} (2.2.2)

   where $j \in [j_1, \ldots, j_k] = [i_1, \ldots, i_{m-k}] \subset [1, \ldots, m]$ \hspace{1cm} ($[i_1, \ldots, i_{m-k}] \cup [i_1, \ldots, i_{m-k}] = [1, \ldots, m]$), $\phi_j(t)$ represents additive faults in the channel of control input, $\rho_j(t)$ represents the multiplicative actuator faults, which, when $0 < \rho_j(t) < 1$, is usually interpreted as loss of effectiveness in the actuator component. Notice that $\rho_j(t) \neq 0$ in this case, that is, no total failure for the actuator component, as $\rho_j(t) = 0$ corresponds to the case of stuck actuator faults that has already been formulated.
Collectively, in view of the formulation of actuator faults, we have

\[ u_i^F(t) = (\rho_l(t)u_l(t) + \phi_l(t)) + \chi_l(\bar{u}_l(t) - \rho_l(t)u_l(t) - \phi_l(t)), \quad l = 1, \ldots, m, \]

(2.2.3)

where \( \chi_l = 1 \) when the \( l \)th actuator component is suffering from the stuck actuator fault, and \( \chi_l = 0 \) when \( u_i^F(t) \) is corrupted with multiplicative and/or additive faults.

In the works mentioned above that deal with stuck actuator faults \([20, 21, 22, 23]\), the authors assume that for up to \( m - 1 \) stuck actuator faults, the remaining actuator components can still achieve the required control objective. This implies actuator redundancy for the systems discussed in these works. In such cases, the stuck actuator faults can be treated as a system disturbance not in the control input channel. Such an assumption about actuator redundancy is usually valid in some industries. For example, in the satellite industry \([24]\), although theoretically only three reaction wheels will be enough to provide torques for the satellite along any direction, it is a common practice to implement four or more reaction wheels, to provide redundancy in case stuck faults happen to any of the actuators. However, other industrial applications may not enjoy the luxury of actuator redundancy, in which any stuck actuator faults may make the system underactuated, making the system possibly not controllable.

In our subsequent presentation, we will not assume actuator redundancy in the systems to be discussed. Multiplicative and additive actuator faults will be modeled and analyzed, and various forms of system uncertainties will be discussed.

### 2.3 System Constraints for Nonlinear Systems

In this section, we will discuss both the state/output constraints and the input constraints and review related material in the literature.
2.3.1 Output and State Constraints for Nonlinear Systems

In the literature, different approaches to study output or state constrained systems have been investigated. For example, the works [25] and [26] are based on the notions of set invariance and admissible set control. Model predictive control that represents an effective control design methodology for handling both constraints and performance issues has been investigated in [27] and [28]. The approach of reference governors has also been proposed to solve the problems of constraints for nonlinear systems in [29]. For linear systems, convex optimization, like in our previous work [30], is also an effective tool to formulate and analyze system output or state constraints. The approaches mentioned above are either numerical in nature or depending heavily on computationally intensive algorithms to solve the constrained control problems. In [31], the problem of stabilization of nonlinear systems subject to state and control constraints has been considered, for cases where the state constraints need to be enforced at all times (hard constraints) and where they can be relaxed for some time (soft constraints). However, the authors assume no system uncertainties in the nonlinear system model. Reference [32] deals with the nonparametric identification of a special class of nonlinear uncertain systems affected by additive bounded disturbances in the state dynamics. The method of dynamic neural networks has been used. Reference [33] treats the problem of constrained control of nonlinear systems subject to parametric uncertainties. The approach combines the adaptive compensator design, an uncertain set estimation, and a constrained control approach. However, only first order nonlinear systems have been considered in these two works.

During the past decade, different forms of Barrier Lyapunov Functions (BLFs) have been proposed in the literature to deal with system output constraints in nonlinear systems [34, 35, 36, 37]. Unlike conventional Lyapunov functions, for example the quadratic forms, which are well-defined over the entire domain and radially unbounded for global stability, a BLF possesses
the special property of approaching infinity whenever its arguments approach some limits over a finite domain. By keeping the BLF bounded through analysis of the closed loop system, its argument will be bounded away from these limits. A graphical illustration is presented in Fig. 2.1, which is adopted from the work [38]. As we can see from the figure, when the system variable approaches the bounds $k_{b1}$ or $-k_{b1}$, the BLF $V_1$ approaches infinity.

![Figure 2.1: Graphic illustration of barrier functions [38].](image)

The following lemma from [37] shows how the BLF works in the closed loop system (Lemma 1 in [37]).

**Lemma 2.3.1** For any positive constants $k_{a1}, k_{b1}$, let $Z_1 := \{z_1 \in \mathcal{R} : -k_{a1} < z_1 < k_{b1}\} \subset \mathcal{R}$ and $N := \mathcal{R}^l \times Z_1 \subset \mathcal{R}^{l+1}$ be open sets. Consider the system

\[
\dot{\eta} = h(t, \eta) \tag{2.3.1}
\]

where $\eta := [w, z_1]^T \in N$, and $h := \mathcal{R}_+ \times N \rightarrow \mathcal{R}^{l+1}$ is piecewise continuous in $t$ and locally Lipschitz in $z$, uniformly in $t$, on $\mathcal{R}_+ \times N$. Suppose that there exist functions $U : \mathcal{R}^l \rightarrow \mathcal{R}_+$ and $V_1 : Z_1 \rightarrow \mathcal{R}_+$, continuously differentiable and positive definite in their respective domains,
such that

\[ V_1(z_1) \to \infty \quad z_1 \to -k_{a1} \quad \text{or} \quad z_1 \to k_{b1} \]  \tag{2.3.2} 

\[ \gamma_1(||w||) \leq U(w) \leq \gamma_2(||w||) \]  \tag{2.3.3} 

where \( \gamma_1 \) and \( \gamma_2 \) are class \( K_\infty \) functions. Let \( V(\eta) := V_1(z_1) + U(w) \), and \( z_1(0) \) belongs to the set \( z_1 \in (-k_{a1}, k_{b1}) \). If the inequality holds:

\[ \dot{V} = \frac{\partial V}{\partial \eta} h \leq 0 \]  \tag{2.3.4} 

then \( z_1(t) \) remains in the open set \( z_1 \in (-k_{a1}, k_{b1}) \) \( \forall t \in [0, \infty) \).

The detailed proof can be seen in [37]. Basically, this lemma implies that by constructing an overall Lyapunov function \( V \) which includes the BLF \( V_1 \), if \( V \) can be shown to be bounded, or its derivative to be negative definite in the closed loop analysis, then \( V_1 \) will also be bounded, and the signal \( z_1 \) can remain in the designed set without violating the constraint requirement.

In these above mentioned works, a log-type BLF has been proposed, which is of the from

\[ V(e) = \frac{1}{2} \log \frac{k_b^2}{k_b^2 - e^2}, \quad |e(0)| < k_b, \]  \tag{2.3.5} 

where \( k_b > 0 \) is the constraint on the system variable \( e \in \mathcal{R} \), which can be the system state, output tracking error, etc. \( |e(0)| < k_b \) means the absolute value of the initial value of \( e \) is within the constraint, so that the BLF is well defined. By keeping the BLF \( V \) bounded in the closed loop analysis, it can be guaranteed that the constraint requirement on the system variable \( e \)
will not be violated, that is, $|e| < k_b$ is guaranteed at all time during operation. Similarly, another BLF is proposed in [39] with the form

$$V(e) = \frac{k_b}{\pi} \tan^2\left(\frac{\pi e}{2k_b}\right), \quad |e(0)| < k_b.$$ \hfill (2.3.6)

In our previous works [40, 41, 42], a new $\tan$-type BLF has been proposed to deal with output constrained iterative learning control problems, where the constraint requirement is constant. The BLF is in the form of

$$V(e) = \frac{k_b^2}{\pi} \tan\left(\frac{\pi e^2}{2k_b^2}\right), \quad |e(0)| < k_b,$$ \hfill (2.3.7)

where, as $|e| \to k_b$, $V(e) \to \infty$. Notice that when there are no constraint requirements on $e$, we can let $k_b \to \infty$, By L’Hospital’s rule, we get

$$\lim_{k_b \to \infty} \frac{k_b^2}{\pi} \tan\left(\frac{\pi e^2}{2k_b^2}\right) = \frac{1}{2} e^2,$$ \hfill (2.3.8)

which means if we do not have constraint requirement on $e$, we can simply replace the BLF with the quadratic form. In such circumstances, the analysis approach remains the same as the cases without any constraint requirements on $e$. Therefore, the analysis with this particular BLF is a general approach that would also work for systems without constraint requirement on $e$. In comparison, notice that the two forms of BLFs proposed in the earlier literature (namely (2.3.5) and (2.3.6)) converge to zero when $k_b \to \infty$, instead of the quadratic form $\frac{1}{2} e^2$. Hence they cannot be used in a unified scheme which can address systems with or without constraint requirements.

In works including [44, 43, 45], the authors extend the use of $\log$-type BLFs to solve prob-
lems with state constrained nonlinear systems. In [45], the authors discuss the state constrained nonlinear switched systems using the log-type BLF. If we do not include the switching signals, since switched systems are not of primary interest in this thesis, the system formulation discussed in [45] can be represented as

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + x_2, \\
\vdots & \\
\dot{x}_i &= f_i(x_i) + x_{i+1}, \\
\vdots & \\
\dot{x}_{n-1} &= f_{n-1}(x_{n-1}) + x_n, \\
\dot{x}_n &= f_n(x_n) + g(x_n)u, \\
y &= x_1,
\end{align*}
\]

(2.3.9)

where, using the same notation from [45], \(x_1, \cdots, x_n\) are state variables, \(u\) is the control signal, \(\bar{x}_i = [x_1, \cdots, x_i]^T\), \(y\) is the system output. \(f_i(\bar{x}_i)\), \(g(\bar{x}_n)\) are smooth functions with \(g(\bar{x}_n) \neq 0\).

The control objective is to design a proper control algorithm so that the system output could track the desired output trajectory. Using the backstepping approach, the authors in [45] denote \(z_1 = x_1 - y_d\) and \(z_i = x_i - \phi_{i-1}, \ i = 2, \cdots, n\), where \(y_d\) is the desired output trajectory, \(\phi_{i-1}\) are stabilizing functions to be designed in the backstepping process. The BLFs used in [45] are designed as (as Eq. (6) in [45])

\[
V_i(z_i) = \frac{1}{2} \log \frac{b_i^2}{b_i^2 - z_i^2}, \quad i = 1, 2, \cdots, n,
\]

(2.3.10)
so that by properly designing the control algorithm, $|z_i| < b_i$ can be guaranteed. A typical result (Theorem 1 of [45]) obtained under certain assumptions and control law design is the following.

**Theorem 2.3.1** Let $A_i$ be an upper bound for $\phi_i$ in compact set $\Omega_i$:

$$A_i \geq \sup_{(\bar{x}_i, \bar{z}_i, \bar{y}_d)} |\phi_i(\bar{x}_i, \bar{z}_i, \bar{y}_d)|, \quad i = 1, \ldots, n - 1,$$

where $\Omega_i = \{ \bar{x}_i \in \mathcal{R}^i, \bar{z}_i \in \mathcal{R}^i, \bar{y}_d \in \mathcal{R}^{i+1} : |x_j| \leq D_{z_j} + A_{j-1}, |z_j| \leq D_{y_d}, |y_d| < A_0, |y_d^{(j)}| \leq B_j, j = 1, \ldots, i \}, \quad D_{z_j} = b_j \sqrt{1 - \prod_{k=1}^{i} b_k^2 - \prod_{k=1}^{i} b_k^2(0)}, i = 1, \ldots, n - 1.$

Given that the following conditions are satisfied:

1. $c_{i+1} > A_i + b_{i+1}$ holds for $\forall i = 1, 2, \ldots, n - 1$;
2. the initial conditions $\bar{z}(0)$ belong to the set $\Omega_{z0} = \{ \bar{z}_n \in \mathcal{R}^n : |z_i| < b_i, i = 1, \ldots, n \}$.

Then, the closed-loop system has the following properties.

(i) The signals $z_i(t)$ $i = 1, 2, \ldots, n$ remain in the compact set $\Omega_z = \{ \bar{z}_n \in \mathcal{R}^n : |z_i| < b_i, i = 1, \ldots, n \}$.

(ii) Every state $x_i(t)$ remains in the set $\Omega_x = \{ \bar{x}_n \in \mathcal{R}^n : |x_i| < D_{z_i} + A_{i-1} < c_i, i = 1, \ldots, n \}, \forall t \geq 0$, i.e., the full state constraints are never violated.

(iii) All closed-loop signals are bounded.

(iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.

The details of this theorem and the analysis presented in [45] will not be discussed here. Similar discussion and analysis is also presented in other works [44, 43]. However, regarding this analysis approach, there are several drawbacks:

1. The stabilizing function at the $i$-th step $\phi_{i-1}$ consists of the derivative of the stabilizing
function at the $i - 1$-th step, $\dot{\phi}_{i-2}$, as shown in the backstepping design in [45]. However, $\dot{\phi}_{i-2}$ involves taking the derivative of different system variables and adaptive laws. As the order of the system increases, the problem of “explosion of complexity” will make the computation and analysis of the stabilizing functions rather difficult. This is a well-known drawback about the backstepping technique [46], where the complexity of the controller grows drastically as the order of the system increases, caused by the repeated differentiations of certain nonlinear functions;

(2) It requires an assumption that there is an $A_i$ such that $A_i \geq \sup_{(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})} |\phi_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})|$, $i = 1, \ldots, n - 1$. However, since the stabilizing function $\phi_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$ involves different system state variables and adaptive laws, it is quite hard, in practice, to effectively estimate the value of $A_i$ beforehand.

(3) The theorem proposed in [45] also assumes that $c_{i+1} > A_i + b_{i+1}$ holds for $\forall i = 1, 2, \ldots, n-1$, so that the result (ii) $|x_i| < D_{z_i} + A_{i-1} < c_i$ can hold, thereby bounding the state variables by the finite constants $c_i$. However, since it is difficult to determine $A_i$ in advance, the value of $c_{i+1}$ is also difficult to obtain. In many cases, the value of $c_{i+1}$ cannot be predetermined. This means that if we have predefined system specifications regarding the state constraints, it would be quite difficult to ensure such requirements are met with suitable values of $c_{i+1}$.

In view of these drawbacks presented in the existing literature, how to propose an effective structure of analyzing predefined state constraint requirements, while avoiding the common difficulties such as “explosion of complexity”, is of practical importance and research interest.

### 2.3.2 Input Constraints for Nonlinear Systems

Over the years, many control structures have been proposed to deal with system input constraints, among which only a small part will be reviewed in this thesis. For linear systems,
convex optimization is an effective approach with mature numerical solving tools to solve the problems associated with input constraints, or actuator rate saturations [47]. For nonlinear systems, the authors in [48, 49, 50] use a nonquadratic cost function in the context of optimal control to deal with input constraints. Anti-windup methods have been investigated in [51, 52] to deal with actuator saturations. Different forms of saturation functions have been proposed in [53, 54, 55, 56] to handle input constraints. In [57], the authors propose neural network control methods to adaptively estimate the discrepancy between the actual control signal and the saturated control input, in order to compensate the saturation effects. In [58], the input constraints are taken into consideration by using the robust bounded terms in the controller design. By tuning certain control gain parameters, the control signal is guaranteed to be limited in any domain. In [59], the authors propose a novel structure with a Nussbaum function to deal with system input constraints, where the Nussbaum function is introduced to compensate for the nonlinear term arising from the input saturation. Unlike some existing control schemes for systems with input saturation, the developed controllers in [59] do not require assumptions on the uncertain parameters within a known compact set and a priori knowledge on the bound of the external disturbance.

Among the various methods proposed in the literature to deal with system input constraints, the auxiliary system, proposed in works including the two widely cited papers [60, 61], is of particular interests with respect to our work. The basic idea about this structure is to introduce auxiliary systems, such that the dynamics of these auxiliary systems will respond to the discrepancy between the actual and saturated control inputs. One can regard the discrepancy between the actual and saturated control inputs as the “control input” to the auxiliary system, and the state variable of the auxiliary system will be used in the real control law design. A brief structure about the auxiliary system may look like the following
For example, in [60], the authors study a class of uncertain multi-input-multi-output (MI-MO) nonlinear systems in the form of (Eq.(1) in [60])

\[
\dot{x}_i = F_i(\bar{x}_i)\theta_i + (G_i(\bar{x}_i) + \Delta G_i(\bar{x}_i))x_{i+1} + D_i(\bar{x}_i, t), \quad i = 1, 2, \cdots, n - 1 \\
\cdots \\
\dot{x}_n = F_n(\bar{x}_n)\theta_n + (G_n(\bar{x}_n) + \Delta G_n(\bar{x}_n))u + D_n(\bar{x}_n, t) \\
y = x_1,
\]

where, using the same notations from [60], \(x_i \in \mathcal{R}^m, i = 1, 2, \cdots, n\) are the state vectors; \(\theta_i \in \mathcal{R}^{q_i}, i = 1, 2, \cdots, n\) are the uncertain parameter vectors; \(F_i \in \mathcal{R}^{m \times q_i}, i = 1, 2, \cdots, n\) are known nonlinear functions; \(G_i \in \mathcal{R}^{m \times m}, i = 1, 2, \cdots, n\) are known control coefficient matrices; \(D_i \in \mathcal{R}^m, i = 1, 2, \cdots, n\) are unknown time-varying disturbances; \(u \in \mathcal{R}^m\) is the control input vector; \(y \in \mathcal{R}^m\) is the system output vector; \(q_i\) are positive integers and \(\Delta G_i \in \mathcal{R}^{m \times m}, i = 1, 2, \cdots, n\) are unknown bounded perturbations of control coefficient matrices.

The auxiliary system in [60] is designed as (Eq.(45) in [60])

\[
\dot{e} = \begin{cases} 
-K_{n2}e - \frac{1}{\|u\|^2}f(u, \Delta u, z_n, \bar{x}_n)e + (G_n(\bar{x}_n) + \gamma_n I_{m \times m})(v - u), & \text{if } \|e\| \geq \sigma; \\
0, & \text{if } \|e\| < \sigma,
\end{cases}
\]

(2.3.13)
where \( f(u, \Delta u, z_n, \bar{x}_n) \) is a nonlinear function, \( \Delta u = u - v \) (Eq. (45) in [60]), \( v \) is the control law to be designed, while \( u \) is the saturated actuator signal, \( K_{n2}, \gamma_n \) are positive design constants such that \( G_n(\bar{x}_n) + \gamma_n I_{m \times m} \) is invertible, and \( e \in \mathbb{R}^m \) is the state of the auxiliary design system, which is then incorporated in the control law design. The design parameter \( \sigma \) is a positive constant which should be chosen as an appropriate value in accordance with the requirement of the tracking performance. In the overall closed loop analysis, the quadratic function \( \frac{1}{2} e^T e \) will be incorporated in the overall Lyapunov function design. Similar design is also shown in [61].

It is worth pointing out that if saturation occurs after \( e \) converges into the set \( ||e|| < \sigma \), then the initial state of the auxiliary system will be reset so that (2.3.13) will continue to respond to any discrepancy between the saturated and designed control input. This structure of the auxiliary system will also be used in our design.

### 2.4 Consensus Problems for Multiagent Systems

The first system we are going to consider in this thesis is about multiagent system, which will be presented in Chapter 3. In the past two decades, networked cooperative systems (or multiagent systems) have attracted considerable attention from the research community due to their widespread applications [62]-[63]. A multiagent system is a system consisting of two or more intelligent agents, which could be ground vehicles, surface vessels, satellites, aircrafts, etc. Agents in a multiagent system may or may not have the ability to communicate with each other directly. Each agent may have exactly the same system or totally different system dynamics with each other. The objective is usually to achieve some goals collectively as a team, such as to achieve a desired formation among all the agents.

Consensus is an important class of multiagent systems coordination problems. According to [64], in the networks of agents, consensus means to reach an agreement regarding certain
quantities of interest that are associated with the agents, which can be either constant or time-varying. The control signal of each agent only depends on the information received from its neighbours, or measurement with respect to its neighbourhood. Since such control algorithm is a kind of distributed algorithm, it is more robust compared to centralized control algorithms [65]. There are many promising applications of consensus algorithms, among which formation control is one of the commonly encountered. The task is to control a group of agents to follow a predefined path or trajectory, while maintaining a desired formation pattern. In [66], consensus algorithms are used to solve the formation control problem of linear multi-agent systems with intermittent communications. Reference [73] introduces a consensus algorithm for systems modeled by second-order dynamics, and apply the algorithm to formation control problems by appropriately choosing the information states on which consensus is reached. The distributed formation control problem for multiple nonholonomic mobile robots using consensus-based approach is considered in [68]. Reference [69] presents two protocols which can make agents reach consensus while achieving and preserving the desired formation in fixed topology with and without communication time-delay for multi-agent network.

Generally speaking, the control problem of networked cooperative systems can be categorized into the cooperative regulator problem and the cooperative tracking problem. For the first type of problem, the objective is such that each agent is eventually driven to an unprescribed common value, which is generally a function of the initial states of the agents in the network [70]. This problem is usually referred to as leaderless consensus, or synchronization, or rendezvous in the literature [71]. In contrast, in coordinate tracking problems, there is a leader in the network. The leader may physically exist in the network, or it may be just a virtual leader. In such cases, tracking the trajectory of a leader/virtual leader node by all follower nodes is desirable. In such problems, the leader/virtual leader node serves as a command generator, which
generates the desired trajectories and is not influenced by all the other nodes in the network. Furthermore, the information from the leader/virtual leader is only available to a subset of the follower nodes. In the literature, this is usually called leader-following consensus [72], model reference consensus [73], leader-following control [74] and so on. We will study this category of consensus problem in Chapter 3. A comprehensive review of the state of the art in consensus problems can be found in [75], in which the results on some other interesting research topics such as finite-time consensus and consensus under limited communication conditions with time delays and quantization are also discussed.

2.4.1 Fault Tolerant Multiagent Systems

Most of the above mentioned works do not consider the effects of actuator faults. With the development of modern technology, control systems these days have become increasingly complex. This is particularly true for multiagent systems, which involve a number of different agents with an integration of various sensors, actuators and other components, which are all subject to faults that can lead to instability and performance deterioration. In the case that faults occur to certain agents in the network, it may be possible that these agents in the network may not be able to follow the leader to achieve the desired system performance. The issue of fault tolerance has been rarely discussed in the literature about networked cooperative tracking systems. In [76], fault-tolerant consensus in multi-agent system using distributed adaptive protocol is investigated. In [77], the problem of fault tolerant consensus control of linear multi-agent systems is investigated using the fixed-gain control protocol and the adaptive-gain control protocol. Only constant multiplicative actuator fault is considered in these works. In [78, 79], actuator faults are considered for a team of unmanned vehicles. However, these works only consider simple second order systems. The work [80] extends the discussion to higher order systems, however only additive actuator faults are considered in the problem formulation. How to address
both time-varying additive and multiplicative actuator faults for nonlinear multiagent systems subject to system uncertainties and disturbances has not been studied in detail yet.

2.4.2 Multiagent Systems with Input and Output Constraints

Few works in the literature on the topic of multiagent systems have considered the issue of system input constraints. In [81], the authors investigate the synchronization problem for a group of agents with identical linear dynamics subject to input saturation. Two classes of linear systems, asymptotically null controllable with bounded control systems and double-integrator systems, are studied. The works [82, 83] study the leader-following consensus problem for a group of agents with identical linear systems subject to control input saturation. Reference [84] studies the observer-based leader-following consensus of a linear multiagent system on switching networks, in which the input of each agent is subject to saturation. However, system uncertainties and disturbances are not considered in the problem formulation in these above mentioned works, and the dynamics of different agents have to be exactly the same. In [85], the problem of leader-following consensus of single-integrator agents subject to saturation constraints has been studied. In [86], the authors propose and analyze consensus algorithms when the velocity (second state) is not available for feedback and the control inputs are constrained by input saturations. Only single or double-integrator dynamics have been considered in these works, and again system uncertainties and disturbances are not addressed in the work. In [87], the authors propose and analyze flocking algorithms in a network of second-order agents with fixed directed network topology with a directed spanning tree. However, system uncertainties and disturbances are not considered, and only second-order dynamics have been addressed. In sum, there is a lack of research to address higher-order nonlinear systems with both system uncertainties and input saturation in the literature on the topic of multiagent systems. Furthermore,
Chapter 2. Background and Literature Review

multiagent systems with output constraints have not been addressed in the previous literature.

2.5 Backstepping Control Design

Since we will be using the technique of backstepping control design in the next two chapters, we will describe briefly this technique here. Backstepping is a control design technique developed by Kokotovic [88] and others for designing stabilizing controls for a special class of nonlinear dynamical systems, which is usually known as the lower-triangle form. These systems are built from subsystems that radiate out from an irreducible subsystem that can be stabilized using some other method. Because of this recursive structure, the designer can start the design process at the known stable system and “back out” new controllers that progressively stabilize each outer subsystem. The process stops when the final external control signal is reached.

To briefly illustrate how backstepping works, let us consider a second order nonlinear system as follows

\[
\begin{align*}
\dot{z} &= f(z) + g(z)\xi \\
\dot{\xi} &= u,
\end{align*}
\]

where \(u\) is the control input signal, \(f(z), g(z)\) are some nonlinear functions of \(z\). The objective is to design a state feedback control law to stabilize the origin. Suppose we can asymptotically stabilize the first subsystem

\[
\dot{z} = f(z) + g(z)\xi, \quad (2.5.2)
\]

by designing a stabilizing function \(\xi = \Phi(z)\), with \(\Phi(0) = 0\), such that the origin of the
subsystem

\[ \dot{z} = f(z) + g(z)\Phi(z) \quad (2.5.3) \]

is asymptotically stable. Suppose also that we can design a Lyapunov function \( V(z) \) such that

\[ \frac{\partial V(z)}{\partial z}(f(z) + g(z)\Phi(z)) \leq -W(z) \quad (2.5.4) \]

for some positive definite \( W(z) \). Rewrite (2.5.2) as

\[ \dot{z} = (f(z) + g(z)\Phi(z)) + g(z)(\xi - \Phi(z)) \]

\[ \dot{\xi} = u. \quad (2.5.5) \]

Denote \( y = \xi - \Phi \), we can see that

\[ \dot{y} = \dot{\xi} - \dot{\Phi} = u - \dot{\Phi}, \]

hence

\[ \dot{z} = (f(z) + g(z)\Phi(z)) + g(z)y \]

\[ \dot{y} = u - \dot{\Phi}. \quad (2.5.6) \]

Denote \( v = u - \dot{\Phi} \), we can see that the system becomes

\[ \dot{z} = (f(z) + g(z)\Phi(z)) + g(z)y \]

\[ \dot{y} = v, \quad (2.5.7) \]

which is in the same form of the original system, with the exception that we now know the first subsystem is asymptotically stable at the origin. The overall Lyapunov function can be
designed as

\[ V_c(z, \xi) = V(z) + \frac{1}{2}y^2. \]  

(2.5.8)

The derivative of \( V_c \) leads to

\[
\dot{V}_c = \frac{\partial V(z)}{\partial z} (f(z) + g(z)\Phi(z)) + \frac{\partial V(z)}{\partial z} g(z)y + yv \\
\leq -W(z) + \frac{\partial V(z)}{\partial z} g(z)y + yv.
\]  

(2.5.9)

The control variable \( v \) can then be designed as

\[
v = -\frac{\partial V(z)}{\partial z} g(z) - ky,
\]  

(2.5.10)

where \( k \) is positive design constant. Hence

\[
\dot{V}_c \leq -W(z) - ky^2,
\]  

(2.5.11)

which implies that the origin is asymptotically stable at \( z = 0, y = 0 \), and since \( \Phi(0) = 0 \), this further implies that \( z = 0, \xi = 0 \) is also asymptotically stable.

We consider a small example here to further illustrate the ideas.

**Example 1** Consider

\[
\begin{align*}
\dot{z} &= z^2 - z^3 + \xi \\
\dot{\xi} &= u.
\end{align*}
\]  

(2.5.12)
Consider the first subsystem \( \dot{z} = z^2 - z^3 + \xi \), we can design a stabilizing function as

\[
\xi = \Phi(z) = -z^2 - z,
\]  

(2.5.13)

hence

\[
\dot{z} = -z - z^3.
\]  

(2.5.14)

Then, the Lyapunov function can be designed as

\[
V(z) = \frac{1}{2}z^2.
\]  

(2.5.15)

Take the derivative with respect to time, we can have

\[
\dot{V} = -z^2 - z^4 \leq -z^2,
\]  

(2.5.16)

which implies that \( z = 0 \) can be made asymptotically stable. Denote \( y = \xi - \Phi = \xi + z^2 + z \), such that

\[
\dot{z} = -z - z^3 + y
\]

\[
\dot{y} = u + (1 + 2z)(-z - z^3 + y).
\]  

(2.5.17)

Design the overall Lyapunov function as \( V_c = \frac{1}{2}z^2 + \frac{1}{2}y^2 \), the derivative of \( V_c \) leads to

\[
\dot{V}_c = z(-z - z^3 + y) + y(u + (1 + 2z)(-z - z^3 + y))
\]

\[
= -z^2 - z^4 + y(u + (1 + 2z)(-z - z^3 + y) + z).
\]  

(2.5.18)
Choose the control law as $u = -z - (1 + 2z)(-z - z^3 + y) - y$, hence

$$\dot{V}_c = -z^2 - z^4 - y^2. \quad (2.5.19)$$

The actual control law in the original coordinate is

$$u = -z - (1 + 2z)(-z - z^3 + \xi + z^2 + z) - \xi - z^2 - z$$

$$= -z - (1 + 2z)(-z^3 + \xi + z^2) - \xi - z^2 - z. \quad (2.5.20)$$
Chapter 3

AFTCTC of Nonlinear Systems with Input and Output Constraints

In this Chapter, we present a novel adaptive fault tolerant cooperative tracking control (AFTCTC) scheme for a class of control input and system output constrained nonlinear multiagent systems with a time-varying active leader, whose dynamics is unknown to all the follower nodes. The follower nodes have distinct higher-order nonlinear dynamics with both parametric and nonparametric uncertainties. The input constraints for each agent can be asymmetric, and the agents’ output constraint can be time-varying. Both additive actuator faults and multiplicative actuator faults are considered in this work. We show that under the proposed distributed control scheme, exponential convergence of the cooperative output tracking error into a small set around zero is guaranteed, while the constraint on the system output will not be violated during operation. State vectors of the agents and all other closed loop signals are bounded.

This Chapter is organized as follows. In Section 3.1, we present the basic graph theory and notations widely adopted in the literature. In Section 3.2, we introduce the problem formulation and control objective. Output and input constraints, as well as actuator faults will
be described. In Section 3.3, the tan type BLF and the selection of bounds will be presented and discussed. By introducing backstepping design analysis, we propose the distributed control scheme and adaption laws to achieve our control objectives in Section 3.4. An illustrative example modified from one discussed in the literature is introduced to demonstrate the effectiveness of the proposed method in Section 3.5, with concluding remarks in Section 3.6.

3.1 Basic graph theory and notations

The basic graph theory and notations used in this Chapter are fairly standard, which have been used widely in the literature [80] [89]. A weighted graph is represented by $G = (V, E)$, where $V = \{v_1, \cdots, v_N\}$ is a nonempty set of nodes/agents, $E \subseteq V \times V$ is the set of edges/arcs. $(v_i, v_j) \in E$ indicates that there is an edge from node $i$ to node $j$. The topology of a weighted graph $G$ is often represented by the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ij} > 0$ if $(v_i, v_j) \in E$, otherwise $a_{ij} = 0$. Throughout this Chapter, it is assumed that $a_{ii} = 0$, $i = 1, \cdots, N$, and the topology is fixed, i.e., $A$ is time-invariant. $G$ is a directed graph, or digraph in short.

Define $d_i = \sum_{j=1}^{N} a_{ij}$ as the weighted in-degree of node $i$ and $D = diag(d_1, \cdots, d_N) \in \mathbb{R}^{N \times N}$ as the in-degree matrix. The graph Laplacian matrix is $L = [l_{ij}] = D - A \in \mathbb{R}^{N \times N}$. Let $1 = [1, \cdots, 1]^T$ with appropriate dimension; then $L1 = 0$. The set of neighbors of node $i$ is denoted as $N_i = \{j | (v_j, v_i) \in E\}$. If node $j$ is a neighbor of node $i$, then node $i$ can get information from node $j$, but not necessarily vice versa for a digraph. For undirected graphs, neighbor is a mutual relation. A directed path from node $i$ to node $j$ is a sequence of successive edges in the form $\{(v_i, v_k), (v_k, v_l), \cdots, (v_r, v_j)\}$. If there exists a node (called the root) such that there is a directed path from the root to every other node in the graph, then we say that the digraph has a spanning tree. A digraph is strongly connected, if for any ordered pair of nodes $[v_i, v_j]$ with $i \neq j$, there is a direct path from node $i$ to node $j$. 
In this Chapter, $| \cdot |$ is the absolute value of a real number; $\| \cdot \|$ is the Euclidean norm of a vector; $\text{tr}(\cdot)$ is the trace of a matrix; $\sigma(\cdot)$ is the set of singular values of a matrix, with the maximum singular value $\bar{\sigma}(\cdot)$ and the minimum singular value $\underline{\sigma}(\cdot)$; matrix $P > 0 (P \geq 0)$ means $P$ is positive definite (positive semidefinite); $I$ represents the identity matrix with appropriate dimension; and $\mathcal{N} = \{1, \cdots, N\}$.

### 3.2 Problem formulation

Consider $N (N > 2)$ agents with distinct high-order nonlinear dynamics. Dynamics of the $j$th agent is in the Brunovsky canonical form as

\begin{align*}
\dot{x}_{j,i}(t) &= x_{j,i+1}(t), \quad i = 1, \cdots, n - 1 \\
\dot{x}_{j,n}(t) &= u_j^F(t) + \varepsilon_j(\bar{x}_j, t) + \theta_j^T(t) F_j(\bar{x}_j), \\
y_j(t) &= x_{j,1}(t),
\end{align*}

where $x_{j,i}(t) \in \mathcal{R}$ is the $i$th state variable of the node $j$, $\bar{x}_j = [x_{j,1}, \cdots, x_{j,n}]^T$ is the state vector of the node $j$. $\varepsilon_j(\bar{x}_j, t) \in \mathcal{R}$ are bounded nonparametric uncertainties for $j = 1, \cdots, N$. $\theta_j(t) \in \mathcal{R}^{m \times 1}$ are the unknown time-varying functions, $F_j(\bar{x}_j) \in \mathcal{R}^{m \times 1}$ are the known nonlinear state dependant functions. $y_j(t)$ is the system output of the node $j$. $u_j^F(t) \in \mathcal{R}$ is the actuator signal of the node $j$, which may be faulty. In particular, we are dealing with both multiplicative and additive actuator faults, which, as mentioned in Chapter 2, can be formulated as

\begin{align*}
u_j^F(t) = \rho_j(t) u_j(t) + \phi_j(t),
\end{align*}
where $u_j(t)$ is the constrained control input signal formulated as

$$u_j(t) = \begin{cases} 
    u_{j,\text{max}}, & \text{if } v_j(t) > u_{j,\text{max}}; \\
    v_j(t), & \text{if } u_{j,\text{min}} \leq v_j(t) \leq u_{j,\text{max}}; \\
    u_{j,\text{min}}, & \text{if } v_j(t) < u_{j,\text{min}},
\end{cases} \quad (3.2.3)$$

$v_j(t)$ is the signal generated by the distributed control law to be designed, $\rho_j(t)$ represents the multiplicative actuator faults, and $\phi_j(t)$ represents the additive fault in the control input channel. If $\rho_j(t) = 1$ and $\phi_j(t) = 0$, we say that the $j$th agent is free from actuator faults. In short, we represent (3.2.3) as $u_j = \text{sat}(v_j)$, $j = 1, \cdots, N$, or collectively $u = \text{sat}(v)$.

Define $x_i(t) = [x_{1,i}(t), \cdots, x_{N,i}(t)]^T$, $y(t) = [y_1(t), \cdots, y_N(t)]^T$, then one will get

$$\begin{align*}
    \dot{x}_i(t) &= x_{i+1}(t), \quad i = 1, \cdots, n-1 \\
    \dot{x}_n(t) &= u^F(t) + \varepsilon(x,t) + \theta^T(t)F(x), \\
    y(t) &= x_1(t), \quad (3.2.4)
\end{align*}$$

where $u^F(t) = [u_1^F(t), \cdots, u_N^F(t)]^T \in \mathcal{R}^N$, $\varepsilon(x,t) = [\varepsilon_1(\bar{x}_1,t), \cdots, \varepsilon_N(\bar{x}_N,t)]^T \in \mathcal{R}^N$, $\theta(t) = \text{diag}[\theta_1(t), \cdots, \theta_N(t)] \in \mathcal{R}^{Nm \times N}$, $F(x) = [F_1(\bar{x}_1)^T, \cdots, F_N(\bar{x}_N)^T]^T \in \mathcal{R}^{Nm}$. Notice that when $n = 3$, $x_1(t)$, $x_2(t)$ and $x_3(t)$ can be the global position vector, global velocity vector and global acceleration vector, respectively [89].

**Remark 3.2.1** Notice that in (3.2.1), or equivalently in (3.2.4), the formulation of nonparametric term $\varepsilon(x,t)$ and parametric term $\theta^T(t)F(x)$ represents a wide range of system uncertainties. In the adaptive fuzzy control literature, it is often assumed that a continuous function $f(x)$ defined on a compact set can be represented to an arbitrary degree of precision as $f(x) = \theta^T F(x) + \varepsilon$, where in this case $F(x)$ is called the base function which is known, $\theta$ is the
ideal neural network weight vector that is unknown and constant, and $\varepsilon$ is the neural network approximation error which is unknown and can be arbitrarily small [80] [99]. Compared with such an uncertainty description, the formulation presented in (3.2.1) or (3.2.4) is more general, as the parametric unknown term can be time-varying, and the nonparametric unknown term can be both time and state dependent.

The leader node, denoted as node $0$, has the time-varying dynamics as

$$\begin{align*}
\dot{x}_{0,i}(t) &= x_{0,i+1}(t), \quad i = 1, \ldots, n - 1 \\
\dot{x}_{0,n}(t) &= f_0(t, \bar{x}_0), \\
y_0(t) &= x_{0,1}(t),
\end{align*}$$

(3.2.5)

where $\bar{x}_0 = [x_{0,1}, \ldots, x_{0,n}]^T$, $f_0(t, \bar{x}_0) \in \mathcal{R}$ is piecewise continuous in $t$ and locally Lipschitz in $\bar{x}_0$ with $f_0(t, 0) = 0$ for all $t \geq 0$ and $\bar{x}_0 \in \mathcal{R}^n$, and it is unknown to all other nodes. System (3.2.5) is assumed to be forward complete, that is, for every initial condition, the solution $\bar{x}_0(t)$ exists for all $t \geq 0$. In other words, there is no finite escape time. The dynamics of the leader node can be considered as an exosystem that generates a desired command trajectory.

Define the $i$th order tracking error, or disagreement variable for node $j$ ($j = 1, \ldots, N$) as $\delta_{j,i}(t) = x_{j,i}(t) - x_{0,i}(t), \; i = 1, \ldots, n$. Let $\delta_i(t) = [\delta_{1,i}(t), \ldots, \delta_{N,i}(t)]^T \in \mathcal{R}^N$, $\bar{x}_{0,i}(t) = [x_{0,i}, \ldots, x_{0,i}]^T \in \mathcal{R}^N$, then we have

$$\delta_i(t) = x_i(t) - \bar{x}_{0,i}(t).$$

(3.2.6)

The agents’ output $y(t) = [y_1(t), \ldots, y_N(t)]^T$ has to satisfy the time-varying output con-
constraint requirement

\[ ||y(t)|| < \bar{k}(t), \quad (3.2.7) \]

with \( \bar{k}(t) \in \mathcal{R} \) a time-varying real-valued function.

**The control objective** is to design a distributed control algorithm so that the agent outputs \( y_j(t), \ j = 1, \cdots, N \), can track the leader output \( y_0(t) \) with an arbitrarily small bounded tracking error, when (3.2.1) is subject to control input constraints (3.2.3) and actuator faults (3.2.2), while ensuring that the output constraint requirement (3.2.7) is not violated.

Notice that in this Chapter, it is assumed that only relative state information can be used for the follower’s controller design. More precisely, for the node \( j \), the only available information is the neighborhood synchronization error

\[ e_{j,i}(t) = \sum_{r \in \mathcal{N}_j} a_{jr}(x_{r,i}(t) - x_{j,i}(t)) + b_j(x_{0,i}(t) - x_{j,i}(t)), \quad (3.2.8) \]

where \( i = 1, \cdots, n \), \( b_j \geq 0 \) are the weight of edge from the leader node to node \( j \), \( b_j > 0 \) if there is an edge from the leader node to node \( j \). Define \( e_i(t) = [e_{1,i}(t), \cdots, e_{N,i}(t)]^T, \ f_0 = [f_0(t, \bar{x}_0), \cdots, f_0(t, \bar{x}_0)]^T \in \mathcal{R}^N \), and \( B = diag(b_1, \cdots, b_N) \in \mathcal{R}^{N \times N} \). Similar to equation (7) in [80], one can easily get

\[ \dot{e}_i(t) = e_{i+1}(t), \quad i = 1, \cdots, n - 1 \]

\[ \dot{e}_n(t) = -(L + B)(u^F(t) + \varepsilon(x, t) + \theta^T(t)F(x) - f_0). \quad (3.2.9) \]

Define the augmented graph as \( \bar{G} = (\bar{V}, \bar{E}) \), where \( \bar{V} = \{v_0, v_1, \cdots, v_N\} \) and \( \bar{E} \subseteq \bar{V} \times \bar{V} \). The following assumption on the graph topology is widely adopted in the literature for cooperative
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tracking problems.

**Assumption 3.2.1** The augmented graph $\bar{G}$ contains a spanning tree with the root node being the leader node $0$.

**Remark 3.2.2** As pointed out in [89], if Assumption 3.2.1 does not hold, synchronization to the leader node $0$ cannot be achieved.

The following assumptions are presented to facilitate the discussion of our main result.

**Assumption 3.2.2** (a) There exists $k_d(t) \in \mathcal{R}$, a known continuous function, such that $||x_{0,1}(t)|| \leq k_d(t)$ and $\bar{k}(t) > k_d(t)$. The function $k_d(t)$ is differentiable up to $n$-th order, and the derivatives are bounded.  
(b) The initial neighborhood output synchronization error satisfies $||e_1(0)||/\sigma(L + B) < \bar{k}(0) - k_d(0)$.

**Remark 3.2.3** An assumption similar to Assumption 3.2.2(a) is presented in the adaptive control literature on problems with output constraints [36]. Notice that $\bar{k}(t) > k_d(t)$ is always true in practice if perfect cooperative output tracking is ever possible to be achieved, since the constraint requirement on the agents’ output cannot be smaller than the bound on the leader’s output trajectory. The functions $k_d(t)$, $\bar{k}(t)$ being differentiable up to $n$-th order is a commonly adopted assumption in adaptive control literature using backstepping analysis that will be presented in the discussion of the main result. More discussion on Assumption 3.2.2(b) will be presented in the next section and Remark 3.3.1.

**Assumption 3.2.3** For the multiplicative actuator faults, $0 < \rho_{j,\text{min}} \leq \rho_j(t) \leq 1$, where $\rho_{j,\text{min}}$ is known, $j = 1, \cdots, N$. The derivative of the multiplicative actuator fault is bounded, in particular, $\text{tr}(\dot{\rho}^T(t)\dot{\rho}(t)) \leq \Xi < \infty$, where $\rho(t) = \text{diag}[\rho_1(t), \cdots, \rho_N(t)] \in \mathcal{R}^{N \times N}$. The additive actuator faults are bounded, that is, $|\phi_j(t)| \leq \Psi_{\phi,j}$. 
Remark 3.2.4 This assumption implies that for each node, the actuator will not totally fail under faulty situations, that is, \( \rho_j(t) = 0 \) will not happen. The multiplicative and additive actuator faults, together with their derivatives, are bounded. Notice that in Assumption 3, the bounds \( \Xi \) and \( \bar{\Psi}_{\phi,j} \) can be unknown.

Assumption 3.2.4 (a) The nonparametric uncertainties \( \varepsilon_j(x_j, t) \) are bounded, that is, \( |\varepsilon_j(x_j, t)| \leq \bar{\Psi}_{\varepsilon,j} \), where \( \bar{\Psi}_{\varepsilon,j} \in \mathbb{R} \) is unknown. The unknown time-varying function \( \theta(t) \) and its derivative are bounded, in particular, \( \theta^T(t)\theta(t) \leq \Theta < \infty \), and \( \dot{\theta}^T(t)\dot{\theta}(t) \leq \bar{\Theta} < \infty \), and both \( \Theta \) and \( \bar{\Theta} \) are unknown. Further, \( F(x) \) is bounded.

(b) There exists positive unknown constants \( \bar{\Psi}_0 \) and \( \bar{\Psi}_{f_0} \) such that \( ||x_0(t)|| \leq \bar{\Psi}_0, \ |f_0(x_0, t)| \leq \bar{\Psi}_{f_0}, \forall t \geq t_0. \) This is similar to Assumption 2 used in [80].

The following two technical lemmas are adopted from [89].

Lemma 3.2.1 Define \( q = [q_1, \cdots, q_N] = (L + B)^{-1}1, \ P = \text{diag}\{p_i\} = \text{diag}\{1/q_i\}, \) then \( P > 0. \)

Lemma 3.2.2 \( ||\delta_i(t)|| \leq ||e_i(t)||/\sigma(L + B), \ i = 1, \cdots , n. \)

To simplify the notation, from this point onwards for the rest of this chapter, the time and state dependence of the system will be omitted whenever no confusion would arise.

3.3 \( \tan \)-type barrier Lyapunov function and selection of bounds

3.3.1 \( \tan \)-type barrier Lyapunov function

To facilitate the discussion about the system output constraints, we introduce the time-varying \( \tan \)-type Barrier Lyapunov Function, mentioned in Chapter 2, as follows,

\[
V = \frac{k_{b,j}^2}{\pi} \tan \left( \frac{\pi e_{j,1}^2}{2k_{b,j}^2} \right), \quad |e_{j,1}(0)| < k_{b,j}(0), \quad j = 1, \cdots , N, \tag{3.3.1}
\]
where \( k_{b,j} \triangleq k_{b,j}(t) \) is a time varying positive continuous function, such that it requires \( |e_{j,1}| < k_{b,j}, \forall t \geq 0 \). Notice that as \( |e_{j,1}| \rightarrow k_{b,j}, V \rightarrow \infty \).

### 3.3.2 Selection of bounds

From (3.2.6), we can get \( ||x_1|| \leq ||\delta_1|| + ||z_{0,1}|| \). In view of (3.2.7), Assumption 3.2.2 and Remark 3.2.4, we can see that as long as

\[
||\delta_1|| < \bar{k} - k_d
\]  

is satisfied \( \forall t \geq 0 \), we can guarantee that the agent output constraint requirement (3.2.7) will be satisfied. From Lemma 3.2.2, we can transform the discussion of output constraint requirement (3.2.7) into the discussion of constrained neighborhood synchronization error, such that

\[
\frac{||e_1||}{\sigma(L + B)} < \bar{k} - k_d. \tag{3.3.3}
\]

Recall that \( e_1 = [e_{1,1}, \cdots, e_{N,1}]^T \). We can achieve the inequality (3.3.3) by designing \( k_{b,j} \triangleq k_{b,j}(t), j = 1, \cdots, N \) and a distributed control algorithm, such that \( |e_{j,1}| < k_{b,j}, \forall t \geq 0 \), and

\[
\sqrt{k_{b,1}^2 + \cdots + k_{b,N}^2} \sigma(L + B) \leq (\bar{k} - k_d). \tag{3.3.4}
\]

**Remark 3.3.1** Notice that the only obtainable information is the neighborhood synchronization error, not the state information of each agent. Besides, not every node in the network can have access to the state information of the leader. The direct tracking error, or disagreement variable is not measurable for all the agents. Therefore, the output constrained problem in the context of multiagent system is more challenging than other output tracking problems with single systems.
like the ones discussed in \cite{36, 35, 40, 92}. In order to guarantee that (3.3.2) is satisfied, we design a control algorithm such that (3.3.3) is satisfied, where $e_1$ can be measured and its trajectory can be controlled. Hence, and also in view of Remark 3.2.3, Assumption 3.2.2 (b) allows the construction of the controller that will be presented later, where by confining the synchronization error, the output tracking error can be constrained. If Assumption 3.2.2 (b) is not met for the initial condition of $e_1$, we can either choose a more stringent initial condition $k_d(0)$, or modify the initial output constraint requirement $\tilde{k}(0)$.

### 3.4 Main Result

We present the design procedure, which is a backstepping approach that will lead to our controller design and main theorem of this chapter.

**Step 1:**

To facilitate the discussion, we introduce the fictitious synchronization errors $z_{j,i}$, $j = 1, \cdots, N$, $i = 1, \cdots, n$, where $z_{j,1} = e_{j,1}$ and $z_{j,h} = e_{j,h} - \alpha_{j,h-1}$, $h = 2, \cdots, n$, and $\alpha$’s are the stabilizing functions to be designed. In view of the discussion regarding the selection of bounds in the last section, design the barrier Lyapunov function at this step as

$$V_1 = \sum_{j=1}^{N} \left\{ \frac{2k_{b,j} \dot{k}_{b,j}}{\pi} \tan \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) - \left( \frac{\dot{k}_{b,j}}{k_{b,j}} \right) \frac{z_{j,1}^2}{\cos^2 \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right)} + \frac{\dot{z}_{j,1}(z_{j,2} + \alpha_{j,1})}{\cos^2 \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right)} \right\}, \quad (3.4.1)$$

The stabilizing function $\alpha_{j,1}$ can then be designed as

$$\alpha_{j,1} = -K_{j,1} \frac{1}{z_{j,1}} \frac{k_{b,j}^2}{\pi} \sin \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) \cos \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) - K_{j,10}z_{j,1}, \quad (3.4.2)$$

where $K_{j,1}$, $K_{j,10}$ are positive, $K_{j,1}$ is a design constant, $K_{j,1} > 2K_{j,10}$, $K_{j,10} = \sqrt{\left( \frac{k_{b,j}}{k_{b,j}} \right)^2 + \epsilon}$,
Remark 3.4.1  By L’Hospital’s Rule, we can see that,

\[
\lim_{z_{j,1} \to 0} \frac{1}{z_{j,1}} k_{b,j}^2 \frac{\sin \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right)}{\pi} \cos \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) \to 0,
\]

therefore singularity will not occur.

However, in a digital computer, since \( \frac{0}{0} \) cannot be evaluated, we can take (3.4.3) to be zero when \( ||z_{j,1}|| < \epsilon_0 \) for some small \( \epsilon_0 > 0 \).

Substituting (3.4.2) into (3.4.1), we can see that

\[
- z_{j,1} \cos \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) K_{j,1} + \frac{1}{z_{j,1}} k_{b,j}^2 \frac{\sin \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right)}{\pi} \cos \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) = -K_{j,1} k_{b,j}^2 \frac{\pi}{\pi} \tan \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right),
\]

and notice that in (3.4.1) that

\[
- K_{j,10} z_{j,1}^2 k_{b,j}^2 \cos^2 \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) - k_{b,j} \frac{z_{j,1}^2}{k_{b,j}} \cos^2 \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) \leq - K_{j,10} z_{j,1}^2 k_{b,j}^2 \cos^2 \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) + K_{j,10} z_{j,1}^2 k_{b,j}^2 \cos^2 \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) \leq 0,
\]

and therefore, we obtain

\[
\dot{V}_1 \leq \sum_{j=1}^{N} -(K_{j,1} - 2K_{j,10}) \frac{k_{b,j}^2}{\pi} \tan \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) + \frac{z_{j,1} z_{j,2}}{\cos^2 \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right)},
\]
where \( \frac{z_{j,1}z_{j,2}}{\cos^2(\frac{\pi z_{j,1}}{2k_{b,j}})} \) is to be cancelled in the next step of the backstepping design.

**Step i** \((2 \leq i \leq n - 1)\):

Design the Lyapunov function in this step as

\[
V_i = \sum_{j=1}^{N} z_{j,i}^2
\]

so that

\[
\dot{V}_i = \sum_{j=1}^{N} z_{j,i} \dot{z}_{j,i} = \sum_{j=1}^{N} z_{j,i}(z_{j,i+1} + \alpha_{j,i} - \dot{\alpha}_{j,i-1}). \tag{3.4.8}
\]

The stabilizing function \(\alpha_{j,i}\) can then be designed as

\[
\alpha_{j,2} = \dot{\alpha}_{j,1} - \frac{z_{j,1}}{\cos^2\left(\frac{\pi z_{j,1}}{2k_{b,j}}\right)} - K_{j,2}z_{j,2},
\]

\[
\alpha_{j,h} = \dot{\alpha}_{j,h-1} - z_{j,h-1} - K_{j,h}z_{j,h}, \quad h = 3, \ldots, n - 1, \quad (3.4.9)
\]

where \(K_{j,2} > 0, K_{j,h} > 0\) are design constants, and \(\dot{\alpha}_{j,i} = \sum_{l=1}^{i} \frac{\partial \alpha_{j,i}}{\partial z_{j,l}} \dot{z}_{j,l} + \frac{\partial \alpha_{j,i}}{\partial k_{b,j}} \dot{k}_{b,j}\) for \(i = 1, \ldots, n - 1\).

Putting (3.4.9) into (3.4.8), we see that when \(i = 2\), we have

\[
\dot{V}_2 = \sum_{j=1}^{N} z_{j,2}(z_{j,3} + \alpha_{j,2} - \dot{\alpha}_{j,1})
\]

\[
= \sum_{j=1}^{N} z_{j,2}z_{j,3} - \frac{z_{j,1}z_{j,2}}{\cos^2\left(\frac{\pi z_{j,1}}{2k_{b,j}}\right)} - K_{j,2}z_{j,2}^2, \tag{3.4.10}
\]

where \(- \frac{z_{j,1}z_{j,2}}{\cos^2\left(\frac{\pi z_{j,1}}{2k_{b,j}}\right)}\) cancels the same term with opposite sign in (3.4.7), which is the previous
step of the backstepping process. When \( i = h, h = 3, \cdots, n - 1 \), we have

\[
\dot{V}_h = \sum_{j=1}^{N} z_{j,h}(z_{j,h+1} + \alpha_{j,h} - \dot{\alpha}_{j,h-1}) = \sum_{j=1}^{N} z_{j,h}(z_{j,h+1} - z_{j,h-1} - K_{j,h}z_{j,h})
\]

\[
= \sum_{j=1}^{N} z_{j,h}z_{j,h+1} - z_{j,h}z_{j,h-1} - K_{j,h}z_{j,h}^2,
\] (3.4.11)

where the term \(-z_{j,h}z_{j,h-1}\) will cancels with the same term but opposite sign from the previous step. Therefore, in an iterative manner, we can have

\[
\sum_{i=1}^{n-1} \dot{V}_i \leq \sum_{j=1}^{N} \left\{ -(K_{j,1} - 2K_{j,10}) \frac{k_{b,j}^2}{\pi} \tan \left( \frac{\pi z_{j,1}^2}{2k_{b,j}^2} \right) - \sum_{h=2}^{n-1} K_{j,h}z_{j,h}^2 \right\} + \sum_{j=1}^{N} z_{j,n-1}z_{j,n}.
\] (3.4.12)

**Step n:**

Design \( V_n = \frac{1}{2} z_n^T P z_n \), where \( P \) is defined in Lemma 3.2.2, \( z_n = [z_{1,n}, \cdots, z_{N,n}]^T \). Recall from (3.2.2) that \( u^F = \rho u + \phi \). The derivative of \( V_n \) can be obtained as

\[
\dot{V}_n = z_n^T P \dot{z}_n = z_n^T P (\dot{e}_n - \dot{\alpha}_{n-1}) = z_n^T P [-L + B](u^F + \varepsilon + \theta^T F - f_0) - \dot{\alpha}_{n-1}
\]

\[
= z_n^T P [-L + B](\rho u + \phi + \varepsilon + \theta^T F - f_0) - \dot{\alpha}_{n-1},
\] (3.4.13)

where \( \alpha_{n-1} = [\alpha_{1,n-1}, \cdots, \alpha_{N,n-1}]^T \), \( \phi = [\phi_1, \cdots, \phi_N]^T \).

Recall that \( u = sat(v) \) where \( sat \) is defined in (3.2.3), \( v = [v_1, \cdots, v_N]^T \) is the unconstrained input signal. Design

\[
v = \hat{\rho}^{-1} \phi,
\] (3.4.14)

where \( \hat{\rho} \), the estimator of \( \rho \), and \( \phi \) will be designed shortly. Write \( \rho u + \phi = \rho u - \hat{\rho}u + \hat{\rho}u + \phi = \)
−\tilde{\rho}u + \phi + \hat{\rho}(u - v) + \varphi, \text{ where } \tilde{\rho} = \hat{\rho} - \rho. \text{ From (3.4.13), we obtain }

\[ \dot{V}_n = z_n^T P\left[ -(L + B)(-\tilde{\rho}u + \phi + \hat{\rho}(u - v) + \varphi + \varepsilon + \theta^T F - f_0) - \dot{\alpha}_n - 1 \right]. \]  

(3.4.15)

Define \( \Psi = [\Psi_1, \cdots, \Psi_N]^T \), where \( \Psi_j = \Psi_{f_0} + \Psi_{\phi,j} + \Psi_{\varepsilon,j}, \ j = 1, \cdots, N \). Also, if \( \varpi = [\varpi_1, \cdots, \varpi_N] \in \mathbb{R}^{1 \times N} \) is a row vector, we define \( \text{sgn}(\varpi) = \text{diag}(\text{sgn}(\varpi_1), \cdots, \text{sgn}(\varpi_N)) \in \mathbb{R}^{N \times N} \), where \( \text{sgn} \) is the sign function. Therefore, for the term \( -z_n^T P(L + B)(\phi + \varepsilon - f_0) \) on the right hand side (RHS) of (3.4.15), we can derive that

\[ -z_n^T P(L + B)(\phi + \varepsilon - f_0) = -z_n^T P(D + B)(\phi + \varepsilon - f_0) + z_n^T PA(\phi + \varepsilon - f_0) \leq -z_n^T P(D + B)\text{sgn}(z_n^T P(D + B))\tilde{\Psi} + \frac{1}{4\gamma_1} \frac{\tilde{\delta}(P)\tilde{\sigma}(A)}{\tilde{\sigma}(P)} z_n^T Pz_n + \gamma_1 \tilde{\delta}(P)\tilde{\sigma}(A)\tilde{\Psi}^T \tilde{\Psi}, \]  

(3.4.16)

where \( \gamma_1 > 0 \). Besides, for the term \( -z_n^T P(L + B)\theta^T F \) on the RHS of (3.4.15) we have

\[ -z_n^T P(L + B)\theta^T F = -z_n^T P(D + B)\theta^T F + z_n^T PA\theta^T F \leq -z_n^T P(D + B)\theta^T F + \frac{1}{4\gamma_2} \frac{\tilde{\delta}(P)\tilde{\sigma}(A)}{\tilde{\sigma}(P)} z_n^T Pz_n + \gamma_2 \tilde{\delta}(P)\tilde{\sigma}(A)F^T \theta \theta^T F, \]  

(3.4.17)

where \( \gamma_2 > 0 \), and \( \gamma_2 \tilde{\delta}(P)\tilde{\sigma}(A)F^T \theta \theta^T F \) is bounded. Notice also on the RHS of (3.4.15) that

\[ z_n^T P(L + B)\hat{\rho}u \leq z_n^T P(L + B)\hat{\rho}(u - v) - z_n^T P(L + B)\varphi \]

\[ = z_n^T P(D + B)\hat{\rho}u - z_n^T P(D + B)\hat{\rho}(u - v) + z_n^T PA\rho u - z_n^T P(D + B)\varphi, \]  

(3.4.18)
and

\[ z_n^T PA u \leq \frac{1}{4\gamma_3} \sigma(P)\sigma(A) z_n^T P z_n + \gamma_3 \sigma(P) \sigma(A) u^T u, \quad (3.4.19) \]

where \( \gamma_3 > 0, u_j \in [u_{j,min}, u_{j,max}], j = 1, \cdots, N \), hence \( u^T u \) is bounded.

Design the distributed control law \( \varphi \in \mathcal{R}^N \) mentioned in (3.4.14) as

\[ \varphi = h_1(D + B)^{-1}(z_n - \chi) - (D + B)^{-1}\dot{\alpha}_{n-1} - \hat{\theta}^T F + \text{sgn}(z_n^T P(D + B))\hat{\Psi} \]
\[ + \frac{1}{\sigma(P)}(D + B)^{-1} \Upsilon z_{n-1} + \frac{h_2^2 \sigma^2(P)}{2\sigma(P)}(D + B)^{-1} z_n, \quad (3.4.20) \]

where \( z_{n-1} = [z_{1,n-1}, \cdots, z_{N,n-1}]^T, \Upsilon = \text{diag}(\text{sgn}(z_{1,n-1}z_{1,n}), \cdots, \text{sgn}(z_{N,n-1}z_{N,n})) \in \mathcal{R}^{N \times N}, \) and \( h_1 \) is a design gain such that \( h_1 > \left( \frac{1}{4\gamma_1} + \frac{1}{4\gamma_2} + \frac{1}{4\gamma_3} \right) \frac{\sigma(P)\sigma(A)}{\sigma(P)}. \hat{\theta}, \hat{\Psi} \) are the estimators for \( \theta \) and \( \Psi \), respectively. \( \chi = [\chi_1, \cdots, \chi_N]^T \in \mathcal{R}^{N \times 1} \) is the state vector of the auxiliary system

\[ \dot{\chi}_j = \begin{cases} -h_2 \chi_j + f_{\chi_j} \chi_j - \beta_j (u_j - v_j), & \text{if } |\chi_j| > v_j; \\ 0, & \text{if } |\chi_j| \leq v_j, \end{cases} \quad (3.4.21) \]

where \( j = 1, \cdots, N, h_2 > 1, \beta_j > 0 \) are design constants,

\[ f_{\chi_j} = z_{j,n} \rho_j (D_j + B_j) (u_j - v_j) - \frac{1}{2} \beta_j^2 (u_j - v_j)^2, \quad v_j > 0 \text{ is a small number.} \]

**Remark 3.4.2** As we will see in the derivation, in (3.4.20), the term \( h_1(D + B)^{-1}z_n \) is to create a negative definite term \( -z_n^T P z_n \) in the closed loop analysis, \( \chi \) is to incorporate the auxiliary system into controller design, \( -(D + B)^{-1}\dot{\alpha}_{n-1} \) is to compensate the term \( \dot{\alpha}_{n-1} \) from (3.4.15), \( -\hat{\theta}^T F, \text{sgn}(z_n^T P(D + B))\hat{\Psi} \) are to compensate \( \dot{\theta}^T F, \dot{\phi} + \varepsilon - f_0 \) in (3.4.15), respectively, and \( \frac{1}{\sigma(P)}(D + B)^{-1} \Upsilon z_{n-1} \) is to cancel the term \( z_{n-1} z_n \) from the previous backstepping step.

Considering (3.4.20), for the last term \( -z_n^T P(D + B)\varphi \) on the RHS of (3.4.18), which
incorporates the control design, we can get

\[-z_n^T P (D + B) \dot{\varphi} = -h_1 z_n^T P z_n + h_1 z_n^T P \alpha_n + z_n^T P (D + B) \hat{\theta}^T F - z_n^T P (D + B) \text{sgn}(z_n^T P (D + B)) \hat{\Psi} - \frac{1}{\sigma(P)} z_n^T P \chi_{z_{n-1}} + z_n^T P (D + B) \tilde{\theta}^T F - z_n^T P z_n,\]

where, the third term on the RHS of (3.4.22) is designed to compensate the same term but with opposite sign on the RHS of (3.4.15). Also, keeping in mind that $P$ is a diagonal matrix that is positive definite, we have $-\frac{1}{\sigma(P)} z_n^T P \chi_{z_{n-1}} \leq -|z_n z_{n-1}|$. Hence, combining (3.4.13)-(3.4.20), we have

\[
\dot{V}_n \leq z_n^T P (D + B) \tilde{\rho} u - z_n^T P (D + B) \tilde{\rho} (u - v) - z_n^T P (D + B) \text{sgn}(z_n^T P (D + B)) \hat{\Psi} - \frac{1}{\sigma(P)} z_n^T P \chi_{z_{n-1}} + \tilde{\theta}^T F - \frac{h_1^2 \sigma^2(P)}{2 \sigma(P)} z_n^T P z_n + C_0,
\]

where $\tilde{\Psi} = \hat{\Psi} - \tilde{\Psi}$, $\tilde{\theta} = \hat{\theta} - \tilde{\theta}$, $C_0$ is a constant such that $\gamma_1 \tilde{\sigma}(P) \tilde{\sigma}(A) \hat{\Psi} \tilde{\Psi} + \gamma_2 \tilde{\sigma}(P) \tilde{\sigma}(A) F^T \tilde{\theta} \tilde{\theta}^T + \gamma_3 \tilde{\sigma}(P) \tilde{\sigma}(A) u^T u \leq C_0$. Adaptive law for $\hat{\rho}$ is designed as

\[
\dot{\hat{\rho}}_j = \begin{cases} 
0, & \text{if } \hat{\rho}_j = 1 \text{ and } \vartheta_j(t) > 0; \\
0, & \text{if } \hat{\rho}_j = \rho_{j, \text{min}} \text{ and } \vartheta_j(t) < 0; \\
\vartheta_j(t), & \text{else},
\end{cases}
\]

where $j = 1, \cdots, N$, $\vartheta_j(t) = -\Omega_1 \hat{\rho}_j - \xi_1 z_{j,n} P_j (D_j + B_j) u_j$, $\Omega_1 > 1$, $\xi_1 > 0$ are design constants, $Q_j$ is the $j$th diagonal entry of a diagonal matrix $Q$. The distributed adaptive laws for $\hat{\theta}$ and
\( \hat{\Psi} \) are constructed such that

\[
\dot{\hat{\Psi}} = -\Omega_2 \hat{\Psi} + \xi_2 \text{sgn}(z_n^T P(D + B))(D + B)Pz_n, \tag{3.4.25}
\]

and

\[
\dot{\theta}_j = -\Omega_3 \hat{\theta}_j - \xi_3 F_j z_{j,n} P_j (D_j + B_j), \tag{3.4.26}
\]

where \( j = 1, \cdots, N, \Omega_2 > 0, \Omega_3 > 1, \xi_2, \xi_3 > 0 \) are design constants, \( F_j = F_j(\bar{x}_j) \).

Henceforth, design the overall Lyapunov function as

\[
V = \sum_{i=1}^{n} V_i + \sum_{j=1}^{N} \frac{1}{2} \chi_j^2 + \sum_{j=1}^{N} \frac{1}{2\xi_1} \dot{\rho}_j^2 + \frac{1}{2\xi_2} \hat{\Psi}^T \hat{\Psi} + \sum_{j=1}^{N} \frac{1}{2\xi_3} \dot{\theta}_j^T \dot{\theta}_j. \tag{3.4.27}
\]

Take the derivative of \( \sum_{j=1}^{N} \frac{1}{2} \chi_j^2 \) for \( |\chi_j| > \nu_j \), we have

\[
\sum_{j=1}^{N} \chi_j (-h_2 \chi_j + f_{\chi_j} \chi_j - \beta_j (u_j - v_j))
\]

\[
= \sum_{j=1}^{N} -h_2 \chi_j^2 + z_{j,n} P_j (D_j + B_j) \hat{\rho}_j (u_j - v_j) - \frac{1}{2} \beta_j^2 (u_j - v_j)^2 - \chi_j \beta_j (u_j - v_j)
\]

\[
\leq \sum_{j=1}^{N} -h_2 \chi_j^2 + z_{j,n} P_j (D_j + B_j) \hat{\rho}_j (u_j - v_j) - \frac{1}{2} \beta_j^2 (u_j - v_j)^2 + \frac{1}{2} \beta_j^2 (u_j - v_j)^2 + \frac{1}{2} \chi_j^2
\]

\[
= \sum_{j=1}^{N} -h_2 \chi_j^2 + z_{j,n} P_j (D_j + B_j) \hat{\rho}_j (u_j - v_j) + \frac{1}{2} \chi_j^2, \tag{3.4.28}
\]

where the term \( z_{j,n} P_j (D_j + B_j) \hat{\rho}_j (u_j - v_j) \) cancels the term with opposite sign in (3.4.18).

Take the derivative of \( \frac{1}{2\xi_1} \dot{\rho}_j^2 \), we can see that if \( \hat{\rho}_j = 1 \) and \( \vartheta_j > 0 \) as in (3.4.24), we have
\[ \dot{\rho}_j = 0. \] Hence
\[ \frac{1}{\xi_1} \dot{\rho}_j \dot{\rho}_j = \frac{1}{\xi_1} \dot{\rho}_j (\dot{\rho}_j - \dot{\rho}_j) = -\frac{1}{\xi_1} \dot{\rho}_j \dot{\rho}_j. \] (3.4.29)

Consider also the term \( z_n^T P(D + B) \dot{\rho} u \) from (3.4.23), since \( \dot{\rho}_j = 1 \), we have \( \dot{\rho}_j = \dot{\rho}_j - \rho_j \geq 0 \), and since \( \vartheta_j = -\Omega_1 \dot{\rho}_j - \xi_1 z_{j,n} P_j (D_j + B_j) u_j > 0 \), we get
\[ -\xi_1 z_{j,n} P_j (D_j + B_j) u_j \geq \Omega_1 \dot{\rho}_j, \] (3.4.30)
which implies that
\[ z_{j,n} P_j (D_j + B_j) \dot{\rho}_j u_j \leq -\frac{\Omega_1}{\xi_1} \dot{\rho}_j \dot{\rho}_j, \] (3.4.31)

hence, consider the \( j \)th component of the term \( z_n^T P(D + B) \dot{\rho} u \) from (3.4.23) together with the derivative of \( \frac{1}{2 \xi_1} \dot{\rho}_j^2 \), we have
\[ z_{j,n} P_j (D_j + B_j) \dot{\rho}_j u_j + \frac{1}{\xi_1} \dot{\rho}_j \dot{\rho}_j \leq -\frac{\Omega_1}{\xi_1} \dot{\rho}_j \dot{\rho}_j - \frac{1}{\xi_1} \dot{\rho}_j \dot{\rho}_j. \] (3.4.32)

Similarly, consider the case when \( \dot{\rho}_j = \rho_{j,\text{min}} \) and \( \vartheta_j = -\Omega_1 \dot{\rho}_j - \xi_1 z_{j,n} P_j (D_j + B_j) u_j < 0 \), we get
\[ \xi_1 z_{j,n} P_j (D_j + B_j) u_j \geq -\Omega_1 \dot{\rho}_j, \] (3.4.33)
and since \( \dot{\rho}_j = \dot{\rho}_j - \rho_j \leq 0 \), we can obtain
\[ z_{j,n} P_j (D_j + B_j) \dot{\rho}_j u_j \leq -\frac{\Omega_1}{\xi_1} \dot{\rho}_j \dot{\rho}_j, \] (3.4.34)
which is the same as \((3.4.31)\). Hence the analysis thereafter follows the same path.

For the last case in \((3.4.24)\), taking the derivative of \(\frac{1}{\xi_1} \tilde{\rho}_j \dot{\tilde{\rho}}_j\), we have

\[
\frac{1}{\xi_1} \tilde{\rho}_j \dot{\tilde{\rho}}_j = \frac{1}{\xi_1} \tilde{\rho}_j (\dot{\hat{\rho}}_j - \hat{\rho}_j) = \frac{1}{\xi_1} \tilde{\rho}_j (-\Omega_1 \hat{\rho}_j - \xi_1 z_{j,n} P_j (D_j + B_j) u_j - \hat{\rho}_j). \tag{3.4.35}
\]

Similarly, consider the term \(z_n^T P(D + B) \hat{\rho} u\) from \((3.4.23)\), this leads to

\[
\frac{1}{\xi_1} \tilde{\rho}_j \dot{\tilde{\rho}}_j = \frac{1}{\xi_1} \tilde{\rho}_j \hat{\rho}_j = - \frac{\Omega_1}{\xi_1} \hat{\rho}_j \hat{\rho}_j - \frac{1}{\xi_1} \tilde{\rho}_j \dot{\tilde{\rho}}_j. \tag{3.4.36}
\]

Therefore, it is always true that

\[
z_{j,n} P_j (D_j + B_j) \hat{\rho}_j u_j + \frac{1}{\xi_1} \tilde{\rho}_j \dot{\rho}_j \leq - \frac{\Omega_1}{\xi_1} \hat{\rho}_j \hat{\rho}_j - \frac{1}{\xi_1} \tilde{\rho}_j \dot{\tilde{\rho}}_j. \tag{3.4.37}
\]

If we take the derivative of \(\frac{1}{\xi_2} \tilde{\Psi}^T \tilde{\Psi}\) on the RHS of \((3.4.27)\), we have

\[
\frac{1}{\xi_2} \tilde{\Psi}^T (\dot{\tilde{\Psi}} - \tilde{\Psi}) = - \frac{1}{\xi_2} \tilde{\Psi}^T \tilde{\Psi} + sgn(z_n^T P(D + B)) \tilde{\Psi}^T (D + B) P z_n - \frac{1}{\xi_2} \tilde{\Psi}^T \dot{\tilde{\Psi}}, \tag{3.4.38}
\]

where the term \(sgn(z_n^T P(D + B)) \tilde{\Psi}^T (D + B) P z_n\) cancels the same term but with opposite sign on the RHS of \((3.4.23)\).

Lastly, taking the derivative of the last term \(\sum_{j=1}^N \frac{1}{\xi_3} \tilde{\theta}_j^T \dot{\tilde{\theta}}_j\) on the RHS of \((3.4.27)\), this yields

\[
\frac{1}{\xi_3} \tilde{\theta}_j^T \dot{\tilde{\theta}}_j = \frac{1}{\xi_3} \tilde{\theta}_j^T (\dot{\hat{\theta}}_j - \hat{\theta}_j) = \frac{1}{\xi_3} \tilde{\theta}_j^T (-\Omega_3 \hat{\theta}_j - \xi_3 F_j z_{j,n} P_j (D_j + B_j) - \dot{\hat{\theta}}_j) = - \frac{\Omega_3}{\xi_3} \tilde{\theta}_j^T \tilde{\theta} - \frac{1}{\xi_3} \tilde{\theta}_j^T \dot{\tilde{\theta}}_j, \tag{3.4.39}
\]

where the term \(-\tilde{\theta}_j^T F_j z_{j,n} P_j (D_j + B_j)\) cancels with the fourth term \(z_n^T P(D + B) \dot{\theta}^T F = \)
\[
\sum_{j=1}^{N} z_{j,n}^T P_j (D_j + B_j) \bar{\theta}^T_j F_j \text{ on the RHS of (3.4.23).}
\]

Notice also that in (3.4.23), we have \( h_1 z_n^T P \chi \leq \frac{h_2^2 \theta^2(P)}{2 \sigma(P)} z_n^T P z_n + \frac{1}{2} \chi^T \chi \). Therefore, take the derivative of (3.4.27), we can derive

\[
\dot{V} \leq \sum_{j=1}^{N} \left\{ -(K_{j,1} - 2K_{j,10}) \frac{k_{b,j}^2}{\pi} \tan \left( \frac{\pi z_j^2}{2k_{b,j}^2} \right) - \sum_{h=2}^{n-1} K_{j,h} \frac{2z_{j,h}^2}{\pi} \right\} - \left( h_1 - \frac{1}{4\gamma_1} + \frac{1}{4\gamma_2} \right) + \frac{1}{2\gamma_3} \bar{\sigma}(P) \bar{\sigma}(A) \right) z_n^T P z_n - (h_2 - 1) \chi^T \chi - \Omega_1 \xi_1 \text{tr}(\bar{\rho}^T \rho) + \frac{1}{\xi_1} \text{tr}(\bar{\rho}^T \rho) \right.
\]

\[
- \Omega_2 \bar{\Psi}^T \bar{\Psi} - \sum_{j=1}^{N} \frac{\Omega_3}{\xi_3} \bar{\theta}_j^T \bar{\theta}_j - \sum_{j=1}^{N} \frac{\Omega_3}{\xi_3} \bar{\theta}_j^T \bar{\theta}_j + C_0. \tag{3.4.40}
\]

Since we have

\[
- \frac{\Omega_1}{\xi_1} \text{tr}(\bar{\rho}^T \rho) \leq - \frac{\Omega_1}{2\xi_1} \text{tr}(\bar{\rho}^T \rho) + \frac{\Omega_1 N}{2\xi_1}, \tag{3.4.41}
\]

\[
- \frac{1}{\xi_1} \text{tr}(\bar{\rho}^T \rho) \leq - \frac{1}{2\xi_1} \text{tr}(\bar{\rho}^T \rho) + \frac{1}{2\xi_1} \Xi, \tag{3.4.42}
\]

\[
- \frac{\Omega_2}{\xi_2} \bar{\Psi}^T \bar{\Psi} \leq - \frac{\Omega_2}{2\xi_2} \bar{\Psi}^T \bar{\Psi} + \frac{\Omega_2}{2\xi_2} \bar{\Psi}^T \bar{\Psi}, \tag{3.4.43}
\]

\[
\sum_{j=1}^{N} \frac{\Omega_3}{\xi_3} \bar{\theta}_j^T \bar{\theta}_j \leq \sum_{j=1}^{N} \frac{\Omega_3}{2\xi_3} \bar{\theta}_j^T \bar{\theta}_j + \frac{\Omega_3}{2\xi_3} \Theta, \tag{3.4.44}
\]

from (3.4.40) we can further obtain

\[
\dot{V} \leq \sum_{j=1}^{N} \left\{ -(K_{j,1} - 2K_{j,10}) \frac{k_{b,j}^2}{\pi} \tan \left( \frac{\pi z_j^2}{2k_{b,j}^2} \right) - \sum_{h=2}^{n-1} (2K_{j,h}) \frac{1}{2} \frac{z_j^2}{\pi} \right\} - \left( 2h_1 - \frac{1}{2\gamma_1} + \frac{1}{2\gamma_2} \right) \right.
\]

\[
- \frac{\Omega_2}{2\xi_2} \bar{\Psi}^T \bar{\Psi} + \sum_{j=1}^{N} \frac{\Omega_3}{2\xi_3} \bar{\theta}_j^T \bar{\theta}_j + C_1. \tag{3.4.46}
\]
where \( C_1 = C_0 + \frac{\Omega_1}{\xi_1} + \frac{1}{2\xi_1} \Xi + \frac{\Omega_3}{\xi_3} \Theta + \frac{1}{2\xi_3} \bar{\Theta} \) is a finite constant. Denote \( C_2 > 0 \) such that
\[
C_2 = \min(K_{j,1} - 2K_{j,10}, 2K_{j,h}, 2h_1 - (\frac{1}{2\gamma_1} + \frac{1}{2\gamma_2} + \frac{1}{2\gamma_3}) \frac{\Omega(P)\bar{\sigma}(A)}{\Omega(A)}, 2h_2 - 2, \Omega_1 - 1, \Omega_2, \Omega_3 - 1),
\]
where \( j = 1, \cdots, N, h = 2, \cdots, n - 1 \). From (3.4.46) we have
\[
\dot{V} \leq -C_2 V + C_1. \tag{3.4.47}
\]

**Remark 3.4.3** Equation (3.4.47) presents some guidelines on how to expedite the convergence speed of the system output, as well as how to modify the size of the set into which the system output cooperative tracking error will converge. To make the size of the set as small as possible, we need to select large \( C_2 \) and small \( C_1 \). A large \( C_2 \) is also desirable to increase the convergence rate. To make \( C_1 \) small, we need small \( \gamma_1, \gamma_2, \gamma_3, \frac{\Omega_1}{\xi_1}, \frac{1}{2\xi_1}, \frac{\Omega_2}{\xi_2}, \frac{\Omega_3}{\xi_3}, \frac{1}{2\xi_3} \). To make \( C_2 \) large, we need to select large \( h_1, h_2, K_{j,i}, \Omega_1, \Omega_2, \Omega_3, \) where \( j = 1, \cdots, N, i = 1, \cdots, n - 1 \). The choice of these parameters are independent from each other. Notice that to choose small \( \frac{\Omega_1}{2\xi_1} \) and large \( \Omega_1, \) for example, are not conflicting objectives, as \( \frac{\Omega_1}{2\xi_1} \) can turn out to be small despite \( \Omega_1 \) being large, by selecting large \( \xi_1 \).

The above backstepping analysis leads to the following theorem, the main result of this chapter.

**Theorem 3.4.1** Consider the multiagent system given by (3.2.1) (or equivalently (3.2.4)) with the leader node (3.2.5) under Assumption 3.2.1-3.2.4 subject to actuator faults (3.2.2) and input constraints (3.2.3). Construct the distributed adaptive control scheme (3.4.14), (3.4.20), (3.4.21), (3.4.24), (3.4.25), (3.4.26). We then have the following results:

1. The output tracking error between agents and the leader \( ||\delta_1|| < \bar{k} - k_d \) is guaranteed, and hence (3.2.7) will not be violated during operation.

2. The output tracking error between agents and the leader \( \delta_1 \) will exponentially converge to
a set defined by \( \delta_1 \ \| \delta_1 \| \leq \frac{1}{\sigma(L+B)} \sqrt{\frac{2C_1}{C_2}} \), which can be designed as an arbitrarily small neighbourhood of zero.

3. The state vectors \( \bar{x}_j, j = 1, \cdots, N \) are bounded for \( \forall t \geq 0 \).

4. Closed-loop signals \( \tilde{\rho}, \chi, \tilde{\theta}, \tilde{\Psi} \) are all bounded.

**Proof:** From (3.4.47), we obtain \( V \leq (V(0) - \frac{C_1}{C_2})e^{-C_2t} + \frac{C_1}{C_2} \), therefore \( V \) is bounded. Hence the closed-loop signals \( \tilde{\rho}, \chi, \tilde{\theta}, \tilde{\Psi} \) are all bounded. The boundedness of \( V \) implies that the BLFs are also bounded. Moreover, for \( j = 1, \cdots, N, \frac{k_{b,j}^2}{\pi} \tan(\frac{\pi z_{2,j,1}}{2k_{b,j}^2}) \leq V \leq (V(0) - \frac{C_1}{C_2})e^{-C_2t} + \frac{C_1}{C_2} \), hence

\[
\sum_{j=1}^{N} \frac{1}{2} z_{2,j,1}^2 \leq \sum_{j=1}^{N} \frac{k_{b,j}^2}{\pi} \tan\left( \frac{\pi z_{2,j,1}}{2k_{b,j}^2} \right) \leq (V(0) - \frac{C_1}{C_2})e^{-C_2t} + \frac{C_1}{C_2},
\]

Therefore, in view of (3.3.4), we have

\[
\| \delta_1 \| \leq \frac{\| e_1 \|}{\sigma(L+B)} \leq \frac{\sqrt{k_{b,1}^2 + \cdots + k_{b,N}^2}}{\sigma(L+B)} \leq (\bar{k} - k_d). \tag{3.4.49}
\]

Furthermore,

\[
\sum_{j=1}^{N} \frac{1}{2} z_{2,j,1}^2 \leq \sum_{j=1}^{N} \frac{k_{b,j}^2}{\pi} \tan\left( \frac{\pi z_{2,j,1}}{2k_{b,j}^2} \right) \leq (V(0) - \frac{C_1}{C_2})e^{-C_2t} + \frac{C_1}{C_2},
\]

therefore \( \| e_1 \| \) will be exponentially convergent to the set \( \| e_1 \| \leq \sqrt{\frac{2C_1}{C_2}} \). Hence the output tracking error between agents and the leader \( \| \delta_1 \| \) will exponentially converge to \( \| \delta_1 \| \leq \frac{1}{\sigma(L+B)} \sqrt{\frac{2C_1}{C_2}} \).

Next, the boundedness of \( V \) implies that \( z_{j,i}, j = 1, \cdots, N, i = 1, \cdots, n \) are bounded. Recall that the fictitious synchronization errors \( z_{j,i} \) are defined as \( z_{j,1} = e_{j,1} \) and \( z_{j,h} = e_{j,h} - \alpha_{j,h-1}, \) \( h = 2, \cdots, n \), and in view of (3.4.2) and (3.4.9), it can be easily seen that the stabilizing
functions $\alpha_{j,h}$, $h = 1, \ldots, n - 1$ are bounded. Hence the neighborhood synchronization errors $e_{j,i}$, $i = 1, \ldots, n$ are bounded. In view of Lemma 3.2.2, the $i$th order tracking error, or disagreement variables $\delta_i$ are bounded, which implies that the state vectors $\bar{x}_j$, $j = 1, \ldots, N$ are bounded for $\forall t \geq 0$.

### 3.5 Simulation

In this section we present a simulation study based on a previously discussed case in the literature [80] [89] with some modifications. Consider a 5-node digraph $G$ and a leader node described in Fig. 3.1, where the numbers on the edge denote the weights between the corresponding two nodes. Let the dynamics of the leader node be in the form of (3.2.5) with

$$
\dot{x}_{0,3} = -5x_{0,2} - 10x_{0,3} + 15\sin(2t) + 30\cos(2t) - \frac{5}{3}(x_{0,1} + x_{0,2} - 1)^2(x_{0,1} + 4x_{0,2} + 3x_{0,3} - 1).
$$

(3.5.1)

The follower nodes are described by third-order nonlinear systems in the form of (3.2.1) with

$$
\dot{x}_{1,3} = u_1^F + 0.1\sin(x_{1,1}) + \sin^2(2t), \quad \dot{x}_{2,3} = u_2^F + 0.5\cos(x_{2,2}) + \sin(2t),
$$

$$
\dot{x}_{3,3} = u_3^F + 0.4\sin^2(x_{3,2}) + \cos^3(2t), \quad \dot{x}_{4,3} = u_4^F + 0.3\cos^2(x_{4,2}) + \cos(2t),
$$

$$
\dot{x}_{5,3} = u_5^F + 0.2\sin^2(x_{5,2}) + \sin(2t),
$$

(3.5.2)

where $u_j^F = \rho_j u + \phi_j$, $j = 1, \ldots, 5$, $\rho_j = 0.5 + 0.5e^{-1.5t}$, $\phi_j = 0.1(1 - e^{-2t})$. Initially we have $\rho = 1$, and $\phi = 0$, meaning there are no actuator faults. The input saturation as in (3.2.3) is $u_{min} = [-50, -50, -50, -50, -50]^T$, and $u_{max} = [30, 30, 30, 30, 30]^T$. The initial condition for the leader is $x_0 = [0, 0, 0]^T$. For the nodes, initially we have $x_1 = [1.5, 0, 0]^T$, $x_2 = [4.8, 0, 0]^T$, 

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\[ x_3 = [5,0,0]^T, \quad x_4 = [1.5,0,0]^T, \quad x_5 = [-0.1,0,0]^T. \] Hence \(|\delta_1(0)|| = 7.249\). Suppose it is required that \(|\delta_1|| < 10 at all time. Notice that initially \(e_{1,1} = -0.5, \ e_{2,1} = -3.3, \ e_{3,1} = -0.8, \ e_{4,1} = 0, \ e_{5,1} = 0.5\), therefore we may choose \(k_{b,1} = 1.2, \ k_{b,2} = 4, \ k_{b,3} = 1.5, \ k_{b,4} = 0.7, \ k_{b,5} = 1.2\). Control gains are selected as \(K_{j,1} = K_{j,2} = 1.5, \ j = 1, \cdots, 5, \ h_1 = 6, \ h_2 = 1.1, \ \Omega_1 = 1.1, \ \Omega_2 = 1, \ \Omega_3 = 1.1, \ \xi_1 = \xi_2 = \xi_3 = 0.1, \ \beta = 0.1\). The profile for \(|\delta_1|| is shown in Fig. 3.2, notice that despite the large initial output tracking error and the presence of actuator faults, \(\delta_1\) can eventually converge into a small neighborhood of zero, while the agents' output constraint is not violated. The individual output tracking error for each agent is shown in Fig. 3.3. The control profiles for each node are shown in Fig. 3.4-3.8.
3.6 Discussion

This Chapter considers the adaptive fault tolerant cooperative tracking control (AFTCTC) problem for a class of control input and agents’ output constrained networked nonlinear multi-
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agent systems with distinct unknown dynamics and both parametric and nonparametric uncertainties. A novel distributed control structure is proposed such that exponential convergence...
of the cooperative output tracking error into a small set around zero is guaranteed, while the
constraint on the agents’ output will not be violated during operation. State vectors of the
agents and all other closed loop signals are bounded.

Several interesting aspects arise in connection with extending the results in this Chapter:

(1) In this Chapter, a class of integrator-type nonlinear systems have been studied. A natural
question to ask is: is it possible to extend the result to other types of generic MIMO nonlinear
systems?

(2) In the problem formulation (3.2.1), the control input gain is 1. What if the control input
gain is a function of the state vector, and furthermore, what if the control input gain function
is not totally known? Under such situations, how can we extend the results in this Chapter?

(3) In this Chapter, to calculate the stabilizing function at each step $\alpha_{j,h}$, where $j = 1, \cdots, N,$
$h = 1, \cdots, n - 1$, we have to iteratively calculate the derivative of the stabilizing function from
the previous step of the backstepping analysis. As the system is only of integrator type, this is
relatively easy. However, for generic nonlinear systems where the order is high, or for general
nonlinear MIMO systems that are not of the integrator-type, there may be the problem of
“explosion of complexity” associated with the calculation of the stabilizing function. Can such
a problem be avoided, while still preserving the main results of this Chapter?

(4) The problem of system output constraint has been discussed in this Chapter. Is there an
effective way to further extend the result in this Chapter to address system state constraints?

These interesting and practically important considerations will be addressed in the next
Chapter.
Chapter 4

AFTC for Input and State Constrained MIMO Nonlinear Systems

In this chapter, we present a novel adaptive FTC (AFTC) scheme for a class of control input and system state constrained multi-input multi-output (MIMO) nonlinear systems with actuator faults, where the input constraints can be asymmetric, and the state constraints can be time-varying. To address system state constraint requirements, we use \( \tan \)-type Barrier Lyapunov functions (BLFs) to ensure the boundedness on the fictitious state tracking errors that arise in control design using the backstepping method, and use command filters to ensure the boundedness on the stabilizing functions. To analyze the effects of control input constraints, the idea of an auxiliary system, introduced in [60, 61, 90, 91], is adopted, and its state is used in the control algorithm design. We show that under the proposed adaptive FTC scheme, exponential convergence of the output tracking error into a small set around zero is guaranteed, while the constraints on the system state will not be violated during operation. Estimation
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errors for actuator faults are also bounded in the closed loop.

This Chapter is organized as follows. In Section 4.1, we introduce the problem formulation and control objective. State and input constraints, as well as actuator faults will be modeled. In Section 4.2, the \textit{tan} type BLF will be presented. In Section 4.3, by introducing backstepping design analysis, we propose the control scheme and adaption laws to achieve our control objectives. Two illustrative examples are introduced to demonstrate the effectiveness of the proposed method in Section 4.4, with discussion presented in Section 4.5.

Some of the notations introduced in this chapter may be different from the previous one. All the notations used in this chapter will be introduced in a self-contained manner. A condensed version of the material presented in this Chapter appears in [92].

4.1 Problem Formulation

Consider the following class of MIMO nonlinear systems with actuator faults subject to both control input and system state constraints

\begin{align*}
\dot{x}_i(t) &= \theta_i^T(t)F_i(\bar{x}_i) + (B_i(\bar{x}_i) + \Delta B_i(\bar{x}_i))x_{i+1}(t) + D_i(\bar{x}_i, t), \\
\dot{x}_n(t) &= \theta_n^T(t)F_n(\bar{x}_n) + (B_n(\bar{x}_n) + \Delta B_n(\bar{x}_n))u(t) + D_n(\bar{x}_n, t), \\
y(t) &= x_1(t),
\end{align*}

(4.1.1)

where \(i = 1, \cdots, n-1\), \(x_j(t) \in \mathbb{R}^m\), \(j = 1, \cdots, n\) are the system state variables, \(\theta_j(t) \in \mathbb{R}^{p \times m}\) are the unknown time-varying functions, \(F_j(\bar{x}_j) \in \mathbb{R}^p\) are the known nonlinear functions, \(\bar{x}_j \triangleq [x_1, \cdots, x_j]^T\). \(B_j(\bar{x}_j) \in \mathbb{R}^{m \times m}\) are the known parts of the control input functions, and \(\Delta B_j(\bar{x}_j) \in \mathbb{R}^{m \times m}\) are unknown. \(u(t) \in \mathbb{R}^m\) is the control input signal. \(D_j(\bar{x}_j, t) \in \mathbb{R}^m\) are unknown bounded disturbances.
Remark 4.1.1 Plant model (4.1.1) is a widely discussed formulation in adaptive control and nonlinear system literature like [60, 93, 94]. For example, dynamics of certain aircraft [95], ships [96, 97] and rigid robots and motors [98] formulations can be converted to the plant model as in (4.1.1).

Remark 4.1.2 The form (4.1.1) is usually referred to as lower triangle structure in the literature. For each subsystem \( \dot{x}_j \), the system nonlinear functions depend on \( \bar{x}_j \triangleq [x_1, \cdots, x_j]^T \). For example, as will be discussed in the simulation part at the end of this chapter, robot manipulators are often described as

\[
\dot{x}_1 = x_2,
\]

\[
D(x_1)\dot{x}_2 + C(x_1, x_2)x_2 + G(x_1) + F(x_1, x_2, t) = u, \tag{4.1.2}
\]

where the dynamics of \( \dot{x}_2 \) depend on the function of \( x_1 \) and \( x_2 \).

Remark 4.1.3 Notice that in (4.1.1), for \( i = 1, \cdots, n - 1 \), if \( \theta_i(t) = 0 \), \( F_i(\bar{x}_i) = 0 \), \( B_i(\bar{x}_i) = I_{m \times m} \), \( \Delta B_i(\bar{x}_i) = 0 \), \( D_i(\bar{x}_i, t) = 0 \), that means, if there are no disturbances and uncertainties for these subsystems, we will have the \( \dot{x}_i(t) = x_{i+1}(t) \), which is the integrator form as in Chapter 3. Hence the plant formulation in this Chapter is more general in this sense.

The system state variables \( x_i(t), i = 1, \cdots, n \), have to satisfy time-varying state constraint requirements, which are

\[
\|x_i(t)\| < \bar{k}_i(t), \tag{4.1.3}
\]

where \( \|x\| \) denotes the Euclidean norm of the vector \( x \), \( \bar{k}_i(t) \in \mathcal{R} \) are the time-varying constraints on the system state variables.
The control input constraint is formulated as

\[
  u_i(t) = \begin{cases} 
    u_{i,\text{max}}, & \text{if } v_i(t) > u_{i,\text{max}}; \\
    v_i(t), & \text{if } u_{i,\text{min}} \leq v_i(t) \leq u_{i,\text{max}}; \\
    u_{i,\text{min}}, & \text{if } v_i(t) < u_{i,\text{min}},
  \end{cases} \tag{4.1.4}
\]

where \( i = 1, \cdots, m \), \( v(t) = [v_1(t), \cdots, v_m(t)]^T \in \mathcal{R}^m \) is the unconstrained control signal, subject to actuator faults. In particular, we are dealing with both multiplicative and additive actuator faults, which can be formulated as

\[
  v(t) = \rho(t)v_0(t) + \phi(t), \tag{4.1.5}
\]

where \( v_0(t) \in \mathcal{R}^m \) is the signal generated by the control law to be designed, \( \rho(t) = \text{diag}[\rho_1(t), \cdots, \rho_m(t)] \in \mathcal{R}^{m \times m} \) represents the time-varying uncertain gain of the actuator signal, and \( \phi(t) \in \mathcal{R}^m \) represents the additive fault in the control input channel. If \( \rho(t) = I_{m \times m} \) and \( \phi = 0 \), we say that the system is free from actuator faults.

**Remark 4.1.4** The actuator system (4.1.4) and (4.1.5) can be interpreted as follows. If there are no actuator faults and the actuator signal is within the input constraint, we will have \( u(t) = v(t) = v_0(t) \). In practice, when the actuator is corrupted by multiplicative and additive faults, the control signal without constraint would be \( v(t) = \rho(t)v_0(t) + \phi(t) \). Furthermore, due to hardware limitations, the real actuator signal supplied is constrained, so the real actuator signal supplied to the plant is \( u(t) \) as in (4.1.4).

**The control objective** is to design a control algorithm so that the system output \( y(t) \) tracks a desired trajectory \( y_d(t) \), when (4.1.1) is subject to control input constraints (4.1.4) and actuator faults (4.1.5), while ensuring that the state constraint requirements (4.1.3) are not
To develop our main results, we make the following assumptions.

**Assumption 4.1.1** There exists \( ||x_{1d}(t)|| \leq k_d(t) \), where \( k_d(t) \in \mathcal{R} \) is a continuous function, and \( \bar{k}_1(t) > k_d(t) \). The signals \( k_d(t) \) and \( x_{1d}(t) \) are differentiable up to \( n \)-th order, and the derivatives are bounded. \( \bar{k}_i(t), i = 1, \ldots, n \) are derivable up to \( (n + 1 - i) \)-th order, and the derivatives are bounded.

**Remark 4.1.5** A similar assumption is presented in the literature on problems with output constraints [37]. Notice that \( \bar{k}_1(t) > k_d(t) \) is always true in practice for the requirement of perfect output tracking, since the constraint requirement on the system output cannot be smaller than the bound on the desired output trajectory. The trajectories \( k_d(t) \) and \( x_{1d}(t) \) being differentiable up to \( n \)-th order, together with \( \bar{k}_i(t), i = 1, \ldots, n \) being differentiable up to \( (n + 1 - i) \)-th order is a commonly adopted assumption in adaptive control literature using backstepping analysis.

**Assumption 4.1.2** For the multiplicative actuator fault, \( 0 < \rho_{i,\text{min}} \leq \rho_i(t) \leq \rho_{i,\text{max}} \), where \( \rho_{i,\text{min}} \) and \( \rho_{i,\text{max}} \) are known constants, \( i = 1, \ldots, m \). The multiplicative actuator fault is continuous with bounded first order derivative, in particular, \( \text{tr}(\dot{\rho}^T(t)\dot{\rho}(t)) \leq \Theta < \infty \), where “\( \text{tr} \)” means trace operation. The additive actuator fault and its derivative are bounded, in particular, \( ||\phi(t)|| \leq \bar{\phi} < \infty \) and \( ||\dot{\phi}(t)|| \leq \bar{\phi}_0 < \infty \).

**Remark 4.1.6** This assumption implies that every actuator component will not totally fail under faulty situations. The multiplicative and additive actuator faults, together with their derivatives, are bounded.

**Assumption 4.1.3** The uncertain control input gain functions \( \Delta B_i(\bar{x}_i) \), \( i = 1, \ldots, n \) are bounded, that is, there are known constants \( \bar{B}_i > 0, \bar{B}_i \in \mathcal{R} \) such that \( ||\Delta B_i(\bar{x}_i)|| \leq \bar{B}_i \). Also, \( B_i(\bar{x}_i) \) are nonsingular and hence invertible.
Remark 4.1.7 The assumption about $B_i(\bar{x}_i)$ being nonsingular and invertible is not restrictive, and is commonly made in the literature [39, 93, 97]. $B_i(\bar{x}_i)$ being nonsingular insures the control direction is unchanged during operation. According to [60], if a square matrix $B \in \mathbb{R}^{m \times m}$ has spectral radius $\varsigma(B)$, there exists a positive constant $\Lambda > 0$ such that $B + (\varsigma(B) + \Lambda)I_{m \times m}$ is nonsingular. Let $\tilde{\gamma}_i = \varsigma(B_i(\bar{x}_i)) + \Lambda_i$. If $B_i(\bar{x}_i)$ is singular, we can also have $B'_i(\bar{x}_i) = B_i(\bar{x}_i) + \tilde{\gamma}_i I_{m \times m}$ and $\Delta B'_i(\bar{x}_i) = \Delta B_i(\bar{x}_i) - \tilde{\gamma}_i I_{m \times m}$, so that $B'_i(\bar{x}_i)$ is nonsingular, and $\Delta B'_i(\bar{x}_i)$ is still bounded in the sense of Euclidean norm.

Assumption 4.1.4 The disturbances $D_i(\bar{x}_i, t)$, $i = 1, \ldots, n$ are bounded, that is, $||D_i(\bar{x}_i, t)|| \leq \bar{D}_i$, where $\bar{D}_i \in \mathcal{R}$ can be unknown. The unknown time-varying functions $\theta_i(t)$, $i = 1, \ldots, n$ and their derivatives are bounded, in particular, $\text{tr}(\theta_i^T(t)\theta_i(t)) \leq \Theta_i < \infty$, and $\text{tr}(\dot{\theta}_i^T(t)\dot{\theta}_i(t)) \leq \bar{\Theta}_i < \infty$.

Assumption 4.1.5 The control signal $v(t)$ defined in (4.1.5) is measurable.

Remark 4.1.8 Notice that this assumption is not restrictive in practice. Even though $v(t)$ is measurable, it cannot be directly controlled due to unknown actuator faults.

To facilitate the presentation, we state the following lemma.

Lemma 4.1.1 [21] For any $\delta > 0$ and $\eta \in \mathcal{R}$, the following inequality always holds

$$0 \leq |\eta| - \eta \tanh\left(\frac{\eta}{\delta}\right) \leq k_p \delta,$$

(4.1.6)

where $|\eta|$ is the absolute value of $\eta$, and $k_p = 0.2785$.

From this point onwards for the rest of this Chapter, to simplify the notation, the time and state dependence of the system will be omitted whenever no confusion would arise.
4.2 *tan*-type Barrier Lyapunov Function

To facilitate the discussion about the system state constraints, we introduce the time-varying *tan*-type Barrier Lyapunov Function as follows,

\[ V_i^* = \frac{k_i^2}{\pi} \tan \left( \frac{\pi z_i^T z_i}{2k_i^2} \right), \]  

(4.2.1)

where \( i = 1, \cdots, n \), \( z_1 = x_1 - x_{1,d} \) is the output tracking error, \( z_j = x_j - \alpha_{j-1} \) for \( j = 2, \cdots, n \) are the fictitious state tracking errors with the stabilizing functions \( \alpha_{j-1} \) to be designed, and \(||\alpha_{j-1}|| < \bar{\alpha}_{j-1,0}\), with \( \bar{\alpha}_{j-1,0} \) being the bound on the stabilizing functions. Here \( k_1 = \bar{k}_1 - k_d > 0 \) and \( k_i = \bar{k}_i - \bar{\alpha}_{i-1,0} > 0, i = 2, \cdots, n \).

**Remark 4.2.1** When there are no constraint requirements on the system state, from (4.1.3) we will have \( \bar{k}_i \to \infty \), hence \( k_i \to \infty, i = 1, \cdots, n \). By L’Hospital’s rule, we get

\[ \lim_{k_i \to \infty} \frac{k_i^2}{\pi} \tan \left( \frac{\pi z_i^T z_i}{2k_i^2} \right) = \frac{1}{2} z_i^T z_i, \]  

(4.2.2)

which means if we do not have constraints on the system state, we can simply replace the BLF with the quadratic form. In such circumstances, the analysis approach remains the same as the cases without system state constraints. Therefore our *tan*-type BLF analysis for systems with state constraints is a general approach that would also work for systems without state constraint requirements.

A conventional BLF adopted for analysis of SISO system in [34, 35, 36, 37] has the form

\[ V = \frac{1}{2} \log \frac{k_b^2}{k_b^2 - \gamma^2}, \]  

(4.2.3)

where \( \gamma \) is a scalar to be constrained. Recently another tangent BLF is proposed in [39] with
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the form

\[ V = \frac{k_b}{\pi} \tan^2\left(\frac{\pi \gamma}{2k_b}\right). \]  

(4.2.4)

Notice that both the forms converge to zero when \( k_b \to \infty \), instead of the quadratic form \( \frac{1}{2} \gamma^2 \), hence they cannot be used in a unified scheme which can address systems with or without state constraints. It is also unclear how to extend (4.2.4) to vector constraint case.

4.3 Main Result

We present the design procedure, which is a backstepping approach that will lead to our controller design and main theorem of this chapter.

Remark 4.3.1 The structure of the discussion in each of the steps is as follows. First, we analyze the dynamics of the fictitious tracking errors \( z_i, i = 1, \cdots, n \). Then, to analyze the time derivative of the BLF, we propose the nominal stabilizing function \( \alpha_{i0} \), or the controller \( v_0 \), together with the auxiliary system \( e_i \). After that, we are then in a position to propose the overall Lyapunov function in each of the steps, followed by the adaptive laws for estimators. Finally, at the end of each step, the time derivative of the overall Lyapunov functions will be shown.

Step 1:

From Section 4.2 we have \( z_1 = x_1 - x_{1,d} \) and \( z_2 = x_2 - \alpha_1 \), where \( \alpha_1 \) is produced by passing the nominal stabilizing function \( \alpha_{10} \) through a command filter as shown in Fig. 4.1 [60, 61, 90]. \( z_1 \) is the true output tracking error, and \( z_2 \) is regarded as the fictitious state tracking error. \( \alpha_1 \) and its derivative \( \dot{\alpha}_1 \) are magnitude, rate and bandwidth limited, both of which are within the operating envelope of the system.
Figure 4.1: Configuration of the command filter, with $i = 1, \cdots, n-1$, where $\alpha_{i0}$ are the nominal stabilizing function, $\alpha_i$ are the virtual control law, $\xi_{i1}$ and $\omega_{i1}$ are the bandwidth parameters.

The dynamics of $z_1$ can be represented as

$$
\dot{z}_1 = \theta_1^T F_1 + B_1(z_2 + \alpha_1) + \Delta B_1 x_2 + D_1 - \dot{x}_{1,d}.
$$

Setting $\tau_i = \frac{z_i}{\cos^2(\frac{\pi z_i^T z_i}{2k_1})}$, $i = 1, \cdots, n$, and taking the derivative of $V_i^*$ with respect to time, we have

$$
\dot{V}_i^* = \frac{2k_1 \dot{k}_1}{\pi} \tan \left( \frac{\pi z_i^T z_i}{2k_1^2} \right) - \left( \frac{\dot{k}_1}{k_1} \right) \tau_1^T z_1 \\
+ \tau_1^T (\theta_1^T F_1 + B_1(z_2 + \alpha_1) + \Delta B_1 x_2 + D_1 - \dot{x}_{1,d}).
$$

The nominal stabilizing function $\alpha_{10}$ is designed as

$$
\alpha_{10} = B_1^{-1} \left( -K_1 \frac{z_1}{z_i^T z_1} \frac{k_1^2}{\pi} \sin \left( \frac{\pi z_i^T z_i}{2k_1^2} \right) \cos \left( \frac{\pi z_i^T z_i}{2k_1^2} \right) + K_1 e_1 - K_{10} z_1 - \dot{\theta}_1^T F_1 + \dot{x}_{1,d} \\
- \dot{D}_1 \tanh \left( \frac{\tau_1}{\delta_1} \right) - \frac{K_2^2}{2} \tau_1 \right),
$$

where $\delta_1 > 0$ is a small constant, $K_1$, $K_{10}$ are positive, $K_1$ is a design constant, $K_1 > 2K_{10}$, $K_{10} = \sqrt{\left( \frac{k_1}{k_1} \right)^2 + \epsilon}$, $\epsilon > 0$ is a small constant, $\dot{\theta}_1$ and $\dot{D}_1$ are adaptive estimators of $\theta_1$ and $\dot{D}_1$, respectively.

**Remark 4.3.2** The terms in the nominal stabilizing function (4.3.3) can be interpreted as...
follows. First, notice that

\[-K_1 \tau_1^T \frac{z_1}{z_1^T z_1} \frac{k_1^2}{\pi} \sin \left( \frac{\pi z_1^T z_1}{2k_1^2} \right) \cos \left( \frac{\pi z_1^T z_1}{2k_1^2} \right) = -K_1 \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1^T z_1}{2k_1^2} \right),\]

this creates the negative definite term in the form of BLF in the time derivative of \( V_1^* \). \(-K_{10} z_1, -\hat{\theta}_1^T F_1, -\hat{D}_1 \tanh \left( \frac{\tau_1}{\delta_1} \right) \), \( \dot{x}_{1,d} \) are designed to compensate the terms \(-\frac{k_1^2}{\pi} \) \( \tau_1^T z_1, \tau_1^T \theta_1^T F_1, \tau_1^T D_1 \) and \( \tau_1^T \dot{x}_{1,d} \) in (4.3.2), respectively. Notice also that \( K_1 \tau_1^T e_i \leq e_i^T e_i + K_2 \frac{1}{2} \tau_1^T \tau_1 \), therefore \(-K_2 \frac{1}{2} \tau_1 \) is introduced in (4.3.3) to compensate the term \( -K_2 \frac{1}{2} \tau_1 \).

**Remark 4.3.3** By L’Hospital’s Rule, we can see that,

\[
\lim_{z_1 \to 0} \frac{z_1}{z_1^T z_1} \sin \left( \frac{\pi z_1^T z_1}{2k_1^2} \right) \cos \left( \frac{\pi z_1^T z_1}{2k_1^2} \right) = 0,
\]

therefore singularity will not occur in (4.3.3). However, in a digital computer, since \( 0 \) cannot be evaluated, we can take (4.3.4) to be zero when \( ||z_1|| < \epsilon_0 \) for some small \( \epsilon_0 > 0 \).

\( e_i \in \mathbb{R}^m \) is the state of the auxiliary system, designed as

\[
\dot{e}_i = \begin{cases} 
-K_{1i} e_i - f_i e_i + \gamma_{1i} (\alpha_i - \alpha_{i0}), & \text{if } ||e_i|| > v_i; \\
0, & \text{if } ||e_i|| \leq v_i,
\end{cases} \quad (4.3.5)
\]

where \( i = 1, \ldots, n - 1, v_i > 0 \) is a small number, \( K_{1i} > 1, \gamma_{1i} > 0 \) are design constants,

\[
f_i = \frac{|\tau_i^T B_i \Delta_i| + \frac{1}{2} \gamma_{1i}^2 \Delta_i^T \Delta_i}{||e_i||^2}, \quad \Delta_i = \alpha_i - \alpha_{i0}.
\]

Notice that in (4.3.2),

\[
2k_1 \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1^T z_1}{2k_1^2} \right) = 2k_1 \frac{k_1^2}{\pi} \tan \left( \frac{\tau_1^T z_1}{2k_1^2} \right) < 2K_{10} \frac{k_1^2}{\pi} \tan \left( \frac{\tau_1^T z_1}{2k_1^2} \right). \quad (4.3.6)
\]
Besides,

$$- \left( \frac{k_1}{k_1} \right) \tau_1^T z_1 \leq \sqrt{\left( \frac{k_1}{k_1} \right)^2 \tau_1^T z_1} < K_{10} \tau_1^T z_1. \quad (4.3.7)$$

Also \( \tau_1^T K_1 e_1 \leq \frac{k_2^2}{2} \tau_1^T \tau_1 + \frac{1}{2} e_1^T e_1 \). For the term \( \tau_1^T B_1 \alpha_1 \) in (4.3.2), we have

\[
\tau_1^T B_1 \alpha_1 = \tau_1^T B_1 \alpha_{10} + \tau_1^T B_1 \Delta \alpha_1 \\
= -K_1 \tau_1^T \frac{z_1}{\tau_1} \frac{k_1}{\pi} \sin \left( \frac{\pi z_1}{2k_1^2} \right) \cos \left( \frac{\pi z_1}{2k_1^2} \right) + \tau_1^T e_1 - K_{10} \tau_1^T z_1 \\
\leq -K_1 \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1}{2k_1^2} \right) + \frac{K_1^2}{2} \tau_1^T \tau_1 + \frac{1}{2} e_1^T e_1 - K_{10} \tau_1^T z_1 \\
\leq -K_1 \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1}{2k_1^2} \right) + \frac{1}{2} e_1^T e_1 - K_{10} \tau_1^T z_1 \\
\leq -K_1 \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1}{2k_1^2} \right) + \frac{1}{2} e_1^T e_1 - K_{10} \tau_1^T z_1 \\
-\tau_1^T \tilde{\theta_1}^T f_1 + \tau_1^T \tilde{\dot{x}}_{1,d} - \tau_1^T \tilde{D}_1 \tan \left( \frac{\gamma_1}{\delta_1} \right) - \frac{K_1^2}{2} \tau_1^T \tau_1 + \tau_1^T B_1 \Delta \alpha_1 \\
\leq -K_1 \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1}{2k_1^2} \right) + \frac{1}{2} e_1^T e_1 - K_{10} \tau_1^T z_1 \\
-\tau_1^T \tilde{\theta_1}^T f_1 + \tau_1^T \tilde{x}_{1,d} - \tau_1^T \tilde{D}_1 \tan \left( \frac{\gamma_1}{\delta_1} \right) + |\tau_1^T B_1 \Delta \alpha_1|. \quad (4.3.8)
\]

Let \( K_1^* = K_1 - 2K_{10} \), we then have

\[
\bar{V}_1^* \leq -K_1^* \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1}{2k_1^2} \right) - \tau_1^T \tilde{\theta_1}^T f_1 + \tau_1^T B_1 \tilde{z}_2 + \frac{1}{2} e_1^T e_1 + \tau_1^T \tilde{D}_1 + B_1 ||\gamma_1|| ||x_2|| \\
-\tau_1^T \tilde{D}_1 \tan \left( \frac{\gamma_1}{\delta_1} \right) + |\tau_1^T B_1 \Delta \alpha_1|, \quad (4.3.9)
\]

where \( \tilde{\theta_1} = \hat{\theta_1} - \theta_1 \). Denote \( \tilde{D}_1 = \hat{D}_1 - \tilde{D}_1 \), we can construct the Lyapunov function at this step

\[
V_1 = V_1^* + \frac{1}{2\gamma_1} \text{tr}(\tilde{\theta_1}^T \tilde{\theta_1}) + \frac{1}{2\xi_1} \tilde{D}_1^2 + \frac{1}{2} e_1^T e_1. \quad (4.3.10)
\]
The adaptive laws for $\hat{\theta}_i$ and $\hat{D}_i$, $i = 1, 2, \cdots, n$ are designed as

$$
\dot{\hat{\theta}}_i = \gamma_i F_i \tau_i^T - \beta_i \hat{\theta}_i, \quad \dot{\hat{D}}_i = \xi_i (\tau_i^T \tanh(\frac{\tau_i}{\delta_i}) - \lambda_i \hat{D}_i), \quad i = 1, \cdots, n, \quad (4.3.11)
$$

where $\gamma_i > 0$, $\xi_i > 0$, $\lambda_i > 0$, $\beta_i > 1$ are design constants.

Now we take the derivative of (4.3.10), by analyzing the terms in (4.3.10). For the term $\frac{1}{2\gamma_1} \text{tr}(\tilde{\theta}_1^T \dot{\tilde{\theta}}_1)$, notice the equality that $\text{tr}(ab^T) = b^T a$ for $a, b \in \mathbb{R}^m$, we have

$$
\frac{1}{\gamma_1} \text{tr}(\tilde{\theta}_1^T \dot{\tilde{\theta}}_1) = \frac{1}{\gamma_1} \text{tr}(\tilde{\theta}_1^T (\dot{\tilde{\theta}}_1 - \tilde{\theta}_1))
= \frac{1}{\gamma_1} \text{tr}(\tilde{\theta}_1^T (\gamma_1 F_1 \tau_1^T - \beta_1 \hat{\theta}_1 - \dot{\theta}_1))
= \text{tr}(\tilde{\theta}_1^T F_1 \tau_1^T) - \frac{1}{\gamma_1} \text{tr}(\tilde{\theta}_1^T (\beta_1 \hat{\theta}_1 + \dot{\theta}_1))
= \tau_1^T \tilde{\theta}_1^T F_1 - \frac{1}{\gamma_1} \text{tr}(\tilde{\theta}_1^T (\beta_1 \hat{\theta}_1 + \dot{\theta}_1)), \quad (4.3.12)
$$

where the term $\tau_1^T \tilde{\theta}_1^T F_1$ cancels the same term with opposite sign in (4.3.9). Take the derivative of $\frac{1}{\xi_1} \dot{\tilde{D}}_1^2$, we can get

$$
\frac{1}{\xi_1} \dot{\tilde{D}}_1 \dot{\tilde{D}}_1 = \frac{1}{\xi_1} \dot{\tilde{D}}_1 \dot{\tilde{D}}_1 = \frac{1}{\xi_1} \dot{\tilde{D}}_1 \xi_1 (\tau_1^T \tanh(\frac{\tau_1}{\delta_1}) - \lambda_1 \dot{D}_1)
= \dot{D}_1 \tau_1^T \tanh(\frac{\tau_1}{\delta_1}) - \lambda_1 \dot{D}_1 \dot{\tilde{D}}_1, \quad (4.3.13)
$$

therefore, we obtain

$$
V_1 \leq -K_1^1 \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1^T z_1}{2k_1^2} \right) + \tau_1^T B_1 z_1 + B_1 ||\tau_1|| ||x_2|| + \tau_1^T D_1 - \tau_1^T \dot{D}_1 \tanh \left( \frac{\tau_1}{\delta_1} \right)
+ |\tau_1^T B_1 \Delta \alpha_1| + \frac{1}{2} e_1^T e_1 - \frac{1}{\gamma_1} \text{tr}(\tilde{\theta}_1^T (\beta_1 \hat{\theta}_1 + \dot{\theta}_1)) + \dot{D}_1 \tau_1^T \tanh \left( \frac{\tau_1}{\delta_1} \right)
- \lambda_1 \dot{D}_1 \dot{\tilde{D}}_1 + e_1^T (-K_1 e_1 - f_1 e_1 + \gamma_1 \Delta \alpha_1). \quad (4.3.14)
$$
Notice that in (4.3.14)

\[ e_1^T \gamma_{11} \Delta \alpha_1 \leq \frac{\gamma_{11}^2}{2} \Delta \alpha_1^T \alpha_1 + \frac{1}{2} e_1^T e_1, \]  

(4.3.15)

\[-\lambda_1 \hat{D}_1 \hat{D}_1 = -\lambda_1 \hat{D}_1 (\hat{D}_1 + \bar{D}_1) = -\lambda_1 \hat{D}_1^2 - \lambda_1 \bar{D}_1 \hat{D}_1 \]

\[ \leq -\lambda_1 \hat{D}_1^2 + \frac{\lambda_1}{2} \bar{D}_1^2 + \frac{\lambda_1}{2} \bar{D}_1^2 = -\frac{\lambda_1}{2} \hat{D}_1^2 + \frac{\lambda_1}{2} \bar{D}_1^2, \]  

(4.3.16)

also, similar with (3.4.41) and (3.4.42), we get

\[-\frac{1}{\gamma_1} \text{tr} \left( \tilde{\theta}_1^T (\beta_1 \hat{\theta}_1 + \hat{\theta}_1) \right) \leq -\beta_1 - \frac{1}{2 \gamma_1} \text{tr}(\tilde{\theta}_1^T \tilde{\theta}_1) + \frac{\beta_1}{2 \gamma_1} \Theta_1 + \bar{\Theta}_1, \]  

(4.3.17)

besides, recall that \( D_1 \in \mathbb{R}^m \) is a vector and \( ||D_1|| \leq \bar{D}_1, \tilde{D}_1 \in \mathcal{R} \), with \( \hat{D}_1 \in \mathcal{R} \) being the estimate of \( \bar{D}_1 \), then

\[ \tau_1^T D_1 - \tau_1^T \bar{D}_1 \tanh \left( \frac{\tau_1}{\bar{\delta}_1} \right) + \bar{D}_1 \tau_1^T \tanh \left( \frac{\tau_1}{\bar{\delta}_1} \right) \]

\[ = \tau_1^T D_1 - \bar{D}_1 \tau_1^T \tanh \left( \frac{\tau_1}{\bar{\delta}_1} \right) - \bar{D}_1 \tau_1^T \tanh \left( \frac{\tau_1}{\bar{\delta}_1} \right) + \bar{D}_1 \tau_1^T \tanh \left( \frac{\tau_1}{\bar{\delta}_1} \right) \]

\[ \leq \sum_{j=1}^{m} |\tau_{1j}| ||D_{1j}|| - \bar{D}_1 \tau_{1j}^T \tanh \left( \frac{\tau_{1j}}{\bar{\delta}_1} \right) \]

\[ \leq \sum_{j=1}^{m} |\tau_{1j}| \sqrt{D_{1j}^2 + \cdots + D_{1m}^2} - \bar{D}_1 \tau_{1j}^T \tanh \left( \frac{\tau_{1j}}{\bar{\delta}_1} \right) \]

\[ \leq \sum_{j=1}^{m} |\tau_{1j}| \bar{D}_1 - \bar{D}_1 \tau_{1j} \tanh \left( \frac{\tau_{1j}}{\bar{\delta}_1} \right) \]

\[ \leq \sum_{j=1}^{m} 0.2785 \bar{D}_1 \bar{\delta}_1 = 0.2785 \bar{D}_1 \bar{\delta}_1 m. \]  

(4.3.18)
Therefore, from (4.3.14) we can have

\[
\hat{V}_1 \leq -K_1^* \frac{k_1^2}{\pi} \tan \left( \frac{\pi z_1^T z_1}{2k_1^2} \right) - \frac{\beta_1 - 1}{2\gamma_1} \text{tr}(\tilde{\theta}_1^T \tilde{\theta}_1) - \frac{\lambda_1}{2} \tilde{D}_1^2 \\
+ \tau_1^T B_1 z_2 + \tilde{B}_1 ||\tau_1|| ||x_2|| - (K_{11} - 1)e_1^T e_1 + C_1, \tag{4.3.19}
\]

where \( C_1 = 0.2785 \tilde{D}_1 \delta_1 m + \frac{\lambda_1}{2} \tilde{D}_1^2 + \frac{\beta_1}{2\gamma_1} \Theta_1 + \frac{1}{2\gamma_1} \Theta_1 \) is a constant. The term \( \tau_1^T B_1 z_2 \) is to be cancelled in the next design step, and \( \tilde{B}_1 ||\tau_1|| ||x_2|| \) is to be cancelled in the last design step.

**Step i** (\( 2 \leq i \leq n - 1 \)):

Define \( z_{i+1} = x_{i+1} - \alpha_i \), where the stabilizing function \( \alpha_i \) is produced from the nominal stabilizing function \( \alpha_{i0} \) through the command filter. The dynamics of \( z_i \) can be represented as

\[
\dot{z}_i = \theta_i^T F_i + B_i x_{i+1} + \Delta B_i x_{i+1} + D_i - \dot{\alpha}_{i-1}, \tag{4.3.20}
\]

where \( \dot{\alpha}_{i-1} \) can be got directly from the command filter in the previous step. The derivative of (4.2.1) at this step leads to

\[
\dot{V}_i^* = \frac{2k_i \dot{k}_i}{\pi} \tan \left( \frac{\pi z_i^T z_i}{2k_i^2} \right) - \left( \frac{\dot{k}_i}{k_i} \right) \tau_i^T z_i \\
+ \tau_i^T (\theta_i^T F_i + B_i (z_{i+1} + \alpha_i) + \Delta B_i x_{i+1} + D_i - \dot{\alpha}_{i-1}). \tag{4.3.21}
\]

The nominal stabilizing function \( \alpha_{i0} \) is designed as

\[
\alpha_{i0} = B_i^{-1} \left[-K_i \frac{z_i}{z_i^T z_i} \frac{k_i^2}{\pi} \sin \left( \frac{\pi z_i^T z_i}{2k_i^2} \right) \cos \left( \frac{\pi z_i^T z_i}{2k_i^2} \right) + K_i e_i - K_{i0} z_i - \tilde{\theta}_i^T F_i \right. \\
- \cos^2 \left( \frac{\pi z_i^T z_i}{2k_i^2} \right) B_{i-1}^T \tau_{i-1} + \dot{D}_i \tanh \left( \frac{\tau_i}{\delta_i} \right) - \frac{K_i^2}{2} \tau_i \right], \tag{4.3.22}
\]
Chapter 4. AFTC for Input and State Constrained MIMO Nonlinear Systems

where $\delta_i > 0$ is a small constant, $K_i, K_{i0}$ are positive, $K_i > 0$ is a designed constant, $K_i > 2K_{i0}$, $K_{i0} = \sqrt{\left(\frac{K_i}{K_1}\right)^2 + \epsilon}$, $\hat{\theta}_i$ and $\hat{D}_i$ are adaptive estimators of $\theta_i$ and $\bar{D}_i$, respectively.

**Remark 4.3.4** The design of $\alpha_{i0}$ is similar with the design of $\alpha_{10}$ in view of Remark 4.3.2, except that $-\cos^2\left(\frac{\pi z_i T}{2k_i^2}\right) B_{i-1}^T \tau_{i-1}$ in (4.3.22) is constructed to compensate the term $\tau_{i-1}^T B_{i-1} z_i$ from the previous step. Therefore, similar with the derivation that leads to (4.3.9), from (4.3.21) we have

$$V_i = V_{i-1} + V_i^* + \frac{1}{2\gamma_i} \text{tr}(\bar{\theta}_i^T \bar{\theta}_i) + \frac{1}{2\xi_i} \bar{D}_i^2 + \frac{1}{2} e_i^T e_i. \tag{4.3.24}$$

Henceforth, taking the derivative of (4.3.24), we can get

$$\dot{V}_i \leq -\sum_{j=1}^{i} K_i \frac{k_i^2}{\pi} \tan\left(\frac{\pi z_i^T z_i}{2k_i^2}\right) - \sum_{j=1}^{i} (K_{ji} - 1) e_j^T e_j + \tau_i^T B_i z_{i+1} + \sum_{j=1}^{i} \frac{\beta_j - 1}{2\gamma_j} \text{tr}(\tilde{\theta}_j^T \tilde{\theta}_j)$$

$$+ \sum_{j=1}^{i} \bar{B}_j ||\tau_j|||x_{j+1}|| - \sum_{j=1}^{i} \frac{\lambda_j}{2} \bar{D}_j^2 + \sum_{j=1}^{i} C_j, \tag{4.3.25}$$

where $C_i = 0.2785 \bar{D}_i \delta_m + \frac{\lambda_i}{2} \bar{D}_i^2 + \frac{\beta_i}{2\gamma_i} \Theta_i + \frac{1}{2\gamma_i} \tilde{\theta}_i$ is a constant.

**Remark 4.3.5** To calculate $\dot{\alpha}_i$ is analytically tedious in traditional backstepping approaches. In some conventional structures of analysis to deal with state constraints, such as [43] and [44], discussed in the background and literature review chapter, the boundedness of the stability
functions $\alpha_i$ is discussed by analyzing the boundedness of each individual argument of $\alpha_i$, in particular $\dot{\alpha}_{i-1}$. To calculate $\dot{\alpha}_{i-1}$ is analytically inconvenient and computationally tedious in these traditional backstepping approaches. By using the command filter, the magnitude on stabilizing functions can be predefined, and the derivative of the stabilizing functions can be obtained easily. A brief structure about the backstepping process for the first $n-1$ steps will look like as Figure 4.2 suggests.

**Step n:**

We can obtain the dynamics of $z_n$ as

$$
\dot{z}_n = \theta_n^T F_n + (B_n + \Delta B_n)(\Delta u + v) + D_n - \dot{\alpha}_{n-1},
$$

(4.3.26)

where $\Delta u = u - v$. Recall from (4.1.5) that $v = \rho v_0 + \varphi$. Construct

$$
v_0 = \hat{\rho}^{-1} \varphi,
$$

(4.3.27)

where $\varphi$ is to be designed, and $\hat{\rho}$ is the estimator of $\rho$ to be defined later. Notice that

$$
B_n v = B_n (\rho v_0 + \varphi) = B_n \varphi + B_n (\hat{\rho} \hat{\rho}^{-1} \varphi - \hat{\rho} \hat{\rho}^{-1} \varphi + \rho v_0) = B_n \varphi + B_n \varphi - B_n \rho v_0,
$$

(4.3.28)
where \( \tilde{\rho} = \hat{\rho} - \rho \) is the estimation error for multiplicative actuator fault. Hence (4.3.26) becomes

\[
\dot{z}_n = \theta_n^T F_n + B_n \phi + B_n \varphi - B_n \hat{\rho} v_0 + B_n \Delta u + \Delta B_n u + D_n - \dot{\alpha}_{n-1},
\]  

(4.3.29)

Design the control law \( \varphi \) as

\[
\varphi = -\hat{\phi} + B_n^{-1} \left( -K_n \frac{z_n}{\tau_n^T \tau_n} \kappa_n^2 \sin \left( \frac{\pi z_n^T \tau_n}{2k_n^2} \right) \cos \left( \frac{\pi z_n^T \tau_n}{2k_n^2} \right) - K_{n0} z_n \right. \\
- \cos^2 \left( \frac{\pi z_n^T \tau_n}{2k_n^2} \right) B_n^T \tau_{n-1} - \hat{\theta}_n^T F_n + \hat{\phi} - \hat{D}_n \tanh \left( \frac{\tau_n}{\delta_n} \right) \\
- \frac{\tau_n g(Z)}{\vartheta^2 + \tau_n^T \tau_n} - \frac{K_n^2}{2} \tau_n + K_n e_n \bigg),
\]  

(4.3.30)

where \( K_n > 0 \) is a design constant, \( K_n > K_{n0}, K_{n0} = \sqrt{\left( \frac{k_n}{k_n} \right)^2 + \epsilon} \), \( \delta_n > 0 \) is a small constant, \( \hat{\phi}, \hat{\theta}_n \) and \( \hat{D}_n \) are the estimators of \( \phi, \theta_n \) and \( \bar{D}_n \), respectively. \( e_n \in \mathbb{R}^m \) is the state of the auxiliary system

\[
\dot{e}_n = \left\{ \begin{array}{ll}
-K_{n1} e_n - f_n e_n + \gamma_{n1} (u - v), & \text{if } |e_n| > \upsilon_n; \\
0, & \text{if } |e_n| \leq \upsilon_n,
\end{array} \right.
\]  

(4.3.31)

where \( \upsilon_n > 0 \) is a small number, \( K_{n1} > 1, \gamma_{n1} > 0 \) are design constants, \( f_n = \frac{\tau_n^T B_n \Delta u + \frac{1}{2} \gamma_{n1}^2 \Delta u^T \Delta u + B_n \|\tau_n\| \|u\|}{\|e_n\|^2} \).

\( \vartheta \) is designed in the same way in [60] as

\[
\dot{\vartheta} = \left\{ \begin{array}{ll}
- \frac{g(Z)}{\vartheta^2 + \tau_n^T \tau_n} - k_\vartheta \vartheta, & \text{if } ||\tau_n|| > \varsigma; \\
0, & \text{if } ||\tau_n|| \leq \varsigma,
\end{array} \right.
\]  

(4.3.32)
where \( \varsigma > 0 \) is a small constant, \( k_\phi > 0 \) is a design constant. The function \( g(Z) \) in (4.3.30),

(4.3.32) is designed as \( g(Z) = \sum_{j=1}^{n-1} \bar{B}_j ||x_j|| ||x_{j+1}|| \), with \( Z = [\tau_j, x_j]^T \).

Hence, let \( K_n^* = K_n - 2K_0 \), the derivative of \( V_n^* \) leads to

\[
\begin{align*}
V_n^* &= \frac{2k_n^2}{\pi} \tan \left( \frac{\pi z_n^T z_n}{2k_n^2} \right) - \left( \frac{k_n}{k_n^*} \right) z_n^T \\
&+ \tau_n^T (\theta_n^T F_n + B_n \phi + B_n \dot{\rho}v_0 + B_n \Delta u + \Delta B_n u + D_n - \dot{\alpha}_n) \\
&\leq 2K_n^* k_n^2 \tan \left( \frac{\pi z_n^T z_n}{2k_n^2} \right) + K_n \theta_n^T z_n + \tau_n^T \theta_n^T F_n + \tau_n^T B_n \phi - \tau_n^T B_n \dot{\rho}v_0 + \tau_n^T B_n \Delta u \\
&+ \tau_n^T B_n \Delta u + \tau_n^T D_n - \tau_n^T \dot{\alpha}_n - \tau_n^T \phi - K_n^* \theta_n^T \phi - \frac{K_n^2}{2} \tau_n^T \phi - K_n^* \tau_n^T e_n \\
&\leq -K_n^* \frac{k_n^2}{\pi} \tan \left( \frac{\pi z_n^T z_n}{2k_n^2} \right) - \frac{1}{2} \tau_n^T \phi - \frac{1}{2} e_n^T e_n + \frac{1}{2} \tau_n^T ||\tau_n|| ||u|| + \tau_n^T D_n - \tau_n^T \dot{D}_n \tanh \left( \frac{\tau_n}{\delta_n} \right) \\
&+ \frac{1}{2} \tau_n^T g(Z) - \frac{1}{2} \tau_n^2 - \tau_n^T e_n - \tau_n^T \dot{D}_n \tanh \left( \frac{\tau_n}{\delta_n} \right) \\
&\leq -K_n^* \frac{k_n^2}{\pi} \tan \left( \frac{\pi z_n^T z_n}{2k_n^2} \right) \tag{4.3.33}
\end{align*}
\]

where \( \tilde{\theta}_n = \theta_n - \theta_n, \tilde{\phi} = \phi - \phi \). Construct the overall Lyapunov candidate function as

\[
V_n = V_{n-1} + V_n^* + \frac{1}{2} e_n^T e_n + \frac{1}{2} \gamma_n \text{tr}(\tilde{\theta}_n^T \tilde{\theta}_n) + \frac{1}{2} \xi_n \tilde{D}_n^2 + \frac{1}{2} \phi^2 + \frac{1}{2} \kappa_1 \text{tr}(\rho^T \tilde{\rho}) + \frac{1}{2} \kappa_2 \phi^T \phi, \tag{4.3.34}
\]

where \( \kappa_1, \kappa_2 > 0 \) are design constants, \( \tilde{D}_n = \tilde{D}_n - \tilde{D}_n \), and \( V_{n-1} \) from the \((n-1)\)-th step is

\[
V_{n-1} = \sum_{i=1}^{n-1} \frac{k_i^2}{\pi} \tan \left( \frac{\pi z_i^T z_i}{2k_i^2} \right) + \frac{1}{2} \gamma_i \text{tr}(\tilde{\theta}_i^T \tilde{\theta}_i) + \frac{1}{2} \xi_i \tilde{D}_i^2 + \frac{1}{2} e_i^T e_i. \tag{4.3.35}
\]

Adaptive laws for \( \tilde{\rho} \) is designed as

\[
\dot{\tilde{\rho}}_i = \begin{cases} 
0, & \text{if } \tilde{\rho}_i = \rho_{i,\text{min}} \text{ and } \chi_i(t) < 0; \\
\chi_i(t), & \text{else,}
\end{cases} \tag{4.3.36}
\]
where \( i = 1, \ldots, m, \chi_i(t) = \kappa_1 \tau_i^T B_n v_0 - \Omega_i \hat{\rho}_i \), \( \Omega_i > 1 \) is a design constant. Let \( \varpi > 1 \) be a design constant, the estimator for the additive actuator fault is designed as

\[
\dot{\hat{\phi}} = \kappa_2 B_n^T \tau_n - \varpi \hat{\phi}.
\]

\[(4.3.37)\]

**Remark 4.3.6** At this last step of the backstepping design process, the structure is as Figure 4.3.

As mentioned in [60] (Eq. (51) in [60]), we have

\[
\frac{\partial^2 g(Z)}{\partial^2 + \|\tau_n\|^2} + \vartheta \dot{\vartheta} = -k_\vartheta \vartheta^2.
\]

\[(4.3.38)\]

Therefore, from (4.3.34) we have

\[
\dot{V}_n \leq - \sum_{j=1}^n K_j^* \frac{k_j^2}{\pi} \tan \left( \frac{\pi z_j z_j^*}{2k_j^2} \right) - \sum_{j=1}^n \frac{\beta_j - 1}{2\gamma_j} \text{tr}(\hat{\theta}_j^T \hat{\theta}_j) - \sum_{j=1}^n (K_{j1} - 1) e_j^T e_j - \frac{\Omega}{\kappa_1} \text{tr}(\hat{\rho}^T \hat{\rho})
\]

\[
- \frac{1}{\kappa_1} \text{tr}(\hat{\rho}^T \hat{\rho}) - \sum_{j=1}^n \frac{\lambda_j}{2} \tilde{D}_j^2 + \sum_{j=1}^n C_j - \frac{\varpi}{\kappa_2} \phi^T \phi - \frac{1}{\kappa_2} \phi^T \phi - k_\vartheta \vartheta^2,
\]

\[(4.3.39)\]
where \( C_n = \frac{\beta_n}{2n} \Theta_n + \frac{1}{2n} \tilde{\Theta}_n + \frac{\lambda_n}{2} \tilde{D}_n^2 + 0.2785 \tilde{D}_n \delta_n m \). Notice that

\[
-\frac{\Omega}{\kappa_1} \text{tr}(\hat{\rho}^T \hat{\rho}) \leq -\frac{\Omega}{2\kappa_1} \text{tr}(\hat{\rho}^T \hat{\rho}) + \frac{\Omega}{2\kappa_1} \sum_{i=1}^{m} \rho_{i,max}^2, \tag{4.3.40}
\]

\[
\frac{1}{\kappa_1} \text{tr}(\hat{\rho}^T \hat{\rho}) \leq \frac{1}{2\kappa_1} \text{tr}(\hat{\rho}^T \hat{\rho}) + \frac{1}{2\kappa_1} \Theta, \tag{4.3.41}
\]

\[
-\frac{\varpi}{\kappa_2} \tilde{\phi}^T \hat{\phi} \leq -\frac{\varpi}{2\kappa_2} \tilde{\phi}^T \hat{\phi} + \frac{\varpi}{2\kappa_2} \tilde{\phi}^2, \tag{4.3.42}
\]

\[
-\frac{1}{\kappa_2} \tilde{\phi}^T \hat{\phi} \leq \frac{1}{2\kappa_2} \tilde{\phi}^T \hat{\phi} + \frac{1}{2\kappa_2} \tilde{\phi}_0^2, \tag{4.3.43}
\]

From (4.3.39), we then have

\[
\dot{V}_n \leq -\sum_{j=1}^{n} K_j^* k_j^2 \tan \left( \frac{\pi z_j^T z_j}{2k_j^2} \right) - \sum_{j=1}^{n} \frac{\beta_j - 1}{2\gamma_j} \text{tr}(\tilde{\theta}_j^T \tilde{\theta}_j) - \sum_{j=1}^{n} \frac{\lambda_j}{2} \tilde{D}_j^2 - \sum_{j=1}^{n} (K_j - 1) e_j e_j^T - k_0 \phi^2 - \frac{\Omega}{2\kappa_1} \text{tr}(\hat{\rho}^T \hat{\rho}) - \frac{\varpi}{2\kappa_2} \tilde{\phi}^T \hat{\phi} + \sum_{j=1}^{n} C_j + \frac{\Omega}{2\kappa_1} \sum_{i=1}^{m} \rho_{i,max}^2 + \frac{\varpi}{2\kappa_2} \tilde{\phi}_0^2 + \frac{1}{2\kappa_2} \tilde{\phi}^2 + \frac{1}{2\kappa_1} \Theta. \tag{4.3.44}
\]

Denote \( C_0 = \sum_{j=1}^{n} C_j + \frac{1}{2\kappa_1} \Theta + \frac{\Omega}{2\kappa_1} \sum_{i=1}^{m} \rho_{i,max}^2 + \frac{\varpi}{2\kappa_2} \tilde{\phi}_0^2 + \frac{1}{2\kappa_2} \tilde{\phi}^2 \) which is a finite constant, and

\[
c = \min(K_j^*, \beta_j - 1, \xi_j \lambda_j, 2k_0, 2(K_j - 1), \Omega, \varpi - 1), \]

where \( j = 1, \ldots, n \), and \( c > 0 \). From (4.3.44) we have

\[
\dot{V}_n \leq -c V_n + C_0. \tag{4.3.45}
\]

The above backstepping design leads to the following theorem.

**Theorem 4.3.1** With control laws (4.3.27), (4.3.30) and adaptive laws (4.3.11), (4.3.36), (4.3.37), the system (4.1.1) under Assumption 4.1.1-4.1.5 subject to actuator faults (4.1.5) and input constraints (4.1.4) has the following properties:
1. The system state constraint requirements will be satisfied, that is, (4.1.3) will not be violated during operation.

2. The system output tracking error will exponentially converge to the set defined by \( \{ z_1 \mid ||z_1|| \leq \sqrt{\frac{2C_0}{c}} \} \), which can be designed as an arbitrarily small neighbourhood of zero. Also, the intermediate fictitious state tracking errors will exponentially converge to \( ||z_i|| \leq \sqrt{\frac{2C_0}{c}} \), \( i = 2, \cdots, n \).

3. The estimation error of the actuator fault signals \( \tilde{\phi} = \hat{\phi} - \phi \) and \( \tilde{\rho} = \hat{\rho} - \rho \) will be bounded.

4. Closed-loop signals \( e_i, \vartheta, \tilde{\theta}_i, \tilde{D}_i \) are all bounded, where \( i = 1, \cdots, n \).

**Proof:**

We can obtain from (4.3.45) that \( V_n \leq (V_n(0) - \frac{C_0}{c})e^{-ct} + \frac{C_0}{c} \), therefore \( V_n \) is bounded. The boundedness of \( V_n \) implies that the BLF is also bounded. Moreover,

\[
\frac{k_i^2}{\pi} \tan\left(\frac{\pi z_i^T z_i}{2k_i^2}\right) \leq V_n \leq (V_n(0) - \frac{C_0}{c})e^{-ct} + \frac{C_0}{c},
\]

(4.3.46)

hence

\[
||z_i||^2 \leq \frac{2k_i^2}{\pi} \tan^{-1}\left(\frac{\pi}{k_i^2}(V_n(0) - \frac{C_0}{c})e^{-ct} + \frac{\pi C_0}{k_i^2} \frac{C_0}{c}\right) < \frac{2k_i^2}{\pi} \frac{\pi}{2} = k_i^2,
\]

(4.3.47)

this means \( ||z_i|| < k_i \). Since \( x_1 = z_1 + x_{1d} \) and \( x_i = z_i + \alpha_{i-1} \), \( i = 2, \cdots, n \), we can get

\[
||x_1|| \leq ||z_1|| + ||x_{1d}|| < \tilde{k}_1 - k_d + k_d = \tilde{k}_1, \text{ and } ||x_i|| \leq ||z_i|| + ||\alpha_{i-1}|| < \tilde{k}_i - \tilde{\alpha}_{i-1} + \tilde{\alpha}_{i-1} = \tilde{k}_i,
\]

hence the constraint on system state will not be violated during operation. Furthermore,

\[
\frac{1}{2} z_i^T z_i \leq \frac{k_i^2}{\pi} \tan\left(\frac{\pi z_i^T z_i}{2k_i^2}\right) \leq (V_n(0) - \frac{C_0}{c})e^{-ct} + \frac{C_0}{c},
\]

(4.3.48)
therefore $z_i$ will be exponentially convergent to the set $||z_i|| \leq \sqrt{\frac{2C_0}{c}}$. The boundedness of $V_i$ implies boundedness of estimation errors of the actuator fault signals $\tilde{\phi}$ and $\tilde{\rho}$, as well as closed-loop signals $e_i, \vartheta, \tilde{\theta}_i, \tilde{D}_i, i = 1, \cdots, n$.

**Remark 4.3.7** Equation (4.3.45) presents some guidelines on how to expedite the convergence speed of the system output, as well as how to modify the size of the set into which the system output will converge. To make the size of the set as small as possible, we need to select large $c$ and small $C_0$. A large $c$ is also desirable to increase the convergence rate. To make $C_0$ small, we need small $\delta_i, \lambda_i, \frac{\Omega}{\kappa_1}, \frac{\varpi}{\kappa_2}, \frac{\beta_i}{\gamma_i}, \frac{1}{\gamma_i}, i = 1, \cdots, n$. To make $c$ large, we need to select large $K_i, \beta_i, \xi_i, \lambda_i, k_\theta, K_{i1}, \Omega, \varpi$. Notice that to choose small $\frac{\Omega}{\kappa_1}$ and large $\Omega$, for example, are not conflicting objectives, as $\frac{\Omega}{\kappa_1}$ can turn out to be small despite $\Omega$ being large, by selecting large $\kappa_1$.

**Remark 4.3.8** From (4.3.45), we can see that using $\frac{1}{2} z_i^T z_i$ instead of a BLF in analysis, the system state variables can still be bounded during operation. However, the bound on the state variables in this case, which depends on various system quantities, can be unknown. Therefore it is very hard to for the system to meet predefined constraint requirements on the system state variables. Whereas by introducing a BLF, the bounds can be specified according to the requirements.

**Remark 4.3.9** Notice that the scheme presented in this Chapter can also work for systems without system uncertainties and actuator faults. For example, a double integrator system without any constraint requirements is described by

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u,
\end{align*}$$

(4.3.49)
where $u$ is the control signal. Define the fictitious state tracking errors as $z_1 = x_1 - x_{1,d}$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is the stabilizing function to be designed. First, design $V_1 = \frac{1}{2} z_1^T z_1$, which is a special case of the tan type BLF $\frac{k_i^2}{\pi} \tan \left( \frac{z_1^T z_1}{2k_i} \right)$ when the bound $k_i$ tends to infinity, and the derivative leads to

$$\dot{V}_1 = z_1^T \dot{z}_1 = z_1^T (z_2 + \alpha_1 - \dot{x}_{1,d}).$$  \hspace{1cm} (4.3.50)$$

Design the stabilizing function $\alpha_1 = \dot{x}_{1,d} - K_1 z_1$, hence

$$\dot{V}_1 = z_1^T z_2 - K_1 z_1^T z_1.$$  \hspace{1cm} (4.3.51)$$

Next, design $V_2 = \frac{1}{2} z_2^T z_2$, and for the derivative we have

$$\dot{V}_2 = z_2^T \dot{z}_2 = z_2^T (\dot{x}_2 - \dot{\alpha}_1) = z_2^T (u - \dot{\alpha}_1).$$  \hspace{1cm} (4.3.52)$$

The controller is designed as $u = \dot{\alpha}_1 - K_2 z_2 - z_1$. Therefore,

$$\dot{V}_1 + \dot{V}_2 = -K_1 z_1^T z_1 - K_2 z_2^T z_2 \leq 0,$$  \hspace{1cm} (4.3.53)$$

and the system is asymptotically stable.

### 4.4 Simulation

#### 4.4.1 Example 1

In this subsection, we will demonstrate the effectiveness of our proposed scheme, and compare it with the performance of the controller in [60], which is designed to handle input constraints.
Consider the MIMO system subject to input and state constraints with constant actuator faults

\[
\begin{align*}
\dot{x}_1 &= 0.1x_1 + x_2, \\
\dot{x}_2 &= \theta_2^T F_2 + (B_2 + \Delta B_2)u
\end{align*}
\]

where \( \theta_2 = (1 + 0.1 \sin^3(t) + 0.5e^{-t}) \times [1, 1] \in \mathcal{R}^{1 \times 2}, \)
\( F_2 = \sin(x_1^T x_2) \in \mathcal{R}, \)
\( B_2 = I_{2 \times 2}, \)
\( \Delta B_2 = [0.1 \cos(x_1^T x_1), 0; 0, 0.1 \sin(x_2^T x_2)]\). Desired system output trajectory is \( x_{1d} = [\sin(t) - e^{-t} + 1, \sin(0.2t) - e^{-t} + 1]^T \). Notice that \( x_{1d} = [0, 0]^T, x_{2d} = [2, 1]^T \). The initial condition is \( x_1 = [0.1, 0.1]^T, x_2 = [2.1, 1.1]^T \), indicating that there is some initial misalignment. For the constraint requirement on \( x_1 \), the bound is \( \bar{k}_1 = ||x_{1d}|| + 0.7 \). For the constraint requirement on \( x_2 \), we have \( \bar{k}_2 = 1.4 \). For the control input constraint, we have \( u_{max} = [10, 10]^T \) and \( u_{min} = [-9, -9]^T \). The gains are selected as \( K_1 = K_2 = 3.5, \) \( \kappa_1 = \kappa_2 = 0.1, \) \( \Omega = 20, \) \( \gamma_2 = 1.1, \) \( \beta_2 = 1.1, \) \( K_{11} = K_{21} = 1.2, \) \( \gamma_{11} = \gamma_{21} = 1 \). The multiplicative actuator fault is \( \rho = diag[0.85, 0.7] \), and initial condition \( \hat{\rho}(0) = diag[1, 1] \). The additive actuator fault is \( \phi = [4.5, 4.5]^T \), meaning that there is some constant offset in the control input channel, and initial condition \( \hat{\phi}(0) = [0, 0]^T \). In Fig. 4.4 and 4.5, case 1 refers to the controller developed in this work, and case 2 refers to the control scheme in [37]. Despite the initial misalignment, it can be seen that under the proposed scheme, the norm \( ||x_1(t)|| \) satisfies the constraint requirement on \( x_1(t) \), which also tracks the desired trajectory very well, whereas other control scheme like the one in [37] could violate the state constraint, and the tracking performance is not satisfactory as the controller is not fault tolerant. Similarly, \( ||x_2(t)|| \) under the proposed scheme tracks the desired trajectory \( ||x_{2d}(t)|| \) better than the performance of the controller in [37]. In addition, the \( ||x_2(t)|| \) trajectory under the controller in [37] violates the state constraint during transient. Fig. 4.6-4.9 show the profile of actuator faults and their adaptive estimators. Fig. 4.10 and
Fig. 4.11 show the profile of the actuator components $u_1(t)$ and $u_2(t)$ and the unconstrained signal $v_1(t)$ and $v_2(t)$ under the proposed control scheme. Initially there are large spikes in the control signal $v(t) = [v_1(t), v_2(t)]^T$, and can be effectively filtered out by the input constraints without compromising the control performance.

Figure 4.4: The profile of $||x_{1d}(t)||$ and $||x_1(t)||$, as well as the constraint requirement $\bar{k}_1(t)$.

Figure 4.5: The profile of $||x_{2d}(t)||$ and $||x_2(t)||$, as well as the constraint requirement $\bar{k}_2(t)$. 
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Figure 4.6: The profile of additive actuator fault $\phi_1(t)$ and its adaptive estimator $\hat{\phi}_1(t)$ under the proposed control scheme.

Figure 4.7: The profile of additive actuator fault $\phi_2(t)$ and its adaptive estimator $\hat{\phi}_2(t)$ under the proposed control scheme.

Figure 4.8: The profile of multiplicative fault $\rho_1$ and the adaptive estimator.
4.4.2 Example 2

In this subsection, we will consider a two degree-of-freedom robot manipulator with time-varying multiplicative and additive actuator faults,
\[ \dot{x}_1 = x_2, \]
\[ D(x_1)\dot{x}_2 + C(x_1, x_2)x_2 + G(x_1) + F(x_1, x_2, t) = u, \tag{4.4.1} \]

The joint position and velocity vectors are \( x_1 = [x_{11}, x_{12}]^T \in \mathbb{R}^2 \) and \( x_2 = [x_{21}, x_{22}]^T \in \mathbb{R}^2 \) respectively. The inertia matrix \( D(x_1) = [D_{ij}]_{2 \times 2} \) is given by \( D_{11} = m_1 l_1^2 + m_2 (l_1^2 + l_2^2 + 2l_1 l_2 \cos(x_{12})) \), \( D_{12} = D_{21} = m_2 (l_2^2 + l_1 l_2 \cos(x_{12})) \) and \( D_{22} = m_2 l_2^2 \). The centripetal-Coriolis matrix \( C(x_1, x_2) = [C_{ij}]_{2 \times 2} \) is given as \( C_{11} = -m_2 l_1 l_2 x_{22} \sin(x_{12}) \), \( C_{12} = -m_2 l_1 l_2 \sin(x_{12}) (x_{21} + x_{22}) \), \( C_{21} = m_2 l_1 l_2 x_{21} \sin(x_{12}) \) and \( C_{22} = 0 \). The gravity vector \( G(x_1) = [G_1; G_2] \) is given as \( G_1 = (m_1 + m_2) l_1 g \cos(x_{11}) + m_2 l_2 g \cos(x_{11} + x_{12}) \), and \( G_2 = m_2 l_2 g \cos(x_{11} + x_{12}) \). \( u \) is the control input to the system, which is subject to actuator faults. The parameters are given as \( l_1 = 1.2 m, l_2 = 1 m, g = 9.81 m/s^2 \). The mass elements are \( m_1 = 0.9 kg, m_2 = 1 kg \). The unmodeled structure vector \( F(x_1, x_2, t) = 0.1 \sin(x_{21}) (1 - e^{-0.1t}) [1,1]^T \in \mathbb{R}^2 \).

The desired output trajectory is \( x_{11,d} = \sin(t) - e^{-t} + 11, x_{12,d} = \sin^3(0.2t) - e^{-t} + 11 \).

The initial condition is \( x_1 = [10.1, 10.1]^T, x_2 = [1,1]^T \), indicating that there is some initial misalignment, as \( x_{1d} = [10,10]^T, x_2 = [2,1]^T \). For the constraint requirement on \( x_1 \), the bound is \( \tilde{k}_1 = ||x_{1d}|| + 1.3 \). For the constraint requirement on \( x_2 \), we have \( \tilde{k}_2 = 9 \). For the control input constraint, we have \( u_{max} = [90,70]^T \) and \( u_{min} = [-90,-70]^T \). The gains are selected as \( K_1 = K_2 = 2.5, \rho_1 = \rho_2 = 0.1, \Omega = 0.1, \gamma_2 = 1.1, \beta_2 = 1.1, K_{11} = K_{21} = 1, \gamma_1 = \gamma_{21} = 0.1 \).

We consider time-varying actuator faults here. The multiplicative actuator fault is \( \rho = diag[0.5 + 0.5e^{-1.5t}, 0; 0, 0.4 + 0.6e^{-1.5t}] \in \mathbb{R}^{2 \times 2} \). The additive actuator fault is \( \phi = [-0.45(1 - e^{-2t}), 1 - e^{-2t}]^T \in \mathbb{R}^2 \). Initially we have \( \rho(0) = diag[1,1] \), and \( \phi(0) = [0,0]^T \), meaning there are no actuator faults. Initial condition \( \dot{\rho}(0) = diag[1,1], \dot{\phi}(0) = [0,0]^T \). Fig. 4.12 and Fig 4.13.
show that the norm $||x_1(t)||$ and $||x_2(t)||$ satisfy the constraint requirements on $x_1(t)$ and $x_2(t)$, with good tracking performance. Fig. 4.14-4.17 show the profile of actuator faults and their adaptive estimators. The estimation errors are bounded as the proof shows. Fig. 4.18 and Fig. 4.19 show the profile of the actuator components $u_1(t)$ and $u_2(t)$ and the unconstrained signal $v_1(t)$ and $v_2(t)$.

Figure 4.12: The profile of $||x_{1d}(t)||$ and $||x_1(t)||$, as well as the constraint requirement $\bar{k}_1(t)$ under time-varying actuator faults.

Figure 4.13: The profile of $||x_{2d}(t)||$ and $||x_2(t)||$, as well as the constraint requirement $\bar{k}_2(t)$ under time-varying actuator faults.
Figure 4.14: The profile of additive actuator fault $\phi_1(t)$ and the adaptive estimator under time-varying actuator faults.

Figure 4.15: The profile of additive actuator fault $\phi_2(t)$ and the adaptive estimator under time-varying actuator faults.

Figure 4.16: The profile of multiplicative fault $\rho_1$ and the adaptive estimator under time-varying actuator faults.
Figure 4.17: The profile of multiplicative fault $\rho_2$ and the adaptive estimator under time-varying actuator faults.

Figure 4.18: The profile of unconstrained input $v_1(t)$ and constrained input $u_1(t)$ under time-varying actuator faults.

Figure 4.19: The profile of unconstrained input $v_2(t)$ and constrained input $u_2(t)$ under time-varying actuator faults.
4.5 Discussion

In this Chapter, we discuss the control input and system state constrained problem for a class of MIMO nonlinear systems with actuator faults. To address this problem, we introduce tan type barrier Lyapunov functions together with auxiliary systems in the controller design, and command filters in the design of stabilizing functions in the backstepping analysis. We show that under the proposed novel adaptive FTC scheme, exponential convergence of the output tracking error into a small set around zero is guaranteed, while the constraint requirement on the system state will not be violated during operation. Estimation errors for actuator faults and unknown time-varying quantities are also bounded in closed loop.
Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this thesis, we discuss the output tracking problem for different classes of constrained nonlinear systems with actuator faults. For the nonlinear system formulation, both parametric and nonparametric system uncertainties have been taken into consideration. For the actuator faults, both time-varying multiplicative and additive actuator faults have been considered. To address time-varying system output constraints for a class of multiagent systems, we propose the control design based on Lyapunov techniques using a tan-type Barrier Lyapunov Function (BLF). To address time-varying system state constraints for general multi-input multi-output (MIMO) nonlinear systems, we integrate the tan-type BLFs together with the technique of command filter into the control scheme, where the tan-type BLFs guarantee the boundedness of the fictitious state tracking error, and the command filter can effectively limit the magnitude of the stabilizing function, so that the system state can be confined within a predefined limit. To address system input constraint, the technique of auxiliary system is introduced. We show that under the proposed novel adaptive FTC schemes, for both generic MIMO nonlinear sys-
tems and a class of integral-type multi-agent systems, exponential convergence of the output tracking error into a small set around zero is guaranteed, while the constraint requirement on the system state or output will not be violated during operation. Estimation errors for actuator faults and unknown time-varying quantities are also bounded in closed loop.

5.2 Future Work

We list several topics that can be the direction of future research:

- As far as the system actuator faults are concerned, we have discussed both time varying multiplicative and additive actuator faults. These actuator faults are assumed to be continuous and bounded, so that that the first order derivative with respect to time is bounded, and the magnitude of these faults is finite. In some real world industrial applications, we may encounter situations where the faults in the actuator may not be continuous, which means that the change in magnitude may be abrupt. Spike-type actuator faults with large magnitude may also be possible. How to address this type of actuator faults is also of vital importance in both the research and industrial communities.

- In terms of system faults, we have considered actuator faults in this work. In some applications, the system measurement may be subject to faults as well, which may imply that the system state variables may not be measured precisely. In such cases, usually system observers have to be integrated into the control approach. How to incorporate observer-based control schemes into the result of this thesis is also an interesting direction for further study.

- For the actuator nonlinearities, we have addressed input saturation in the thesis. Other types of actuator nonlinearities, such as deadzone, hysteresis, which are practical consid-
erations in some industrial applications, could be interesting research topics that deserve further attention.

- For the multiagent systems that we have discussed in this thesis, we assume that there is no time delay in the measurement. In reality, the information about neighborhood synchronization error, such as relative position or velocity information, may be collected by different sensors such as camera, infra-red or ultrasonic sensors. Such sensors usually give rise to a certain time delay, as it takes time for the signal to travel to the neighbours and then be bounced back. How to analyze time delay effects is a challenging research topic in the study of multi-agent systems.
Bibliography


