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Complete List of Authors:	López Torres, Ignacio; Universidad Politécnica de Madrid Fullana Belda, Carmen; Universidad Pontificia de Comillas
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1 **Dimensionless numbers for the sustainable harvesting of a**
2 **monospecific uneven-aged forest**

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4 **Ignacio López Torres^{1*}, Carmen Fullana Belda²**

5 ¹*Universidad Politécnica de Madrid (UPM). ETSI de Montes. Ciudad Universitaria, 28040*
6 *Madrid, Spain*

7 ²*Universidad Pontificia de Comillas. Alberto Aguilera, 23. 28015 Madrid, Spain*

8 *Corresponding author: Tel: +00 34913367123; Fax: +00 34913366386; Email: i.lopez@upm.es

9 **Abstract**

10 This study proposes a simple and direct method based on dimensionless
11 numbers to provide reliable approximations of the population growth rate,
12 the “sustainable/stable” harvest rate, the proportion of trees that has to
13 remain unharvested to retain the stable diameter distribution, and the
14 stable diameter distribution, of a forest stand. Those numbers, obtained
15 under conditions of stable equilibrium from a matrix model, could also
16 serve to estimate boundaries between sustainable and unsustainable
17 harvesting. In order to exemplify and test the results, the model uses data
18 from uneven-aged managed *Pinus nigra* stands, considering three levels of
19 tree diameter growth, 6 levels of basal area, and 33 levels of recruitment,
20 creating a total of 594 planning scenarios. The best approximation of all the
21 variables observed occurred in any case for the lowest level of diameter
22 growth, the lowest basal area, and the highest recruitment scenarios.
23 Furthermore, the study reveals the existence of a strong positive linear
24 correlation between those variables and their respective approximations,
25 as well as a small distance between the stable diameter distribution of the
26 stand and its approximation. Finally, we incorporate natural disturbances
27 into the dimensionless numbers and criteria.

28
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31 *Keywords:* Dimensionless numbers; Equilibrium; Sustainable harvesting; Matrix model; Natural
32 disturbances.

33 *Running Title:* Dimensionless numbers for sustainable harvesting.

34 1. Introduction

35 In fluid mechanics and aerodynamics, the Reynolds number is a dimensionless number computed as
36 the ratio of inertial forces to viscous forces, that quantifies the relative importance of these two
37 forces for given flow conditions. A critical value of this number (in general experimentally
38 determined, and that should not be interpreted narrowly) is often used to estimate the transition
39 between laminar and turbulent flow.

40 Similarly, in hydrology and fluid dynamics, the Froude number is a dimensionless number defined as
41 the ratio of inertial to gravitational forces, which can describe different flow regimes. In free-surface
42 flow, the nature of the flow, supercritical (fast flow) or subcritical (slow flow), depends upon
43 whether the Froude number is greater than or less than unity.

44 In general, dimensionless numbers (which are numbers representing a property of a system, but
45 without any physical units, such as length L , area L^2 , mass, or time, associated with them), help in the
46 understanding of certain problems for which no analytic mathematical formulation exists, or for
47 which the mathematical formulation is too complex to be easily comprehended. In case of existence
48 of such numbers for a specific system, they allow to extract certain conclusions about the behavior
49 of the system without the need of a more detailed mathematical formulation relating all the
50 different variables involved in the problem at hand. Critical values of these numbers are important
51 to identify bifurcations between different states of the system. Thus, they are widely used in many
52 fields, such as mathematics, physics, biology, engineering, and economics (e.g. see Szirtes (2006)).

53 In this study, we demonstrate that in the particular case of uneven-aged monospecific stand
54 dynamics, where we can also find a “demographic flow” (defined by the stems evolution through the
55 diameter classes over time), there are dimensionless numbers which have a strong influence on the
56 behaviour of the system. Moreover, critical values of those numbers identify a bifurcation in the
57 stand management practices switching between sustainable and unsustainable harvesting.

58 The main factors influencing this flow (in the absence of external disturbances) are recruitment,
 59 transition probabilities between diameter classes, and mortalities (natural or induced by harvesting)
 60 (Schütz, 2006). These processes of evolution over time can easily be modeled by a matrix population
 61 model (e.g. Buongiorno and Michie, 1980; Liang and Picard, 2012; Vanclay, 2012) for which the
 62 population growth rate is the dominant eigenvalue λ of the transition matrix. By asymptotic analysis
 63 (long-term behaviour), we know that (Caswell, 2001), independent of the initial conditions, when
 64 $\lambda > 1$ the total number of stems/ha of the tree population increases exponentially over time (unless
 65 harvests are conducted), when $\lambda < 1$ the population is decaying until extinction, and when $\lambda = 1$ a
 66 stable distribution proportional to the right eigenvector W_0 of the transition matrix corresponding to
 67 λ is obtained. Gotelli (2001) refers to the special case of the stable distribution when $\lambda = 1$ as the
 68 “stationary distribution”; we consider the same case in reference to the stable diameter distribution
 69 of the stand W_0 .

70 Therefore, in this study we have used the matrix model applied to uneven-aged managed *Fagus*
 71 *sylvatica* L. and *Pinus nigra* stands by López et al. (2007, 2008, 2012, 2013), which describes
 72 accurately the stand dynamics. In particular, in order to exemplify and test the results, the model
 73 uses the transition probabilities obtained in López et al. (2012, 2013) for uneven-aged managed
 74 *Pinus nigra* stands, considering three levels of tree diameter growth (Quality I, faster; Quality II,
 75 medium; and Quality III, slower diameter growth) (see Table 1). The stands are located in the
 76 Spanish Iberian System, a mountain range extending about 400 km along the north-eastern edge of
 77 the central plateau, concentrating 60% of the area occupied by *Pinus nigra* in Spain (Grande et al.
 78 2005). Based on data from those stands, the model simulates outcomes for 594 different forest
 79 management planning scenarios: 3 diameter growth levels (Qualities I–III) x 6 basal area levels ($G =$
 80 21 to $G = 26 \text{ m}^2/\text{ha}$) x 33 recruitment levels ($R = 200$ to $R = 840 \text{ stems/ha}$).

81 Calculations were performed using Maple version 18 (Maplesoft, 2014).

82

83 2. Methods

84 2.1. Matrix model

85 The matrix model for the recruitment, harvesting and stem flow throughout the dbh-classes over
 86 time is defined by means of a finite difference homogeneous linear system of equations, which
 87 accurately describes transitions from time t (initial) to time $t+1$ (final). Since the harvesting
 88 operations in the study area generally took place every 10 years, this was the time step adopted in
 89 the model as an example (thus the time interval $(t, t+1)$ represents 10 years). Considering this time
 90 step, trees were grouped into n diameter classes of equal width $w = 6$ cm (the parameter w
 91 represents the width of the diameter classes): $(0,6]$, $(6,12]$, $(12,18]$, ..., $(30,36]$, ..., (i.e., $(6,12]$ means a
 92 diameter interval which contains the diameter 12 cm but not the diameter 6 cm), with the last class
 93 being $(48, \rightarrow)$ (more than 48 cm) for Quality I, $(42, \rightarrow)$ (more than 42 cm) for Quality II and $(36, \rightarrow)$
 94 (more than 36 cm) for Quality III. Therefore, an individual tree in class k can remain in class k or
 95 progress to class $k+1$ during the projection interval $(t, t+1)$. The number of trees in each class
 96 changes in each projection interval, because some are harvested, some remain in the same diameter
 97 class and others grow past the boundary to the next diameter class. In such conditions, p_k is the
 98 probability that an individual tree in class k at time t (initial) will appear in class $k+1$ at time $t+1$ (final
 99 time of projection); r_k , recruitment coefficient, is the number of offspring (stems/ha) living at time
 100 $t+1$ produced in the interval of projection $(t, t+1)$ by an average tree in class k at time t (similar to
 101 many standard size classified matrix models (Caswell, 2001), $r_1 = 0$); $h_k(t)$ defines the proportion of
 102 harvested trees in class k , natural mortalities included; $x_k(t)$ and $x_k(t+1)$ describe the stem densities in
 103 class k at the initial and final times of projection. Finally, $R = r_2 x_2 + r_3 x_3 + \dots + r_n x_n$ will denote the
 104 global amount of recruitment at each time step. Notice that $0 < p_k < 1$ for $k = 1, 2, \dots, n-1$; $r_k \geq 0$ for
 105 $k = 2, 3, \dots, n$, with $R = 0$ if and only if $r_k = 0$ for all k ; and $x_k > 0$ for $k = 1, 2, \dots, n$.

106 By analysing the dynamics of the projections, we find that the model is described accurately by the
 107 matrix model

$$X(t+1) = A(I - H(t)) X(t), \quad (1)$$

where

$$A = \begin{pmatrix} 1-p_1 & r_2 & r_3 & \cdots & r_{n-1} & r_n \\ p_1 & 1-p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 1-p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-p_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & p_{n-1} & 1 \end{pmatrix}$$

I is the identity matrix of order n , $H(t) = \text{diag}(h_1(t), h_2(t), \dots, h_n(t))$ is a diagonal matrix with the harvest rates $h_k(t)$ (including natural mortalities), and $X(t)$ and $X(t+1)$ are column vectors indicating the stem densities at the initial and final times of projection, respectively. Note that the model only supports transitions between consecutive diameter classes. Therefore, the time step and the width of the diameter classes should be chosen according to this prerequisite.

2.2. Sustainable/stable harvesting strategy

As we can see in (1), harvesting induces a perturbation in the model of natural growth of trees (i.e. growth model without harvesting), defined by

$$X(t+1) = A X(t), \quad (2)$$

modifying the natural transition matrix A toward the perturbed $A(I-H)$. Thus, whenever the dominant eigenvalue of matrix A be $\lambda_0 > 1$, the long-term sustainable harvest rates can be determined as the proportion of trees harvested in each class so that the dominant eigenvalue of matrix $A(I-H)$ be $\lambda = 1$. There are many different harvesting strategies to satisfy this boundary condition $\lambda = 1$, and many other unable to achieve this condition. In particular, the “sustainable/stable” harvesting strategy is based on the following conditions (C1, C2, and C3) (López et al., 2007, 2008):

(C1) A necessary condition to carry out the harvesting operations is $\lambda_0 > 1$.

(C2) For reasons of sustainability, the population growth rate λ of matrix $A(I-H)$ should be $\lambda = 1$.

(C3) Each harvest should lead to the stable diameter distribution of the stand, which is given by the right eigenvector W_0 corresponding to the population growth rate λ_0 of the matrix A .

Note that $\lambda_0 \geq 1$, and $\lambda_0 = 1$ for $R = 0$. In fact, from $\lambda_0 = 1$ for $R = 0$ (as we can see in the matrix A for $r_2 = r_3 = \dots = r_n = 0$), and $\frac{\partial \lambda_0}{\partial R} \geq 0$, as we know by sensitivity analysis (Caswell, 1978), it follows that $\lambda_0 \geq 1$.

Moreover, assuming that condition (C1) holds, conditions (C2) and (C3) can be rewritten by means of the linear system $AHW_0 = (\lambda_0 - 1)W_0$. Thus, we obtain the “sustainable/stable” harvest rates s by solving this determinate compatible linear system, yielding the following rate

$$s = \frac{\lambda_0 - 1}{\lambda_0}, \quad (3)$$

for all the diameter classes. In this equation (3), $s' = \frac{1}{\lambda_0} = 1 - s$ has the interpretation of the proportion of trees that has to remain unharvested (for all the diameter classes) to retain the stable diameter distribution (with sustainability, since $\lambda = 1$).

Let us suppose a harvesting strategy such that $h_1 = h_2 = \dots = h_{n-1} = h_n = h$, with $h < s$. Then, $\lambda > 1$, and h is a sustainable harvest rate, although the resources are under harvested and the stable diameter distribution of the stand is not reached at each time step. Conversely, if $h_1 = h_2 = \dots = h_{n-1} = h_n = h$, with $h > s$, then, $\lambda < 1$, h is not a sustainable harvest rate and the harvest does not lead to the stable diameter distribution of the stand either. Hence, the “sustainable/stable” harvest rate s (and, by extension, s') defines a boundary between sustainable and unsustainable harvesting.

It would therefore be useful to find easy expressions, involving the main variables of the problem, which help identify this bifurcation between sustainable and unsustainable harvesting. For this we will manipulate the equation for the dynamic stand equilibrium introduced by López et al. (2007) and generalized in López et al. (2008).

2.3. Equilibrium equation. Dimensionless numbers

Let us consider first the growth model without harvesting (2), then the harvesting model (1), but bearing in mind that both models are related in the bifurcation corresponding to the “sustainable/stable” harvesting strategy by means of their transition matrices

$$A(I-H) = (1-s)A. \quad (4)$$

From (4) it follows that the corresponding population growth rates λ_0 and λ are related by the equation

$$\lambda = (1-s) \lambda_0. \quad (5)$$

We now recall (López et al., 2007, 2008) that, by solving $AW_0 = \lambda_0 W_0$, we obtain the components of the stable diameter distribution of the stand W_0 (which is simultaneously the right eigenvector of the matrices A and $A(I-H)$ corresponding respectively to the dominant eigenvalues λ_0 and $\lambda = 1$), yielding (proportional to) the vector $W_0 = (N_1, N_2, N_3, \dots, N_{n-1}, N_n)$, where

$$N_1 = (\lambda_0 - 1) \prod_{k=2}^{n-1} (\lambda_0 - 1 + p_k), \quad N_i = (\lambda_0 - 1) \left(\prod_{k=1}^{i-1} p_k \right) \left(\prod_{k=i+1}^{n-1} (\lambda_0 - 1 + p_k) \right) \text{ for } 1 < i < n-1,$$

$$N_{n-1} = (\lambda_0 - 1) \prod_{k=1}^{n-2} p_k, \text{ and } N_n = \prod_{k=1}^{n-1} p_k.$$

By dividing consecutive components, we arrive at

$$N_{i+1} = \frac{p_i}{\lambda_0 - 1 + p_{i+1}} N_i = l_i N_i, \text{ for } i = 1, 2, \dots, n-2, \text{ and } N_n = \frac{p_{n-1}}{\lambda_0 - 1} N_{n-1} = l_{n-1} N_{n-1}, \quad (6)$$

$$\text{where } l_i = \frac{p_i}{\lambda_0 - 1 + p_{i+1}}, \text{ for } i = 1, 2, \dots, n-2 \text{ and } l_{n-1} = \frac{p_{n-1}}{\lambda_0 - 1}.$$

Substituting N_i (for $i = 1, 2, \dots, n-1, n$) from (6) into the equation of the basal area of the stand G (prior to harvest), we get

$$G = \frac{\pi N_1}{4} \left[\frac{w^2}{4} + \sum_{i=2}^n \left(\frac{(2i-1)^2 w^2}{4} \prod_{k=1}^{i-1} l_k \right) \right]. \quad (7)$$

But, in the stable position, we know that $X(t+1) = \lambda_0 X(t)$ which, for the first component, gives

$$(1 - p_1) N_1 + \sum_{i=2}^n r_i N_i = (1 - p_1) N_1 + R = \lambda_0 N_1 \quad (8)$$

where R is the global amount of recruitment. By solving (8) for N_1 , we obtain

$$N_1 = \frac{R}{\lambda_0 - 1 + p_1}$$

Substituting N_1 into (7), we finally arrive at the equilibrium equation

$$G = \frac{\pi R}{4(\lambda_0 - 1 + p_1)} \left[\frac{w^2}{4} + \sum_{i=2}^n \left(\frac{(2i-1)^2 w^2}{4} \prod_{k=1}^{i-1} l_k \right) \right]. \quad (9)$$

Rearranging (9), we get the polynomial equation, expanded around $\lambda_0 = 1$, given by

$$P_n(\lambda_0) = a_n(\lambda_0 - 1)^n + a_{n-1}(\lambda_0 - 1)^{n-1} + a_{n-2}(\lambda_0 - 1)^{n-2} + \dots + a_1(\lambda_0 - 1) + a_0 = 0. \quad (10)$$

We recall that the dominant (positive) root of this polynomial $P_n(\lambda_0) = a_n(\lambda_0 - 1)^n + \dots + a_1(\lambda_0 - 1) + a_0$ is the population growth rate λ_0 .

The coefficients a_k of the polynomial $P_n(\lambda_0)$ are specified in Appendix 1 in the general case, and in

Appendix 2 for $n = 7$ as an example. Note that, except for the constant term a_0 , the factor $\frac{16G}{\pi w^2 R}$ is

present in all the coefficients.

In particular, as we show in Sections 3 and 4, the leading coefficient and the constant term of $P_n(\lambda_0)$,

$$a_n = \frac{16G}{\pi w^2 R} \quad (11)$$

186
$$a_0 = -(2n-1)^2 \prod_{i=1}^{n-1} p_i, \tag{12}$$

187 have a strong influence on λ_0 . In fact, by applying the Descartes' rule of signs, considering that the
188 number of changes of sign in the list of coefficients of $P_n(\lambda_0)$ was 1 for any of the 594 examined
189 scenarios (with $a_n > 0$ and $a_0 < 0$ in any case), $P_n(\lambda_0)$ has exactly one (positive) root greater than 1.
190 Moreover, a rough approximation $\hat{\lambda}_0$ of the root λ_0 may be obtained from

191
$$P_n(\lambda_0) = a_n(\lambda_0-1)^n + a_0 = 0, \tag{13}$$

192 yielding

193
$$\lambda_0 \approx \hat{\lambda}_0 = 1 + n \sqrt[n]{-\frac{a_0}{a_n}} = 1 + n \sqrt[n]{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16G}}. \tag{14}$$

194 Therefore, a dimensionless number for the growth model without harvesting (2) is given by

195
$$\hat{\lambda}_0 = 1 + n \sqrt[n]{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16G}} \tag{15}$$

196 As equation (14) suggests and we show in Section 3, the variables λ_0 and $\hat{\lambda}_0$ have a strong positive
197 linear correlation.

198 Regarding the growth model with harvesting (1), a rough approximation $\hat{\lambda}$ of the root $\lambda = 1$
199 corresponding to the “sustainable/stable” harvesting strategy may be obtained from (5) and (15). In
200 fact, as we show in Section 3, the dimensionless number $\hat{\lambda} = (1-s)\hat{\lambda}_0$,

$$\hat{\lambda} = (1-s) \left(1 + \sqrt[n]{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16G}} \right) = \frac{\lambda_0}{\lambda_0}, \quad (16)$$

is very close to 1 for any of the 594 examined scenarios. Therefore, for a harvesting strategy such that $h_1 = h_2 = \dots = h_{n-1} = h_n = h$, with $h < s$, we would obtain

$$(1-h) \left(1 + \sqrt[n]{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16G}} \right) > 1, \quad (17)$$

thus h is a sustainable harvest rate. Conversely, if $h_1 = h_2 = \dots = h_{n-1} = h_n = h$, with $h > s$, then

$$(1-h) \left(1 + \sqrt[n]{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16G}} \right) < 1, \quad (18)$$

so h is not a sustainable harvest. Hence, the dimensionless number $\hat{\lambda}$ defines a boundary between sustainable (equation (17)) and unsustainable (equation (18)) harvesting. However, since s (and, by extension, λ_0) is needed for calculating $\hat{\lambda}$, it would be more useful to find another boundary, which is not based on those variables. This can be done by considering equations (16), (17) and (18), from which we define a rough approximation $\hat{s}$ to the “sustainable/stable” harvest rate s by means of

$$(1-\hat{s}) \left(1 + \sqrt[n]{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16G}} \right) = 1,$$

yielding

$$\hat{s} = 1 - \left(1 + n \sqrt{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16 G}} \right)^{-1} = \frac{\lambda_0 - 1}{\lambda_0}. \quad (19)$$

In this case, a harvesting strategy such that $h_1 = h_2 = \dots = h_{n-1} = h_n = h$, with $h < \hat{s}$, should be sustainable and, conversely, if $h_1 = h_2 = \dots = h_{n-1} = h_n = h$, with $h > \hat{s}$, then the harvesting strategy should be unsustainable.

Nevertheless, for reasons which will be explained in Section 3, in general it would be better to use in this case s' , rather than $\hat{s}$, defined as a rough approximation to $s' = \frac{1}{\lambda_0} = 1-s$ (proportion of trees that has to remain unharvested to retain the stable diameter distribution) given by

$$s' = 1 - \hat{s} = \frac{1}{\lambda_0} = \left(1 + n \sqrt{\frac{(2n-1)^2 \pi w^2 R \prod_{i=1}^{n-1} p_i}{16 G}} \right)^{-1}. \quad (20)$$

Finally, as we show in Section 3, a rough approximation W_0 to the stable diameter distribution of the stand W_0 may be obtained by considering the expressions of W_0 and λ_0 given above, through which we define W_0 as (proportional to) the vector $(N_1, N_2, \dots, N_{n-1}, N_n)$, where

$$N_1 = (\lambda_0 - 1) \prod_{k=2}^{n-1} (\lambda_0 - 1 + p_k), \quad N_i = (\lambda_0 - 1) \left(\prod_{k=1}^{i-1} p_k \right) \left(\prod_{k=i+1}^{n-1} (\lambda_0 - 1 + p_k) \right) \text{ for } 1 < i < n-1,$$

$$N_{n-1} = (\lambda_0 - 1) \prod_{k=1}^{n-2} p_k, \text{ and } N_n = \prod_{k=1}^{n-1} p_k \quad (21)$$

2.4. Extension to the case of natural perturbations.

In the context of matrix models, natural disturbances can be addressed in several ways (see Liang et al., 2005; Liang and Picard, 2012), and abundant evidence suggests that considering natural perturbations can make huge differences in stand dynamics and forest equilibrium (e.g. Zhou and Buongiorno, 2004). In a broad sense, a system is considered to be stable if it always returns to a reference position (equilibrium) after small perturbations (i.e., natural disturbances in forests). As shown in the Introduction, in the case of the discrete dynamic system (2), we say that the equilibrium position defined by W_0 is stable because the trajectories (that is, solutions represented in the phase space, where all possible states of the system are represented) with initial condition selected in a sufficiently small neighborhood of that equilibrium, always remain near the equilibrium.

However, in applications of Perturbations Theory and Structural Stability of Dynamical Systems (see, for example, Hirsch and Smale (1974)), we require our mathematical model to be more robust (not just stable equilibrium). By this we mean that its qualitative properties (geometric shape of trajectories in the phase space, type of stability of equilibrium points and critical solutions, etc.) should not change significantly when the model is subjected to small perturbations (i.e., small changes on the right hand side of the system of linear difference equations (2)). In such a case, we say that the system is structurally stable. In particular, if (2) is structurally stable and it has an attractor (as the stable diameter distribution W_0), any nearby perturbed system will have a nearby attractor (we say that the attractor persists).

A simple necessary and sufficient condition for the structural stability of a linear dynamical system is that all the eigenvalues of the associated matrix have non-zero real parts. In general, this condition is easily met in the case of linear systems of difference equations modeling forest dynamics, in particular in the equation (2) for all scenarios.

In our study, the perturbation on the model of natural growth of trees (2) due to harvesting, is given by

$$-A H X(t), \quad (22)$$

where

$$H = \text{diag}(h_1, h_2, \dots, h_n)$$

is a diagonal matrix with the harvest rates (including natural mortalities) for each time step (see also equation (1)). Since all the eigenvalues of the matrix A have non-zero real parts, (2) is structurally stable. In particular, the attractor given by the stable diameter distribution is persistent under small perturbations caused by harvesting.

Note that (22) is a linear perturbation of system (2), which is structurally stable. Let us now prove that the results from Sections 2.2. and 2.3. can be extended to the case of any linear perturbation E of (2), defined by

$$E = -A \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) X(t),$$

where ε_i represents the perturbation rate for the i th diameter class, comprising harvest rates (μ_i) (excluding in this case natural mortalities) and natural disturbance rates (δ_i) (natural mortalities included), that is, $\varepsilon_i = \mu_i + \delta_i$. In fact, the model (2) of natural growth of trees including natural perturbations is defined by means of

$$X(t+1) = A(I-\Delta) X(t), \quad (23)$$

where $\Delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$ is a diagonal matrix with the natural disturbance rates. The model of natural growth of trees including natural perturbations and harvesting is given by

$$X(t+1) = A(I-\Delta-H) X(t),$$

where $H = \text{diag}(\mu_1, \mu_2, \dots, \mu_n)$ is a diagonal matrix with the harvest rates. Let λ_0 be the dominant eigenvalue of matrix $A(I-\Delta)$, with $\lambda_0 > 1$ (otherwise condition (C1) is not verified), and W_0 the

associated eigenvector (stable diameter distribution corresponding to the matrix $A(I-\Delta)$). By substituting h_i for μ_i in (22), conditions (C2) and (C3) can be rewritten by means of the linear system

$$A \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_n) W_0 = (\lambda_0 - 1) W_0.$$

Since $AW_0 \approx \lambda_0 W_0$ (let us recall that (23) is the perturbed system of (2) and that the attractors of structurally stable systems persist under small perturbations), a fairly good approximation to the solution of this system is the “sustainable/stable harvest rate” μ (natural perturbations excluded), defined by means of

$$\mu = \frac{\lambda_0 - 1}{\lambda_0},$$

for all the diameter classes (that is, $\mu_1 = \mu_2 = \dots = \mu_n = \mu$). Note that in this equation λ_0 is assuming the natural disturbances δ_i from the matrix $A(I-\Delta)$ (internally, since δ_i form part of the coefficients of the matrix $A(I-\Delta)$). Moreover, the results of section 2.3. can be easily extended by carrying out the following changes:

1. Throughout Section 2.3. the transition probabilities are given by $p_k(1 - \delta_k)$.
2. By solving (8) for N_1 (considering the model (23)), we obtain

$$N_1 = \frac{R - M_R}{\lambda_0 - (1 - p_1)(1 - \delta_1)},$$

where $M_R = \delta_2 r_2 x_2 + \delta_3 r_3 x_3 + \dots + \delta_n r_n x_n$ is the global mortality for tree recruitment due to natural disturbances (stems/ha). We recall that λ_0 is the dominant eigenvalue of matrix $A(I-\Delta)$.

3. The equilibrium equation (9) for the model (23) becomes

$$G = \frac{\pi(R - M_R)}{4(\lambda_0 - (1 - p_1)(1 - \delta_1))} \left[\frac{w^2}{4} + \sum_{i=2}^n \left(\frac{(2i-1)^2 w^2}{4} \prod_{k=1}^{i-1} l_k \right) \right],$$

294 assuming in l_k the new transition matrix coefficients.

295 4. The expressions (11) and (12) for the model (23) are given by

296
$$a_n = \frac{16G}{\pi w^2 (R - M_R)} \tag{24}$$

297
$$a_0 \approx -(2n-1)^2 \prod_{i=1}^{n-1} p_i (1-\delta_i) \tag{25}$$

298 (the equality for a_0 holds if $\delta_n = 0$).

299 5. The expressions (11) to (20) should be adapted to include the new a_0 and a_n given by (24) and
300 (25). For example, the expression (14) becomes

301
$$\lambda_0 = 1 + n \sqrt{-\frac{a_0}{a_n}} \approx 1 + n \sqrt{\frac{(2n-1)^2 \pi w^2 (R - M_R) \prod_{i=1}^{n-1} p_i (1-\delta_i)}{16G}}$$

302 (the equality holds if $\delta_n = 0$).

303 6. Renaming and redefining variables, s as μ , $\hat{s}$ as μ' , and s' as μ'' (defined by $\mu'' = 1-\mu$). For
304 example,

305
$$\mu = 1 - \left(1 + n \sqrt{\frac{(2n-1)^2 \pi w^2 (R - M_R) \prod_{i=1}^{n-1} p_i (1-\delta_i)}{16G}} \right)^{-1} = \frac{\lambda_0 - 1}{\lambda_0}.$$

306 The natural mortality rate was included into the harvesting rate in Section 2.1. considering that all
307 dead trees along the decade are harvested subsequently, but it's clear that for higher stand densities
308 than the values considered in our scenarios, the natural disturbances rate (natural mortalities

included) could be high, in particular for the lower diameter classes. . In such cases, it should be used the model (23) and the associated results.

With regard to nonlinear perturbations, the results from Sections 2.2. and 2.3. cannot be directly extended, and each case must be analyzed individually. However, the strong property of structural stability of system (2) may be applied, obtaining a first order approximation to the solutions of the (nonlinear) perturbed system of (2) by means of the corresponding linearization, which might be addressed in the same manner as the linear perturbation E described above.

3. Results

Since the results from Sections 2.2. and 2.3. can be easily extended to the case of the linear perturbation introduced in Section 2.4., simulations were conducted using only the first case. By substituting the transition probabilities p_i from Table 1 into (10) and solving the corresponding polynomial equation for each scenario, we obtained the population growth rate λ_0 and, subsequently, the “sustainable/stable” harvest rate s (equation (3)). Moreover, by substituting p_i , recruitment R , basal area G , together with w and n , into the equations (15), (16), (19) and (20), we obtained $\hat{\lambda}_0$, $\hat{\lambda}$, $\hat{s}$ and $\hat{s}'$, as summarized in Tables 2, 3 and 4 for selected scenarios.

The relative error between λ_0 and $\hat{\lambda}_0$, defined as

$$\eta_{\lambda_0} = \frac{\lambda_0 - \hat{\lambda}_0}{\lambda_0} = 1 - \hat{\lambda},$$

ranged in absolute value from 1.85% to 19.42% for Quality I (averaging 9.66%), from 2.47% to 17.47% for Quality II (averaging 9.24%), and from 1.89% to 14.45% for Quality III (averaging 7.66%), always with $\lambda_0 < \hat{\lambda}_0$. The maximum relative errors between λ_0 and $\hat{\lambda}_0$, occurred in any case for the

329 $G = 26 \text{ m}^2/\text{ha}$ and $R = 200 \text{ stems/ha}$ scenarios, and the minimum for the $G = 21 \text{ m}^2/\text{ha}$ and $R = 840$
 330 stems/ha scenarios. Between λ (with $\lambda = 1$) and $\hat{\lambda}$ we have the same relative errors, because

331
$$\eta_{\lambda} = 1 - \hat{\lambda} = \eta_{\lambda_0}.$$

332 With regard to the relative error between s and $\hat{S}$, s' and $\hat{S}'$, defined respectively as

333
$$\eta_s = \frac{s - \hat{S}}{s} = \frac{\lambda_0 - \hat{\lambda}_0}{\hat{\lambda}_0(\lambda_0 - 1)} = \frac{1 - \hat{\lambda}}{\hat{\lambda}(\lambda_0 - 1)} = \frac{\eta_{\lambda_0}}{\hat{\lambda}(\lambda_0 - 1)},$$

334
$$\eta_{s'} = \frac{s' - \hat{S}'}{s'} = \frac{\frac{1}{\lambda_0} - \frac{1}{\hat{\lambda}_0}}{\frac{1}{\lambda_0}} = 1 - \frac{1}{\hat{\lambda}} = \frac{\eta_{\lambda_0}}{\eta_{\lambda_0} - 1},$$

335 we note that, while η_s expands the error η_{λ_0} , $\eta_{s'}$ reduces it by a factor of $\left| \frac{1}{\eta_{\lambda_0} - 1} \right| < 1$. In fact, the

336 relative error between s and $\hat{S}$, ranged in absolute value from 3% to 57.81% for Quality I (averaging
 337 21.41%), from 4.58% to 61.61% for Quality II (averaging 23.77%), and from 4.12% to 62.25% for
 338 Quality III (averaging 23.69%), always with $s < \hat{S}$. Conversely, the relative error between s' and $\hat{S}'$,
 339 ranged in absolute value from 1.82% to 16.26% for Quality I (averaging 8.65%), from 2.41% to
 340 14.87% for Quality II (averaging 8.35%), and from 1.85% to 12.62% for Quality III (averaging 7.03%),

341 always with $s' > \hat{S}'$. Therefore, in general it would be better to use $\hat{S}'$, rather than $\hat{S}$. Just as for

342 η_{λ_0} and η_{λ} , the maximum relative errors between s and $\hat{S}$, s' and $\hat{S}'$, occurred in any case for the
 343 $G = 26 \text{ m}^2/\text{ha}$ and $R = 200 \text{ stems/ha}$ scenarios, and the minimum for the $G = 21 \text{ m}^2/\text{ha}$ and $R = 840$
 344 stems/ha scenarios.

345 Additionally, we obtain the following results:

1. As mentioned in the previous section, the variables λ_0 and $\hat{\lambda}_0$ have a strong positive linear correlation. Table 5 shows the regression parameters and Figure 1 depicts the points $(\hat{\lambda}_0, \lambda_0)$ (198 points for each Quality level) and the fitted linear regression line. In particular, the total proportion of variance in λ_0 explained by $\hat{\lambda}_0$ was $r^2 = 0.9961$ for Quality I, 0.9964 for Quality II, and 0.9966 for Quality III.

2. The variables s and $\hat{s}$ have a strong positive linear correlation (see Table 5). The total proportion of variance in s explained by $\hat{s}$ was $r^2 = 0.9995$ for Quality I, 0.9993 for Quality II, and 0.9991 for Quality III.

3. The variables s' and $\hat{s}'$ have a strong positive linear correlation, with the same r^2 values as between s and $\hat{s}$ (see Table 5).

4. By substituting the transition probabilities p_i from Table 1 and the previously calculated $\hat{\lambda}_0$ values into (21), we obtained $\hat{W}_0$, as summarized in Table 6 together with W_0 for some scenarios. We evaluated the distance between W_0 and $\hat{W}_0$ with Keyfitz's Δ ,

$$\Delta(W_0, \hat{W}_0) = \frac{1}{2} \sum_{i=1}^n |N_i - \hat{N}_i|,$$

where N_i and $\hat{N}_i$ are the components of the diameter distributions W_0 and $\hat{W}_0$, scaled to 1 (i.e.

$\sum_{i=1}^n N_i = \sum_{i=1}^n \hat{N}_i = 1$) (Keyfitz, 1968). The maximum value of Δ is 1, and the minimum of 0 occurs

when the diameter distributions W_0 and $\hat{W}_0$ (scaled to 1) are identical. Smaller values of Δ result

when the distributions W_0 and $\hat{W}_0$ (scaled to 1) become more similar. As Table 6 shows, Keyfitz's Δ

returns low values for all the scenarios and thus suggests that $\hat{W}_0$ is a good approximation of W_0 . The minimum distance between W_0 and $\hat{W}_0$ occurred in any case for the $G = 21 \text{ m}^2/\text{ha}$ and $R = 840$ stems/ha scenarios, and the maximum for the $G = 26 \text{ m}^2/\text{ha}$ and $R = 200$ stems/ha scenarios.

4. Discussion and Conclusion

This study proposes a simple and direct method based on dimensionless numbers, obtained under conditions of stable equilibrium from a matrix model, to provide reliable approximations of the population growth rate, the “sustainable/stable” harvest rate, the proportion of trees that has to remain unharvested to retain the stable diameter distribution, and the stable diameter distribution, of a monospecific uneven-aged stand. Moreover, $\hat{s}$ and $\hat{s}'$ could also serve to estimate boundaries between sustainable and unsustainable harvesting. In fact, these dimensionless numbers are based on the following variables: (a) the transition probabilities $p_{k,t}$, which can be calculated from counts of observed transitions between adjacent diameter classes from t to $t+1$. In the absence of a detailed census of the trees in the study area, they could be estimated using the methods proposed by López et al. (2007, 2008, 2012); (b) the basal area of the stand G , and the global amount of (natural) recruitment R (defined as the total number of trees entering into the first diameter class from t to $t+1$, in general above a certain threshold diameter). Although *Pinus nigra* regenerates well in the study area under its own canopy (Retana et al., 2002; Tiscar, 2007), there is a critical basal area to allow sufficient light transmittance for regeneration. This maximum value is around 20-30 m^2/ha (Schütz, 1989; Serrada et al., 1994; Monge, 1997). On the other hand, basal areas clearly below 20 m^2/ha (before harvesting) could compromise natural regeneration (Serrada et al., 1994). Consequently, we have considered, in addition to three levels of diameter growth (Qualities I–III), 6 levels of stand basal area from $G = 21$ to $G = 26 \text{ m}^2/\text{ha}$, and 33 levels of global recruitment, from $R = 200$ (scarce recruitment) to $R = 840$ (abundant recruitment) (stems/ha), creating a total of 594

planning scenarios. The corresponding problems simulate a wide variety of typical management cases in the study area; and (c) the number n and width w of the diameter classes.

Disregarding those scaling factors n and w and the constant terms, the core of these dimensionless numbers is given by the quotient

$$\gamma_G = \frac{R \prod_{i=1}^{n-1} p_i}{G}$$

which represents the product of the stand growth inputs, R and $\prod_{i=1}^{n-1} p_i$, per unit of basal area (since γ_G is measured in stem/m², γ_G has dimension L^{-2}). In this regard, the higher the level of recruitment and the faster the flow of the stems through the diameter classes, for a given basal area, the greater the population growth rate. Furthermore, there is also a strong positive linear correlation between λ_0 and γ_G , though a little bit weaker than that seen between λ_0 and λ_0^* , with r^2 being 0.9874 for Quality I, 0.9876 for Quality II, and 0.9879 for Quality III. This suggests that the scaling factors n and w could have limited influence on λ_0 and λ_0^* .

As shown in Section 3, the relative errors between λ_0 and λ_0^* , λ and $\hat{\lambda}$, s and $\hat{s}$, s' and $\hat{s}'$, in general were small, in particular for Quality III scenarios, with the lowest basal area and the highest recruitment scenarios providing the most accurate approximations. This result is even more noteworthy given the fact that possibly the most likely scenarios would be those with basal areas between 21-22 m²/ha and abundant recruitment, especially bearing in mind that the highest population growth rates λ_0 (and, by extension, “sustainable/stable” harvest rates s) were obtained for the lowest basal area and the highest recruitment scenarios.

Those small relative errors between λ_0 and λ_0^* are not due to partial or total self-cancellation of the polynomial $P_n(\lambda_0)$ terms between a_n and a_0 , as can be determined by direct calculations, but is likely

409 attributable to the fact that the factor $a_n = \frac{16G}{\pi w^2 R}$ is present in all the coefficients of $P_n(\lambda_0)$ on the
410 one hand, and the impact of the products of successive transition probabilities on those coefficients
411 (which, accumulating more transition probabilities at each product as the order of the coefficient
412 diminishes, show a decreasing influence) on the other. Hence the scenarios with the lowest relative
413 errors were those corresponding to Quality III (where the transition probabilities are lowest), with
414 low basal area and high recruitment values enhancing those effects. In worst-case scenarios (Quality
415 I, high basal area and low recruitment), the coefficient a_{n-1} would also have some influence on λ_0 ,
416 although less than a_n . In either case, as depicted in Figure 2, $P_n(\lambda_0)$ provides a good approximation to
417 obtain the dominant root of $P_n(\lambda_0)$ in the vicinity of $\lambda_0 = 1$, especially in the lowest basal area and the
418 highest recruitment scenarios. Conversely, the convergence of the successive approximations to the
419 root greater than 1 given by the linear (or tangent line) approximation of $P_n(\lambda_0)$, is very slow if we
420 start the process with $\lambda_0 = 1$, due to the shape of the graph of $P_n(\lambda_0)$ in the vicinity of $\lambda_0 = 1$.
421 Furthermore, the strong positive linear correlation between λ_0 and λ_0' means (among other things)
422 that stratifying the forest into uniform areas based upon tree diameter growth, the same linear
423 regression line can be used in each stratum to estimate the population growth rate (this also applies
424 to s and s'). Moreover, only two homogeneous sample plots are required to obtain the parameters
425 of those linear regressions for each quality site.
426 As shown in the Introduction, population growth rates from matrix models are calculated to assess,
427 over the long-term, whether a population is expected to grow, remain stable or decline. With this
428 broader view, the population growth rate and the asymptotic stock recovery rate that is used in
429 tropical forests to assess logging sustainability (Picard et al., 2009) (dimensionless quantities),
430 probably also might be considered dimensionless numbers, but there are not simple and direct
431 expressions relating them to the inputs of the problem. In this regard, this study shows a simple and
432 direct method based on dimensionless numbers to provide reliable approximations $\lambda_0, \hat{s}, s'$ and

$\hat{W}_0$ of the population growth rate λ_0 , the “sustainable/stable” harvest rate s , the proportion of trees
 that has to remain unharvested to retain the stable diameter distribution s' , and the stable diameter
 distribution W_0 . Moreover, $\hat{s}$ and $\hat{s}'$ could also serve to estimate boundaries between sustainable
 and unsustainable harvesting. The best approximations occurred in any case for the lowest level of
 diameter growth, the lowest basal area, and the highest recruitment scenarios. Furthermore, the
 study reveals the existence of a strong positive linear correlation across different stand attributes
 between λ_0 and $\hat{\lambda}_0$, s and $\hat{s}$, s' and $\hat{s}'$, as well as a small distance between the stable diameter
 distribution of the stand W_0 and its approximation $\hat{W}_0$.

Conflict of interest statement

None.

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Table and Figure captions

Figure 1: Linear regression between λ_0 and $\hat{\lambda}_0$ (left: Quality I; middle: Quality II; right: Quality III).

Figure 2: Graph of the polynomials $P_n(\lambda_0)$ (introduced in equation (10)) and $\hat{P}_n(\lambda_0)$ (as defined in equation (13)) (left: Quality I; middle: Quality II; right: Quality III; first row: $G = 21 \text{ m}^2/\text{ha}$ and $R = 840$ stem/ha scenarios; second row: $G = 26 \text{ m}^2/\text{ha}$ and $R = 200$ stem/ha scenarios).

Table 1: Transition probabilities between diameter classes for each quality (n = number of diameter classes; w = width of the diameter classes = 0.06 meters).

Table 2: Numerical values of λ_0 (as defined in equation (10)), s (as defined in equation (3)), s' (with $s' = 1-s$), $\hat{\lambda}_0$ (as defined in equation (15)), $\hat{\lambda}$ (as defined in equation (16)), $\hat{s}'$ (as defined in equation (20)) and $\hat{s}$ (as defined in equation (19)), for Quality I.

Table 3: Numerical values of λ_0 (as defined in equation (10)), s (as defined in equation (3)), s' (with $s' = 1-s$), $\hat{\lambda}_0$ (as defined in equation (15)), $\hat{\lambda}$ (as defined in equation (16)), $\hat{s}'$ (as defined in equation (20)) and $\hat{s}$ (as defined in equation (19)), for Quality II.

Table 4: Numerical values of λ_0 (as defined in equation (10)), s (as defined in equation (3)), s' (with $s' = 1-s$), $\hat{\lambda}_0$ (as defined in equation (15)), $\hat{\lambda}$ (as defined in equation (16)), $\hat{s}'$ (as defined in equation (20)) and $\hat{s}$ (as defined in equation (19)), for Quality III.

Table 5: Regressions between: λ_0 (as defined in equation (10)) and $\hat{\lambda}_0$ (as defined in equation (15)); s (as defined in equation (3)) and $\hat{s}$ (as defined in equation (19)); s' (with $s' = 1-s$) and $\hat{s}'$ (as defined in equation (20)) (r^2 = coefficient of determination; r_{adj}^2 = adjusted coefficient of determination; Linear regressions: a = slope with its associated 95% confidence interval, b = intercept with its associated 95% confidence interval; Number of points = 198 in each case).

Table 6: Stable diameter distributions (W_0) and approximation ($\hat{W}_0$) (as defined in equation (21)), scaled to 1, along with Keyfitz's Δ .

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TABLES

Table 1: Transition probabilities between diameter classes for each quality (n = number of diameter classes; w = width of the diameter classes = 0.06 meters).

	Quality I	Quality II	Quality III
n	9	8	7
$(0,6] \rightarrow (6,12]$	$p_1 = 0.7697$	$p_1 = 0.5951$	$p_1 = 0.4564$
$(6,12] \rightarrow (12,18]$	$p_2 = 0.8602$	$p_2 = 0.6824$	$p_2 = 0.5326$
$(12,18] \rightarrow (18,24]$	$p_3 = 0.7913$	$p_3 = 0.6200$	$p_3 = 0.4697$
$(18,24] \rightarrow (24,30]$	$p_4 = 0.6828$	$p_4 = 0.5190$	$p_4 = 0.3692$
$(24,30] \rightarrow (30,36]$	$p_5 = 0.5533$	$p_5 = 0.3971$	$p_5 = 0.2475$
$(30,36] \rightarrow (36,42]$	$p_6 = 0.4106$	$p_6 = 0.2618$	$p_6 = 0.1119$
$(36,42] \rightarrow (42,48]$	$p_7 = 0.2587$	$p_7 = 0.1171$	-
$(42,48] \rightarrow (48, \rightarrow)$	$p_8 = 0.1000$	-	-
$\prod_{i=1}^{n-1} p_i$	0.002104	0.001590	0.001168

Table 2: Numerical values of λ_0 (as defined in equation (10)), s (as defined in equation (3)), s' (with $s' = 1-s$), $\hat{\lambda}_0$ (as defined in equation (15)), $\hat{\lambda}$ (as defined in equation (16)), $\hat{s}'$ (as defined in equation (20)) and $\hat{s}$ (as defined in equation (19)), for Quality I.

		$R = 200$ stem/ha	$R = 520$ stem/ha	$R = 840$ stem/ha
$G = 22$ m^2/ha	λ_0	1.305091	1.477671	1.594240
	s	0.233770	0.323259	0.372742
	$s' = \lambda_0^{-1}$	0.766230	0.676741	0.627258
	$\hat{\lambda}_0$	1.540047	1.600537	1.633405
	$\hat{\lambda}$	1.180031	1.083148	1.024566
	$\hat{s}'$	0.649331	0.624790	0.612218
	$\hat{s}$	0.350669	0.375210	0.387782
$G = 24$ m^2/ha	λ_0	1.292499	1.458959	1.571264
	s	0.226305	0.314580	0.363570
	$s' = \lambda_0^{-1}$	0.773695	0.685420	0.636430
	$\hat{\lambda}_0$	1.534851	1.594759	1.627310
	$\hat{\lambda}$	1.187507	1.093080	1.035669
	$\hat{s}'$	0.651529	0.627054	0.614511
	$\hat{s}$	0.348471	0.372946	0.385489
$G = 26$ m^2/ha	λ_0	1.281305	1.442342	1.550881
	s	0.219546	0.306683	0.355205
	$s' = \lambda_0^{-1}$	0.780454	0.693317	0.644795
	$\hat{\lambda}_0$	1.530115	1.589493	1.621756
	$\hat{\lambda}$	1.194185	1.102022	1.045700
	$\hat{s}'$	0.653546	0.629132	0.616616
	$\hat{s}$	0.346454	0.370868	0.383384

565 **Table 3:** Numerical values of λ_0 (as defined in equation (10)), s (as defined in equation (3)), s' (with $s'=1-s$), $\hat{\lambda}_0$ (as
566 defined in equation (15)), $\hat{\lambda}$ (as defined in equation (16)), $\hat{s}'$ (as defined in equation (20)) and $\hat{s}$ (as defined in equation
567 (19)), for Quality II.

		$R = 200 \text{ stem/ha}$	$R = 520 \text{ stem/ha}$	$R = 840 \text{ stem/ha}$
$G = 22$ m^2/ha	λ_0	1.262093	1.412527	1.514472
	s	0.207665	0.292049	0.339704
	$s' = \lambda_0^{-1}$	0.792335	0.707951	0.660296
	$\hat{\lambda}_0$	1.467953	1.527320	1.559897
	$\hat{\lambda}$	1.163111	1.081267	1.029994
	$\hat{s}'$	0.681221	0.654742	0.641068
	$\hat{s}$	0.318779	0.345258	0.358932
$G = 24$ m^2/ha	λ_0	1.251140	1.396188	1.494358
	s	0.200729	0.283764	0.330816
	$s' = \lambda_0^{-1}$	0.799271	0.716236	0.669184
	$\hat{\lambda}_0$	1.462891	1.521615	1.553841
	$\hat{\lambda}$	1.169247	1.089835	1.039805
	$\hat{s}'$	0.683578	0.657196	0.643567
	$\hat{s}$	0.316422	0.342804	0.356433
$G = 26$ m^2/ha	λ_0	1.241406	1.381684	1.476521
	s	0.194462	0.276245	0.322732
	$s' = \lambda_0^{-1}$	0.805538	0.723755	0.677268
	$\hat{\lambda}_0$	1.458283	1.516422	1.548327
	$\hat{\lambda}$	1.174702	1.097518	1.048632
	$\hat{s}'$	0.685738	0.659447	0.645858
	$\hat{s}$	0.314262	0.340553	0.354142

Table 4: Numerical values of λ_0 (as defined in equation (10)), s (as defined in equation (3)), s' (with $s' = 1-s$), $\hat{\lambda}_0$ (as defined in equation (15)), $\hat{\lambda}$ (as defined in equation (16)), $\hat{s}'$ (as defined in equation (20)) and $\hat{s}$ (as defined in equation (19)), for Quality III.

		$R = 200$ stem/ha	$R = 520$ stem/ha	$R = 840$ stem/ha
$G = 22$ m^2/ha	λ_0	1.220604	1.350816	1.439536
	s	0.180733	0.259707	0.305332
	$s' = \lambda_0^{-1}$	0.819267	0.740293	0.694668
	$\hat{\lambda}_0$	1.385635	1.442037	1.473383
	$\hat{\lambda}$	1.135205	1.067531	1.023512
	$\hat{s}'$	0.721691	0.693463	0.678710
	$\hat{s}$	0.278309	0.306537	0.321290
$G = 24$ m^2/ha	λ_0	1.211164	1.336630	1.422004
	s	0.174348	0.251850	0.296767
	$s' = \lambda_0^{-1}$	0.825652	0.748150	0.703233
	$\hat{\lambda}_0$	1.380872	1.436577	1.467535
	$\hat{\lambda}$	1.140119	1.074775	1.032019
	$\hat{s}'$	0.724180	0.696099	0.681415
	$\hat{s}$	0.275820	0.303901	0.318585
$G = 26$ m^2/ha	λ_0	1.202780	1.324044	1.406468
	s	0.168593	0.244738	0.288999
	$s' = \lambda_0^{-1}$	0.831407	0.755262	0.711001
	$\hat{\lambda}_0$	1.376541	1.431613	1.462219
	$\hat{\lambda}$	1.144466	1.081243	1.039639
	$\hat{s}'$	0.726458	0.698513	0.683892
	$\hat{s}$	0.273542	0.301487	0.316108

Table 5: Regressions between: λ_0 (as defined in equation (10)) and $\hat{\lambda}_0$ (as defined in equation (15)); s (as defined in equation (3)) and $\hat{S}$ (as defined in equation (19)); s' (with $s'=1-s$) and $\hat{S}'$ (as defined in equation (20)) (r^2 = coefficient of determination; r^2_{adj} = adjusted coefficient of determination; Linear regressions: a = slope with its associated 95% confidence interval, b = intercept with its associated 95% confidence interval; Number of points = 198 in each case).

		r^2	r^2_{adj}	a	b
Q I	regression of λ_0 on $\hat{\lambda}_0$	0.9961	0.9961	3.0823 (3.0552, 3.1093)	-3.4514 (-3.4944,-3.4084)
	regression of s on $\hat{S}$	0.9995	0.9995	3.7471 (3.7350, 3.7593)	-1.0821 (-1.0866,-1.0776)
	regression of s' on $\hat{S}'$	0.9995	0.9995	3.7471 (3.7350, 3.7593)	-1.6651 (-1.6727,-1.6575)
Q II	regression of λ_0 on $\hat{\lambda}_0$	0.9964	0.9964	2.7319 (2.7087, 2.7551)	-2.7564 (-2.7917,-2.7211)
	regression of s on $\hat{S}$	0.9993	0.9993	3.2894 (3.2771, 3.3018)	-0.8429 (-0.8471,-0.8387)
	regression of s' on $\hat{S}'$	0.9993	0.9993	3.2894 (3.2771, 3.3018)	-1.4465 (-1.4546,-1.4384)
Q III	regression of λ_0 on $\hat{\lambda}_0$	0.9966	0.9966	2.4835 (2.4631, 2.5039)	-2.2275 (-2.2567,-2.1983)
	regression of s on $\hat{S}$	0.9991	0.9991	2.8989 (2.8868, 2.9110)	-0.6281 (-0.6318,-0.6245)
	regression of s' on $\hat{S}'$	0.9991	0.9991	2.8989 (2.8868, 2.9110)	-1.2708 (-1.2792,-1.2623)

586 **Table 6:** Stable diameter distributions (W_0) and approximation ($\hat{W}_0$) (as defined in equation (21)), scaled to 1, along with
 587 Keyfitz's Δ .

			$R = 200$ stem/ha	$R = 840$ stem/ha
Q I	$G = 21$ m^2/ha	W_0	[0.2884, 0.1894, 0.1477, 0.1175, 0.0927, 0.0710, 0.0511, 0.0321, 0.0103]	[0.4409, 0.2313, 0.1423, 0.0873, 0.0514, 0.0279, 0.0133, 0.0049, 0.0008]
		$\hat{W}_0$	[0.4136, 0.2269, 0.1463, 0.0944, 0.0588, 0.0341, 0.0175, 0.0070, 0.0013]	[0.4527, 0.2328, 0.1402, 0.0841, 0.0483, 0.0255, 0.0117, 0.0041, 0.0006]
		Δ	0.1626	0.0133
	$G = 26$ m^2/ha	W_0	[0.2677, 0.1805, 0.1447, 0.1188, 0.0972, 0.0777, 0.0591, 0.0401, 0.0143]	[0.4313, 0.2353, 0.1508, 0.0967, 0.0598, 0.0005, 0.0175, 0.0069, 0.0013]
		$\hat{W}_0$	[0.4078, 0.2258, 0.1470, 0.0959, 0.0604, 0.0355, 0.0185, 0.0076, 0.0014]	[0.4468, 0.2321, 0.1413, 0.0857, 0.0498, 0.0267, 0.0124, 0.0045, 0.0007]
		Δ	0.1877	0.0417
Q II	$G = 21$ m^2/ha	W_0	[0.3106, 0.1945, 0.1494, 0.1177, 0.0918, 0.0688, 0.0468, 0.0204]	[0.4690, 0.2310, 0.1376, 0.0817, 0.0460, 0.0232, 0.0094, 0.0021]
		$\hat{W}_0$	[0.4416, 0.2279, 0.1426, 0.0893, 0.0534, 0.0290, 0.0129, 0.0032]	[0.4862, 0.2323, 0.1340, 0.0768, 0.0415, 0.0200, 0.0077, 0.0016]
		Δ	0.1645	0.0185
	$G = 26$ m^2/ha	W_0	[0.2886, 0.1859, 0.1473, 0.1201, 0.0976, 0.0770, 0.0563, 0.0273]	[0.4447, 0.2283, 0.1421, 0.0885, 0.0526, 0.0283, 0.0125, 0.0031]
		$\hat{W}_0$	[0.4351, 0.2270, 0.1436, 0.0911, 0.0553, 0.0305, 0.0139, 0.0035]	[0.4795, 0.2319, 0.1354, 0.0787, 0.0432, 0.0212, 0.0083, 0.0018]
		Δ	0.1875	0.0384
Q III	$G = 21$ m^2/ha	W_0	[0.3310, 0.1992, 0.1525, 0.1204, 0.0939, 0.0688, 0.0341]	[0.4960, 0.2306, 0.1336, 0.0767, 0.0406, 0.0179, 0.0045]
		$\hat{W}_0$	[0.4597, 0.2278, 0.1414, 0.0877, 0.0509, 0.0252, 0.0073]	[0.5108, 0.2310, 0.1300, 0.0722, 0.0368, 0.0155, 0.0036]
		Δ	0.1573	0.0152
	$G = 26$ m^2/ha	W_0	[0.3076, 0.1909, 0.1512, 0.1242, 0.1018, 0.0801, 0.0442]	[0.4711, 0.2289, 0.1391, 0.0843, 0.0476, 0.0227, 0.0063]
		$\hat{W}_0$	[0.4521, 0.2269, 0.1428, 0.0900, 0.0532, 0.0270, 0.0080]	[0.5032, 0.2308, 0.1319, 0.0745, 0.0388, 0.0167, 0.0040]
		Δ	0.1805	0.0340

588

FIGURES

Figure 1: Linear regression between λ_0 and $\hat{\lambda}_0$ (left: Quality I; middle: Quality II; right: Quality III).

Figure 2: Graph of the polynomials $P_n(\lambda_0)$ (introduced in equation (10)) and $\hat{P}_n(\lambda_0)$ (as defined in equation (13)) (left: Quality I; middle: Quality II; right: Quality III; first row: $G = 21 \text{ m}^2/\text{ha}$ and $R = 840 \text{ stem/ha}$ scenarios; second row: $G = 26 \text{ m}^2/\text{ha}$ and $R = 200 \text{ stem/ha}$ scenarios).

621 **Appendix 1: Coefficients of the polynomial $P_n(\lambda_0)=a_n(\lambda_0-1)^n+\dots+a_1(\lambda_0-1)+a_0$ for the general case.**

622
$$a_n = \frac{16G}{\pi w^2 R}.$$

623
$$a_{n-1} = \frac{16G}{\pi w^2 R} \left(\sum_{i=1}^{n-1} p_i \right) - 1^2.$$

624
$$a_{n-2} = \frac{16G}{\pi w^2 R} \left(\sum_{\substack{i=1 \\ (i < j)}}^{n-2} p_i p_j \right) - 1^2 \left(\sum_{i=2}^{n-1} p_i \right) - 3^2 p_1.$$

625
$$a_{n-3} = \frac{16G}{\pi w^2 R} \left(\sum_{\substack{i=1 \\ (i < j < k)}}^{n-3} p_i p_j p_k \right) - 1^2 \left(\sum_{\substack{i=2 \\ (i < j)}}^{n-2} p_i p_j \right) - 3^2 p_1 \sum_{i=3}^{n-1} p_i - 5^2 p_1 p_2.$$

626
$$a_{n-4} = \frac{16G}{\pi w^2 R} \left(\sum_{\substack{i=1 \\ (i < j < k < l)}}^{n-4} p_i p_j p_k p_l \right) - 1^2 \left(\sum_{\substack{i=2 \\ (i < j < k)}}^{n-3} p_i p_j p_k \right) - 3^2 p_1 \left(\sum_{\substack{i=3 \\ (i < j)}}^{n-2} p_i p_j \right) - 5^2 p_1 p_2 \sum_{i=4}^{n-1} p_i - 7^2 p_1 p_2 p_3.$$

627

628
$$a_1 = a_{n-(n-1)} = \frac{16G}{\pi w^2 R} \prod_{i=1}^{n-1} p_i - 1^2 \prod_{i=2}^{n-1} p_i - 3^2 p_1 \prod_{i=3}^{n-1} p_i - 5^2 p_1 p_2 \prod_{i=4}^{n-1} p_i - 7^2 p_1 p_2 p_3 \prod_{i=5}^{n-1} p_i -$$

$$- 9^2 p_1 p_2 p_3 p_4 \prod_{i=6}^{n-1} p_i - 11^2 p_1 p_2 p_3 p_4 p_5 \prod_{i=7}^{n-1} p_i - 13^2 p_1 p_2 p_3 p_4 p_5 p_6 \prod_{i=8}^{n-1} p_i - \dots - (2n-3)^2 \prod_{i=1}^{n-2} p_i$$

629 with $\prod_{i=a}^b p_i = 0$ if $a > b$

630 (note that $\prod_{i=1}^{n-1} p_i = \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_{n-1}}}^{n-(n-1)} p_{i_1} p_{i_2} p_{i_3} \dots p_{i_{n-1}}, \prod_{i=2}^{n-1} p_i = \sum_{\substack{i_2 < i_3 < i_4 < \dots < i_{n-1}}}^{n-(n-2)} p_{i_2} p_{i_3} p_{i_4} \dots p_{i_{n-1}},$ and so on).

631
$$a_0 = a_{n-n} = -(2n-1)^2 \prod_{i=1}^{n-1} p_i.$$

632

633

634 **Appendix 2: Coefficients of the polynomial $P_n(\lambda_0)=a_n(\lambda_0-1)^n+\dots+a_1(\lambda_0-1)+a_0$ for $n = 7$.**

635
$$a_7 = \frac{16G}{\pi w^2 R}.$$

636
$$a_6 = \frac{16G}{\pi w^2 R} (p_1 + p_2 + p_3 + p_4 + p_5 + p_6) - 1.$$

637
$$a_5 = \frac{16G}{\pi w^2 R} (p_4 p_5 + p_3 p_5 + p_3 p_4 + p_5 p_6 + p_3 p_6 + p_2 p_5 + p_2 p_3 + p_4 p_6 + p_2 p_6 + p_2 p_4 +$$

 $+ p_1 p_6 + p_1 p_5 + p_1 p_4 + p_1 p_3 + p_1 p_2) - p_2 - p_3 - p_4 - p_5 - p_6 - 9 p_1 .$

638
$$a_4 = \frac{16G}{\pi w^2 R} (p_2 p_5 p_6 + p_4 p_5 p_6 + p_1 p_5 p_6 + p_2 p_3 p_6 + p_3 p_4 p_5 + p_3 p_4 p_6 + p_2 p_4 p_5 + p_2 p_4 p_6 +$$

 $+ p_2 p_3 p_5 + p_1 p_2 p_5 + p_1 p_2 p_6 + p_1 p_3 p_4 + p_1 p_3 p_5 + p_1 p_3 p_6 + p_1 p_4 p_5 + p_1 p_4 p_6 + p_1 p_2 p_3 +$
 $+ p_1 p_2 p_4 + p_3 p_5 p_6 + p_2 p_3 p_4) - p_2 p_3 - p_3 p_4 - p_2 p_4 - p_2 p_5 - p_3 p_5 - p_4 p_5 - p_2 p_6 - p_3 p_6 -$
 $- p_4 p_6 - p_5 p_6 - 9(p_1 p_3 + p_1 p_4 + p_1 p_5 + p_1 p_6) - 25 p_1 p_2 .$

639
$$a_3 = \frac{16G}{\pi w^2 R} (p_1 p_4 p_5 p_6 + p_1 p_3 p_4 p_5 + p_2 p_3 p_5 p_6 + p_1 p_2 p_3 p_5 + p_1 p_3 p_4 p_6 + p_1 p_2 p_3 p_6 +$$

 $+ p_1 p_2 p_3 p_4 + p_2 p_4 p_5 p_6 + p_1 p_2 p_5 p_6 + p_1 p_3 p_5 p_6 + p_1 p_2 p_4 p_5 + p_1 p_2 p_4 p_6 + p_2 p_3 p_4 p_6 +$
 $+ p_3 p_4 p_5 p_6 + p_2 p_3 p_4 p_5) - p_2 p_3 p_5 - p_2 p_4 p_5 - p_2 p_3 p_4 - p_3 p_4 p_5 - p_2 p_3 p_6 - p_3 p_4 p_6 -$
 $- p_2 p_4 p_6 - p_2 p_5 p_6 - p_3 p_5 p_6 - p_4 p_5 p_6 - 9(p_1 p_3 p_4 + p_1 p_3 p_5 + p_1 p_4 p_5 + p_1 p_3 p_6 +$
 $+ p_1 p_4 p_6 + p_1 p_5 p_6) - 25(p_1 p_2 p_4 + p_1 p_2 p_5 + p_1 p_2 p_6) - 49 p_1 p_2 p_3 .$

640
$$a_2 = \frac{16G}{\pi w^2 R} (p_2 p_3 p_4 p_5 p_6 + p_1 p_2 p_4 p_5 p_6 + p_1 p_2 p_3 p_5 p_6 + p_1 p_2 p_3 p_4 p_6 + p_1 p_2 p_3 p_4 p_5 +$$

 $+ p_1 p_3 p_4 p_5 p_6) - p_2 p_3 p_4 p_5 - p_2 p_3 p_5 p_6 - p_2 p_4 p_5 p_6 - p_2 p_3 p_4 p_6 - p_3 p_4 p_5 p_6 -$
 $- 9(p_1 p_3 p_4 p_5 + p_1 p_3 p_4 p_6 + p_1 p_3 p_5 p_6 + p_1 p_4 p_5 p_6) - 25(p_1 p_2 p_4 p_5 + p_1 p_2 p_4 p_6 + p_1 p_2 p_5 p_6) -$
 $- 49(p_1 p_2 p_3 p_6 + p_1 p_2 p_3 p_5) - 81 p_1 p_2 p_3 p_4 .$

641
$$a_1 = \frac{16G}{\pi w^2 R} p_1 p_2 p_3 p_4 p_5 p_6 - p_2 p_3 p_4 p_5 p_6 - 9 p_1 p_3 p_4 p_5 p_6 - 25 p_1 p_2 p_4 p_5 p_6 - 49 p_1 p_2 p_3 p_5 p_6 -$$

 $- 81 p_1 p_2 p_3 p_4 p_6 - 121 p_1 p_2 p_3 p_4 p_5 .$

642
$$a_0 = -169 p_1 p_2 p_3 p_4 p_5 p_6 .$$