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Examination of the estimation of relative permeability for unsaturated soils

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Abstract:

The unsaturated permeability function is an important soil property function used in the numerical modeling of saturated-unsaturated soil systems. The permeability function is generally predicted by integrating along the soil-water characteristic curve (SWCC) starting at saturated soil conditions. The integration is based on a particular integral formula. The Fredlund-Xing-Huang permeability function is a flexible integration techniques used for calculating the unsaturated permeability function. The original permeability theory published by Fredlund et al., (1994) specified that the air-entry value, ψ_{aev} (AEV), be used as the lower limit of the integration when calculating the permeability function. However, since there was no analytical procedure available for the calculation of the air-entry value on the SWCC, it became common practice to start the integration procedure from a value near zero. The assumption was made that the error associated with starting the integration from an arbitrary low value was minimal. While this might be the case in some situations, the error can be quite substantial in other situations. This paper undertakes a study of the effect of the lower limit of integration on the calculation of the permeability function. Comparisons are made between starting the integration from various values below the AEV and starting the integration from the calculated air-entry value, ψ_{aev} . A mathematical algorithm is also proposed for the calculation of the AEV for integration purposes. The results show that the relative coefficient of permeability can be significantly under-estimated when the lower limit of integration is smaller than the AEV. The recommendation is that the AEV always be used as the lower limit of integration in the Fredlund-Xing-Huang permeability equation.

Key words: Permeability function, unsaturated coefficient of permeability, soil suction, unsaturated soil property functions, air-entry value.

Introduction

The unsaturated coefficient of permeability function is required when modeling saturated-unsaturated soil systems. Direct measurement of the unsaturated permeability function is costly, technically-demanding, and time-consuming. As a result, the measurement of the unsaturated permeability function is reserved for research studies or large projects where substantial risk may be involved. Considerable research has been directed towards the estimation of the unsaturated coefficient of permeability function. There are four categories of models used for the estimation of unsaturated coefficient of permeability functions (Fredlund et al, 2012); namely, i.) empirical models, ii.) statistical models, iii.) correlation models and iv.) regression models. Empirical models and statistical models appear to be most extensively used in geotechnical engineering. The past decades have witnessed a rapid increase in the combined modeling of the saturated-unsaturated portions as a soil continuum (Fredlund et al. 2012). Considerable effort is expended in measuring the saturated coefficient of permeability of each soil layer and then the unsaturated soil permeability functions are generally estimated based on one of the above-mentioned models. Often the numerical modeling is followed by a parametric study or a probabilistic analysis which quantifies the effect of variations in the permeability function on the final outcome of the analysis. In any case, the estimation of the permeability function has become an integral part of assessing the hydraulic soil properties associated with seepage analyses.

Empirical models utilize the similar character of the soil-water characteristic curve, (SWCC), and the permeability function to estimate the unsaturated coefficient of permeability function. The Brooks and Corey (1964) equation is one example of an empirical model. Statistical models make use of the fact that the permeability function and the soil-water characteristic curve are mainly controlled by the pore-size distribution of the soil. Consequently, the permeability function was developed based on the interpretation and application of the SWCC. Childs and

Collis-George (1950), Burdine (1953) and Mualem (1976) are three commonly used integral formulas of relative permeability based on different physical models.

The van Genuchten equation (1980) and the Fredlund and Xing equation (1994) are two well-known mathematical equations for the SWCC. The van Genuchten SWCC equation was introduced into the Burdine (1953) equation and the Mualem (1976) integral formulas to obtain a permeability function. This gave rise to two closed-form solutions for the unsaturated soil permeability equation. The Fredlund and Xing (1994) SWCC equation was also introduced into the Childs and Collis-George (1950) integral formula, yielding an integral solution for the permeability equation. These combinations have given rise to three unsaturated soil permeability functions commonly used in geotechnical engineering. The three methodologies for the relative permeability function are referred to as: i.) the van Genuchten-Burdine (1980) equation, ii.) the van Genuchten-Mualem (1980) equation, and iii.) the Fredlund, Xing and Huang (1994) permeability function. In each of the above cases, the unsaturated soil permeability function is obtained by combining the saturated coefficient of permeability and the relative coefficient of permeability. The Fredlund, Xing and Huang (1994) permeability function has the advantage that the integral permeability function retains the independence of the SWCC fitting variables when estimating the coefficients of permeability. On the other hand, the van Genuchten permeability functions are closed-form and simpler to use in engineering practice.

The original relative permeability theory published by Fredlund et al., (1994) specified the air-entry value, ψ_{aeV} , as the lower limit of the integration. However, implementations in engineering practice appear to have used other values between zero and ψ_{aeV} as the starting point of integration when calculating the relative coefficient of permeability. It does not appear that any study has been undertaken to assess whether the choice for the lower limit of integration influences the calculation of the Fredlund, Xing and Huang (1994) permeability function.

This paper investigates the error caused by using various values for the lower limit of integration. The effect of the lower limit of integration is examined in terms of the effect of each of the SWCC fitting parameters, (i.e., " a_f , n_f , m_f , m_f' "), on the resulting error. An empirical procedure for the determination of the air-entry value is also described. The definition of the "permeability error" is described followed by a study of the impact of the fitting parameters on the magnitude of the error in the permeability function.

Determination of the Air-Entry Value (AEV) from the Degree of Saturation SWCC, (S-SWCC)

The SWCC for a soil is defined as the relationship between the water content and soil suction (Williams 1982), and is commonly used as the basis for the estimation of unsaturated soil properties (e.g., the permeability function for an unsaturated soil). Different designations for the amount of water in the soil generate different forms of SWCC, such as gravimetric water content SWCC, volumetric water content SWCC, instantaneous volumetric water content SWCC, and degree of saturation SWCC. The volumetric water content is the water content with the volume of water referenced to the original total volume of the soil specimen. The instantaneous volumetric water content is the water content with the volume of water referenced to the instantaneous total volume of the soil specimen. Each form of the SWCC provides similar information to the geotechnical engineer if the soil does not undergo volume change as soil suction is increased. When soil undergoes volume change, as is the case for soft clays and slurry soils, the gravimetric water content SWCC, instantaneous volumetric water content SWCC and degree of saturation SWCC are distinctly different from one another. Volumetric water content SWCC is not of significance when soil undergoes high volume change. Conventional permeability functions (e.g., Fredlund et al. (1994) equation; van Genuchten-Burdine equation; van Genuchten-Mualem equation) produce reasonable estimations using

volumetric water content SWCC when there is no volume change during drying. Volumetric water content SWCC is no longer appropriate in the estimation of the relative permeability function when soil undergoes volume change. It is important to know that the relative coefficient of permeability function, as well as the air-entry value must be estimated from degree of saturation SWCC (Fredlund et al. 2011). This paper uses the degree of saturation SWCC to calculate the appropriate estimation of the relative permeability function.

Various forms of mathematical equations have been suggested to characterize the SWCC. The equation proposed by Fredlund and Xing (1994) has been shown to have sufficient flexibility to best-fit laboratory data reasonably well over the entire soil suction range from near zero to 10^6 kPa provided the material behaves in a mono-modal manner. The form of the Fredlund and Xing (1994) equation written in terms of degree of saturation, (i.e., S-SWCC) is shown in equation (1).

$$S(\psi) = \frac{S_0 \left(1 - \ln(1 + \psi/\psi_r) / \ln(1 + 10^6/\psi_r) \right)}{\left(\ln \left(\exp(1) + (\psi/a_f)^{n_f} \right) \right)^{m_f}} \quad (1)$$

where,

ψ = soil suction;

$S(\psi)$ = degree of saturation at a soil suction of ψ ;

S_0 = initial degree of saturation at zero soil suction; and

a_f, n_f, m_f, ψ_r = four best-fitting parameters controlling the shape of the SWCC.

The shape of the SWCC (e.g., described by the air-entry value, the slope, the residual conditions) are influenced by the four fitting parameters, (i.e., a_f, n_f, m_f and ψ_r) in a combined

and complex manner. There is no simple one-on-one connection between the fitting parameters and the features of the curve, although a_f affects the air-entry value in a significant way, while n_f significantly influences the slope of SWCC. Bharat and Sharma, (2012) studied the validity limits of the Fredlund-Xing parameters and found that small values of ψ_r influenced the SWCC near saturation and m_f also influenced the residual portion of the SWCC. In other words, these variables affect the shape of an SWCC in a coupled manner.

The air-entry value, (AEV), of the soil is the suction at which air begins to enter the largest pores in the soil (Fredlund and Xing, 1994). Vanapalli et al., (1998) proposed an empirical, graphical construction technique to estimate the air-entry value from the SWCC. The air-entry value must be determined from the degree of saturation SWCC (Fredlund et al. 2011).

A mathematical algorithm is proposed in this paper for the determination of the AEV based on the graphical construction suggested by Vanapalli et al. (1998). The following steps are outlined with respect to the analysis for the AEV.

Step 1. Find the best-fitting SWCC for the degree of saturation SWCC using the Fredlund and Xing (1994) equation (Figure 1).

Step 2. Through use of a variable substitution technique, the Fredlund-Xing (1994) best-fitting equation can be transformed into a substitution equation (i.e., Eq. 2). The substitution equation describes the relationship between the degree of saturation and the logarithm of soil suction to the base 10 (Figure 2). The shape of the curve for the substitution equation plotted using arithmetic coordinates is the same as the shape of the curve for the best-fitting equation plotted

using a semi-logarithmic coordinate system. The arithmetic plot of the substitution equation has the same inflection point as the semi-logarithmic plot of the best-fitting equation.

$$SS(\xi) = \frac{S_0 \left(1 - \ln \left(1 + 10^{\xi} / \psi_r \right) \right) / \ln \left(1 + 10^6 / \psi_r \right)}{\left(\ln \left(\exp(1) + \left(10^{\xi} / a_f \right)^{n_f} \right) \right)^{m_f}} \quad (2)$$

where,

$\xi = \log_{10}(\psi)$; $SS(\xi)$ = the degree of saturation at a soil suction of ψ ;

ψ = soil suction.

Step 3. Determine the point of maximum slope (or the inflection point) on the arithmetic plot of the substitution equation. The point of maximum slope is also a point of zero curvature. Therefore, the second derivative of equation (2) can be set equal to zero as shown in equation (3).

$$\frac{d^2 SS(\xi)}{d\xi^2} = 0 \quad (3)$$

Solving equation (3) for the ξ value of zero curvature point and substituting the ξ value into equation (2) yields the corresponding term, $SS(\xi)$. The determined point of zero curvature has coordinates $(\xi_i, SS(\xi_i))$ (Figure 2).

Step 4. Draw a line tangent to the curve through the point of maximum slope (Figure 2). The point of maximum slope is $(\xi_i, SS(\xi_i))$ and the maximum slope is $SS'(\xi_i)$. The equation for the tangent line is as shown in equation (4).

$$TL(\xi) = SS'(\xi_i)(\xi - \xi_i) + SS(\xi_i) \quad (4)$$

Where, $TL(\xi)$ represents the function of the tangent line.

Step 5. Draw a horizontal line through the maximum degree of saturation. The intersection of the two lines indicates the air-entry value (Figure 2). The horizontal line is given by equation (5).

$$HL(\xi) = S_0 \quad (5)$$

Where, $HL(\xi)$ represents the function of the horizontal line.

The intersection point can be obtained mathematically by solving equations (5) and (4). The

intersection point is, $\left(\frac{S_0 - SS(\xi_i)}{SS'(\xi_i)} + \xi_i, S_0 \right)$, on the arithmetic plot.

Step 6. Back calculate the AEV through use of the relationship, $\xi = \log_{10}(\psi)$. The air-entry value for the soil can be written as follows.

$$\psi_{AEV} = 10^{\frac{S_0 - SS(\xi_i)}{SS'(\xi_i)} + \xi_i} \quad (6)$$

Statement of the Integration Problem Associated with the Fredlund, Xing and Huang (1994) Permeability Function

Fredlund, Xing and Huang (1994) suggested a mathematical function for the estimation of the relative coefficient of permeability based on a physical model proposed by Childs and Collis-George (1950) (see equation (7)).

$$k_r^s(\psi) = \frac{\int_{\ln(\psi)}^b \frac{S(e^y) - S(\psi)}{e^y} S'(e^y) dy}{\int_{\ln(\psi_{aev})}^b \frac{S(e^y) - S(\psi_{aev})}{e^y} S'(e^y) dy} \quad (7)$$

Where,

$k_r^s(\psi)$ = relative coefficient of permeability at soil suction of ψ . The superscript s means that the degree of saturation-SWCC is used for the estimation of the relative permeability in Equation (7);

b = upper limit of integration [i.e., $\ln(1000000)$];

y = dummy variable of integration representing the logarithm of suction;

S = degree of saturation-SWCC equation;

S' = derivative of the degree of saturation-SWCC equation;

e^y = natural number raised to the dummy variable power.

The denominator of equation (7) is an integral, the lower limit of the integration of which is the air-entry value, ψ_{aev} . Although the original theory (Fredlund et al, 1994) specified the air-entry value, ψ_{aev} , as the lower limit of integration, other values between a value close to zero and ψ_{aev} have been used as the starting point for integration while estimating the relative permeability function. The arbitrarily selected small value for the starting point of integration appears to have been used because no closed-form analytical procedure had been proposed for the calculation of the AEV. Details on how the integration using Fredlund et al (1994) permeability is to be carried out can be found in the original paper. In addition, the importance of using the degree of saturation SWCC for calculating the permeability function has not been clearly emphasized in the research literature.

If a suction value ψ_i between (near) zero and ψ_{aev} is used as the lower limit of integration, the permeability function of equation (7) takes on the form shown in equation (8)

$$k_{ri}^S(\psi) = \frac{\int_{\ln(\psi)}^b \frac{S(e^y) - S(\psi)}{e^y} S'(e^y) dy}{\int_{\ln(\psi_i)}^b \frac{S(e^y) - S(\psi_i)}{e^y} S'(e^y) dy} \quad (8)$$

where, $k_{ri}^S(\psi)$ = relative coefficient of permeability at soil suction of ψ , when a suction value ψ_i is used as the lower limit of integration for the integral in the denominator of the Equation (8).

Childs and Collis-George (1950) proposed the use of a statistical model. There are three common assumptions for a methodology characterizing the statistical models: (a) The porous medium may be regarded as a set of interconnected pores randomly distributed in the sample. The pores are characterized by their length scale called "the pore radius". (b) The Hagen-Poiseuille equation is assumed valid at the level of the single pore and thus used to estimate the hydraulic conductivity of the elementary pore unit. The total hydraulic conductivity has to be determined by integration over the contributions of the filled pores. (c) The soil-water characteristic curve is considered analogous to the pore radius distribution function. The capillary law is used to uniquely relate the pore radius to the capillary head (Mualem, 1986). The air-entry value of the soil corresponds to the largest pore radius. The change of the lower limit integration implies a change in the largest pore radius of the soil and thus the change in the pore radius distribution function.

The relative coefficient of permeability obtained using equation (7) is theoretically correct and is used as the reference value in the present study. An error in the estimation of the relative permeability is introduced when using equation (8) along with a variety of the lower limits of integration in the denominator. The slope on the soil-water characteristic curve, (SWCC), prior to the AEV (as defined by the degree of saturation SWCC), contributes to the error in the computed permeability function.

The Fredlund, Xing and Huang (1994) permeability function was developed based on the interpretation of the SWCC. Figure 3 illustrates a situation where the effect of the starting point for integration is small. Starting integration at any point from 0.1 kPa to the AEV results in the computation of essentially the same relative permeability function. Figure 4, on the other hand, shows how the starting point for integration can have a significant effect on the computed

permeability function. The difference between the results shown in Figures 3 and 4 appears to be mainly due to a change in the n_f (or steepness of the SWCC) variable.

The shape of the SWCC greatly influences the errors that could be caused in the estimation results for the permeability function. Therefore, it is important to study the effect of each of the four fitting parameters a_f, n_f, m_f, ψ_r on the errors in the permeability function that is introduced by using a small value as the lower limit of integration. The objective of this paper is to examine the effect of each of the fitting variables, a_f, n_f, m_f, ψ_r on errors in the relative permeability function that is caused by using various small values for the lower limit of integration.

Definition of the Error Introduced by Using an Inappropriate Lower Limit of Integration

$ERR(\psi, \psi_i)$ is the mathematical function used to quantify the error introduced as a result of selecting various values for the lower limit of integration. More specifically, it is the change in permeability introduced by using equation (8) with a lower limit of integration other than the air-entry value, (AEV), in the denominator. The comparison is made to the permeability obtained when using equation (7) with the AEV as the lower limit of integration in the integral of denominator. The mathematical form of the error $ERR(\psi, \psi_i)$ is given by equation (9).

$$ERR(\psi, \psi_i) = \begin{cases} 0 & , 0 < \psi < \psi_i < \psi_{aev} \\ \text{Log}_{10} \left(\frac{\int_{\ln(\psi_i)}^b \frac{S(e^y) - S(\psi_i)}{e^y} S'(e^y) dy}{\int_{\ln(\psi)}^b \frac{S(e^y) - S(\psi)}{e^y} S'(e^y) dy} \right) & , 0 < \psi_i \leq \psi \leq \psi_{aev} \\ \text{Log}_{10} \left(\frac{\int_{\ln(\psi_i)}^b \frac{S(e^y) - S(\psi_i)}{e^y} S'(e^y) dy}{\int_{\ln(\psi_{aev})}^b \frac{S(e^y) - S(\psi_{aev})}{e^y} S'(e^y) dy} \right) & , 0 < \psi_i < \psi_{aev} < \psi \end{cases} \quad (9)$$

The error is defined in terms of orders of magnitude. $ERR(\psi, \psi_i)$ in equation (9) is the common logarithm of the ratio of the permeability at any soil suction, ψ , estimated by equation (7) to the permeability estimated by equation (8) with ψ_i set at various lower limits of integration in the denominator. The lower limit of integration ψ_i in equation (8) is a suction value between the AEV and a lower suction. The definition of $ERR(\psi, \psi_i)$ implies that the closer the value, $ERR(\psi, \psi_i)$ is to 0, the smaller the error. The error $ERR(\psi, \psi_i)$ remains at a constant value equal to $ERR(AEV, \psi_i)$ for soil suctions greater than the AEV. $ERR(AEV, \psi_i)$ is the upper bound of the error $ERR(\psi, \psi_i)$ and it represents the largest error across the entire soil suction range when using various ψ_i values as the lower limit of integration rather than the AEV in equation (8) when calculating the relative permeability. The error at the AEV, $ERR(AEV, \psi_i)$ rather than the error, $ERR(\psi, \psi_i)$, across the entire suction range is studied in a parametric manner. Figure 5 illustrates the meaning of the error at the AEV, $ERR(AEV, \psi_i)$ in terms of orders of magnitude caused by using ψ_i as the lower limit of integration in equation (8).

The sensitivity of $ERR(AEV, \psi_i)$ to changes in the best-fitting parameters, “ a_f , n_f , m_f , ψ_r ” for the SWCC

A parametric study was undertaken to investigate the empirical relationships between the best-fitting parameters “ a_f , n_f , m_f , ψ_r ” of the SWCC and the error of $ERR(\psi, \psi_i)$, associated with the Fredlund et al. (1994) permeability function. The error at the AEV, $ERR(AEV, \psi_i)$ rather than the error, $ERR(\psi, \psi_i)$, across the entire suction range is studied for simplification. Table 1 summarizes the parametric study in matrix form.

The influence of the n_f value on $ERR(AEV, \psi_i)$

The sensitivity of the error in the permeability function at the AEV to the change of the n_f value on the SWCC is studied for permeability functions obtained using equation (8). Seven different lower limits of integration ψ_i were selected for the integral in the denominator. These seven different lower limits of integration are 10, 4, 2, 1, 0.5, 0.2 and 0.1 Log_{10} cycles less than the empirical AEV. Figure 6 and Figure 8 show the errors in the estimation of the relative coefficient of permeability caused by using various lower limits of integration when $a_f = 10$ kPa and $a_f = 100$ kPa, respectively (Note: $n_f = 1$; $m_f = 1$; $\psi_r = 2000$ kPa).

The empirical relationships between $ERR(AEV, \psi_i)$ and the corresponding n_f value for various ψ_i are plotted in Figure 7 and Figure 9. A value of 1 on the vertical coordinate refers to one order of magnitude change in the coefficient of permeability at the AEV, and a value of 4 would mean 4 orders of magnitude. Figure 7 reveals the influence of n_f on the errors when $a_f = 10$ kPa; $m_f = 1$; $\psi_r = 2000$ kPa. Figure 9 shows the influence of n_f on the errors when $a_f = 100$ kPa; $m_f = 1$; $\psi_r = 2000$ kPa. In Figures 6 to 9, the curve denoted by SP1 in the legend is related to the

error caused by using a value 4 Log_{10} cycles less than the empirical AEV as the lower limit of integration. The curve denoted by SP2 in the legend is for the error caused by using a lower limit of integration that is 2 Log_{10} cycles less than the empirical AEV. The other notations (e.g., SP3, SP4, SP5 and SP6) can be interpreted in the same way as interpreted for SP1 and SP2.

Figures 7 and 8 show a similar pattern when different a_f values are selected. The results show the errors in the estimation of the relative permeability at the AEV when using equation (8) with different lower limits of integration ψ_i instead of using equation (7) with the AEV as the starting integration point. The results in Figures 7 and Figure 8 reveal that the error decreases with an increase in the n_f value, particularly when the n_f value is smaller than 2. The slope of the change of the error versus the n_f value becomes much steeper at small n_f values. This is particularly true for errors caused by using a lower limit of integration which is beyond 2 Log_{10} cycles less than the AEV. The results also show that using a value of more Log_{10} cycles separated from the AEV as the lower limit of integration produces a greater error in the estimated permeability function for a particular SWCC. This phenomenon is more apparent when the n_f value is smaller than 2. In this case, the estimated relative permeability is significantly influenced by the selected lower limit of integration for a particular SWCC. It is important to use the correct lower limit of integration (i.e., the computed AEV), in the estimation of the permeability function

Figures 10 present errors caused by using a lower limit of integration of 4 Log_{10} cycles less than the AEV for permeability functions obtained from SWCCs with various a_f values. The purpose of arranging the results in this manner is to show how the a_f value affects the error in the estimation of the permeability function when an inappropriate lower limit of integration is used. The starting point for integration is denoted in terms of the Log_{10} cycles less than the AEV. It was found that the a_f value does not have much influence on the error caused by using the inappropriate lower limit of integration. However, the error is more sensitive to the a_f value when

it is combined with small n_f values. Table 2 presents the range of the magnitude of the error in the estimation of permeability when the n_f value changes from 0.5 to 12, with $m_f = 1$ and $\psi_f = 2000$ kPa. The Table shows that if a_f is equal to 1 kPa and the integration starts from a value of 10 Log_{10} cycles less than the AEV, the error would range from 0.1 to 10 orders of magnitude when the n_f value changes from 0.5 to 12, respectively.

The influence of the m_f value on $ERR(AEV, \psi_i)$

The sensitivity of the error in the permeability function (at the AEV) to changes in the m_f value is studied for permeability functions obtained using equation (8) with various lower limits of integration. The results are shown in Figures 11 to 13. Figures 11 and 12 are for different a_f values and show the errors in the estimation of permeability at AEV caused by using equation (8) with different lower limits of integration ψ_i instead of the AEV. Figure 13 presents the errors in a different manner to show the effect of the a_f value on the error in the estimation of the permeability function when an inappropriate lower limit of integration is used. The errors in the comparison at particular m_f value are for permeability functions obtained from SWCCs with varying a_f values.

The results show that the error caused by a lower limit of integration of several Log_{10} cycles less than the AEV does not change much with changing m_f values for the SWCCs. In other words, the m_f value of the SWCC has limited influence on the errors in the estimation of the permeability function that may be caused by a low starting point of integration. The greater difference the lower limit of integration has from the AEV, the larger the error for the permeability function for a particular SWCC. Figure 13 shows also that the influence of the a_f value of the SWCC having on the error is small when " n_f, m_f, ψ_r " are fixed. The smaller the a_f value, the less the error caused by using a lower limit of integration below the AEV. The

influence of the a_f value on the error is relatively apparent at small m_f values. Table 3 shows the range of the magnitude of the error in the estimation of permeability when the m_f value changes from 0.5 to 4 with $n_f = 2$ and $\psi_f = 2000$ kPa.

The influence of the “ ψ_f / a_f ” value on $ERR(AEV, \psi_i)$

The influence of the ψ_f / a_f value on the error in the permeability function at the AEV was also studied using equation (8) with different lower limits of integration. The results show that the magnitude of the error caused by a small value for the lower limit of integration (i.e., $\log_{10}$ cycles less than the AEV) does not significantly change with the ψ_f / a_f value except when the ψ_f / a_f value is smaller than 10. Also, the influence of the a_f value on the error is negligible.

Comparison of the influences of the n_f , m_f , ψ_f / a_f values on the magnitude of the error

The error in terms of orders of magnitude caused by using an inappropriate lower limit of integration that is 10 $\log_{10}$ cycles less than the AEV can vary from 0.1 to 10 when the n_f value changes from 12 to 0.5 with $m_f = 1$, $\psi_f = 2000$ kPa. The change in the magnitude of error is within 0.05 orders of magnitude when the m_f value changes between 0.5 to 4 with $n_f = 2$, $\psi_f = 2000$ kPa. There is change of about 0.5 orders of magnitude in the error when the ψ_f / a_f value changes from 1 to 1000 kPa with $n_f = 2$, $m_f = 1$ kPa.

The analysis reveals that the influence of the n_f on the error caused by using too low a lower limit of integration is much greater than the influence of the m_f and ψ_f / a_f values. The a_f has limited influence on the error. The lower the starting point of integration below the AEV is, the greater the calculation error.

Conclusions

Following is a summary of the conclusions that can be drawn from the study related to the starting point of integration for the Fredlund et al., (1994) permeability function.

- I. If a lower limit of integration used in the integral of Fredlund et al (1994) is smaller than the AEV, the computed results will under-estimate the relative coefficient of permeability. The smaller the value used for the starting point of integration compared to the AEV, the greater will be the difference between the computed results and the relative permeability.
- II. The error caused by using a small value for the lower limit of integration is influenced by the fitting parameters of the Fredlund-Xing (1994) SWCC equation, namely a_f , n_f , m_f , ψ_r . The analysis reveals that the influence of the n_f value is much greater than the influence of a_f , m_f and ψ_r / a_f values.
- III. The difference caused by a particular lower limit of integration, defined in terms of a particular number of Log_{10} cycles less than the AEV, decreases with an increase in the n_f value when the values of a_f , m_f , ψ_r are fixed. This is particularly true when the n_f value is smaller than 2.
- IV. The m_f value for the SWCC has limited influence on the difference in the estimation of the permeability function that may be caused by a low starting point of integration.
- V. The difference in the estimation of the relative coefficient of permeability caused by using a particular low starting point of integration usually does not change much with the change in the a_f value. However, the difference becomes more sensitive to the a_f value when it is combined with a small n_f values and small m_f values.

- VI. It is recommended that the AEV always be used as the lower limit of integration when estimating the relative permeability function with the Fredlund et al., (1994) estimation procedure.
- VII. Further studies regarding the importance of AEV in the estimation of the relative permeability function are recommended to be undertaken where other physical models are used along with other SWCCs.

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LIST OF FIGURES

Figure 1. S-SWCC for a hypothetical soil plotted using semi-logarithmic coordinate

Figure 2. The arithmetical plot of the substitution equation

Figure 3. Relative coefficient of permeability obtained using equation (8) with different lower limits of integration for a soil with $a_f = 500$ kPa, $n_f = 4$, $m_f = 1$, $\psi_r = 10,000$ kPa for SWCC

Figure 4. Relative coefficient of permeability obtained using equation (8) with different lower limits of integration for a soil with $a_f = 500$ kPa, $n_f = 1$, $m_f = 1$, $\psi_r = 10000$ kPa for SWCC

Figure 5. Error at AEV in terms of orders of magnitude caused by using ψ_i equal to 1 kPa as the lower limit of integration in equation (8)

Figure 6. The error in the estimation of the relative coefficient of permeability caused by various lower limits of integration ($a_f = 10$ kPa; $n_f = 1$; $m_f = 1$; $\psi_r = 2000$ kPa)

Figure 7. The influence of n_f on the errors caused by using various lower limits of integration

Figure 8. The error in the estimation of the relative coefficient of permeability caused by various lower limits of integration ($a_f = 100$ kPa; $n_f = 1$; $m_f = 1$; $\psi_r = 2000$ kPa)

Figure 9. The influence of n_f on the errors caused by using various lower limits of integration ($a_f = 100$ kPa; $m_f = 1$; $\psi_r = 2000$ kPa)

Figure 10. The influence of n_f on the errors caused by using a low lower limit of integration of 4 Log_{10} cycles less than the AEV in cases of various a_f values

Figure 11. The influence of m_f on the errors caused by using various lower limits of integration ($a_f = 10$ kPa; $n_f = 2$; $\psi_r = 2000$ kPa)

Figure 12. The influence of m_f on the errors caused by using various lower limits of integration ($a_f = 100$ kPa; $n_f = 2$; $\psi_r = 2000$ kPa)

Figure 13. The influence of m_f on the errors caused by using a low lower limit of integration of 4 Log_{10} cycles less than the AEV in cases of various a_f values

LIST OF TABLES

Table 1. Matrix of fitting parameters used in the parametric study

Table 2. Range of the magnitude of the error in the estimation of permeability when the n_f value changes from 0.5 to 12. ($m_f = 1$; $\psi_f = 2000$ kPa)

Table 3. Range of the magnitude of the error in the estimation of permeability when the m_f value changes from 0.5 to 4. ($n_f = 2$; $\psi_f = 2000$ kPa)

Draft

Table 1. Matrix of fitting parameters used in the parametric study.

Figure #	Designated values				
	n_f	m_f	$a_f(\text{kPa})$	$\psi_r(\text{kPa})$	Lower limit of integration (in terms of Log_{10} cycles less than the AEV)
Figure 6	1	1	10	2000	various
Figure 7	0.5 - 12	1	10	2000	various
Figure 8	1	1	100	2000	various
Figure 9	0.5 - 12	1	100	2000	various
Figure 10	0.5 - 12	1	various	2000	4
Figure 11	2	0.5 - 4	10	2000	various
Figure 12	2	0.5 - 4	100	2000	various
Figure 13	2	0.5 - 4	various	2000	4

Draft

Table 2. Range of the magnitude of the error in the estimation of permeability when the n_f value changes from 0.5 to 12. ($m_f = 1$; $\psi_f = 2000$ kPa)

Integration starts at # Log ₁₀ cycles less than the AEV	Error when a_f is a designated value				
	$a_f = 1$ kPa	$a_f = 10$ kPa	$a_f = 50$ kPa	$a_f = 100$ kPa	$a_f = 200$ kPa
10	0.107~10.424	0.107~10.434	0.110~10.495	0.113~10.527	0.124~10.521
4	0.107~4.418	0.107~4.427	0.109~4.487	0.112~4.519	0.119~4.512
2	0.107~2.367	0.107~2.375	0.109~2.425	0.111~2.451	0.117~2.443
1	0.107~1.262	0.107~1.267	0.109~1.301	0.110~1.316	0.114~1.309
0.5	0.107~0.659	0.107~0.662	0.108~0.681	0.108~0.689	0.110~0.683
0.2	0.106~0.272	0.106~0.273	0.106~0.281	0.105~0.284	0.104~0.281
0.1	0.091~0.137	0.091~0.138	0.090~0.142	0.090~0.143	0.089~0.142

Table 3. Range of the magnitude of the error in the estimation of permeability when the m_f value changes from 0.5 to 4. ($n_f = 2$; $\psi_f = 2000$ kPa)

Integration starts at # Log ₁₀ cycles less than the AEV	Error when a_f is a designated value			
	$a_f = 10$ kPa	$a_f = 50$ kPa	$a_f = 100$ kPa	$a_f = 200$ kPa
10	0.317~0.323	0.321~0.326	0.325~0.339	0.333~0.374
4	0.317~0.323	0.320~0.324	0.324~0.333	0.331~0.353
2	0.317~0.322	0.320~0.324	0.323~0.330	0.329~0.344
1	0.309~0.315	0.311~0.316	0.313~0.317	0.317~0.324
0.5	0.262~0.270	0.263~0.270	0.264~0.271	0.266~0.271
0.2	0.156~0.164	0.156~0.164	0.156~0.164	0.156~0.164
0.1	0.090~0.096	0.090~0.096	0.090~0.096	0.089~0.096

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Figure 1. S-SWCC for a hypothetical soil plotted using semi-logarithmic coordinate

177x111mm (96 x 96 DPI)

Figure 2. The arithmetical plot of the substitution equation

182x126mm (96 x 96 DPI)

Figure 3. Relative coefficient of permeability obtained using equation (8) with different lower limits of integration for a soil with $a_r = 500$ kPa, $n_r = 4$, $m_r = 1$, $\psi_r = 10000$ kPa for SWCC.

180x125mm (96 x 96 DPI)

Figure 4. Relative coefficient of permeability obtained using equation (8) with different lower limits of integration for a soil with $a_f= 500$ kPa, $n_f= 1$, $m_f= 1$, $\psi_r= 10000$ kPa for SWCC

181x126mm (96 x 96 DPI)

Figure 5. Error at AEV in terms of orders of magnitude caused by using ψ_i equal to 1 kPa as the lower limit of integration in equation (8).

181x127mm (96 x 96 DPI)

Figure 6. The error in the estimation of the relative coefficient of permeability caused by various lower limits of integration ($a_f = 10$ kPa; $n_f = 1$; $m_f = 1$; $\psi_r = 2000$ kPa).

180x127mm (96 x 96 DPI)

Figure 7. The influence of n_f on the errors caused by using various lower limits of integration ($a_f = 10$ kPa; $m_f = 1$; $\psi_r = 2000$ kPa).

182x126mm (96 x 96 DPI)

Figure 8. The error in the estimation of the relative coefficient of permeability caused by various lower limits of integration ($a_f = 100$ kPa; $n_f = 1$; $m_f = 1$; $\psi_r = 2000$ kPa).

181x127mm (96 x 96 DPI)

Figure 9. The influence of n_f on the errors caused by using various lower limits of integration ($a_f = 100$ kPa; $m_f = 1$; $\psi_r = 2000$ kPa).

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Figure 10. The influence of n_f on the errors caused by using a low lower limit of integration of 4 Log_{10} cycles less than the AEV in cases of various a_f values.

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Figure 11. The influence of m_f on the errors caused by using various lower limits of integration ($a_f = 10 \text{ kPa}$; $n_f = 2$; $\psi_f = 2000 \text{ kPa}$).

181x126mm (96 x 96 DPI)

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Figure 13. The influence of m_f on the errors caused by using a low lower limit of integration of 4 Log₁₀ cycles less than the AEV in cases of various a_f values.

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