Modeling Multibody Frictional Contact Problem in a CANDU Fuel Rod

by

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Abstract

A three-dimensional multibody frictional contact problem in a CANDU nuclear fuel rod is studied in this thesis. To account for the effect of friction on fuel rod deformations, a computational model is developed based on an existing model. For the contact algorithm, a set of complementary equations involving complementarity variables inequality is introduced to deal with the three-dimensional frictional contact. The algorithm is implemented into a computer program. Three cases are studied to show the feasibility and efficiency of the numerical scheme for handling frictional contact among multiple bodies. The application of the research work offers the assessment of deflection as to safety evaluation for the nuclear industry under different scenarios.
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Chapter 1. Introduction

This chapter serves as background information of the CANDU nuclear fuel rod structure and literature review on the frictional contact methodology.

1.1 Background

In a CANDU fuel channel, the fuel bundles are lined up horizontally for operation. Figure 1.1 shows a typical CANDU fuel bundle which contains four rings of concentric fuel rods, two endplates, and three arrays of bearing pads on the outer ring rods. Each fuel rod is consist of three components: a slender hollow Zircaloy sheath, a number of short UO\textsubscript{2} pellets, and two Zircaloy endcaps that are used for sealing the fuel pellets [1]. The scheme of the fuel rod can be understood from Figure 1.2 [2].

![Figure 1.1: Models of a Standard CANDU 6Fuel Bundle and a Fuel Rod attached to an Endplate](image)

During operation, nuclear fission takes place in the fuel pellets and generates the heat. The induced heat is then transferred from the pellets to their surrounding sheath and makes the fuel rods and fuel bundles hot. The high temperatures in the fuel pellets cause the solids to thermally expand in all directions. As a result, all solids would have contact with their neighboring bodies depending on their initial status. With a shape of perfect cylinder for the fuel pellets inside the fuel rod, simulation shows the local stresses and strains around the pellet-to-pellet interfaces after expansion is significantly higher than those close to the pellet mid-plane: this phenomenon is known as the hourglass effect [3]. Serious hourglass effect will shorten the fuel bundle’s service life and can even lead the fuel rod to cracking. In order to reduce the hourglass behavior and avoid the material fail, the fuel pellet is manufactured with dishes around the center line and chamfers at the edges as shown in Figure 1.2. The dish is designed to accommodate the thermal expansion while the chamfer is cut to minimize the chipping effect. Accordingly, the fuel rod is assembled as a porous structure.

The contact problem within a fuel rod is complicated and it is primarily driven by the temperature [3]. Each fuel pellet can be considered as one solid body that has potential axial contact with its two neighbor pellets (except pellets at two ends that has axial contact with their neighbor endcaps); meanwhile, all pellets are designed to have radial contact with their surrounding sheath to maximize the heat transfer. In addition, the pellets are exposed to high fission gas pressure from the holes in the rod under operation. In fact, gaps at different locations in the fuel rod may be closed or open up depending on the operation time and temperature distribution in the system.
Although the contact pressure is determined by the temperature distributions within contact bodies, it also affects the temperature distribution through the thermal contact conductance, where the coefficient of heat transfer is increased by the large contact pressure \[4\]. Therefore, the thermal behavior and the contact phenomenon are coupled in the real situation. To create any application for this purpose, a contact model needs to be developed first.

A numerical model based on the finite element method for a fuel rod structure has been developed by Yu et al. \[2\], where the thermal expansion and the contact without considering potential slippage associated with insufficient frictional force have been studied. The contact problem was modeled with an assumption of no slippages for the nodal contact pairs; in other words, this model handles the problem by assuming presence of sufficient frictional force. However, to make the model be more realistic with the capability to predict the behavior of this system (i.e. bending of the fuel rod and material loss of the pellet from accumulated fretting), a frictional model without the stick assumption is desired. Accordingly, this research develops the frictional contact package based on the existing model.

### 1.2 Literature review

The three-dimensional frictional contact problem represents a branch of nonlinear problems in solid mechanics \[5\]. The nature of contact nonlinearities involves dry friction, unilateral constraint condition, and the nonlinearity of contact materials. Challenges for modeling this problem involve the formulation for the frictional contact conditions and the solution methods.

Dry friction, or the friction between two solid contact bodies, is a joint name of static friction and kinetic friction. The static friction is the friction between two contact faces when they are in a relative static state, while the kinetic friction is the friction with respect to the relative moving status. To enable the two contact bodies changing their relative motion from stationary to moving, a maximum static friction (stiction) needs to be overcome. By contact, a kinetic friction (dynamic frictional force) needs to be overcome between the two relative moving bodies. According to the Coulomb’s law of friction, the coefficient of static friction is defined as the ratio between the maximum possible frictional force before sliding begins and the normal contact
force \( (\mu_s = \frac{F_{\text{stick friction}}}{F_{\text{normal}}}) \), whereas the coefficient of kinetic friction is the ratio of the kinetic friction to the normal contact force when two bodies are relatively moving \( (\mu_k = \frac{F_{\text{sliding friction}}}{F_{\text{normal}}}) \). In general, for a given pair of contact faces, the coefficient of static friction is larger than that of kinetic friction, and the coefficient of kinetic friction is dependent on the sliding velocity [6]. For most materials, the maximum frictional force can only be reached in the moment of shifting the status from sticking to slipping. As this research studies the static contact problem, an assumption of a consistent coefficient of friction is made for simplification so that the Coulomb’s Law of Friction can be directly applied.

The constraint of the friction within an elastic domain can be expressed by \( \|t\| - \mu p_N \leq 0 \) [5], where \( t \) and \( p \) refers to the frictional stress and normal pressure respectively, the subscripts \( T \) and \( N \) indicate the tangential and the normal direction respectively, and the symbol \( \| \| \) represents the magnitude of the vector. Figure 1.3 visualizes the relation between the frictional force and the normal force under the classical Coulomb’s Law.

![Figure 1.3: Coulomb Frictional Cone](image)

The nonlinear constraint function for the frictional force can be replaced by a set of linear constraint functions, which creates a corresponding polyhedral cone to approximate the frictional cone [7]. Usually the research works [7][8][9][10][11] chose the circumscribed polygon (as shown in Figure 1.4) instead of the inscribed polygon to reduce the error range. However, this
approximation enlarges the stick possibility. Our research deals with the Coulomb’s frictional cone as it is and does not modify the stiction or slippage regions.

![Figure 1.4: Four-sized and Eight-sized Polyhedral Approximation of Coulomb’s Frictional Cone](image)

Contact material nonlinearity involves material inhomogeneous, anisotropic, and deformation inelasticity. The inelastic deformation has been studied in elastic-plastic contact [11][12][13][14], where the elasto-plastic constitutive relations in the contact interface are formulated to regularize Coulomb’s Law of Friction and carry experimental observation [5].

According to this theory, the tangential slippage component \( g_f \) can be decomposed into an elastic part \( g_f^e \) and a plastic part \( g_f^p \). The elastic and plastic slippages are defined with respect to the relative stick and slip status between two contact faces. This theory is known as the classical elasto-plastic relation, which was initially concluded from a friction experiment between two platinum surfaces [15]. In our application, a perfect elastic stick and a perfect plastic slip are applied to describe the stick and slip status respectively. Meanwhile, the materials for all the components in the fuel rod are considered as homogeneous. The general methodology in this thesis is developed based on isotropic material assumption, but it is also compatible with anisotropic materials.
Because of the varying temperature distribution in the fuel pellets, contact occurs on their surfaces after thermal expansion. In addition, for a surface-to-surface contact problem, neither the direction of the nodal frictional force nor the slippage direction is known before the solution is obtained. In such situation, specific method has to be performed to deal with the contact search.

Traditionally, researchers use the trial-error procedures to search for the contact between two solids in every time step for the numerical simulation [16][17]. However, to meet the contact condition over the contact surfaces simultaneously, large computation time is required on contact searching since this method is based on explicit algorithms. Therefore, efficient techniques for detecting contact are desired for multibody contact problems.

Some Newton-based iterative methods have been developed and optimized to solve the contact problem without undergoing the trial-error guess [18][19][20]. This type of method is compared with other methods in several test cases, i.e. beam contact example from Li et al.[19], to be the most computational efficient. However, the convergence of iteration is problematic when handling large degree of contact freedoms involving multiple contact bodies. One Newton-type iterative method has been applied into our single pellet-to-sheath contact model without success in convergence within the solution procedure. Therefore, as this research deals with multibody contact with complex geometry, this method is not desirable.

Some nonlinear programming methods [9][21][22] have been introduced and succeeded in applications recently due to the powerful CPU in calculation process. This method is accurate and reliable in problems with small degree of contact freedom. Nevertheless, to solve the proposed multibody contact problem which requires finding the solutions for all nodes on the large contact surfaces simultaneously, experience suggests that the nonlinear programming technique would become very inefficient.

Compared with the nonlinear programming, the linear programming method is much faster and relative extensively studied. For the two-dimensional contact problem, Xu et al. have developed variable inequalities to represent the contact condition and solve the contact through the linear complementarity equations [23]. Some methods combining variable inequality and the complementarity equation have been used in the three-dimensional contact problem
where the frictional cone constrain is approximated by a polygon boundary. Bazaraa et al. have proved that the solution for a linear complementarity equation is guaranteed if the contact matrix satisfies the semi-positive condition [24].

Combining all the factors discussed above, this research formulated a non-smooth equation for the frictional contact problem in the nuclear fuel rod and proposes to apply the linear complementarity equation method to solve it. This equation can be solved through the Lemke’s algorithm [25]. The solution of multibody frictional contact can be obtained through a correcting procedure on finding the slippage directions without any approximation on the frictional cone constrain. Detailed discussion related will be presented in Chapter 5.

Despite some special algorithms having been developed to integrate mechanical system with Coulomb friction and unilateral contact [26][27][28][29], this research employs the Coulomb’s Law of Friction to study static contact based on the dry friction simplification. This model is useful and accurate in the fuel rod system that has relatively high surface roughness.

1.3 Objective

The contact effect is determined by various factors including the increased temperature, initial gaps between potential contact faces and the coefficient of friction. The influence of these parameters identify the need for a thorough analysis. To understand the effect the frictional contact, a numerical model is desired.

The objective of this research is to develop a computational module that is capable to calculate the frictional effect in the fuel rod. The deflection of fuel rod is of most interest as well as the interior contact pressure variation caused by the frictional force. The computational module will be coupled with other existing modules to conduct safety analysis in various operating scenarios.

To establish such a computer module, an effective three-dimensional frictional contact algorithm need to be developed to solve the multibody contact problem. The algorithm for the three-dimensional friction contact will be useful for researchers in both industries and academia.
Chapter 2. Formulation of Contact Problem

This chapter describes a special three-dimensional finite element model that is developed by Yu et al. [2] for all the components in a CANDU6 nuclear fuel rod. The equations for contact between two bodies and contact among multiple bodies are also introduced.

2.1 Finite element method

Because of the complex geometry shape within the fuel rod and unpredictable gap locations in this porous structure, the analytical method is not desired to deal with this multibody problem. Therefore, a numerical approach in terms of the finite element method is needed.

The CANDU nuclear fuel rod can be considered as a composite cylinder containing components all in cylindrical sizes. By taking this advantage, all components in the fuel rod can be modeled using an annulus-type finite element method to generate a 3D mesh [2]. For instance, the radial-axial plane employs a nine-node Lagrangian element, while a $2\pi$-periodic function in circumferential direction can be adopted. The periodic function can further expand into Fourier series to be more convenience to deal with [30][31].

For convenience, all variable will be expressed in cylindrical coordinate system and such a 3D element can be understood from Figure 2.1.

Within this typical element, a variable of the coordinate $(r, \theta, z)$ can be expressed by a function of the nodal variables. For example, the displacement $u$ in the element $e$ can be written as

$$u^e(\theta,r,z) = \left[N_1(\theta)\right]\left[N_2(r,z)\right]\{u\}^e$$

(2.1)

where $\{u\}^e$ is the nodal displacement vector, $\left[N_1(\theta)\right]$ and $\left[N_1(\theta)\right]$ are the shape function matrices for the circumferential direction and the radial-axial plane.
If the circumferential $2\pi$-periodic function is expanded into $m$th order of harmonic terms, the two shape function matrices would become

$$[N_1(\theta)] = \begin{bmatrix} 1 & \cos \theta & \sin \theta & \cdots & \cos m\theta & \sin m\theta \end{bmatrix}$$

$$[N_2(r,z)] = \text{diag} \begin{bmatrix} L(r,z) & \cdots & L(r,z) \end{bmatrix}_{(2m+1)\text{ submatrices}}$$

where the submatrix $L(r,z)$ can be found by nine-node Lagrange element. In our case, the first order Fourier series is chosen so that three harmonic terms are applied to conduct the analysis.

The shape function matrix $[N_2(r,z)]$ with global coordinates $(r,z)$ could further be converted into the non-dimensional local coordinates $(\xi, \eta)$ and a Jacobian matrix $[J]$[32]. For a given nine-node element as shown in Figure 2.2, the shape function can be defined as [33]:

$$L_1 = \frac{\xi \eta}{4} (1-\xi)(1-\eta)$$
$$L_2 = -\frac{\xi \eta}{4} (1+\xi)(1-\eta)$$
$$L_3 = \frac{\xi \eta}{4} (1+\xi)(1+\eta)$$
$$L_4 = -\frac{\xi \eta}{4} (1-\xi)(1+\eta)$$
$$L_5 = -\frac{\eta}{2} (1-\xi^2)(1-\eta)$$
$$L_6 = \frac{\xi}{2} (1+\xi)(1-\eta^2)$$
$$L_7 = \frac{\eta}{2} (1-\xi^2)(1+\eta)$$
$$L_8 = -\frac{\xi}{2} (1-\xi)(1-\eta^2)$$
$$L_9 = (1-\xi^2)(1-\eta^2)$$

(2.2)
Figure 2.1: A Typical 3D Element

Figure 2.2: A Nine-node Lagrange Element
In the contact problem studied for this research, the number of nodes on the potential contact surface is relatively small when it compared with the total number of nodes on all solid components. Therefore, calculations can be reduced on solving the contact through condensing all involved variables onto the potential contact nodes [19].

Given a category of contact nodes on the potential contact surface and interior nodes with respect to the potential contact surface, the general equation of equilibrium for a solid body with contact force can be structured as

\[
\begin{bmatrix}
K_{oo} & K_{oc} \\
K_{co} & K_{cc}
\end{bmatrix}
\begin{bmatrix}
u_o \\
u_c
\end{bmatrix}
= \begin{bmatrix}
Q_o \\
Q_c
\end{bmatrix} + \begin{bmatrix}
0 \\
F_c
\end{bmatrix}
\]  \hspace{1cm} (2.3)

where \( K, u, Q, \) and \( F \) are stiffness, displacement, load and the contact force vectors respectively, while the subscript \( c \) indicates the nodes on the contact surface and the subscript \( o \) indicates the interior nodes.

The above equation can be condensed to the contact nodes only as

\[
\begin{bmatrix}
K'_{cc}
\end{bmatrix}
\begin{bmatrix}
u_c
\end{bmatrix}
= \begin{bmatrix}
Q'_{c}
\end{bmatrix} + \begin{bmatrix}
F_c
\end{bmatrix}
\]  \hspace{1cm} (2.4)

where

\[
K'_{cc} = K_{cc} - K_{co} [K_{oo}]^{-1} K_{oc}
\]

\[
\begin{bmatrix}
Q'_{c}
\end{bmatrix} = \begin{bmatrix}
Q_c
\end{bmatrix} - K_{co} [K_{oo}]^{-1} \begin{bmatrix}
Q_o
\end{bmatrix}
\]

\subsection{2.2 Equation for contact between two bodies}

According to Newton’s Third Law, contact forces on two potential contact surfaces are in opposite directions. Therefore, based on equation (2.4), a general equation set for contact between two contact solids can be restructured by the following:
\[
\begin{bmatrix}
K_1 & 0 \\
0 & K_2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = \begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix} + \begin{bmatrix}
-F \\
F
\end{bmatrix}
\]

(2.5)

where the subscripted number refers to the contact bodies.

Instead of solving equation (2.5) to find the displacements for two bodies respectively, this study proposed to solve for the relative displacement first. Therefore, a new variable \( \{ u_{2,1} \} \) is introduced to represent the relative nodal displacement on the two potential contact surfaces. The relation between this variable and the initial two displacement variables can be expressed as

\[
\{ u_{2,1} \} = \{ u_2 \} - \{ u_1 \}
\]

where the subscript “2,1” indicates the direction of the relative displacement which is oriented from body 1 to body 2.

Accordingly, the following equation can be derived from equation (2.5):

\[
\begin{bmatrix}
\bar{K}_{2,1}
\end{bmatrix}\{ u_{2,1} \} = \{ Q_{2,1} \} + \{ F \}
\]

(2.6)

where

\[
\begin{bmatrix}
\bar{K}_{2,1}
\end{bmatrix} = [K_2] - [K_2]([K_1] + [K_2])^{-1}[K_2]
\]

\[
\{ Q_{2,1} \} = \{ Q_2 \} - [K_2]([K_1] + [K_2])^{-1}(\{ Q_1 \} + \{ Q_2 \})
\]

Vectors in equation (2.6) can further be decomposed into three sub-vectors in radial, circumferential and axial directions as follows:

\[
\begin{align*}
\{ u_{2,1} \} &= \begin{bmatrix} u_{2,1}^r \\ u_{2,1}^\theta \\ u_{2,1}^z \end{bmatrix}, \\
\{ Q_{2,1} \} &= \begin{bmatrix} Q_{2,1}^r \\ Q_{2,1}^\theta \\ Q_{2,1}^z \end{bmatrix}, \\
\{ F \} &= \begin{bmatrix} F^r \\ F^\theta \\ F^z \end{bmatrix}
\end{align*}
\]
where the superscript $r$, $\theta$ and $z$ refer to the radial, circumferential and axial directions, respectively.

Equation (2.6) can be directly applied to solve the two-body contact problems, i.e. contact between a single fuel pellet and its surrounding sheath, or contact between two fuel pellets. However, extra efforts on assembling are needed when dealing with multibody contact problems.

2.3 Equation for contact among multiple bodies

2.3.1 Model introduction

Since the symmetric assumption is made on the fuel rod, calculation can be reduced by modelling only half of the rod structure, which contains half number of the fuel pellets, half length of the hollow sheath and one endcap. This research takes the advantage of the existing contact modelling technique [2] to create the three-dimensional contact model for the frictional contact study.

For the modelling techniques, each pellet or endcap is modeled as one contact body; the long sheath is modeled as several subdivided bodies designing to have radial contact with their own covering fuel pellets. These sheath modules are connected each other by the continuity condition. The length of the each so-obtained sheath component is the length of the fuel pellet plus half of the axial gap on each side of the pellet.

The equation for the multibody contact problem in the fuel rod is created through assembling individual contact equation for each contact body into the global contact equation, which is an equation set with respect to the radial contact displacements and the axial contact displacements for contact surfaces. In order to obtain this target equation, equations for the exact displacement on contact surfaces will be developed first.

The exact contact displacement in the fuel rod can be grouped into four vectors: the radial displacement vector $\{\bar{u}_r^s\}$ for the sheath inner surface, the radial displacement vector $\{\bar{u}_r^p\}$ for the out surface of all pellets, the axial displacement vector $\{\bar{u}_R^p\}$ for the right side of all pellets, and
the vector \( \{ \bar{u}_L \} \) involving the axial displacement for the endcap and the axial displacement for the left side of all pellets except the pellet right to the fuel rod midplane.

### 2.3.2 Contact components assembling equations

Equation (2.6) can be restructured for a fuel pellet containing one set of radial contact and two sets of axial contact as

\[
\begin{bmatrix}
K^p_{rr} & K^p_{rL} & K^p_{rR} \\
K^p_{Lr} & K^p_{LL} & K^p_{LR} \\
K^p_{Rr} & K^p_{RL} & K^p_{RR}
\end{bmatrix}
\begin{bmatrix}
u^p_r \\
u^p_L \\
u^p_R
\end{bmatrix}
= 
\begin{bmatrix}
Q^p_r \\
Q^p_L \\
Q^p_R
\end{bmatrix}
+ 
\begin{bmatrix}
F^p_r \\
F^p_L \\
F^p_R
\end{bmatrix}, \quad (i = 1, 2, ..., n_p)
\] (2.7)

where \( K, u, Q, \) and \( F \) refer to the stiffness, displacement, equivalent load, and contact force respectively; the superscript \( p \) refers to the “pellet”; the subscript \( i \) indicates the pellet number; the subscript \( r, L, \) and \( R \) refer to the radial, left, and right degree of freedom responsible for radial and axial contact. Figure 2.3 illustrates the contact displacement field for a fuel pellet.

![Figure 2.3: Contact Displacement Field for a Fuel Pellet](image)
For the first fuel pellet located right to the fuel rod midplane, no axial displacement on the pellet left surface is assumed due to the symmetry assumption. Accordingly, the restructured equation for this specific pellet is

\[
\begin{bmatrix}
K_{rr}^p & K_{rl}^p & K_{rR}^p \\
K_{lr}^p & K_{ll}^p & K_{lR}^p \\
K_{rr}^p & K_{RL}^p & K_{RR}^p
\end{bmatrix}
\begin{bmatrix}
u_r^p \\
u_L^p \\
u_R^p
\end{bmatrix} =
\begin{bmatrix}
Q_r^p \\
Q_L^p \\
Q_R^p
\end{bmatrix} +
\begin{bmatrix}
F_r^p \\
F_L^p \\
F_R^p
\end{bmatrix}
\] (2.8)

where the left degree of freedoms becomes interior degree of freedoms, and the subscript number \( I \) indicates the first pellet.

The equation for a sheath component can be restructured as

\[
\begin{bmatrix}
K_{rr}^s & K_{rl}^s & K_{rR}^s \\
K_{lr}^s & K_{ll}^s & K_{lR}^s \\
K_{rr}^s & K_{RL}^s & K_{RR}^s
\end{bmatrix}
\begin{bmatrix}
u_r^s \\
u_L^s \\
u_R^s
\end{bmatrix} =
\begin{bmatrix}
Q_r^s \\
Q_L^s \\
Q_R^s
\end{bmatrix} +
\begin{bmatrix}
F_r^s \\
F_L^s \\
F_R^s
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
H_R^s
\end{bmatrix}, (i = 1, 2, ..., n_p)
\] (2.9)

where \( H \) indicates the constraining force responsible for sheath connectivity, and the superscript \( s \) refers to the “sheath”.

Similarly, the first sheath module covering the first fuel pellet has the follow equation of equilibrium:

\[
\begin{bmatrix}
K_{rr}^c & K_{rl}^c & K_{rR}^c \\
K_{lr}^c & K_{ll}^c & K_{lR}^c \\
K_{rr}^c & K_{RL}^c & K_{RR}^c
\end{bmatrix}
\begin{bmatrix}
u_r^c \\
u_L^c \\
u_R^c
\end{bmatrix} =
\begin{bmatrix}
Q_r^c \\
Q_L^c \\
Q_R^c
\end{bmatrix} +
\begin{bmatrix}
F_r^c \\
F_L^c \\
F_R^c
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
H_R^c
\end{bmatrix}
\] (2.10)

The equation for the endcap can be restructured as simple as

\[
\begin{bmatrix}
K_{pp}^c & K_{ps}^c \\
K_{cp}^c & K_{ss}^c
\end{bmatrix}
\begin{bmatrix}
u_p^c \\
u_s^c
\end{bmatrix} =
\begin{bmatrix}
Q_p^c \\
Q_s^c
\end{bmatrix} +
\begin{bmatrix}
F_p^c \\
H_s^c
\end{bmatrix}
\] (2.11)
where the superscript $c$ indicates the “endcap”, and the subscript $s$ and $p$ refer to the connection to the “sheath” and “pellet”, while the vector $u_j^c$ and the vector $H$ refer to the constraining displacement and force responsible for endcap connectivity to the sheath.

Since all fuel pellets in a CANDU fuel rod are identically manufactured and the temperature distribution under normal operation is indistinguishable among pellets, the stiffness matrix $K$ and the equivalent load vector $Q$ for all fuel pellets are identical. This simplification saves the computational memory to store the data for contact components.

Equation (2.7) contains $3n_p$ sub-equations, which can be categorized into three sets of equations in radial, left-axial, and right-axial components. The $n_p$ subsets of radial equations can be written as

$$
\begin{bmatrix}
\mathbf{K}_{rr}^p \\
\mathbf{K}_{rl}^p \\
\mathbf{K}_{rr}^p
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_r^p \\
\mathbf{u}_L^p \\
\mathbf{u}_R^p
\end{bmatrix}
+ 
\begin{bmatrix}
\mathbf{K}_{rr}^p \\
\mathbf{K}_{rl}^p \\
\mathbf{K}_{rr}^p
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_L^p \\
\mathbf{u}_R^p
\end{bmatrix}
+ 
\begin{bmatrix}
\mathbf{K}_{rr}^p
\end{bmatrix}
\mathbf{Q} = 
\begin{bmatrix}
\mathbf{F}_r^p \\
\mathbf{F}_L^p \\
\mathbf{F}_R^p
\end{bmatrix}
$$

(2.12)

where

$$
\begin{bmatrix}
\mathbf{K}_{rr}^p
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{K}_{rr}^p_1 \\
\mathbf{K}_{rr}^p_2 \\
\vdots \\
\mathbf{K}_{rr}^p_{n_p}
\end{bmatrix},
\begin{bmatrix}
\mathbf{K}_{rl}^p
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{K}_{rl}^p_1 \\
\mathbf{K}_{rl}^p_2 \\
\vdots \\
\mathbf{K}_{rl}^p_{n_p}
\end{bmatrix},
\begin{bmatrix}
\mathbf{K}_{rr}^p
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{K}_{rr}^p_1 \\
\mathbf{K}_{rr}^p_2 \\
\vdots \\
\mathbf{K}_{rr}^p_{n_p}
\end{bmatrix},
\begin{bmatrix}
\mathbf{K}_{rl}^p
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{K}_{rl}^p_1 \\
\mathbf{K}_{rl}^p_2 \\
\vdots \\
\mathbf{K}_{rl}^p_{n_p}
\end{bmatrix}.
$$
\[
\{ \vec{u}_r^p \} = \begin{bmatrix}
\{ u_r^p \}_1 \\
\vdots \\
\{ u_r^p \}_n \\
\end{bmatrix}, \quad \{ \vec{u}_L^p \} = \begin{bmatrix}
\{ u_L^p \}_1 \\
\vdots \\
\{ u_L^p \}_n \\
\end{bmatrix}, \quad \{ \vec{u}_R^p \} = \begin{bmatrix}
\{ u_R^p \}_1 \\
\vdots \\
\{ u_R^p \}_n \\
\end{bmatrix}, \quad \{ \vec{Q}_r^p \} = \begin{bmatrix}
\{ Q_r^p \}_1 \\
\vdots \\
\{ Q_r^p \}_n \\
\end{bmatrix}, \quad \{ \vec{F}_r^p \} = \begin{bmatrix}
\{ F_r^p \}_1 \\
\vdots \\
\{ F_r^p \}_n \\
\end{bmatrix}.
\]

Similarly, the \( n_p \) subsets of left-axial and right-axial equations can be composed as

\[
\begin{align*}
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
\begin{bmatrix}
\vec{u}_r^p \\
\vec{u}_L^p \\
\vdots \\
\vec{u}_R^p \\
\end{bmatrix}
+ 
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
\begin{bmatrix}
\vec{u}_L^p \\
\vec{u}_L^p \\
\vdots \\
\vec{u}_L^p \\
\end{bmatrix}
+ 
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
\begin{bmatrix}
\vec{u}_R^p \\
\vec{u}_R^p \\
\vdots \\
\vec{u}_R^p \\
\end{bmatrix}
= 
\begin{bmatrix}
\vec{Q}_L^p \\
\vec{Q}_L^p \\
\vdots \\
\vec{Q}_L^p \\
\end{bmatrix} 
+ \begin{bmatrix}
\vec{F}_L^p \\
\vec{F}_L^p \\
\vdots \\
\vec{F}_L^p \\
\end{bmatrix} 
\tag{2.13}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\vec{K}_R^p \\
\vec{K}_R^p \\
\vdots \\
\vec{K}_R^p \\
\end{bmatrix}
\begin{bmatrix}
\vec{u}_r^p \\
\vec{u}_L^p \\
\vdots \\
\vec{u}_R^p \\
\end{bmatrix}
+ 
\begin{bmatrix}
\vec{K}_R^p \\
\vec{K}_R^p \\
\vdots \\
\vec{K}_R^p \\
\end{bmatrix}
\begin{bmatrix}
\vec{u}_L^p \\
\vec{u}_L^p \\
\vdots \\
\vec{u}_L^p \\
\end{bmatrix}
+ 
\begin{bmatrix}
\vec{K}_R^p \\
\vec{K}_R^p \\
\vdots \\
\vec{K}_R^p \\
\end{bmatrix}
\begin{bmatrix}
\vec{u}_R^p \\
\vec{u}_R^p \\
\vdots \\
\vec{u}_R^p \\
\end{bmatrix}
= 
\begin{bmatrix}
\vec{Q}_R^p \\
\vec{Q}_R^p \\
\vdots \\
\vec{Q}_R^p \\
\end{bmatrix} 
+ \begin{bmatrix}
\vec{F}_R^p \\
\vec{F}_R^p \\
\vdots \\
\vec{F}_R^p \\
\end{bmatrix} 
\tag{2.14}
\end{align*}
\]

where

\[
\begin{align*}
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
&= \begin{bmatrix}
K_{L_1}^p \\
K_{L_2}^p \\
\vdots \\
K_{L_n}^p \\
\end{bmatrix}, \\
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
&= \begin{bmatrix}
K_{LL_1}^p \\
K_{LL_2}^p \\
\vdots \\
K_{LL_n}^p \\
\end{bmatrix}, \\
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
&= \begin{bmatrix}
K_{Lr_1}^p \\
K_{Lr_2}^p \\
\vdots \\
K_{Lr_n}^p \\
\end{bmatrix}, \\
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
&= \begin{bmatrix}
K_{LR_1}^p \\
K_{LR_2}^p \\
\vdots \\
K_{LR_n}^p \\
\end{bmatrix}, \\
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
&= \begin{bmatrix}
K_{RL_1}^p \\
K_{RL_2}^p \\
\vdots \\
K_{RL_n}^p \\
\end{bmatrix}, \\
\begin{bmatrix}
\vec{K}_L^p \\
\vec{K}_L^p \\
\vdots \\
\vec{K}_L^p \\
\end{bmatrix}
&= \begin{bmatrix}
K_{RR_1}^p \\
K_{RR_2}^p \\
\vdots \\
K_{RR_n}^p \\
\end{bmatrix}.
\end{align*}
\]
The relative displacement for axial contact refers to the difference in displacement between the right contact surface of \(i\)th pellet and the left contact surface of \((i+1)\)th pellet, where \(i = 1, 2, ..., n_p - 1\), and the difference in displacement between the right contact surface of \(n_p\)th pellet and the contact surface of the right endcap. Therefore, equation is required to formulate based on the exact displacement for right-axial surface displacement from pellet 1 to \(n_p\), and the exact displacement for left-axial surface displacement from pellet 2 to surface displacement of the right endcap. Accordingly, by combining equation (2.8),(2.11),(2.12),(2.13),and (2.14), the following equations can be obtained:

\[
\left[ \bar{K}_{rr}^p \right] \{ \bar{u}_r^p \} + \left[ \bar{K}_{rL}^p \right] \{ \bar{u}_L^p \} + \left[ \bar{K}_{rR}^p \right] \{ \bar{u}_R^p \} = \{ \bar{Q}_r^p \} + \{ \bar{F}_r^p \} \tag{2.15}
\]

\[
\left[ \bar{K}_{Lr}^p \right] \{ \bar{u}_r^p \} + \left[ \bar{K}_{LL}^p \right] \{ \bar{u}_L^p \} + \left[ \bar{K}_{LR}^p \right] \{ \bar{u}_R^p \} = \{ \bar{Q}_L^p \} + \{ \bar{F}_L^p \} \tag{2.16}
\]

\[
\left[ \bar{K}_{Rr}^p \right] \{ \bar{u}_r^p \} + \left[ \bar{K}_{RL}^p \right] \{ \bar{u}_L^p \} + \left[ \bar{K}_{RR}^p \right] \{ \bar{u}_R^p \} = \{ \bar{Q}_R^p \} + \{ \bar{F}_R^p \} \tag{2.17}
\]

where

\[
\left[ \bar{K}_{rL}^p \right] = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}, \quad \left[ \bar{K}_{rR}^p \right] = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
K_{LL}^p & & \\
& \ddots & \\
& & K_{LL}^p_{n_p}
\end{bmatrix}
- 
\begin{bmatrix}
K_{LL}^p & \\
& \ddots \\
& & K_{LL}^p_{n_p}
\end{bmatrix} =
\begin{bmatrix}
0 & \\
& \ddots \\
& & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
K_{LR}^p & \\
& \ddots \\
& & K_{LR}^p_{n_p}
\end{bmatrix}
- 
\begin{bmatrix}
K_{LR}^p & \\
& \ddots \\
& & K_{LR}^p_{n_p}
\end{bmatrix} =
\begin{bmatrix}
0 & \\
& \ddots \\
& & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
K_{ps}^c & 0 \\
& \ddots \\
& & K_{ps}^c_{n_p}
\end{bmatrix} =
\begin{bmatrix}
0 & \\
& \ddots \\
& & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
K_{RL}^p & \\
& \ddots \\
& & K_{RL}^p_{n_p}
\end{bmatrix}
- 
\begin{bmatrix}
K_{RL}^p & \\
& \ddots \\
& & K_{RL}^p_{n_p}
\end{bmatrix} =
\begin{bmatrix}
0 & \\
& \ddots \\
& & 0
\end{bmatrix},
\]

Equation (2.15) accounts for radial contact, while equation (2.16) & (2.17) are the assembled equation for the axial contact between pellets and the endcap.

To develop the contact equation for the radial contact between the sheath and the pellets, the continuity for the sheath components is considered first. The condition of continuity for all sheath modules and the right endcap can be expressed as

\[
\left\{ u^p_L \right\}_2 - \left\{ u^p_L \right\}_1 = \left\{ u^s_L \right\}_1 \\
\vdots \\
\left\{ u^p_L \right\}_{n_p} - \left\{ u^p_L \right\}_{n_p-1} = \left\{ u^s_L \right\}_{n_p} \\
\left\{ u^p_L \right\}_{n_p} = \left\{ u^s_L \right\}_{n_p}
\]

\[
(2.18)
\]
\[
\left\{ H^s_R \right\}_1 + \left\{ H^s_L \right\}_2 = \{0\}
\]
\[
\left\{ H^s_R \right\}_2 + \left\{ H^s_L \right\}_3 = \{0\}
\]
\[\vdots\]
\[
\left\{ H^s_R \right\}_{n_p-1} + \left\{ H^s_L \right\}_{n_p} = \{0\}
\]
\[
\left\{ H^s_R \right\}_{n_p} + \left\{ H^s_L \right\} = \{0\}
\]  \hspace{1cm} (2.19)

The \( n_p \) sets of equation (2.9) can be rewritten into three equations as

\[
\begin{bmatrix}
\bar{K}^s_{rr} \\
\bar{K}^s_{rl} \\
\bar{K}^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_r \\
\bar{u}^s_l \\
\bar{u}^s_r
\end{bmatrix}
+
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_l \\
\bar{u}^s_l \\
\bar{u}^s_l
\end{bmatrix}
+
\begin{bmatrix}
\bar{K}^s_{rr} \\
\bar{K}^s_{rl} \\
\bar{K}^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_r \\
\bar{u}^s_r \\
\bar{u}^s_r
\end{bmatrix}
=
\begin{bmatrix}
\bar{Q}^s_r \\
\bar{Q}^s_l \\
\bar{Q}^s_r
\end{bmatrix}
+
\begin{bmatrix}
\bar{F}^s_r \\
\bar{F}^s_l \\
\bar{F}^s_r
\end{bmatrix}
\]  \hspace{1cm} (2.20)

\[
\begin{bmatrix}
\bar{K}^s_{rr} \\
\bar{K}^s_{rl} \\
\bar{K}^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_r \\
\bar{u}^s_l \\
\bar{u}^s_r
\end{bmatrix}
+
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_l \\
\bar{u}^s_l \\
\bar{u}^s_l
\end{bmatrix}
+
\begin{bmatrix}
\bar{K}^s_{rr} \\
\bar{K}^s_{rl} \\
\bar{K}^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_r \\
\bar{u}^s_r \\
\bar{u}^s_r
\end{bmatrix}
=
\begin{bmatrix}
\bar{Q}^s_r \\
\bar{Q}^s_l \\
\bar{Q}^s_r
\end{bmatrix}
+
\begin{bmatrix}
\bar{F}^s_r \\
\bar{F}^s_l \\
\bar{F}^s_r
\end{bmatrix}
\]  \hspace{1cm} (2.21)

\[
\begin{bmatrix}
\bar{K}^s_{rr} \\
\bar{K}^s_{rl} \\
\bar{K}^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_r \\
\bar{u}^s_l \\
\bar{u}^s_r
\end{bmatrix}
+
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_l \\
\bar{u}^s_l \\
\bar{u}^s_l
\end{bmatrix}
+
\begin{bmatrix}
\bar{K}^s_{rr} \\
\bar{K}^s_{rl} \\
\bar{K}^s_{rr}
\end{bmatrix}
\begin{bmatrix}
\bar{u}^s_r \\
\bar{u}^s_r \\
\bar{u}^s_r
\end{bmatrix}
=
\begin{bmatrix}
\bar{Q}^s_r \\
\bar{Q}^s_l \\
\bar{Q}^s_r
\end{bmatrix}
+
\begin{bmatrix}
\bar{H}^s_r \\
\bar{H}^s_l \\
\bar{H}^s_r
\end{bmatrix}
\]  \hspace{1cm} (2.22)

where

\[
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix}_{n_p},
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix},
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix}_{n_p}
\]

\[
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix}_{n_p},
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix},
\begin{bmatrix}
K^s_{rr} \\
K^s_{rl} \\
K^s_{rr}
\end{bmatrix}_{n_p}
\]
\[\begin{align*}
\begin{bmatrix}
K_{LL}^s \\
K_{LR}^s \\
K_{RL}^s \\
K_{RR}^s
\end{bmatrix} &= 
\begin{bmatrix}
K_{LL}^L \\
K_{LR}^L \\
K_{RL}^L \\
K_{RR}^L
\end{bmatrix} \\
\begin{bmatrix}
K_{LL}^p \\
K_{LR}^p \\
K_{RL}^p \\
K_{RR}^p
\end{bmatrix} &= 
\begin{bmatrix}
K_{LL}^P \\
K_{LR}^P \\
K_{RL}^P \\
K_{RR}^P
\end{bmatrix} \\
\end{align*}\]
Combing the force continuity equation (2.19), equation (2.21) and (2.22) can be summed as

\[
\begin{bmatrix}
\vec{K}_{Lr}^s \\
\end{bmatrix} \{\vec{u}_r\} + \begin{bmatrix}
\vec{K}_{LL}^s \\
\vec{K}_{LR}^s
\end{bmatrix} \{\vec{u}_L\} + \begin{bmatrix}
\vec{K}_{LR}^s \\
\vec{K}_{RR}^s
\end{bmatrix} \{\vec{u}_R\} = \{\vec{Q}_L^s\} + \{\vec{H}_L^s\}
\]

(2.23)

where

\[
\begin{bmatrix}
\vec{K}_{Lr}^s \\
\end{bmatrix} = \begin{bmatrix}
K_{Lr}^1 & \ldots & 0 \\
K_{Rr}^1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
K_{Rr}^{n_p-1} & \ldots & K_{Lr}^{n_p}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vec{K}_{LL}^s \\
\vec{K}_{RL}^s \\
\vec{K}_{LR}^s
\end{bmatrix} = \begin{bmatrix}
K_{LL}^1 & \ldots & 0 \\
K_{RL}^1 & \ldots & 0 \\
K_{LR}^1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
K_{RL}^{n_p-1} & \ldots & K_{LL}^{n_p}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vec{K}_{LR}^s \\
\vec{K}_{RR}^s \\
\vec{K}_{LR}^s
\end{bmatrix} = \begin{bmatrix}
K_{LR}^1 & \ldots & 0 \\
K_{RR}^1 & \ldots & 0 \\
K_{LR}^1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
K_{RR}^{n_p-1} & \ldots & K_{LR}^{n_p}
\end{bmatrix}
\]
Considering the displacement continuity equation (2.18) and the endcap equation (2.11), equation (2.23) can be updated to the following containing the continuity condition with the endcap:

\[
\left[ \bar{K}^{e}_{Lr} \right] \{ \bar{u}_r \} + \left[ \bar{K}^{i}_{LL} \right] \{ \bar{u}_L \} + \left[ \bar{K}^{c}_{sp} \right] \{ \bar{u}_L \} = \{ \bar{Q}_L \}
\]

(2.24)

where

\[
\bar{K}^{i}_{LL} = \\
\begin{bmatrix}
K_{LL}^i & K_{LR}^i \\
K_{RL}^i & K_{LL}^i + K_{LR}^i \\
K_{RL}^i & K_{LL}^i + K_{LR}^i \\
& \ddots \\
K_{RL}^i & K_{LL}^i + K_{LR}^i \\
K_{RL}^i & K_{LL}^i + K_{LR}^i \\
K_{RL}^i & K_{LL}^i + K_{LR}^i \\
K_{RL}^i & K_{LL}^i + K_{LR}^i \\
K_{RL}^i & K_{LL}^i + K_{LR}^i \\
K_{RL}^i & K_{LL}^i + K_{LR}^i
\end{bmatrix}
\]

\[
\bar{K}^{c}_{sp} = \\
\begin{bmatrix}
0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
\bar{Q}_L = \\
\begin{bmatrix}
\{ Q_L \}_1 \\
\{ Q_L \}_2 + \{ Q_R \}_1 \\
\{ Q_L \}_3 + \{ Q_R \}_2 \\
\vdots \\
\{ Q_L \}_{n_p} + \{ Q_R \}_{n_p-1} \\
\{ Q_L \}_{n_p} + \{ Q_R \}_{n_p}
\end{bmatrix}
\]
Equation (2.24) describes the connectivity for sheath modules as well as the endcap. However, the radial contact equation (2.20) also needs to be reformulated as follows with respect to the same displacement vectors as used for the connectivity equation:

\[
\begin{bmatrix}
K_{sp}^r \\
K_{rp}^s \\
\end{bmatrix}
\begin{bmatrix}
\bar{u}_r \\
\bar{u}_L \\
\end{bmatrix}
+ \begin{bmatrix}
\bar{K}_{sp}^r \\
\bar{K}_{rp}^s \\
\end{bmatrix}
\begin{bmatrix}
\bar{u}_r \\
\bar{u}_L \\
\end{bmatrix}
= \begin{bmatrix}
\bar{Q}_r \\
\bar{F}_r \\
\end{bmatrix}
\]

(2.25)

where

\[
\begin{bmatrix}
\bar{K}_{sp}^r \\
\bar{K}_{rp}^s \\
\end{bmatrix} = 
\begin{bmatrix}
[K_{sp}^r]_1 \\
[K_{rp}^s]_1 \\
[K_{sp}^r]_2 \\
[K_{rp}^s]_2 \\
\vdots \\
\vdots \\
[K_{sp}^r]_{n_p} \\
[K_{rp}^s]_{n_p} \\
0
\end{bmatrix}
\]

Equation (2.23) and (2.25) are equations for modules connectivity and radial contact.

Since the assembling between the pellets and the endcap is finished as well as the assembling between the sheath and the endcap, the global contact equation can be achieved only after assembling equation (2.15), (2.16), (2.17), (2.24) and (2.25).

Because the contact displacement vectors are \(\{\bar{u}_r^p\}, \{\bar{u}_r^s\}, \{\bar{u}_r^p\},\) and \(\{\bar{u}_L^i\},\) the displacement \(\{\bar{u}_L^i\}\) appeared in the above equations shall be considered as interior displacement and should be eliminated. The displacement vector \(\{\bar{u}_L^i\}\) can be determined by equation (2.24) as follows:

\[
\begin{bmatrix}
\bar{u}_L^i \\
\end{bmatrix} = \left[K_{LL}^s \right]^{-1} \left(\bar{Q}_L^s \right) - \left[K_{Lr}^s \right] \left[\bar{u}_r^i \right] - \left[K_{sp}^c \right] \left[\bar{u}_L^p \right]
\]

(2.26)

Substituting equation (2.26) into equation (2.16) & (2.25) and combining equation (2.15) & (2.17), the following equation may be acquired:
Equation (2.27) corresponds to equation (2.17), (2.15), (2.16) and (2.25), respectively.

Equation (2.27) is of the most useful for describing the contact problem. However, as discussed in the previous sections, the relative displacement and the contact force are proposed to solve in the contact algorithm. Therefore, equation based on the relative displacement is needed.

2.3.3 Final equation for the multibody contact problem

Equation (2.27) can be rewritten as follows for simplification:

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
O_Q \\
O_F
\end{bmatrix} +
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

(2.28)

where

\[
\begin{bmatrix}
K_{11} = 
\begin{bmatrix}
\bar{K}_{RR}^p & \bar{K}_{Rr}^p \\
\bar{K}_{rR}^p & \bar{K}_{rr}^p
\end{bmatrix},
K_{12} = 
\begin{bmatrix}
\bar{K}_{RL}^p & \bar{K}_{rL}^p \\
\bar{K}_{Lr}^p & \bar{K}_{lr}^p
\end{bmatrix},
K_{21} = 
\begin{bmatrix}
\bar{K}_{RL}^c & \bar{K}_{rL}^c \\
\bar{K}_{Lr}^c & \bar{K}_{lr}^c
\end{bmatrix},
K_{22} = 
\begin{bmatrix}
\bar{K}_{LL}^c & \bar{K}_{Ll}^c \\
\bar{K}_{lL}^c & \bar{K}_{ll}^c
\end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
O_Q \\
O_F
\end{bmatrix} +
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\bar{K}_{RR}^p & \bar{K}_{Rr}^p \\
\bar{K}_{rR}^p & \bar{K}_{rr}^p
\end{bmatrix}
\begin{bmatrix}
\bar{K}_{RL}^p & \bar{K}_{rL}^p \\
\bar{K}_{Lr}^p & \bar{K}_{lr}^p
\end{bmatrix}
\begin{bmatrix}
\bar{K}_{RL}^c & \bar{K}_{rL}^c \\
\bar{K}_{Lr}^c & \bar{K}_{lr}^c
\end{bmatrix}
\begin{bmatrix}
\bar{K}_{LL}^c & \bar{K}_{Ll}^c \\
\bar{K}_{lL}^c & \bar{K}_{ll}^c
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
O_Q \\
O_F
\end{bmatrix} +
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

(2.27)
Introducing the relative displacement vector:

\[ \{u_{2/1}\} = \{u_2\} - \{u_1\} \]

Equation (2.28) can be reformulated as

\[ \{\bar{K}_{2/1}\}\{u_{2/1}\} = \{Q_{2/1}\} + \{F\} \]  

(2.29)

where

\[
[\bar{K}_{1/2}] = [K_{2/2}] - ([K_{2/1}] + [K_{2/2}]) ([K_{1/1}] + [K_{1/2}] + [K_{2/1}])^{-1} ([K_{1/2}] + [K_{2/2}]),
\]

\[\{Q_{2/1}\} = \{Q_2\} - ([K_{2/1}] + [K_{2/2}]) ([K_{1/1}] + [K_{1/2}] + [K_{2/1}])^{-1} (\{Q_1\} + \{Q_2\}),\]

\[\{F\} = \{F_2\} = -\{F_1\}.\]

Equation (2.29) is the final equation of multibody contact problem in the fuel rod with respect to the relative displacement and the contact force, which serves as the input information for the frictional contact solver.
Chapter 3. Solution of Frictional Contact

In this chapter, the target equation for contact derived from the previous chapter will be employed directly. This equation is solved through variable inequality techniques and the linear complementarity equation method.

3.1 Three-dimensional frictional contact algorithm

Since the directions of the frictional forces for all the contact pairs are unknown before the solution is obtained, this research introduces a correcting procedure to find the right slippage directions.

Initially the direction of the friction forces would be assumed by the load vector in the contact equation (2.6) or (2.29). Based on this assumption, a solution could be found for the contact equation. This so-obtained solution may not be accurate because it is based on the previous assumption. Therefore, the solution needs to be examined back to the initial contact equation (2.6) or (2.29) for accuracy concern. If this solution failed the examination, a slack displacement vector can be calculated. This displacement vector is applied to update the assumed slippage directions and is signed as the end of one correcting procedure. The above correcting procedure would continue until the slippage directions were found, where the slack displacement vector is approaching to zero.

Equation (2.6) and (2.29) are in the same format so that we can use equation (2.6) as representative for the proposed method.

By this approach, the solution of a three-dimensional contact problem can be obtained through solving a few times of two-dimensional-like contact, which has high efficiency and reliability.
Figure 3.1: Local Coordinate System for Each Node

Based on the assumption of the slippage directions, the curved plane coordinates could be transformed from $e_t e_{\theta} e_z$ to $e_t e_{\theta} e_{\phi}$, where $e_t$ is the assumed slippage direction and $e_{\phi}$ is the direction corresponding to an angle $\phi$, which is the angle between the original direction $e_{\theta}$ and the new direction $e_t$, shown in Figure 3.1. Therefore, equation (2.6) or (2.29) can be transformed by a connection matrix $[A]$ into a new equation with respect to the so-obtained new coordinate system, where

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix}$$

Note: $\cos \phi$ and $\sin \phi$ in the above matrix are identical matrices containing ‘nodal $\cos \phi$’s and ‘nodal $\sin \phi$’s.

A new equation can be obtained accordingly through coordinate transformation:
\[
\begin{bmatrix}
\mathbf{K}_{2,1}
\end{bmatrix}\{u_{2,1}\} = \{\mathbf{Q}_{2,1}\} + \{\mathbf{F}\} 
\]

(3.1)

where

\[
\begin{bmatrix}
\mathbf{K}_{2,1}
\end{bmatrix} = [A]\begin{bmatrix}
\mathbf{K}_{2,1}
\end{bmatrix}[A]^T = \begin{bmatrix}
K_{rr}^{rr} & K_{rr}^{r\phi} & K_{rr}^{\phi r} & K_{rr}^{\phi\phi} \\
K_{rt}^{rr} & K_{rt}^{r\phi} & K_{rt}^{\phi r} & K_{rt}^{\phi\phi} \\
K_{rt}^{rr} & K_{rt}^{r\phi} & K_{rt}^{\phi r} & K_{rt}^{\phi\phi} \\
K_{rt}^{rr} & K_{rt}^{r\phi} & K_{rt}^{\phi r} & K_{rt}^{\phi\phi}
\end{bmatrix}
\]

\[
\{\mathbf{Q}_{2,1}\} = [A]\{Q_{2,1}\} = \begin{bmatrix}
Q_{r}^{r} \\
Q_{t}^{r} \\
Q_{\phi}^{r}
\end{bmatrix}, (q = u, Q)
\]

\[
\{\mathbf{F}\} = [A]\{F\} = \begin{bmatrix}
F^{r} \\
F^{t} \\
F^{\phi}
\end{bmatrix} = \begin{bmatrix}
0
\end{bmatrix}.
\]

Note: there is no frictional force in \( e_\phi \) direction by the assumption on the slippage direction.

Introducing the submatrices, equation (3.1) may be reduced to

\[
\begin{bmatrix}
K_{rr} & K_{rt} \\
K_{tr} & K_{tt}
\end{bmatrix}\begin{bmatrix}
u_{2,1}^{r} \\
u_{2,1}^{t}
\end{bmatrix} = \begin{bmatrix}
Q_{r}^{r} \\
Q_{t}^{r}
\end{bmatrix} + \begin{bmatrix}
F^{r} \\
F^{t}
\end{bmatrix} 
\]

(3.2)

where the submatrices are

\[
\begin{bmatrix}
K_{rr}
\end{bmatrix} = \begin{bmatrix}
K_{rr}^{rr} & K_{rr}^{r\phi} \end{bmatrix}^{-1} \begin{bmatrix}
K_{rr}^{r\phi} \\
K_{rr}^{\phi r}
\end{bmatrix} \quad \begin{bmatrix}
K_{rt}
\end{bmatrix} = \begin{bmatrix}
K_{rt}^{rr} & K_{rt}^{r\phi} \end{bmatrix}^{-1} \begin{bmatrix}
K_{rt}^{r\phi} \\
K_{rt}^{\phi r}
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_{tr}
\end{bmatrix} = \begin{bmatrix}
K_{tr}^{rr} & K_{tr}^{r\phi} \end{bmatrix}^{-1} \begin{bmatrix}
K_{tr}^{r\phi} \\
K_{tr}^{\phi r}
\end{bmatrix} \quad \begin{bmatrix}
K_{tt}
\end{bmatrix} = \begin{bmatrix}
K_{tt}^{rr} & K_{tt}^{r\phi} \end{bmatrix}^{-1} \begin{bmatrix}
K_{tt}^{r\phi} \\
K_{tt}^{\phi r}
\end{bmatrix}
\]

\[
\{Q_{r}\} = \{Q_{r}^{r}\} - \begin{bmatrix}
K_{rr}^{r\phi} & K_{rr}^{\phi r} \end{bmatrix}^{-1} \{Q_{2,1}\} \quad \{Q_{t}\} = \{Q_{t}^{r}\} - \begin{bmatrix}
K_{rt}^{r\phi} & K_{rt}^{\phi r} \end{bmatrix}^{-1} \{Q_{2,1}\}
\]

\[
\{Q_{\phi}\} = \{Q_{\phi}^{r}\} - \begin{bmatrix}
K_{rr}^{r\phi} & K_{rr}^{\phi r} \end{bmatrix}^{-1} \{Q_{2,1}\} \quad \{Q_{t}\} = \{Q_{t}^{r}\} - \begin{bmatrix}
K_{rt}^{r\phi} & K_{rt}^{\phi r} \end{bmatrix}^{-1} \{Q_{2,1}\}
\]
3.2 Linear complementarity equation

Equation (3.2) is the one proposed to be solved by the linear complementarity problem method. This research applied the method introduced by Xu and Yu [23] to create the linear complementarity equation.

To build such equation, the frictional force and the relative displacement is replaced by four new variables as follows:

\[
\begin{align*}
\{\bar{u}\} &= \max(\{u_{2,1}^i\}, 0) \\
\{\bar{\bar{u}}\} &= \max(-\{u_{2,1}^i\}, 0) \\
\{\bar{s}\} &= \mu \{F^r\} + \{F^i\} \\
\{\bar{\bar{s}}\} &= \mu \{F^r\} - \{F^i\}
\end{align*}
\]

which has the complementarity condition

\[
\begin{align*}
\{\bar{u}\} &\geq 0, \{\bar{\bar{u}}\} \geq 0, \{\bar{s}\} \geq 0, \{\bar{\bar{s}}\} \geq 0 \\
\{\bar{u}\}^T \{\bar{s}\} &= 0 \\
\{\bar{\bar{u}}\}^T \{\bar{\bar{s}}\} &= 0
\end{align*}
\]

and relation with the original variables

\[
\begin{align*}
\{u_{2,1}^i\} &= \{\bar{u}\} - \{\bar{\bar{u}}\} \\
\{F^i\} &= \{\bar{s}\} - \mu \{F^r\} \\
\{F^r\} &= -\{\bar{\bar{s}}\} + \mu \{F^r\}
\end{align*}
\]

New variables introduced above are created to describe the relation between the frictional force and the slippage displacement.
With the initial gap \( g_0 \) between the contact pairs, a gap vector is employed to represent the radial gaps after contact as

\[
\{ \mathbf{g} \} = \{ \mathbf{g}_0 \} + \{ \mathbf{u}^r_{2,1} \}.
\]

The complementarity condition with respect to this gap vector is

\[
\begin{align*}
\{ \mathbf{g} \} &\geq 0 \\
\{ \mathbf{F}_r \} &\geq 0 \\
\{ \mathbf{g} \}^T \cdot \{ \mathbf{F}_r \} &\geq 0
\end{align*}
\]

Accordingly, equation (3.2) can be reformulated by the above new variables as

\[
\begin{bmatrix}
K_{rr} & K_{rt} & -K_{rt} \\
K_{rt} & K_{tt} & -K_{tt} \\
K_{rt} & K_{tt} & -K_{tt}
\end{bmatrix}
\begin{bmatrix}
\mathbf{g} \\
\mathbf{u}^r \\
\mathbf{u}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{Q}_r + K_{rr} \mathbf{g}_0 \\
\mathbf{Q}_r + K_{rt} \mathbf{g}_0 + -\mu \mathbf{I} \mathbf{0} \\
\mathbf{Q}_r + K_{rt} \mathbf{g}_0 + \mu \mathbf{I} - \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\mathbf{I} & 0 & 0 \\
-\mathbf{I} & 0 & 0 \\
\mathbf{I} & -\mathbf{I} & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{F}_r \\
\mathbf{s} \\
\mathbf{s}
\end{bmatrix}
\]

Pre-multiplying the above equation by \( \begin{bmatrix}
-\mu & \mathbf{I} & 0 \\
\mu & 0 & -\mathbf{I}
\end{bmatrix}^{-1} \), a standard form of linear complementarity equation can be achieved:

\[
\begin{align*}
\{ \mathbf{y} \} &= \begin{bmatrix} M \end{bmatrix} \{ \mathbf{x} \} + \{ \mathbf{q} \} \\
\{ \mathbf{x} \} &\geq 0, \{ \mathbf{y} \} \geq 0 \\
\{ \mathbf{y} \}^T \cdot \{ \mathbf{x} \} &= 0
\end{align*}
\]

where

\[
\begin{bmatrix} M \end{bmatrix} =
\begin{bmatrix}
K_{rr} & K_{rt} & -K_{rt} \\
K_{rt} + \mu K_{rr} & K_{tt} + \mu K_{rt} & -K_{tt} - \mu K_{tt} \\
-K_{rt} + \mu K_{tt} & -K_{tt} + \mu K_{rt} & K_{tt} - \mu K_{tt}
\end{bmatrix},
\]
\[
\begin{align*}
\{x\} &= \begin{bmatrix} g \\ \tilde{u} \end{bmatrix}, \quad \{y\} = \begin{bmatrix} \tilde{s} \end{bmatrix}, \quad \{q\} = \begin{bmatrix} -Q_r - K_{rr}g_0 \\ -Q_t - K_{rr}g_0 - \mu(Q_r + K_{rr}g_0) \\ Q_t + K_{rr}g_0 - \mu(Q_r + K_{rr}g_0) \end{bmatrix}.
\end{align*}
\]

Equation (3.4) can be solved by Lemke’s algorithm and after this solution is found, a slack force can be found from equation (3.1). The slack force is designed to calculate the slack displacement which helps to correct the assumption on the slippage direction.

The slack force \(\{\tilde{Q}_{2,1}^\varphi\}\) and the slack displacement \(\{\tilde{u}_{2,1}^\varphi\}\) can be calculated as

\[
\begin{align*}
\{\tilde{Q}_{2,1}^\varphi\} &= \{Q_{2,1}^\varphi\} - \left(\begin{bmatrix} K_{2,1}^\varphi \end{bmatrix}\{u_{2,1}^r\} + \begin{bmatrix} K_{2,1}^\varphi \end{bmatrix}\{u_{2,1}^t\}\right),
\end{align*}
\]

\[
\begin{align*}
\{\tilde{u}_{2,1}^\varphi\} &= \left[\begin{bmatrix} K_{2,1}^\varphi \end{bmatrix}\right]^{-1}\{\tilde{Q}_{2,1}^\varphi\}.
\end{align*}
\]

The slack force and the slack displacement determine the iteration status on the correcting procedure. In other words, they are the detectors of the right slippage directions.
Chapter 4. Case Studies

This chapter presents three case studies of three-dimensional frictional contact problem and comparable results solved by the proposed method.

The first case study is a contact problem between one pellet and the surrounding sheath. It is studied as a reliability test for the frictional contact algorithm. The second case study is the benchmark for the developed multiple body contact system which contains four fuel pellets, two endcaps and one hollow sheath. The third case studies the bending of the fuel rod under specific temperature distribution.

Results in the first two test cases are compared with ABAQUS simulation models and the results in the third one are compared with the existing contact model [2].

4.1 Contact between a single pellet and its surrounding sheath

A fundamental example of a two-body contact problem is tested for the proposed methodology of three-dimensional frictional contact algorithm. Results on contact pressure are compared both with analytical solution in frictionless case and frictional simulation by ABAQUS/CAE 6.13 with standard/explicit model.

In this example, a surface-to-surface contact model is created for frictional contact problem. The fuel pellet is described with thermal expansion and radial contact to its surrounding sheath. The sheath has the same length as the fuel pellet. Loads in this system are driven by the increased temperature in the fuel pellet. The expansion in this system is assumed axisymmetric under uniform temperature distribution. Both materials of pellet and sheath are linear-elastic.

Figure 4.1 shows a cross-section of the sheath with a pellet inside it.

Table 4-1 summaries the geometry for the pellet and sheath as well as other related parameters.
Figure 4.1 Side View of Axisymmetric Contact between Sheath and Pellet

Table 4-1 Description of a Single Pellet-to-sheath Contact Problem

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Test case 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pellet radius ( R_p ) (mm)</td>
<td>6</td>
</tr>
<tr>
<td>Pellet length ( l_p ) (mm)</td>
<td>16</td>
</tr>
<tr>
<td>Radial between pellet and sheath ( g_0 ) (mm)</td>
<td>0~0.4</td>
</tr>
<tr>
<td>Sheath inner radius ( R_{s,i} ) (mm)</td>
<td>6+ ( g_0 )</td>
</tr>
<tr>
<td>Sheath outer radius ( R_{s,o} ) (mm)</td>
<td>6.4+ ( g_0 )</td>
</tr>
<tr>
<td>Pellet Chamfer height(mm)</td>
<td>0</td>
</tr>
<tr>
<td>Pellet Chamfer width(mm)</td>
<td>0</td>
</tr>
<tr>
<td>Pellet dish depth(mm)</td>
<td>0</td>
</tr>
</tbody>
</table>
### Table 4-2: Material Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pellet dish radius (mm)</td>
<td>0</td>
</tr>
<tr>
<td>Pellet Young’s modulus $E_p$ (GPa)</td>
<td>170</td>
</tr>
<tr>
<td>Pellet Poisson’s ratio $\nu_p$</td>
<td>0.3</td>
</tr>
<tr>
<td>Sheath Young’s modulus $E_s$ (GPa)</td>
<td>84.1</td>
</tr>
<tr>
<td>Sheath Poisson’s ratio $\nu_s$</td>
<td>0.33</td>
</tr>
<tr>
<td>Pellet coefficient of expansion $\alpha$ (K$^{-1}$)</td>
<td>9.25$\times 10^{-6}$</td>
</tr>
<tr>
<td>Pellet initial temperature $T_0$ (K)</td>
<td>293.15</td>
</tr>
<tr>
<td>Pellet increased temperature $\Delta T$ (K)</td>
<td>200.0</td>
</tr>
</tbody>
</table>

Xu [34] has formulated the contact pressure for axisymmetric contact without considering the friction, which gives as follows:

$$
p = \begin{cases} 
(\alpha_p \Delta T R_p - g_0) & \left( \frac{R_s}{R_p} \right)^2 + 1 + \nu_s \left( \frac{R_s}{R_p} \right)^2 - 1 \end{cases} \right) \right)^{-1}

\begin{align*}
(\alpha_p \Delta T R_p & \geq g_0) \\
(\alpha_p \Delta T R_p & < g_0) 
\end{align*}

The expression (4.1) is employed as the analytical solution for contact pressure and it will be compared by the calculated results from the developed code.

Table 4-2 shows the nodal contact force calculated from the developed code under a mesh size of 4mm, under which the pellet axial direction is divided into four elements. Each surface element has three contact nodes. The mesh of this geometry can be understood from Figure 4.2.
Figure 4.2: Mesh Cross-section under Size of 4mm for the Pellet and the Sheath

Table 4-2 Nodal Contact Force under Zero Initial Gap

<table>
<thead>
<tr>
<th>Axial coordinate (mm)</th>
<th>Nodal force (N) ($\mu=0$)</th>
<th>Nodal force (N) ($\mu=0.2$)</th>
<th>Nodal force (N) ($\mu=0.4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>84.50</td>
<td>76.28</td>
<td>72.16</td>
</tr>
<tr>
<td>6</td>
<td>321.98</td>
<td>328.82</td>
<td>335.13</td>
</tr>
<tr>
<td>4</td>
<td>160.99</td>
<td>166.86</td>
<td>172.52</td>
</tr>
<tr>
<td>2</td>
<td>321.98</td>
<td>345.12</td>
<td>369.49</td>
</tr>
<tr>
<td>0</td>
<td>160.99</td>
<td>180.30</td>
<td>201.59</td>
</tr>
<tr>
<td>-2</td>
<td>321.98</td>
<td>345.12</td>
<td>369.49</td>
</tr>
<tr>
<td>-4</td>
<td>160.99</td>
<td>166.86</td>
<td>172.52</td>
</tr>
<tr>
<td>-6</td>
<td>321.98</td>
<td>328.82</td>
<td>335.13</td>
</tr>
<tr>
<td>-8</td>
<td>80.50</td>
<td>76.28</td>
<td>72.16</td>
</tr>
</tbody>
</table>
In general, the nodal forces at four corners \( F_i (i = 1, 2, 3, 4) \) of the nine-node element (see Figure 2.2) are one quarter of the nodal forces at the midpoint of four edges \( F_j (j = 5, 6, 7, 8) \). However, since two neighboring elements have conjunction on the same element boundary, the nodal forces calculated on these shared edges will be the summation of the two different nodal forces from the two neighboring element. As a result, the nodal contact forces at the conjunct corners \( (z = 0, \pm 4) \) are around one half of their neighboring nodal \( (z = \pm 2, \pm 6) \) contact forces, while the nodal contact forces at the two ends of the pellet \( (z = \pm 8) \) are about one quarter of their neighboring nodal \( (z = \pm 6) \) contact force. These characters are clearly shown under the frictionless case (2\textsuperscript{nd} column in Table 4-2), where the distribution of the contact pressure is uniform.

The contact pressure within each element can be calculated from the contact force. Figure 4.3 shows the contact pressures in the pellet midplane calculated by the code under the frictionless case are identical to those analytical solutions. From the results comparison under different coefficients of friction, it is clear that the contact pressure increases along with the frictional force.

![Contact Pressure for Pellet Midplane](image)

**Figure 4.3: Comparison of the Contact Pressure for Pellet Midplane**
A further comparison of the results from ABAQUS simulation is presented here. The general contact algorithm used in most ABAQUS analysis has shown excellence performance in many examples. However, the software requires users to define the contact surfaces before it starts the simulation, and these contact surfaces have to be in physical contact. In other words, it cannot simulate a problem that is to determine whether two surfaces would be in contact or not when it is started from a surface status of separation. Therefore, in the case study conducted in ABAQUS, a contact status between sheath inner surface and pellet outer surface is predefined.

Since the expansion is symmetric with respective to pellet midplane, only half of the pellet is modeled. A boundary condition of zero axial displacement at the right surface of the pellet and the sheath is applied to represent the symmetric condition. Both components are controlled by the wedge mesh with 1mm mesh size.

Figure 4.4 shows the contours of the contact pressure from the simulation under zero initial radial gaps with coefficient of friction 0.4.

![Figure 4.4: Contour of the Contact Pressure under Zero Initial Gap with Coefficient of Frictional 0.4](image-url)
The pressure data from ABAQUS simulation is exported at surface nodal locations $z = 1, 3, 5, 7$ so that it is used to compare with the average contact pressure within each contact element (see Figure 4.2) calculated by the code.

Figure 4.5 shows the comparison of contact pressure under different coefficients of friction with the obtained data described above.

![Contact Pressure along Pellet Axial Direction](image)

**Figure 4.5: Comparison between Results from the Code and ABAQUS**

The result displayed in Figure 4.5 illustrates the identity between the code and ABAQUS simulation. This plot also implies that the frictional force increases the contact pressure.

To understand the reason behind this phenomenon, a test case with zero Poisson’s ratio is studied. In other words, the new problem can be described as Table 4-1 except for the Poisson’s ratios for the pellet and the sheath, where both of them are defined as zero.
The same procedure is conducted for this problem as well as the mesh identical to the pellet and the sheath as Figure 4.2. The so-obtained results on the contact pressure and corresponding comparison are presented in Figure 4.6.

![Figure 4.6: Contact Pressure Comparison between Zero and Nonzero Poisson’s Ratio](image)

Considering the results with zero Poisson’s ratio and the coefficient of friction 0, 0.2, and 0.4 respectively, Figure 4.6 tells that the contact pressures are consistent under these combinations. This finding implies that the contact pressure is affected by the frictional force or the frictional shear stress through the Poisson’s ratio. This outcome could be further validated by the experiment.

The example of contact between one pellet and its surrounding sheath is of significant importance because it verifies the methodology for the finite element model of the axisymmetric bodies and the algorithm for the three-dimensional frictional contact.

### 4.2 Contact within a fuel rod structure

To prove the assembling methodology in chapter 4.2 and test the contact solution for multibody frictional contact problems in the fuel rod, a relative short rod containing a hollow sheath, eight
fuel pellets and two endcaps is studied. Such a system can be understood from Figure 1.2. Table 4-3 summarizes the fuel rod geometry and other related parameters.

<table>
<thead>
<tr>
<th>Table 4-3 Description of Fuel Rod Frictional Contact Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
</tr>
<tr>
<td>Rod length (mm)</td>
</tr>
<tr>
<td>Rod diameter (mm)</td>
</tr>
<tr>
<td>Sheath length (mm)</td>
</tr>
<tr>
<td>Sheath inner radius $R_{s,i}$ (mm)</td>
</tr>
<tr>
<td>Sheath outer radius $R_{s,o}$ (mm)</td>
</tr>
<tr>
<td>Sheath Young’s modulus $E_s$ (GPa)</td>
</tr>
<tr>
<td>Sheath Poisson’s ratio $\nu_s$</td>
</tr>
<tr>
<td>Radial gap between sheath and pellet (mm)</td>
</tr>
<tr>
<td>Number of pellet</td>
</tr>
<tr>
<td>Pellet radius $R_p$ (mm)</td>
</tr>
<tr>
<td>Pellet length $l_p$ (mm)</td>
</tr>
<tr>
<td>Pellet Chamfer height(mm)</td>
</tr>
<tr>
<td>Pellet Chamfer width(mm)</td>
</tr>
<tr>
<td>Pellet dish depth(mm)</td>
</tr>
<tr>
<td>Pellet dish radius (mm)</td>
</tr>
<tr>
<td>---------------------------------</td>
</tr>
<tr>
<td>Pellet Young’s modulus $E_p$ (GPa)</td>
</tr>
<tr>
<td>Pellet Poisson’s ratio $\nu_p$</td>
</tr>
<tr>
<td>Pellet coefficient of expansion $\alpha$</td>
</tr>
<tr>
<td>Pellet initial temperature $T_0$ (K)</td>
</tr>
<tr>
<td>Pellet increased temperature $\Delta T$ (K)</td>
</tr>
<tr>
<td>Axial gap between pellets (mm)</td>
</tr>
<tr>
<td>Axial gap between pellet and endcap (mm)</td>
</tr>
<tr>
<td>Total axial gap (mm)</td>
</tr>
<tr>
<td>Endcap radius (mm)</td>
</tr>
<tr>
<td>Endcap thickness (mm)</td>
</tr>
<tr>
<td>Endcap Young’s modulus $E_p$ (GPa)</td>
</tr>
<tr>
<td>Endcap Poisson’s ratio $\nu_p$</td>
</tr>
<tr>
<td>Contact coefficient of friction</td>
</tr>
<tr>
<td>Mesh size (mm)</td>
</tr>
</tbody>
</table>

For the two cases described in the above table, calculation can be reduced by modelling only half of the fuel rod by taking the advantage of symmetry. Results were calculated using the code and ABAQUS. In the case without axial gap, the contact surface between neighboring pellets and pellet to endcap are predefined in the ABAQUS simulation, while in the case with open axial...
gap, these potential axial contact surfaces are not predefined because they would not in axial contact after thermal expansion.

The general shape after deformation can be understood from Figure 4.7, where the left end represent the rod midplane while the right chamfered end represents the high stiffness endcap.

![Figure 4.7: Rod Axial displacement Distribution](image)

Once the nodal contact force is obtained from the code, the following procedure is applied to calculate the contact pressure distribution within an element.

Assume that node 4, 7 and 3 in Figure 2.2 are the nodes responsible for the contact on the surface, we can apply equation (2.2) with $\eta = 1$ to calculate the shape functions. According to the finite element method, if a polynomial function $P$ is assumed to express the contact force distribution function, where $P = a\xi^2 + b\xi + c$, the relation between the force distribution and the equivalent nodal force can be written as

$$\int_{-1}^{1} \begin{bmatrix} L_4 \\ L_7 \\ L_3 \end{bmatrix} Pd\xi = \begin{bmatrix} F_4 \\ F_7 \\ F_3 \end{bmatrix}$$  \hspace{1cm} (4.2)
where the contact force vector \( \{F_4\}, \{F_7\}, \{F_3\} \) can be obtained from the code.

Equation (4.2) can be solved to find the unknown parameter \( a, b, c \) for the contact force distribution function \( P \) within each element as

\[
\begin{bmatrix}
    a \\
    b \\
    c 
\end{bmatrix} = \begin{bmatrix}
    15 & 15 & 15 \\
    22 & 22 & 22 \\
    -3 & 0 & 3 \\
    2 & 2 & 2 \\
    24 & 9 & 24 \\
    22 & 22 & 22 
\end{bmatrix} \begin{bmatrix}
    F_4 \\
    F_7 \\
    F_3 
\end{bmatrix}
\]

If the force distribution \( P \) is known, the pressure distribution can be calculated as

\[
\text{pressure} = \sum_{i=1}^{nh} \left( P \right)_i / (2\pi R_{i,i})
\]

where \( (P)_i \) represents the contact force distribution for any harmonic position.

Figure 4.8: Radial Contact Pressure along the Axial Direction
Figure 4.8 shows the radial contact pressure on several specific axial locations. The comparison indicates that the results from the developed code have a good match with the results from ABAQUS, and both of them show the significant high contact pressure at regions near the endcap.

4.3 Bending of fuel rod

In this example, the fuel rod is studied with the temperature distribution for each fuel pellet as

\[ T = T_0 + \Delta T + \Delta T' \frac{r}{R_p} \cos \theta \]

where \( \Delta T' = 25K \) is a prescribed temperature variation value.

Other parameters are the same as the Case 1 in Table 4-3.

Accordingly the fuel pellets have no temperature gradient in the axial direction, while they have constant temperature gradient in the \( y \) direction, which can be described as the following figure:

![Temperature Gradient in the Pellet Cross-section](image)

**Figure 4.9: Temperature Gradient in the Pellet Cross-section**
Two scenarios with zero initial gap and totally 2mm initial gap are studied for bending comparison. The deflections of the sheath inner surface are calculated from the developed code (with coefficient of friction 0.0, 0.1 and 0.4, respectively) and they are compared with the initial code developed by Yu et al. [2]. The results are shown in Figure 4.10 and Figure 4.11.

**Figure 4.10: Deflections of Sheath inner Surface without initial axial gap**

**Figure 4.11: Deflections of Sheath inner Surface with initial axial gap 2mm**
Figure 4.10 tells that without the initial axial gap, the frictional force affect little on the sheath deflection. This result matches our experience because in this scenario the fuel pellets have very limited space to move inside the fuel rod structure.

With 2mm initial axial gap, the fuel pellets have some space to expand axially and they are not in contact axially after expansion. Under this scenario, the dashed lines in Figure 4.11 shows that the frictional force decrease the deflection of the sheath, and they are above the bottom line of the compared results, which is calculated based on the assumption of sufficient of frictional force.
Chapter 5. Conclusions

A numerical study on the frictional contact problem in CANDU nuclear fuel rod is conducted. The model is developed through an annulus-type finite element. The contact problem is construed by the contact force and relative displacement. The algorithm for the frictional contact is formulated as a linear complementarity problem.

A computational code is developed from the proposed algorithm. The main objective of the code is to assess the bending behavior of fuel rods after coupling the contact package with the previously developed packages on heat transfer and thermal expansion.

Case studies in the research indicate that the frictional force inside the fuel rod raises the contact pressure between the contact surfaces and decreases the sheath deflection of the sheath surface, and these effects are caused by the Poisson’s ratio from the material.

The computational speed for the tested cases by means of the developed code is faster than that by the commercial software. Since the code is designed for the specific purpose of application, it has high accuracy and reliability.

The methodology established for the static three-dimensional frictional contact can be further applied to the dynamic problems.
References


[34] Xu, S., 2000, “Three dimensional thermal stress and contact problems in a CANDU fuel element”, thesis for degree of Master of Engineering Science, the University of Western Ontario, London, Ontario, Canada.