Network Calculus Analysis of Feedback Systems with Random Service

by

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Abstract

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Feedback systems are deployed in many network related applications and various analytical approaches have tried to evaluate the performance of these systems. Network calculus is one such approach that uses (min,plus)-algebra for the analysis of networks. Although network calculus has successfully analyzed feedback systems under deterministic assumptions, the analysis of random feedback systems remains an open problem.

In this thesis, we use network calculus to extend the deterministic analysis of feedback systems to stochastic settings. We provide statistical service bounds for the service function of a feedback system modelled by a window flow control scheme. We find an exact characterization for the stochastic service of special cases in window flow control and provide the upper and lower bounds for the service function of the feedback system. We also derive backlog bounds based on our analysis and compare these bounds with the simulated experiments.
Dedication

I dedicate this dissertation to my family, my beloved mother and father, whom I am forever indebted to.
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I would like to thank my supervisors Jörg Liebeherr and Almut Burchard for their support in every step I took during my graduate studies. Their help, patience and professionalism made me understand the true meaning of research.
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Chapter 1

Introduction

Feedback control algorithms are used in a variety of systems designed to adjust themselves to changes of the system. In communication networks and applications feedback is mostly used in traffic control algorithms. Rate control and congestion control algorithms are used in the networks to prevent buffer overflow and avoid congestion. They deploy feedback mechanisms to reach stability [12]. Congestion control algorithms such as the Transmission Control Protocol (TCP) [17], are used to fully utilize the network while maintaining the stability. Different TCP variants such as [21], [22], [23], have the same window-based congestion control algorithm that uses feedback acknowledgments from the receiver to adjust its window size and determine the amount of traffic allowed inside the network. Congestion and rate control algorithm are also used in wireless sensor networks [20] where feedback helps the wireless connection to perform well over a noisy channel. Feedback is also used in smart grid applications, where the closed loop feedback created between energy providers and customers could cause price instability. Dynamic pricing algorithms in smart grids, [18], [19], take this feedback loop into account to adaptively adjust the power price.

When we look at the wide deployment of feedback systems in network related algorithms and applications, an analysis of feedback systems seems essential. A first step for per-
forming an analysis of feedback systems is to choose a suitable methodology that can exploit network properties and could result in a comprehensive characterization of the system under study.

1.1 Methodology

Control theory studies feedback algorithms in linear and non-linear systems [24], [25]. When feedback systems were introduced to the networks as window flow control algorithms, they were studied using different methodologies. Queuing theory methods were applied in [14], [26], [27], while [28], [10] used control theory for the analysis of such systems. Network calculus [5], [13], applies system-theoretic methods under an alternative algebra called (min,plus)-algebra (based on (max,plus)-algebra in [2]) which is suitable for representation of network systems. In network calculus the departures from a network element are characterized by the (min,plus)-convolution of the arrival traffic and a network function. Using this method for a tandem of network elements, an end-to-end service function can be obtained by convolving the service function of the elements. This methodology is widely used for performance analysis of networks resulting in bounds for delay and buffer requirements of the network. There are two approaches in the area of network calculus, deterministic [4], and stochastic [3], [11]. While the deterministic approach assumes non-random network functions, stochastic network calculus performs its analysis under probabilistic assumptions where arrival traffic and service function of a network can be random. The assumption of deterministic arrival and service traffic leads to worst-case end-to-end results for the network under study, and pessimistic performance bounds are obtained for the delay and buffer requirements. By using stochastic network calculus we exploit the random properties of traffic flows and network functions. Stochastic network calculus results in statistical bounds for the random functions of the network. The advantage of statistical bounds over deterministic bounds is that they
allow probabilistic violations to avoid overestimation of network requirements.

### 1.2 Feedback Systems

The analysis of feedback systems in the context of deterministic network calculus in [1], [4], [6], addressed a window flow control model. Fig. 1.1 shows the window flow control system which was studied in [1] and [4]. In this model a throttle is placed at the entrance of the network element. It uses the feedback from the end-point of the network to control the amount of traffic allowed into the system, so that the data in transit is limited to a certain amount of traffic called window size. The feedback traffic is the acknowledgments specifying the amount of departed traffic. Note that the feedback loop may introduces a feedback delay, which is the time that it takes for the acknowledgments to reach the network entrance. This feedback system is fully characterized by network calculus methods under deterministic assumptions and equivalent service functions are derived using the concept of sub-additive closure. However, no analysis performed on this system accounts for randomness of traffic and network elements, which is an inevitable property of real network systems.
1.3 Problem Statement and Contribution

This thesis addresses the problem of feedback analysis under probabilistic assumptions using stochastic network calculus. The existing results for a feedback system under deterministic assumptions raise the question of the feasibility of an extension to stochastic settings, especially since most of the work in the area of deterministic network calculus has already been extended to probabilistic settings. The intrinsic properties of feedback loops in a network system introduce enormous difficulties to the extension. Especially the time correlations caused by feedback traffic in the network appears to eliminate the possibility of using conventional methods for stochastic extension. This problem is stated as an open problem in the area of network calculus in several survey papers, [8], [9], [11]. In this thesis we present results on the analysis of feedback systems using stochastic network calculus.

For our analysis, we choose the window flow control feedback system shown in Fig. 1.1 which is the same model that was studied under deterministic assumptions. We use the moment-generating function (MGF) network calculus introduced in [7] and [9] for our analysis of this feedback system. The derivations in MGF calculus help us overcome the difficulty of expressing feedback systems, specifically by providing bounds for iteratively convolved random service functions.

Our contributions are the following:

- We present derivations that exactly characterize a feedback system with a feedback delay equal to one time unit, and a service function with independent increments.

- We provide upper and lower bounds for the equivalent service of a general feedback system where the lower bound corresponds to the equivalent service of a feedback system with a delay of one time unit.

- We derive a simple and an improved equivalent service bound for a feedback system with independent increments. We later use our lower and upper bounds to evaluate
the accuracy of these bounds.

\section*{1.4 Thesis Organization}

The remainder of the thesis is structured as follows:

In Chapter 2, we provide a thorough review of network calculus. We introduce (min,plus)-algebra for two categories of functions, one suitable for deterministic analysis and one for stochastic analysis. We review moment-generating function network calculus and its existing methods.

In Chapter 3, we provide the-state-of-the-art of feedback system analysis under deterministic network calculus. In Chapter 4, we present exact results achieved for a special case in feedback systems. Then we provide upper and lower bounds for the equivalent service of a general feedback system which we later use as a benchmark for our numerical examples.

In Chapter 5, we consider a feedback system where the service function is exponentially distributed. We provide two bounds for the equivalent service of a such system.

In Chapter 6, we evaluate derived equivalent service bounds as well as upper and lower bounds. We also provide backlog bounds for the case where we derived exact results for the equivalent service of the system. We also compare backlog bounds with simulation results.

Finally, in Chapter 7, we conclude the thesis and briefly discuss future work.
Chapter 2

Review of network calculus

In this chapter, we introduce the network calculus methodology for the analysis of networks. In this methodology a network element is characterized by a network service function. The arrival traffic enters the network element and creates the departure traffic. To understand how the network is analyzed by network calculus, let us look at input-output relations in system theory. In system theory the output function of a linear time invariant (LTI) system is described by the convolution of an input function and the impulse response of the system. Network calculus introduces another algebra called (min,plus)-algebra where the convolution works in a similar fashion for the network elements. In this algebra the departure traffic is described by the (min,plus)-convolution of the arrival traffic and network function. In the context of network calculus, functions are univariate or bivariate. Most of the work in network calculus is done in a univariate representation, where \( f(t) \) describes a network function (which can be arrival, departure, or service function) in the time interval \([0,t)\). In the bivariate representation we use \( f(s,t) \) to describe events that occur in the time interval \([s,t)\). The elements of the (min,plus)-algebra for univariate and bivariate network calculus have a close connection and similarity in the essence, however, the representation of the definitions and properties are different. The univariate version has more powerful algebraic properties such
as commutativity. However, it is not suitable to be used under probabilistic assumptions, which is the reason for using a bivariate representation. In the following sections we discuss the elements of (min,plus)-algebra for both univariate and bivariate network calculus. Then we introduce a method in stochastic network calculus to analyze the networks under probabilistic assumption, particularly when we have random processes as the arrival traffic and service function of the network element.

We will address the functions of network system such as arrival and departure traffic and service function as network processes throughout this thesis. We also consider a discrete-time domain for the processes with $t = 0, 1, 2, \ldots$ to describe the time unit.

### 2.1 Univariate network calculus

The univariate network calculus works with univariate processes such as $f(t)$ describing a function in time interval $[0, t)$. These processes are expressed and characterized in the context of (min,plus)-algebra [5]. In the (min,plus)-algebra the conventional summation is replaced by a minimization, and the conventional multiplication is replaced by a summation. The minimization ($\wedge$) is defined as the point-wise minimum of two processes,

$$f(t) \wedge g(t) \triangleq \min\{f(t), g(t)\},$$

while the definition of the summation ($+$) remains same as in the conventional algebra. The minimum ($\wedge$) and the summation ($+$) have the following properties,

1. **Associativity:** For univariate functions $A, B, C$ we have the associativity of both $\wedge$ and $+$ operation as follows

   $$(A \wedge B) \wedge C)(t) = (A \wedge (B \wedge C))(t),$$

   $$((A + B) + C)(t) = (A + (B + C))(t).$$
2. Commutativity of both $\land$ and $+$: For univariate functions $A, B$ we have

\[
((A \land B)(t) = (B \land A)(t),
\]

\[
(A + B)(t) = (B + A)(t).
\]

3. Distributivity of $\land$ with respect to $+$: For univariate functions $A, B, C$ the minimum distributes over the summation and we have

\[
(A + (B \land C))(t) = ((A + B) \land (A + C))(t).
\]

In the view of discrete-time system theory, the convolution operation is defined (within the regular algebra) as

\[
f(t) \ast g(t) = \sum_{\tau=0}^{\infty} f(\tau)g(t - \tau).
\]

This operation is used to describe the output of an LTI system as the convolution of the input signal and the impulse response of the system. (The impulse response in system theory is the function that characterizes the LTI system). In network calculus, the converted version of this convolution to $(\land, +)$-algebra (the summation changes to a minimization and the multiplication changes to a summation) can be used in the same way to describe the departure of a network element. The $(\land, +)$-convolution is defined as

\[
f(t) \otimes g(t) \triangleq \min_{0 \leq \tau \leq t} \{f(\tau) + g(t - \tau)\},
\]

which we will call convolution from now on.

Let $\mathcal{F}$ be the univariate, non-decreasing, non-negative processes $f(t)$ where $f(0) \geq 0$ and for all $s \leq t$ we have $f(s) \leq f(t)$. The $\land$ and $\otimes$ operations form a commutative dioid called $(\land, \otimes)$-dioid over $\mathcal{F}$ with the following properties,

1. Associativity: For $A, B, C \in \mathcal{F}$ we have the associativity of both $\land$ and $\otimes$ operation
as follows

\[( (A \land B) \land C)(t) = ((A \land (B \land C))(t), \]
\[( (A \otimes B) \otimes C)(t) = (A \otimes (B \otimes C))(t) . \]

2. Commutativity of both \( \land \) and \( \otimes \): For \( A, B \in \mathcal{F} \) we have

\[(A \land B)(t) = (B \land A)(t), \]
\[(A \otimes B)(t) = (B \otimes A)(t) . \]

3. Distributivity of \( \otimes \) with respect to \( \land \): For \( A, B, C \in \mathcal{F} \) the convolution distributes over minimum and we have

\[(A \otimes (B \land C))(t) = ((A \otimes B) \land (A \otimes C))(t) . \]

4. Zero element: The zero element is defined as

\[ \epsilon(t) \triangleq \infty , \]

and \( A \land \epsilon = A \) holds for any \( A \in \mathcal{F} \).

5. Absorbing property of zero element: For \( A \in \mathcal{F} \) we have \( A \otimes \epsilon = \epsilon \otimes A = \epsilon \).

6. Neutral element: \( A \otimes \delta = \delta \otimes A = A \) holds for any \( A \in \mathcal{F} \) where the neutral element \( \delta \) is defined as

\[ \delta(t) \triangleq \begin{cases} 0 & t \leq 0, \\ \infty & t > 0 . \end{cases} \]

Causal processes are a sub-class of \( \mathcal{F} \) which are functions \( f \in \mathcal{F} \) that \( f(0) = 0 \). Causal processes are denoted by \( \mathcal{F}_0 \). In the context of the \((\land, +)-\)algebra another operation is used for the performance metrics of the network. This operation is called deconvolution
and is defined as

\[ f(t) \otimes g(t) \triangleq \max_{\tau \in \mathbb{N}_0} \{ f(\tau + t) - g(\tau) \}. \]

### 2.1.1 Sub-additivity and sub-additive closure

A process \( f \in \mathcal{F} \) is called a sub-additive process if \( f(t + \tau) \leq f(t) + f(\tau) \) holds for all \( t, \tau \in \mathbb{N}_0 \). We define the sub-additive closure of \( f \in \mathcal{F} \), denoted by \( f^* \), as

\[ f^*(t) \triangleq \delta(t) \wedge f(t) \wedge f^{(2)}(t) \wedge f^{(3)}(t) \wedge \ldots = \bigwedge_{n=0}^{\infty} f^{(n)}(t). \]

where \( f^{(n+1)}(t) = f^{(n)}(t) \otimes f(t) \) for \( n \geq 1 \), \( f^{(0)}(t) = \delta(t) \), and \( f^{(1)}(t) = f(t) \). The following properties hold for sub-additive functions and sub-additive closure of a function:

1. For every sub-additive process in \( \mathcal{F} \) we have \( f(t) \otimes f(t) = f(t) \).
2. For every sub-additive process in \( \mathcal{F} \) we have \( f(t) \otimes f(t) = f(t) \).
3. For sub-additive processes \( f, g \in \mathcal{F} \), \( f(t) + g(t) \), \( f(t) \otimes g(t) \), and \( f(t) \otimes g(t) \) are also sub-additive processes in \( \mathcal{F} \).
4. The sub-additive closure of a causal function \( f \in \mathcal{F}_0 \) is given by \( f^*(t) = \lim_{n \to \infty} f^{(n)}(t) \).
5. If \( f \in \mathcal{F} \) is a sub-additive process then we have \( f^*(t) = f(t) \).

### 2.2 Bivariate network calculus

The family of bivariate functions as defined in [4] is

\[ \tilde{\mathcal{F}} \triangleq \{ F(\cdot,\cdot) : F(s,t) \geq 0, F(s,t) \leq F(s,t+1), \text{for all } 0 \leq s \leq t \}. \]
For $f, g \in \tilde{F}$, the minimum ($\wedge$) and convolution ($\otimes$) operations are defined as
\[
\begin{align*}
    f \wedge g (s, t) & \triangleq \min \{f(s, t), g(s, t)\}, \\
    f \otimes g (s, t) & \triangleq \min_{s \leq \tau \leq t} \{f(s, \tau) + g(\tau, t)\}.
\end{align*}
\]
Causal processes in $\tilde{F}$ are denoted by $\tilde{F}_0$ and defined as $f \in \tilde{F}$ that $f(t, t) = 0$. $(\tilde{F}, \wedge, \otimes)$ forms a complete non-commutative dioid which has the following properties,

1. Associativity for both $\wedge$ and $\otimes$ in $\tilde{F}$,

2. Commutativity of $\wedge$: In the bivariate settings only $\wedge$ has commutativity and convolution operation $\otimes$ does not commutes.

3. Distributivity of $\otimes$ with respect to $\wedge$: For $A, B, C \in \tilde{F}$ we have $A \otimes (B \wedge C) = (A \otimes B) \wedge (A \otimes C)$.

4. Zero element: The zero element is defined as
\[
    \epsilon(s, t) \triangleq \infty,
\]
and $A \wedge \epsilon = A$ holds for any $A \in \tilde{F}$.

5. Absorbing property of zero element: For $A \in \tilde{F}$ we have $A \otimes \epsilon = \epsilon \otimes A = \epsilon$.

6. Neutral element: This element is defined as
\[
\delta(s, t) \triangleq \begin{cases}
    0 & s \geq t, \\
    \infty & s < t,
\end{cases}
\]
where $A \otimes \delta = \delta \otimes A = A$ holds for any $A \in \tilde{F}$.
7. Monotonicity: For $A_1, A_2, B_1, B_2 \in \tilde{F}$ where $A_1 \leq A_2, B_1 \leq B_2$ we have

$$A_1 \land B_1 \leq A_2 \land B_2,$$

$$A_1 \otimes B_1 \leq A_2 \otimes B_2.$$

For two processes $f, g \in \tilde{F}$, the deconvolution is defined as

$$f \ominus g(s, t) \triangleq \max_{0 \leq \tau \leq s} \{f(\tau, t) - g(\tau, s)\}.$$

A bivariate function $f \in \tilde{F}$ is sub-additive if $f(s, t) \leq f(s, \tau) + f(\tau, t)$. The sub-additive closure of $f \in \tilde{F}$ is defined (same as univariate definition) by

$$f^* \triangleq \delta \land f \land f^{(2)} \land f^{(3)} \land \ldots = \bigwedge_{n=0}^{\infty} f^{(n)}.$$

where $f^{(n+1)} = f^{(n)} \otimes f$ for $n \geq 1$, $f^{(0)} = \delta$, and $f^{(1)} = f$.

### 2.3 Arrival, departure and service processes

In this section we introduce network processes using the bivariate representation. Unless otherwise is stated, the univariate version is easily obtainable by setting the first variable equal to zero.

The traffic arrivals of a network element are denoted by $A(s, t)$ describing the cumulative arrivals to the network in the time interval $[s, t]$,

$$A(s, t) = \sum_{k=s}^{t-1} a_k.$$
The traffic departures are denoted by $D(s, t)$ describing the departures of network in time interval $[s, t)$,

$$D(s, t) = \sum_{k=s}^{t-1} d_k.$$  

We characterize a network element by a service process $S(s, t) \in \tilde{F}_0$ where the departure process of the element is described by convolution of the arrival process and the service process

$$D(0, t) \geq (A \otimes S)(0, t). \quad (2.1)$$

This is called dynamic server in [5] and [7]. By the concept of dynamic server, a tandem of network elements can easily be characterized since the convolution has similar properties of system theoretic convolution. As shown is Fig. 2.1 the equivalent of a network that consists of $n$ elements with service processes $S_1, S_2, \ldots, S_n$ is given by their convolution $S_1 \otimes S_2 \otimes \ldots \otimes S_n$. The backlog traffic is the amount of arrival traffic which is buffered waiting to be served by the network element. Backlog is defined as

$$B(t) \triangleq A(0, t) - D(0, t),$$
with arrival process $A$ and departure $D$.

**Lemma 1** (Theorem 2 in [7]). For a network element with arrival process $A \in \tilde{F}$ and service process $S \in \tilde{F}_0$, for every $t \geq 0$ a bound on the backlog can be expressed in terms of the deconvolution as

$$B(t) \leq (A \circ S)(t,t), \quad (2.2)$$

Note that in the univariate setting, the backlog bound is given by $B(t) \leq (A \circ S)(0)$.

**Proof.**

$$B(t) = A(0, t) - D(0, t)$$

$$\leq A(0, t) - A \otimes S(0, t)$$

$$= A(0, t) - \min_{s \leq \tau \leq t} \{A(0, \tau) + S(\tau, t)\}$$

$$= \max_{0 \leq \tau \leq t} \{A(0, t) - A(0, \tau) - S(\tau, t)\}$$

$$= \max_{0 \leq \tau \leq t} \{A(\tau, t) - S(\tau, t)\}$$

$$= (A \circ S)(t, t).$$

In the second line we use Eq. (2.1). The third line is written with the definition of $\otimes$. The fourth line is written by taking the minimum to the left which changes the minimum to a maximum due to the minus sign. In the last line we use the definition of deconvolution to complete the proof.

\[\square\]

### 2.4 Stochastic network calculus

In this section we introduce methods to analyze network systems with randomness. Among different approaches of network calculus for dealing with randomness, the moment-
generating functions (MGF) network calculus [7] gives a straightforward analysis. In this setting the random processes of a network are characterized by deterministic bounds which are based on the moment-generating function of those random processes. As we already mentioned, we will use bivariate processes in this section since it is the best choice for the function representation in random settings.

### 2.4.1 MGF network calculus

By MGF calculus we can analyze random systems through characterization of the network processes with their moment-generating function. The moment-generating function of a random variable $X$ for any $\theta \in \mathbb{R}$ is defined as

$$M_X(\theta) \triangleq E\left[e^{\theta X}\right]. \quad (2.3)$$

If $X$ and $Y$ are two random variables, the moment-generating function has the following properties,

1. For $\theta > 0$ we have $M_{\min\{X,Y\}}(\theta) \leq \min\{M_X(\theta), M_Y(\theta)\}$.

2. For $\theta > 0$ we have $M_{\max\{X,Y\}}(-\theta) \leq \min\{M_X(-\theta), M_Y(-\theta)\}$.

3. If $X$ and $Y$ are independent random variables then

$$M_{X+Y}(\theta) = M_X(\theta) \cdot M_Y(\theta),$$

$$M_{X-Y}(\theta) = M_X(\theta) \cdot M_Y(-\theta).$$

MGF calculus introduces bounds on the convolution and deconvolution of two processes which are given in the next lemma.
Lemma 2 (Lemma 1 in [7]). The following inequalities hold for two independent random processes $f, g \in \tilde{F}$

\[ M_{f \otimes g}(-\theta, s, t) \leq \sum_{\tau = s}^{t} M_f(-\theta, s, \tau)M_g(-\theta, \tau, t), \quad (2.4) \]

\[ M_{f \otimes g}(\theta, s, t) \leq \sum_{\tau = 0}^{s} M_f(\theta, \tau, t)M_g(-\theta, \tau, s). \quad (2.5) \]

Proof.

\[ M_{f \otimes g}(-\theta, s, t) = E \left[ e^{-\theta(f \otimes g(s,t))} \right] \]

\[ = E \left[ e^{-\theta(\min_{s \leq \tau \leq t\{ f(s,\tau)+g(\tau,t) \})} \right] \]

\[ = E \left[ \max_{s \leq \tau \leq t\{ e^{-\theta(f(s,\tau)+g(\tau,t))} \} \right] \]

\[ \leq E \left[ \sum_{\tau = s}^{t} e^{-\theta(f(s,\tau)+g(\tau,t))} \right] \]

\[ \leq \sum_{\tau = s}^{t} E \left[ e^{-\theta f(s,\tau)} e^{-\theta g(\tau,t)} \right] \]

\[ = \sum_{\tau = s}^{t} M_f(-\theta, s, \tau)M_g(-\theta, \tau, t). \]

The first line is written by the definition of moment-generating functions. In the second line we use the definition of convolution. In the third line the monotonicity of exponential function is used. The fourth line is written by bounding the maximum over a set by the total summation of that set, and the last line is given by the independency of $f$ and $g$. 

The second inequality is proven in the same fashion as follows,

\[
M_{f \odot g}(\theta, s, t) = E \left[ e^{\theta(f \odot g(s, t))} \right] \\
= E \left[ e^{\theta(\max_{0 \leq \tau \leq s} \{f(\tau, t) - g(\tau, s)\})} \right] \\
\leq E \left[ \max_{0 \leq \tau \leq s} \{e^{\theta(f(\tau, t) - g(\tau, s))}\} \right] \\
\leq E \left[ \sum_{\tau=0}^{s} e^{\theta(f(\tau, t) - g(\tau, s))} \right] \\
\leq \sum_{\tau=0}^{s} E \left[ e^{\theta f(\tau, t)} e^{-\theta g(\tau, s)} \right] \\
= \sum_{\tau=s}^{t} M_f(\theta, \tau, t) M_g(-\theta, \tau, s).
\]

2.4.2 Statistical service bound for service processes

A statistical service bound \( S^\varepsilon(s, t) \) of a service process \( S \in \tilde{F}_0 \) for a given \( \varepsilon > 0 \), and for \( 0 \leq s \leq t \) is defined as

\[
Pr \left( S(s, t) \leq S^\varepsilon(s, t) \right) \leq \varepsilon. \tag{2.6}
\]

\( S^\varepsilon(s, t) \) is a deterministic bound for the service process of the network that can be violated at most by the probability of \( \varepsilon \). This bound can be computed by the Chernoff bound, where for a random variable \( X \), it is given by

\[
Pr(X > x) \leq e^{-\theta x} M_X(\theta).
\]

**Lemma 3** (see [7]). The service bound \( S^\varepsilon(s, t) \) is given by

\[
S^\varepsilon(s, t) = \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \log M_S(-\theta, s, t) \right\}. \tag{2.7}
\]
where \( \varepsilon > 0 \) is the violation probability, \( \theta > 0 \), and \( S^\varepsilon(s,t) \) satisfies Eq. (2.6).

Note that in the lemma we can replace \( M_S(-\theta, s, t) \) by any of its upper bounds.

\[ e^{\theta S^\varepsilon(s,t)} M_S(-\theta, s, t) \leq \varepsilon, \]

where the second line is written by the Chernoff bound. From the last line of the above equations we have

\[ e^{\theta S^\varepsilon(s,t)} M_S(-\theta, s, t) \leq \varepsilon, \]

which is equivalent to

\[ S^\varepsilon(s,t) \leq \frac{1}{\theta} \left\{ \log \varepsilon - \log M_S(-\theta, s, t) \right\}, \]

for all \( \theta > 0 \). We choose the optimal value for \( \theta \) and the proof is complete.

2.4.3 Effective capacity for service processes

Analyzing long-term rate of network processes is important when we are dealing with randomness. Since we are using the moment-generating function of network processes from the MGF calculus, the concept of effective capacity [15] which is expressed by the moment-generating function of random processes can be helpful. The effective capacity
of a service process $S(s,t)$ is defined as

$$
\gamma_S(-\theta) \triangleq \lim_{t \to \infty} -\frac{1}{\theta t} \log M_S(-\theta, 0, t) \quad \text{(2.9)}
$$

for $\theta > 0$. Since the limit in the definition is $t \to \infty$, the effective capacity can be used to measure long-term rates. In particular

$$
\gamma_S(0) \triangleq \lim_{\theta \to 0} \gamma_S(-\theta),
$$
gives us the long-term average service rate.

### 2.4.4 Stochastic backlog bound for network elements

We use the same technique that we used for statistical service bound to get a stochastic bound for the backlog. A stochastic backlog bound $b^*(t)$ for a given violation probability $\varepsilon > 0$, and $t \geq 0$ is defined as

$$
Pr\left(B(t) \geq b^*(t)\right) \leq \varepsilon. \quad \text{(2.10)}
$$

**Lemma 4** (see [7]). *A stochastic bound on the backlog at a network element with service process $S$ is given by*

$$
b^*(t) = \min_{\theta > 0} \frac{1}{\theta} \left\{ \log \left( \sum_{\tau=0}^{t} M_A(\theta, \tau, t)M_S(-\theta, \tau, t) \right) - \log \varepsilon \right\}. \quad \text{(2.11)}
$$

where $\varepsilon > 0$ is the violation probability and $\theta > 0$. 
Proof.

\[
Pr( B(t) \geq b^*(t) ) = Pr\left( A \odot S(t, t) \geq b^*(t) \right) \\
\leq e^{-b^*(t)\theta} M_{A \odot S}(\theta, t, t) \\
\leq e^{-b^*(t)\theta} \sum_{\tau=0}^{t} M_f(\theta, \tau, t) M_g(-\theta, \tau, t) \\
\leq \varepsilon,
\]

where the first line is written by Eq. (2.2). The second line is derived by the Chernoff bound. The third line used Eq.(2.5), and the last line is by the definition of \( b^*(t) \) in Eq. (2.10). From the last line of above equation we have

\[
e^{-b^*(t)\theta} \sum_{\tau=0}^{t} M_f(\theta, \tau, t) M_g(-\theta, \tau, t) \leq \varepsilon,
\]

which is equivalent to

\[
b^*(t) \geq \frac{1}{\theta} \left\{ \log \left( \sum_{\tau=0}^{t} M_A(\theta, \tau, t) M_S(-\theta, \tau, t) \right) - \log \varepsilon \right\},
\]

for \( \theta > 0 \), and we choose the optimal value for \( \theta \) which completes the proof. \( \square \)
Chapter 3

State-of-the-art in feedback systems analysis

An analysis of a deterministic feedback system in the network calculus was done by Cruz in [1] and Chang in [4] and [6]. Their work is based on similar feedback systems with slight differences in the model and method of computation. In this chapter we introduce the models and we analyze them via bivariate settings following the analysis in [4].

![Figure 3.1: Window flow control with deterministic service function.](image-url)
3.1 Model description

In this section we introduce the window flow control model in Fig. 3.1. This model uses feedback traffic to prevent the backlog of the service element to exceed the window size $w$. The feedback traffic is consist of the departure process which is added to the window size $w$. For adding the window size, we define $\delta^{+w}$ as

$$\delta^{+w}(s,t) \triangleq \begin{cases} w & s \geq t, \\ \infty & s < t. \end{cases}$$  \hspace{1cm} (3.1)

Using the above definition we can write $f + w = f \otimes \delta^{+w}$. Note that since $\delta^{+w}$ is a summation service process, it commutes in bivariate settings, meaning $f \otimes \delta^{+w} = \delta^{+w} \otimes f$.

An element is placed at the entrance of the network that takes the minimum of the arrival process $A$ and the feedback traffic $D'$ which is equal to $D \otimes \delta^{+w}$. So we have

$$A' = \min\{A, D'\}$$

$$= \min\{A, D + w\},$$

this element ensures that traffic inside the system never exceeds $w$ since we have

$$A' - D = \min\{A, D + w\} - D$$

$$\leq D + w - D$$

$$= w.$$

The exceeding traffic is stored in a FIFO buffer outside of the system. The network service in this model has a service process $S(s,t)$ which means

$$D \geq A' \otimes S,$$ \hspace{1cm} (3.2)
Chapter 3. State-of-the-art in feedback systems analysis

3.2 Bivariate feedback system

In order to analyze the window flow control model given in Fig. 3.1, we first consider its closed loop system called generic feedback server. It is illustrated in Fig. 3.2. The following lemma helps us to describe a generic feedback server in network calculus settings.

**Lemma 5** (Lemma 2.2 in [6]). For $A, D \in \tilde{F}_0$ and $S \in F$ we have

- $D = A \otimes S^*$ is the maximal solution of $D = (D \otimes S) \land A$.

- The above solution is the unique solution of the equation, if $\inf_t S(t, t) > 0$.

- If $\inf_t S(t, t) > 0$ and $D \geq (D \otimes S) \land A$, then $D \geq A \otimes S^*$.

**Proof.** To show that $D = A \otimes S^*$ is a maximal solution, we first show that it satisfies the equation $D = (D \otimes S) \land A$,

$$
((A \otimes S^*) \otimes S) \land A = (A \otimes (S^* \otimes S)) \land A
$$

$$
= (A \otimes (S^* \otimes S)) \land (A \otimes \delta)
$$

$$
= A \otimes ((S^* \otimes S) \land \delta)
$$

$$
= A \otimes S^*.
$$
In the first line, we use the associative property of bivariate functions. In the second line, the identity element is used to replace $A$ with $A \otimes \delta$, and in the third line we use distributivity of $\otimes$ with respect to $\land$. The last line is shown below

\[
(S^* \otimes S) \land \delta = \left( \bigwedge_{n=0}^{\infty} S^{(n)} \right) \otimes S \land \delta
\]

\[
\quad = \left( \bigwedge_{n=1}^{\infty} S^{(n)} \right) \land \delta
\]

\[
\quad = \left( \bigwedge_{n=0}^{\infty} S^{(n)} \right)
\]

\[
\quad = S^*,
\]

where the first line uses the definition of sub-additive closure. The second line uses the distributivity of $\otimes$ with respect to $\land$ and in the third line the definition of sub-additive closure is recreated.

Now that we have verified $D = A \otimes S^*$ is a solution, we need to show that this solution is the maximal solution. By inserting the equation $D = (D \otimes S) \land A$ into itself, $n$ times repeatedly, we get

\[
D = A \land (A \otimes S) \land (A \otimes S^{(2)}) \land ... \land (A \otimes S^{(n)}) \land (D \otimes S^{(n+1)})
\]

\[
\quad = \left( A \land (A \otimes S) \land (A \otimes S^{(2)}) \land ... \land (A \otimes S^{(n)}) \right) \land (D \otimes S^{(n+1)}).
\]

Then we have

\[
D \leq A \land (A \otimes S) \land (A \otimes S^{(2)}) \land (A \otimes S^{(3)}) \land ... \land (A \otimes S^{(n)})
\]

\[
\quad = A \otimes (\delta \land S \land S^{(2)} \land S^{(3)} \land ... \land S^{(n)})
\]

\[
\quad \Rightarrow_{n \to \infty} A \otimes S^*.
\]

The first line is derived from Eq. (3.3) and the fact that the minimum of two functions
is less or equal to any of the functions. In the second line, the distributivity of $\otimes$ with respect to $\land$ is used. By $n \to \infty$, the right side of the inequality yields the sub-additive closure. We showed that $D \leq A \otimes S^*$, which means $A \otimes S^*$ is the maximal solution.

To show that $D = A \otimes S^*$ is the unique solution of equation $D = (D \otimes S) \land A$ under the condition that $\inf_t S(t, t) = \alpha > 0$, we can say for $S \in \bar{F}$ and $0 \leq s \leq t$

$$S(s, t) \geq S(s, s) \geq \inf_{s} S(s, s) = \alpha.$$

Then for every $0 \leq s \leq \tau \leq t$

$$S(s, \tau) + S(\tau, t) \geq 2\alpha,$$

which yields

$$S \otimes S = \inf_{s \leq \tau \leq t} \{S(s, \tau) + S(\tau, t)\} \geq 2\alpha.$$

By induction we get $S^{(k)} \geq k\alpha$. Then for $D \in \bar{F}_0$ and $0 \leq s \leq \tau \leq t$

$$D(s, \tau) + S^{(k+1)}(\tau, t) \geq D(s, \tau) + (k + 1)\alpha \geq (k + 1)\alpha,$$

which yields

$$D \otimes S^{(k+1)} \geq (k + 1)\alpha. \quad (3.4)$$

By Eq. (3.4) for a fixed value of $t$ where $0 \leq s \leq t$, since we can arbitrarily increase
Chapter 3. State-of-the-art in feedback systems analysis

\[(k + 1)\alpha, \text{there exists a finite } n \text{ such that}\]

\[A(s, t) \leq (n + 1)\alpha \leq D \otimes S^{(n+1)}(s, t),\]

having \(A(s, t) \leq D \otimes S^{(n+1)}(s, t)\) for a fixed value of \(t\), we can re-write Eq. (3.3) as

\[D(s, t) = (A \wedge (A \otimes S) \wedge (A \otimes S^{(2)}) \wedge ... \wedge (A \otimes S^{(n)}))(s, t)\]

\[= (A \otimes (\delta \wedge S \wedge S^{(2)} \wedge ... \wedge S^{(n)}))(s, t)\]

\[\geq (A \otimes S^*)(s, t).\]

Since we already verified that \(A \otimes S^*\) is the maximal solution, we have \(D(s, t) = (A \otimes S^*)(s, t)\), and hence the solution is unique.

To show that \(D \geq A \otimes S^*\) holds under the conditions \(\inf_t S(t, t) > 0\) and \(D \geq (D \otimes S) \wedge A\), we create the following inequality for \(D \geq (D \otimes S) \wedge A\), the same way it was created for the equality in \(D = (D \otimes S) \wedge A\) in Eq. (3.3),

\[D \geq A \wedge (A \otimes S) \wedge ... \wedge (A \otimes S^{(n)}) \wedge (D \otimes S^{(n+1)}). \quad (3.5)\]

By the same argument used previously, for a fixed value of \(t\) where \(0 \leq s \leq t\) there exists a finite \(n\) such that \(A(s, t) \leq D \otimes S^{(n+1)}(s, t)\). Using this for Eq. (3.5) for a fixed value of \(t\), we obtain

\[D(s, t) \geq (A \wedge (A \otimes S) \wedge ... \wedge (A \otimes S^{(n)}))(s, t)\]

\[= (A \otimes (\delta \wedge S \wedge S^{(2)} \wedge ... \wedge S^{(n)}))(s, t)\]

\[\geq (A \otimes S^*)(s, t),\]

which completes the proof. \(\square\)

The generic feedback server characterized in Lemma 5 is the equivalent element for
the part for system that takes the minimum of the arrivals and feedback traffic. Using this lemma we get the following expression for Fig. 3.1

\[ A' \geq A \otimes (S \otimes \delta^+)^* . \]

Then we have

\[ D \geq A' \otimes S \]
\[ \geq A \otimes (S \otimes \delta^+)^* \otimes S , \]

which means the equivalent service process of the whole system can be represented as

\[ S_{\text{win}} = (S \otimes \delta^+)^* \otimes S . \]

### 3.3 Threshold window size

In the previous section we obtained \( S_{\text{win}} \), the equivalent service of a window flow control system shown in Fig. 3.1. Increasing the window size means more traffic is allowed to enter the system. When a threshold window size, which we denote by \( w_{\text{th}} \), is reached the feedback will no longer effect the departure traffic, and the service process is equal to of a system without feedback (\( S_{\text{win}} = S \)). In order to find \( w_{\text{th}} \), we first re-write the
expression of the equivalent service process of the system

\[
S_{\text{win}} = (S \otimes \delta^{+w})^* \otimes S
\]

\[
= \left( \bigwedge_{n=0}^{\infty} (S \otimes \delta^{+w})^{(n)} \right) \otimes S
\]

\[
= \bigwedge_{n=0}^{\infty} \left( (S^{(n)} \otimes \delta^{+nw}) \otimes S \right)
\]

\[
= \bigwedge_{n=0}^{\infty} (S^{(n+1)} \otimes \delta^{+nw})
\]

\[
= (S) \wedge (S^{(2)} \otimes \delta^{+w}) \wedge (S^{(3)} \otimes \delta^{+2w}) \wedge \ldots .
\]  

In the second line we used the definition of sub-additive closure. The third line is written by commutativity of \(\delta^{+w}\). (Note that in general, processes do not commute in bivariate settings). We also used the distributive property of \(\otimes\) with respect to \(\wedge\) in the third line. And the last line is simply the expansion of the minimum over \(n\) from 0 to \(\infty\). So we have

\[
S_{\text{win}} = (S) \wedge (S^{(2)} + w) \wedge (S^{(3)} + 2w) \wedge \ldots .
\]  

The following lemma helps us find the condition in which we get \(w_{th}\) for.

**Lemma 6** (see [1]). *If \((S) \leq (S^{(2)} + w)\) in Eq. (3.7), then \(S_{\text{win}} = S\).*

**Proof.** \((S) \leq (S^{(2)} + w)\) yields \(S^{(k)} \leq (S^{(k+1)} + w)\) by convolving \(S^{(k-1)}\) with both sides of the inequality using the monotonic property. If we add \((k - 1)w\) to both sides, we get

\[
(S^{(k)} + (k - 1)w) \leq (S^{(k+1)} + kw).
\]  

Having Eq. (3.8) under the condition that \((S) \leq (S^{(2)} + w)\), we can indicate by Eq. (3.7) that we have \(S_{\text{win}} = S\).

If the window size is large enough to satisfy \((S) \leq (S^{(2)} + w)\) then \(S_{\text{win}} = S\) and the
equivalent service process is equal to of a system without feedback. Using this condition $w_{th}$ (see [1]) is obtained as

$$w_{th} \geq \max_{s, t, s \leq t} \{ (S(s, t) - S^{(2)}(s, t)) \}. \tag{3.9}$$

### 3.4 Constant Bit Rate Server

In this section we will analyze a deterministic feedback system with a constant bit rate server. The service process of a constant rate server with rate $C$ is given by

$$S(s, t) = (C(t - s))I_{t-s \geq 0}$$

In the following lines we show that this service process is additive and therefore also sub-additive:

$$S(s, \tau) + S(\tau, t) = (C(\tau - s))I_{\tau-s \geq 0} + (C(t - \tau))I_{t-\tau \geq 0}$$

$$= C((\tau - s)I_{\tau \geq s} + (t - \tau)I_{t \geq \tau})$$

$$= (C(t - s))I_{t-s \geq 0}$$

$$= S(s, t).$$

For a sub-additive process we have $S^{(2)} = S$. By Eq. (3.9) we have $w_{th} \geq 0$. This means that for any value of $w$, the system is already in the threshold region and feedback has no effect. In order to overcome this issue for the analysis of a constant bit rate server we consider the model illustrated in Fig. 3.3. In this model a feedback delay element $\delta_d$ is added to the feedback loop. The delay element is offering a delay of $d \geq 0$, the service process of this element is defined as

$$\delta_d(s, t) \triangleq \delta(s, t - d),$$
such that \( f(s, t - d) = f \otimes \delta_d(s, t) \) for every \( f \in \tilde{F} \).

Since we are in a deterministic settings we can use univariate representation for the constant bit rate server. By setting \( s = 0 \) for the service process we switch to univariate network calculus to continue our analysis

\[
S(t) = (C(t)) I_{t \geq 0}.
\]

Using the derivations for the previous model we can get \( S_{\text{win}} \) in Fig. 3.3 as follows

\[
S_{\text{win}} = \left( S \otimes \delta_d \otimes \delta^{+w} \right)^* \otimes S(t)
\]

\[
= \bigwedge_{n=0}^{\infty} \left( S \otimes \delta_d \otimes \delta^{+w} \right)^{(n)} \otimes S(t)
\]

\[
= \bigwedge_{n=0}^{\infty} \left( S^{(n)} \otimes \delta_{nd} \otimes \delta^{+nw} \right) \otimes S(t)
\]

\[
= \bigwedge_{n=0}^{\infty} \left( S^{(n+1)}(t - nd) + nw \right) \quad (3.10)
\]

\[
= \bigwedge_{n=0}^{\infty} \left( S(t - nd) + nw \right)
\]

\[
= \bigwedge_{n=0}^{\infty} \left( (C(t - nd)) I_{t \geq nd} + nw \right).
\]

The second line is written by the definition of sub-additive closure. Since we are in the univariate network calculus we have commutativity for \( \otimes \) which is used in the third and
third line of the above equations. The fourth line is written by $S(n) = S$ which holds for sub-additive processes. In the last line we plugged in the constant bit rate server.

So the service process for a constant bit rate server with feedback delay $d$ is

$$S_{\text{win}}(t) = \left( (C(t))_{t \geq 0} \right) \wedge \left( (C(t-d))_{t \geq d} + w \right) \wedge \left( (C(t-2d))_{t \geq 2d} + 2w \right) \wedge \ldots . \quad (3.12)$$

Fig. 3.4 shows the service function of a constant rate server with $C = 2$, $d = 1$ and $w = 1$.

### 3.4.1 Threshold window size for bit rate server

Using Eq. (3.9) for the constant bit rate server, we get threshold window size as

$$w_{th} \geq \sup_{0 \leq t} \left\{ ((Ct)_{t \geq 0} - ((C(t-d))_{t \geq 0})))(t) \right\}$$

$$= Cd . \quad (3.13)$$
Figure 3.5: Service of the feedback system with a constant rate server for $C = 2$ and $d = 1$

The average rate of a feedback system with a constant bit rate server is $w/d$. From Eq. (3.13) the threshold window size of such a system is given by $w_{th} \geq Cd$, which means for any $w$ greater than $w_{th}$ the feedback service is equal to the system without feedback. Also in order for the window size to influence the service, we need to have $w < Cd$. Then the ratio $w/d$ should satisfy the following inequality

$$0 < \frac{w}{d} < C.$$

Therefore the constant bit rate server has an average rate equal to $\min\{w/d, C\}$. Fig. 3.5 shows the equivalent service process of constant bit rate server for $C = 2$ and $d = 1$. Then by the above inequality we have an average rate of $\min\{2, w\}$ where Fig. 3.5 illustrates the effect of feedback for different values of $w$. 

![Service of the feedback system with a constant rate server](image)
Chapter 4

Stochastic Feedback Analysis

In this chapter we first discuss difficulties and challenges of feedback analysis under stochastic assumptions. Then we introduce a case where the feedback delay is equal to one time unit. In this case, we are able to fully characterize stochastic feedback system by obtaining an equivalent service process for the whole feedback system. Then we consider a feedback system in general and provide upper and lower bounds for its equivalent service process.

4.1 Challenges of stochastic feedback systems

We introduced two models in Chapter 3 for analysis of feedback systems, both based on window flow control scheme. The first model is shown in Fig 3.1. Its equivalent service is derived as follows

\[ S_{\text{win}} = (S \otimes \delta^{+w})^* \otimes S. \]
In this model, let us consider a sub-additive service process $S$ and obtain the equivalent service of the feedback system

$$S_{\text{win}} = (S \otimes \delta^+)^* \otimes S$$

$$= \bigwedge_{n=0}^{\infty} (S \otimes \delta^+)^{(n)} \otimes S$$

$$= \bigwedge_{n=0}^{\infty} (S^{(n+1)} + nw)$$

$$= \bigwedge_{n=0}^{\infty} (S + nw)$$

$$= S.$$  \hfill (4.1)

The second line is written by the definition of the sub-additive closure. The third line is due to the fact that $\delta^+$ has commutative property (both in univariate and bivariate settings), and the last line is written since we have $S^{(n)} = S$ when $S$ is a sub-additive process.

We observe that a sub-additive service process for feedback system results in a trivial case where $S_{\text{win}} = S$, and feedback has no impact on the equivalent service. This was also the case for the constant bit rate server in Section 3.4. In order to prevent these cases, we focus our analysis on the model given in Fig. 3.3. The equivalent service of this model is given as

$$S_{\text{win}} = (S \otimes \delta_d \otimes \delta^+)^* \otimes S.$$  

To discuss the difficulties of stochastic analysis of this model, let us take a look back to Chapter 3 in the deterministic domain where we derived $S_{\text{win}}$ for this model in Eq. (3.11) as follows

$$S_{\text{win}} = \bigwedge_{n=0}^{\infty} (S^{(n+1)}(t - nd) + nw).$$
the above derivation is based on Eq. (3.10) where we used the commutativity of $\otimes$ in the univariate settings. Since we use bivariate network calculus in our stochastic analysis in this chapter, we can not use the above derivation. Knowing that $\delta^{+w}$ commutes in bivariate settings, we have for the model in Fig. 3.3

$$S_{\text{win}} = (S \otimes \delta_d \otimes \delta^{+w})^* \otimes S$$
$$= \bigwedge_{n=0}^{\infty} (S \otimes \delta_d \otimes \delta^{+w})^{(n)} \otimes S$$
$$= \bigwedge_{n=0}^{\infty} (S \otimes \delta_d)^{(n)} \otimes S \otimes \delta^{+nw}.$$  \hspace{1cm} (4.2)

The second line is given by the definition of sub-additive closure. The third line is written by commutativity of $\delta^{+w}$. If $S \otimes \delta_d$ is sub-additive, by the same steps as in Eq. (4.1) we end up in a trivial case. In the following we show that this is never the case and $S \otimes \delta_d$ is not sub-additive.

$$S \otimes \delta_d(s, \tau) + S \otimes \delta_d(\tau, t) = S(s, \tau - d) + S(\tau, t - d)$$
$$< S(s, t - d)$$
$$= S \otimes \delta_d(s, t).$$

The difficulty of the analysis of this model introduces itself when we look at the term $(S \otimes \delta_d)^{(n)}$ in Eq. (4.2), the n-fold convolution of a non-sub-additive process. Furthermore the minimum is ranging over an infinite number of convolutions $(S \otimes \delta_d)$ with itself, which makes the analysis even harder.
Let us look at \((S \otimes \delta_d)^{(n)}\) for \(n = 2\),

\[
(S \otimes \delta_d)^{(2)}(s, t) = (S \otimes \delta_d) \otimes (S \otimes \delta_d)(s, t)
\]
\[
= \min_{s \leq \tau \leq t} \{(S \otimes \delta_d)(s, \tau) + (S \otimes \delta_d)(\tau, t)\} 
= \min_{s \leq \tau \leq t} \{S(s, \tau - d) + S(\tau, t - d)\} ,
\]

where the first line is written by the definition of convolution. The second line is written by the definition of \(\delta_d\). Fig. 4.1 illustrates intervals appearing in the last line. By the same argument for \((S \otimes \delta_d)^{(n)}\) we have

\[
(S \otimes \delta_d)^{(n)}(s, t) = \min_{s = \tau_0 \leq \cdots \leq \tau_n \leq t} \left\{ \sum_{i=1}^{n} (S(\tau_{i-1}, \tau_i - d)) \right\} ,
\]

where the minimum ranges over all non-decreasing sequences \(\tau_0, \ldots, \tau_n\) with \(\tau_0 = s\) and \(\tau_n \leq t\). In the above equation we can observe that for a stochastic analysis of \(S_{\text{win}}\) we need to determine the joint distribution of \(S(\tau_{i-1}, \tau_i - d)\) which is not straightforwardly obtainable from \(S(s, t)\). In order to get \(S_{\text{win}}\) from Eq. (4.2) we convolve Eq. (4.4) with
\( S \otimes \delta^{+nw} \) resulting in

\[
(S \otimes \delta_d)^{(n)} \otimes S \otimes \delta^{+nw}(s, t) = \min_{s = \tau_0 \leq \cdots \leq \tau_n \leq t} \left\{ \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_{n}, t) \right) \right\} + nw.
\]

(4.5)

The corresponding intervals of the above expression are illustrated in Fig. 4.2.

### 4.2 Exact service process of a stochastic feedback system

In the following theorem we introduce a case for feedback system in Fig. 3.3 in which we can provide the exact stochastic characterization of the system.

**Theorem 1** (Lemma 2 in [16]). For a feedback system shown in Fig. 3.3 with service process \( S(s, t) = \sum_{k=s}^{t-1} c_k \) where \( c_k \)'s are independent and identically distributed random variables, if \( d = 1 \) then

\[
S_{\text{win}}(s, t) = \sum_{k=s}^{t-1} \min \{c_k, w\}
\]

**Proof.** Setting \( d = 1 \) and inserting Eq. (4.5) into the expression for \( S_{\text{win}} \) given in Eq. (4.2) we get

\[
S_{\text{win}}(s, t) = \bigwedge_{n=0}^{\infty} \left\{ \min_{s = \tau_0 \leq \cdots \leq \tau_n \leq t} \left( \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_{n}, t) \right) \right) + nw \right\}.
\]

(4.6)

In the above equation, for the term in the braces, we consider the following cases:

- \( n = 0 \): In this case the term in the braces equals to \( S(s, t) \).
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Figure 4.3: Exact $S_{\text{win}}(s,t)$ given in Theorem 1.

- $n = 1$: In this case we have

$$
\min_{s \leq \tau_1 \leq t} \{S(s, \tau_1 - 1) + S(\tau_1, t)\} + w = \sum_{k=s}^{\tau_1 - 2} c_k + \sum_{k=\tau_1}^{t-1} c_k + w
$$

$$
= S(s, t) + w - \max_{s \leq k < t} c_k .
$$

where the largest $c_k$ in $S(s, t)$ is replaced by $w$.

- $n > 1$: Similar to the previous case, we get $S(s, t)$ where the $n$ largest values of $c_k$ are replaced by $w$.

In order to get $S_{\text{win}}$ we need to find the optimal value for $n$. We claim the optimal value $n^*$ is equal to the number of $c_k$'s in $S(s, t)$ which are greater than $w$

$$
n^* = \#\{k \mid s \leq k < t, c_k > w\} .
$$

This is illustrated in Fig. 4.3. To see that $n^*$ is the optimal value, let us choose a greater value for $n^*$. This will increase the objective function because of the term $+nw$. On the other hand, if we reduce $n^*$ by 1, its corresponding $c_k$ (which is greater than $w$) will be added to the objective function.
4.3 General service process of stochastic feedback system

In this section we consider the general service process of a feedback system shown in Fig. 3.3. In the following theorem we will show that the exact case where \( d = 1 \) can be used as a lower bound for the service process of a feedback system in general. We also provide an upper bound in the theorem for the equivalent service process.

**Theorem 2** (Theorem 1 in [16]). Suppose \( S_{\text{win}}(s, t) \) is the equivalent service process of the feedback system shown in Fig. 3.3 with service process \( S(s, t) \), feedback delay \( d \) and window size \( w \). We have

\[
S^*_{\text{win}}(s, t) < S_{\text{win}}(s, t) < \min \left\{ S(s, t), \left\lceil \frac{t-s}{d} \right\rceil w \right\}
\]

where \( S^*_{\text{win}}(s, t) \) is the equivalent service process of the same feedback system with \( d^* = 1 \) and \( w^* = w/d \).

**Proof.** For the upper bound, we use Eq. (4.6) to bound \( S_{\text{win}} \) by the minimum of two of its terms. For \( n = 0 \), we get \( S(s, t) \), and for \( n = \left\lceil \frac{t-s}{d} \right\rceil \) we get

\[
\min_{\tau_0 \leq \cdots \leq \tau_n} \left( \sum_{i=1}^{\left\lfloor \frac{t-s}{d} \right\rfloor} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + \left\lfloor \frac{t-s}{d} \right\rfloor w = \left\lfloor \frac{t-s}{d} \right\rfloor w. \right.
\]

In the above case, since \( n = \left\lfloor \frac{t-s}{d} \right\rfloor \), the length of the interval \([s, t)\) is exactly equal to \( n \) delay intervals and the minimum is achieved when we eliminate the entire interval by placing the delay intervals.
For the lower bound, let us look at Eq. (4.5) for n=1:

\[
(S \otimes \delta_d) \otimes S \otimes \delta^+ w(s, t) = \min_{s \leq \tau \leq t} \{ S(s, \tau - d) + S(\tau, t) \} + w
\geq \min_{s=\tau_0 \leq \cdots \leq \tau_d = \tau} \left\{ \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - 1) + S(\tau, t) \right) \right\} + dw^*
= (S \otimes \delta_1)^{(d)} \otimes S \otimes \delta^+ dw^* (s, t).
\]

In the second line, if we set \( \tau_1 = \cdots = \tau_d = \tau \) we get the first line. So the first line is a term in the second line, and therefore under minimization it would be greater than the second line. The third line follows from Eq. (4.5) for \( n = d \). Similarly, for \( n > 1 \), since by Eq. (4.2) we have

\[
S_{\text{win}} = \bigwedge_{n=0}^{\infty} (S \otimes \delta_d)^{(n)} \otimes S \otimes \delta^{+n w}, \tag{4.7}
\]

the \( n \)-th term in \( S_{\text{win}} \) is bounded from below by the \( nd \)-term in \( S^*_{\text{win}} \) and the proof is complete. \( \square \)
Chapter 5

Variable Bit Rate Service With Feedback

In this chapter we consider a feedback system as shown in Fig. 3.3 which has a FIFO buffer transmitting with a variable rate as its service element. We call this service element a variable bit rate (VBR) server offering a service process

\[ S(s, t) = \sum_{k=s}^{t-1} c_k, \]  

where \( c_k \) is the random amount of traffic that is transmitted in the \( k \)-th time slot. The \( c_k \)'s are independent and identically distributed (i.i.d.) random variables with \( E[c_k] = C \). If we set \( M_{c_k}(-\theta) = E[e^{-\theta c_k}] \), then the moment generating function of \( S \) is given by

\[ M_{S}(-\theta, s, t) = (M_{c_k}(-\theta))^{(t-s)}. \]
The effective capacity of the VBR server given in Eq. (5.1) is

\[
\gamma_S(-\theta) = \lim_{t \to \infty} -\frac{1}{\theta t} \log M_S(-\theta, 0, t) \\
= \lim_{t \to \infty} -\frac{1}{\theta t} \log (M_{ck}(-\theta))^{(t)} \\
= -\frac{1}{\theta} \log M_{ck}(-\theta). \tag{5.3}
\]

The first line is given by the definition of the effective capacity in Eq. (2.9) and the second line is given by Eq. (5.2).

In the following corollary we provide the upper and lower bounds for the equivalent service process of a VBR feedback system and its effective capacity using general derivation in previous chapter. Then we represent our bounds derived specifically for the VBR feedback system, both as statistical service bounds and effective capacity bounds. Later we evaluate these bounds in Chapter 6 using the following upper and lower bounds as benchmarks.

**Corollary 1** (see [16]). By Theorem 2 the following inequalities hold for a feedback system with VBR server given in Eq. (5.1),

(a) Upper and lower bounds for the equivalent service process \(S_{\text{win}}(s, t)\) are given by

\[
\sum_{s=0}^{t-1} c_k^* \leq S_{\text{win}}(s, t) \leq \min \left\{ S(s, t), \left\lceil \frac{t-s}{d} \right\rceil w \right\}.
\]

(b) Upper and lower bound for the effective capacity of the equivalent service process \(S_{\text{win}}(s, t)\) are given by

\[
-\frac{1}{\theta} \log E\left[e^{-\theta c_k^*}\right] \leq \gamma_{\text{win}}(-\theta) \leq \min \left\{ \gamma_S(-\theta), \frac{w}{d} \right\}.
\]

where \(c_k^* = \min \left\{ c_k, \frac{w}{d} \right\}\) and \(\theta > 0\).
Proof. Part (a) is derived using Theorem 2 where we plugged in the service process of the VBR server given in Eq. (5.1). Then we have that $S_{\text{win}}^*(s,t)$ in Theorem 2 equals $\sum_{k=s}^{t-1} \min \{ c_k, \frac{w_k}{q} \}$ by Theorem 1.

Part (b) straightly results from part (a) using the definition of the effective capacity in Eq. (2.9) and the moment-generating function in Eq. (2.3) for $-\theta$ where $\theta > 0$. \qed

5.1 Service bounds for a feedback system with VBR server

In order to obtain a statistical service bound $S_{\text{win}}^e$ for the equivalent service process $S_{\text{win}}$ of a feedback system we will use Lemma 3 which requires us to provide an upper bound for the moment-generating function of $S_{\text{win}}$. To do that we first present the following lemma which will help us eliminate the minimums in the expression for $S_{\text{win}}$ given in Eq. (4.6) by providing an upper bound for the moment-generating function of minimum of two processes.

**Lemma 7** (Lemma 4 in [16]). For two random processes $f, g \in \tilde{F}$ we have

$$M_{f \wedge g}(-\theta, s, t) \leq M_f(-\theta, s, t) + M_g(-\theta, s, t).$$

Note that this lemma does not require $f$ and $g$ to be independent.

**Proof.**

$$M_{f \wedge g}(-\theta, s, t) = E[e^{-\theta \min \{f(s,t), g(s,t)\}}]$$

$$= E\left[ \max \{ e^{-\theta f(s,t)}, e^{-\theta g(s,t)} \} \right]$$

$$\leq E\left[ e^{-\theta f(s,t)} + e^{-\theta g(s,t)} \right]$$

$$= M_f(-\theta, s, t) + M_g(-\theta, s, t).$$
In the above lines, we used the monotonic property of the exponential function and the fact that the maximum of two non-negative functions is bounded by their summation. □

The next theorem provides an upper bound for the moment-generating function of \( S_{\text{win}} \) using Lemma 7. This upper bound is valid only under a certain convergence condition.

**Theorem 3.** The moment-generating function of the equivalent service process \( S_{\text{win}} \) of a feedback system shown in Fig. 3.3 with \( S(s,t) \) given in Eq. (5.1) is bounded above by

\[
M_{S_{\text{win}}}(-\theta, s, t) \leq \frac{M_{c_k}(-\theta)^{t-s}}{\left(1 - M_{c_k}(-\theta)^{-d}e^{-\theta w}\right)^{t-s+2}}
\]

when \( M_{c_k}(-\theta)^{-d}e^{-\theta w} < 1 \).

**Proof.** We have \( S_{\text{win}} \) from Eq. (4.6) given by

\[
S_{\text{win}}(s, t) = \bigwedge_{n=0}^{\infty} \left\{ \min_{s = \tau_0 \leq \cdots \leq \tau_n \leq t} \left( \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + nw \right) \right\}.
\]

To find the upper bound for the moment-generating function of \( S_{\text{win}} \) we use Lemma 7. Although the processes here are not independent, the disjoint intervals of the VBR server \( S(s,t) \) appearing in the expression of \( S_{\text{win}} \) for \( S(s,t) \) given in Eq. (5.1) are independent. By Lemma 7 the minimums in the above expression would appear as sums in the upper
bound for $M_{\text{swin}}(-\theta, s, t)$.

$$M_{\text{swin}}(-\theta, s, t) \leq \sum_{n=0}^{\infty} \left\{ \sum_{s=\tau_0 \leq \cdots \leq \tau_n \leq t} \left( \prod_{i=1}^{n} \left( M_{S}(-\theta, \tau_{i-1}, \tau_i - d) \right) \cdot M_{S}(-\theta, \tau_n, t) \right) \cdot e^{-\theta n w} \right\}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{s=\tau_0 \leq \cdots \leq \tau_n \leq t} \left( \prod_{i=1}^{n} \left( M_{ch}(-\theta)^{\tau_i - \tau_{i-1}} \cdot M_{ch}(-\theta)^{t-\tau_n} \right) \cdot e^{-\theta n w} \right) \right\}$$

$$= M_{ch}(-\theta)^{t-s} \sum_{n=0}^{\infty} \left\{ \left( M_{ch}(-\theta)^{-d} e^{-\theta w} \right)^{n} \sum_{s=\tau_0 \leq \cdots \leq \tau_n \leq t} (1) \right\}$$

$$= M_{ch}(-\theta)^{t-s} \sum_{n=0}^{\infty} \left\{ \left( M_{ch}(-\theta)^{-d} e^{-\theta w} \right)^{n} \left( t - s + 1 + n \right) \right\}$$

$$= \frac{M_{ch}(-\theta)^{t-s}}{\left( 1 - M_{ch}(-\theta)^{-d} e^{-\theta w} \right)^{t-s+2}}.$$ 

In the second line, we plugged in Eq. (5.2). The third line is written by re-arranging the terms in second line. The fourth line is due to the fact that $\sum_{s=\tau_0 \leq \cdots \leq \tau_n \leq t} (1)$ is a combinatorics problem results in $\left( t - s + 1 + n \right)$ number of counts. The last line is given by the following series under the condition that $M_{ch}(-\theta)^{-d} e^{-\theta w} < 1$.

$$\sum_{n=0}^{\infty} x^n \binom{i + n}{n} = \frac{1}{(1-x)^{i+1}}, \quad (|x| < 1).$$

The main weakness of the bound given in Theorem 3 is its validity under a convergence condition. In order to remove the convergence condition and strengthen the bound, we provide the following lemma which later enables us to provide an improved bound in Theorem 4 without any convergence condition.

**Lemma 8** (Lemma 3 in [16]). For a feedback system shown in Fig. 3.3 with a service process $S \in \tilde{F}_0$ we have

$$S_{\text{win}}(s, t) = \bigwedge_{n=0}^{\left\lceil \frac{t-s}{d} \right\rceil} \left\{ \min_{C_n(s,t)} \left( \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + n w \right) \right\},$$
where $C_n(s,t)$ is given as

$$C_n(s,t) = \{ s = \tau_0 \leq \cdots \leq \tau_n \leq t \mid \forall i = 0, \ldots, n \; \tau_i - \tau_{i-1} \geq d \}.$$  

Note that this lemma holds in general for any service process $S \in \tilde{F}_0$.

**Proof.** Let us first show that why the range of the minimum is limited to $\lceil \frac{t-s}{d} \rceil$. By Eq. (4.2) and Eq. (4.5) we have

$$S_{\text{win}} = \bigwedge_{n=0}^{\infty} (S \otimes \delta_d)^{(n)} \otimes S \otimes \delta^{+nw}$$

$$= \bigwedge_{n=0}^{\infty} \left\{ \min_{s=\tau_0 \leq \cdots \leq \tau_n} \left( \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + nw \right) \right\}.$$  

Suppose $n > \lceil \frac{t-s}{d} \rceil$ then $nd > t - s$ and the delay intervals cover the whole interval $[s,t)$ eliminating the inner minimum. The only term that remains in the brackets is $+nw$ which is increasing in $n$. So we can limit the range of the minimum to $0 \leq n \leq \lceil \frac{t-s}{d} \rceil$.

The second part of the lemma indicates that the range of the minimum in the brackets is over $C_n(s,t)$ instead of $s = \tau_0 \leq \cdots \leq \tau_n \leq t$, which means it is sufficient to take the minimum over $\tau_0 \leq \cdots \leq \tau_n$ where each $\tau_i$ is placed at least with the distance of $d$ from $\tau_{i-1}$. In order to show this we use induction over $n$.

For $n = 0$:

$$(S \otimes \delta_d)^{(n)} \otimes S \otimes \delta^{+nw} \bigg|_{n=0} = S(s,t),$$

and there is nothing to show.
\( n = 1: \)

\[
(S \otimes \delta_d)^{(1)} \otimes S \otimes \delta^{+w} = \min_{s \leq \tau_1 \leq t} \{ S(s, \tau_1 - d) + S(\tau_1, t) \} + w
\]

\[
= \min_{s + d \leq \tau_1 \leq t} \{ S(s, \tau_1 - d) + S(\tau_1, t) \} + w
\]

\[
= \min_{C_1(s, t)} \{ S(s, \tau_1 - d) + S(\tau_1, t) \} + w,
\]

where the third line is due to the fact that for \( s \leq \tau_1 \leq s + d \) the objective function \( S(s, \tau_1 - d) + S(\tau_1, t) \) is a non-increasing function of \( \tau_1 \). So it is sufficient to take the minimum over \( s + d \leq \tau_1 \leq t \).

Assume the lemma holds for \( n = k \),

\[
(S \otimes \delta_d)^{(k)} \otimes S \otimes \delta^{+kw}
\]

\[
= \left\{ \min_{C_k(s, t)} \left( \sum_{i=1}^{k} \left( S(\tau_{i-1}, \tau_i - d) \right) + S(\tau_k, t) \right) + kw \right\}.
\]

So we also have

\[
(S \otimes \delta_d)^{(k)} \otimes S = \left\{ \min_{C_k(s, t)} \left( \sum_{i=1}^{k} \left( S(\tau_{i-1}, \tau_i - d) \right) + S(\tau_k, t) \right) \right\}.
\]

For \( n = k + 1 \) we can write

\[
(S \otimes \delta_d)^{(k+1)} \otimes S \otimes \delta^{+(k+1)w} = \left( (S \otimes \delta_d)^{(k)} \otimes S \right) \otimes \delta_d \otimes S \otimes \delta^{+(k+1)w}
\]

\[
= \min_{s \leq \tau_{k+1} \leq t} \left\{ \left( (S \otimes \delta_d)^{(k)} \otimes S \right)(s, \tau_{k+1} - d) + \left( (S \otimes \delta_d)^{(k)} \otimes S \right)(\tau_{k+1}, t) \right\} + (k + 1)w
\]

\[
= \min_{s + d \leq \tau_{k+1} \leq t} \left\{ \left( (S \otimes \delta_d)^{(k)} \otimes S \right)(s, \tau_{k+1} - d) + \left( (S \otimes \delta_d)^{(k)} \otimes S \right)(\tau_{k+1}, t) \right\} + (k + 1)w,
\]

where the first line is due to the associativity of \( \otimes \). The second line is written by expansion of the convolution for function \( (S \otimes \delta_d)^{(k)} \otimes S \). The last line is derived by the same argument we used for the inductive basis. Since the minimum in \( (S \otimes \delta_d)^{(k)} \otimes S \)
takes over \( C_k(s, t) \) for \( n = k + 1 \) we have

\[
(S \otimes \delta_d)^{(k+1)} \otimes S \otimes \delta^+(k+1)w
\]

\[
= \left\{ \min_{s = \tau_0 \leq \cdots \leq \tau_{k+1} \leq t} \left( \sum_{i=1}^{k+1} S(\tau_{i-1}, \tau_i - d) + S(\tau_{k+1}, t) + (k + 1)w \right) \right\}
\]

\[
= \left\{ \min_{C_{k+1}(s, t)} \left( \sum_{i=1}^{k+1} S(\tau_{i-1}, \tau_i - d) + S(\tau_{k+1}, t) + (k + 1)w \right) \right\},
\]

and the proof is complete. \( \square \)

Now in the following theorem, we enhance the bound given in Theorem 3 using the above lemma.

**Theorem 4** (Theorem 2 in [16]). The moment-generating function of the equivalent service process \( S_{\text{win}} \) of a feedback system shown in Fig. 3.3 with \( S(s, t) \) given in Eq. (5.1) is bounded above by

\[
M_{S_{\text{win}}}(-\theta, s, t) \leq \left( M_{c_k}(-\theta)d + e^{-\theta w}d \right) \left[ \frac{t-s}{d} \right]
\]

**Proof.** By Lemma 8 we have

\[
S_{\text{win}}(s, t) = \bigwedge_{n=0}^{\left\lceil \frac{t-s}{d} \right\rceil} \left\{ \min_{C_n(s, t)} \left( \sum_{i=1}^{n} S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + nw \right\},
\]

By the same steps as in the proof of Theorem 3, we use Lemma 7 to bound the moment-generating function of \( S_{\text{win}} \) from above. The minimums in the above expression appear
as summations in the upper bound for \( M_{\text{win}}(-\theta, s, t) \).

\[
M_{\text{win}}(-\theta, s, t) \leq \sum_{n=0}^{\lceil \frac{t-s}{d} \rceil} \left\{ \sum_{C_{n}(s, t)} \left( \prod_{i=1}^{n} \left( M_{S}(-\theta, \tau_{i-1}, \tau_{i} - d) \cdot M_{S}(-\theta, \tau_{n}, t) \right) \cdot e^{-\theta nw} \right) \right\}
\]

\[
= \sum_{n=0}^{\lceil \frac{t-s}{d} \rceil} \left\{ \sum_{C_{n}(s, t)} \left( \prod_{i=1}^{n} \left( M_{c,k}(-\theta)^{\tau_{n} - d - \tau_{i-1}} \cdot M_{c,k}(-\theta)^{t - \tau_{n}} \right) \cdot e^{-\theta nw} \right) \right\}
\]

\[
= \sum_{n=0}^{\lceil \frac{t-s}{d} \rceil} \left\{ M_{c,k}(-\theta)^{t - s - nd} e^{-\theta nw} \sum_{C_{n}(s, t)} (1) \right\}
\]

\[
\leq \sum_{n=0}^{\lceil \frac{t-s}{d} \rceil} \left\{ \left( M_{c,k}(-\theta)^{d} \right)^{\left\lceil \frac{t-s}{d} \right\rceil - n} e^{-\theta nw} \binom{t-s-nd+n}{n} \right\}
\]

\[
\leq \sum_{n=0}^{\lceil \frac{t-s}{d} \rceil} \left\{ \left( M_{c,k}(-\theta)^{d} \right)^{\left\lceil \frac{t-s}{d} \right\rceil - n} \left( e^{-\theta w d} \right)^{n} \left( \frac{\left\lceil \frac{t-s}{d} \right\rceil}{n} \right) \right\}
\]

\[
= \left( M_{c,k}(-\theta)^{d} + e^{-\theta w d} \right)^{\left\lceil \frac{t-s}{d} \right\rceil}.
\]

In the second line, we plugged in Eq. (5.2). The third line is written by re-arranging the terms in the second line. The fourth line is due to the fact that \( \sum_{C_{n}(s, t)}(1) \) is a combinatorics problem results in \( \binom{t-s-nd+n}{n} \) number of counts. The fifth line is the results of following derivation. And the last line is given by binomial theorem.

\[
\binom{t-s-nd+n}{n} = \frac{1}{n!} \prod_{i=1}^{n} (t-s-nd+i)
\]

\[
\leq \frac{1}{n!} \prod_{i=1}^{n} (t-s-nd+id)
\]

\[
\leq \frac{d^{n}}{n!} \prod_{i=1}^{n} \left( \frac{t-s}{d} - n + i \right)
\]

\[
= d^{n} \left( \frac{\left\lceil \frac{t-s}{d} \right\rceil}{n} \right).
\]
5.2 Effective capacity for a feedback system with VBR server

In this section we provide bounds on the effective capacity of the equivalent service of a feedback system with VBR server. The following corollaries correspond to derivations in Theorems 3 and 4.

**Corollary 2** (see [16]). For a feedback system with VBR service $S(s,t)$ given by Eq. (5.1) using the upper bound on $M_{S_{\text{win}}}(-\theta,s,t)$ given in Theorem 3 the effective capacity of $S_{\text{win}}$ is bounded below by

$$
\gamma_{S_{\text{win}}} \geq \gamma_{S}(-\theta) + \frac{1}{\theta} \log \left( 1 - e^{-\theta(w-d\gamma_{S}(-\theta))} \right),
$$

when $e^{-\theta(w-d\gamma_{S}(-\theta))} < 1$.

**Proof.**

$$
\gamma_{S_{\text{win}}}(-\theta) = \lim_{t \to \infty} -\frac{1}{\theta t} \log M_{S_{\text{win}}}(-\theta, 0, t) \\
\geq \lim_{t \to \infty} -\frac{1}{\theta t} \log \left( \frac{M_{c_{k}}(-\theta)^{t}}{(1 - M_{c_{k}}(-\theta) - d e^{-\theta w})^{t+2}} \right) \\
= -\frac{1}{\theta} \log M_{c_{k}}(-\theta) + \frac{1}{\theta} \log (1 - M_{c_{k}}(-\theta) - d e^{-\theta w}) \\
= \gamma_{S}(-\theta) + \frac{1}{\theta} \log \left( 1 - e^{-\theta(w-d\gamma_{S}(-\theta))} \right).
$$

The first line is by the definition of the effective capacity. In the second line we plugged in the result of Theorem 3 which is valid under the condition $e^{-\theta(w-d\gamma_{S}(-\theta))} < 1$. In the third line we used logarithm properties to re-arrange the expressions. And the last line is by Eq. (5.3).

**Corollary 3** (see [16]). For a feedback system with VBR service $S(s,t)$ given by Eq. (5.1) using the upper bound on $M_{S_{\text{win}}}(-\theta,s,t)$ given in Theorem 4 the effective capacity of $S_{\text{win}}$...
is bounded below by

\[ \gamma_{S_{\text{win}}} \geq \gamma_S(-\theta) - \frac{1}{\theta d} \log \left( 1 + e^{-\theta(w - d\gamma_S(-\theta))} \right). \]

**Proof.**

\[ \gamma_{S_{\text{win}}}(-\theta) = \lim_{t \to \infty} -\frac{1}{\theta t} \log M_{S_{\text{win}}}(-\theta, 0, t) \]
\[ \geq \lim_{t \to \infty} -\frac{1}{\theta t} \log \left( M_{c_k}(-\theta)^d + e^{-\theta w d} \right)^\left\lceil \frac{t}{d} \right\rceil \]
\[ = -\frac{1}{\theta d} \log \left( M_{c_k}(-\theta)^d (1 + M_{c_k}(-\theta)^{-d} e^{-\theta w d}) \right) \]
\[ = -\frac{1}{\theta} \log M_{c_k}(-\theta) - \frac{1}{\theta d} \log \left( 1 + M_{c_k}(-\theta)^{-d} e^{-\theta w d} \right) \]
\[ = \gamma_S(-\theta) - \frac{1}{\theta d} \log \left( 1 + e^{-\theta(w - d\gamma_S(-\theta))} \right). \]

The first line is by the definition of the effective capacity. In the second line we plugged in the result of Theorem 4. In the third and fourth line we used logarithm properties to re-arrange the expressions. And the last line is given by Eq. (5.3). \( \square \)
Chapter 6

Numerical Results

In this chapter we provide numerical results for the derivations in the previous chapters. We evaluate the service bounds derived in Theorem 3 and Theorem 4 for feedback system with VBR server with exponential distribution. We compute backlog bounds for this system and compare these bounds with the simulations. The feedback system that we use for numerical results is a VBR server which offers a service process

\[ S(s, t) = \sum_{k=s}^{t-1} c_k, \quad (6.1) \]

where \( c_k \)'s are i.i.d. random variables with exponential distribution with parameter \( 1/C \), or \( c_k \overset{iid}{\sim} Exp(1/C) \). The mean rate of such a server is \( E[c_k] = C \) and since \( M_{c_k}(-\theta) = E[e^{-\theta c_k}] = (1 + C\theta)^{-1} \), its moment-generating function is given by

\[ M_S(-\theta, s, t) = (1 + C\theta)^{-(t-s)}. \]

6.1 Statistical service bounds for exponential VBR

In order to provide statistical service bounds for the equivalent service of a VBR feedback system with violation probability \( \varepsilon \), we will use Lemma 3 and the service bounds we
Let us start with Corollary 1(a) which gives us an upper and a lower bound on $S_{\text{win}}(s, t)$. By Corollary 1(a), we have

$$S_{\text{lower}}(s, t) = \sum_{k=s}^{t-1} c_k^*, \quad S_{\text{upper}}(s, t) = \min\left\{ S(s, t), \left\lceil \frac{t - s}{d} \right\rceil w \right\},$$

where $c_k^* = \min\{c_k, w/d\}$. Since statistical service bounds are lower bounds on the available service, we can use Lemma 3 for the lower bound $S_{\text{lower}}$, but not for the upper bound $S_{\text{upper}}$. To get the statistical service bound for $S_{\text{lower}}$, using Lemma 3, we calculate its moment-generating function as follows,

$$M_{S_{\text{lower}}}(-\theta, s, t) = \left( E\left[ e^{-\theta c_k^*} \right] \right)^{t-s},$$

where $c_k^* = \min\{c_k, w/d\}$. Then the tail probability of $c_k^*$ is

$$Pr(c_k^* > t) = \begin{cases} e^{-C^{-1}t}, & t \leq w/d, \\ 0, & t > w/d. \end{cases}$$

This can be used to compute the moment-generating function of $c_k^*$ as following

$$E[e^{-\theta c_k^*}] = \int_0^\infty Pr(e^{-\theta c_k^*} > t) \, dt$$

$$= \int_0^\infty Pr(c_k^* < s) \theta e^{-\theta s} \, ds$$

$$= \int_0^{w/d} (1 - e^{-C^{-1}s}) \theta e^{-\theta s} \, ds + \int_{w/d}^\infty \theta e^{-\theta s} \, ds$$

$$= \int_0^{w/d} \theta e^{-\theta s} \, ds - \int_0^{w/d} \theta e^{-(\theta + C^{-1})s} \, ds$$

$$= 1 + \frac{\theta}{C^{-1} + \theta} e^{-(\theta + C^{-1})w/d} \bigg|_0^{w/d}$$

$$= 1 + C\theta e^{-(\theta + C^{-1})w/d} \bigg|_0^{w/d}$$

$$= \frac{1 + C\theta}{1 + C\theta}. \quad (6.2)$$
In the second line we have made the substitution \( t = e^{-\theta s} \). By the above derivation, moment-generating function of \( S_{\text{lower}} \) is given by

\[
M_{S_{\text{lower}}}(-\theta, s, t) = \left( \frac{1 + C\theta e^{-(\theta + C^{-1})w/d}}{1 + C\theta} \right)^{t-s} . \tag{6.3}
\]

By Lemma 3 we have

\[
S_{\text{lower}}^\varepsilon (s, t) = \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \log M_{S_{\text{lower}}}(-\theta, s, t) \right\}
= \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \log \left( \frac{1 + C\theta e^{-(\theta + C^{-1})w/d}}{1 + C\theta} \right)^{t-s} \right\}
= \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - (t-s) \log \left( \frac{1 + C\theta e^{-(\theta + C^{-1})w/d}}{1 + C\theta} \right) \right\} . \tag{6.4}
\]

To get the statistical service bound of \( S_{\text{upper}}(s, t) \) we cannot use Lemma 3. Instead we use its \( \varepsilon \)-percentile since we know the distribution of \( S(s, t) \) for an exponential VBR server. In a VBR server with exponential service \( c_k \), the service process \( S(s, t) = \sum_{k=s}^t c_k \) is a Gamma distributed random variable with the following CDF

\[
Pr(S(s, t) \leq \varepsilon) = \frac{\Gamma((t-s), C^{-1}\varepsilon)}{\Gamma(t-s)} ,
\]

where \( \Gamma(n) = (n-1)! \) and

\[
\Gamma(s, x) = \int_0^x t^{s-1}e^{-t}dt .
\]

Using the above CDF of the Gamma distribution, we can get the statistical service bound for \( S_{\text{upper}}(s, t) \) by

\[
S_{\text{upper}}^\varepsilon (s, t) = \min \left\{ \frac{\Gamma((t-s), C^{-1}\varepsilon)}{\Gamma((t-s))}, \left\lfloor \frac{t-s}{d} \right\rfloor w \right\} . \tag{6.5}
\]
Now, in order to evaluate our derived bounds in Theorems 3 and 4, we provide statistical service bounds of them.

Let us first consider the simpler bound given in Theorem 3. Then for the moment-generating function of \( S_{\text{win}} \) with an exponential VBR server we have

\[
M_{S_{\text{win}}}(-\theta, s, t) \leq \frac{M_{c_k}(-\theta)^{t-s}}{\left(1 - M_{c_k}(-\theta)^{-d} e^{-\theta w}\right)^{t-s+2}} \frac{(1 + C\theta)^{-(t-s)}}{\left(1 - (1 + C\theta)^d e^{-\theta w}\right)^{t-s+2}},
\]

under convergence condition \((1 + C\theta)^d e^{-\theta w} < 1\) which is equivalent to

\[
\frac{1}{\theta} \log(1 + C\theta) < \frac{w}{d}.
\]

We use Lemma 3, where for a violation probability \( \varepsilon \) the statistical service bound of such a system is given by

\[
S_{\text{win}}^\varepsilon(s, t) = \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \log M_{S_{\text{win}}}(-\theta, s, t) \right\}
\]

\[
= \max_{\left\{\theta > 0 \mid \frac{1}{\theta} \log(1 + C\theta) < \frac{w}{d}\right\}} \frac{1}{\theta} \left\{ \log \varepsilon - \log \left(1 + C\theta\right)^{-(t-s)} \right\}
\]

\[
= \max_{\left\{\theta > 0 \mid \frac{1}{\theta} \log(1 + C\theta) < \frac{w}{d}\right\}} \frac{1}{\theta} \left\{ \log \varepsilon + (t - s) \log (1 + C\theta)
\]

\[
+ (t - s - 2) \log \left(1 - (1 + C\theta)^d e^{-\theta w}\right) \right\}.
\] (6.6)

Fig. 6.1 illustrates \( S_{\text{win}}^\varepsilon(0, t) \) for a feedback system with exponential VBR server where \( C = 1 \text{ Mbps} \) and \( \varepsilon = 10^{-6} \). We picked a time unit of length 1 ms which gives an average service rate of 1 Gbps. In Fig. 6.1(a) we plot \( S_{\text{win}}^\varepsilon(0, t) \) given in Eq. (6.6) as dashed lines for different values of delay \( d \) and window size \( w \) while the ratio \( w/d \) is fixed at 100 Mbps.
The statistical lower bound $S^\varepsilon_{\text{lower}}(0,t)$ given in Eq. (6.4), and the statistical upper bound $S^\varepsilon_{\text{upper}}(0,t)$ given in Eq. (6.5) are also illustrated as dash-dotted lines. The black dotted lines in the figure specify the two components of $S^\varepsilon_{\text{upper}}(0,t)$ given in Eq. (6.5). Fig. 6.1(b) illustrates the same scenario when $w/d = 500$ Mbps.

To evaluate the bounds derived in Theorem 4 by feedback system with VBR server we have

$$M_{S_{\text{win}}}(-\theta, s, t) \leq \left( M_{c_k}(-\theta)^d + e^{-\theta w}d \right)^{\left\lceil \frac{t-s}{\theta} \right\rceil}$$

$$= \left( (1 + C\theta)^{-d} + e^{-\theta w}d \right)^{\left\lceil \frac{t-s}{d} \right\rceil}. \tag{6.7}$$

Using Lemma 3 for a violation probability $\varepsilon$, the statistical service bound of such a system is given by

$$S^\varepsilon_{\text{win}}(s, t) = \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \log M_{S_{\text{win}}}(-\theta, s, t) \right\}$$

$$= \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \log \left( (1 + C\theta)^{-d} + e^{-\theta w}d \right)^{\left\lceil \frac{t-s}{d} \right\rceil} \right\}$$

$$= \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \left[ \frac{t-s}{d} \right] \log \left( (1 + C\theta)^{-d} + e^{-\theta w}d \right) \right\}. \tag{6.7}$$

Fig. 6.2 illustrates $S^\varepsilon_{\text{win}}(0, t)$ from the above expression, with the same parameters as in Fig. 6.1. In Fig. 6.2(a), we plot $S^\varepsilon_{\text{win}}(0, t)$ given in Eq. (6.7) as solid lines for different values of the delay $d$ and the window size $w$ while the ratio $w/d$ is fixed at 100 Mbps. Dash-dotted lines are $S^\varepsilon_{\text{lower}}(0, t)$ and $S^\varepsilon_{\text{upper}}(0, t)$ given in Eq. (6.4) and Eq. (6.5). Fig. 6.1(b) illustrates the same scenario when $w/d = 500$ Mbps.

In Fig. 6.1 and Fig. 6.2 we evaluated our bounds derived in Theorems 3 and 4. We observe that when we increase $d$ and $w$ while fixing the ratio $w/d$, the statistical service bounds also increase. Note that for the upper bound $S^\varepsilon_{\text{upper}}(0, t)$ in Fig. 6.1(b) and Fig. 6.2(b) the first component of Eq.(6.5) is dominant for $t \leq 50$ ms. In these figures, we observe that the statistical service plots do not fully capture the differences of the bounds in
Theorems 3 and 4. In order to get a more accurate comparison, we turn to effective capacity bounds.

6.2 Effective capacity bounds for exponential VBR

To evaluate long-term rate of a feedback system with exponential VBR, we will provide effective capacity bounds for this feedback system. The effective capacity of a VBR server with exponential distribution is given by

\[
\gamma_S(-\theta) = \lim_{t \to \infty} -\frac{1}{\theta t} \log M_S(-\theta, 0, t) = \lim_{t \to \infty} -\frac{1}{\theta t} \log (1 + C\theta)^{-t} = \frac{1}{\theta} \log (1 + C\theta).
\]

Corollary 1(b) gives us upper and lower bounds on \(\gamma_{\text{win}}(-\theta)\). So for an exponential VBR server we have

\[
-\frac{1}{\theta} \log \left( \frac{1 + C\theta e^{-\theta(C+1)w/d}}{1 + C\theta} \right) \leq \gamma_{\text{win}}(-\theta) \leq \min \left\{ \frac{1}{\theta} \log (1 + C\theta), \frac{w}{d} \right\}, \tag{6.8}
\]

where the lower bound is derived by Eq. (6.2).

The effective capacity of Theorem 3 is given in Corollary 2. By this corollary for a feedback system with exponential VBR server we have

\[
\gamma_{S_{\text{win}}} \geq \gamma_S(-\theta) + \frac{1}{\theta} \log (1 - e^{-\theta(w - d\gamma_S(-\theta))}) = \frac{1}{\theta} \log (1 + C\theta) + \frac{1}{\theta} \log \left( 1 - e^{-\theta \frac{w}{d} \log (1 + C\theta)} \right) = \frac{1}{\theta} \log (1 + C\theta) + \frac{1}{\theta} \log \left( 1 - e^{-\theta w (1 + C\theta)^d} \right). \tag{6.9}
\]
The effective capacity of Theorem 4 as given in Corollary 3 is calculated as follows for an exponential VBR feedback system.

\[
\gamma_{S_{\text{win}}}(-\theta) \geq \gamma_S(-\theta) - \frac{1}{\theta d} \log \left( 1 + e^{-\theta(w - d\gamma_S(-\theta))} \right) \\
= \frac{1}{\theta} \log (1 + C\theta) - \frac{1}{\theta d} \log \left( 1 + e^{-\theta(w - \frac{d}{\theta} \log(1+C\theta))} \right) \\
= \frac{1}{\theta} \log (1 + C\theta) - \frac{1}{\theta d} \log \left( 1 + e^{-\theta w (1 + C\theta)^d} \right). 
\] (6.10)

Fig. 6.3(a) and Fig. 6.3(b) illustrate the effective capacity bounds for a feedback system with exponential VBR server where the ratio \(w/d\) is fixed at 100 Mbps and 500 Mbps. We use the same parameters that we used in Fig. 6.1 and Fig. 6.2.

The solid lines are given by Eq. (6.9) and the dashed lines are given by Eq. (6.10). The upper and lower bounds are shown with dot-dashed lines. They were plotted by Eq. (6.8) for an exponential VBR feedback system.

In Fig 6.3, we observe that for large values of \(\theta\) the plotted bounds converge. The bounds given by Corollaries 2 and 3 appear to approach each other faster than the others when we increase \(\theta\). This is the reason that the statistical service plots in Fig 6.1 and Fig. 6.2 appear to be close to each other (the optimal \(\theta\) for the statistical service bounds was picked in the convergence area). Another observation of Fig. 6.3 is the degradation of bounds from Corollaries 2 and 3. This clearly shows the improvement that has been made from Theorem 3 to Theorem 4. When the degradation happens for the bounds given by Corollaries 2 and 3, they are no longer useful. Instead the bound from Corollary 1(b) should be used.

Fig. 6.4 illustrates the maximum of the effective capacity bounds \(\gamma_{S_{\text{win}}}(-\theta)\) as function of feedback delay with the ratio \(w/d\) kept fixed at 100 Mbps and 500 Mbps. This figure shows that by increasing feedback delay \(d\), the maximum effective capacity tends to the upper bound which in this figure specifies the fixed ratio \(w/d\). The difference between the bounds from Corollaries 2 and 3 also shows the improvement of Theorem 4 in terms
of maximum effective capacity.

6.3 Backlog bound for exponential VBR

In this section we analyze backlog bounds for a feedback system with an exponential VBR server. We consider a feedback system where the feedback delay is equal to 1 time unit and we have the exact expression for the service by Theorem 1. The moment-generating function of the exact case is given in Eq. (6.2) since the lower bound of Theorem 2 represents the exact case. In order to perform a backlog analysis we use the backlog bound from Lemma 4. We consider a stochastic arrival traffic similar to the service process of the feedback system. The arrival process is

\[ A(s, t) = \sum_{k=s}^{t-1} a_k, \quad (6.11) \]

where \( a_k \)'s are i.i.d. random variables with exponential distribution with parameter \( 1/\lambda \), or \( a_k \overset{iid}{\sim} Exp(1/\lambda) \). The mean rate of the arrivals is \( E[a_k] = \lambda \) and since \( M_{a_k}(\theta) = E[e^{\theta a_k}] = (1 - \lambda \theta)^{-1} \), its moment-generating function is given by

\[ M_A(\theta, s, t) = (1 - \lambda \theta)^{-(t-s)}. \]

Since both \( M_A(\theta, s, t) \) and \( M_{S_{\text{win}}}(-\theta, s, t) \) are stationary, we use the univariate version of Eq. (2.11) for backlog bound given by

\[ b^* = \min_{\theta > 0} \frac{1}{\theta} \left\{ \log \left( \left( \sum_{\tau=0}^{\infty} M_A(\theta, 0, t + \tau) M_{S_{\text{win}}}(-\theta, 0, \tau) \right) \bigg|_{t=0} \right) - \log \epsilon \right\}. \quad (6.12) \]
where \( b^* \) is the time-invariant version of backlog bound defined in Eq. (2.10), \( \varepsilon \) is the violation probability and \( \theta > 0 \). We have

\[
\sum_{\tau=0}^{\infty} M_A(\theta, 0, t + \tau) M_{s_{\text{win}}}(-\theta, 0, \tau) = \sum_{\tau=0}^{\infty} (M_{a_k}(\theta))^{t+\tau} (M_{c_k}^*(-\theta))^{\tau}
\]

\[
= (M_{a_k}(\theta))^t \sum_{\tau=0}^{\infty} (M_{a_k}(\theta) M_{c_k}^*(-\theta))^{\tau}
\]

\[
= (M_{a_k}(\theta))^t (1 - M_{a_k}(\theta) M_{c_k}^*(-\theta))^{-1},
\]

where

\[
M_{c_k}^*(-\theta) = \frac{1 + C\theta e^{-((\theta+C^{-1})w/d)}}{1 + C\theta}, \quad M_{a_k}^{(\theta)} = (1 - \lambda\theta)^{-1},
\]

and the stability condition \( M_{a_k}(\theta) M_{c_k}^*(-\theta) < 1 \) must hold. Using Eq. (6.12) we have the backlog bound as follows

\[
b^* = \min_{\theta > 0} \left\{ \frac{1}{\theta} \left\{ \log \left( (1 - M_{a_k}(\theta) M_{c_k}^*(-\theta))^{-1} \right) - \log \varepsilon \right\} \right\}. \quad (6.13)
\]

Fig. 6.5 depicts the backlog bounds for a feedback systems as function of the arrival rate with same the parameters as before with a fixed feedback delay equal to 1 ms. The arrival process is given by Eq. (6.11) and has a average rate of \( \lambda \) Mbps. We plot the figures for two violation probabilities \( \varepsilon = 10^{-3} \) and \( 10^{-6} \), and for a fixed ratio \( w/d = 100 \text{ Mbps and 500 Mbps} \) (in Fig. 6.5(a) and in Fig. 6.5(b)). The solid lines are the bounds derived from Eq. (6.13). We compare these bounds to the simulation results of the same feedback system shown with dashed lines. For the simulations, we simulate a feedback system in MATLAB and run the simulation for \( 10^9 \) time slots (the first \( 10^5 \) ones were discarded). The comparison shows that the derived backlog bounds are above the simulation results,
which means they are pessimistic, however they seem to follow the blow up when the arrival traffic is increased.

Fig. 6.6 illustrates backlog bounds for the same parameters as a semi-log graph. In these figures the logarithm of the violation probability is depicted as a function of backlog bound. (Fig. 6.6(a) is for $w/d = 100$ Mbps and Fig. 6.6(b) is for $w/d = 500$ Mbps). The violation probability $\varepsilon$ varies from $10^0$ to $10^{-8}$ and for each data point we run the same simulation as before. Semi-log plots in Fig. 6.6 show that the derived backlog bounds are less pessimistic when the arrival rate is low. On the other hand when the arrival rate is increased they appear to diverge more from the simulation results, even for high violation probabilities.
Figure 6.1: Statistical service bounds $S_{\text{win}}^{\varepsilon}(0,t)$ derived by Theorem 3 for feedback system with VBR server (Avg. rate: 1 Gbps, $\varepsilon = 10^{-6}$).


Figure 6.2: Statistical service bounds $S_{\text{win}}^\varepsilon(0, t)$ derived by Theorem 4 for feedback system with VBR server (Avg. rate: 1 Gbps, $\varepsilon = 10^{-6}$).
Figure 6.3: Effective capacity $\gamma_{\text{win}}(\theta)$ of VBR server with feedback (Avg. rate: 1 Gbps).
Figure 6.4: Max effective capacity $\gamma_{\text{win}}(\theta)$ of VBR server with feedback (Avg. rate: 1 Gbps) versus delay.
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Figure 6.5: Backlog bounds for VBR server with feedback (Avg. rate: 1Gbps).
Figure 6.6: Semi-log backlog bounds for VBR server with feedback (Avg. rate: 1Gbps).
Chapter 7

Conclusions and future work

In this thesis we presented improvements on characterizing the stochastic feedback systems using network calculus while the problem was stated as an open problem in the network calculus literature.

7.1 Conclusions

We considered a window flow control model to study feedback systems. We presented special cases where we were able to fully characterize the system by providing an equivalent stochastic process. We derived upper and lower bounds for the available service of a general stochastic feedback system. Then we investigated a feedback system with a service process that has independent increments. We provided two service bounds for this system and evaluated the accuracy of them using the derived upper and lower bounds as benchmarks. We also presented backlog bounds as the performance metric of feedback system.
7.2 Future work

In this thesis we investigated an specific window flow control model where the window size and feedback delay was fixed. To extend this work two directions are worth exploring:

- The first direction is to consider the same feedback system, with window flow control scheme. One can explore the stochastic analysis of this system with variable window size and feedback delay. Even random parameters for window size and delay can be assumed for the analysis.

- The other direction is to extend the results of this work to the existing algorithms that use models based on window flow control scheme such as TCP.
Bibliography


