Canadian Journal of Physics
Revue canadienne de physique

Cross-section for (nearly) everyone

Journal:	<i>Canadian Journal of Physics</i>
Manuscript ID	cjp-2015-0344.R1
Manuscript Type:	Article
Date Submitted by the Author:	24-Jun-2015
Complete List of Authors:	Cavinato, Michela; Università degli Studi di Milano, Physics Giliberti, Marco; Università degli Studi di Milano, Physics Perotti, Luca; Texas Southern University, Physics
Keyword:	Total cross-section, Differential cross-section, Perfectly reflecting surfaces, Education, Undergraduate students

SCHOLARONE™
Manuscripts

Only

Cross-section for (nearly) everyone

Michela Cavinato, Marco Giliberti, and Luca Perotti

Abstract: Both total and differential cross-sections for the reflection of light beams off finite solid surfaces obtained by the rotation of a generic derivable convex function are presented. The calculations are developed using elementary notions of mathematics and are therefore suitable for introductory undergraduate university physics courses. Three particular cases are discussed as examples of the general procedure and of the possible considerations about cross-sections in general and the physical properties of specific systems.

PACS Nos.: 01.40.Gm, 01.40.Ej

1. Introduction

Cross-section is a fundamental conceptual tool in modern physics. Nevertheless, except that in physics courses for physicists, this important concept is generally not presented, even in scientific curricula. On the contrary, we believe that at least a first approach to the meaning of cross-section is very important **for the understanding of most of the physics experiments of the last century; understanding which is fundamental for the scientific education of physicists and research oriented chemists, material scientists and some engineers.** As the “effective surface” the target presents to the probing radiation, be it material or electromagnetic, the cross-section contains all the information about the nature of the interactions between probes and target we can extract from the experiment, and which can be used to derive information on the target structure in the inverse scattering problem.

Examples useful to clarify the geometrical and physical meaning of cross-section, if given, are generally very few: typically scattering off rigid spheres, which has the drawback of being a very special case of isotropic scattering. Moreover, the first taste of this concept usually comes to undergraduate students in physics in the case of the Geiger-Marsden experiment on the Rutherford scattering that is itself a very special case, in that, as the Coulomb potential is a long-range one, its total cross section is infinite.

In the following, we present an approach where the concept of cross section is introduced through the study of the reflection (according to the rules of geometrical optics) of light beams off surfaces having rotational symmetry.

Based on simple geometrical considerations, this approach allows an explicit calculation of several cross-sections with very little effort, namely that of finding the inverse function of a derivative (a similar approach, but within the frame of particle mechanics, and using a different procedure, can be found in [1]). The mathematics required is a standard subject in most science university courses **and therefore is not, at least in theory, an obstacle to the presentation of this subject to students attending such courses.** Intuitive geometrical examples as the ones here proposed can be both interesting and useful in

fixing in students’ mind both the basic concepts and the difficulties, such as for example the fact that the inverse scattering problem is an arduous, and often also an ill-posed one ¹.

The paper is organized as follows: Sect. 2 and Sect. 3 introduce the concepts of total and of differential cross-sections, respectively; Sect. 4 outlines the proposed approach to the calculation of the differential cross-section for a particular class of interactions; Sect. 5 presents some particular significant cases. A discussion of our approach and a summary on the advantages of the proposed path are presented in Sect. 6.

2. The concept of total cross-section

Within the framework of geometrical optics, let us suppose we have a target made only of equal perfectly reflecting finite surfaces, held fixed at some points of space (see Fig. 1). Let us imagine to shoot orthogonally at it a well collimated beam of light, of section S smaller than the section of the layer, which is reflected by the individual surfaces.

The fraction of the incident beam that is deflected (that is the ratio between the scattered energy flux and the incoming energy flux) depends:

- a) on the number of reflecting surfaces per unit volume, that is on their density n ;
- b) on the thickness h of the target;
- c) and on the section σ_T that each surface shows to the beam.

This last quantity is what is called the total cross-section for the beam-target interaction.

Let us suppose, for simplicity, that:

- i) the beam section S is much greater than both σ_T and of the square of the mean distance between the surfaces;
- ii) the individual surfaces are sufficiently spaced from each other and h is small enough, so that $nh\sigma_T \ll 1$ and the probability that one surface covers another is negligible (thin target approximation).

In this case, the total area shown to the beam by the surfaces contained in the volume of base S and height h is $nSh\sigma_T$; and the fraction P of the beam hitting the surfaces is given by the ratio between this area and the beam section S

$$P = \frac{nSh\sigma_T}{S} = nh\sigma_T \quad (1)$$

Michela Cavinato and Marco Giliberti. Dipartimento di Fisica dell’Università di Milano, via Celoria 16, 20133 Milano Italy.

Luca Perotti. Department of Physics, Texas Southern University, Houston, Texas 77004 USA.

¹ A simple example for the students is given by the Rutherford scattering itself: repulsive and attractive Coulomb potentials give the same cross section.

Provided the target is thin, in the sense of the approximation ii), (1) may be regarded as a general definition of the total cross-section σ_T .

In the following we always assume approximation ii) to be verified. Moreover, for simplicity, we consider individual surfaces having axial symmetry and assume the beam to be shot at them along the direction of their axis.

3. The concept of differential cross section

We can analyze the scattering process in more detail. For instance we can wonder which is the fraction of the incoming beam that is deflected at a given angle. To answer this question we first study the fraction of the incident beam that is scattered at angles greater than a given one.

We can start considering a beam ray hitting one of the surfaces, at a distance b from the axis of the surface (b is called impact parameter). The ray will be deflected according to the reflection laws: the incident ray, the perpendicular to the reflecting surface at the reflection point and the reflected ray all lie in the same plane and the incidence angle is equal to the reflection angle. Let us call ϕ the scattering angle, that is the angle of deviation from the incident beam direction caused by the reflection.

As can be seen in Fig. 2, if the surface is convex, the smaller the impact parameter b the larger the scattering angle ϕ . That means that all the rays within a disc of area πb^2 , centered on the target axis, and perpendicular to the beam direction, will suffer a scattering angle greater than ϕ . Therefore the fraction of the incoming beam which is scattered through an angle greater than ϕ is

$$P_{>\phi} = nh\pi b^2 \quad (2)$$

In other words and keeping in mind (1), we can define the cross-section for scattering through an angle greater than ϕ as follows

$$\sigma_{>\phi} = \pi b^2 \quad (3)$$

We observe that (3) is not physically useful in that it relates the cross-section to the impact parameter b which, in a scattering experiment, where the positions of the individual surfaces are not known, is not a measurable quantity. Nonetheless this equation will be of great importance for the next considerations.

We now further refine the question to: what fraction of the beam is scattered through an angle between ϕ and $\phi + d\phi$?

The rays scattered through an angle between ϕ and $\phi + d\phi$ are given by those deflected through an angle greater than ϕ minus those deflected more than $\phi + d\phi$. These are the rays that hit the fixed reflecting surface on a section of area $|d\sigma| = \sigma_{>\phi} - \sigma_{>(\phi+d\phi)}$. The required fraction is then

$$dP(\phi) = nh|d\sigma| \quad (4)$$

After the scattering, these rays are contained into a solid angle of amplitude $d\Omega$ given by the ratio between the area of the spherical zone Z of Fig. 3 and r^2 , that is

$$d\Omega = \frac{2\pi(r \sin \phi)rd\phi}{r^2} = 2\pi \sin \phi d\phi \quad (5)$$

and therefore the beam fraction scattered around the angle ϕ , per unit solid angle, is

$$P(\phi) = nh \frac{|d\sigma|}{d\Omega} \quad (6)$$

Equation (6) is of general interest and it is valid for all scattering experiments in the thin target approximation.

The fundamental quantity $|d\sigma|/d\Omega$ is called the differential cross-section. To better understand its physical meaning we can extend our description of the experiment to include the detection process. Consider a beam shot against a fixed target and an ideal detector of effective section A , located at an angle ϕ at a distance R from the target and perpendicular to the scattered beam direction. In this way it detects all the radiation in the solid angle of amplitude A/R^2 . The fraction of scattered radiation contained between ϕ and $\phi + d\phi$ that is collected by the detector is given by

$$P_d(\phi) = \frac{A}{R^2} \frac{1}{d\Omega} dP(\phi) = nh \frac{|d\sigma|}{d\Omega} \frac{A}{R^2} \quad (7)$$

Equation (7) shows a clear way of calculating the differential cross-section from given experimental measures and a comparison between (6) and (7) helps students enlighten the conceptual meaning of this useful quantity.

4. Detailed calculation

Let us consider the reflecting surface produced by the complete rotation around the y axis of the increasing convex function $y = f(x)$ with x between 0 and a (Fig. 2). Our aim is to calculate the total and differential cross-sections for the reflection of a beam, incident on this fixed surface in the direction of the y axis. The calculation of the total cross-section

$$\sigma_T = \pi a^2 \quad (8)$$

is straightforward. For what concerns the differential cross-section we make reference to Fig. 2: since the incident ray, the perpendicular to the surface and the reflection ray lie all on the same plane, the problem can be considered in the $x - y$ plane. Let b be the impact parameter, j the line of incidence, t the tangent to the curve in the collision point, p the perpendicular, and d the line of reflection. Now j is orthogonal to the x axis and p is orthogonal to t , therefore the angle α , between t and the x axis, is equal to the angle i between j and p , which is the incidence angle.

The steps to obtain the differential cross-section are the following:

1. Express b as a function of ϕ . This can be done solving for b the equation

$$\left. \frac{df}{dx} \right|_{x=b} = \tan \alpha = \tan i = \tan \left(\frac{\pi - \phi}{2} \right) = \cot \frac{\phi}{2} \quad (9)$$

which can be easily found looking at Fig. 2.

2. Find the extremes of ϕ from the obvious geometrical limitation $0 \leq b \leq a$ that are due to the finite size of the reflecting surface. This can be performed by substituting for b in equation (9) the value 0 to find the maximum value of ϕ and the value a to find the minimum value of ϕ .

3. Calculate $\sigma(\phi) = \pi b^2$.
4. Differentiate $\sigma(\phi)$ to obtain $|d\sigma(\phi)|$. The absolute value is due to the fact that the maximum of ϕ corresponds to the minimum of b and viceversa.
5. Divide the found expression by $d\Omega$ given by (5).

5. Three examples

We present here, as examples of the previous procedure, three cases we think significant, together with sample discussions of some key points of the theory.

5.1. Ellipsoids, spheres and hyperboloids

Let us consider the ellipsoid obtained by the rotation, around the y axis, of the function shown in Fig. 4a)

$$f(x) = -c\sqrt{1 - \left(\frac{x}{a}\right)^2}, \quad 0 \leq x \leq a \quad (10)$$

with a and c the two semi-axes of the generating ellipse (this example is discussed in detail in [2]).

From step 1. we get

$$b^2 = a^2 \frac{\cot^2\left(\frac{\phi}{2}\right)}{\left(\frac{c}{a}\right)^2 + \cot^2\left(\frac{\phi}{2}\right)} \quad (11)$$

and then (steps 2. to 5.)

$$\begin{cases} \frac{|d\sigma(\phi)|}{d\Omega} = \frac{a^2}{4} \left(\frac{ac}{c^2 \sin^2\left(\frac{\phi}{2}\right) + a^2 \cos^2\left(\frac{\phi}{2}\right)} \right)^2 \\ 0 \leq \phi \leq \pi \end{cases} \quad (12)$$

From the obvious relations: $\int d\Omega = 4\pi$ and $\int \frac{d\sigma}{d\Omega} d\Omega = \sigma_T$, where both integrals are on the whole solid angle (i.e. on all directions) and from (12), one gets immediately the trivial result $\sigma_T = \pi a^2$.

Changing the ratio c/a for the ellipsoid, changes the angular distribution of the scattered rays. In particular, for $c = a$, (12) reduces the well known expression of the differential cross-section of a rigid sphere [3] $|d\sigma(\phi)|/d\Omega = a^2/4$, which depends only on the sphere radius a , which is a constant, but is independent of ϕ . This means that after the collision with a sphere the rays are isotropically scattered. More in general, $|d\sigma(\phi)|/d\Omega$ is strictly decreasing for $c > a$ and increasing for $c < a$.

A simple change of signs in (10) gives the equation of the hyperbola (shown in Fig. 4b))

$$f(x) = c\sqrt{1 + \left(\frac{x}{\tilde{a}}\right)^2}, \quad 0 \leq x \leq a \quad (13)$$

which results first in

$$b^2 = \tilde{a}^2 \frac{\cot^2\left(\frac{\phi}{2}\right)}{\left(\frac{c}{\tilde{a}}\right)^2 - \cot^2\left(\frac{\phi}{2}\right)} \quad (14)$$

and finally in the cross-section

$$\begin{cases} \frac{|d\sigma(\phi)|}{d\Omega} = \frac{\tilde{a}^2}{4} \left(\frac{\tilde{a}c}{c^2 \sin^2\left(\frac{\phi}{2}\right) - \tilde{a}^2 \cos^2\left(\frac{\phi}{2}\right)} \right)^2 \\ \pi - 2 \arctan\left(\frac{c\tilde{a}}{\tilde{a}} \frac{1}{\sqrt{\tilde{a}^2 + a^2}}\right) \leq \phi \leq \pi \end{cases} \quad (15)$$

where, as the curve (13) is not bound in x , we had to distinguish between the target size parameter a (introduced as to avoid an infinite total cross-section) and the curve parameter $\tilde{a}$. Since for (14) to be true it must be $c/\tilde{a} > \cot(\phi/2)$, $|d\sigma(\phi)|/d\Omega$ is always strictly decreasing.

In a scattering experiment we do not usually know the *exact* nature and composition of the target, and the aim of the experiment is to infer the missing information, be it the potential for a target composed of force centers, or the generating curve $f(x)$ in the cases here discussed. Direct reconstruction of the unknown scattering potential, or of the generating curve, is an arduous task, on which whole books have been written [4]; on the other hand, partial information on the nature of the target can help reduce the range of possible interactions. This either makes a trial and error approach, such as was used in the interpretation of early scattering experiments, more feasible, or even allows univocal direct solution of the problem [5].

The simplest example one can give, that of isotropic scattering, already allows for the highlight of some necessary subtleties. If in a scattering experiment the detector gives the same reading at all angles, independently from the physical properties of the incident beam, such as, for example, the energy (such a dependence would suggest a target composed of force centers [3]), or the electric charge carried by the beam, we have strong indications that the interaction between the beam and the target is a reflection off the individual surfaces, and that the target is made of rigid spheres, impenetrable by the beam.

It seems a simple and straightforward procedure, but we must note that both the angular dependence *and its range of validity* are important to determine the scattering surface: a differential cross-section which over a given range of angles has the constant value $a^2/4$ can be caused by surfaces generated by any curve having a derivative of the form ²

$$\frac{df}{dx} = \sqrt{\frac{\gamma^2 + x^2}{\beta^2 - x^2}}, \quad 0 \leq x \leq \beta, \quad \gamma^2 + \beta^2 = a^2 \quad (16)$$

but it is zero outside the range $0 \leq \phi \leq \pi - 2 \arctan(|\gamma/\beta|)$, while the differential cross-section for spheres is constant over the whole range $0 \leq \phi \leq \pi$.

More examples of reconstruction of hard reflecting surfaces of revolution from observed cross sections can be found in [1].

5.2. Paraboloids

Let us consider ([6]) the solid obtained by the rotation of the curve shown in Figure 4c)

$$f(x) = \frac{x^2}{c}, \quad 0 \leq x \leq a \quad (17)$$

²Essentially elliptic integrals of the second kind.

Recalling (8), the total cross section is again $\sigma_T = \pi a^2$. For the differential cross-section, step 1. instead gives:

$$b = \frac{c}{2} \cot\left(\frac{\phi}{2}\right) \quad (18)$$

and from the steps 2. to 5. we obtain

$$\begin{cases} \frac{|d\sigma|}{d\Omega} = \frac{\pi c^2}{4} \frac{\cos(\frac{\phi}{2})}{\sin^3(\frac{\phi}{2})} \frac{1}{2\pi \sin \phi} = \frac{c^2}{16} \operatorname{cosec}^4\left(\frac{\phi}{2}\right) \\ \pi - 2 \arctan\left(2\frac{a}{c}\right) \leq \phi \leq \pi \end{cases} \quad (19)$$

Note that contrary to the previous case where the ratio c/a influences the angular distribution of the scattered beam, here c is just a scale factor: the angular distribution is independent from it.

This result shows some similarities with the Rutherford cross-section [7, 8]

$$\frac{|d\sigma_R|}{d\Omega} = \frac{K^2}{16} \operatorname{cosec}^4\left(\frac{\phi}{2}\right), \quad K = \frac{Ze\rho_q}{4\pi\epsilon_0\rho_{K_0}} \quad (20)$$

where Ze is the target charge, ρ_q and ρ_{K_0} are the charge and kinetic energy densities of the incident beam, and ϵ_0 is the vacuum permittivity³. Clearly, the same angular dependence appears; moreover, both cross sections are not valid for small angles, but for very different reasons. In the present case because **the domain of the function defined in (17) is bound by the cutoff in $x = a$, whereas in the Rutherford case the reason is that when the impact parameter is large the Coulomb potential is shielded by the atoms' electronic cloud, which cannot be neglected when the distance of closest approach to the nucleus is large. This causes the experimentally observed total cross-section to be finite, while it is infinite in the theoretical pure point-charge case. As a further remark we note that the experimental Rutherford cross-section has also an energy dependent cutoff at large scattering angles, due to the finite dimension of nuclei.**

How can we then distinguish the two cases? First of all, (19) is only valid when the incidence direction is that of the symmetry axis of the individual targets; a rotation of the target with respect to the incident beam would produce a different result. More important, the differential cross-section given by (19) shows no dependence from the kinetic energy carried by the incident beam (the variable in the Rutherford experiment which can be most easily changed). We would therefore have to imagine different scattering surfaces, **one for each different kinetic energy, obtained by putting $c = K$ in (19), with K given by (20): a highly unlikely circumstance. However, for a rough numerical estimate, we observe that, in the specific and common case of natural radioactive decay alpha particles thrown upon a golden target, the two differential cross-sections would be approximately indistinguishable if, in (17), we set $c = 10^{-13}$ m and $a = 10^{-10}$ m.**

It might be instructive to view the paraboloid as the special case for $n = 2$ of the solids obtained by rotation of the curves

$$f(x) = \frac{x^n}{c}, \quad 0 \leq x \leq a \quad n = 2, 3, \dots \quad (21)$$

³Describing the beam as composed of point-like particles, K itself acquires the meaning of distance of closest approach to the nucleus for head-on collisions.

The differential cross section in this case reads

$$\begin{cases} \frac{|d\sigma|}{d\Omega} = \frac{(\frac{c}{n})^{\frac{2}{n-1}}}{4(n-1)} \left(\frac{1}{\sin^{\frac{n}{n-1}}(\frac{\phi}{2}) \cos^{\frac{n-2}{n-1}}(\frac{\phi}{2})} \right)^2 \\ \pi - 2 \arctan\left[\left(\frac{n}{c}\right)a^{(n-1)}\right] \leq \phi \leq \pi \end{cases} \quad (22)$$

We see that the paraboloid case ($n = 2$) is the only one for which $|d\sigma|/d\Omega$ does not diverge in $\phi = \pi$; on the contrary, for $n > 2$, $|d\sigma|/d\Omega$ does diverge since the curvature of $f(x)$ in $x = 0$ is zero. It is also worth noting that, while $|d\sigma|/d\Omega$ is a strictly decreasing function for $n = 2$, for $n > 2$ it has a minimum in $\phi = 2 \arctan \sqrt{\frac{n}{n-2}}$. For this minimum to be within the ϕ range it must be

$$c < \sqrt{\frac{n^3 a^{(n-1)}}{n-2}}$$

5.3. The arcsin as the symmetric scatterer to the paraboloids

Let us finally consider the surface obtained by the rotation, around the y axis, of the function shown in Fig. 4d)

$$f(x) = \tilde{c} \arcsin \frac{x}{a}, \quad 0 \leq x \leq a \quad (23)$$

From step 1. we get

$$b^2 = a^2 - \tilde{c}^2 \tan^2\left(\frac{\phi}{2}\right) \quad (24)$$

then (steps 2.- 5.)

$$\begin{cases} \frac{|d\sigma|}{d\Omega} = \frac{\tilde{c}^2}{4} \frac{1}{\cos^4(\frac{\phi}{2})} = \frac{\tilde{c}^2}{4} \sec^4\left(\frac{\phi}{2}\right) \\ 0 \leq \phi \leq \pi - 2 \arctan\left(\frac{\tilde{c}}{a}\right) \end{cases} \quad (25)$$

which is a "mirror" image of our previous example of (19), obtained by the substitutions $\phi \rightarrow \pi - \phi$ and $c \rightarrow 2\tilde{c}$.

6. Discussion and Conclusions

The concept of cross-section in the case of reflection off the surface of a solid has been introduced on the basis only of simple statistical and geometrical considerations. This choice has been made because of didactic reasons: with this approach, the intuition is helped by the material existence of both total and differential cross-sections (which are real parts of the rigid surface of the body).

In physically more significant cases, as e. g. the Rutherford scattering itself, the interaction is between the incoming beam and the force field of the target. It is true that in these cases the total cross-section can be geometrically conceived as the effective surface presented by the target force field to the incoming beam and the differential cross section as the part of that surface scattering the beam in a given direction, but this only in an abstract sense, which usually is not easy for the students to grasp. Even in the simple geometrical case of a light beam randomly scattered by a rough surface, the differential cross section, even if mathematically defined, cannot be associated with any geometrically definite part of the scattering surface.

Besides this difficulty, the general approach, in which the scattering is introduced from the beginning in its abstract meaning, often allows students only to discuss existing experimental data, thus missing important relations and theoretical considerations, an understanding of which can be gained by the direct calculation of several simple cross-sections.

In conclusion, this paper gives a simple formula to calculate with little effort many different cross-sections. Even if the formula itself is of limited utility because of the very simple interaction taken into account, at the introductory level, it can help students understand the link between experiments and theoretical explanations⁴.

An extended case study, about the students' understanding of the cross-section concept, following the ideas outlined in the present paper, is under way and will be presented in a forthcoming paper.

References

1. J.L. Brun and A.F. Pacheco. *Eur. J. Phys.* **26**, 747 (2005).
2. F. Corni, M. Michelini, L. Santi, F. Soramel and A. Stefanel. *Proc. Girep Conference Teaching the Science of Condensed Matter and New Materials*, Forum Editrice, Udine, 192 (1996).
3. see e.g.: L.D. Landau and E.M. Lifshits. *Mechanics*, Pergamon, Oxford, 3rd ed., 48 (1976).
4. One of the most recent ones is: K. Chadán and P.C. Sabatier. *Inverse Problems in Quantum Scattering Theory*, Springer, New York (1989), to whose bibliography we refer for earlier works on the subject.
5. See e.g.: J.A. Blackburn and H.J.T. Smith. *Am. J. Phys.* **72**, 237 (2004); A.H. Carter. *Am. J. Phys.* **68**, 698 (2000); S.C. Gandhi and C.J. Efthimiou. *Am. J. Phys.* **74**, 638 (2006).
6. This particular model has already been presented with similar finalities in: J. Evans. *Am. J. Phys.* **56**, 423 (1988).
7. E. Rutherford. *Phil. Mag. series 6*, **xxi**, 669 (1911).
8. W.S.C. Williams. *Nuclear and Particle Physics*, Oxford University Press, Oxford (1992).

Fig. 1. A collimated beam of light is shot against a thin target of perfectly reflecting solid surfaces, each one of total cross-section σ_T , fixed at some points of the space. Most of the beam will be undeflected but part of it will be scattered by the solid surfaces.

Fig. 2. A sharp-edged solid surface is given by the complete rotation around the y axis of the increasing convex function $y = f(x)$ with x between 0 and a . A beam ray, directed along the y axis, is reflected and scattered by the surface through the angle ϕ : b is the impact parameter, t the tangent and p the perpendicular to the surface at the collision point, j and d are the incident and the reflected rays, respectively.

⁴A discussion of other cases of classical and quantum scattering would be didactically interesting, but it is well beyond the aims of this paper.

Fig. 3. The beam rays scattered between ϕ and $\phi + d\phi$ are contained into a solid angle of amplitude given by the ratio between the area $2\pi r^2 \sin \phi d\phi$ of the spherical zone Z and r^2 .

Fig. 4. The curves generating the reflecting solid surfaces of our examples: a) an ellipse; b) an hyperbola; c) a parabola; d) an inverse sine.