The Frequency and Modal System Identification of the Balloon-borne Imaging Testbed

by

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A thesis submitted in conformity with the requirements for the degree of Master of Applied Science
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Abstract

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The Balloon-borne Imaging Testbed (BIT) is a stratospheric ballooning project for astronomy that was successfully launched on September 18, 2015 from Timmins, Ontario to an altitude of 36 km over the period of one night. The design of the BIT gondola must allow the on-board telescope to maintain a constant position for an extended period of time during which the structure is subjected to external forces. Oscillations created by resonances of the gondola can prove detrimental to the resulting image quality. Therefore, the development of a modal analysis method for the gondola is explored in this thesis. This method combines simulation with measurement results to identify the modes of the system. For the gondola’s main structure, the four lowest natural frequencies and their corresponding mode shapes are identified. BIT represents a new generation of balloon-borne telescopes that can produce images comparable to those of space telescopes but at significantly lower costs.
Acknowledgements

The author would like to thank his thesis supervisor Professor Chris J. Damaren for his tutelage, patience, and support during the course of this research project. The author would also like to thank Professor C. Barth Netterfield for providing the lab space, software, as well as the opportunity to be working on the bit project. Special thanks goes to Javier Romuakdez and John Hartley for their contributions in setting up the experiment and creating the electronics that make the experiment possible. The author must also thank everyone one on the bit team, including Matthew Galloway and Ivan Padilla, for the construction of the gondola over the past two years. Without any of them, this project and thesis would not be possible.
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1 Introduction

Developments in the aerospace industry for scientific research have been phenomenal. In the field of astronomy, the Hubble Space Telescope has offered a new and profound understanding of the universe through its spectacular images. However, the cost of producing space-faring vehicles has been astronomical. On the other hand, although terrestrial or aerial vehicles are significantly cheaper, many of their capabilities to serve as astronomical telescopes are restrained by atmospheric imperfections. A possible solution is provided by the use of balloons. Despite the current ubiquity of scientific ballooning, there is a scarcity of high accuracy control dynamics for these atmospheric vehicles. The Balloon-borne Imaging Testbed (BIT), a project funded by the Canadian Space Agency is designed to demonstrate the unprecedented pointing accuracy of 50 milliarcseconds, the rough equivalent of seeing a disc of 50 m in diameter on the surface of the moon from Earth [1]. A new and innovative structural design is required to satisfy the requirements.

If successful, BIT will represent a new generation of atmospheric telescopes that can produce measurements comparable to those of space telescopes but at significantly lower costs and will be an instrumental step in propelling Canada to the forefronts of atmospheric astronomy and scientific ballooning.

1.1 Background on BIT

BIT’s goal is to showcase an ultra-high-accuracy pointing system. BIT is to be launched at night from a newly built scientific ballooning facility in Timmins, ON in September 2015 to an altitude of approximately 40 km where it will remain for several hours.

The gondola of BIT (Figure 1 on the next page) was designed by the author between the summer of 2012 and 2013. Its construction was achieved by everyone at the University of Toronto High Bay who is on the BIT team between 2013 and 2015. At the time of this thesis, BIT is at its final stages of integration and testing in preparation for the Timmins launch campaign in a few weeks.

1.1.1 Scientific Significance of BIT

The scientific goal of BIT is to create a next generation telescope for astronomy. Specifically, BIT is designed to create diffraction limited optical or near-UV images at a resolution which is better than terrestrial telescopes at these wavelengths and much much cheaper than space based telescopes [1].

1.1.2 Engineering Significance of BIT

The engineering objective of the project is to design, analyse, and construct the BIT structure that allows the on-board telescope to maintain a constant orientation for an extended period of time, during which the Outer Frame is subjected to dynamic forces such as turbulence and wind shear. Due to the high level of pointing accuracy required, the normally imperceptible oscillations created by stepper motors, ball bearings, and frame resonances can prove to be detrimental to the resulting image quality. The mitigation of these concerns requires a multidisciplinary optimization of the system through structural and frequency analyses.

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1Other collaborators include University of Durham for the characterization of the science camera, CNES for providing the launch and launch facility, and included JPL for integrating its own telescope. At the University of Toronto, BIT is a collaborative project between the Institute for Aerospace Studies, Department of Physics, and the Department of Astronomy.

2An optical system is diffraction limited if it has the ability to produce images with angular resolution as good as the instrument’s theoretical limit caused by diffraction.
Moreover, these analyses are integral in the design of the on-board control system for achieving the desired pointing requirement.

Although not a direct focus of this thesis, a potential application of its results is the incorporation of the identified mode shapes and frequencies in the design of a control system. The mathematical development of such a motion model is presented in Appendix A on page 52.

1.2 An Overview of Structural Analysis

With a few minor exceptions, the structural analysis for BIT can be broken down into two major segments — stress analysis and frequency analysis.

1.2.1 The Importance of Structural Analysis

For a large mechanical structure that is designed to traverse the upper stratosphere such as BIT, naturally, it is important to make sure that the structure can survive the journey with a significant margin of safety. The criteria of this analysis are intuitive and direct, as they are usually explicitly specified by the launch provider (in this case, CNES) and their safety factor requirements are very clear-cut and justifiable. The frequency analysis, on the other hand, is not to ensure the mechanical system would not fail, but rather to
ensure that the system can perform the task it is set out to do. The result of such analysis is no longer pass or fail and thus much more care is needed for the interpretation of the results.

As such, the focus of this thesis report will be scrutinizing the nuances of the frequency analysis of the BIT gondola. A detailed methodology will be provided on its system identification. Results from experiments that were carried out will also be presented.

2 Approach and Methodology

This section discusses in great detail the primary experiment conducted towards the creation of this thesis.

2.1 Framing of the Frequency Analysis Problem

As suggested above, the basic frequency analysis question we want to solve is: What is the vibrational behaviour of the BIT gondola and how do we know that it will not adversely affect the pointing accuracy? The first part of the question is a classic system identification problem. As such, the frequency analysis of BIT gondola can be framed as the modal system identification of the BIT gondola. To deconstruct the problem further, first, we want to define specifically what we would like to achieve.

Simply stated, we want to learn the characteristic frequencies and mode shapes of the BIT gondola within a range of frequencies that we care about. There are two ways which this problem can be solved – through finite element analysis (FEA) simulation and through direct measurements. Each method has its benefits and shortcomings.

**FEA simulation** This method of modal system identification has the obvious benefit of versatility – the retrieval of results does not require the physical system, the mode shapes can be easily represented visually, and the information regarding the entire gondola can be retrieved to arbitrary precision. However, the FEA simulation method has several fundamental drawbacks that makes our system identification completely relying on it infeasible. Namely, all of the benefits stated above rely on an accurate FEA model of the physical system, and this is often difficult to achieve due to computational limitations – the greater the accuracy of the model, the greater the computational time. A simplified model requires all sorts of assumptions that, if not chosen properly, may not actually represent the system.\(^3\)

**Physical measurements** This method of modal system identification has the advantage of accuracy\(^4\). In laymen’s terms, what you measure is what you get. However, sensors are also highly constrained because they are physical objects. If every sensor measures a single point on the gondola, information regarding mode shapes is only known at those measured locations. It can get increasing difficult to add more sensors to represent the continuum that is the physical gondola.\(^5\)

**Combining the systems** Due to each of their advantages and limitations, both systems are employed in the system identification of the BIT gondola. Specifically, the procedure can be stated as the following:

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\(^3\)An exact definition on how the gondola is reproduced in the FEA model and its associated assumptions are stated in Section 2.2.4 on page 7.

\(^4\)Up to a certain degree. All sensors are limited by their ranges of operation and noise.

\(^5\)A methodology for choosing the locations of the sensors is presented in Section 2.4 on page 10.
1. Create an FEA model of the bit gondola (Section 2.2.4 on page 7)

2. Simulate the FEA model to retrieve characteristic frequencies and mode shapes (Section 2.3 on page 10)

3. Determine locations of greatest amplitude for modes that fall within the frequency range of interest (Section 2.4 on page 10)

4. Place sensors (accelerometers) onto the bit gondola at the locations of interest (Section 2.5 on page 11)

5. Collect impulse response data from the sensors (in the time domain) (Section 2.6 on page 22)

6. Process the data and create Fast Fourier Transform (FFT) results (Section 2.7 on page 24)

7. Identify peaks to determine
   (a) Natural frequencies (Section 2.8 on page 26)
   (b) Amplitude ratios (between sensors) (Section 2.11 on page 34)

8. Identify spread to determine the damping ratios (Section 2.10 on page 33)

9. Use FFT circle plots (Nyquist plots) to identify phases between sensors (Section 2.9 on page 26)

10. Match measured mode shapes with the simulated mode shapes (Section 2.12 on page 35)

11. Interpret the results (Section 3 on page 40)

Although it can be simply stated, each of the above steps is actually a fairly involved process and is individually explained in the following sections. If the simulation results and the physical measurements corroborate with each other, the “true” (more accurate) frequencies will be represented by the physically measured results and the “true” (more complete) mode shape will be represented by the simulation result. At this point, the system identification can be said to be successfully completed.

2.1.1 A Cantilevered Beam Analysis

A simplified version of the above steps is performed on a cantilevered beam to check for the correctness of the methodology. This beam analysis is presented in Appendix B on page 58. In the analysis, the FEA steps are replaced by an analytical solution of the beam using Euler-Bernoulli beam theory. The identification of the mode shapes and amplitude ratios uses the exact mathematical method as developed in Sections 2.9 and 2.11 on page 26 and on page 34.

2.2 Mechanical Description

The mechanical component of bit can be summarized as a mechanism that accommodates the pointing degrees of freedom, maintains the pointing accuracy, and withstands the various forces imposed on a balloon-borne telescope. The bit gondola is approximately 3 m tall from the base to the pivot joint (Figure 2) and weighs about 800 kg, and it is constructed primarily out of aluminum honeycomb panels [2].
2.2.1 Pointing Degrees of Freedom

The telescope gondola must have three free rotational axes to allow full degrees of freedom. The established mechanical design allows for the following ranges of pointing on the three axes – pitch, roll, and yaw. In the altitude (also termed pitch or elevation) direction, the telescope should point from approximately $20^\circ$ to $55^\circ$ from the horizon. The lower bound is limited by the amount of atmosphere in the view; and the upper bound is limited by the full expansion of the helium balloon at cruise altitude. From analysis of sky movement, for the duration of integration on the order of a few tens of minutes, the telescope must be able to rotate its view by around $\pm4^\circ$ (in the roll direction). Finally, the telescope will not be limited in the azimuth (heading) direction as the balloon is rotationally symmetric about the flight train$^6$ axis.

2.2.2 The Three-Frame Structure

In order to achieve these degrees of freedom, three gimbal frames are used to allow telescope rotation in the three independent axes – the Inner Frame, the Middle Frame, and the Outer Frame as shown in Figure 3 on the following page. Since the balloon is symmetric about the vertical axis, the azimuthal angle degree of freedom will be achieved by a pivot connecting the Outer Frame to the balloon tether. Inside the Outer Frame, the Middle Frame, is connected at the bow and stern of the Outer Frame to allow for the roll motion. And lastly, inside the Middle Frame, the Inner Frame, which rigidly holds the telescope, is connected at the port and starboard corners of the Middle Frame which allows for elevation movement. This equatorial mounting scheme accommodates the full degrees of freedom.

$^6$The flight train is the colloquial term for the tether that connects the balloon to the gondola.
Figure 3: A diagram showing the three frames of the BIT gondola
2.2.3 Construction Materials

As mentioned previously, any low frequency vibrations incurred by structural vibrations could prove detrimental to the pointing accuracy. Consequently, a very stiff and strong material, aluminum honeycomb panels, was chosen as the main structural element for all three frames. Moreover, directly surrounding the telescope, a prismatic baffle used for thermal regulation for the telescope is also structured out of the same aluminum honeycomb panels as the frames. Due to the large sizes of the Outer and Middle Frames, as well as their hollow geometry, they can produce fairly significant vibration effects.

2.2.4 Constructing the Simulation Model

Even though SolidWorks Simulation has the capability of simulating a detailed body structure, a direct simulation of the entire structure with all its intricacies would prove overly complicated. Therefore, the geometry of the structure must be greatly simplified in order to practically perform the simulation. Since both the Inner Frame and the Outer Frame’s main structural elements are honeycomb panel structures, a shell structure would be an appropriate simplification.

Modelling the structure as a shell provides two main benefits: 1, the complexity of the simulation is significantly reduced (compared to a solid body simulation); and 2, unrealistic edge effects due to corner connections that are generally found in solid body simulations are removed. (The solver in SolidWorks Simulation assigns a bending property to the sheet as well as the connections. There are no elements through the thickness of the shell.)

Creating the correct shell  Because the version of SolidWorks Simulation available does not support composite shell definition, the modelling was done manually by defining a shell that has an equivalent bending stiffness $EI$ as the Teklam panels. It can be shown that the $EI$ of a sandwich panel is approximately the product of the area moment $I$ and the Young’s modulus $E$ of the surface sheets [3]. Thus, a shell thickness of 12.4 mm, which would represent the same $I$ as two 0.02” (0.5 mm) sheets kept at 0.98” (24.9 mm) apart, was used for the shell definition.

In addition, the density of the material was recalculated so that a 12.4 mm thick sheet would have the same mass per area as the full 1” (25.4 mm) thick sheet. A custom material was created for this purpose. The final property assignments are documented in Table 1.

<table>
<thead>
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<th>Property</th>
<th>Value</th>
<th>Units</th>
<th>Source</th>
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<td>Elastic Modulus in X</td>
<td>69000</td>
<td>N/mm²</td>
<td>AL3003-H18</td>
</tr>
<tr>
<td>Poisson’s Ration in XY</td>
<td>0.33</td>
<td>0.33</td>
<td>AL3003-H18</td>
</tr>
<tr>
<td>Shear Modulus in XY</td>
<td>25000</td>
<td>N/mm²</td>
<td>AL3003-H18</td>
</tr>
<tr>
<td>Mass Density</td>
<td>523</td>
<td>kg/m³</td>
<td>Computed</td>
</tr>
<tr>
<td>Tensile Strength in X</td>
<td>200</td>
<td>N/mm²</td>
<td>AL3003-H18</td>
</tr>
<tr>
<td>Yield Strength in XY</td>
<td>185</td>
<td>N/mm²</td>
<td>AL3003-H18</td>
</tr>
</tbody>
</table>

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7Ch. 1.4 “Classical Sandwich Analysis,” pp. 42.
8For simulation simplicity, higher order effects such as shear deformation were not considered.
Defining distributed masses  In order to create a high fidelity model, distributed masses were assigned to the shell model. Six split-line areas were created where the batteries are located in the model to represent the batteries. A mass of 40 lb (18 kg) was assigned to each of the areas, representing the weight of the individual batteries. A separate shell plate was created for the reaction wheel, whose diameter is approximately the same as that of the real reaction wheel (100 cm), sitting on a rigid cylinder, whose height is approximately the same as that of the reaction wheel (20 cm). A separate spline-line ring was defined on the plate at which a distributed mass of 100 lb (45 kg) was assigned. To reduce simulation complexity and better represent the actual system, this newly created structure is set to be a rigid surface, meaning that the part would not deform elastically in the simulation. A picture of the mass distribution setup can be seen in Figure 4.

In addition to the above mass assignments, extra masses were added to all edge joints where the extra masses were estimated using the weight of the joining aluminum sheets, fasteners (bolts, nuts, and washers), and epoxy glue. Significant electronics boxes, such as the Outer Frame Computer, were also added by defining new split-lines.

Figure 4: Assigned mass distributions on the Outer Frame
The self-mass of the shell structure is assigned automatically when the density of the material is defined as in Table 1 on page 7. In certain areas, an extra shell layer was added in the model to represent extra reinforcement material. Specifically, this was done to the sheet that joins the mid-plane ring to the lower half of the Outer Frame, which can be clearly seen in Figure 3 on page 6. This is a rather convenient method of “reinforcing” material without redundantly defining a new type of sheet. However, this was not done to the very bottom sheet of the Outer Frame, where there are two layers of 1" aluminium honeycomb panel. A new equivalent shell was defined for a double-layered honeycomb panel.

Mass consistency A crucial consistency check for the integrity of the model definition is that the total mass should be fairly close to the mass of the real structure. This step is critical in the accuracy of the simulation results, because, in a naïve sense, the natural frequencies are proportional to the mass of a structure. Unfortunately, this was not done for the simulation model of the Outer Frame because of the limited access the user has on the simulation data of SolidWorks Simulation. This might have contributed to a portion of the inaccuracies in the results.

One frame vs. multiple frames When creating the simulation model, one of the choices that could have been made is to simulate all three frames of the gondola (as described in Section 2.2.2 on page 5). However, actually defining such a structure in SolidWorks is a non-trivial task. Since the different frames of the gondola are moving independently with respect to each other, flexible connections must be made between the frames. A proof-of-concept setup was made between the Outer Frame and the Middle Frame using an unlocked pin connection (as shown in Figure 5). This pin connection allows for a rotational degree of freedom but is constrained in all other directions. In order to create this pin connection, a separate shell geometry was made to create the necessary contacting surfaces. Again, this extra geometry was set to be a rigid surface.

Figure 5: Pin connection between the Outer and Middle Frame shells

From a few trial simulations, it was noted that whether the Middle Frame shell was included did not
significantly change the elastic body modes of the Outer Frame. It can be assumed that, at least in simulation, the frames are sufficiently decoupled. Moreover, sensor measurements are only made on the Outer Frame. Therefore, to save on computation time, only the Outer Frame shell is simulated for the purposes of this report.

2.3 Simulation

Once a represented shell model is created, the next step is to simulate the system using SOLIDWORKS Simulation.

2.3.1 Defining Boundary Conditions

One of the most crucial aspects of this simulation process is to properly define the boundary conditions for the structure. SOLIDWORKS Simulation allows for the definition of a variety of simulation conditions such as fixed, sliding, pinned, and free. Unfortunately, it does not provide a direct definition for a gondola hanging from a tether, which is the real structure. However, we know that a 20 m long tether (at the University of Toronto High Bay) does not fix the motion of the gondola at any position. Therefore, the closest boundary condition definition would be to define the structure as a free body, free from any constraints.

It is also noted that SOLIDWORKS Simulation allows for the definition of gravity in a simulation. However, it can be shown that a uniform force term in the model of a structure does not affect its elastic body modes. Moreover, a gravity term cannot be defined for a free body structure. Therefore, gravity was not defined in the simulation.

An incorrect definition Just from a visual inspection of the physical system, one might be tempted to define a pinned connection at the top of the gondola representing where the tether is attached to the pivot. This is an incorrect definition as the pinned connection would restrict all translational motion at the top of the gondola whereas a tethered connection does not. Table 2 shows the modal frequencies of the first six elastic modes using the two types of boundary conditions.

<table>
<thead>
<tr>
<th>Elastic mode</th>
<th>1 (^{\text{sim.}})</th>
<th>2 (^{\text{sim.}})</th>
<th>3 (^{\text{sim.}})</th>
<th>4 (^{\text{sim.}})</th>
<th>5 (^{\text{sim.}})</th>
<th>6 (^{\text{sim.}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned connection</td>
<td>46.6 Hz</td>
<td>47.2 Hz</td>
<td>62.4 Hz</td>
<td>87.0 Hz</td>
<td>95.2 Hz</td>
<td>99.5 Hz</td>
</tr>
<tr>
<td>Free connection</td>
<td>36.3 Hz</td>
<td>41.8 Hz</td>
<td>71.6 Hz</td>
<td>80.9 Hz</td>
<td>91.9 Hz</td>
<td>93.2 Hz</td>
</tr>
</tbody>
</table>

Note that due to the large number of natural frequencies presented in this experiment, to minimize confusion, all simulated frequencies will be labelled with the superscript \(^{\text{sim.}}\) and all measured or identified frequencies will be labelled with \(^{\text{mea.}}\). In Appendix B on page 58, the frequencies calculated using a theoretical beam analysis will be labelled with \(^{\text{the.}}\).

2.4 The Identification of Sensor Placement Locations

Given the undamped linear system solved by SOLIDWORKS Simulation, the physical interpretation of a mode is when each point in the structure vibrates at the same frequency and reaches maximum deflections
simultaneously [5]. Assuming that the physical structure can be approximated by the methods described earlier, based on this information, it would be advantageous to place the sensors at the spatial antinodes (or locations of greatest structural deflection) to maximize the signal to noise ratio for the sensors.

2.4.1 Visual Inspection of Simulation Results

Table 3 on the next page shows the mode shapes of the first twenty modes as simulated by SOLIDWORKS SIMULATION. Ignoring the first six rigid body modes, the physical distortions of the elastic modes can be clearly seen and are circled in the diagrams. Based on the observed mode shapes, sensor locations are heuristically chosen to maximize the measured amplitudes. The final choices are shown in Figures 12 and 13 on page 20 and on page 21.

2.5 Sensor Selection, Installation, and Readout

For the modal system identification of the BIT gondola, physical measurements were done using accelerometers. There are several advantages to this particular choice of sensors. Firstly, accelerometers are inexpensive and fairly common place. The miniaturization of microelectromechanical systems (MEMS) have made these inertial sensors much smaller than a fingernail and they can be fastened to anywhere in the form of a breakout board. Secondly, the directions of the accelerometer measurements are the same as those of the mode shapes. As the second derivative of a sine function is another sine function, the acceleration of the modal vibrations maintains the “shape” of the modal vibration with a constant scaling. Thirdly, accelerometers can preserve much more information in the higher frequencies. In the Fourier domain, the amplitude is proportional to frequency squared, so the peaks in the higher frequencies would appear much more prominently. Because of these reasons, a total of 21 accelerometer channels were used in the physical data collection in the system identification process.

2.5.1 Accelerometer Choice

As shown in Figure 6a on page 15, the accelerometer chosen for this experiment is the ADXL335 Triple Axis Analogue Accelerometer from Analog Devices. However, since the accelerometer is actually in the form of a MEMS, the breakout board is designed by Produino which builds electronic peripheral devices for hobbyist development kits such as the Arduino. The accelerometer breakout has a dimension of 0.4 cm×0.4 cm×0.15 cm and weighs around 1 g.9

When making the selection for the accelerometers to use, many considerations were made.

Gyros Before the final results were carried out using accelerometers, the initial proof-of-concept was done using gyroscopes. As part of the BIT control system, three high precision gyros were already available. As such, the first set of experiments was done with these. The mathematical modelling of gyro measurements is presented in Section 2.6.2 on page 24. The results from the gyros measurements are presented in Appendix C on page 62.

9The lightness of the board is actually fairly important so that the measurement device would not contribute to what it’s measuring.
Table 3: The first six modes (rigid body) as solved by SOLIDWORKS SIMULATION

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode (rigid body)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 1</td>
<td></td>
</tr>
<tr>
<td>Mode 2</td>
<td></td>
</tr>
<tr>
<td>Mode 3</td>
<td></td>
</tr>
<tr>
<td>Mode 4</td>
<td></td>
</tr>
<tr>
<td>Mode 5</td>
<td></td>
</tr>
<tr>
<td>Mode 6</td>
<td></td>
</tr>
</tbody>
</table>
Table 4: The next six modes (elastic body) as solved by **SolidWorks Simulation**

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>36.27</td>
</tr>
<tr>
<td>8</td>
<td>41.75</td>
</tr>
<tr>
<td>9</td>
<td>71.56</td>
</tr>
<tr>
<td>10</td>
<td>80.94</td>
</tr>
<tr>
<td>11</td>
<td>91.85</td>
</tr>
<tr>
<td>12</td>
<td>93.20</td>
</tr>
</tbody>
</table>

Mode 7 (36.27 Hz)  

Mode 8 (41.75 Hz)  

Mode 9 (71.56 Hz)  

Mode 10 (80.94 Hz)  

Mode 11 (91.85 Hz)  

Mode 12 (93.20 Hz)
Table 5: The higher frequency modes resulted from the simulation

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz (sim.))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 13</td>
<td>102.17 Hz</td>
</tr>
<tr>
<td>Mode 14</td>
<td>108.47 Hz</td>
</tr>
<tr>
<td>Mode 15</td>
<td>113.71 Hz</td>
</tr>
<tr>
<td>Mode 16</td>
<td>125.74 Hz</td>
</tr>
<tr>
<td>Mode 17</td>
<td>129.15 Hz</td>
</tr>
<tr>
<td>Mode 18</td>
<td>145.43 Hz</td>
</tr>
</tbody>
</table>

Mode 13 (102.17 Hz (sim.))

Mode 14 (108.47 Hz (sim.))

Mode 15 (113.71 Hz (sim.))

Mode 16 (125.74 Hz (sim.))

Mode 17 (129.15 Hz (sim.))

Mode 18 (145.43 Hz (sim.))
Analogue vs. digital  A major consideration for the ADXL335 choice is that it is an analogue accelerometer. There are several advantages for an analogue device over a digital device for the purposes of this experiment. Analogue signals can be more easily interpreted than a digital signal. In this case, an analogue signal is merely a linear mapping of output voltage to acceleration. However, a digital signal requires the setup of a particular communication protocol. The two communication protocol commonly used in MEMS accelerometers and gyros, I²C and SPI, are not setup on any computers used in the BIT experiment.

Conveniently, there are 30 analogue input ports on the BIT Inner Frame Computer designed for thermometer measurements during the BIT flight running at approximately 1260 Hz with timestamps. By taking advantage of these parallel ports, we can make simultaneous accelerometer measurements across the gondola. This is very important for identifying the phase of the gondola vibration modes.

Range of measurement  The measurement range of each of the axes is ±3 g with an accuracy of 10%.

Noise performance  The ADXL335 accelerometer has a noise of 150 μg/√Hz rms. This means that, at a data collection rate of 1260 Hz, the effective resolution of the measurement would be 150 μg/√Hz·√1260 Hz ≈ 4.2 mg or 4.1 cm/s². This value is not significant immediately after the impact but the measurement will become dominated by noise soon after. The requirement to make long duration measurements to identify higher frequency responses is one of the reasons why it is difficult to identify higher frequency modes.

---

10 The resampling to account for jitter in the measurements is explained in Section 2.7 on page 25.
11 The accuracy in amplitude is not that important as we’re more interested in the frequency response results.
**Breakout board**  Like most microelectronic development kits, integrated circuits come in the form of breakout boards. For this Produino accelerometer, the breakout board, seen in Figure 6a, comes in the convenient dimension of a 0.4 cm×0.4 cm square which can be easily mounted to the gondola surface using hot glue.

### 2.5.2 Connectorization

Schematics of the setup can be seen in Figure 7 on this page. Each accelerometer breakout has a set of power cables and a set of signal cables. The power cables are comprised of a 5 V power cable and a ground cable. The signal cables are comprised of either 1 or 3 axes from the breakout, depending on the setup, and a ground cable.\(^{12}\)

![Schematic of the accelerometer readout](image)

**Figure 7: Schematic of the accelerometer readout**

All power cables from the 15 accelerometer breakouts are connected to a jumper block and are powered collectively by a power supply. The geometric schematic of the setup can be seen in Figure 8 on the next page. The final setup of all the accelerometers and wires on the BIT gondola are shown in Figure 9 on page 18.

**Connectors**  A white six-pinned female connector was soldered onto each accelerometer breakout board. Connectorizing between the accelerometer breakouts and the wires offers the freedom to replace faulty accelerometers\(^{13}\) and the freedom to remove the cables without removing the breakouts once they are secured.

\(^{12}\)As it turns out, only one ground cable is needed for all the signals because all the ground in the Inner Frame Computer readout is shared.

\(^{13}\)No faulty accelerometer was actually encountered in the experiment.
Figure 8: Physical layout of the accelerometer wiring. Red lines represent 3-axis accelerometers and green lines represent single-axis accelerometers.

to the gondola.

On the side of the computer readout, a special three-pin Mini-clamp plug/connector is used (which can be seen in Figure 10a on the next page). For power and ground, circle connectors are used to attach to jumper and the power supply.

**Wiring** Due to the low power requirements for these breakouts (on the order of milliamperes), 22 gauges wires were used for all power and signals. They are consistently colour-coded to minimize confusion in the installation process.

**Power** All breakouts were powered using a standalone power supply. The maximum current was set to 0.01 A\(^{14}\) and the voltage was set to 5 V. All power cables for the breakout were connected together and all

\(^{14}\)This is limited by the resolution of the power supply.
Figure 9: Physical setup of the experiment

Figure 10: Wires are connected to the IFC conveniently through the star camera port.
the ground cables were connected together. On the readout side, one ground cable from one signal connector was also connected to the ground of the power supply so that all the grounds are shared.\textsuperscript{15}

**Readout** The Inner Frame Computer is capable of taking in 30 analogue voltage signals. During the actual BIT flight, these will be purposed for the thermometer measurements for temperatures around the gondola. 21 of the 30 ports will be used for simultaneous measurements of the BIT gondola. The plotting is done by an open-source data processing software KST as shown in Figure 11 on this page.

![Figure 11: Readout of the collected data rendered in real time using KST](image)

**2.5.3 Sensor Installation**

Due to the small form factor of the accelerometer breakouts (much unlike the ungainly gyro box, shown in Figure 38b on page 63), they can be easily placed anywhere on the gondola. In the experiment, they are fastened to the gondola surfaces using hot glue because of its convenience in application as well as removal. The installed sensors and their locations on the BIT gondola are shown in Figures 12 and 13 on the following page and on page 21.

\textsuperscript{15}This sharing of ground must be done to prevent the signal voltage to be floating.
Figure 13: Sensor placements (cont.)
2.5.4 Readout and Data Collection

The data is collected by the Inner Frame Computer at the rate of approximately 1260 Hz, which is much greater than the frequency of interest of 50 Hz. The output file is in the form of 31 columns of ASCII data where the first column represents the time stamp of each row of data and the rest of the columns represent the 30 analogue voltage input. Naturally, as only 21 accelerometer outputs are connected to the readout, only the first 22 columns contain useful data. The useful data are loaded into MATLAB for later processing. (See Section 2.7 on page 24.)

2.6 Impulse Response

Now that we have a physical description of our sensor setup, we can begin with the mathematical modelling of the experiment. An easy way (both mathematically and practically) to excite a system across all frequencies is by applying an impulsive input [6]. Mathematically, a perfect impulse (a Dirac-delta function $\delta(t)$) corresponds to a constant value across all frequencies in the Fourier domain. This simplicity will be shown to be algebraically advantageous in the derivations. Practically speaking, an impulse can be approximated simply by an impact from a hammer [6]. To prevent damage to the gondola, a rubber mallet was chosen for this task.

2.6.1 Plant Model

We start with the partial differential equation describing the deformation field $w(r, t)$ of the elastic structure (where $w$ is a function of spatial location $r$ and time $t$) under an impulsive excitation at location $r_a$ with a strength $f$ in the direction $n_a$:

$$M\ddot{w}(r, t) + D\dot{w}(r, t) + Kw(r, t) = n_a\delta(r - r_a)f\delta(t)$$

where $M$, $D$, and $K$ are the mass, damping, and stiffness operators, respectively [6, 8, 9].

The undamped eigenvalue problem (as solved in SOLIDWORKS SIMULATION) is

$$-\omega^2_\alpha M\psi_\alpha + K\psi_\alpha = 0, \quad \alpha = 1, 2, 3, \ldots$$

where $\psi_\alpha$ and $\omega_\alpha$ are the $\alpha$th mode shape (eigenfunction) and undamped natural frequency (root of the

1650 Hz is considered as the effective bandwidth of the fine pointing camera of the telescope.

17It is important to recognize that even though the rest of the columns (23-31) do not contain useful information, when viewed, those data streams can resemble the useful data because of their connectors have floating voltages. This is why keeping track of the connectors and the ports they correspond to is very important.


19An alternative method is to perform a frequency sweep by exciting the system using a frequency generator, such as a woofer or a motor with an offset mass [7]. However, this method is more difficult to implement without being any easier to model (McTavish: ch. II “The Structural System,” pp. 9-3).


21In reality, achieving an infinite value for an infinitesimal period of time is rather difficult. However, if the delta function is approximated by a really sharp rectangular function or Gaussian, the Fourier transforms would have decreasing amplitudes for increasing frequencies (McTavish: ch. II “The Structural System,” pp. 9-9).

22The Dirac delta function is used to represent the point of impact both in space and time.

eigenvalue), respectively. As such, \( w \) can be written as a modal decomposition in the form of
\[
\mathbf{w}(\mathbf{r}, t) = \sum_{\alpha=1}^{\infty} \psi_{\alpha}(\mathbf{r}) \eta_{\alpha}(t).
\]  
(3)

where \( \eta_{\alpha} \) are the modal coordinates.

The mass and stiffness operators have the orthogonality properties of
\[
\int_{V} \psi_{\alpha}^{T} M \psi_{\beta} dV = \delta_{\alpha\beta} \quad \text{and} \quad \int_{V} \psi_{\alpha}^{T} K \psi_{\beta} dV = \omega_{\alpha}^2 \delta_{\alpha\beta}.
\]  
(4)

where \( \delta_{\alpha\beta} \) is the Kronecker delta. In addition, if we make the assumption that damping in the system is relatively small and the modes are orthogonal to the damping operator, we can use the approximation
\[
\int_{V} \psi_{\alpha}^{T} D \psi_{\beta} dV = 2 \zeta_{\alpha} \omega_{\alpha} \delta_{\alpha\beta}
\]  
(5)

where \( \zeta_{\alpha} \) is the \( \alpha \)-th damping ratio [8].

Using modal decomposition as well as the orthogonality conditions described in Eqs. (4) and (5), substituting Eq. (3) into Eq. (1) allows it to be diagonalized and simplified to
\[
\tilde{\eta}_{\alpha}(t) + 2 \zeta_{\alpha} \omega_{\alpha} \tilde{\eta}_{\alpha}(t) + \omega_{\alpha}^2 \eta_{\alpha}(t) = \psi_{a,\alpha} f \delta(t), \ \alpha = 1, 2, 3, \ldots
\]  
(6)

where \( \psi_{a,\alpha} = \psi_{\alpha}^{T}(\mathbf{r}_a) \mathbf{n}_a \) is a constant representing the mode shape evaluated at the impulse location in the direction of the impulse.

Taking the Laplace transform of Eq. (6) on the current page and isolating for \( \eta_{\alpha} \) gives
\[
\eta_{\alpha}(s) = \frac{\psi_{a,\alpha} f}{s^2 + 2 \zeta_{\alpha} \omega_{\alpha} s + \omega_{\alpha}^2}.
\]  
(7)

2.6.2 Sensor Models

Now that we have a full modal description of any elastic system in the frequency domain, the next thing to do is to model the observability of the system. We will start by describing the accelerometer model as it is used in the experiment. However, we will see in the following that the model for the gyroscope has a very similar approach.

**Accelerometer model** Letting \( r_s \) and \( n_s \) be the location and the measurement direction of an accelerometer, the output measurement \( y \) can be expressed as
\[
y(t) = \mathbf{n}_s^{T} \mathbf{\tilde{w}}(\mathbf{r}_s, t)
\]  
(8a)
\[
= \sum_{\alpha=1}^{\infty} \psi_{s,\alpha} \tilde{\eta}_{\alpha}(t)
\]  
(8b)

---

24 For a simple mass-spring-dashpot system, the approximation is exact [10].

25 Note that we abuse notation and denote the Laplace transform by simply changing the time argument \( t \) to the Laplace transform variable \( s \).
where $\psi_{s,\alpha} = n_s^T \psi_{s,\alpha}(r_s)$ is a constant representing the amplitude of the mode shape evaluated at the accelerometer location in the direction of the accelerometer axis. Taking the Laplace transform of Eq. (8b) and substituting Eq. (7) give

$$Y(s) = \sum_{\alpha=1}^{\infty} \psi_{s,\alpha} \eta_\alpha(s) s^2$$  \hspace{1cm} (9a)

$$= \sum_{\alpha=1}^{\infty} \frac{C_\alpha s^2}{s^2 + 2 \zeta_\alpha \omega_\alpha s + \omega_\alpha^2}$$  \hspace{1cm} (9b)

$$= \sum_{\alpha=1}^{\infty} C_\alpha H_\alpha(s)$$  \hspace{1cm} (9c)

where $C_\alpha = \psi_{s,\alpha} \psi_{a,\alpha} f$ is a constant value representing a relative amplitude, and $H_\alpha(s) = s^2 / (s^2 + 2 \zeta_\alpha \omega_\alpha s + \omega_\alpha^2)$ \[6\] 26. Eq. (9b) fully describes sensor measurements as a function of modal parameters in the Laplace domain.

**Gyroscope model**  Although only accelerometers were used for the final results, gyroscopes were used in the initial proof-of-concept for the sensor model, due the availability of the gyros. Letting $r_s$ and $n_s$ be the location and axis of rotation of the gyro, the output measurement $y$ can be expressed as

$$y(t) = \frac{1}{2} n_s^T \nabla^x \mathbf{w}(r_s, t)$$  \hspace{1cm} (10a)

$$= \sum_{\alpha=1}^{\infty} \theta_{s,\alpha} \dot{\eta}_\alpha(t)$$  \hspace{1cm} (10b)

where $\theta_{s,\alpha} = \frac{1}{2} n_s^T \nabla^x \psi_{s,\alpha}(r_s)$ is a constant representing the slope \[27\] of the mode shape evaluated at the gyro location about the gyro axis. Taking the Laplace transform of Eq. (10b) and substituting Eq. (7) on the preceding page give

$$Y(s) = \sum_{\alpha=1}^{\infty} \theta_{s,\alpha} \eta_\alpha(s) s$$  \hspace{1cm} (11a)

$$= \sum_{\alpha=1}^{\infty} \frac{C_\alpha s}{s^2 + 2 \zeta_\alpha \omega_\alpha s + \omega_\alpha^2}$$  \hspace{1cm} (11b)

where $C_\alpha = \theta_{s,\alpha} \psi_{a,\alpha} f$ is a constant value representing a relative amplitude. Eq. (11b) fully describes sensor measurements as a function of modal parameters in the Laplace domain.

**2.7 MATLAB Magic**

All computations in this experiment are carried out using MATLAB. These steps include re-sampling the time series, finding the time of impulse, converting time domain data to frequency domain using the Fast Fourier Transform, creating frequency and amplitude plots, and all miscellaneous functions regarding frequency and phase determination. These processes will be explained in this and following sections.

---

\[26\] Ch. II “The Structural System,” pp. 10-1.

\[27\] Generally, the expression $\frac{1}{2} \nabla^x \mathbf{v}$ represents the “slope” of the displacement field $\mathbf{v}$.
Unit conversion  Analogue voltage data are collected by the Inner Frame Computer (IFC). The voltage measurements can be directly converted to gravitational acceleration by linearly mapping 1.8-3.6 V to $\pm 3g$. It turns out that this remapping is not crucial to the rest of the analysis. However, all gravitational offsets must be removed for proper FFT results.\textsuperscript{28}

Re-sampling  The data that is collected by the IFC are not precisely synchronized to an exact frequency. Rather, there can be a 5\% jitter depending on the operational speed of the computer at the time. When performing frequency analysis, specifically, the Fast Fourier Function, a regular sampling rate is required. Of course, a naive method is to assume that the jitter is small enough and the time series data can be processed directly.\textsuperscript{29} However, due diligence was paid to re-interpolate the data so that the sampling rate is regular. Based on the suggestion of Javier Romualdez, the Akima interpolation\textsuperscript{30} was used \cite{11}. The comparison between an excerpt of the re-interpolated data and the original data time series is shown in Figure 14 on the next page.

Impulse search  Another important piece of data processing needed before performing the Fast Fourier Transform is determining precisely when the impulse occurred.\textsuperscript{31} A clever algorithm was devised that checks every point in the time series until a new value is several standard deviations away from the rest of the values.\textsuperscript{32} This point of transition is then marked as the time when the impulse occurred. The data stream is then truncated to the start of the impulse. This new truncated data stream is then used for the Fast Fourier Transform.

FFT  The Fast Fourier Transform is fairly standard function in MATLAB. Given a vector $v$ of size $N$, it computes the following
\[ \hat{v}(k) = \sum_{j=1}^{N} v(j) \exp \left[ -\frac{2\pi i (j - 1)(k - 1)}{N} \right] \] (12)
where $\hat{v} = \text{fft}(v)$, in which the elements are complex values. The FFT results can then be used to create the amplitude and circle plots used in the analysis \cite{12}.

Smoothing  To plot the FFT results, a slight data manipulation was done to make the curves look slightly smoother. Ideally, the purpose of this smoothing process is to eliminate the Gaussian noise that is inherent in all sensor measurements. One of the ways to accomplish this is to down-sample the time-domain data, then take the FFT of all “sliding-window” permutations of the down-sampling, then averaging the FFT results. However, it turns out that a much simpler method that has almost the same effect is to apply an average filter on the FFT results. Therefore, this method was used to generate all following plots of FFT results. A window of 10 samples was chosen for the down-sampling. Figure 15b shows comparisons between the smoothed data and raw data.

\textsuperscript{28}A constant offset in the time domain translates to a large amplitude for low frequencies in the frequency domain.
\textsuperscript{29}As it turns out, this is not a bad assumption to make as no visually discernible difference can be observed.
\textsuperscript{30}The Akima interpolation method is advantageous for being a very stable interpolation method that minimizes the artificial oscillations that are more prevalent in spline techniques.
\textsuperscript{31}The Fourier Transform of the Delta function is only a constant when the Delta function is centred about zero.
\textsuperscript{32}The point is visually inspected to make sure it’s correct.
2.8 Peak Identification

Plotting the amplitude vs. frequency of the FFT results in a log-log plot, we get a fairly recognizable shape for damped elastic structures, shown in Figure 16 on the following page. The modal frequencies can be easily identified by measuring the frequencies corresponding to the peaks in this plot. Based on the accelerometer measurements, the peaks identified are shown in Table 6 on page 28.

2.9 Circle Plots

Another method of plotting FFT data is by directly plotting the complex values on an imaginary vs. real plot. These circle plots (or Nyquist plots) can have a lot of unique information that cannot be represented by mere amplitude plots.

The derivations for the resultant circle plots for both the accelerometer and gyroscope measurements are stated below.
Figure 15: Comparison of smoothed vs. raw data

Figure 16: Samples of the amplitude vs. frequency data of the accelerometer measurements
Table 6: Three frequencies with peaks that occur across a significant number of sensors. These peaks are identified manually through visual inspection. The prominence of these peaks is arguable. Refer to Figures 12 and 13 on page 20 and on page 21 for locations of the sensors. The labels of ‘x,’ ‘y,’ and ‘z’ represent the directions in the local coordinate system as defined by each triple-axis accelerometer. These directions are also physically labelled on the accelerometer breakout boards and can be seen in Figure 6a on page 15.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>discernible on sensors...</th>
<th>prominent on sensors...</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.9</td>
<td>1z,4x,4-6z,11-15z</td>
<td>4x\textsuperscript{T},11-15z\textsuperscript{T}</td>
</tr>
<tr>
<td>35.0</td>
<td>1x,1-3z,4y,5x,5-6z,11z,15z</td>
<td>1x,1-3z,4y,5z,11z\textsuperscript{T},15z\textsuperscript{T}</td>
</tr>
<tr>
<td>49.7</td>
<td>1z,1x,3z,4x,4-5z,5x,6z-15z</td>
<td>1x,5z</td>
</tr>
<tr>
<td>70.2</td>
<td>1x,2-3z,4y,5z,5x,12z,15z</td>
<td>2z,5x,12z,15z</td>
</tr>
</tbody>
</table>

2.9.1 For Accelerometers

Starting with the description of the sensor model in the Laplace domain, Eq. (9b) can be converted into the Fourier domain by substituting \( s = j\omega \), where \( j = \sqrt{-1} \).

\[
Y_\alpha(j\omega) = \frac{-C_\alpha \omega(-\omega^2 + \omega_n^2 - 2\zeta_\alpha \omega_n j\omega)}{(-\omega^2 + \omega_n^2 + 2\zeta_\alpha \omega_n j\omega)(-\omega^2 + \omega_n^2 - 2\zeta_\alpha \omega_n j\omega)}
\]

\[
= \frac{-C_\alpha \omega \left((-\omega^2 + \omega_n^2) - 2\zeta_\alpha \omega_n j\omega\right)}{(-\omega^2 + \omega_n^2)^2 + 4\zeta_\alpha^2 \omega_n^2 \omega^2}
\]

(13a) \hspace{1cm} (13b)

which, when plotted, looks like a lopsided circle that is “almost” symmetrical about the imaginary axis and “almost” tangent to the real axis [9]\textsuperscript{33}.

Substituting \( \omega = \omega_n \), we get

\[
Y_\alpha(j\omega_n) = \frac{jC_\alpha \omega_n^4 2\zeta_\alpha}{4\zeta_\alpha^2 \omega_n^4}
\]

\[
= \frac{j C_\alpha}{2\zeta_\alpha}
\]

(14a) \hspace{1cm} (14b)

which means that \( Y_\alpha(j\omega) \) evaluated at the natural frequency \( \omega_n \) lies on the imaginary axis [9]\textsuperscript{34}.

In general, the damped natural frequency can be expressed as

\[
\omega_{\alpha,d} = \sqrt{1 - \zeta_\alpha^2} \omega_n
\]

\[
= \left(1 - \frac{1}{2} \zeta_\alpha^2 + O(\zeta_\alpha^3)\right) \omega_n
\]

(15a) \hspace{1cm} (15b)

where \( \zeta_\alpha \) is the damping ratio. Therefore, \( \omega_{nd} \approx \omega_n \) to the order of \( \zeta_\alpha \).\textsuperscript{35}

\textsuperscript{33}Ch. 2 “Single Degree of Freedom Systems,” pp. 15.
\textsuperscript{34}Ch. 2 “Single Degree of Freedom Systems,” pp. 15.
\textsuperscript{35}Intuitively, this can be analogized to the damped natural frequency in the expression \( e^{\alpha t} \sin(\omega_n t) \).
Now, we would like to solve for the frequency that results in an amplitude of -3 dB, or a factor of $\sqrt{\frac{1}{2}}$.\(^{36}\)

$$|Y_\alpha(j\omega_{3\text{dB}})| = \frac{C_\alpha}{2\sqrt{2}\zeta_\alpha}$$

(16a)

$$|Y_\alpha(j\omega_{3\text{dB}})|^2 = \frac{C_\alpha^2}{8\zeta_\alpha^2}$$

(16b)

$$|Y_\alpha(j\omega_{3\text{dB}})|^2 = \frac{C_\alpha^2\omega_{3\text{dB}}^4}{8\zeta_\alpha^2} \left[ (-\omega_{3\text{dB}}^2 + \omega_\alpha^2)^2 + 4\zeta_\alpha\omega_\alpha j\omega_{3\text{dB}} \right]^2$$

(17a)

$$\frac{C_\alpha^2}{8\zeta_\alpha^2} = \frac{C_\alpha^2\omega_{3\text{dB}}^4}{(-\omega_{3\text{dB}}^2 + \omega_\alpha^2)^2 + 4\zeta_\alpha^2\omega_\alpha^2\omega_{3\text{dB}}^2}$$

(17b)

$$\frac{1}{8\zeta_\alpha^2} = \frac{\omega_{3\text{dB}}^4}{(-\omega_{3\text{dB}}^2 + \omega_\alpha^2)^2 + 4\zeta_\alpha^2\omega_\alpha^2\omega_{3\text{dB}}^2}$$

(17c)

Rearranging the above expression by pre multiplying, we get

$$\omega_{3\text{dB}}^4 - 2\omega_{3\text{dB}}^2\omega_\alpha^2 + \omega_\alpha^4 + 4\zeta_\alpha^2\omega_\alpha^2\omega_{3\text{dB}} = 8\zeta_\alpha^2\omega_{3\text{dB}}^4$$

(18a)

$$(1 - 8\zeta_\alpha^2)\omega_{3\text{dB}}^4 + 2\omega_\alpha^2 (2\zeta_\alpha^2 - 1) \omega_{3\text{dB}}^2 + \omega_\alpha^4 = 0$$

(18b)

\(^{36}\)In decibels, $20 \log |Y_\alpha(j\omega_{\alpha})| - 20 \log |Y_\alpha(j\omega_{1,2})| = 20 \log \frac{|Y_\alpha(j\omega_{\alpha})|}{|Y_\alpha(j\omega_{1,2})|} = 20 \log \sqrt{\frac{1}{2}} \approx 3.01.$
which has the form of a reduced quartic. Solving for $\omega_{3\text{dB}}^2$ using the quadratic formula, we get

$$\omega_{3\text{dB}}^2 = \frac{-2\omega_0^2 (2\zeta_\alpha - 1) \pm \sqrt{4\omega_0^4 (2\zeta_\alpha^2 - 1)^2 - 4\omega_0^4 (1 - 8\zeta_\alpha^2)}}{2 (1 - 8\zeta_\alpha^2)}$$

(19a)

$$\omega_{3\text{dB}}^2 = \frac{-2\omega_0^2 (2\zeta_\alpha - 1) \pm \sqrt{4\omega_0^4 (4\zeta_\alpha^4 - 4\zeta_\alpha^2 + 1) - 4\omega_0^4 + 32\omega_0^4 \zeta_\alpha^2}}{2 (1 - 8\zeta_\alpha^2)}$$

(19b)

where the discriminant can be simplified to

$$\sqrt{\Delta} = \sqrt{16\omega_0^4 \zeta_\alpha^2 (1 + \zeta_\alpha)}.$$  

(20)

Using the Taylor expansion approximations $\sqrt{1+x} = 1 + \frac{x}{2}$ and $\frac{1}{1-x} = 1 + x$,

$$\omega_{3\text{dB}}^2 = \left[-4\omega_0^2 \zeta_\alpha^2 + 2\omega_0^2 \pm 4\omega_0^2 \zeta_\alpha \left(1 + \frac{1}{2\zeta_\alpha^2}\right) \left(\frac{1}{2}\right) (1 + 8\zeta_\alpha^2)\right]$$

(21a)

$$= \omega_0^2 (1 \pm 2\zeta_\alpha)$$

(21b)
which leads to

\[ \omega_{3\text{dB}} = \begin{cases} 
\omega_\alpha (1 - \zeta_\alpha) = \omega_1 \\
\omega_\alpha (1 + \zeta_\alpha) = \omega_2 
\end{cases} \]  \quad (22)

where \( \omega_1 \) represents the the frequency smaller than \( \omega_\alpha \) at which the amplitude is 3 dB (or \( \sqrt{2} \) times) lower than the peak amplitude. Similarly, \( \omega_2 \) represents the the frequency greater than \( \omega_\alpha \) at which the amplitude is 3 dB (or \( \sqrt{2} \) times) lower than the peak amplitude.

Rearranging the expressions for \( \omega_1 \) and \( \omega_2 \), we can solve for \( \zeta_\alpha \) with

\[ \zeta_\alpha = \frac{\omega_2 - \omega_1}{2\omega_\alpha}. \]  \quad (23)

Recall that \( \zeta_\alpha \) is defined as the damping ratio in Eq. (5) on page 23. The geometrical interpretation of this damping ratio is presented in Section 2.10 on page 33. An example of the calculations for \( \zeta_\alpha \) is presented in Table 7 [9]\(^{37}\).

Table 7: Damping ratio calculation of the fundamental (34.5 Hz(meas.)) mode. Note that the units of the frequencies cancel as long as they are consistent.

<table>
<thead>
<tr>
<th>Sensor</th>
<th>( \omega_1 \text{(meas.)} ) (Hz)</th>
<th>( \omega_2 \text{(meas.)} ) (Hz)</th>
<th>( \omega_\alpha \text{(meas.)} ) (Hz)</th>
<th>( \zeta_\alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1z</td>
<td>32.02</td>
<td>35.04</td>
<td>34.17</td>
<td>4.4%</td>
</tr>
<tr>
<td>2z</td>
<td>34.33</td>
<td>35.49</td>
<td>35.19</td>
<td>1.6%</td>
</tr>
<tr>
<td>3z</td>
<td>33.50</td>
<td>35.37</td>
<td>34.77</td>
<td>2.7%</td>
</tr>
</tbody>
</table>

**Least square fit for Nyquist plot**  In order to find the best \( C_\alpha \) that fits the experimental data, a least square fit can be performed such that we solve for \( C_\alpha \) that minimizes the cost function

\[ J = \sum_i |Y_\alpha(j\omega_i) - C_\alpha H_\alpha(j\omega_i)|^2 \]  \quad (24a)

\[ = \sum_i [Y_\alpha^*(j\omega_i) - C_\alpha H_\alpha^*(j\omega_i)] [Y_\alpha(j\omega_i) - C_\alpha H_\alpha(j\omega_i)] \]  \quad (24b)

\[ = \sum_i |Y_\alpha(j\omega_i)|^2 - 2C_\alpha \sum_i \text{Re} [Y_\alpha^*(j\omega_i)H_\alpha(j\omega_i)] + C_\alpha^2 \sum_i |H_\alpha(j\omega_i)|^2 \]  \quad (24c)

where \( i \) is the index along the FFT result’s vector. Recall that \( H_\alpha \) is defined in Eq. (9c) on page 24 [6]\(^{38}\).

In order to minimized \( J \), we take the derivative with respect to \( C_\alpha \) and set it to zero:

\[ \frac{dJ}{dC_\alpha} = -2 \sum_i \text{Re} [Y_\alpha^*(j\omega_i)H_\alpha(j\omega_i)] + 2C_\alpha \sum_i |H_\alpha(j\omega_i)|^2 = 0. \]  \quad (25)

Solving for \( C_\alpha \) yields

\[ C_\alpha = \frac{\sum_i \text{Re} [Y_\alpha^*(j\omega_i)H_\alpha(j\omega_i)]}{\sum_i |H_\alpha(j\omega_i)|^2}. \]  \quad (26)

---

\(^{37}\)Ch. 2.1 “Viscous Damping,” pp. 13.

\(^{38}\)Ch. II “The Structural System,” pp. 10-1.
Solving for $C_\alpha$ using the least square fit  To actually solve for $C_\alpha$ using the above method requires several steps. The first step is to take the circles shown in Figure 18 on page 30, and find the -3 dB ranges for each of the curves.\footnote{We would want to find a segment of data that is close to the peak for this process because that is where the mode would be the most decoupled from another mode.} However, the basis for the analysis to solve for $C_\alpha$ requires that the maximum amplitudes are synchronized to lie on the imaginary axis though, as we can see in Figure 18 on page 30, this is clearly not the case. If we assume that the peaks are offset from each other from phase shifts caused by the imperfections in the structure, for this analysis, we can force the curves to have a maximum on the imaginary axes by rotating the curves about the origin of the Laplace domain.

![Figure 19: The same curves in Figure 18 on page 30 rotated so that the maximum lies on the imaginary axis, with the -3 dB ranges highlighted in bold](image)

We can generate $H_\alpha$ with $\omega_\alpha$ and $\zeta_\alpha$ that are already determined in Table 7 on the previous page. Using the ranges of data shown in Figure 19 on this page, we can solve for $C_\alpha$ for each of the curves using Eq. (26). The solved $C_\alpha$ for the three sensors are shown in Table 8. The physical interpretation of this least square method is shown in Figure 20 on the next page, where the best fit lines represent $C_\alpha H_\alpha$ and the measured curves represent $Y_\alpha$.

Table 8: $C_\alpha$ solved using Eq. (26)

<table>
<thead>
<tr>
<th>Sensor</th>
<th>1z</th>
<th>2z</th>
<th>3z</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_\alpha$</td>
<td>0.7832</td>
<td>0.4002</td>
<td>1.4137</td>
</tr>
</tbody>
</table>

39We would want to find a segment of data that is close to the peak for this process because that is where the mode would be the most decoupled from another mode.
2.9.2 For Gyros

Taking the gyro analogy to Eq. (9b), Eq. (11b) can be converted into the Fourier domain the same way by substituting \( s = j\omega \). It can be shown that each term in the summation corresponds to a circle on the complex plane described by the equation \( (x_\alpha - r_\alpha)^2 + y_\alpha^2 = r_\alpha^2 \) where \( x_\alpha = \Re\{Y_\alpha(j\omega)\} \), \( y_\alpha = \Im\{Y_\alpha(j\omega)\} \) and \( r_\alpha = \frac{|C_\alpha|}{4\zeta_\alpha \omega_\alpha} \) [8]. This corresponds to a circle that is symmetrical about the real axis and tangent to the imaginary axis. The theoretical shape of gyro measurements’ Nyquist circle is shown in Figure 21 on the following page. Examples of Nyquist circles generated using actual gyro measurements are shown in Figure 41 on page 65 in Appendix C.

2.10 Spread Identification

With a bit more manipulations, the spread of the peak in the amplitude plot can be used to determine the damping ratio, if it is small enough.

2.10.1 For Accelerometers

The geometry of lopsided circle described by Eq. (13b) on page 28 provides some insights on the parameters in Eq. (9b). Consider several points: at the point \((x, y) = (2r, 0)\), \( \omega = \omega_\alpha \) which implies \( 2r = Y_\alpha(j\omega_\alpha) \); and at the points \((x, y) = (r, \pm r)\), \( \omega = \omega_{3\text{dB}} = \omega_\alpha \left(\frac{\sqrt{1 + \zeta_\alpha^2} + \zeta_\alpha}{\sqrt{1 + \zeta_\alpha^2} - \zeta_\alpha}\right) \approx \omega_\alpha (1 + \zeta_\alpha) \) for small damping ratios. In addition, since \( Y_\alpha(j\omega_\alpha) = \sqrt{2}Y_\alpha(j\omega_{3\text{dB}}) \), this means \( \omega_{3\text{dB}} \) can be retrieved by taking the frequencies at approximately 3 dB below the peak amplitude.
The two expressions we get from these geometrical manipulations are

\[ \zeta_\alpha = \frac{\omega_2 - \omega_1}{2\omega_\alpha} \quad \text{and} \quad C_\alpha = 2\zeta_\alpha \Im\{Y_{\alpha}(j\omega_{\alpha})\} \tag{27} \]

where \( \omega_1 \) and \( \omega_2 \) are defined in Eq. (22) on page 31. Eq. (27) can be used to determine damping ratios \( \zeta_\alpha \) and relative amplitudes\(^{40}\) \( C_\alpha \) from sensor measurements \(^{[9]}\).

### 2.10.2 For Gyros

The only difference for gyros is that \( C_\alpha = 2\zeta_\alpha \omega_\alpha Y_{\alpha}(j\omega_{\alpha}) \) with \( Y_{\alpha}(j\omega_{\alpha}) \) already lying on the real axis. The expression for \( \zeta_\alpha \) remains the same. Examples of actual gyro measurements showing the maximum amplitudes lying near the real axis are shown in Figure 41 on page 65 in Appendix C.

### 2.11 Amplitude Ratios

Recall in Section 2.6.2 on page 23, the combined amplitude coefficient \( C_\alpha = \psi_{s,\alpha}\psi_{a,\alpha}f \) for accelerometer measurements (and \( C_\alpha = \theta_{s,\alpha}\psi_{a,\alpha}f \) for gyroscope measurements). Even though \( C_\alpha \) can be computed from the circle plots (or from the peaks of the amplitude plots) of the physical measurements, we have no knowledge of the terms inside of \( C_\alpha \), namely, the sensor measurement amplitude coefficient \( \psi_{s,\alpha} \) (or \( \theta_{s,\alpha} \) of gyros), the plant amplitude coefficient \( \psi_{a,\alpha} \), and the impact strength \( f \).

In the simulation model, the modal amplitudes are normalized for each mode shape and do not actually represent physical amplitudes of the gondola \(^{[9]}\)\(^{41}\). Therefore, the comparison between \( C_\alpha \) with simulation

\(^{40}\)The sign of \( Y_{\alpha} \) may be negative, which would correspond to a circle being on the other side of the real axis.

\(^{41}\)Ch. 3.1.5 “Normalization of Mode Shapes,” pp. 25.
measurements must be done with ratios where the unknown constant coefficients ($\psi_{a,\alpha}$ and $f$) are cancelled out, leaving only the ratios of the sensor measurements:

$$\frac{C_\alpha (r_1)}{C_\alpha (r_2)} = \frac{\psi_{s,\alpha} (r_1)}{\psi_{s,\alpha} (r_2)}$$

(or $\theta_{s,\alpha} (r_1)/\theta_{s,\alpha} (r_2)$ for gyroscopes), where $r_1$ and $r_2$ are two different sensor locations.

### 2.11.1 Least Square vs. Maximum Amplitude

In Section 2.9.1 on page 32, we presented a method to solve for $C_\alpha$ using a least square fit near the peak of the $\alpha$th mode. Another method to determine $C_\alpha$ is by using Eq. (27) on the previous page – by directly taking the amplitude at maximum (or the value on the imaginary axis), which is a much simpler method. The comparison between the two methods are presented in Table 9. We can see that the differences are very small. Therefore, only the maximum amplitudes will be used to compute the amplitude ratios.

<table>
<thead>
<tr>
<th>Sensor</th>
<th>1z</th>
<th>2z</th>
<th>3z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least square</td>
<td>0.7832</td>
<td>0.4002</td>
<td>1.4137</td>
</tr>
<tr>
<td>Max amplitude</td>
<td>0.8084</td>
<td>0.3946</td>
<td>1.3833</td>
</tr>
<tr>
<td>Difference</td>
<td>3.2%</td>
<td>1.4%</td>
<td>2.2%</td>
</tr>
</tbody>
</table>

Table 9: $C_\alpha$ solved using two different methods

#### Examples of amplitude ratios

A selection of ratios for prominent sensor measurements around the modal peak of 35 Hz$^{(mea.)}$ are presented in Table 10 on this page. It’s noteworthy to point out that all of the ratios between simulated measurements and physical measurements are very good except for the 3z/2z ratio. (More will be discussed on this discrepancy in Section 4.) Based on these measurements, we can heuristically say with a high degree of confidence that the simulated mode shape at 46 Hz$^{(sim.)}$ represents the measured mode shape at 35 Hz$^{(mea.)}$.

<table>
<thead>
<tr>
<th>Ratios (sensor loc.)</th>
<th>Sim. ratios</th>
<th>Measured ratios</th>
<th>Err. relative to 2z</th>
</tr>
</thead>
<tbody>
<tr>
<td>3z/2z</td>
<td>0.488</td>
<td>1.27</td>
<td>78%</td>
</tr>
<tr>
<td>4z/2z</td>
<td>0.444</td>
<td>0.52</td>
<td>7.6%</td>
</tr>
<tr>
<td>6z/2z</td>
<td>0.199</td>
<td>0.16</td>
<td>3.9%</td>
</tr>
<tr>
<td>9z/2z</td>
<td>0.063</td>
<td>0.10</td>
<td>3.7%</td>
</tr>
<tr>
<td>10z/2z</td>
<td>0.430</td>
<td>0.49</td>
<td>6.0%</td>
</tr>
</tbody>
</table>

2.12 Matching Mode Shapes with Natural Frequencies

Now that we have established all of the mathematics behind the modelling of the BIT structure, we have come to the ultimate and most crucial portion of the procedure – the actual identification of the mode shapes of the gondola. Due to the large sets of imperfect data which are collected through the numerous sensors and
repeated impacts and the inability to fully manipulate the simulation model, certain heuristics are developed for this identification procedure.

**Step 1 – extracting modal amplitudes from simulation**  Despite its usability and convenience, SOLIDWORKS SIMULATION actually provides very limited access to its modal simulation results data through its interface. The modal amplitude can be extracted as a net displacement value\(^{42}\) but no information is given about its direction.\(^{44}\) This process has to be done manually for each sensor location and each mode, as shown in Figure 22. It was decided that the six elastic modes\(^{45}\) under 100 Hz be used. All amplitudes measured

![Figure 22: The extraction of the local amplitudes is done by probing the simulated gondola surface at the sensor locations. All amplitudes are manually recorded.](image)

this way are recorded in Table 11 on page 38.

\(^{42}\)This displacement, although normalized to have a certain unit, in this case millimetres, does not represent physical amplitudes but are proportionally correct \(^{9}\). Note that the size of this value actually represents the colour in the display.

\(^{44}\)Unfortunately, the not all post-processing data generated by SOLIDWORKS SIMULATION are accessibly by the user.

\(^{45}\)The mode shapes can be referred to in Tables 3 and 4 on page 12 and on page 13.
Step 2 – determining largest amplitudes Once we have all the amplitudes at the sensor locations, the next step is to find a suitable value from which to calculate the ratio for each mode shape. As suggested by Dr. Damaren, it is preferable to choose a large base value because it is less likely to be dominated by noise. As such, the largest value (and its corresponding sensor) is selected for the denominator in the ratios (Table 12 on the following page). However, it is noticed that the majority of the modes have sensor location number 2 as the largest value. The last two modes also have fairly large values at location number 2. For the convenience of Step 4, it is decided to consistently use the value at location number 2 as the ratio denominator for all of the modes (Table 13 on the next page).

Step 3 – determining the ratios Now that we have determined the common denominators, we can generate the ratios by taking the columns of Table 11 on the following page and dividing them by the columns of Table 13 on the next page. The ratios are generated in Table 14 on the following page. Note that all of the ratios for sensor number 2 are 100% as expected.

Step 4 – calculating measured ratios At this point, we have computed the ratios for the amplitudes in the simulation model. The goal is to match these ratios to the actual measured ratios. There are a couple of immediate problems to this approach – the measured results are continuous spectra, meaning there are no distinct modes to select the ratios; and, the measured results have a lot of the noise of the real structure expressed in them as well as noise from the measurements.

The latter of these problems can be ameliorated by lightly “smoothing” the amplitude vs. frequency curves using a simple average filter, as mentioned previously in Section 2.7 on page 25. From the raw data, there are approximately 1260 indices per Hertz. By linearly averaging windows of 10 samples, the amplitude plots will be much smoother (to a resolution of approximately 0.008 Hz) and less likely to produce erroneous spikes in the ratios.

To solve the former problem, a ratio is taken for every frequency step. To do this, the FFT results of all 21 sensors is converted to amplitudes at the 15 locations. Specifically, 3-axis sensors at locations 1, 4, and 5 are converted using $\sqrt{|x|^2 + |y|^2 + |z|^2}$ where $x$, $y$, and $z$ are complex values (from the FFT results), and the rest of the amplitudes extracted by taking $|z|$ of the FFT results from the single direction sensor measurements.

An important thing to note is that all of the single direction sensors are placed on the surface of the sheet (with the measurement pointing normal to the sheet). This is done based on the assumption that the vibration of the sheet will be dominated by the vibration in the normal direction. This simplification is crucial in reducing the amount of data collected with minimum losses of information[3][46].

Once all of the amplitudes are extracted, they are divided by the amplitudes of the chosen sensor for each frequency to act as the denominator – in this case, the amplitudes of sensor number 2. A example of this process can be seen in Figure 23. This is done for all of the amplitudes of the FFT results for all of the impacts to produce the ratios.

Step 5 – comparing measured ratios with simulated ratios From the previous step, 15 amplitude ratios are generated for each frequency step and for each impact. We would like to know which of these

---

46 Ch. 2.2 “Plate Theory,” pp. 51.
Table 11: Amplitudes for all 15 sensor locations and the first six elastic modes

<table>
<thead>
<tr>
<th>Loc.</th>
<th>36.3 Hz</th>
<th>41.8 Hz</th>
<th>71.6 Hz</th>
<th>80.9 Hz</th>
<th>91.9 Hz</th>
<th>93.2 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>58.49</td>
<td>76.91</td>
<td>23.43</td>
<td>158.6</td>
<td>202.2</td>
<td>103.0</td>
</tr>
<tr>
<td>2</td>
<td>269.1</td>
<td>164.3</td>
<td>30.07</td>
<td>137.7</td>
<td>218.5</td>
<td>35.55</td>
</tr>
<tr>
<td>3</td>
<td>101.6</td>
<td>112.8</td>
<td>53.21</td>
<td>361.2</td>
<td>196.1</td>
<td>36.10</td>
</tr>
<tr>
<td>4</td>
<td>101.3</td>
<td>47.12</td>
<td>38.85</td>
<td>29.07</td>
<td>24.60</td>
<td>46.99</td>
</tr>
<tr>
<td>5</td>
<td>29.07</td>
<td>23.14</td>
<td>44.40</td>
<td>42.42</td>
<td>14.24</td>
<td>8.968</td>
</tr>
<tr>
<td>6</td>
<td>67.47</td>
<td>39.39</td>
<td>40.42</td>
<td>35.19</td>
<td>78.26</td>
<td>52.84</td>
</tr>
<tr>
<td>7</td>
<td>35.81</td>
<td>25.81</td>
<td>41.66</td>
<td>28.9</td>
<td>112.1</td>
<td>68.61</td>
</tr>
<tr>
<td>8</td>
<td>29.60</td>
<td>29.81</td>
<td>40.59</td>
<td>36.52</td>
<td>35.98</td>
<td>50.44</td>
</tr>
<tr>
<td>9</td>
<td>15.66</td>
<td>17.49</td>
<td>44.05</td>
<td>29.43</td>
<td>34.06</td>
<td>48.84</td>
</tr>
<tr>
<td>10</td>
<td>116.3</td>
<td>65.74</td>
<td>30.68</td>
<td>54.87</td>
<td>105.5</td>
<td>71.89</td>
</tr>
<tr>
<td>11</td>
<td>33.80</td>
<td>21.15</td>
<td>98.59</td>
<td>27.26</td>
<td>40.92</td>
<td>44.59</td>
</tr>
<tr>
<td>12</td>
<td>15.79</td>
<td>10.47</td>
<td>82.64</td>
<td>42.72</td>
<td>42.03</td>
<td>49.37</td>
</tr>
<tr>
<td>13</td>
<td>23.38</td>
<td>13.44</td>
<td>89.55</td>
<td>23.34</td>
<td>14.44</td>
<td>63.06</td>
</tr>
<tr>
<td>14</td>
<td>30.42</td>
<td>10.76</td>
<td>70.95</td>
<td>21.01</td>
<td>11.99</td>
<td>52.83</td>
</tr>
<tr>
<td>15</td>
<td>26.84</td>
<td>15.34</td>
<td>142.5</td>
<td>26.22</td>
<td>23.05</td>
<td>35.64</td>
</tr>
</tbody>
</table>

Table 12: Greatest amplitude from each mode

<table>
<thead>
<tr>
<th>Mode</th>
<th>36.3 Hz</th>
<th>41.8 Hz</th>
<th>71.6 Hz</th>
<th>80.9 Hz</th>
<th>91.9 Hz</th>
<th>93.2 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max</td>
<td>269.1</td>
<td>164.3</td>
<td>142.5</td>
<td>361.2</td>
<td>218.5</td>
<td>103.00</td>
</tr>
<tr>
<td>Sensor</td>
<td>2</td>
<td>2</td>
<td>15</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 13: Selected ratio denominators (all selected from sensor number 2 of each mode). The chosen values that are different from the greatest amplitude values are italicized.

<table>
<thead>
<tr>
<th>Mode</th>
<th>36.3 Hz</th>
<th>41.8 Hz</th>
<th>71.6 Hz</th>
<th>80.9 Hz</th>
<th>91.9 Hz</th>
<th>93.2 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude</td>
<td>269.1</td>
<td>164.3</td>
<td>30.07</td>
<td>137.7</td>
<td>218.5</td>
<td>35.55</td>
</tr>
</tbody>
</table>

Table 14: Amplitudes ratios for all 15 sensor locations of the first six elastic modes

<table>
<thead>
<tr>
<th>Loc.</th>
<th>36.3 Hz</th>
<th>41.8 Hz</th>
<th>71.6 Hz</th>
<th>80.9 Hz</th>
<th>91.9 Hz</th>
<th>93.2 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22%</td>
<td>47%</td>
<td>78%</td>
<td>115%</td>
<td>93%</td>
<td>290%</td>
</tr>
<tr>
<td>2</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>3</td>
<td>38%</td>
<td>69%</td>
<td>177%</td>
<td>262%</td>
<td>90%</td>
<td>102%</td>
</tr>
<tr>
<td>4</td>
<td>38%</td>
<td>29%</td>
<td>129%</td>
<td>21%</td>
<td>11%</td>
<td>132%</td>
</tr>
<tr>
<td>5</td>
<td>11%</td>
<td>14%</td>
<td>148%</td>
<td>31%</td>
<td>7%</td>
<td>25%</td>
</tr>
<tr>
<td>6</td>
<td>25%</td>
<td>24%</td>
<td>134%</td>
<td>26%</td>
<td>36%</td>
<td>149%</td>
</tr>
<tr>
<td>7</td>
<td>13%</td>
<td>16%</td>
<td>139%</td>
<td>21%</td>
<td>51%</td>
<td>193%</td>
</tr>
<tr>
<td>8</td>
<td>11%</td>
<td>18%</td>
<td>135%</td>
<td>27%</td>
<td>16%</td>
<td>142%</td>
</tr>
<tr>
<td>9</td>
<td>6%</td>
<td>11%</td>
<td>146%</td>
<td>21%</td>
<td>16%</td>
<td>137%</td>
</tr>
<tr>
<td>10</td>
<td>43%</td>
<td>40%</td>
<td>102%</td>
<td>40%</td>
<td>48%</td>
<td>202%</td>
</tr>
<tr>
<td>11</td>
<td>13%</td>
<td>13%</td>
<td>328%</td>
<td>20%</td>
<td>19%</td>
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<tr>
<td>12</td>
<td>6%</td>
<td>6%</td>
<td>275%</td>
<td>31%</td>
<td>19%</td>
<td>139%</td>
</tr>
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<td>13</td>
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<td>8%</td>
<td>298%</td>
<td>17%</td>
<td>7%</td>
<td>177%</td>
</tr>
<tr>
<td>14</td>
<td>11%</td>
<td>7%</td>
<td>236%</td>
<td>15%</td>
<td>5%</td>
<td>149%</td>
</tr>
<tr>
<td>15</td>
<td>10%</td>
<td>9%</td>
<td>474%</td>
<td>19%</td>
<td>11%</td>
<td>100%</td>
</tr>
</tbody>
</table>
matches best for each of the 15 amplitude ratios for each of the 6 simulated modes (step 3). This is accomplished by taking the least square difference between the simulated 15 amplitudes and the measured 15 amplitudes for every frequency and impact. Figures 24 to 29 on pages 39–42 show the results of such a comparison for frequencies between 10 Hz and 120 Hz. The inverse of the least square value is take to highlight the closer values. The plots are discussed in detail in the following section. At this point, we can conclude the comparison portion of the experiment.

It is assumed that, for each impact, there is only one fundamental frequency per mode, for all of the measurements. However, in reality, this is observed to not be the case. This discrepancy is further explained in Section 4 on page 45.
Figure 25: The inverse least square ratio differences for elastic mode 2 (41.8 Hz$^{\text{sim.}}$). The measured and simulated ratios seem to be highly correlated for this particular mode.

Figure 26: The inverse least square ratio differences for elastic mode 3 (71.6 Hz$^{\text{sim.}}$). No correlation is observed for this mode.

3 Overview of Results

There are many things that can be said about Figures 24 to 29. The immediate thing that can be observed is that there are frequencies with greater than average correlations. Specifically, there are discernible peaks at around 34.5 Hz$^{\text{mea.}}$, 40.9 Hz$^{\text{mea.}}$, 47.7 Hz$^{\text{mea.}}$, 58.3 Hz$^{\text{mea.}}$, 79.2 Hz$^{\text{mea.}}$. The second thing to notice is that some impacts seem much more correlated than others. Specifically, impact number 2 seems to dominate over the other impacts, and impacts 1 and 4 are barely correlated at all. No obvious correlation can be observed in Figure 26 (elastic mode 3), Figure 28 (elastic mode 5), and Figure 29 (elastic mode 6).

The black vertical lines in the plots show the natural frequencies of the corresponding mode as simulated.

\[\text{It is recognized that the correlations at } 34.5 \text{ Hz}^{\text{mea.}} \text{ and } 47.7 \text{ Hz}^{\text{mea.}} \text{ are relatively weak.}\]
Figure 27: The inverse least square ratio differences for elastic mode 4 (80.9 Hz\textsuperscript{sim.}). Slight correlation is observed near 79 Hz\textsuperscript{mea.}.

Figure 28: The inverse least square ratio differences for elastic mode 5 (91.9 Hz\textsuperscript{sim.}). No significant correlation is observed.

by \textsc{SolidWorks Simulation}. It seems like with the exception of Figure 26 (elastic mode 3), Figure 28 (elastic mode 5), and Figure 29 (elastic mode 6), the natural frequencies all correspond to an observable peak.

### 3.1 Interpretation

There are many ways which these resulting data can be interpreted. They consider different levels of detail and suggest different potential uses for the resulting information. Some of these interpretations are listed here.
Figure 29: The inverse least square ratio differences for elastic mode 6 (93.2 Hz$^{(sim.)}$). No significant correlation is observed.

**Mean correlation data for natural frequencies** The most obvious thing to be said about the mean correlation data (thick black curve) is that its peaks correspond to natural frequencies. Specifically, there is a simulated frequency that matches the peaks of 34.5 Hz$^{(mea.)}$, 40.9 Hz$^{(mea.)}$, and 79.2 Hz$^{(mea.)}$. Naturally, by extension, the mode shapes corresponding to the natural frequencies represent those of the gondola expressed at those frequencies.

Although this interpretation is simple and direct, there are several discrepancies that potentially undermine the correctness and usefulness of the results. For instance, there is a peak at around 58.3 Hz that does not match any of the simulated natural frequencies; there is more than one peak per mode shape; the averages over all the impacts are significantly different from the results of the individual impacts; and so on. These discrepancies will be addressed in Section 4 on page 45.

**Observation of mode shapes** It is noted that the first and second elastic modes (modes 6 and 7 in Table 4 on page 13, reproduced in Figure 30) are quite similar. Namely, the motion in both modes is dominated by the L-corners at the front of the gondola bellowing in and out. Moreover, their corresponding natural frequencies of 36.3 Hz$^{(sim.)}$ and 41.8 Hz$^{(sim.)}$ are also close, with a difference of around 5 Hz.

The only significant difference between the two mode shapes is that the bellowing motion of the two L-corners (on port side and starboard side on the gondola) are out of phase in the first mode and are in phase in the second mode. The reason for the similarity between these two modes is the fact that the BIT gondola has bilateral symmetry, and the first two modes are modes with (nearly) repeated natural frequencies.\(^{49}\) This symmetry is actually broken by the difference in the distributed mass assignment on the two sides of the gondola, specifically, the assignment of the heavy Outer Frame computer on the port side and assignment of the relatively light motor controllers on the starboard side.

\(^{49}\)If the structure did have perfectly repeated natural frequencies, the a set of mode shapes corresponding to those frequencies can form another orthogonal set of mode shapes through linear combinations. Since the real structure can actualize any of these linear combinations, it would be very difficult to determine whether the solved mode shapes actually match the measured ones. Luckily, this problem is not present for the modes of interest.
(a) First elastic mode (36.3 Hz\textsuperscript{sim.}) \hspace{1.5cm} (b) Second elastic mode (41.8 Hz\textsuperscript{sim.})

Figure 30: Nearly identical modes

It is also noted that the second elastic mode and fourth elastic mode (Figure 31a on the current page) are similar in the bellowing motion of the L-corners. The main difference is that the third elastic mode also experiences oscillation of the reaction wheel floor.

(a) Third elastic mode (41.8 Hz\textsuperscript{sim.}) \hspace{1.5cm} (b) Fourth elastic mode (80.9 Hz\textsuperscript{sim.})

Figure 31: Next two similar mode shapes

Assuming that the 47.7 Hz\textsuperscript{mea.} peak actually corresponds to the third elastic mode, a simple check could be made to see if the sensors placed on the reaction wheel floor of the gondola register a significant increase in amplitude going from 34.5 Hz\textsuperscript{mea.} to 47.7 Hz\textsuperscript{mea.}.

Referring back to Table 6 on page 28, we can clearly see that near 47.7 Hz\textsuperscript{mea.}, all of the sensors on the bottom floor (11z-15z) registered peaks whereas near 34.5 Hz\textsuperscript{mea.}, only two of the sensors on the bottom floor (11z and 15z, see Figure 8 on page 17) registered peaks. The amplitude plots showing the discernible peaks are shown in Figure 32. Although not entirely elegant, this result corroborates with the hypothesis that the third mode shape is dominated by the oscillation of the bottom reaction wheel floor.
Figure 32: Sample peaks at the reaction wheel floor, delimited at 34.5 Hz\(^{\text{mea.}}\) and 47.7 Hz\(^{\text{mea.}}\).

The same argument can be made about the last peak at 79.2 Hz\(^{\text{mea.}}\). As shown in Figure 33 on this page, the amplitude measurement at the L-corner (sensor 2z) has a much weaker peak at 79.2 Hz\(^{\text{mea.}}\) than the amplitude measurement at the sensor directly above it (sensor 3z).

Figure 33: Comparing measurement at the L-corner and the reaction wheel floor, delimited at 79.2 Hz\(^{\text{mea.}}\).

With the above arguments, we can say with a high degree of confidence that the first four elastic modes produced by \textsc{SolidWorks Simulation} correspond to the measured natural frequencies of approximately 34.5 Hz\(^{\text{mea.}}\), 40.9 Hz\(^{\text{mea.}}\), 47.7 Hz\(^{\text{mea.}}\), and 79.2 Hz\(^{\text{mea.}}\). A final comparison between the simulated frequencies and the measured frequencies is presented in Table 16.
3.2 Applications of Methodologies and Results

The goal of this experiment is to discover a methodology to determine the natural frequencies and mode shapes of a large mechanical structure. This was completed through the system identification of the BIT gondola. This methodology can be used in the development of future balloon-borne experiments where it is important to determine structural vibrations for high fidelity control systems. It had been the idea for the control system of BIT to incorporate the identified frequencies and mode shapes. However, it was decided that this is not necessary for the successful operation of BIT. It would be incredibly interesting, even purely on an academic level, to have something like this done for a future experiment. (More information on modelling the dynamics of the BIT gondola is suggested in Appendix A on page 52.)

Despite the lack of usefulness of the mode shapes, the natural frequencies determined in this experiment might still be important for the control system of BIT. Due to the highly sensitive nature of the pointing controls, it might be necessary for the motor controllers to actively avoid being driven at those frequencies. Depending on the situation, this avoidance might be achieved through the implementation of band-pass filters or notch filters in the controller. Appendix A on page 52 shows the mathematical development of such a comprehensive motion model for BIT’s elastic multibody system. This motion model would be the connecting step between the natural frequencies and mode shapes identified in this thesis and the suggested control system.

4 Error Characterization and Areas for Improvement

For a large experiment such as this, involving lots of engineering heuristics, there exists many areas for improvement. Several of these areas are presented here that might facilitate potential future experiments.

Multiple peaks in the least square plots In Section 2.12 on page 35, the goal was to match the simulated modal amplitude ratios with the measured modal amplitude ratios. This was done by taking the sum of the squared difference between the two for each frequency. Ideally, if the modal amplitude ratios matches perfectly (or very closely) for that mode, the sum of the squared difference would be near zero. Since all of the mode shapes are different, that would also mean the sum of the least squared difference would not be near zero for all other frequencies. If it were a perfect setup, we would expect a single peak in each of Figures 24 to 29 representing the natural frequency corresponding to the mode shape. Instead, we see many peaks, even worse for the individual (unaveraged) impacts.

The explanation for this phenomenon is actually quite simple. As suggested in Section 3 on page 40, the mode shapes are not entirely independent. For instance, the first two elastic modes have very similar modal amplitudes at the same sensor locations. Between modes 2, 3, and 4, the bellowing motion of the L-corners at the front of the gondola are all observed. Between the modes chosen for analyses, the differences between
them are observably fewer than the similarities between them. This is what causes the average (black bold) curves in Figures 24 to 27 on pages 39–41 to look very similar from each other.

A method is suggested in Section 3 on page 40 to overcome this problem. Rather than comparing all of the ratios together, individual ratios must be selected out to characterize the differences and similarities between every two mode shapes. Great care would be needed to avoid confirmation bias when selecting measurements for comparison. The downside to this method is that the process can no longer be automated, at least it will not be as simple as finding the least squared difference.

**Unsigned ratios** In addition to the similar mode shapes, this problem is exacerbated by the fact that the amplitude ratios are generated using only the amplitudes of the peaks, rather than using the least square circle fits. This would also mean that the ratios are unsigned – it would be impossible to tell if geometries are moving in-phase or out-of-phase with each other. Consequently, this problem can be ameliorated by using the $C_\alpha$ ratios solved by taking the least square fit and comparing them to the signed amplitude ratios measured from a signed measurement of the simulated amplitudes. As mentioned in Section 2.12 on page 37, the latter was not done due to the difficulty in extracting the directional amplitudes from SOLIDWORKS SIMULATION.

An example of signed ratios can be seen in Table 17 on page 62 of Appendix B. In the beam analysis, the signs of the $C_\alpha$'s generated using the least square fit are preserved to calculate the ratios. Since the theoretical mode shapes can be analytically determined, their signs are also preserved when calculating the ratios.

**Unmatched peak(s) in the least square plots** Looking at Figures 24 to 27 on pages 39–41, several peaks can be distinguished in the plots – at 34.5 Hz(meas.), 40.9 Hz(meas.), 47.7 Hz(meas.), 58.3 Hz(meas.), 79.2 Hz(meas.). However, in Section 2.12 on page 35, only four of the frequencies were used to match to the simulated mode shapes. This leaves the orphaned peak at 58.3 Hz(meas.).

There are several potential explanations for this discrepancy. This mode could be one of the first two modes, whose frequencies might not actually be adjacent to each other due to the differences between the real and simulated structures. This peak might not represent a mode but rather an artefact of taking the least squared and averaging processes. This peak could actually correspond to one of the higher simulated modes due the imperfections in the structure. This peak could represent an unmodelled object in the structure. The method to definitely determine the mode shape behind the peak would be the same as above – by specifically selecting ratios for comparison.

**Impact locations** Even though that, theoretically, the measurement results should be independent from the impact location, observations show that the results are actually highly dependent on the location of impact. As can be clearly seen in Figures 24 to 29 on pages 39–42, different impacts resulted in vastly different matches with amplitude ratios although general trends are maintained.

Although the discrepancy is undesirable, there exists a simple explanation for this phenomenon. Take for instance the difference between impact number 1 and impact number 2 – 2 resulted in highly correlated data whereas 1 resulted in really flat data. This is because impact number 2 was performed at the bow of

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50Or, with more confidence.
the gondola where the modal amplitude is maximum for most of the simulated whereas impact number 1 was performed at the stern of the gondola where the modal amplitude is the least for most of the modes. This would mean that the structural response for impact number 2 is much greater than that of impact number 1, whose minuscule response would be dominated by noise. Unfortunately, a method to quantify the differences due to impact locations has not yet been determined and warrants future investigation.

**Multiple peaks per mode**  An interesting observation that can be made about the modal amplitude plots is that all of the peaks are slightly shifted from one another. An example of this phenomenon is shown in Figure 34 on the current page. In this particular example, the differences are not large. However, the peaks at 34.5 Hz are prominent peaks. The same thing might not be true for higher frequency, less prominent peaks.

![Figure 34: Peak offsets occurring near 34.5 Hz](image)

Due to the significance of the misalignment, the most likely explanation for this is actually a systematic error – the synchronization of data and measurement. It can be argued from Figure 18 on page 30 (and mentioned in Section 2.9.1 on page 32) that the FFT results are actually shifted in phase. We know that the Fourier transform of the delta function (impulse) shifted by a period of $\tau$ represents, among other things, a linear stretch of $\tau$ in the frequency domain \[51\]. Or, in equation form: \[52\]

$$\mathcal{F}_t[\delta(t-\tau)](\omega) \propto e^{-j\omega \tau}. \quad (29)$$

Eq. (29) on this page means that the results can be phase-shifted by a significant amount for large frequencies even if there is only a small offset in when the impulse actually takes place. This does not even take into account that the impact itself does not occur at one instance in time. This problem is exacerbated by the fact that the time stamps of the accelerometer measurements are generated in the hard drive of the IFC after the data from each of the analogue inputs is collected. The synchronization between the sensors is spurious.

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As the analogue signal from each accelerometer is read sequentially into the computer, their synchronization is limited by the duration of one time step, or approximately 80 ms.

The effects of this asynchronism has been tested by manually selecting the time index of when the impulse takes place. As shown in Figure 35 on the current page, a slight shift of just a few sample periods can result in a phase shift of a few tens of degrees. This would have easily happened during the experiment. A better experimental setup would mitigate this problem by collecting data that are much more synchronized.

![Figure 35](image)

Figure 35: The solid curves are the same as those of Figure 18 on page 30. The dashed curves represent the FFT of the time series shifted by 1 sample period (0.8 ms). The resulted net rotation is around 10°.

**Other possible explanations**  There are several other explanations for this phenomenon. All of them boil down to the theoretical model of impulse response as described in Section 2.6 on page 22 being not entirely representative of the real structure. For instance, it is already observed that impacting the gondola at different locations results in different responses. The propagation of the impulse through the gondola can cause unpredictable nonlinear behaviours. Moreover, since the aluminum honeycomb panels are a material that is known for its damping properties, the observed nonlinearities are actually expected.

Another plausible cause is that the impulse response is not actually an impulse. The impact of the mallet might have been smeared across several collection periods of the sensors. It is noted from past experiments that the phases of the FFT solutions are highly dependent on the selections of where the time streams began. If the impulse is experienced by the different sensors at different times, the FFT solutions would be shifted from each other in phase.

An important thing to note is that, as seen in Figure 18 on page 30, even though the peaks do not occur on the imaginary axis, the overall shapes of the curves resemble that of the theoretical solution. That is,
the curve lies above the real axis is approximately centred about the imaginary axis with a slight stretch towards the positive real axis near base. Because of this, it is conceivable that the maximum not occurring on the imaginary axis could be caused by the large spread and a close adjacent peak.

**Damping**  Directly following from above, many of the imperfections of the results are caused by working with a highly damped structure. In Section 2.10 on page 33, a method was determined to solve for the damping ratio $\zeta_{\alpha}$. Using Eq. (27) on page 34, $\zeta_{\alpha}$ can be approximated using the formula $(\omega_2 - \omega_1)/(2\omega_n)$, where $\omega_1$ and $\omega_2$ are the frequencies surrounding the natural frequency that correspond to approximately 3 dB below the peak amplitude. For the peaks shown in Figure 34 on page 47, the $\zeta_{\alpha}$'s are calculated for each of them and are presented in Table 7 on page 31. All of the damping ratios seem to be below 10%, which is consistent with what is expected of honeycomb structures such as the BIT gondola.

It is important to note that the calculations in Table 7 on page 31 are done for some of the most prominent peaks in the results. For the less prominent (more spread out) peaks, the damping ratios would be much higher. The greater the damping ratio, the greater the difference between real and theoretical results. Most of the discrepancies in the results can be attributed to this fact.

**Discrepancies in damping ratios**  In Table 7 on page 31, one of the immediate things to notice is that the damping ratios $\zeta_{\alpha}$ solved using $(\omega_2 - \omega_1)/(2\omega_n)$ are actually different between different sensors of the same mode. Also, from observing Figure 34 on page 47, we can see that the “sharpness” of the peaks are different. For instance, the spread of the of the $1z$ peak is much greater than that of the $2z$ peak. However, from theory, there should be one damping ratio for each mode.

A possible explanation for this discrepancy is the fact that an adjacent mode or modes are too close to the peak of interest, and the spreads of different peaks are “bleeding” into one another. This is coupled with the fact that, for peaks with smaller amplitudes, the noise floor is also bleeding into the actual amplitude signal [9].

It would be of significant interest to future experiments to characterize the various dependencies of the damping ratio, and actually measuring the damping ratio using a better technique. For the same setup, however, we may be able to improve the identification of $\zeta_{\alpha}$, along with other parameters such as $C_{\alpha}$ and $\omega_{\alpha}$ using a nonlinear regression method to solve for all of them together.

**Discrepancies in sensor ratios**  As first noted in Table 10 on page 35, there can be significant differences in certain amplitude ratios even though everything else matches nicely. Again, this can be attributed to the imperfections of the physical system (or the imperfect modelling of the physical system). Results can be improved by taking multiple measurements and averaging them.

It is also noted that some sensors are not perfectly attached to the honeycomb panels (either through a thin aluminium sheet or the glue connection is weak from the dirt). Systematics can be improved by securing sensors at better locations and/or better preparing surface conditions.

**Single shell simulation**  Due to time constraints, the SOLIDWORKS SIMULATION was performed on the shell model for the Outer Frame only, even though the measurements were done on the gondola with all three

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53Ch. 5.1.2 “Close Resonances,” pp. 56-58.
frames. This simplification slightly undermines the accuracy of the results. However, since there is actually minimum constraints between the frames, they can be treated as independent systems. It is also confirmed from my undergraduate thesis that simulating the Outer Frame individually would give very similar results as simulation of all three frames together [13]. A future iteration would include an accurate modelling of all three frames of the gondola for simulation.

5 Conclusion

A method of combining simulation results with measurement results for the purposes of modal system identification was developed in this thesis. For the BIT gondola’s Outer Frame, the four lowest modes are identified at approximately 34.5 Hz, 40.9 Hz, 47.7 Hz, and 79.2 Hz. Their corresponding mode shapes are also identified and are shown in Figures 30 and 31 on page 43. The mode shapes were generated using SOLIDWORKS SIMULATION and were compared with measured values through amplitude ratios. The frequencies were generated from the peaks in the amplitude vs. frequency curves derived from the physical measurements, which were made using analogue accelerometers placed throughout the gondola.

A potential use for the fully identified system is to incorporate the results in advanced control algorithms. Future iterations of the experiment may involve the analyses of the measurement results in greater detail, creating a better simulation model, and refining the measurement systematics.
References


A Developing the BIT Motion Model

The modelling of the BIT structure can be broken down to two main categories – developing a mathematical model for the BIT dynamics and creating a simulation model [13].

A.1 Developing Equations of Motion for the BIT Gondola

Since BIT has three frames that are joined together, the full motion model should include both the rigid body motion as well as the elastic body motion.

A.1.1 The Unconstrained Elastic Body

The overall setup of BIT can be described as dangling a large elastic structure by the end of a rope. Ignoring the low frequency pendulations created by the 100 m long balloon tether, BIT can be fairly accurately approximated as an unconstrained elastic body. To derive the equations of motion of such a system, consider a body with its body-fixed frame $F_b$ attached at some point $O$ on the body, floating in space with an inertial frame $F_i$. The relative linear velocity $v$ and rotational velocity $\omega$ between the two frames can be described by

$$\begin{align*}
\dot{r}(t) &= v(t) = F_b^T v(t) \\
\omega &= F_b^T \omega(t)
\end{align*} \tag{30a} \tag{30b}$$

both as functions of time $t$. The body is composed of infinitesimal masses $dm$, whose position in $F_b$ can be described as $\rho$ such that

$$\rho = F_b^T \rho(t) \tag{31}$$

and its deformation can be described as

$$u_e = F_b^T u_e(\rho, t) \tag{32}$$

as a function of the undeformed coordinates $\rho$ and time $t$. And thus, the absolute position $R$ of the elemental mass $dm$ can be expressed as

$$R(\rho, t) = r_e(t) + \rho(\rho, t) + u_e(\rho, t). \tag{33}$$

Taking the derivative, we can retrieve the absolute velocity of $dm$

$$\dot{R} = \dot{r} + \dot{\rho} + \dot{u}_e \tag{34a}$$

$$= v + (\dot{\rho} + \omega \times \rho) + (\dot{u}_e + \omega \times u_e) \tag{34b}.$$

The period of a small-angle pendulum can be determined by the gravitational acceleration $g$ and the length of the rope $l$ using the formula $T = 2\pi \sqrt{l/g}$. For $l = 100$ m and $g = 9.8$ m/s$^2$, the period $T$ is determined to be approximately 20 s, which is significantly larger than the period of the controller, which is around 100 ms.
Since $\rho$ is the position with respect to the body frame, its time derivative with respect to the body frame must be zero. Expressing everything in $F_b$ and assuming small motion\textsuperscript{55}, we can write

$$V = v - \rho^\times \omega + \dot{u}_e$$

(35a)

$$= [1 - \rho^\times] \begin{bmatrix} v \\ \omega \end{bmatrix} + \dot{u}_e$$

(35b)

The elastic deformation $u_e$ can be expressed as

$$u_e(\rho, t) = \sum_{\alpha=1}^{N_e} \psi_{ea}(\rho) q_{ea}(t)$$

(36)

where $\psi_{ea}$'s are the basis functions and $N_e$ is the total number of degrees of freedom. From these definitions, the system can be solved using a Lagrangian approach, which utilizes the kinetic and potential energy of the system.

**Kinetic energy** Letting a density function $\sigma(\rho)$ denote the mass density per unit volume, the total kinetic energy of the entire body can be written as

$$T = \frac{1}{2} \int_V V^T V \sigma dV.$$  

(37)

Through a non-trivial series of manipulations, the kinetic energy equation can be rewritten as

$$T = \frac{1}{2} \left[ \nu^T q_e^T \right] \begin{bmatrix} M_{rr} & M_{re} \\ M_{re}^T & M_{ee} \end{bmatrix} \begin{bmatrix} \nu \\ \dot{q}_e \end{bmatrix}$$

(38)

where

$$\nu = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

(39a)

$$q_e = \begin{bmatrix} q_{e1} & \cdots & q_{ea} & \cdots & q_{eN_e} \end{bmatrix}$$

(39b)

$$M_{rr} = \int_V \begin{bmatrix} 1 & -\rho^\times \\ -\rho^\times & -\rho^\times \rho^\times \end{bmatrix} dm$$

(39c)

$$M_{ee} = \int_V \begin{bmatrix} \psi_{e1}^T \psi_{e1} & \cdots & \psi_{ea}^T \psi_{e1} \\ \vdots & \ddots & \vdots \\ \psi_{ea}^T \psi_{e1} & \cdots & \psi_{eN_e}^T \psi_{e1} \end{bmatrix} dm$$

(39d)

$$M_{re} = \int_V \begin{bmatrix} \psi_{e1} & \cdots & \psi_{ea} & \cdots & \psi_{eN_e} \\ \rho^\times \psi_{e1} & \cdots & \rho^\times \psi_{ea} & \cdots & \rho^\times \psi_{eN_e} \end{bmatrix} dm$$

(39e)

in which $dm = \sigma dV$ represents an infinitesimal mass element.

\textsuperscript{55}Since both the rotation rates $\omega$ and the elastic deformations $u_e$ are small, their product would be negligible.
Potential energy  For the strain potential energy, it can be shown that the solution exists in the form

\[ U = \frac{1}{2} q_e^T K_{ee} q_e \]  

(40)

where \( K_{ee} \) is the stiffness matrix of the elastic body. At this point, we can define generalized coordinates \( q \) such that

\[ q = \begin{bmatrix} q_r \\ q_e \end{bmatrix} \]  

(41)

where for small angles, the approximation \( q_r \approx \nu \) can be made. From this, we can finally write

\[ T = \frac{1}{2} q^T M \dot{q} \]  

(42a)

\[ K = \frac{1}{2} q^T K q \]  

(42b)

where

\[ M = \begin{bmatrix} M_{rr} & M_{re} \\ M_{re}^T & M_{ee} \end{bmatrix} \]  

(43a)

\[ K = \begin{bmatrix} 0 & 0 \\ 0 & K_{ee} \end{bmatrix} \]  

(43b)

Note that in the stiffness matrix in Eq. (43b), the terms relating to the rigid body motions are zero as expected.

Applied force  It can be shown that for small angles, the virtual work \( \delta W_e \) performed on the system by force per unit volume \( f_e \) can be expressed as

\[ \delta W_e = \int_V \mathbf{f}_e \cdot \delta \mathbf{R} \, dV \]  

(44a)

\[ = \begin{bmatrix} \delta q_r^T \\ \delta q_e^T \end{bmatrix} \begin{bmatrix} \dot{\mathbf{f}}_r \\ \dot{\mathbf{f}}_e \end{bmatrix} \]  

(44b)

where

\[ \dot{\mathbf{f}}_r = \int_V \begin{bmatrix} \mathbf{f}_e \\ \rho \times \mathbf{f}_e \end{bmatrix} \, dV \]  

(45a)

\[ \dot{\mathbf{f}}_e = \int_V \begin{bmatrix} \psi_{e1}^T f_e \\ \vdots \\ \psi_{eN}^T f_e \end{bmatrix} \, dV \]  

(45b)
**Lagrange’s equation** Substituting the kinetic energy and potential energy equations into Lagrange’s equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q^l} \right) - \frac{\partial L}{\partial q^l} = \dot{f}
\]  

(46)

where

\[
L = T - U \\
\dot{f} = \begin{bmatrix} \dot{\hat{f}}_r \\ \dot{\hat{f}}_e \end{bmatrix}
\]

(47a, 47b)

the following familiar equation can be derived:

\[
M\ddot{q} + Kq = \dot{f}. 
\]

(48)

### A.1.2 Application to BIT

Now that a general mathematical description of a free elastic body has been developed, it shall be modified to describe the BIT system.

**Elastic body terms** Since no force is directly applied to the elastic body, \( \hat{f}_e = 0 \).

**Rigid body terms** Since the BIT gondola is a system where three frames are rotationally connected to each other, only angular information is needed to describe the rigid body motion of the gondola. And thus, the terms relating to the rigid body motions can be reduced to

\[
q_r = \theta \\
M_{rr} = M_{\theta\theta} = -\int_V \rho^x \rho^y \, dm \\
M_{re} = M_{\theta e} = \int_V \left[ \rho^x \psi_{e1} \cdots \rho^x \psi_{eN} \right] \, dm \\
\dot{\hat{f}}_r = \int_V \rho^x f_e \, dV.
\]

(49a, 49b, 49c, 49d)

In addition, a further reduction can be made by noting that only three particular angular vectors, correspond to the angular rotations of the three frames, are needed. Since the rotations of the three frames correspond to the rotations of the three orthogonal directions (each about the centre of mass in that direction), the rigid mass matrix \( M_{\theta\theta} \) can be reduced to

\[
M_{\theta\theta} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}
\]

(50)

\[56\text{It can be assumed that all the gravitational forces are applied to the rigid body.}\]
where \( I_x, I_y, \) and \( I_z \) are the moments of inertia about the three axes of rotation.\(^{57}\) Accordingly, the angular momentum matrix\(^{58}\) \( M_{\theta e} \) can be rewritten as

\[
M_{\theta e} = \left[ \begin{array}{ccc}
X_T \int_V \left[ \rho^x \psi_{e1} \cdots \rho^x \psi_{eN_e} \right] \, dm \\
y_T \int_V \left[ \rho^y \psi_{e1} \cdots \rho^y \psi_{eN_e} \right] \, dm \\
z_T \int_V \left[ \rho^z \psi_{e1} \cdots \rho^z \psi_{eN_e} \right] \, dm
\end{array} \right]
\]

(51)

where \( x, y, \) and \( z \) are the unit vectors representing the three orthogonal axes. Likewise, the rigid body force terms can be reduced to the torques \( \hat{\tau} \) applied by the motors in those three axes. And finally, the resultant matrix equation of motion of \( \text{BIT} \) can be written as

\[
\begin{bmatrix}
M_{\theta \theta} & M_{\theta e} \\
M_{e \theta}^T & M_{e e}
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\ddot{\eta}_e
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & K_{ee}
\end{bmatrix}
\begin{bmatrix}
\theta \\
\eta_e
\end{bmatrix}
= \begin{bmatrix}
\hat{\tau} \\
0
\end{bmatrix}
\]

(52)

A.1.3 Converting to Modal Space

Even though Eq. (52) can fully describe the \( \text{BIT} \) system, this is not a very efficient form because the mass and stiffness matrices must be fully populated in order to solve the system. Therefore, it would be desirable to convert the system into modal space. In that case, both the mass matrix and the stiffness matrix are diagonalized, which would be significantly easier to solve. This can be achieved by solving the eigenvalue problem

\[
(K - \omega^2 q_{\alpha}) q_{\alpha} = 0
\]

(53)

where the \( \omega_{\alpha} \)'s are the natural frequencies of the entire system. Once solved, the modal equation of motion of the \( \text{BIT} \) system can be written as

\[
\begin{bmatrix}
M_{\theta \theta} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta} \\
\ddot{\eta}_e
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & \Omega^2
\end{bmatrix}
\begin{bmatrix}
\theta \\
\eta_e
\end{bmatrix}
= Q^T \begin{bmatrix}
\hat{\tau} \\
0
\end{bmatrix}
\]

(54)

where \( \Omega^2 = \text{diag} \{\omega_{\alpha}^2\} \) is a diagonal matrix of eigenvalues corresponding to the elastic portion of the system and \( Q = \text{row} \{q_{\alpha}\} \) is a matrix whose columns are the eigenvectors of the eigenproblem. The spatial coordinates can be retrieved from the modal coordinates using the relationship

\[
\begin{bmatrix}
\theta \\
q_e
\end{bmatrix}
= Q \begin{bmatrix}
\theta \\
\eta_e
\end{bmatrix}
\]

(55)

Another advantage of having the system in modal space is having the ability to select parts of the matrix equation corresponding to frequencies of interest. This cannot be done in the spatial coordinates.

\(^{57}\)It is important to realize that rigid body system presented here is an over-simplified version. In reality, the entire description of the rigid body portion of the \( \text{BIT} \) gondola requires ten rigid bodies which corresponds to ten independent degrees of freedom. The single rigid body with three independent degrees of freedom described here serves as proof-of-concept and is not the focus of this thesis. The description of the elastic components is, however, accurate.

\(^{58}\)An interpretation of the terms in this matrix is the projections of the angular momenta from the elastic deformations onto the axes of interest.
A.2 Simulating BIT in SolidWorks

Now that there is a mathematical model of the BIT gondola, in order for it to be useful, however, the mass and stiffness matrices have to be fully populated. Namely, the four matrices $M_{\theta\theta}$, $M_{\theta e}$, $M_{ee}$, and $K_{ee}$ have to be determined. As mentioned previously, the first of these matrices can be determined easily based on the geometry and the mass distribution of the system. The last three matrices must be determined through simulation.

A.2.1 Using SolidWorks Simulation

The simulation package used in this project is SolidWorks Simulation which allows the frequency simulation of various types of models under many types of conditions. In addition, the user has the freedom to specify the mesh density of the model as well as the number of modes the solution returns. For results, the $x$-$y$-$z$ deformations of the nodes of the mesh for all the modes as well as the modal frequencies are returned.

Inadequacies of SolidWorks Simulation As there is always a trade-off between the user-friendliness and the versatility of a software, SolidWorks Simulation has certain limitations. From the documentations of SolidWorks Simulation, we know that the solution is determined by solving the eigenvalue problem

\[
(K_{ee} - \omega^2 M_{ee}) q_\alpha = 0
\]

[14]. However, the mass and stiffness matrices, which are exactly the elastic matrices $M_{ee}$ and $K_{ee}$ in Eq. (52) on the previous page that we are seeking, are inaccessible to the user. Thus, a method must be developed to reconstruct these matrices from the solution.

Retrieving the mass matrix Recall in Eq. (39d) on page 53, the elastic mass matrix can be expressed as

\[
M_{ee,\alpha\beta} = \int_V \psi_{e\alpha}^T \psi_{e\beta} \, dm
\]

where $\psi_{e\alpha}$'s are the shape functions. From the documentations of SolidWorks Simulation, we know that the solver uses the lumped-mass approximation [14]. Accordingly, the mass matrix can be discretized as

\[
M_{ee,\alpha\beta} = \sum_{l=1}^{M} \psi_{e\alpha}^T l \psi_{e\beta} \Delta m_l
\]

(56)

where $\psi_{e\alpha} = [\Delta x \  \Delta y \  \Delta z]^T$ is the displacement of the $l$th node in the $\alpha$th mode and $\Delta m_l$ is the lumped mass at that node, which can be approximated, for regions with uniform mass distributions and sufficiently large mesh densities, as $\Delta m_l = \frac{1}{M} \int_V dm$ with $M$ being the total number of nodes in that region.

Retrieving the stiffness matrix Recall that in the eigenvalue problem described by Eq. (53) on the previous page has the solution in the form

\[
K_{ee} Q_\alpha = M_{ee} Q_\alpha \Omega^2
\]

(57)

where $\Omega^2 = \text{diag} \{\omega_{e\alpha}^2\}$ is the eigenvalue diagonal matrix and $Q_\alpha = \text{row} \{\hat{q}_{e\alpha}\}$ is the $M_{ee}$-orthonormalized eigenvector matrix. The eigenvector $\hat{q}_{e\alpha}$ represents the modal basis function of the elastic structure and can be expressed as

\[
\hat{q}_{e\alpha} = \text{col} \{\psi_{e\alpha}\}
\]

(58)

\[\text{The simulation models can be defined as body, truss, shell, or combined.}\]
Thus, the stiffness matrix $K_{ee}$ can be retrieved as

$$K_{ee} = M_{ee}Q_{e}\Omega^2Q_{e}^{-1}. \quad (59)$$

**Retrieving the angular momentum matrix**  As shown in Eq. (49c) on page 55, the angular momentum matrix has the closed form $M_{\theta e} = [a_k^T \int_V \rho^x \psi_{\epsilon a} dm]$, where $a_k$ is a unit vector representing the $k$th axis of interest. Similar to the mass matrix, the angular momentum matrix can be discretized as

$$M_{\theta e} = \left[ a_k^T \sum_{l}^{M} \rho_{kl} \hat{\psi}_{\epsilon a} \Delta m_l \right] \quad (60)$$

where $\rho_{kl}$ is the relative position from the rotation origin of $a_k$ to the $l$th node. Since in the setup of the BIT gondola, the rotational axes also correspond to the origin of the model, meaning that $\rho_{kl}$ is just the absolute position of the $l$th node. With these derived matricies, the terms needed to solve Eq. (52) on page 56 are fully defined.

**B  The Analysis of a Cantilevered Beam**

The theoretical solution for the frequencies and mode shapes of a vibrating uniform cantilevered beam is well established. As such, it can be used to test the correctness of the methodology developed for identifying the BIT gondola.

**B.1 Euler-Bernoulli Beam Model**

In linear elastic beam theory, the dynamics of a generic elastic thin beam undergoing small deflection can be described by

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) = -\mu \frac{\partial^4 w}{\partial t^4} + q \quad (61)$$

where $\mu$ is the linear mass density of the beam, $w(x,t)$ describes the shape of the beam and $q(x,t)$ describes the applied force per unit length on the beam. For a uniform beam without any external force, the beam equation can be simplified to $^6$0

$$EI \frac{\partial^4 w}{\partial x^4} = -\mu \frac{\partial^2 w}{\partial t^2}. \quad (62)$$

For a cantilevered beam, the differential equation is subject to the boundary conditions $w(0,t) = 0$, $w_x(0,t) = 0$, $w_{xx}(L,t) = 0$ and $w_{xxx}(L,t) = 0$, where $L$ represents the length of the beam.

---

$^6$0Note that $q$ is not present if the beam is setup such that the vibration we care about is orthogonal to the direction of gravitational loading. However $q$ will be used later for the identification of $EI$. 

58
B.1.1 Natural Frequencies

Assuming that the solution is of the form \( w(x, t) = \Re[\hat{w}_\alpha(x) e^{j\omega_\alpha t}] \), then for each natural frequency\(^{61}\) \( \omega_\alpha \), the ODE

\[
EI \frac{d^4 \hat{w}_\alpha}{dx^4} - \mu \omega_\alpha^2 \hat{w}_\alpha = 0
\]  

(63)

can only have non trivial solutions if \( \cosh \beta_\alpha \cos \beta_\alpha + 1 = 0 \) where \( \beta_\alpha = L \left( \frac{m_\alpha^2}{EI} \right)^{\frac{1}{4}} \). Numerically solving for \( \beta_\alpha \), we get

\[
\begin{array}{cccccc}
\alpha & 1 & 2 & 3 & 4 & \cdots \\
\beta_\alpha & 1.875 & 4.694 & 7.855 & 10.9955 & \cdots \\
\end{array}
\]

The corresponding natural frequencies can be calculated by rearranging to get

\[
\omega_\alpha = \frac{\beta_\alpha^2}{L^2} \sqrt{\frac{EI}{\mu}}. 
\]  

(64)

The theoretical mode shape is\(^{15}\)\(^{62}\)

\[
w_\alpha(x) = \cos \beta_\alpha x - \cosh \beta_\alpha x + \frac{\sin \beta_\alpha L - \sinh \beta_\alpha L}{\cos \beta_\alpha L + \cosh \beta_\alpha L} (\sin \beta_\alpha x - \sinh \beta_\alpha x). 
\]  

(65)

B.1.2 Parameter Identification for Calculating Natural Frequencies

In a realistic situation, some of the parameters in Eq. (64) may be difficult to determine. Linear dimensions are easy to measure. However, determining \( E \) and \( I \) may not be trivial. For the purpose of this beam experiment, \( E \) and \( I \) do not need to be determined independently. Since Eq. (61) on the preceding page has been solved for static, cantilevered beam under gravity (uniform loading), we can express the deflection of the tip (free end) of the cantilever as

\[
w_{\text{max}} = \frac{PL^3}{8EI} 
\]  

(66)

where \( w_{\text{max}} \) is the deflection of the beam at its free end and \( P \) is the weight of the beam, such that \( P = \mu L g \). We then take the square of the expression for the natural frequency in Eq. (64) and substitute it with Eq. (66), we get

\[
\omega_\alpha^2 = \frac{\beta_\alpha^4 EI}{L^4} \mu = \frac{\beta_\alpha^4 EI L g}{P} 
\]  

(67a)

\[
= \frac{\beta_\alpha^3 EI g}{L^3} \cdot \frac{1}{w_{\text{max}}} \cdot w_{\text{max}} = \frac{\beta_\alpha^3 EI g}{L^3} \cdot \frac{1}{w_{\text{max}}} \cdot \frac{PL^3}{8EI} 
\]  

(67b)

\[
= \frac{\beta_\alpha^4 g}{8w_{\text{max}}} 
\]  

(67c)

\(^{61}\) \( \alpha \) denotes the mode number.

\(^{62}\) A significant amount of trouble was gone through to find the correct form of the mode shape equation. Some sources presented in literature are unreliable in getting the signs right.
in which \( w_{\text{max}} \) can be easily measured. The natural frequency \( \omega_\alpha \) can then be calculated using

\[
\omega_\alpha = \beta_\alpha^2 \sqrt{\frac{g}{8w_{\text{max}}}}.
\] 

(68)

### B.2 Experimental Setup

All equipment used in this experiment was found in the lab. A 144 cm aluminum alloy ruler was used as the cantilevered beam. The beam was clamped such that the cantilevered portion was exactly 1 m for easy calculation. Two ADXL355 (detailed in Section 2.5.1) accelerometers were attached to the beam at \( x = 0.5 \) m and \( x = 1 \) m, such that the direction of measurement was along the flexible direction of the beam. For identifying \( w_{\text{max}} \) the beam is clamped horizontally to a table (Figure 36a) and the deflection at the tip was measured against a rigid beam extended from the table (not shown), and were determined to be 10.5 cm. The \( \omega_\alpha \)'s were calculated using Eq. (68). The first three natural frequencies where computed to be 12.01 rad/s (1.912 Hz), 75.30 rad/s (11.98 Hz), and 210.86 rad/s (33.56 Hz).

Once theoretical natural frequencies were identified, the beam was then rotated to a vertical position (Figure 36b) for the modal identification, so that the force of gravity would not be a factor.\(^{63}\)

\(^{63}\) It can be shown that for a strictly linear system, the force of gravity would not affect its elastic modes. Practically, however, the beam is flexible enough that it can be bent so far that it can no longer be considered as one dimensional. We do not want that to happen so we choose to remove the effects of gravity all together.
B.3 Experimental Results

An impulse was applied to the beam at the free end using a hard object.\textsuperscript{64} The data were collected using BIT’s Inner Frame Computer’s analogue inputs.

![Time series of the impulse response](image1)

![FFT of the impulse response](image2)

Figure 37: The measured results of the beam experiment

B.4 Identification of Frequencies and Mode Shapes

The frequency response plot of Figure 37b is created using the FFT result of the time response. Three clear peaks can be identified, occurring at 2.0 Hz\textsuperscript{mea.}, 12.6 Hz\textsuperscript{mea.}, and 35.4 Hz\textsuperscript{mea.}. The comparison between the measured natural frequencies and the theoretical natural frequencies is presented in Table 16 on the next page.\textsuperscript{65} Using Eq. (23) on page 31, the damping ratios for the three modes are calculated to be 0.323%,

\textsuperscript{64}As can be seen in Figure 37a, the motion saturated the detection range (±3 g) for one of the sensors for a short while (< 0.5 s). This would not adversely affect the identified frequencies but might affect the identification of the mode shapes.

\textsuperscript{65}Ideally, we would also want to compare results generated using SOLIDWORKS SIMULATION with the theoretical results. Unfortunately, this was not accomplished due to time constraints.
0.365%, and 0.682%, respectively.

Table 16: Comparing experimental and theoretical modal frequencies

<table>
<thead>
<tr>
<th>α</th>
<th>ω_α^{mea.}</th>
<th>ω_α^{(thc.)}</th>
<th>Relative diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0 Hz</td>
<td>1.912 Hz</td>
<td>4.6%</td>
</tr>
<tr>
<td>2</td>
<td>12.6 Hz</td>
<td>11.98 Hz</td>
<td>5.2%</td>
</tr>
<tr>
<td>3</td>
<td>35.4 Hz</td>
<td>33.56 Hz</td>
<td>5.5%</td>
</tr>
</tbody>
</table>

Table 17: Measured results from the beam modal identification

<table>
<thead>
<tr>
<th>α</th>
<th>ω_α^{(mea.)}</th>
<th>ζ_α</th>
<th>C_α(r_{end})</th>
<th>C_α(r_{mid})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0 Hz</td>
<td>0.323%</td>
<td>10.29</td>
<td>3.81</td>
</tr>
<tr>
<td>2</td>
<td>12.6 Hz</td>
<td>0.365%</td>
<td>1.64</td>
<td>-1.51</td>
</tr>
<tr>
<td>3</td>
<td>35.4 Hz</td>
<td>0.682%</td>
<td>-2.52</td>
<td>-0.04</td>
</tr>
</tbody>
</table>

Using the least squares method presented in Section 2.9.1 on page 32, the C_α’s were identified and summarized in Table 17. Ratios were taken with C_α(r_{end}) in the denominator because of its larger value. Theoretical amplitude ratios were computed using Eq. (65) on page 59 at the locations of the sensors. The comparison between the identified ratios and theoretical ratios are presented in Table 18. Both Tables 16 and 18 show a high degree of agreement between the theoretical and experimental results.

Table 18: Comparing experimental and theoretical ratios

<table>
<thead>
<tr>
<th>α</th>
<th>C_α(r_{mid})/C_α(r_{end})</th>
<th>w_α(x_{mid})/w_α(x_{end})</th>
<th>Relative diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.370</td>
<td>0.3396</td>
<td>9.0%</td>
</tr>
<tr>
<td>2</td>
<td>-0.921</td>
<td>-0.7133</td>
<td>29%</td>
</tr>
<tr>
<td>3</td>
<td>0.016</td>
<td>0.0196</td>
<td>18%</td>
</tr>
</tbody>
</table>

B.5 Conclusion

Although the agreement between the theoretical ratios and the experimental ratios is not perfect, they are quite close. At the very least, the signs and the ordering are correct. This beam analysis shows the methodology developed in this thesis for modal system identification being applied to a simple, well-understood system. The results of this analysis corroborate the validity of this methodology.

C Examples of Gyroscope Results

Prior to using accelerometers, gyroscopes were used to make the measurements of the impulse response of the gondola. Three single-axis fibre-optic gyroscopic sensors (Kvh dsp-1750, shown in Figure 6b on page 15) are rigidly mounted into a 12.5 cm×12.5 cm×12.5 cm aluminum box (Figure 38b on the following page) aligned with three orthogonal axes. These gyro's have a sampling rate of 1000 Hz and an accuracy of 476.8 µ°/s.

The gyro box is movable and can be theoretically be placed anywhere on the gondola. However, its large size and weight preclude this possibility. Moreover, even though the gyro's are highly accurate compared
to the accelerometers, there are only three gyros fixed together at a single location, making the identification of multiple locations on the gondola impossible with a single impact. Since the data cannot be synchronized across multiple impacts, it would be impossible to identify the mode shapes of the gondola using the gyroscopes. Another reason that made the identification process using gyroscopes difficult is that gyroscopes measure rotational velocity of the shell surface and, thus, is not proportional to the simulation model as produced by SolidWorks Simulation. However, accelerometer measurements are. This would mean that in order to identify the mode shapes, the SolidWorks Simulation would have to have been post-processed to match the gyroscope measurements. That being said, gyroscope measurements were used as a proof-of-concept for the whole process which was later adapted for accelerometer measurements.

![Image](image.png)

(a) Bit Outer Frame  
(b) Gyro box with a lead brick and paper padding to ensure sufficient contact with bottom plate of the Outer Frame

Figure 38: Setup at the time for gyroscope measurements

C.1 Impulse Response

At the time when the gyroscope measurements were made, the geometrical structure of the gondola (Figure 38a) was slightly different, but not by much. The resultant frequency peaks are slightly higher than those of the accelerometer measurements due to the slightly lighter structure. The impulse was achieved the same way using a rubber mallet, and the gondola was hung by the pivot the same way as for the accelerometer measurement. Figure 39 shows the time series collected for the impulse response. By taking the FFT of the time series, amplitude vs. frequency data can be generated, and are shown in Figure 40.
C.2 Circle Plots

Analogous to Figure 18 on page 30 for accelerometer measurements, circle plots for gyroscope measurements are presented in Figure 41 on the following page. It is immediately noticed that the curves for the gyroscopes are much smoother than those of the accelerometers. This is because the signal-to-noise ratio for the $3,000$ gyroscopes is much greater than that of the $10$ accelerometers. Moreover, with the exception of the $z$-gyro, the maximum amplitudes (marked by a ‘+’) lie closely to the real axis, consistent with the theoretical circle (Figure 21 on page 34). The circles are also symmetrical about the real axis, as expected.
Figure 41: Circle plots of the first elastic mode (38.5 Hz(measured)) measured by the gyroscopes, with the maximum amplitude marked by ‘+’.

C.3 Conclusion

The analyses performed using gyroscope measurement served as a crucial step in establishing the methodology developed in this thesis. Despite not being used, the results from the analyses were able to validate the underlying mathematical theory.