ESSAYS IN MARKET MICROSTRUCTURE

by

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Abstract

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This thesis examines the impact of various financial market innovations on trading in limit order markets, with a focus on financial market quality and investor welfare.

Chapter 1 is a joint work with Katya Malinova. We model a financial market where privately informed investors trade in a limit order book monitored by low-latency liquidity providers. Price competition between informed limit order submitters and low-latency market makers allows us to capture trade-offs between informed limit and market orders in a methodologically simple way.

In Chapter 2, I extend the model from Chapter 1 to examine the impact of dark pool trade-at rules. Dark pools—trading systems that do not publicly display orders—fill orders at a price better than the prevailing displayed quote, but do not guarantee execution. This improvement is known as the “trade-at rule”. In my model, investors, who trade on private information or liquidity needs, can elect to trade on a visible market, or a dark market where limit orders are hidden. A competitive liquidity provider participates in both markets. The dark market accepts market orders from investors, and if a limit order is available, fills the order at a price better than the displayed quote by a percentage of the bid-ask spread (the trade-at rule). The impact of dark trading on measures of market quality and social welfare depends on the trade-at rule, relative to the price impact of visible limit orders. A dark market with a large (but not too large) trade-at rule improves market quality and welfare; a small trade-at rule, however, impacts market quality and social welfare negatively. Price efficiency declines with either dark market. For a trade-at rule at midpoint or larger, no liquidity is provided to the dark market.

Chapter 3 is also a joint work with Katya Malinova. We study a financial market where investors trade a security for liquidity reasons. Investors pay a ‘take’ fee for trading with market orders, or a ‘make’ fee for limit orders—so-called ‘maker-taker pricing’. When all investors face the same fee schedule, only the total exchange fee per transaction has an economic impact, consistent with previous literature. However, when a subset of investors pay only the average exchange fee through a flat fee per trade—a common practice in the industry—maker-taker fees have an impact beyond the total fee. In comparison to a single-tier fee system, a ‘two-tiered’ fee system leads to a fall in trading volume and
investor welfare; investors who pay maker-taker fees directly earn higher average profits than investors that pay an average flat fee per trade. Under this “two-tiered pricing”, increasing or decreasing the maker rebate can improve trading volume and welfare; however, only a reduction in the maker fee maximizes volume and welfare, and reduces differential profits to zero.
In memory of Alan G. Green.
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Chapter 1

Informed Trading in a Low-Latency Limit Order Market
1.1 Introduction

Equity trading around the world is highly automated. Exchanges maintain limit order books where orders to trade pre-specified quantities at pre-specified prices are arranged in a queue, according to a set of priority rules.\(^1\) A trade occurs when an arriving trader finds the terms of a limit order at the top of the queue sufficiently attractive, and fills the limit order by posting a marketable order.

With the rise of algorithmic trading, exchanges have adopted technology that offers extremely high-speed, or “low-latency” market data transmission, in order to appeal to speed-sensitive participants. The increased speed of trading systems has given rise to a “new type of professional liquidity provider”: proprietary trading firms that “take advantage of low-latency systems” when providing liquidity.\(^2\)

The role of new low-latency computerized traders remains controversial. Proponents maintain that the new trading environment benefits all market participants through increased competition. Opponents argue that the increased competition for liquidity provision makes it difficult for long-term investors to trade via limit orders and that it compels them to trade with more expensive marketable orders.

The key aim of this paper is to understand how to model the decisions of long-term investors who choose between market and limit orders, in limit order books where professional liquidity providers act as de facto market makers. It is particularly important to understand these trade-offs in the presence of private information — where some traders have a speed advantage, others arguably need an informational advantage to compete. Existing models typically study markets where all available liquidity is provided by competitive market makers, or assume that all traders strategically choose between supplying and demanding liquidity, and that they have temporal market power in liquidity provision.\(^3\)

Analyzing a trader’s choice between market and limit orders is methodologically challenging. When liquidity providers have market power, a limit order submitter must optimally choose the limit order price, while accounting for the impact of the chosen price on the probability that the limit order will be filled. The resulting dynamic optimization problem is especially difficult with informed liquidity provision, as the limit order price may reveal the liquidity provider’s private information.

In this paper, we build on Kaniel and Liu (2006) and provide a model of a limit order book where privately informed traders (who we refer to as “investors”) trade with market and limit orders, and, when submitting a limit order, compete with uninformed low-latency market makers. Price competition

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\(^1\)Most exchanges sort limit orders first by price, and then by time of arrival (so-called price-time priority).


in liquidity provision between informed and uninformed (but fast) traders is a key methodological insight in our paper — it allows us to circumvent the complexity of the optimization problem, because all limit orders are posted at prices that yield zero-profits to professional liquidity providers.

Our setup captures the professional liquidity providers’ speed advantage in interpreting market data, such as trades and quotes. In practice, the speed advantage comes at a cost and professional liquidity providers are arguably at a disadvantage (relative to humans or sophisticated algorithms) when processing more complex information, such as news reports. We capture this difference in information processing skills by allowing investors an informational advantage with respect to the security’s fundamental value. Additionally, investors have private valuations (e.g., liquidity needs) for the security.\footnote{Assuming that traders have liquidity needs is common practice in the literature on trading with asymmetric information, to avoid the no-trade result of Milgrom and Stokey (1982); modelling these needs as private valuations allows use to derive welfare implications.}

In equilibrium, an investor’s behavior is governed by his valuation, which is the sum of his private valuation of the security and his expected value of the security. Investors with extreme valuations optimally choose to submit market orders, investors with moderate valuations submit limit orders, and investors with valuations close to the public expectation of the security’s value abstain from trading.

Our analysis of a limit order market with competitive informed liquidity provision contributes to the broader theoretical literature on specialist and limit order markets, see, e.g., Glosten and Milgrom (1985), Kyle (1985), Easley and O’Hara (1987), and Glosten (1994), for competitive uninformed liquidity provision; Parlour (1998), Foucault (1999), Goettler, Parlour, and Rajan (2005), Rosu (2009), Back and Baruch (2013), and Baruch and Glosten (2013) for limit order books with strategic uninformed liquidity provision; Kaniel and Liu (2006), Goettler, Parlour, and Rajan (2009), and Rosu (2011), for strategic informed liquidity provision.\footnote{See also the survey by Parlour and Seppi (2008) for further related papers.} The pricing rule model is very closely related to the equilibrium pricing rule in Kaniel and Liu (2006); differently to them, all traders in our model behave strategically. We complement the theoretical literature that focuses on the trading strategies of professional liquidity providers, see e.g., Biais, Foucault, and Moinas (2012), Foucault, Hombert, and Rosu (2013), Hoffmann (2012), and McInish and Upson (2012).

The role of professional liquidity providers as competitive liquidity providers is supported empirically by, e.g., Hasbrouck and Saar (2013), Hendershott, Jones, and Menkveld (2011), Hendershott and Riordan (2012), and Jovanovic and Menkveld (2011).
1.2 The Model

We model a financial market where risk-neutral investors enter the market sequentially to trade a single risky security for informational or liquidity reasons (as in Glosten and Milgrom (1985)). Trading is conducted via limit order book. Investors choose between posting a limit order to trade at a pre-specified price, and submitting a market order to trade immediately with a previously posted limit order. Additionally, we assume the presence of professional liquidity providers who choose to only submit limit orders, effectively acting as market makers. These traders react to changes in the limit order book before the arrival of subsequent investors. We assume that they are uninformed and have no liquidity needs. Professional liquidity providers compete in prices, are continuously present in the market, and ensure that the limit order book is always full.\footnote{See Figure 1.1 in the Appendix for a diagrammatic illustration of the model’s timing.}

Security. There is a single risky security with an unknown liquidation value. This value follows a random walk, and at each period $t$ experiences an innovation $\delta_t$, drawn independently and identically from a density function $g$ on $[-1,1]$. Density $g$ is symmetric around zero on $[0,1]$. The fundamental value in period $t$ is given by,

$$V_t = \sum_{\tau \leq t} \delta_\tau$$  \hspace{1cm} (1.1)

Market Organization. Trading is organized via limit order book. A trader in period $t$ may choose between posting a price at which they are willing to trade by submitting a limit order, and trading against the best-priced limit order that was posted in period $t - 1$ by submitting a market order. We denote the highest price of all period $t - 1$ buy limit orders by $\text{bid}_t$, and we denote the lowest price of all period $t - 1$ sell limit orders by $\text{ask}_t$. Period $t - 1$ limit orders that remain unexecuted, or unfilled, in period $t$ are cancelled (as in Foucault (1999)). Similar to Foucault (1999), we assume that trading occurs throughout a “trading day” where, with probability $(1 - \rho) > 0$, the trading process ends after period $t$ and the payoff to the security is realized. The history of transactions, limit order submissions and cancellations is observable to all market participants. We denote this history up to (but not including) period $t$ by $H_t$. The structure of the model is common knowledge among all market participants.

Investors. There is a continuum of risk-neutral investors. Each period, a single investor randomly arrives at the market. Upon entering the market in period $t$, the investor is informed with probability $\mu \in (0,1)$, and if so, he learns the period $t$ innovation $\delta_t$ to the fundamental value.\footnote{We refer to investors as male, and we refer to professional liquidity providers as female.} Otherwise, the investor is uninformed, and he is endowed with liquidity needs, which we quantify by assigning him a
private value for the security, \( y_t \), uniformly distributed on \([-1, 1]\). Informed investors have \( y_t = 0 \).

An investor can submit a single order upon arrival and only then. He can buy or sell a single unit (round lot) of the risky security, using a market or a limit order, or he can abstain from trading. If the investor chooses to buy with a limit order, he posts it at the bid price \( \text{bid}^{\text{inv}}_{t+1} \), for execution in period \( t+1 \); similarly for sell limit orders.

**Professional Liquidity Providers.** There is continuum of professional liquidity providers who are always present in the market. They post limit orders on both sides of the market, and they update their limit orders in response to the period \( t \) investor’s order submission or cancellation before the arrival of the period \( t+1 \) investor.9

Professional liquidity providers are risk-neutral, they do not receive any information about the security’s fundamental value, and they do not have liquidity needs. We assume that liquidity providers compete in prices, and post limit orders at prices that yield zero expected profits, conditional on execution. A liquidity provider who submits a buy limit order in period \( t \) posts it at the price \( \text{ask}^{\text{LP}}_{t+1} \); a sell limit order posts at the price \( \text{bid}^{\text{LP}}_{t+1} \). Professional liquidity providers ensure that, upon arrival of an investor, the limit order book always contains at least one buy limit order and one sell limit order.

**Investor Payoffs.** The payoff to an investor who buys one unit of the security in period \( t \) is given by the difference between the security’s fundamental value in period \( t \), \( V_t \), and the price that the investor pays for the unit; similarly for a sell decision. We normalize the payoff to a non-executed order to 0. Investors are risk-neutral, and they aim to maximize their expected payoffs. The period \( t \) investor has the following expected payoffs to submitting, respectively, a market buy order to trade immediately at the prevailing ask price \( \text{ask}_t \) and a limit buy order at price \( \text{bid}^{\text{inv}}_{t+1} \):

\[
\pi^{\text{MB}}_{t, \text{inv}}(y_t, \text{info}_t, H_t) = y_t + E[V_{t+1} \mid \text{info}_t, H_t] - \text{ask}_t
\]

\[
\pi^{\text{LB}}_{t, \text{inv}}(y_t, \text{info}_t, H_t, \text{bid}^{\text{inv}}_{t+1}) = \rho \cdot \Pr(\text{fill} \mid \text{info}_t, H_t, \text{bid}^{\text{inv}}_{t+1}) \times \left( y_t + E[V_{t+1} \mid \text{info}_t, H_t, \text{fill at } \text{bid}^{\text{inv}}_{t+1}] - \text{bid}^{\text{inv}}_{t+1} \right)
\]

where \( \text{info}_t \) is the period \( t \) investor’s information about the innovation \( \delta_t \), (the investor knows \( \delta_t \) if informed and does not if uninformed); \( \Pr(\text{fill} \mid \text{bid}^{\text{inv}}_{t+1}, \text{info}_t, H_t) \) is the probability that an investor’s period \( t \) limit order is filled in period \( t+1 \) (by a market sell order) given the order’s price, \( \text{bid}^{\text{inv}}_{t+1} \); \( E[V_{t+1} \mid \text{info}_t, H_t, \text{fill at } \text{bid}^{\text{inv}}_{t+1}] \) is the period \( t \) investor’s expectation of the fundamental, conditional on the fill of their limit order. Payoffs to sell orders are analogous.

---

8Assuming that traders have liquidity needs is common practice in the literature on trading with asymmetric information, to avoid the no-trade result of Milgrom and Stokey (1982).

9The professional liquidity providers’ ability to identify the type of a limit order submitter (i.e., investor or liquidity provider) can be justified, for instance, by the ability to differentiate traders based on reaction times. Alternatively, one may assume a single liquidity provider who acts competitively.
Professional Liquidity Provider Payoffs. A professional liquidity provider observes the period $t$ investor’s action before posting her period $t$ limit order. A professional liquidity provider in period $t$ has the following payoff to submitting a limit buy order at price $\text{bid}_{t+1}^{LP}$ given by

$$
\pi_{t,1,\text{LB}}^{LP}(\text{bid}_{t+1}^{LP}) = \rho \cdot \Pr(\text{fill} \mid \text{investor action at } t, \text{bid}_{t+1}^{LP}, H_t) \\
\times (E[V_{t+1} \mid H_t, \text{investor action at } t, \text{fill at } \text{bid}_{t+1}^{LP}] - \text{bid}_{t+1}^{LP})
$$

where $\Pr(\text{fill} \mid \text{investor action at } t, \text{bid}_{t+1}^{LP}, H_t)$ is the probability that a liquidity provider’s period $t$ limit order is filled in period $t + 1$ given the order’s price, $\text{bid}_{t+1}^{LP}$, and period $t$ investor’s action; $E[V_{t+1} \mid H_t, \text{investor action at } t, \text{fill at } \text{bid}_{t+1}^{LP}]$ is the liquidity provider’s expectation of the fundamental, conditional on the fill of her limit order; analogously for sell orders.

1.3 Equilibrium

We search for a symmetric, stationary perfect Bayesian equilibrium in which the best bid and ask prices in period $t$ are set competitively with respect to information that is available to professional liquidity providers just prior to the arrival of the period $t$ investor.

1.3.1 Pricing and Decision Rules

Equilibrium Pricing Rule. We denote the equilibrium bid and ask prices in period $t$ by $\text{bid}_t^*$ and $\text{ask}_t^*$, respectively, and we use $\text{MB}_t^*$ and $\text{MS}_t^*$ denote, respectively, the period $t$ investor’s decisions to submit a market buy order against the price $\text{ask}_t^*$ and a market sell order against the price $\text{bid}_t^*$.

The professional liquidity provider payoffs, given by equation (1.4), then imply the following competitive equilibrium pricing rules:

$$
\text{bid}_t^* = E[V_t \mid H_t, \text{MS}_t(\text{bid}_t^*)] \\
\text{ask}_t^* = E[V_t \mid H_t, \text{MB}_t(\text{ask}_t^*)]
$$

where we use the fact that history $H_{t-1}$ together with the period $t - 1$ investor’s action yield the same information about the security’s value $V_t$ as history $H_t$ (because information about $V_t$ is only publicly revealed through investors’ actions).

Investor Decisions with Competitive Liquidity Provision. An investor can choose to submit a market order, a limit order, or he can choose to abstain from trading. In what follows, we focus on investor choices to buy; sell decisions are analogous.
Because an investor is always able to obtain zero profits by abstaining from trade, we restrict attention to limit orders posted at prices that cannot be improved upon by professional liquidity providers. We thus search for an equilibrium where an investor that posts a buy limit order in period \( t - 1 \) does so at the price \( \text{bid}_t^{\text{inv}} = \text{bid}_t^* \), and where a limit order that is posted at a price other than \( \text{bid}_t^* \) either yields the submitter negative profits in expectation, or an execution probability of zero.

**Non-Competitive Limit Orders.** Formally, the zero probability of execution for limit orders posted at non-competitive prices is achieved by defining appropriate beliefs of market participants, regarding the information content of a limit order that is posted at an “out-of-equilibrium” price (e.g., when the period \( t - 1 \) investor posts a limit order to buy at a price different from \( \text{bid}_t^* \)) — so-called out-of-equilibrium beliefs. The appropriate definition of out-of-equilibrium beliefs is frequently necessary to formally describe equilibria with asymmetric information. To see the role of these beliefs in our model, observe first that when an order is posted at the competitive equilibrium price, market participants derive the order’s information content by Bayes’ Rule, using their knowledge of equilibrium strategies. The knowledge of equilibrium strategies, however, does not help market participants to assess the information content of an order that cannot occur in equilibrium — instead, traders assess such an order’s information content using out-of-the-equilibrium beliefs. We describe these beliefs in Appendix 1.5, and we focus on prices and actions that occur in equilibrium in the main text.

**Investor Equilibrium Payoffs.** Because innovations to the fundamental value are independent across periods, all market participants interpret the transaction history in the same manner. A period \( t \) investor decision then does not reveal any additional information about innovations \( \delta_\tau \), for \( \tau < t \), and equilibrium pricing conditions (1.5)-(1.6) can be written as

\[
\begin{align*}
\text{bid}_t^* &= \mathbb{E}[V_{t-1} \mid H_t] + \mathbb{E}[\delta_t \mid H_t, MS_t(\text{bid}_t^*)] \quad (1.7) \\
\text{ask}_t^* &= \mathbb{E}[V_{t-1} \mid H_t] + \mathbb{E}[\delta_t \mid H_t, MB_t(\text{ask}_t^*)] \quad (1.8)
\end{align*}
\]

The independence of innovations across time further allows us to decompose investors’ expectations of the security’s value, to better understand investor equilibrium payoffs. The period \( t \) investor’s expectation of the security’s value in period \( t \) is given by

\[
\mathbb{E}[V_t \mid \text{info}_t, H_t] = \mathbb{E}[\delta_t \mid \text{info}_t] + \mathbb{E}[V_{t-1} \mid H_t],
\]

where \( \mathbb{E}[\delta_t \mid \text{info}_t] \) equals \( \delta_t \) if the investor is informed, and zero otherwise. When the period \( t \) investor submits a limit order to buy, his order will be executed in period \( t + 1 \) (if ever), and we thus need to understand this investor’s expectation of the period \( t + 1 \) value, conditional on his private and public
information, and on the order execution, $E[V_{t+1} | \text{info}_t, H_t, MS_{t+1}(\text{bid}_{t+1}^*)]$. Since the decision of the period $t + 1$ investor reveals no additional information regarding past innovations, we obtain

$$E[V_{t+1} | \text{info}_t, H_t, MS_{t+1}(\text{bid}_{t+1}^*)] = E[V_{t-1} | H_t] + E[\delta_t | \text{info}_t]$$

+ $E[\delta_{t+1} | \text{info}_t, H_t, MS_{t+1}(\text{bid}_{t+1}^*)]$ (1.10)

Further, the independence of innovations implies that, conditional on the period $t$ investor submitting a limit buy order at price $\text{bid}_{t+1}^*$, the period $t$ investor’s private information of the innovation $\delta_t$ does not afford him an advantage in estimating the innovation $\delta_{t+1}$ or the probability of a market order to sell in period $t+1$, relative to the information $H_{t+1}$ that will be publicly available in period $t+1$ (including the information that will be revealed by the period $t$ investor’s order). Consequently, the period $t$ investor’s expectation of the innovation $\delta_{t+1}$ coincides with the corresponding expectation of the professional liquidity providers, conditional on the period $t$ investor’s limit buy order at price $\text{bid}_{t+1}^*$.

The above insight, together with conditions (1.7)-(1.8) on the equilibrium bid and ask prices, allows us to rewrite the investor payoffs given by expressions (1.2)-(1.3) as:

$$\pi_t^{MB}(y_t, \text{info}_t, H_t) = y_t + E[\delta_t | \text{info}_t] - E[\delta_t | H_t, MB_t(\text{ask}_t^*)]$$

(1.11)

$$\pi_t^{LB}(y_t, \text{info}_t, H_t) = \rho \cdot \Pr(MS_{t+1}(\text{bid}_{t+1}^*) | LB_t(\text{bid}_{t+1}^*), H_t)$$

$$\times (y_t + E[\delta_t | \text{info}_t] - E[\delta_t | LB_t(\text{bid}_{t+1}^*), H_t])$$

(1.12)

**Investor Equilibrium Decision Rules.** An investor submits an order to buy if, conditional on his information and on the submission of his order, his expected profits are non-negative. Moreover, conditional on the decision to trade, an investor chooses the order type that maximizes his expected profits. An investor abstains from trading if he expects to make negative profits from all order types.

Expressions (1.11)-(1.12) illustrate that the period $t$ investor payoffs, conditional on the order’s execution, are determined by this investor’s informational advantage with respect to the period $t$ innovation to the fundamental value (relative to the information content revealed by the investor’s order submission decision) or by the investor’s private valuation of the security. Our model is stationary, and in what follows, we restrict attention to investor decision rules that are independent of the history and are solely governed by an investor’s private valuation or his knowledge of the innovation to the security’s value.

When the decision rules in period $t$ are independent of the history $H_t$, the public expectation of the period $t$ innovation, conditional on the period $t$ investor’s action, does not depend on the history either. Expressions (1.11)-(1.12) reveal that neither do investor equilibrium payoffs. Our setup is thus
internally consistent in the sense that the assumed stationarity of the investor decision rules does not preclude investors from maximizing their payoffs.

The expected payoffs of a period $t$ investor are affected by the sum of the realizations of his private value $y_t$ and his expectation of $\delta_t$, conditional on the period $t$ investor’s information. We thus focus on decision rules with respect to this sum, which we refer to as investor’s valuation. We denote the period $t$ investor’s valuation by

$$z_t = y_t + \mathbb{E}[\delta_t | \text{info}_t]$$ (1.13)

Informed investors in our model have no liquidity needs; thus $z_t$ equals $\delta_t$ if the investor is informed and it equals $y_t$ if the investor is uninformed. Since $y_t$ and $\delta_t$ are symmetrically distributed on $[-1, 1]$, the valuation $z_t$ is symmetrically distributed on $[-1, 1]$.

### 1.3.2 Equilibrium Characterization

We first derive properties of market and limit orders that must hold in equilibrium.

Our setup is symmetric, and we focus on decision rules that are symmetric around the zero valuation, $z_t = 0$. We focus on equilibria where investors use both limit and market orders. Appendix 1.5 establishes the following result on the market’s reaction to market and limit orders.

**Lemma 1 (Informativeness of Trades and Quotes)** In an equilibrium where investors use both limit and market orders, both trades and investors’ limit orders contain information about the security’s fundamental value; a buy order increases the expectation of the security’s value and a sell order decreases it.

Lemma 1 implies that a price improvement stemming from a period $t$ investor’s limit buy order at the equilibrium price $\text{bid}^*_t > \text{bid}^*_t$ increases the expectation of a security’s value. In our setting, such a buy order will be immediately followed by a cancellation of a sell limit order at the best period $t$ price $\text{ask}^*_t$ and a placement of a new sell limit order at the new ask price $\text{ask}^*_t > \text{ask}^*_t$ by a professional liquidity provider.

**Lemma 2 (Equilibrium Market and Limit Order Submission)** In any equilibrium with symmetric time-invariant strategies, investors use threshold strategies: investors with the most extreme valuations submit market orders, investors with moderate valuations submit limit orders, and investors with valuations around zero abstain from trading.

---

10 Any equilibrium where professional liquidity providers are the only liquidity providers closely resembles equilibria in market maker models in the tradition of Glosten and Milgrom (1985).
To understand the intuition behind Lemma 2, observe first that, conditional on order execution, an investor’s payoff is determined, loosely, by the advantage that his valuation provides relative to the information revealed by his order (see expressions (1.11)-(1.12)). Second, since market orders enjoy guaranteed execution, whereas limit orders do not, for limit orders to be submitted in equilibrium, the payoff to an executed limit order must exceed that of an executed market order. Consequently, the public expectation of the innovation $\delta_t$, conditional on, say, a limit buy order in period $t$, must be smaller than the corresponding expectation, conditional on a market buy order in period $t$ (in other words, the price impact of a limit buy order must be smaller than that of a market buy order). For this ranking of price impacts to occur, investors who submit limit orders must, on average, observe lower values of the innovation than investors who submit market buy orders. With symmetric distributions of both, the innovations and investor private values, we arrive at the previous lemma.

### 1.3.3 Equilibrium Existence

Utilizing Lemmas 1 and 2, we look for threshold values $z^M$ and $z^L < z^M$ such that investors with valuations above $z^M$ submit market buy orders, investors with valuations between $z^L$ and $z^M$ submit limit buy orders, and investors with valuations between $-z^L$ and $z^L$ abstain from trading. Symmetric decisions are taken for orders to sell. Investors with valuations of $z^M$ and $z^L$ are marginal, in the sense that the investor with the valuation $z^M$ is indifferent between submitting a market buy order and a limit buy order, and the investor with the valuation $z^L$ is indifferent between submitting a limit buy order and abstaining from trading. Using (1.11)-(1.12), and the definition of the valuation (1.13), thresholds $z^M$ and $z^L$ must solve the following equilibrium conditions

\[
\begin{align*}
  z^M - E[\delta_t | MB_t^*] &= \rho \cdot Pr(MS_{t+1}^*) \times (z^M - E[\delta_t | LB_t^*]) \quad (1.14) \\
  z^L &= E[\delta_t | LB_t^*] \quad (1.15)
\end{align*}
\]

where the stationarity assumption on the investor’s decision rule allows us to omit conditioning on the history $H_t$; $MB_t^*$ denotes an equilibrium market buy order in period $t$, which occurs when the period $t$ investor valuation $z_t$ is above $z^M$ ($z_t \in [z^M, 1]$), $LB_t^*$ denotes an equilibrium limit buy order in period $t$ ($z_t \in [z^L, z^M]$), and $MS_{t+1}$ denotes a market order to sell in period $t+1$ ($z_{t+1} \in [-1, -z^M]$). Given thresholds $z^M$ and $z^L$, these expectations and probabilities are well-defined and can be written out explicitly, as functions of $z^M$ and $z^L$ (and independent of the period $t$).

Further, when investors use thresholds $z^M$ and $z^L$ to determine their decision rules, the bid and ask prices that yield zero profits to professional liquidity providers, given by the expressions in (1.5)-(1.6),
can be expressed as

\[ \text{bid}_t^* = p_{t-1} + \mathbb{E}[\delta_t | z_t \leq -z^M] \tag{1.16} \]

\[ \text{ask}_t^* = p_{t-1} + \mathbb{E}[\delta_t | z_t \geq z^M] \tag{1.17} \]

where \( p_{t-1} \equiv \mathbb{E}[V_{t-1}|H_t] \). The choice of notation for the public expectation of the security’s value recognizes that this expectation coincides with a transaction price in period \( t-1 \) (when such a transaction occurs). Since the innovations are distributed symmetrically around zero, the public expectation of the period \( t \) value of the security at the very beginning of period \( t \), \( \mathbb{E}[V_t|H_t] \), also equals \( p_{t-1} \).

For completeness, investors who submit limit orders to buy or sell in period \( t \), in equilibrium, will post them at prices \( \text{bid}_{t+1}^* \) and \( \text{ask}_{t+1}^* \), respectively, given by:

\[ \text{bid}_{t+1}^* = p_{t-1} + \mathbb{E}[\delta_t | z_t \in [z^L, z^M]] + \mathbb{E}[\delta_{t+1} | z_{t+1} \leq -z^M] \tag{1.18} \]

\[ \text{ask}_{t+1}^* = p_{t-1} + \mathbb{E}[\delta_t | z_t \in (-z^M, -z^L)] + \mathbb{E}[\delta_{t+1} | z_{t+1} \geq z^M] \tag{1.19} \]

For an equilibrium to exist, we require that the bid-ask spread is positive, which holds as long as market orders are informative. Finally, as discussed above, the equilibrium is supported by out-of-the-equilibrium beliefs such that professional liquidity providers outbid all non-competitive prices. We prove the following existence theorem in Appendix 1.5; we include with it, a discussion on out-of-the-equilibrium beliefs that support the equilibrium prices and decision rules.

**Theorem 1 (Equilibrium Characterization and Existence)** There exist values \( z^M \) and \( z^L \), with \( 0 < z^L < z^M < 1 \), that solve indifference conditions (1.14)-(1.15). These threshold values constitute an equilibrium for any history \( H_t \), given competitive equilibrium prices, \( \text{bid}_t^* \) and \( \text{ask}_t^* \) in (1.16)-(1.17), for the following investor decision rules. The investor who arrives in period \( t \) with valuation \( z_t \):

- places a market buy order if \( z_t \geq z^M \),
- places a limit buy order at price \( \text{bid}_{t+1}^* \) if \( z^L \leq z_t < z^M \),
- abstains from trading if \( -z^L < z_t < z^L \).

Investors’ sell decisions are symmetric to buy decisions. In this equilibrium, a buy limit order in period \( t \) at a price different to \( \text{bid}_{t+1}^* \) is executed with zero probability. Investors’ sell decisions are analogous.

In the case where \( g(\delta) \sim U \), the equilibrium in Theorem 1, we can produce the following corollary, which says that the equilibrium is unique.

**Corollary 1 (Uniformly Distributed Innovations and Uniqueness)** If \( g(\delta) \sim U \), then the equilibrium described by Theorem 1 is unique.
1.4 Conclusion

We provide a model to analyze a financial market where investors trade for informational or liquidity reasons in a limit order book that is monitored by professional liquidity providers. These liquidity providers are endowed with a speed advantage in reacting to trade and quote information.

The presence of professional liquidity providers that act as de facto market makers plays a key role in the limit order pricing decision. Price competition between informed investors and uninformed market makers allows us to simplify the investor’s pricing decision, such that the model is analytically tractable. We believe that the tractability of our model provides a useful platform to study various innovations to market structure. Chapter 2 focuses on one such innovation, the introduction of a dark pool.

1.5 Appendix

1.5.1 Proofs of Lemmas 1 and 2

Proof. In the main text, we present the two lemmas separately, for the sake of exposition. Here we establish the two results simultaneously. We restrict attention to an equilibrium where investors use symmetric, time-invariant strategies and trade with both, market and limit orders. Since we search for an equilibrium with competitive pricing, an investor’s equilibrium action does not affect the price that he pays or the probability of his limit order execution. We show, in 5 steps, that in any such equilibrium investors must use decision rules that lead to Lemmas 1 and 2.

Step 1: In any equilibrium, an investor with the valuation $z_t$ prefers a market (limit) buy order to a market (limit) sell order if and only if $z_t \geq 0$.

Proof: Using (1.11), an investor’s payoff to a market buy order is $z_t - E[\delta_t \mid H_t, MB_t(ask_t^*)]$. When innovations $\delta_t$ are independent across time and investors’ equilibrium strategies are time-invariant functions of $z_t$, the expectation $E[\delta_t \mid H_t, MB_t(ask_t^*)]$ does not depend on the history $H_t$ or on the ask price $ask_t^*$. With symmetric decision rules, $E[\delta_t \mid MB_t] = -E[\delta_t \mid MS_t]$; investor payoff (1.11) and an analogous payoff for sell orders then yield Step 1 for market orders. Similarly, symmetry, expression (1.12) and an analogous expression for limit sell orders yield the result for limit orders.

Step 2: In any equilibrium, there must exist $z^* \in (0, 1)$ such that an investor with valuation $z_t$ prefers a market buy order to a limit buy order if and only if $z_t \geq z^*$, with indifference if and only if $z_t = z^*$. 
Proof: Comparing investor equilibrium payoffs (1.11) and (1.12), an investor with valuation $z_t$ prefers a market buy order to a limit buy order if and only if

$$z_t \geq \frac{\mathbb{E}[\delta_t | MB_t] - \Pr(MS_t)\mathbb{E}[\delta_t | LB_t]}{1 - \Pr(MS_t)} \equiv z^*.$$  \hspace{1cm} (1.20)

The fraction in (1.20) is well-defined in an equilibrium where investors submit both market and limit orders, since $0 < \Pr(MS_t) < 1$. Next, for investors to submit limit orders with positive probability, there must exist $z$ such that for the investor with the valuation $z_t = z$, the payoff to a limit buy order (i) exceeds that to the market buy order and (ii) is non-negative. For this $z$, we then have

$$z - \mathbb{E}[\delta_t | MB_t] \leq \Pr(MS_t)(z - \mathbb{E}[\delta_t | LB_t]) \leq z - \mathbb{E}[\delta_t | LB_t]$$  \hspace{1cm} (1.21)

Hence, $\mathbb{E}[\delta_t | MB_t] \geq \mathbb{E}[\delta_t | LB_t]$. Since $0 < \Pr(MS_t) < 1$, the following inequalities are strict: $\mathbb{E}[\delta_t | MB_t] > \Pr(MS_t)\mathbb{E}[\delta_t | LB_t] \geq 0$.

**Step 3:** In any equilibrium, submitting the market buy order is strictly optimal for an investor with valuation $z_t > z^*$.

*Proof:* By Steps 1 and 2, an investor with valuation $z_t$ such that $z_t > z^* > 0$ strictly prefers a market buy order to a market sell order and to a limit buy order (and, consequently, by Step 1, to a limit sell order). Finally, an investor with valuation $z_t > z^*$ strictly prefers submitting a market order to abstaining from trade, as:

$$z_t - \mathbb{E}[\delta_t | MB_t] > \frac{\mathbb{E}[\delta_t | MB_t] - \Pr(MS_t)\mathbb{E}[\delta_t | LB_t]}{1 - \Pr(MS_t)} - \mathbb{E}[\delta_t | MB_t] \geq 0,$$

where the last inequality follows since $\mathbb{E}[\delta_t | LB_t] \leq \mathbb{E}[\delta_t | MB_t]$ by Step 2.

**Step 4:** In any equilibrium, an optimal action for an investor with valuation $z_t \in (0, z^*)$ must be either a limit buy order or a no trade.

*Proof:* This investor prefers a limit buy order to a market buy order by Step 2, and the investor prefers a limit buy order to a limit sell order by Step 1, which in turn is preferred by a market sell order by symmetry and Step 2.

**Step 5:** There exists $z^* \in (0, z^*)$ such that an investor with the valuation $z_t = z^*$ is indifferent between submitting a limit buy order and abstaining from trade; it is strictly optimal for an investor with valuation $z_t \in (z^*, z^*)$ to submit a limit buy order, and it is strictly optimal for an investor with valuation $z_t \in [0, z^*)$ to abstain from trading.
Chapter 1: Informed Trading in a Low-Latency Limit Order Market

Proof: In an equilibrium where investors submit both market and limit orders the probability of a limit order is strictly positive, consequently, the limit buy order is preferred to abstaining from trade if and only if an investor’s valuation \( z_t > E[\delta_t \mid LB_t] \) (and, by Step 4, the limit buy order is then the optimal action for this investor, and abstaining from trade is optimal for an investor with \( z_t < E[\delta_t \mid LB_t] \)). For investors to submit both market and limit orders with non-zero probability, in equilibrium we must have \( E[\delta_t \mid LB_t] < z^* \) (otherwise, by Step 3, any investor, except for the zero-probability case of \( z_t = z^* \) that prefers the limit order to abstaining from trade also strictly prefers the market buy order to the limit buy order). We are looking for a stationary equilibrium and the distribution of \( \delta_t \) does not depend on \( t \), hence \( E[\delta_t \mid LB_t] \) does not depend on \( t \) and we can thus set \( z^{**} = E[\delta_t \mid LB_t] \).

What remains to be shown is that \( E[\delta_t \mid LB_t] > 0 \). We proceed by contradiction. Suppose not and \( E[\delta_t \mid LB_t] \leq 0 \). Then, by Steps 1-4, in a symmetric equilibrium, the limit buy is strictly optimal for an investor with \( z \in (0, z^*) \); it is strictly optimal for an investor with \( z > z^* \) to submit the market buy order; it is strictly optimal for an investor with valuation \( z_t < 0 \) to submit either the market or the limit sell orders; finally, investors with \( z_t = 0 \) and \( z_t = z^* \) are indifferent between the limit buy and a different action (the limit sell and the market buy, respectively) and they occur with zero probability. This implies that limit buy orders are only submitted by investors whose valuations are (weakly or strictly) in the interval of \([0, z^*]\) and only by these investors. But then \( E[\delta_t \mid LB_t] = E[\delta_t \mid z_t \in (0, z^*)] > 0 \), a contradiction.\(^{11}\)

Steps 1-5 show that threshold rules are optimal in any symmetric, time-invariant equilibrium where traders submit both market and limit orders, and that investors with the more extreme valuations submit market orders, investors with moderate valuations submit limit orders, and investors with valuations close to zero abstain from trade. Given threshold rules described in these steps, (investors’) quotes are informative because \( E[\delta_t \mid LB_t] = E[\delta_t \mid z \in (z^{**}, z^*)] > 0 \) and trades are informative because \( E[\delta_t \mid MB_t] = E[\delta_t \mid z \in (z^*, 1)] > 0 \). Furthermore, by the proof of Step 2, a trade has a higher price impact than a quote.

\(^{11}\)The inequality follows because \( z = y_t + \delta_t \), where \( y_t \) and \( \delta_t \) are independent and symmetrically distributed on \([-1, 1]\); the explicit derivation of this expectation is in the Internet Appendix.

1.5.2 Proof of Theorem 1

Proof. To prove existence of a symmetric, stationary equilibrium in threshold strategies, we show that there exist a \( z^L \) and \( z^M \), such that \( z^L \leq z^M \). We then prove the optimality of the threshold strategy.

The equilibrium conditions (1.14) and (1.15) can be written as:
where an investor submits a market buy over a limit buy as long as $z_t \geq z^M$, submits a limit buy if $z^M > z_t \geq z^L$, and abstains from trading otherwise.

The probability of a market sell order $\Pr(\text{MS})$ is a function of $z^M$: \[ \Pr(\text{MS}) = \mu \int_{-1}^{z^M} g(\delta) d\delta + (1 - \mu)(1 - z^M), \] (1.24) where as in the main text, $\mu \in (0, 1)$ is the probability that an investor is informed, $g$ denotes the density function of the period-$t$ innovation $\delta_t$ and it is symmetric on $[-1, 1]$; private values are distributed uniformly on $[-1, 1]$. The price impacts of market and limit buy orders, $E[\delta_t \mid \text{MB}^*_t]$ and $E[\delta_t \mid \text{LB}^*_t]$, are functions of $z^M$ and of $z^M$ and $z^L$, respectively:

\[ E[\delta_t \mid \text{MB}^*_t] = \mu \int_{-1}^{z^M} \frac{\delta g(\delta) d\delta}{\mu g(z^M) + (1 - \mu)(1 - z^M)}, \] (1.25)

\[ E[\delta_t \mid \text{LB}^*_t] = \mu \int_{-1}^{z^L} \frac{\delta g(\delta) d\delta}{\mu g(z^M) + (1 - \mu)(z^M - z^L)}, \] (1.26)

We proceed in 3 steps. In step 1, we show that for any $z^M \in [0, 1]$, there exists a unique $z^L = z^{L^*}$ that solves (1.23). Defining function $z^*(\cdot)$ for each $z^M$ as $z^*(z^M) = z^{L^*}$, we show that $z^*(\cdot)$ is continuous.

In Step 2, we show that there exists a $z^M \in (0, 1)$ that solves (1.22). Finally, in Step 3, we argue the optimality of the threshold strategy.

**Step 1: Existence and Uniqueness of $z^{L^*}(z^M)$**

Denote the left-hand side of (1.23) by $\Delta^{LB}(z^M, z^L)$. First, using (1.25), $\Delta^{LB}(z^M, 0) < 0$. Second, when $z^L = z^M$, by L’Hospital’s Rule,

\[ \Delta^{LB}(z^M, z^M) = z^M - \frac{\mu g(z^M) z^M}{\mu g(z^M) + (1 - \mu) z^M} = \frac{(1 - \mu) z^M}{\mu g(z^M) + (1 - \mu) z^M} > 0. \] (1.27)

Function $\Delta^{LB}(\cdot, \cdot)$ is continuous, and the above two observations imply that there exists $z^{L^*} \in (0, z^M)$ that solves equation (1.23).

Next, we will show that at $z^L = z^{L^*}$, the derivative of $\Delta^{LB}(z^M, \cdot)$ with respect to the second argument is $> 0$ for all $z^M$. This step ensures uniqueness of $z^{L^*}$ and also, by the Implicit Function Theorem, the existence of a differentiable (and therefore continuous) function $z^*(\cdot)$ such that $z^*(z^M) = z^{L^*}$. Denoting the probability of an equilibrium limit order (which is given by the denominator of the right-hand-side
Similarly, we denote the left-hand side of equation (1.26) by \( \Pr(LB^*) \) and denote the partial derivative of \( \Delta^L_{z^L}(\cdot,\cdot) \) with respect to the second argument by \( \Delta^L_{z^L}(\cdot,\cdot) \), we obtain:

\[
\Delta^L_{z^L}(z^M, z^L)_{|z^L = z^L^*} = 1 - \frac{1}{\Pr(LB^*)} \times \left[-\mu z^L^* g(z^L^*) - \mathbb{E}[\delta_t | LB^*_t] \times (-\mu g(z^L^*) - (1 - \mu)) \right] = 1 - (1 - \mu) \frac{\mathbb{E}[\delta_t | LB^*_t]}{\Pr(LB^*)},
\]

(1.28)

where the last equality follows from \( \mathbb{E}[\delta_t | LB^*_t] = z^L^* \). Hence, the desired inequality given by \( \Delta^L_{z^L}(z^M, z^L)_{|z^L = z^L^*} > 0 \) holds if and only if \( \Pr(LB^*) > (1 - \mu) \mathbb{E}[\delta_t | LB^*_t] \). To show the latter inequality, we use \( z^L^* = \mathbb{E}[\delta_t | LB^*_t] \) and rewrite (1.26) as follows:

\[
z^L^* = \frac{\int_{z^L}^{z^M} \delta g(\delta) d\delta}{\int_{z^L}^{z^M} g(\delta) d\delta} \times \frac{\mu \int_{z^L}^{z^M} g(\delta) d\delta}{\mu \int_{z^L}^{z^M} g(\delta) d\delta + (1 - \mu)(z^M - z^L^*)} < z^M \times \frac{\mu \int_{z^L}^{z^M} g(\delta) d\delta}{\Pr(LB^*)},
\]

(1.29)

where the inequality obtains because \( z^L^* \in (0, z^M) \). Subtracting \( z^M \) from both sides of (1.29), and rearranging:

\[
z^M - z^L^* > z^M \times \left(1 - \frac{\mu \int_{z^L}^{z^M} g(\delta) d\delta}{\Pr(LB^*)} \right) = z^M \times \frac{(1 - \mu)(z^M - z^L^*)}{\Pr(LB^*)} > z^L^* \times \frac{(1 - \mu)(z^M - z^L^*)}{\Pr(LB^*)},
\]

(1.30)

where the last inequality follows from \( z^L^* < z^M \). The above inequality implies the desired inequality \( \Pr(LB^*) > (1 - \mu) \mathbb{E}[\delta_t | LB^*_t] \) and thus \( \Delta^L_{z^L}(z^M, z^L)_{|z^L = z^L^*} > 0 \).

Thus, there exists a unique \( z^L^* \) that solves indifference equation (1.23) for any \( z^M \in [0, 1] \), and a continuous function \( z^*(\cdot) \) such that \( z^*(z^M) = z^L^* \).

**Step 2: Existence of \( z^M \)**

Similarly, we denote the left-hand side of equation (1.22) as \( \Delta^M_{z^M}(z^M, z^L) \). Function \( \Delta^M_{z^M}(\cdot,\cdot) \) is continuous in both arguments, and therefore function \( \Delta^M_{z^M}(\cdot, z^*(\cdot)) \) is continuous in \( z^M \). We have:

\[
\Delta^M_{z^M}(0, z^*(0)) = 0 - \mathbb{E}[\delta | MB^*] - \rho \Pr(MS^*) \times (0 - 0) < 0 \quad (1.31)
\]

\[
\Delta^M_{z^M}(1, z^*(1)) = 1 - \mathbb{E}[\delta | MB^*] - 0 \times (z^M - z^L^*) > 0, \quad (1.32)
\]

where the inequalities follow directly from expressions (1.24)-(1.26).
1.5.3 Proof of Corollary 1

**Proof.** Continuing from Step 2 of the proof of Theorem 1, we substitute \( g(\delta) \sim U \) into function \( \Delta^{MB}(z^M, z^{L*}) \), which yields:

\[
\Delta^{MB}(z^M, z^{L*}) = z^M - \frac{\mu(1 + z^M)}{2} - \rho \times \frac{1 - z^M}{2} \times \left( z^M - \frac{\mu z^M}{2 - \mu} \right) \tag{1.33}
\]

where \( z^{L*} = \frac{\mu z^M}{2 - \mu} \) from our substitution of the uniform distribution for \( g(\delta) \). Then, because \( z^{M*} \) exists by the proof of Theorem 1, we only need to show that it is unique. We do so by differentiating \( \Delta^{MB}(z^M, z^{L*}) \) by \( z^M \), and showing that it is increasing in \( z^M \).

\[
\frac{\partial \Delta^{MB}(z^M, z^{L*})}{\partial z^M} = 1 - \frac{\mu}{2} + \frac{\rho}{2} \times \left( z^M - \frac{\mu z^M}{2 - \mu} \right) - \rho \times \frac{1 - z^M}{2} \times \left( 1 - \frac{\mu}{2 - \mu} \right) \tag{1.34}
\]

\[
> \left( 1 - \frac{\mu}{2} \right) \times \left( 1 - \rho \times \frac{1 - z^M}{2} \right) + \frac{\rho}{2} \times \left( z^M - \frac{\mu z^M}{2 - \mu} \right) > 0 \tag{1.35}
\]

Therefore, \( z^{M*} \) is unique, thus completing the proof. ■

**Step 3: Optimality of the Threshold Strategies**

The intuition for the optimality of the threshold strategies stems from competitive pricing and stationarity of investor decisions. An investor’s deviation from one equilibrium action to another will not affect equilibrium bid and ask prices or probabilities of the future order submissions. Consequently, it is possible to show that the difference between a payoff to a market order and a payoff to a limit order at the equilibrium price to an investor with a valuation above \( z^M \) is strictly greater than 0.

**Out-of-Equilibrium Beliefs.** A more complex scenario arises when an investor deviates from his equilibrium strategy by submitting a limit order at a price different to the prescribed competitive equilibrium price. Whether or not this investor expects to benefit from such a deviation depends on the reaction to this deviation by the professional liquidity providers and investors in the next period. For instance, can an investor increase the execution probability of his limit buy order by posting a price that is above the equilibrium bid price?

We employ a perfect Bayesian equilibrium concept. This concept prescribes that investors and professional liquidity providers update their beliefs by Bayes rule, whenever possible, but it does not place any restrictions on the beliefs of market participants when they encounter an out-of-equilibrium action.

To support competitive prices in equilibrium we assume that if a limit buy order is posted at a price different to the competitive equilibrium bid price \( \text{bid}_{t+1}^* \), then market participants hold the following beliefs regarding this investor’s knowledge of the period \( t \) innovation \( \delta_t \).
If a limit buy order is posted at a price $\hat{\text{bid}} < \text{bid}^*_t + 1$, then market participants hold the same beliefs regarding the investor’s information as they would in equilibrium (i.e., they assume that this investor followed the equilibrium threshold strategy, but “made a mistake” when pricing his orders). A professional liquidity provider then updates his expectation about $\delta_t$ to the equilibrium value and posts a buy limit order at $\text{bid}^*_t + 1$. The original investor’s limit order then executes with zero probability.

If a limit buy order is posted at a price $\hat{\text{bid}} > \text{bid}^*_t + 1$, then we posit that market participants believe that this order stems from an informed investor with the highest possible valuation (i.e., $\delta_t = 1$). They then update their expectation to $E[\delta_t | \hat{\text{bid}}] = 1$. The new posterior expectation of $V_t$ equals to $p_{t-1} + E[\delta_t | \hat{\text{bid}}] = p_{t-1} + 1$. A professional liquidity provider is then willing to post a bid price $\text{bid}^{**}_t + 1 \leq p_{t-1} + 1 + E[\delta_{t+1} | \text{MS}_{t+1}]$. A limit order with the new price $\text{bid}^{**}_t + 1$ outbids any limit buy order that yields investors positive expected profits (for an investor to make positive profits, their valuation would need to exceed 1).

The beliefs upon an out-of-equilibrium sell order are symmetric. The above out-of-equilibrium beliefs ensure that no investor deviates from his equilibrium strategy. We want to emphasize that these beliefs and actions do not materialize in equilibrium. Instead, they can be loosely thought of as a “threat” to ensure that investors do not deviate from their prescribed equilibrium strategies.
Figure 1.1: Entry and Order Submission Timeline

This figure illustrates the timing of events upon the arrival of an investor at an arbitrary period, $t$, until their departure from the market. Value $y_t$ is the private valuation of the period $t$ investor and $\delta_t$ is the innovation to the security’s fundamental value in period $t$. 

```
Professional liquidity providers
post limit orders

Period $t$ investor
enters market,
learns $y_t$ and $\delta_t$

Period $t - 1$ investor
leaves market

Period $t - 1$ limit orders either
trade against the period $t$ market order
or get cancelled

Period $t + 1$ investor
enters market,
learns $y_{t+1}$ and $\delta_{t+1}$

Period $t$ investor
enters market,
leaves market

Period $t + 1$ limit orders either
trade against the period $t + 1$ market order
or get cancelled
```
Chapter 2

Should Dark Pools Improve Upon Visible Quotes? The Impact of Trade-at Rules
2.1 Introduction

In recent years, concern has arisen over the impact of dark trading on equity markets. While dark markets are not new, this concern comes as they begin to match the electronic organization of visible markets, and gain a significant share of global trading activity. The CFA Institute estimates that dark markets in the U.S. generate approximately one third of all volume\(^1\). Because dark markets benefit from visible market transparency, there is concern from regulators that these benefits come at the expense of market quality (see Securities and Exchange Commission (2010)).

Equity trading primarily conducts through two types of markets: visible limit order markets, and dark markets. In visible markets, investors submit limit orders at pre-specified (limit) prices; a trade occurs when a subsequent investor submits a market order. Limit orders are visible to all market participants. In a dark market, liquidity is hidden, and orders sent to the dark market are filled only if sufficient liquidity is present. To entice investors to submit orders to trade against their hidden liquidity, orders sent to a dark market usually fill at a better price than the prevailing quote offered by the visible market.

In many dark markets, orders receive only marginal improvement on displayed quotes. Marketplaces generally give first-fill priority to visible orders over dark orders at the same price, which provides an incentive for liquidity providers to display their trading intentions. However, a marginal price improvement moves the dark order to the front of the queue, according to price-visibility-time priority. To more adequately incentivize liquidity providers to display their orders, some countries now require dark liquidity to provide “meaningful price improvement”—the so-called “trade-at rule”. Canada and Australia now have trade-at rules in place that require dark liquidity to provide a price improvement of at least one trading increment (i.e., one cent in most major markets).\(^2\) For securities with bid-offer spreads of one or two trading increments, orders fill at the midpoint of the spread (see Investment Industry Regulatory Organization of Canada (2011) and Australian Securities and Investments Commission (2012)). By imposing a minimum trade-at rule, regulators suggest that dark markets may have a place in the financial landscape, but only if liquidity providers pay a premium for hiding their trading intentions. But how do trade-at rules impact market quality and investor welfare? Is more improvement always better?

To analyze an investor’s optimal order placement decision, I model an investor’s order placement strategy with three components: their valuation of the asset, the price of the order, and its probability.

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\(^1\)CFA Institute (2012): “Dark Pools, Internalization, and Equity Market Quality”.
\(^2\)The immediate impact of the trade-at rule in Canada was a 50% drop in dark trading volume. Rosenblatt Securities Inc. (2013)
of execution. The choice of order type focuses on the trade-off between price and execution risk. In a market where a competitive liquidity provider ensures the limit order book is full, market orders face lower execution risk than visible limit orders or dark orders. As compensation, visible limit orders and dark orders offer better prices. The trade-off between dark orders and visible limit orders, however, is more complex, as both orders face execution uncertainty. To compensate for execution risk, dark orders receive price improvement on the prevailing quote, while visible limit orders permit investors to set their own quote. Hence, a dark market’s trade-at rule plays a key role in how dark orders compete with visible limit orders. In this paper, I analyze how trade-at rules impact overall market quality and investor welfare. I then discuss my results in the context of trade-at rule requirements by regulators.

I propose a dynamic trading model where investors trade for either informational or liquidity reasons. They may trade at a visible market using either a limit order or market order, or send an order to the dark market. Both the visible and dark markets are monitored by a competitive, uninformed liquidity provider that acts as a market maker. The liquidity provider possesses a monitoring advantage towards interpreting and reacting to market data, an advantage they use to ensure that their limit orders are priced competitively. Moreover, the liquidity provider ensures that the visible limit order book always contains limit orders on either side of the book. Consequently, investors who submit market orders are guaranteed execution at the available quote. Limit orders, however, are subject to execution risk, as they may trade only if the subsequent investor submits the appropriate market order.

In the dark market, orders fill at a price derived from the visible market: the prevailing quote, improved by a percentage of the bid-ask spread.\(^3\) Because the liquidity provider cannot set the price of their dark orders directly, they instead choose the probability with which they post orders to the dark market to ensure that they earn zero expected profits in equilibrium. As a result, investor dark orders fill with a probability less than one.

I analyze the impact of introducing a dark market alongside a visible limit order market, by first solving the case where all investors are indifferent between visible limit orders and dark orders. I assume that when indifferent between visible and dark orders, investors use visible orders. In this way, this setting serves as the “visible market only” benchmark. I find that there exists a trade-at rule where investors are indifferent to visible limit orders and dark orders, which occurs when trade-at rule equates price of a dark order and the price impact of a visible limit order. I refer to this as the “benchmark” trade-at rule. In equilibrium, investors use dark orders when the trade-at rule is above or below the benchmark level. I refer to these trade-at rules as a “large trade-at rule” and a “small trade-at rule”,

\(^3\)For instance, prior to minimum price improvement regulation in Canada, the dark pools MatchNow and Alpha IntraSpread used trade-at rules of 20% and 10%, respectively.
I find that a dark market with a large trade-at rule improves market liquidity. By offering a “discount” trading opportunity, investors who would otherwise be pushed out of the market by the costs of visible orders can participate. The trade-off, however, is higher execution risk of dark orders relative to visible limit orders. Because of this, only low-valuation visible limit order investors migrate to the dark. This leads to an increase in the relative attractiveness of visible market orders, and investors who submit limit orders migrate to visible market orders. The result is an increase in volume both on the exchange, and overall. Quoted spreads also narrow. However, if the trade-at rule is at the midquote (or better), then no liquidity is provided to the dark market, and investors remain with the visible market.

Conversely, a dark market with a small trade-at rule incentivizes the liquidity provider to ensure an investor’s dark order has lower execution risk than a visible limit order. Then, because dark orders have lower execution risk than visible limit orders, the dark market attracts investors away from both visible order types: the visible market order submitters with the lowest valuations, and the visible limit order submitters with the highest valuations. Hence, investors who submit visible market orders have higher average valuations, which, because some investors are informed, implies that visible market order submitters have a larger price impact, on average. This leads to an increase in the trading costs of visible market orders, thus negatively impacting visible market liquidity. The result is an increase in quoted spreads, and a reduction in visible market and total volume.

By modelling both informed investors, and uninformed investors with private values, I can study price efficiency and social welfare, respectively. I study price efficiency by measuring the difference between the change in the fundamental value of the security and the price impact of an investor’s order placement. I measure welfare in the sense of allocative efficiency, similar to Bessembinder, Hao, and Lemmon (2012): the expected private valuation realized by trade counterparties, discounted by the probability of a trade. I find that price efficiency declines regardless of the dark market’s trade-at rule. Social welfare increases with a large trade-at rule, but falls with a small trade-at rule.

My model suggests that the impact of dark market trade-at rules on equity markets is dichotomous, depending crucially on the price impact of visible limit orders. My results suggest that there may be a role for minimum trade-at rule requirements, but that a midpoint trade-at rule is not welfare-maximizing. By measuring the price impact of visible limit orders, a trade-at rule that yields a discount for dark orders greater than the price impact of limit orders (but less than midpoint) would be welfare-improving to the current minimum trade-at rules in markets such as Canada and Australia.

**Related Literature.** The prevalence of dark trading generates a need to understand the impact of dark trading on market quality, price efficiency and investor welfare. Several papers theoretically examine
the impact of crossing networks. In these models, dark orders fill at the prevailing visible market quote, or midquote of the spread: Zhu (2014), Hendershott and Mendelson (2000) and Degryse, van Achter, and Wuyts (2009) fill orders at the midquote, whereas Ye (2011) assumes orders fill at the prevailing quote. Buti, Rindi, and Werner (2014) examine both periodic and continuous crossing networks where trades occur at the midquote. The literature on continuous dark pools, however, is relatively new. I contribute to the literature by studying an increasingly common dark pool setup: continuous dark pools where orders may be priced away from the midquote, depending on the pricing rule set by the dark pool. In particular, I investigate how the pricing mechanism (the “trade-at rule”) affects visible market quality and price efficiency.

My work is most closely related to Buti, Rindi, and Werner (2014), who also study the impact of dark pools on market quality. They model the dark pool as a crossing network that executes trades at the midquote of the visible market spread. In their model, uninformed traders trade either a large (institutional) or a small (retail) order, at a visible limit order book or a dark pool. Buti, Rindi, and Werner (2014) find that exchange orders migrate to the dark pool, but that there is volume creation overall. They predict that this order migration to the dark pool intensifies for stocks with narrower spreads or greater market depth at the prevailing quotes. For periodic-crossing dark pools, they find that all traders benefit from the availability of a periodic-crossing dark pool when stocks are liquid. When stocks are illiquid, however, institutional traders see welfare gains, while retail traders suffer reduced welfare. For dark pools with continuous crossing, welfare effects are amplified.

Complementing their work, I model a dark pool as a venue where informed traders may trade with a liquidity provider, at a price better than the displayed quote, according to a trade-at rule. A competitive liquidity provider submits limit orders to both the visible and dark market, and investors submit only market orders to the dark pool. Similarly to Buti, Rindi, and Werner (2014), I find that orders migrate to the dark pool, but that additional volume is created only when the trade-at rule demands sufficient improvements over the prevailing visible quotes. If the trade-at-rule is too loose, volume declines.

Zhu (2014) models traders with correlated information who choose between a visible market and a dark pool. He finds that access to a dark pool improves price discovery, because informed traders concentrate their orders on the visible market. The increased adverse selection on the visible exchange then leads to worsening exchange liquidity. Ye (2011) models informed trading in the sense of Kyle (1985), and contrary to Zhu (2014), finds that the availability of a dark pool harms price discovery. My model predicts that a dark market negatively impacts price efficiency, regardless of the trade-at rule.

Menkveld, Yueshen, and Zhu (2014) develop a “pecking order hypothesis” that predicts that when choosing a trading venue (whether a lit or dark venue), investors prefer low-cost (low immediacy) venues,
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but move down to higher-cost (higher immediacy) venues as their immediacy needs dictate. They find support for their hypothesis empirically. My model predicts a similar cost-immediacy trade-off, where investors with more extreme trading needs choose order types with greater immediacy than investors with more moderate trading needs.

I also contribute to a wider theoretical literature on dark trading. Recent works by Boulatov and George (2013), Buti and Rindi (2013) and Moinas (2011) study dark trading within a limit order market via the use of hidden orders. Boulatov and George (2013) examine the differences in market quality between a fully-displayed limit order book, and a fully-hidden limit order book, where informed trading is modelled in the tradition of Kyle (1985). They find that a fully-hidden limit order book entices informed traders to limit orders, and moreover, the increased competition in liquidity provision lowers transaction costs for uninformed traders, in turn improving market quality. Buti and Rindi (2013) and Moinas (2011) consider a limit order market that permits both visible and hidden orders. Both studies find that traders choose to hide their orders to reduce exposure costs.

My predictions may also explain some of the seemingly contradictory results in the empirical literature. Comerton-Forde and Putniński (2015) analyze Australian exchange and dark pool data. They find that dark trades contain information, but that those who migrate to the dark are less informed than those traders that remain on the exchange. The increase in informed trading on the exchange worsens liquidity. In the context of my model, their results are consistent with a dark pool that sets a small trade-at rule. Comerton-Forde and Putniński (2015) state that a large majority of dark trades execute at the bid-offer quotes, where liquidity providers trade against incoming investor and institutional orders. Similar to Comerton-Forde and Putniński (2015), Nimalendran and Ray (2012) study U.S crossing networks and find that dark trades have an informational impact on visible market prices. Foley and Putniński (2014) study the impact of a minimum trade-at rule, using the recent minimum price improvement restriction in Canada that requires all dark orders to improve upon the current visible market prices by one tick. They examine their data in the context of one-sided versus two-sided dark trading, and find that a relative increase of two-sided dark trading is beneficial for market quality. Two parallel studies, Comerton-Forde, Malinova, and Park (2015) and Anderson, Devani, and Zhang (2015), produce conflicting results to Foley and Putniński (2014). Both studies find weak or no evidence for a change in overall market quality. In particular, Comerton-Forde, Malinova, and Park (2015) find no evidence of a change in overall market price efficiency or bid-ask spreads, and only weak evidence for declines in 5-minute and 10 minute price impact.

Degryse, de Jong, and van Kervel (2015), Weaver (2014) and Foley, Malinova, and Park (2012) analyze Dutch, U.S. and Canadian data, respectively, and conclude that dark trading negatively impacts market
quality. Using U.S. data, Buti, Rindi, and Werner (2011) find that dark trading improves liquidity. In a laboratory experiment, Bloomfield, O’Hara, and Saar (2014) conclude that dark trading has no impact on liquidity and price efficiency. Degryse, Tombeur, and Wuyts (2014) examine the interrelations between two types of opaque trading: hidden orders on lit venues, and orders on dark venues. They provide evidence that dark trading and hidden orders are substitutes, and that an investor’s choice of an opaque order type depends on the prevailing market conditions.

2.2 Model

I model a financial market where risk-neutral investors enter a market sequentially to trade for either informational or liquidity reasons (as in Glosten and Milgrom (1985)). Investors have access to a visible limit order book and a dark market, at which a competitive, (uninformed) professional liquidity provider supplies limit orders, similar to Brolley and Malinova (2014). The price at which investors trade in the dark market is given by a “trade-at rule” that improves upon the prevailing quote at the visible market. The dark market in my model is similar to several types of dark pools: dark limit order books where liquidity is provided solely by outside liquidity providers (e.g., Alpha Intraspread\(^4\)), ping destinations (dark pools that accept immediate-or-cancel orders from investors; e.g., Citadel, Getco), and internalization pools populated by outside liquidity partners (e.g., Sigma X, Crossfinder).

**Security.** There is a single risky security with an unknown fundamental value, \(V\). The fundamental value follows a random walk, and at each period \(t\) an innovation \(\delta_t\) to the fundamental value occurs, which is uniformly distributed on \([-1, 1]\). The fundamental value in period \(t\) is given by,

\[
V_t = \sum_{\tau \leq t} \delta_{\tau}
\]  

\(2.1\)

**Market Organization.** Traders can access two trading venues: a visible limit order book, or a dark market. The visible limit order book accepts market orders and limit orders. In the dark market, liquidity is supplied by a professional liquidity provider. I denote dark orders submitted by the liquidity provider as those that are ‘posted’ to the dark market. Investors may send their order to the dark market to trade against posted liquidity (if any). In period \(t\), I denote the price of the best-priced buy limit order (submitted in period \(t - 1\)) as \(\text{bid}_t\); for the analogous sell limit order, I denote the best price as \(\text{ask}_t\). Visible limit orders and liquidity posted to the dark market live for one period, after which any unfilled orders are cancelled.\(^5\)


\(^5\)Figure 2.5 in the Appendix diagrammatically illustrates the timing of the model.
An order sent to the dark market by an investor fills with some probability, at a pre-specified price. The probability that an order is filled in the dark is determined endogenously by the probability that the professional liquidity provider posts orders to the dark market in the previous period. The dark market sets the price based on an exogenous trade-at rule, \( \lambda \in (0, 1) \), measured as a fraction of the spread, \( \text{ask}_t - \text{bid}_t \), in the visible market in period \( t \). For example, an investor’s buy order in the dark trades at: \( \text{ask}_t - \lambda \times (\text{ask}_t - \text{bid}_t) \). By modelling the trade-at rule as an improvement on the visible spread, the model effectively imposes price-visibility-time priority, a common practice in industry.

Similar to Foucault (1999), I assume that the security is traded throughout a “trading day” where the trading process ends after period \( t \) with probability \( (1 - \rho) > 0 \), at which point the payoff to the asset is realized. All participants observe the history of all transactions on both markets, as well as quotes and cancellations on the visible market when at the market. I denote this history up to (but not including) period \( t \) as \( H_t \). The public expectation of the security’s fundamental value at period \( t \) conditional on the public history is denoted by \( \nu_t \). The structure of the model is common knowledge among all participants.

**Investors.** At each period \( t \), a single investor randomly arrives at the market from a continuum of risk-neutral investors. With probability \( \mu \in (0, 1) \) the investor is informed, and privately learns the period \( t \) innovation \( \delta_t \). Uninformed investors are endowed with liquidity needs, \( y_t \), uniformly distributed on \([-1, 1]\). Informal investors have no liquidity needs. Upon arriving at the market in period \( t \) and only then, an investor may submit an order for a single unit (round lot) of the security. I assume that, if indifferent between a visible order or a dark order, the investor chooses the visible order. An investor leaves the market forever upon the execution or cancellation of their order.

**Professional Liquidity Provider.** There is a single professional liquidity provider that participates in both markets. The liquidity provider is risk-neutral, does not receive information about the security’s fundamental value, and has no liquidity needs. Within any period \( t \), the liquidity provider updates their orders in response to market information (i.e. changes to the public history) before the arrival of the next investor. I assume that the professional liquidity provider acts competitively, and thus earns zero expected profits, conditional on execution. The liquidity provider maintains a “full” limit order book at

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6Assuming that some investors have liquidity needs is common practice in the literature on trading with asymmetric information, to avoid the no-trade result of Milgrom and Stokey (1982).

7Numerical simulations show that the results are qualitatively robust to densities of the innovation \( \delta_t \), where the innovation density first-order stochastically dominates the density for the private values (i.e., as the average valuations increase, so does the price impact of the order).

8As dark trading has been historically popular for executing large orders, trading in a single unit may seem unnatural. The CFA Institute notes, however, that order and transaction sizes in today’s dark pools are similar to those on visible markets (CFA, 2012). Data from FINRA from July 21-27 2014 finds that for 41 alternative trading systems (ATS), the average trade size is 209 shares.

9See Brolley and Malinova (2014) for a justification of this assumption.
any period $t$ by submitting limit orders to fill any vacancies on either side of the limit order book, at prices $\text{bid}_{t}^{LP}$ and $\text{ask}_{t}^{LP}$ for buy and sell limit orders, respectively.

The professional liquidity provider also posts orders to the dark market. However, because prices for posted dark orders are derived from visible market prices and the trade-at rule, $\lambda$, the liquidity provider does not set the price of their posted dark order. Instead, the liquidity provider chooses the intensity with which they post orders to the dark market, such that their expected payoffs are zero. Dark order prices are denoted by the superscript ‘Dark’.

**Investor Payoffs.** I focus on the payoff to buyers at period $t$; sell orders are symmetrically defined. As notational shorthand, period $t$ market and limit buy orders are denoted $\text{MB}_{t}$ and $\text{LB}_{t}$, respectively. An investor’s dark buy order in period $t$ is denoted $\text{DB}_{t}$. The superscript ‘inv’ denotes limit prices submitted by investors. An investor’s payoff to any order type is the difference between their valuation (their private value $y_{t}$, plus their assessment of the security’s value) and the price paid, discounted by the execution probability. Investors who abstain from trading receive a payoff of zero. The payoffs to each order type are:

$$\pi_{t,\text{inv}}^{\text{MB}} = y_{t} + E[V_{t+1} | \text{info}_{t}, H_{t}] - \text{ask}_{t}^{\text{Dark}}$$  \hfill (2.2)$$

$$\pi_{t,\text{inv}}^{\text{LB}} = \rho \cdot \text{Pr}(\text{fill} | \text{info}_{t}, H_{t}, \text{bid}_{t+1}^{\text{inv}}) \times \left( y_{t} + E[V_{t+1} | \text{info}_{t}, H_{t}, \text{fill at bid}_{t+1}^{\text{inv}}] - \text{bid}_{t+1}^{\text{inv}} \right)$$  \hfill (2.3)$$

$$\pi_{t,\text{inv}}^{\text{DB}} = \text{Pr}(\text{dark fill} | \text{info}_{t}, H_{t}) \times \left( y_{t} + E[V_{t+1} | \text{info}_{t}, H_{t}, \text{fill at bid}_{t+1}^{\text{inv}}] - \text{ask}_{t}^{\text{Dark}} \right)$$  \hfill (2.4)$$

where $\text{info}_{t}$ is the period $t$ investor’s information about the innovation $\delta_{t}$; $\text{Pr}(\text{fill} | \text{info}_{t}, H_{t}, \text{bid}_{t+1}^{\text{inv}})$ and $\text{Pr}(\text{dark fill} | \text{info}_{t}, H_{t})$ are the respective probabilities that a visible limit order is filled, and that a dark order is filled; $E[V_{t+1} | \text{info}_{t}, H_{t}, \text{fill at bid}_{t+1}^{\text{inv}}]$, is the period $t$ investor’s expectation of the fundamental value, conditional on the fill of their limit order.

**Professional Liquidity Provider Payoffs.** I focus on buy orders, with sell orders similarly defined. In any period $t$, the professional liquidity provider observes the period $t$ investor’s action, and then may submits limit orders to the visible market, or post orders to the dark market. At period $t$, a visible buy limit order at price $\text{bid}_{t+1}^{LP}$ earns the payoff,

$$\pi_{t,LP}^{\text{LB}} = \rho \cdot \text{Pr}(\text{fill} | \text{investor action at } t, H_{t}, \text{bid}_{t+1}^{LP}) \times \left( y_{t} + E[V_{t+1} | \text{investor action at } t, H_{t}, \text{fill at bid}_{t+1}^{LP}] - \text{bid}_{t+1}^{LP} \right)$$  \hfill (2.5)$$

I denote period $t$ dark buy orders posted by the liquidity provider as $\text{DLB}_{t}$, and $\text{DLS}_{t}$ for dark sell orders. Given a trade-at rule $\lambda$, the liquidity provider chooses $\text{Pr}(\text{DLS}_{t})$ to satisfy the zero-profit condition,

$$\text{ask}_{t+1}^{\text{Dark}} = E[V_{t+1} | \text{investor action at } t, H_{t}, \text{fill at ask}_{t+1}^{\text{Dark}}]$$  \hfill (2.6)$$
The condition for \( \Pr(DLB) \) is similar.

### 2.3 Equilibrium

I search for a symmetric, stationary perfect Bayesian equilibrium in pure strategies in which the bid and ask prices at the visible market in period \( t \) are competitive with respect to information available to the professional liquidity provider just prior to the arrival of the period \( t \) investor.

#### 2.3.1 Order Pricing Rules

In equilibrium, the professional liquidity provider posts competitive limit orders to the visible market. I use \( ^* \) to denote “in equilibrium”. I denote the equilibrium bid and ask prices by \( \text{bid}^*_t \) and \( \text{ask}^*_t \), respectively, which, by the competitive pricing assumption, are given by:

\[
\text{bid}^*_t = E[V_t \mid H_t, \text{MS}^*_t] = v_t + E[\delta_t \mid \text{MS}^*_t] \tag{2.7}
\]

\[
\text{ask}^*_t = E[V_t \mid H_t, \text{MB}^*_t] = v_t + E[\delta_t \mid \text{MB}^*_t] \tag{2.8}
\]

A limit order posted by an investor in period \( t \) is posted competitively if the period \( t+1 \) investor compensates the period \( t+1 \) investor for the fact that they may have been informed. To compensate the period \( t+1 \) investor, a competitive limit price includes the expected price impact, \( E[\delta_t \mid \text{LB}^*_t] \) of the submitting investor. Hence, an investor’s limit order is posted at:

\[
\text{bid}^*_{t+1} = E[V_{t+1} \mid H_t, \text{MS}^*_{t+1}, \text{LB}^*_t] = v_t + E[\delta_t \mid \text{LB}^*_t] + E[\delta_{t+1} \mid \text{MS}^*_{t+1}] \tag{2.9}
\]

\[
\text{ask}^*_{t+1} = E[V_{t+1} \mid H_t, \text{MB}^*_{t+1}, \text{LS}^*_t] = v_t + E[\delta_t \mid \text{LS}^*_t] + E[\delta_{t+1} \mid \text{MB}^*_{t+1}] \tag{2.10}
\]

Prices in the dark market are derived from (a) the quote in the visible market, and, (b) the trade-at rule, \( \lambda \). Hence, the professional liquidity provider does not choose the price of the dark orders they post, and their prices, \( \text{bid}^{\text{Dark}}_t \) and \( \text{ask}^{\text{Dark}}_t \) are given by,

\[
\text{bid}^{\text{Dark}}_t = \text{bid}^*_t + \lambda \times (\text{ask}^*_t - \text{bid}^*_t) \tag{2.11}
\]

\[
\text{ask}^{\text{Dark}}_t = \text{ask}^*_t - \lambda \times (\text{ask}^*_t - \text{bid}^*_t) \tag{2.12}
\]

Equations (2.11)-(2.12), by symmetry, simplify to \((1 + 2\lambda) \times \text{bid}^*_t \) and \((1 - 2\lambda) \times \text{ask}^*_t \), respectively.

At the dark market, quotes are derived from the visible market, so the liquidity provider chooses only to post or not post a limit order to the dark. However, the prices in the dark must still satisfy the
competitive pricing assumption, and so the liquidity provider chooses the intensity with which they post orders to the dark market such that they earn zero profits. That is, the liquidity provider chooses this intensity such that the price that they must post for their dark limit order is equal to the public value of the security at period \( t-1 \), \( v_{t-1} \), plus the price impact of trading with an investor in the dark at period \( t \). Hence, the price of orders posted to the dark must satisfy:

\[
\text{bid}_t^{\text{Dark}}(\lambda) = \mathbb{E}[V_t \mid H_t, DS_t^*] = v_t + \mathbb{E}[\delta_t \mid DS_t^*] \\
\text{ask}_t^{\text{Dark}}(\lambda) = \mathbb{E}[V_t \mid H_t, DB_t^*] = v_t + \mathbb{E}[\delta_t \mid DB_t^*]
\] (2.13) (2.14)

It is not immediate, however, that an investor’s dark order has a price impact, even post-trade. Because the market only reveals that a trade has occurred in the dark (and not which side is active or passive), it would be intuitive that an investor’s dark order would have no price impact post-trade. However, the liquidity provider participates in both markets, and upon the fill of an order in the dark, they use this information to update their limit orders on the visible market. In this way, trades in the dark have a post-trade price impact. In what follows, unless otherwise noted, the term ‘price impact’ refers to post-trade price impact.

Lastly, all investors form a common prior, \( v_t \) from \( H_t \); it does not, however, appear in the expectations about future innovations, as they are independent of past price history.

### 2.3.2 Investor Decision Rules

An investor will submit an order (to either market) only if, conditional on their information and on the submission of the order, their expected profits are non-negative. Moreover, an investor who submits an order chooses the order type that maximizes their expected profits. Similar to Brolley and Malinova (2014), I restrict my attention to equilibria where investors submit visible limit orders that cannot be improved upon by the professional liquidity provider. That is, where \( \text{bid}_t^{\text{inv}} = \text{bid}_t^{\text{LP}} = \text{bid}_t^* \). Orders with non-competitive prices yield the investor negative profits, or an execution probability of zero. In the Appendix, I discuss the off-the-equilibrium-path beliefs about non-competitive limit orders such that competitive limit order pricing is supported in equilibrium.

Investors choose their order based on available public information, plus: (a) their private valuation \( y_t \) (if uninformed), or; (b) their knowledge of \( \delta_t \), (if informed). Because these valuations enter investor payoff functions identically, I can summarize investor decisions in terms of an investor’s “valuation”, regardless of type. I denote the period \( t \) investor’s valuation by, \( z_t = y_t + \mathbb{E}[\delta_t \mid \text{info}_t] \). \( z_t \) is symmetrically distributed on the interval \([−1, 1]\). Using this notation, I depict the decision tree diagrammatically in Figure 2.1.
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Figure 2.1: Investor Decision Tree: Given their valuation, $z_t$, the period $t$ investor chooses to submit an order to the visible market, dark market, or abstain from trading. If the investor sends an order to the visible market, the investor chooses either a market buy, or a limit buy (with the appropriate competitive price). A market order is filled automatically in period $t$, while a limit order is filled in period $t+1$ with probability $Pr(MS_{t+1})$. An order sent to the dark market, is filled with probability $Pr(DLS_{t-1})$.

Because innovations $\delta_t$ are time-independent, I can decompose an investor’s valuation into two parts: the expected value of the security based on the public information, plus $z_t$,

$$E[V_t | info_t, H_t] = y_t + E[\delta_t | info_t] + E[V_{t-1} | H_t] = v_t + z_t$$

In addition, when submitting a limit buy order to the visible market, an investor incurs an adverse selection cost of trading with a potentially informed investor in period $t+1$, equal to $E[\delta_{t+1} | H_t, LB^*_t, MS^*_{t+1}]$. This cost is identical for informed and uninformed investors, because information about period $\tau \leq t$ innovations does not inform about future innovations. I summarize all the costs of submitting an order as “transaction costs”, which includes: order prices, and, if trading in the following period (with a visible limit order), the expected adverse selection associated from being picked off.

Hence, the payoff expressions (2.2)-(2.4) can be reduced to three components: (a) the investor’s valuation, (b) the order’s transaction costs, and (c) the order’s execution probability. The payoff of any order type, $k$, is then generally characterized by,

$$\pi^k_t = \text{execution probability} \times (z_t - \text{transaction costs})$$

In a symmetric equilibrium, visible market buy orders are guaranteed execution. Limit buy orders submitted at period $t$, however, are triggered by incoming market sell orders submitted by investors, and
thus, their execution probability is dependent on the likelihood that a market sell order is submitted and that the market does not close, $\rho \times \Pr(MS_{t+1})$. Dark buy orders at period $t$ trade against orders posted at period $t-1$, which are posted by the liquidity provider with probability $\Pr(DLS_{t-1})$.

### 2.3.3 Equilibrium Characterization

In this section, I derive properties for visible and dark order types that must hold in a symmetric, stationary equilibrium, governed by the aforementioned pricing and investor decision rules. Moreover, I focus my attention primarily to the cases where all order types—visible market orders, visible limit orders, and dark orders—are used in equilibrium. Each of these properties, in turn, shape the type of equilibria sought, and successively reduce the set of candidate equilibria. Proofs are in the appendix.

**Lemma 1 (Transaction Costs)** *In any symmetric, stationary equilibrium where investors use all order types, an order’s transaction costs are equal to its post-trade price impact.*

When an investor submits an order, their payoff simplifies to expression (2.16), where the term “transaction costs” aggregates the ask or bid price of the order, and all adverse selection costs they expect to incur. In equilibrium, competitive liquidity provision in the visible and dark markets leads transaction costs to equal to the post-trade price impact of the order.

On the visible market, the liquidity provider posts competitive prices. On the dark market, because prices are predetermined, the professional liquidity provider posts “competitive” (zero-expected profit) prices by choosing the probability with which they submit orders to the dark market, such that the investors who submit dark orders have a post-trade price impact that offsets the price at which their order would trade.

**Lemma 2 (Buy, Sell or Abstain)** *In any symmetric, stationary equilibrium where investors use all order types, investors do not sell if $z_t \geq 0$ and do not buy if $z_t \leq 0$. Moreover, any buy (sell) order used in equilibrium has a positive (negative) post-trade price impact.*

Intuitively, investors do not trade in the opposite direction of their valuation (though some investors with non-zero valuations may abstain from trading because transaction costs are too high). It then follows that a buy order moves the mid-quote upward, in the direction of the investors’ valuations, by the post-trade price impact (the average informativeness) of the order; likewise, a sell order pushes the mid-quote downward.

Investors develop their optimal order placement strategy by considering both components of an order type’s payoff—transaction costs and execution probability—given generally in equation (2.16). As
alluded to in Lemma 2, the payoff to any order type must be such that an investor’s valuation is greater than an order type’s post-trade price impact, so that the payoff to that order type is positive. As an order’s post-trade price impact is defined by the expected informativeness of the order, the value of the post-trade price impact acts as a minimum threshold such that investors with valuations below this value will not submit that order type, in equilibrium. Hence, investors consider only order types for which their valuation meets (or exceeds) the post-trade price impact threshold.

Of these order types with post-trade price impacts less than their valuation, investors then weigh the post-trade price impact of an order (i.e., how “cheap” a filled order is) against its execution probability (i.e., how likely their payoff is realized). The result of this process is a threshold strategy, governed by investors’ valuations, $z_t$. In the context of this environment, I define a threshold strategy in the following way.

**Definition 1 (Threshold Strategy)** Let $I$ and $J$ be two order types. A threshold strategy is defined as a set of valuation thresholds $z^I$ and $z^J$, such that for $z^I < z^J$:

1. an investor with valuation $z_t > z^J$ chooses order type $J$;
2. an investor with valuation $z^I \leq z_t \leq z^J$ chooses order type $I$;
3. an investor with valuation $z_t < z^I$ chooses neither.

If an investor’s valuation exceeds the post-trade price impact of the order, then increasing an order’s fill rate scales up the investor’s expected profit from using that order type, as in expression 2.16. Then, investors with the most extreme valuations (those away from zero) have the most to gain from increased execution probability, but consequently, are the most informed (on average), and hence contribute to high price impact. This generates a type of “order submission hierarchy”: those with the most extreme valuations opt for orders with the highest fill rate, but in turn, increase the post-trade price impact of the order type such that investors with a valuation below some threshold prefer to take a chance on order types with lower execution probability, in exchange for lower post-trade price impact. Lemma 3 formalizes this intuition.

**Lemma 3 (Threshold Strategy and Execution Probability)** In any symmetric, stationary equilibrium where investors use all order types, investors use a threshold strategy, as described in Definition 1. Moreover, the (absolute) valuation thresholds $|z^J|$ are increasing with the order’s execution probability.

Lemma 3 is a similar result to Hollifield, Miller, and Sandás (2004), who predict that the more extreme an investor’s valuation, the higher the execution probability of the order they submit. They test this
prediction empirically, as a monotonicity restriction on investors’ order submission strategies, and find support when the choices of buy and sell orders are considered separately.

Taken together, Lemmas 1-3 describe how investors weigh order fill rates and price impact when optimally choosing an order type. The higher an investor’s valuation, the larger price impact they can absorb in exchange for better execution.

**Corollary 1 (Execution Probability and Price Impact)** In any symmetric, stationary equilibrium where investors use all order types, if an order $I$ has higher execution probability than order $J$, then $I$ must also have a greater (absolute) post-trade price impact than $J$.

Corollary 1 provides an intuition similar to the pecking order hypothesis of dark order types, coined by Menkveld, Yueshen, and Zhu (2014). They generate a pecking-order result in a model with symmetric information: investors with high private valuations trade against quotes on the visible market (using market orders) where they are guaranteed execution, whereas investors with lower private valuations opt to send orders to a dark market to receive a price improvement, at the expense of higher execution uncertainty. Here, I show that the result is preserved in a model with asymmetric information; an investor weighs an order’s execution likelihood against its transaction cost—the latter of which is equivalent to the order’s post-trade price impact.

### 2.3.4 Equilibrium Existence

In what follows, I drop all time subscripts, as I focus on stationary equilibria. Lemmas 2-3 reduce the set of candidate equilibria to threshold strategies that are increasing in order execution probability: investors with higher valuations choose order types with higher likelihood of execution. It also follows from Lemma 3 that investors with the most extreme valuations submit visible market orders, because visible market orders are guaranteed execution, by virtue of the professional liquidity provider ensuring the visible limit order market is always full. Investors with valuations closest to zero abstain from trading, which loosely fits the narrative where, the closer an investor’s valuation is to the public prior, the less likely they are to trade. In the case of these investors who abstain from trading, not submitting an order is, in a sense, facing an “execution probability” of zero.

The remaining candidate equilibria classify the behaviours of investors with valuations not too close to zero, but not too far from zero. In keeping with the previous narrative, the two possibilities are: (a) the execution probability of a dark order is lower than that of a visible limit order, implying that the equilibrium thresholds are $0 \leq z^D \leq z^L \leq z^M \leq 1$, and; (b) the execution probability of a
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Abstain Limit Sell Market Buy Market Sell Limit Sell Abstain Limit Buy Market Buy

Figure 2.2: Benchmark Equilibrium: In the benchmark equilibrium described by Theorem 1, an investor with valuation \( z_t \in [-1, 1] \) optimally chooses their order type (market order, limit order, abstain from trade) based on the region their valuation falls into, as illustrated by the figure above.

dark order is higher than that of a visible limit order, implying that the equilibrium thresholds are \( 0 \leq z^L < z^D < z^M \leq 1 \).

Benchmark Equilibrium

I examine the model from the perspective of introducing a dark market alongside a visible limit order market. In the absence of a dark market, the model follows Brolley and Malinova (2014) which has unique equilibrium where investors use both market order and limit orders, when the innovations \( \delta \) are drawn from a uniform distribution. I refer to this as the benchmark equilibrium.

**Theorem 1 (Benchmark Equilibrium)** If investors may only access the visible market, there exist values \( z^M_B \) and \( z^L_B \), where \( 0 < z^L_B < z^M_B < 1 \) that constitute a symmetric, stationary equilibrium in threshold strategies (as in Definition 1).

Figure 2.2 illustrates how an investor with valuation \( z \) behaves in the benchmark equilibrium.

Now, consider introducing a dark market with a trade-at rule \( \lambda \) to the model. The benchmark equilibrium can be an equilibrium for any trade-at rule \( \lambda \in (0, 1) \), if the professional liquidity provider chooses not to provide liquidity to the dark (\( \Pr(DLS^*) = \Pr(DLB^*) = 0 \)). To see how, conjecture that an equilibrium with a dark market exists where \( z^L_* = z^L_B, z^M_* = z^M_B \) (as in Theorem 1), \( z^D_* = z^L_B \), and \( \Pr(DLS^*) = \Pr(DLB^*) = 0 \).

Since the professional liquidity provider leaves the dark market empty, any order submitted to the dark would earn zero profits. Then, investors who submit visible limit orders or market orders (\( z \in [z^L_*, 1] \)) do not have an incentive to deviate from their strategy. Moreover, investors who abstain from trading, \( z \in [0, z^D_*] \), also earn zero profits, and thus will not deviate from their strategy.

An investor with \( z = z^D_* = z^L_B \) also has no incentive to deviate from the conjectured strategy, as they are indifferent to visible limit orders, dark orders and not trading, earning zero profits from any of the three actions. Then, because the group of investors that sends orders to the dark is of measure zero,
any limit orders sent to the dark have an execution probability of zero. Hence, the professional liquidity provider does not benefit from changing his liquidity provision strategy in the dark (he earns zero profit in either case).

The conjectured equilibrium above is hence equivalent to the benchmark equilibrium, as all investors behave as in Theorem 1, except for investors with \( z = z^{D^*} \), who are of measure zero. The following corollary summarizes the discussion.

**Corollary 2** For any \( \lambda \in (0,1) \), an equilibrium exists where the professional liquidity provider chooses \( \Pr(DLS^*) = \Pr(DLB^*) = 0 \), and the equilibrium thresholds satisfy \( z^{M^*} = z^M_B, \ z^{L^*} = z^L_B, \) and \( z^{D^*} = z^L_B \).

Because I focus on equilibria where investors use the dark market, it is useful to determine for which trade-at rules—if any—that the benchmark equilibrium may be the unique equilibrium (i.e., investors who send orders to the dark are of measure zero). First, applying symmetry to the prices of orders that the liquidity provider posts to the dark in (2.11)-(2.12), we can rewrite the price of a sell order posted to the dark as, \( \text{ask}^{\text{Dark}} = (1 - 2\lambda) \times \text{ask}^* \). The range of prices that the liquidity provider might post a dark sell order at, given trade-at rule \( \lambda \), is then bounded by \((-\text{ask}, \text{ask})\).

Consider any trade-at rule \( \lambda \geq \lambda = 1/2 \). These trade-at rules imply \( \text{ask}^{\text{Dark}} \in (-\text{ask}, 0) \), which prices sell orders posted to the dark at midquote or better (i.e., equal to or lower than the public value \( v \)). But, by Lemma 2, an investor’s dark buy order has a positive post-trade price impact, and hence, the liquidity provider would expect to earn a loss when trading against it. Thus, the liquidity provider chooses \( \Pr(DLS^*) = \Pr(DLB^*) = 0 \), when \( \lambda \in (\lambda, 1) \).

The set of trade-at rules that may yield an equilibrium with dark trading reduces to \((0, \lambda)\). Now consider the investor’s problem. Is there a trade-at rule (or rules) where we can introduce a dark market with a trade-at rule that prices dark orders such that investors have no incentive to deviate from the equilibrium strategy in Theorem 1, even with the liquidity provider sending orders with some positive frequency to the dark?

If we introduce a dark market with a trade-at rule \( \lambda^* \) such that an order sent to the dark by an investor is identical in every payoff-relevant aspect to submitting a visible limit order, investors have no incentive to submit dark orders by the assumption that investors choose to send orders to the visible market when they are indifferent between a visible order and a dark order. This scenario is equivalent to a trade-at rule \( \lambda^* \) such that the transaction costs from sending an order to the dark market versus submitting a visible limit order satisfy,
(1 − 2λ∗) × E[δt | MB∗, zM∗ = zB∗] = E[δt | LB∗, zM∗ = zB∗, zL∗ = zB∗] (2.17)
\iff λ∗ = \frac{1 - zL∗}{2(1 + zB∗)} (2.18)

When the transaction costs of a dark order and a visible limit order are equal, any
Pr(DLS∗) ≤ ρPr(MSB) chosen by the liquidity provider implies that dark orders are as attractive (or less attractive)
than visible limit orders. To ensure the zero expected profit condition of the liquidity provider is satisfied
for liquidity provided to the dark market, we require that
(1 − 2λ∗) × E[δt | MB∗, zM∗ = zB∗] = E[δ | DB∗].

But, because no investors would submit orders to the dark market on-the-equilibrium-path, the post-
trade price impact of a dark order, E[δ | DB∗], is defined only by off-equilibrium path beliefs. I assume
that these beliefs are such that E[δ | DB∗] = (1 − 2λ∗) × E[δt | MB∗, zM∗ = zB∗], which satisfies the
liquidity provider’s zero profit condition. Hence, if the liquidity provider posts dark limit orders with
the frequency, Pr(DLS∗) = Pr(DLB∗) ≤ Pr(MS∗), the resulting equilibrium actions by investors is as in
the benchmark equilibrium.

We can now use the fact that λ = λ∗ yields the benchmark equilibrium to say something about the
equilibrium types possible for λ ∈ (0, λ∗) and λ ∈ (λ∗, 3).

**Lemma 4 (Trade-at Rule)** In any equilibrium where all order types are used, equilibrium thresholds
must satisfy:

1. λ ∈ (0, λ∗) ⇒ 0 ≤ zL∗ ≤ zB∗ ≤ zD∗ ≤ zM∗ ≤ 1
2. λ ∈ (λ∗, 3) ⇒ 0 ≤ zD∗ ≤ zB∗ ≤ zL∗ ≤ zM∗ ≤ 1

Lemma 4 illustrates that, when the trade-at rule is large enough, the only possible equilibrium with
dark trading is one in which investors with valuations closer to zero (and thus, with smaller post-trade
price impact) trade in the dark, compared to those who use visible limit orders. For a smaller trade-at
rule, investors who prefer dark trading have a larger post-trade price impact than those who use visible
limit orders.

Finally, there exists a λ ∈ (0, λ∗) such that for any λ ∈ (0, 3), there is no positive liquidity provision
strategy that satisfies the liquidity provider’s zero profit condition.

In the following proposition, I summarize the discussion on the regions of λ for which no equilib-
rium with dark trading exists, and thus, the unique equilibrium (in pure strategies) is the benchmark
equilibrium described in Theorem 1.

**Proposition 1 (No Dark Trading)** There exist unique λ ∈ (0, λ∗) such that for,

1. λ ∈ (0, λ), the liquidity provider chooses Pr(DLS∗) = Pr(DLB∗) = 0
(2) \( \lambda = \lambda^* \), the liquidity provider chooses \( \Pr(DLS^*) = \Pr(DLB^*) = 0 \)

(3) \( \lambda \in (\lambda^*, 1) \), the liquidity provider chooses \( \Pr(DLS^*) = \Pr(DLB^*) = 0 \)

Moreover, in each case, \( z_M^* = z_B^* \), and \( 0 \leq z_D^* \leq z_L^* = z_B^* \).

**Dark Orders with a Small Trade-at Rule.**

If the dark market has a trade-at rule \( \lambda \in (\lambda^*, 1) \)—a “small trade-at rule”—then dark orders have higher transaction costs than visible limit orders, by Lemma 4 and Corollary 1. For dark orders to be used by investors in equilibrium, the professional liquidity provider must compensate investors for these higher transaction costs by ensuring that the fill rate of an investor’s dark order is greater than the fill rate of a visible limit order, \( \Pr(DLS^*) \geq \rho \Pr(MS^*) \). Lemma 4 further dictates that equilibrium thresholds must satisfy \( 0 < z_L^* \leq z_D^* \leq z_M^* \leq 1 \). Thus, I look for equilibrium valuation thresholds where an investor with valuation: i) \( z_M^* \) is indifferent to market orders and dark orders; ii) \( z_D^* \) is indifferent to dark orders and limit orders, and; iii) \( z_L^* \) is indifferent to limit orders and abstaining from trade. The professional liquidity provider must also earn zero expected profits, implying that the premium he charges (in excess of the public value) for his dark limit orders must equal the post-trade price impact that he expects to incur. These equilibrium indifference equations are respectively given by,

\[
Z_{\lambda<\lambda^*}^M \equiv z_M - \mathbb{E}[\delta | MB^*] - \Pr(DLS) \times (z_M - (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*]) \tag{2.19}
\]

\[
Z_{\lambda<\lambda^*}^L \equiv \Pr(DLS) \times (z_D - (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*]) - \rho \cdot \Pr(MS^*) \times (z_D - \mathbb{E}[\delta | LB^*]) \tag{2.20}
\]

\[
Z_{\lambda<\lambda^*}^D \equiv z_L - \mathbb{E}[\delta | LB^*] \tag{2.21}
\]

\[
Z_{\lambda<\lambda^*}^D \equiv (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*] - \mathbb{E}[\delta | DB^*] \tag{2.22}
\]

Setting equations (2.19)-(2.22) equal to zero, I find that there exist \( z_M^* \), \( z_L^* \) and \( z_D^* \) such that \( 0 \leq z_L^* \leq z_D^* \leq z_M^* \leq 1 \), and a dark order fill rate \( \Pr(DLS^*) \geq \rho \Pr(MS^*) \), that solve the system (2.19)-(2.22), yielding the following equilibrium.

**Theorem 2 (Small Trade-at Rule)** There exist values \( z_M^* \), \( z_L^* \) and \( z_D^* \), where

\[
0 \leq z_L^* \leq z_D^* \leq z_M^* \leq 1, \quad \text{and} \quad \Pr(DLS^*) > \rho \Pr(MS^*)
\]

that constitute a symmetric, stationary equilibrium in threshold strategies (as in Definition 1), if and only if \( \lambda \in (\lambda^*, 1) \).

Figure 2.3 illustrates how an investor with valuation \( z \) behaves in equilibrium, when choosing between a visible market and a dark market with a small trade-at rule.
Figure 2.3: Small Trade-at Rule: In an equilibrium where a dark market offers a trade-at rule $\lambda \in (\lambda^*, \lambda)$ as described by Theorem 2, an investor with valuation $z \in [-1,1]$ optimally chooses their order type (market order, limit order, dark order, abstaining from trade) based on the region their valuation falls into, as illustrated by the figure above.

**Dark Orders with a Large Trade-at Rule.**

If the trade-at rule is equal to $\lambda \in (\lambda^*, \lambda)$—a “large trade-at rule”—then dark orders have lower transaction costs in equilibrium than visible limit orders, from Lemma 4 and Corollary 1. When the trade-at rule is large, dark orders provide substantial price improvement over posted prices, making them attractive relative to visible limit orders in terms of transaction costs. Because the transaction costs of dark orders are determined in equilibrium to ensure that the professional liquidity provider breaks even, they can only break even when posting to the dark if they disincentivize moderate-valuation investors (who have, on average, larger post-trade price impact) from sending orders to the dark, instead attracting only low-valuation investors. To do so, the professional liquidity provider offers a lower fill rate for dark orders relative to limit orders, $\Pr(\text{DLS}^*) \leq \rho \Pr(\text{MS}^*)$.

Lemma 4 dictates that equilibrium thresholds must satisfy $0 \leq z^{D*} \leq z^{L*} \leq z^{M*} \leq 1$. Similar in method to a small trade-at rule, I look for valuation thresholds where, in equilibrium, an investor with valuation: i) $z^M$ is indifferent to market orders and limit orders; ii) $z^L$ is indifferent to limit orders and dark orders, and; iii) $z^D$ is indifferent to dark orders and abstaining from trade. Again, the professional liquidity provider’s zero profit condition must hold. These equilibrium indifference equations are respectively given by,

\begin{align}
Z^{\gamma}_{\lambda > \lambda^*} &\equiv z^M - E[\delta \mid \text{MB}^*] - \rho \cdot \Pr(\text{MS}^*) \times (z^M - E[\delta \mid \text{LB}^*]) \\
Z^{\gamma}_{\lambda > \lambda^*} &\equiv \rho \cdot \Pr(\text{MS}^*) \times (z^L - E[\delta \mid \text{LB}^*]) - \Pr(\text{DLS}) \times (z^L - (1 - 2\lambda) \times E[\delta \mid \text{MB}^*]) \\
Z^{\delta}_{\lambda > \lambda^*} &\equiv z^D - (1 - 2\lambda) \times E[\delta \mid \text{MB}^*] \\
Z^{\delta}_{\lambda > \lambda^*} &\equiv (1 - 2\lambda) \times E[\delta \mid \text{MB}^*] - E[\delta \mid \text{DB}^*]
\end{align}

(2.23) (2.24) (2.25) (2.26)

Similar to the small trade-at rule case, by equating the indifference equations in (2.23)-(2.26) to zero, I find that there exist $z^{M*}$, $z^{L*}$ and $z^{D*}$ such that $0 \leq z^{D*} \leq z^{L*} \leq z^{M*} \leq 1$, and a dark order fill rate
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Figure 2.4: Large Trade-at Rule: In an equilibrium where a dark market offers a trade-at rule $\lambda \in (\lambda^*, \bar{\lambda})$ as described by Theorem 3, an investor with valuation $z_0 \in [-1, 1]$ optimally chooses their order type (market order, limit order, dark order, abstaining from trade) based on the region their valuation falls into, as illustrated by the figure above.

Pr(DLS$^*$) $\leq$ $\rho$Pr(MS$^*$), that solve the system $(2.23)$-$(2.26)$, yielding the following equilibrium.

Theorem 3 (Large Trade-at Rule) There exist values $z^{M^*}$, $z^{L^*}$ and $z^{D^*}$, where $0 \leq z^{D^*} \leq z^{L^*} \leq z^{M^*} \leq 1$ and Pr(DLS$^*$) $\leq$ $\rho$Pr(MS$^*$) that constitute a symmetric, stationary equilibrium in threshold strategies (as in Definition 1), if and only if $\lambda \in (\lambda^*, \bar{\lambda})$.

Figure 2.4 illustrates how an investor with valuation $z$ behaves in equilibrium, when choosing between a visible market and a dark market with a small trade-at rule.

2.4 The Impact of Trade-at Rules

In this section, I analyze the impact of dark trading on market quality and investor welfare. I do so by comparing market quality and welfare measures when investors use a dark market (with either a small or a large trade-at rule) to the levels under the benchmark equilibrium. As before, the classification of trade-at rule level as large or small is relative to the benchmark trade-at rule, $\lambda^*$. The following subsection focuses on aspects of market quality; subsections 2.4.2 and 2.4.3 discuss price efficiency and welfare, the results for which are numerical.

2.4.1 Market Quality Measures

Trading volume in the context of my model has two components: visible and dark market volume. Visible market volume is measured by the probability that an investor submits either a buy or sell market order to the visible market (these orders are always filled). Because dark orders have some level of execution risk, dark market volume is therefore measured by the probability that an investor submits an order to the dark market and that the order will be filled. Total trading volume at period $t$ is thus measured as,

\[
\text{Trading Volume}_t = 2 \times (\text{Pr(MB}_t) + \text{Pr(DLS}_{t-1}) \times \text{Pr(DB}_t)) \tag{2.27}
\]
where the factor of 2 accounts for sell orders. Overall market participation measures the likelihood that an investor who arrives at the market in period \( t \) submits an order of some type (i.e., does not abstain from trading), which is measured by:

\[
\text{Market Participation}_t = 2 \times (\text{Pr}(MB_t) + \text{Pr}(DB_t) + \text{Pr}(LB_t))
\]  

(2.28)

**Proposition 2 (Visible Market Volume and Market Participation)** Compared to the visible market only equilibrium, introducing a dark market with a small trade-at rule, decreases volume on the visible market; a large trade-at rule increases visible market volume. Market participation (weakly) increases for any trade-at rule.

**Numerical Observation 1 (Total Volume)** Compared to the visible market only equilibrium, introducing a dark market with a small trade-at rule reduces total volume; a large trade-at rule increases total volume.

Figure 2.7 summarizes the findings of Proposition 2 and Numerical Observation 1 graphically.

I find that a dark venue with a large trade-at rule improves market quality by introducing an attractive (i.e., affordable) order type for low valuation investors. Investors who would be indifferent to visible limit orders and not trading without dark trading, now elect to trade with dark orders, increasing the price impact of visible limit orders. As a result, visible limit order submitters who would be indifferent to market and limit orders without a dark market, now submit visible market orders with the availability of a dark market (See Figure 3). Thus, there is net order and volume creation.

A dark market with a small trade-at rule offers lower execution risk versus visible limit orders. Dark orders then induce visible market orders submitters with the lowest valuations, and visible limit orders submitters with the highest valuations (in a ‘visible market only’ setting) to migrate to the dark market. The result is volume migration from the visible market to the dark market. Because dark orders fill less often than visible market orders, the result is a reduction in total volume.

My results on visible market volume and total volume provide some guidance for further empirical research. In their 2010 Concept Release, the SEC takes as given that dark trading siphons volume away from visible markets, as a premise to their concern about price efficiency loss from increased dark trading. Proposition 2 suggests, however, that the impact of dark trading on volume is dependent on the role that dark orders play in the “cost-immediacy pecking order”—to borrow from Menkveld, Yueshen, and Zhu (2014)—alluded to by Lemma 3 and Corollary 1. My model predicts that, if dark orders offer a higher fill rate than visible limit orders (i.e., with a small trade-at rule), then the dark

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market will lead investors to migrate to the dark. However, if dark orders offer a lower fill rate than visible limit orders (i.e., with a large trade-at rule), the model predicts increased visible market volume, as limit order submitters migrate to visible market orders (as well as dark orders).

I also consider the impact of a dark market on visible market transaction costs, measured in my model by the quoted (half) spread.¹¹ The results are illustrated graphically in Figure 2.8.

**Proposition 3 (Quoted Spread / Visible Market Order Price Impact)** Compared to the visible-market-only equilibrium, introducing a dark market with a small trade-at rule widens the quoted spread (market order price impact increases). A large trade-at rule, narrows the quoted spread (market order price impact decreases).

Dark orders subject to a large trade-at rule act as a lower (transaction) cost alternative to visible limit orders for low-valuation investors, who migrate to dark orders. Visible limit orders become more informative on average, increasing their price impact, and making them less attractive to those with the highest valuations who would submit limit orders in the case with no dark market. These marginal limit order submitters now prefer market orders, decreasing the average informativeness of visible market orders. The end result is a decline in the price impact of market orders, and a narrowing of the bid-ask spread.

Conversely, a small trade-at rule dark order provides an alternative for marginal market order submitters that has a higher execution probability; marginal market order submitters then prefer dark orders, increasing the average informativeness of visible market orders, thereby widening the quoted spread and increasing the price impact of visible market orders.

### 2.4.2 Price Efficiency

Price impact measures the post-trade impact on the public expectation of the security. However, because of noise from uninformed investors, the change in the public expectation does not coincide with the innovation to the true value of the security. Thus, one can think of price efficiency in this framework as the ex-ante the mean squared difference between the innovation’s value and the investor’s expected price impact (per period). More formally,

\[
\text{Price Efficiency} = E[(\delta - (1 - \mu) \times E[\delta | \text{uninformed}]) - \mu \times E[\delta | \text{informed}])^2]
\]

(2.29)

That is, for each possible innovation draw, \(\delta_t\) there is an expected price impact from the incoming

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¹¹In this framework, the bid-ask spread is synonymous with the price impact of a market order, as the spread measures the adverse selection costs due to information of a trade on the visible market, which occurs when a visible market order is placed. For better illustration, I use the quoted half-spread measure (symmetry implies that the bid-ask spread is equal to twice the quoted half-spread)
investor’s action, and this price impact may or may not be equal to the value of the innovation (and in the case of uninformed investors, may not even be in the same direction). I then measure price efficiency as the expected deviation from the value of the innovation each period. Because, for any innovation value, an uninformed trader’s expected price impact is zero, the term \( E[\delta | \text{uninformed}] = 0 \). Then, we can re-write (2.29) as,

\[
\text{Price Efficiency} = 2 \times \int_{MB} \left( \delta - \mu \times \frac{E[\delta | MB^*]}{2} \right)^2 d\delta + 2 \times \int_{LB} \left( \delta - \mu \times \frac{E[\delta | LB^*]}{2} \right)^2 d\delta \\
+ 2 \times \int_{DB} \left( \delta - \mu \cdot \Pr(DLS^*) \times \frac{E[\delta | DB^*]}{2} \right)^2 d\delta \tag{2.30}
\]

where the factors of two account for symmetry. The following result is numerical. Figure 2.9 presents it graphically.\(^{12}\)

**Numerical Observation 2 (Price Efficiency)** *Compared to the visible market only equilibrium, price efficiency worsens with any trade-at rule where investors use the dark market.*

As the general usage of dark orders increases, price efficiency falls because investors migrate from visible orders, to an order that only impacts prices when it is filled. Moreover, in both cases, the mass of investors that migrate to the dark market is larger than those who would participate only in a market with dark market trading (they previously made no contribution to price efficiency). The combined result is a reduction in price efficiency.

### 2.4.3 Social Welfare

Uninformed investors earn gains from realizing their private valuation. Following the work of Bessembinder, Hao, and Lemmon (2012), I use private valuations to define social welfare as a measure of allocative efficiency. If a transaction occurs in period \( t \), then the social welfare gain is given by the private valuation of a buyer minus the private valuation of a seller.

A trade occurs when the period \( t \) investor submits either a market order to the visible market, or an order to the dark market and the order is filled. I measure the total expected gains from trade in period \( t \), by the expected private valuation of the investor submitting the market order and the expected welfare of the counter-party, discounted by the probability of an order being filled. Thus, the total expected

\(^{12}\text{Figure 2.9 uses zero as the informationally efficient benchmark, and larger deviations from zero, correspond with less informationally efficient prices.}\)
welfare in period $t$ is given by,

$$W_t = 2 \cdot \Pr(MB_t) \cdot (\mathbb{E}[y_t \mid H_t, MB_t] - \Pr(LS_{t-1}) \cdot \mathbb{E}[y_{t-1} \mid H_{t-1}, MB_t, LS_{t-1}])$$

Expression (2.31) describes total expected welfare per period as the expected private value realized in period $t$, conditional on a visible or dark trade, and discounted by the likelihood of each trade type. The first term describes the (discounted) total expected welfare of a visible market trade: the expected private valuation from the market order submitter, and the expected private valuation of the limit order submitter (which is non-zero in expectation for the investor, but zero for the professional liquidity provider). The second term describes the (discounted) total expected welfare of a dark trade. The factors of two in expression (2.31) account for the symmetry of buyer and seller trades.

Numerically, I find that introducing dark trading alongside the visible market has the following effect on expected social welfare, $W_t$ (see Figure 2.10).

**Numerical Observation 3 (Social Welfare)** Compared to the visible market only equilibrium, introducing a dark market with a small trade-at rule reduces social welfare. A large trade-at rule improves social welfare.

A dark market with a large trade-at rule improves welfare by encouraging greater use of visible market orders from investors who use visible limit orders (implying more frequent trades). This occurs because the transaction costs of a limit order increases when the low-value limit order submitters migrate to the dark market. Moreover, the dark market attracts investors who would otherwise not trade in a purely visible market. Hence, private valuations are realized more frequently per order (more visible market orders) and more frequently per investor (higher market participation). This result suggests that when the transaction costs of dark orders are low, an intermediated dark market is socially beneficial.

I find that not all dark markets are welfare-improving, however. A dark market with a small trade-at rule leads to fewer visible market orders in equilibrium. Despite more limit order submitters now migrating to the dark market (where execution risk is lower), the likelihood that an investor’s private valuation is realized, declines. While more investors are able to participate in the market because of the availability of dark trading, the decline in visible market volume exceeds the increase in market participation, leading to the decline in social welfare.
2.5 Policy Implications and Empirical Predictions

In Section 2.4, my model predicts that the impact of dark trading on visible market quality and social welfare depends on the trade-at rule in the dark market. More importantly, a minimum trade-at rule has the potential to improve welfare and visible market quality. A minimum trade-at rule of $\lambda = \lambda^*$ would make it unprofitable for a dark market to attract moderate valuation investors with a small trade-at rule (which implies low execution risk). Instead, the dark market may attract low valuation investors, contributing positively to both market quality and investor welfare.

The minimum trade-at rule discussion has important implications for equity markets in Canada and Australia. On October 15th, 2012, the Investment Industry Regulatory Organization of Canada (IIROC) implemented a trade-at rule that requires most dark orders to provide a meaningful price improvement of one trading increment (or in the case of a one tick spread, half the spread). Australia followed suit with a similar rule on May 26th, 2013. Since many liquid securities operate at a spread of one or two ticks, the minimum trade-at rule effectively implies dark trades must occur at midpoint. The security that I model is reflective of this type of security, as investors trade in single units (i.e., never walk the book), and the professional liquidity provider ensures that the book is always full.

My model predicts that a minimum trade-at rule requiring dark pools to execute orders at midpoint would eliminate all market-making activities in dark markets, a result that is supported empirically by Comerton-Forde, Malinova, and Park (2015). Setting a less-restrictive minimum trade-at rule would lead to liquidity provision that results in the dark market being beneficial to market quality and welfare. To determine what this type of trade-at rule would be, my model suggests that we need to understand the price impact of limit orders on the visible market. Given an estimate of visible limit order price impact, any minimum trade-at rule that ensures dark orders have a smaller price impact would (at least weakly) improve market quality and welfare.

2.6 Concluding Remarks

In this paper, I study how trade-at rules in dark pools impact market quality and investor welfare in equity markets. I construct a model where both informed and uninformed investors can access a dark market and a visible limit order market. Moreover, the pricing decision for visible limit orders is simplified, as a professional liquidity provider ensures competitive visible limit order pricing.

The main result is that a dark market’s trade-at rule matters for market quality and welfare, but that

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13See Investment Industry Regulatory Organization of Canada (2011)
the price impact of visible limit orders play a key role in determining their impact. A dark market with a trade-at rule that prices dark orders better the price impact of visible limit orders improves market quality and social welfare. The dark market trade-offs their discount pricing with higher execution risk. The “high discount” dark market attracts investors from both the pool of investors that submit visible limit orders, as well as investors who would otherwise abstain from trading. A small trade-at rule that prices a dark order above the price impact of a visible limit order incentivizes the professional liquidity provider to offer reduced execution risk, relative to the dark market with a large trade-at rule. The execution risk of dark orders facing a small trade-at rule is lower than the execution risk of visible limit orders, and this leads to orders of investors with moderate valuations migrating to the dark market. Orders migrate from both the pools of investors submitting visible market orders and visible limit orders. Consequently, market quality and social welfare decline. I find that while adding a dark market improves market participation in all cases, price efficiency declines, as fewer investors submit order types that have guaranteed price impact (i.e., dark orders).

These results have implications for minimum trade-at rule regulation. My model predicts that the impact of dark trading on visible market quality and social welfare depends on the trade-at rule of the dark market, and as such, a minimum trade-at rule has the potential to improve welfare and market quality. A minimum trade-at rule of equal to the benchmark trade-at rule would prevent dark markets from attracting investors with moderate valuations, as the liquidity provider would not be willing to provide liquidity with that level of intensity (it would be unprofitable). However, it would be profitable for the dark market to attract low valuation investors by setting a large trade-at rule, an as such, the dark market would contribute positively to both market quality and welfare. To determine what trade-at rules would be beneficial, we must first understand the price impact of visible limit orders—the measurement of which would allow regulators to discern a large from a small trade at rule.

2.7 Appendix

The appendix contains all proofs and figures not presented in the text.

2.7.1 Proofs: Lemmas

Proof (Lemma 1). Recall that the general payoff function is given by,

\[ \text{execution probability} \times (z_t - \text{transaction costs}) \]
In this proof, I show that for each buy order type, the transaction costs are equal to the price impact. The result for sell order types hold by symmetry.

For a market buy order, we take the investor’s payoff to a market buy order from (2.2), and substitute the equilibrium \( \text{ask}^*_t \) from equation (2.8).

\[
\pi_{MB} = v_t + z_t - \text{ask}_t \\
= v_t + z_t - v_t - \mathbb{E}[\delta_t | MB^*_t] \\
= z_t - \mathbb{E}[\delta_t | MB^*_t] 
\]

and hence the transaction cost of a market buy order is equal to its post-trade price impact, \( \mathbb{E}[\delta_t | MB^*_t] \).

For a limit buy orders, simplifying the investor’s payoff function (2.3) by inputting the equilibrium value for \( \text{bid}^*_t+1 \) yields,

\[
\pi_{LB} = \rho \mathbb{Pr}(MS^*)(\mathbb{E}[V_{t+1} | LB_t, MS^*_{t+1}] - \text{bid}^*_t+1) \\
= \rho \mathbb{Pr}(MS^*)(v_t + z_t + \mathbb{E}[\delta_{t+1} | LB_t, MS^*_{t+1}] - v_t - \mathbb{E}[\delta_t | LB_t] - \mathbb{E}[\delta_{t+1} | LB_t, MS^*_{t+1}]) \\
= \rho \mathbb{Pr}(MS^*)(z_t - \mathbb{E}[\delta_t | LB^*_t])
\]

and hence the transaction cost of a limit buy order is equal to its post-trade price impact, \( \mathbb{E}[\delta_t | LB^*_t] \).

The post-trade price impact of dark order follows from the liquidity provider’s zero-profit condition. In equilibrium, the price of an order filled in the dark at period \( t \), \( (1 - 2\lambda) \times \text{ask}^*_t \), is equal to \( \mathbb{E}[V_t | DB^*_t, H_t] \), which simplifies to,

\[
\text{ask}_t - \lambda \times (\text{ask}_t - \text{bid}_t) = \mathbb{E}[V_t | DB^*_t, H_t] \\
\iff v_t + (1 - 2\lambda) \times \mathbb{E}[\delta_t | MB^*_t] = v_t + \mathbb{E}[\delta_t | DB^*_t] 
\]

I can now write the investor’s payoff to a dark order (2.4) as,

\[
\pi_{DB} = \mathbb{Pr}(DLS^*)(v_t + z_t - v_t - (1 - 2\lambda) \times \mathbb{E}[\delta_t | MB^*_t]) \\
= \mathbb{Pr}(DLS^*)(z_t - \mathbb{E}[\delta_t | DB^*_t])
\]

which implies that the (post-trade) price impact of a dark order, \( \mathbb{E}[\delta_t | DB^*_t] \) is equal to the transaction costs of a dark buy order. ■

**Proof (Lemma 2).** I prove this lemma by showing that, in a stationary, symmetric equilibrium it must be the case that any investor who submits a buy order has \( z_t \geq 0 \), and symmetrically for sell orders. Time subscripts are dropped, as I focus on stationary equilibria.
Let $\gamma_I$ denote the fill rate of order type, $I$, and let $p_I$ denote order type $I$’s price impact. Further, let investor $t$ have a valuation equal to $z_t \geq 0$. For any order type $I$, there is a buy and a sell option, IB, and IS, respectively. Then, for any investor $z_t$, a buy order of type $I$ is preferred to a sell order of type $I$ if and only if:

$$
\gamma_{IB} \times (z_t - p_{IB}) \geq \gamma_{IS} \times (z_t - p_{IS})
$$

(2.39)

In any symmetric equilibrium, $\gamma_{IB} = \gamma_{IS}$, and $p_{IB} = -p_{IS}$. Then, equation (2.39) becomes $z_t \geq 0$, implying that no investor with $z_t \geq 0$ would prefer IS to IB.

It follows, then, that for any buy order type IB to be used in equilibrium, the price impact $p_{IB}$ must be positive. To see this, suppose instead that $p_{IB} < 0$. It must be then, by symmetry, that $p_{IS} > 0$. Now, because $p_{IB}$ describes the average informativeness of investors who submit orders of type IB in equilibrium, it must be the case that some investor with $z_t < 0$ submits orders of type IB. But from the previous argument, any investor with $z_t < 0$ must prefer IS to IB. A contradiction. Thus, in any equilibrium, buy orders that are used by some investors must have a positive price impact, $p_{IB} > 0$, and symmetrically for sell orders.

Lastly, I show that investors who choose buy orders of type $I$ do not prefer any other sell order type $J$, in equilibrium. Consider then two order types, $I$ and $J$. Again, symmetry allows us to consider a buy order of type $I$, and a sell order of type $J$, with the reverse following analogously.

That investors with $z_t \geq 0$ will not use any sell order type, in equilibrium, follows from the argument above. Suppose that an investor $z_t \geq 0$ prefers a sell order of type JS to a buy order, IS. By the argument above, this investor must prefer order type JB to JS. Note, finally, that we have only shown that investors with $z_t \geq 0$ do not submit sell orders in equilibrium. Because $p_{IB} > 0$, any investor with $z_t \in (0, p_{IB})$ will prefer to abstain from trading, or prefer another buy order type $J$, with $p_{JB} < p_{IB}$.

The argument for investors not using buy orders if $z_t \leq 0$ follows by symmetry.

**Proof (Lemma 3).** Once again, let $\gamma_I$ denote the fill rate of order type, $I$, and let $p_I$ denote order type $I$’s price impact. In this proof, the buy and sell notation is dropped, as I focus on the decisions between buy order types. The decisions for sell order types proceeds analogously. I will show that in any symmetric, stationary equilibrium where all order types are used, investors use a threshold strategy such that an investor with valuation $z_t \geq z_J > z_K$ prefers order type $J$ to order type $K$. Moreover, if $\gamma_J \geq \gamma_K$, then $z_J \geq z_K$ for all $p_J, p_K$.

Suppose there is an order type $J$ such that, in equilibrium, $\gamma_J \geq \gamma_K$ and $p_J < p_K$. Suppose order type $K$ be used by some investor, $t$, in equilibrium (investor $t$ earns non-zero profits). Then, the profit of investor $t$ from using order type $K$ is,

$$
\pi_t^K = \gamma_K \times (z_t - p_K) < \gamma_J \times (z_t - p_J) = \pi_t^J
$$

(2.40)
But we supposed that investor $t$ chooses order type $K$ in equilibrium; a contradiction. Thus, for an order to be used in equilibrium, there must be no order that has both i) a higher fill rate, and, ii) a lower price impact.

Thus, we must have that for three order types to be used in equilibrium, we must have that
\[ \gamma_I \geq \gamma_J \geq \gamma_K \geq 0, \]
Thus, we must have that for three order types to be used in equilibrium, we must have that
\[ \frac{\gamma_I p_I}{2} \geq \frac{\gamma_J p_J}{2} \geq \frac{\gamma_K p_K}{2} \geq 0 \] (2.41)
Recall that the profit function of order type $I$ for some investor $t$, takes the form,
\[ \pi_I = \gamma_I \times (z_t - p_I) \]
The profit function is linear in $z_t$; moreover, the order with the highest fill rate also has the lowest intercept, $-\gamma_I p_I$. The linearity of $\pi_I$ in $z_t$ implies that since $\gamma_I \geq \gamma_J \geq \gamma_K \geq 0$, if $\pi_I$ crosses $\pi_J$ at some $z_T$, it remains above $\pi_J$ for all $z_t > z_T$. Similarly for $\pi_K$. Moreover, because $\gamma p_I \geq \gamma_J p_J \geq \gamma_K p_K$, for order type $I$ to be used in equilibrium, $\pi_I$ must cross both $\pi_J$ and $\pi_K$. Thus, $\max\{z_J, z_K\} \leq z_I \leq 1$.

Then, the relation in 2.41 implies that for $I$ to be used in equilibrium, it must cross $\pi_J$ and $\pi_K$ at some $z_I < 1$. Likewise, since $\gamma_J \geq \gamma_K \geq 0$, for $J$ to be used in equilibrium, $\pi_J$ must cross $\pi_K$, before $\pi_I$ crosses $\pi_J$. Thus, $z_J \leq z_I$. Lastly, $\pi_K$ crosses the no-trade threshold, $\pi_{NT} = 0$ before $\pi_J$ and $\pi_I$, but at a positive $z_K = \gamma_K p_K$. Hence, $z_K > 0$.

**Proof (Lemma 4).** I prove this lemma in four steps: step 1 shows that, in an equilibrium where all order types are used, given equilibrium valuation thresholds satisfy $0 \leq z^{L*} \leq z^{D*} \leq z^{M*} \leq 1$, then it must be the case that the valuation thresholds also satisfy $0 \leq z^{L*} \leq z^{L_B} \leq z^{D*} \leq z^{M_B} \leq z^{M*} \leq 1$. Then, in step 2, I show that these valuation thresholds can only be an equilibrium for $\lambda \geq \lambda^*$. Steps 3 and 4 follow similarly, showing that for $\lambda \leq \lambda^*$, equilibrium valuation thresholds must satisfy $0 \leq z^{D*} \leq z^{L_B} \leq z^{M*} \leq z^{M_B} \leq z^{M*} \leq 1$.

**Step 1.** To show $0 \leq z^{L*} \leq z^{D*} \leq z^{M*} \leq 1 \Rightarrow 0 \leq z^{L*} \leq z^{L_B} < z^{D*} \leq z^{M_B} \leq z^{M*} \leq 1$, I first make the following simplifying notational substitution.
\[ E[\delta | z_I \leq \delta \leq z_J] \equiv f(z_I, z_J) = \frac{\mu(z_I + z_J)}{2} \] (2.42)

From (2.42), it is immediate that $f(z_I, z_J)$ is increasing in both $z_I$ and $z_J$, a fact I will use throughout the proof. Using this notation, I rewrite the indifference conditions given by equations (2.19)-(2.22) in...
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the following way:

\[
\begin{align*}
z^{M*} - f(z^{M*}, 1) & - \Pr(\text{DLS}) \times (z^{M*} - f(z^{D*}, z^{M*})) = 0 \quad (2.43) \\
\Pr(\text{DLS}^*) \times (z^{D*} - f(z^{D*}, z^{M*})) & - \Pr(\text{MS}^*) \times (z^{D*} - f(z^{L*}, z^{D*})) = 0 \quad (2.44) \\
z^{L*} - f(z^{L*}, z^{D*}) & = 0 \quad (2.45)
\end{align*}
\]

Let \( z^{M*} < z^M_B \) and \( z^{D*} < z^L_B \). Then, \( \Pr(\text{DLS}^*) > \Pr(\text{MS}^*) > \Pr(\text{MS}^B) \), and \( f(z^{D*}, z^{M*}) > f(z^L_B, z^M_B) \).

Moreover,

\[
\Pr(\text{DLS}^*) \times (z^{M*} - f(z^{D*}, z^{M*})) > \Pr(\text{MS}^B) \times (z^{M*} - f(z^L_B, z^M_B)) \quad (2.46)
\]

This implies that any \( z^{M*} \) that solves (2.43) satisfies \( z^{M*} > z^M_B \), \( \Rightarrow \). Moreover, \( f(z^{L*}, z^{D*}) \leq f(z^L_B, z^M_B) \) implies that to satisfy \( \pi_M(z^{M*}) \geq \pi_L(z^{M*}) \), \( z^{M*} \geq z^M_B \).

Thus, \( z^{M*} \geq z^M_B \). Now, let \( z^{D*} > z^M_B \). If we solve for \( z^{D*} \) in equilibrium, we have that \( z^{D*} = \frac{\mu z^{M*}}{2 - \mu} \).

Since \( z^M_B = \frac{\mu}{2 - \mu} \), it must be that any \( z^{M*} \in (z^M_B, 1) \), \( z^{D*} < z^M_B \). Thus, \( z^{M*} \geq z^M_B \geq z^{D*} \geq z^L_B \). Finally, because \( z^{D*} \leq z^M_B \), by condition (2.45), \( z^{L*} \leq z^L_B \).

**Step 2.** To prove that in any equilibrium with thresholds, \( 0 \leq z^{L*} \leq z^{D*} \leq z^{M*} \leq 1 \), can only exist for \( \lambda < \lambda^* \), note that the following must hold in equilibrium:

\[
E[\delta \mid DB^*] = (1 - 2\lambda) \times E[\delta \mid MB^*]
\]

\[
\iff \lambda = \frac{1 - z^{D*}}{2(1 + z^{M*})} < \frac{1 - z^L_B}{2(1 + z^M_B)} = \lambda^*
\]

(2.48)

which follows from the result in step 1 that \( z^{D*} > z^L_B \), and \( z^{M*} > z^M_B \).

**Step 3.** Now I show that given that equilibrium thresholds satisfy \( 0 \leq z^{D*} \leq z^{L*} \leq z^{M*} \leq 1 \), in an equilibrium where all order types are used, equilibrium thresholds must also satisfy \( 0 \leq z^{D*} \leq z^L_B \leq z^{L*} \leq z^M_B \leq z^{M*} \leq 1 \).

To show \( 0 \leq z^{D*} \leq z^{L*} \leq z^{M*} \leq 1 \Rightarrow 0 \leq z^{D*} \leq z^L_B \leq z^{L*} \leq z^M_B \leq z^{M*} \leq 1 \), the equilibrium conditions must satisfy the following, via (2.23)-(2.26):

\[
\begin{align*}
z^{M*} - f(z^{M*}, 1) & - \Pr(\text{MS}^*) \times (z^{M*} - f(z^{L*}, z^{M*})) = 0 \quad (2.49) \\
\Pr(\text{MS}^*) \times (z^{L*} - f(z^{L*}, z^{M*})) & - \Pr(\text{DLS}^*) \times (z^{L*} - f(z^{D*}, z^{L*})) = 0 \quad (2.50) \\
z^{D*} - f(z^{L*}, z^{D*}) & = 0 \quad (2.51)
\end{align*}
\]

Let \( z^{M*} > z^M_B \) and \( z^{L*} < z^L_B \). Then \( \Pr(\text{MS}^*) < \Pr(\text{MS}^B) \), and moreover,

\[
z^{L*} > \frac{\mu z^{M*}}{2 - \mu} > \frac{\mu z^M_B}{2 - \mu} > z^L_B, \text{ a contradiction.}
\]

(2.52)
Instead, suppose then that \( z^L > z^L_B \). Then, \( \pi_L(z^{M*}) < \pi_L(z^M_B) \), implying that the \( z^{M*} \) that solves (2.49) must be such that \( z^{M*} < z^M_B \), a contradiction. Hence, \( z^{M*} \leq z^M_B \).

Then, given \( z^{M*} > z^M_B \), let \( z^L > z^L_B \). Because \( f(z^L, z^{M*}) < f(z^L_B, z^{M*}) \), \( \pi_L(z^{M*}) > \pi_L(z^M_B) \) \( \Rightarrow \) \( z^{M*} > z^M_B \), a contradiction. Thus, \( z^L > z^L_B \). Finally, for any \( z^M_B \geq z^{M*} \geq z^L_B \), in equilibrium, solving (2.51) for \( z^{D*} \) yields,

\[
\frac{\mu z^L}{2 - \mu} = \frac{\mu z^M_B}{2 - \mu} = z^L_B \tag{2.53}
\]

Thus, \( z^{D*} \leq \frac{z^L_B}{2 - \mu} \).

Step 4. To prove that in any equilibrium with thresholds, \( 0 \leq z^{D*} \leq z^L \leq z^{M*} \leq 1 \), can only exist for \( \lambda > \lambda^* \), note that the following must hold in equilibrium:

\[
E[\delta | DB] = (1 - 2\lambda) \times E[\delta | MB] \tag{2.54}
\]

\[
\iff \lambda = \frac{1 + z^{M*} - z^{L*} - z^{D*}}{2(1 + z^{M*})} \tag{2.55}
\]

Then, applying the condition that \( E[\delta | LB] > (1 - 2\lambda) \times E[\delta | MB] \)

\[
E[\delta | LB] > (1 - 2\lambda) \times E[\delta | MB] \tag{2.56}
\]

\[
\iff \lambda > \frac{1 + z^{M*} - z^{L*} - z^{D*}}{2(1 + z^{M*})} > \frac{1 - z^{D*}}{2(1 + z^{M*})} > \frac{1 - z^L_B}{2(1 + z^M_B)} = \lambda^* \tag{2.57}
\]

which we know from the argument above that \( z^{D*} < z^L_B \), and \( z^{M*} < z^M_B \).

### 2.7.2 Proofs: Theorems and Propositions

This section contains the proofs of the existence theorems and proposition 1. I conduct the existence proofs in similar steps. I select a single threshold to be the reference threshold, showing that all other thresholds exist and are unique for all values of the reference threshold, making use of the intermediate value theorem, Lemma (3), and the implicit function theorem. Then, using the intermediate value theorem, I show that there exists a value of the reference threshold that such that the equilibrium exists.

As a remark on notation, I drop the time subscripts in all proofs, because of the stationarity condition.

**Proof (Theorem 1).** To prove existence of a symmetric, stationary equilibrium in threshold strategies of this form, I first prove the existence of an equilibrium in an environment without a dark market, and then show that introducing a dark market that satisfies \( \lambda = \lambda^* \) to trading on the visible market has no effect on the equilibrium thresholds.

The market order and limit orders indifference conditions are given by,
where an investor submits a market buy over a limit buy as long as \( z_t \geq z^M \), submits a limit buy if \( z^M > z_t \geq z^L \), and abstains from trading otherwise. To show existence of a threshold equilibrium, I need to show existence of thresholds \( z^M \) and \( z^L \).

I proceed in 2 steps. In step 1, I show that for any given \( z^M \in [0,1] \) there exists a unique \( z^L \) that solves (2.59). In Step 2, I show that there exists a unique \( z^M \) that solves (2.58).

**Step 1: Existence and Uniqueness of** \( z^L(z^M) \)

Take the function \( Z^l_B \equiv z^L - E[\delta | LB^*] \). To show that the solution \( z^L(z^M) \) to \( Z^l_B \) is unique, and \( z^L < z^M \) for all \( z^M \), we solve \( Z^l_B = 0 \) to yield \( z^L = \frac{\mu z^M}{2-\mu} \). Thus, \( z^L \) is unique.

**Step 2: Existence and Uniqueness of** \( z^M \)

Now, take the function First, I show that the function, \( Z^m_B \equiv z^M - E[\delta | MB^*] - \rho Pr(\text{MS}^*) \times (z^M - E[\delta | LB^*]) \) (2.60) crosses zero from below.

\[
Z^m_B(0) = 0 - E[\delta | MB^*] - \rho Pr(\text{MS}^*) \times (0 - 0) < 0
\]

\[
Z^m_B(1) = 1 - E[\delta | MB^*] - 0 \times (1 - z^L) > 0
\]

Because \( Z^m \) is continuous in \( z^M \), the intermediate value theorem implies the existence of a \( z^M \in [0,1] \) that crosses zero from below for all \( z^L(z^M) \). This holds for \( z^L(z^M) \) by the implicit function theorem, as \( Z^l_B \) is continuous for all \( z^M \). Then, taking the first derivative of \( Z^m_B \) with respect to \( z^M \),

\[
\frac{\partial Z^m_B}{\partial z^M} = 1 - \frac{\mu}{2} + \frac{\rho}{2} \times (z^M - E[\delta | LB^*]) - \rho Pr(\text{MS}^*) \times \left( 1 - \frac{\mu}{2-\mu} \right) \]

\[
\Leftrightarrow \frac{\partial Z^m_B}{\partial z^M} = \frac{2(1 - \mu) + \mu^2 + 4\rho(1 - \mu)z^M + 2(1 - \mu) \cdot (1 - \rho)}{2(2 - \mu)} > 0
\]

Therefore, a unique equilibrium exists in threshold strategies. ■

**Proof (Proposition 1).**

This proof details the critical points that define on which region(s) of \( \lambda \) that the benchmark equilibrium (from Theorem 1) is unique. Case 1 examines \( \lambda \in [1/2, 1) \); Case 2, \( \lambda(0, \lambda^* \), and; Case 3, \( \lambda = \lambda^* \).

**Case 1.** When \( \lambda \) is large, any liquidity provided to the dark must be at a large discount (i.e.,
narrower spread) from liquidity posted to the visible market. The expected payoff for the liquidity provider sending sell limit orders to the dark market (conditional on execution) satisfies:

\[ \pi_{DLS} = (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*] - \mathbb{E}[\delta | DB^*] = 0 \] (2.65)

By Lemma 2, \( \mathbb{E}[\delta | DB^*] > 0 \) and \( \mathbb{E}[\delta | MB^*] > 0 \). Then, for all \( \lambda \geq 1/2 \), \( (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*] \leq 0 \). Therefore, any filled dark limit sell order earns the liquidity provider a negative payoff. Thus, in equilibrium, no limit sell orders are sent to the dark market (i.e., \( \Pr(DLS^*) = 0 \)). The argument for dark limit buy orders is analogous. I denote \( \lambda = 1/2 \) as \( \lambda^* \).

**Case 2.** I begin by recalling the result from Lemma 4 that if \( \lambda < \lambda^* \), then the valuation thresholds must satisfy \( 0 \leq z_{L^*} \leq z_B^* \leq z_{D^*} \leq z_B^* \leq z_{M^*} \leq 1 \). Then, from the professional liquidity provider’s zero profit condition (2.6), we know that:

\[ (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*] = \mathbb{E}[\delta | DB^*] \] (2.66)

\[ \iff \lambda = \frac{1 - z_{D^*}}{2(1 + z_{M^*})} \] (2.67)

Then, we see that \( \lambda = \frac{1 - z_B^*}{2(1 + z_{M^*})} \) which has a unique minimum where \( z_{D^*} = z_B^* \) and \( z_{M^*} = 1 \), which yields:

\[ \lambda = \frac{1 - z_B^*}{4} = \Delta > 0 \] (2.68)

Thus, there exists a region, \( (0, \Delta] \), on which investors do not use dark trading in pure threshold strategies.

**Case 3.** Conjecture that an equilibrium exists where the strategy profile of investors and the liquidity provider is given by,

\[ (z_{D^*}, z_{L^*}, z_{M^*}, \Pr(DLS^*)) = (z_{D^*} \leq z_B^*, z_B^*, z_{M^*} = 1, \Pr(MS^B)) \] (2.69)

Where ‘B’ denotes the value the threshold takes in the benchmark equilibrium. Further conjecture that \( \lambda = \lambda^* \), the parameter value of \( \lambda \) that solves:

\[ (1 - 2\lambda^*) \times \mathbb{E}[\delta_t | MB_t^*, z_{M^*} = z_B^*] = \mathbb{E}[\delta_t | LB_t^*, z_{M^*} = z_B^*, z_{L^*} = z_B^*] \] (2.70)

To show that such a trade-at rule exists, rewrite equation (2.70) as the function \( h(\lambda) \),

\[ h(\lambda) = (1 - 2\lambda^*) \times \mathbb{E}[\delta_t | MB_t^*, z_{M^*} = z_B^*] - \mathbb{E}[\delta_t | LB_t^*, z_{M^*} = z_B^*, z_{L^*} = z_B^*] \] (2.71)

\[ = (1 - 2\lambda) \times \frac{(1 + z_B^*)}{2} - \frac{(z_M^* + z_B^*)}{2} \] (2.72)

Now evaluate \( h(\lambda) \) at \( \lambda = \{0, 1/2\} \). We then have,
which implies, by the intermediate value theorem, that there exists a \( \lambda^* \) such that (2.71) holds given \( z_B^L \) and \( z_B^M \). Moreover, \( h(\lambda) = - (1 + \frac{M_B^L}{M_B^R}) \mu \) \( < 0 \) implies that \( \lambda^* \) is unique.

Then, given \( \lambda^* \) as in (2.71), the level of liquidity provision to the dark pool must be equal to \( \Pr(DLS^*) = \Pr(DLB^*) \leq \Pr(MS^*) \). To see this, note that in the benchmark equilibrium (no dark orders), \((z_L^*, z^M, \Pr(MS^*)) = (z_B^L, z_B^M, \Pr(MS^B))\). Without dark orders, \( \Pr(MS^B) \) is the equilibrium fill rate for limit orders with with transaction costs equal to \( E[\delta \mid LB^*] = \frac{(M_B^M + z_B^L)}{2} \). Then, if dark orders have identical transaction costs, as in equation (2.70), any \( \Pr(DLS^*) > \Pr(MS^B) \) means that all investors with \( z_t \in [z_B^L, z_B^M] \) strictly prefer dark orders, and investors with \( z_t = z_B^M + \epsilon \) for some \( \epsilon > 0 \) prefer dark orders to market orders. Thus, \( z^M > z_B^M \). A contradiction.

I now show that \( \Pr(DLS^*) \leq \Pr(MS^B) \) can sustain the equilibrium initially posited. Because \((1 - 2\lambda^*) \times E[\delta \mid MB^*] = E[\delta \mid LB^*] \) and \( \Pr(DLS^*) \leq \Pr(MS^B) \) implies that \( \pi_L = \pi_D \) for any \( z_t \), and since investors choose visible orders over dark orders when payoffs are identical (by assumption), no investor has an incentive to deviate from the benchmark equilibrium strategy profile.

Moreover, as no investors submit dark orders on-the-equilibrium path, the liquidity provider’s payoff for all dark limit orders is zero given the appropriate set of off-the-equilibrium path beliefs. I assume that these beliefs are such that \( E[\delta \mid DB^*] = (1 - 2\lambda^*) \times E[\delta \mid MB^*, z^M = z_B^M] \), which satisfies the liquidity provider’s zero profit condition. Thus, the liquidity provider has no incentive to deviate from the equilibrium liquidity provision strategy. Hence, \((z^D, z^L^*, z^M^*, \Pr(DLS^*)) = (z^D, z_B^L, z_B^M, \Pr(MS^B))\) is an equilibrium strategy profile for \( \lambda = \lambda^* \).

To prove that no other equilibrium exists in which \( \lambda \) satisfies equation (2.70),

Lastly, \( \Delta \leq \lambda^* < 1/2 \) if and only if,

\[
\frac{1 - z_B^M}{4} \leq \frac{(1 - z_B^L)}{2(1 + z_B^M)} \leq \frac{1}{2}
\]

which, because \( z_B^L < z_B^M \), holds trivially. ■

**Proof (Theorem 2).** Similar to the proof of Theorem 1, I proceed by showing there exist unique \( z^L^*, z^D^* \) and \( \Pr(DLS^*) \) for all \( z^M \), and then show that a unique \( z^M^* \) exists for all \( \lambda \in (\Delta, \lambda^*) \). In doing
so, I reference the functions from (2.19)-(2.22) (reproduced below),

\[
Z^M_{\lambda^*} \equiv z^M - \mathbb{E}[\delta | MB^*] - \text{Pr}(\text{DLS}) \times (z^M - (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*])
\] (2.76)

\[
Z^m_{\lambda^*} \equiv \text{Pr}(\text{DLS}) \times (z^D - (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*]) - \rho \times \text{Pr}(\text{MS}^*) \times (z^D - \mathbb{E}[\delta | LB^*])
\] (2.77)

\[
Z^l_{\lambda^*} \equiv z^L - \mathbb{E}[\delta | LB^*]
\] (2.78)

\[
Z^d_{\lambda^*} \equiv (1 - 2\lambda) \times \mathbb{E}[\delta | MB^*] - \mathbb{E}[\delta | DB^*]
\] (2.79)

Preliminaries.

For an equilibrium to exist where thresholds satisfy \(0 < z^L < z^D \leq z^M < 1\), it must be true that \((1 - 2\lambda) \times \mathbb{E}[\delta | MB^*] > \mathbb{E}[\delta | LB^*]\), by Lemma 3. Moreover, Lemma 4 gives us the “only if”, as any equilibrium where \(0 < z^L < z^D \leq z^M < 1\) can only exist for \(\lambda \in (\lambda^*, \lambda^*)\). We can also apply Lemma 4 to restrict the range of \(z^M\) to \([z^M_B, 1]\).

**Step 1: Existence and Uniqueness of** \(z^L^*(z^M)\)

I now show that there exists a unique \(z^L^* \in [0, z^D]\) that solves (2.79) for all \(z^M \in [z^M_B, 1]\). Simple rearranging yields, \(z^L^* = \frac{mD}{2 - \rho}\). Thus, a unique \(z^L^*\) exists.

**Step 2: Existence and Uniqueness of** \(z^D^*(z^M)\)

I now show that there exists a unique \(z^D \in [z^M_B, z^M]\) that solves (2.78) for all \(z^M \in [z^M_B, 1]\). I achieve the bounding on \(z^D\) and \(z^M\) through Lemma 4. Simplifying (2.78),

\[
Z^d_{\lambda^*} = \frac{M}{2} \times ((1 - 2\lambda) \times (1 + z^M) - z^M - z^D)
\] (2.80)

Now solve for \(\lambda\), yielding \(\lambda = \frac{1 - z^D}{2 + z^M}\). From Lemma 4, we know that \(\lambda \in (\lambda^*, \lambda^*)\), which equals the range \((\frac{1 - z^M_B}{4}, \frac{1 - z^M_B}{1 + z^M_B})\). Then, for any \(\lambda \in (\lambda^*, \lambda^*)\), and any \(z^M \in (z^M_B, 1)\), there exists a unique \(z^D^* \in (z^D_B, z^M)\) that solves (2.78).

**Step 3: Existence and Uniqueness of** \(\text{Pr}(\text{DLS}^*)(z^M)\)

I now show that there exists a unique \(\text{Pr}(\text{DLS}^*) \in [\rho \text{Pr}(\text{MS}^*), 1]\) that solves (2.76) for all \(z^M \in [z^M_B, 1]\).

First, I show that, by evaluating \(Z^r_{\lambda^*}\) at the lower bound \(\text{Pr}(\text{DLS}^*) = \rho \text{Pr}(\text{MS}^*)\), we have that \(Z^r_{\lambda^*} > 0\).
Using this fact, let $Z_{\lambda<\lambda^*} = Z^M - E[\delta | MB^*] - \rho \Pr(MS^*) \times (z^M - (1 - 2\lambda)E[\delta | MB^*])$ \hspace{1cm} (2.81)

$> z^M - E[\delta | MB^*] - \rho \Pr(MS^B) \times (z^M - (1 - 2\lambda)E[\delta | MB^*])$ \hspace{1cm} (2.82)

$> z^M - E[\delta | MB^*] - \rho \Pr(MS^B) \times (z^M - (1 - 2\lambda)E[\delta | MB^*]) = 0$ \hspace{1cm} (2.83)

Where the equality at (2.83) follows by the fact that this is an indifference condition for the benchmark equilibrium, and hence, the expression is zero when evaluated at $(z_B^L, z^M_B)$.

Then, evaluating (2.76) at the upper bound, $\Pr(DLS^*) = 1$:

$$Z^\gamma_{\lambda<\lambda^*} = z^M - \frac{\mu(1 + z^M_B)}{2} - 1 \times \left(z^M - (1 - 2\lambda) \times \frac{\mu(1 + z^M_B)}{2}\right) < 0 \hspace{1cm} (2.84)$$

Thus, by the intermediate value theorem, $\Pr(DLS^*)$ exists. It is immediate from (2.76) that $Z^\gamma_{\lambda<\lambda^*}$ is decreasing in $\Pr(DLS^*)$, implying that $\Pr(DLS^*)$ is unique for all $z^M \in [z_B^L, 1]$.

**Step 4: Existence of $z^M$**

Lastly, I show that there exists a $z^M \in [z_B^L, 1]$ that solves (2.77). First, evaluate (2.77) at $z^M = z_B^L$:

$$Z^m_{\lambda<\lambda^*} = \Pr(DLS^*) \times (z^{D^*} - (1 - 2\lambda) \times \frac{\mu(1 + z^M_B)}{2}) - \rho \Pr(MS^B) \times \left(z^{D^*} - \frac{\mu(z^{L^*} + z^{D^*})}{2}\right) \hspace{1cm} (2.85)$$

We are looking to show that (2.85) is negative. To remove $\Pr(DLS^*)$, substitute $\Pr(DLS^*)$ evaluated at $z_B^L$. Also, substitute $(1 - 2\lambda) \times E[\delta | MB^*] = E[\delta | DB^*]$.

$$Z^m_{\lambda<\lambda^*} = \left(\frac{z_B^L - \frac{\mu(1 + z^M_B)}{2}}{z_B^L - \frac{\mu(z^{D^*} + z^M_B)}{2}}\right) \times \left(z^{D^*} - \frac{\mu(z^{L^*} + z^{D^*})}{2}\right) - \rho \Pr(MS^B) \times \left(z^{D^*} - \frac{\mu(z^{L^*} + z^{D^*})}{2}\right) \hspace{1cm} (2.86)$$

We know by expression (2.58) that in the benchmark equilibrium,

$$\frac{\left(\frac{z_B^L - \frac{\mu(1 + z^M_B)}{2}}{\rho \Pr(MS^B)}\right)}{\left(\frac{z_B^L - \frac{\mu(z^{L^*} + z^M_B)}{2}}{2}\right)} = \left(\frac{z_B^L - \frac{\mu(z^{L^*} + z^M_B)}{2}}{2}\right) \hspace{1cm} (2.87)$$

Using this fact, let $Z^m_{\lambda<\lambda^*} < 0$, and substitute in (2.87) to get:

$$\frac{z_B^L - \frac{\mu(z^{D^*} + z^M_B)}{2}}{z_B^L - \frac{\mu(z^{D^*} + z^M_B)}{2}} < \frac{z_B^L - \frac{\mu(z^{D^*} + z^M_B)}{2}}{2} \hspace{1cm} (2.88)$$

Now, substitute the equilibrium value for $z^{L^*} = \frac{\mu D^*}{2\mu}$, and use the fact that $z_B^L = \frac{\mu M}{2\mu}$ to simplify (2.88) down to:
\[
\frac{z^{D*} - \mu(z^{D*} + z^M)}{z_B^M - \mu(z^{D*} + z^M)} < \frac{z^{D*}}{z_B^M} \iff z^{D*} < z_B^M
\] (2.89)

which holds by Lemma 4. Hence, at \(z^{M*} = z_B^M\), the indifference condition \(Z_{\lambda < \lambda^*}^m < 0\).

Then, we evaluate (2.77) at \(z^M = 1\):

\[
Z_{\lambda < \lambda^*}^m = \Pr(DLS^*)((1 - (1 - 2\lambda) \times \mu)) - 0 \times (1 - E[\delta \mid LB^*]) > 0
\] (2.90)

Thus, \(z^{M*}\) exists, by the intermediate value theorem. ■

**Proof (Theorem 3).** Similar to the proof of Theorem 2, I proceed by showing the existence and uniqueness of \(z^{L*}, z^{D*}\) and \(\Pr(DLS^*)\) for all \(z^M\), and then show that unique \(z^{M*}\) exists, for all \(\lambda \in (\lambda^*, \lambda)\).

In doing so, I reference the functions from (2.23)-(2.26)) (reproduced below),

\[
Z_{\lambda > \lambda^*}^m \equiv z^M - E[\delta \mid MB^*] - \rho \cdot \Pr(MS*) \times (z^M - E[\delta \mid LB^*])
\] (2.91)

\[
Z_{\lambda > \lambda^*}^L \equiv \Pr(DLS) \times ((1 - (1 - 2\lambda) \times E[\delta \mid MB^*]) - \rho \cdot \Pr(MS*) \times (z^L - E[\delta \mid LB^*])
\] (2.92)

\[
Z_{\lambda > \lambda^*}^D \equiv z^D - (1 - 2\lambda) \times E[\delta \mid MB^*]
\] (2.93)

\[
Z_{\lambda > \lambda^*}^L \equiv (1 - 2\lambda) \times E[\delta \mid MB^*] - E[\delta \mid DB^*]
\] (2.94)

**Preliminaries.**

For an equilibrium to exist where threshold values satisfy \(0 < z^D < z^L \leq z^M < 1\), it must be true that \((1 - 2\lambda) \times E[\delta \mid MB^*] < E[\delta \mid LB^*]\), by Lemma 3. As in the proof of Theorem 2, the “only if” follows from Lemma 4, as any equilibrium where \(0 \leq z^D \leq z^L \leq z^M < 1\) can only exist for \(\lambda \in (\lambda^*, \lambda)\). We can also apply Lemma 4 to restrict the range of \(z^M\) to \([\frac{\mu}{2\mu}, z_B^M]\).

**Step 1: Existence and Uniqueness of** \(z^{D*}(z^M)\)

I now show that there exists a unique \(z^D \in [0, z^L]\) that solves (2.93) for all \(z^M \in [\frac{\mu}{2\mu}, z_B^M]\). Rearranging,

\[
z^{D*} = (1 - 2\lambda) \times \frac{(1 + z^M)\mu}{2} < E[\delta \mid LB^*]
\] (2.95)

Thus, a unique \(z^{D*}\) exists if \(z^L > E[\delta \mid LB^*]\), which I will show in step 2.

**Step 2: Existence and Uniqueness of** \(z^{L*}(z^M)\)

To show there exists a unique \(z^L \in [z^{D*}, z^M]\) that solves (2.94) for all \(z^M \in [\frac{\mu}{2\mu}, z_B^M]\), evaluating (2.94) at \(z^{D*}\) and \(z^M\),
Lastly, I show that there exists a $z$ such that $Z_{\lambda, \lambda^*}(z^L = z^D) = -(1 - 2\lambda) \times \left(1 - \frac{\mu}{2}\right) (1 + z^M) < 0$ \hfill (2.96)

$Z'_{\lambda, \lambda^*}(z^L = z^M) = z^M \times \left(1 - (1 - 2\lambda) \times (1 - \frac{\mu}{2}) \right) > 0$

and because $Z_{\lambda, \lambda^*}'$ is increasing in $z^L$, a unique $z^L \in [z^D, z^M]$ exists. Moreover, because $z^L > E[\delta \mid LB^*] \iff z^L > (1 - 2\lambda) \times (1 + z^M) - z^M$, solving $Z_{\lambda, \lambda^*}'$ for $z^L$ yields, $z^L = (1 - 2\lambda) \times (1 + z^M) - z^D > (1 - 2\lambda) \times (1 + z^M) - z^M$, and hence, $z^L > E[\delta \mid LB^*]$. 

**Step 3: Existence and Uniqueness of $Pr(DLS^*)(z^M)$**

I now show that there exists a unique $Pr(DLS^*) \in [0, Pr(MS^*)]$ that solves (2.92) for all $z^M \in \left[\frac{\mu}{2 - \mu}, z_B^M\right]$. By rearranging (2.92)

$$Pr(DLS^*) = \frac{\rho Pr(MS^*) \times (z^L - E[\delta \mid LB^*])}{z^L - (1 - 2\lambda)E[\delta \mid MB^*]} < \rho Pr(MS^*)$$

which is unique for all $z^M \in \left[\frac{\mu}{2 - \mu}, z_B^M\right]$. 

**Step 4: Existence of $z^M$**

Lastly, I show that there exists a $z^M \in \left[\frac{\mu}{2 - \mu}, z_B^M\right]$ that solves (2.76).

$$z^M = \frac{\mu}{2 - \mu} \implies Z_{\lambda, \lambda^*}' = -\rho Pr(MS^*)(E[\delta \mid MB^*] - E[\delta \mid LB^*]) < 0$$ 

$$z^M = z_B^M \implies Z_{\lambda, \lambda^*}' = \frac{\mu(1 + z^M)}{2} - \rho \left(\frac{1 - z_B^M}{2}\right) \times \left(z_B^M - \frac{\mu(z_B^M + z^L)}{2}\right) > 0$$ \hfill (2.98)

Where $Z_{\lambda, \lambda^*}'$ in equation (2.98) is positive by the following argument. We know that for $z^L = z_B^L$, that $Z_{\lambda, \lambda^*}'(z^M = z_B^M)$ is zero, as it satisfies the equilibrium indifference condition for $z^M$ in the equilibrium described by Theorem 1. Then, since $z^L > z_B^M$ by Lemma 4, the second (negative) term is smaller, implying that $Z_{\lambda, \lambda^*}'(z^M = z_B^M) > 0$. Therefore, $z^M$ exists.

Then, taking the first derivative of $Z_{\lambda, \lambda^*}'$

$$\frac{\partial Z_{\lambda, \lambda^*}'}{\partial z^M} = 1 - \frac{\mu}{2} + \rho \times \left(\frac{z^M - E[\delta \mid LB^*]}{2}\right) - \rho Pr(MS^*) \times \left(1 - \frac{\mu}{2 - \mu}\right)$$ \hfill (2.99)

$$\iff \frac{\partial Z_{\lambda, \lambda^*}'}{\partial z^M} = \frac{2(1 - \mu) + \mu^2 + 4\rho(1 - \mu)z^M + 2(1 - \mu) \cdot (1 - \rho)}{2(2 - \mu)} > 0$$ \hfill (2.100)

Therefore, $z^M$ exists and is unique. ■

**Proof (Proposition 2).** The exchange volume is given by $2Pr(MB^*)$, which is a decreasing function of $z^M$. Thus, by Lemma 4, if $\lambda \in (\lambda, \lambda^*)$, then $z^M \geq z_B^M$, and $2Pr(MB^*) < 2Pr(MB^B)$. Further, if $\lambda \in (\lambda^*, 0)$, then $z^M \leq z_B^M$, and $2Pr(MB^*) > 2Pr(MB^B)$. 
Because market participation is defined as the probability of submitting any order, we can write it as \(1 - \Pr(NT^*)\). If \(\lambda \in (\lambda, \lambda^*)\), \(\Pr(NT^*) = z^L_\ast - (-z^L_\ast) = 2z^L_\ast\), and in equilibrium, \(z^L_\ast < z^B_\ast\). Thus, for any small trade-at rule, market participation increases. If \(\lambda \in (\lambda^*, \lambda)\), \(\Pr(NT^*) = z^D_\ast - (-z^D_\ast) = 2z^D_\ast\), and in equilibrium, \(z^D_\ast < z^B_\ast\). Thus, for any large trade-at rule, market participation increases.

**Proof (Proposition 3).** The price impact measure is given by \(E[\delta | MB^*]\), which is an increasing function of \(z^M_\ast\) only. Thus, by Lemma 4, if \(\lambda \in (\lambda, \lambda^*)\), then \(z^M_\ast \geq z^M_B\), and \(E[\delta | MB^*]\) is always greater than in the benchmark case. Further, if \(\lambda \in (\lambda^*, \lambda)\), then \(z^M_\ast \leq z^M_B\), and \(E[\delta | MB^*]\) is always lower than in the benchmark case.

### 2.7.3 Out-of-Equilibrium Limit Orders and Beliefs

The equilibrium concept I employ in this paper is a perfect Bayesian equilibrium. On-the-equilibrium-path, investors submit limit orders with competitive limit prices. However, I require an appropriate set of out-of-equilibrium beliefs to ensure that competitive limit prices strategically dominate any off-equilibrium-path deviations in the limit price. Intuitively, any limit order that is posted at a price worse than the competitive equilibrium price is strategically dominated by the competitive price, as the professional liquidity provider reacts to the non-competitive order by undercutting it. For non-competitive limit orders that undercut the competitive price (i.e., a price inside the competitive spread), however, it is not immediate that the competitive price strategically dominates this set of prices.

Perfect Bayesian equilibrium prescribes that investors and the professional liquidity provider update their beliefs by Bayes rule, whenever possible, but does not place any restrictions on the beliefs of market participants when they encounter an out-of-equilibrium action. To support competitive prices in equilibrium, I assume (similar to Brolley and Malinova (2014)) that if a limit buy order is posted at a price different to the competitive equilibrium bid price \(\hat{\text{bid}}^t_{t+1}\), then market participants hold the following beliefs regarding this investor’s knowledge of the period \(t\) innovation \(\delta_t\).

If a limit buy order is posted at a price \(\hat{\text{bid}} < \text{bid}^t_{t+1}\), then market participants assume that this investor followed the equilibrium threshold strategy, but “made a mistake” when pricing his orders. The professional liquidity provider then updates his expectation about \(\delta_t\) to the equilibrium value and posts a buy limit order at \(\text{bid}^t_{t+1}\). The original investor’s limit order then executes with zero probability.

If a limit buy order is posted at a price \(\hat{\text{bid}} > \text{bid}^t_{t+1}\), then market participants believe that this order stems from an investor with a sufficiently high valuation (e.g., \(z_t = 1\)) and update their expectations about \(\delta_t\) to \(E[\delta_t | \hat{\text{bid}}]\) accordingly. The new posterior expectation of \(V_t\) equals to \(p_{t-1} + E[\delta_t | \hat{\text{bid}}]\). The professional liquidity provider is then willing to post a bid price \(\text{bid}^{t+1}_t \leq p_{t-1} + E[\delta_t | \hat{\text{bid}}] + E[\delta_{t+1} | \hat{\text{bid}}]\)
MS_{t+1}]. With the out-of-the-equilibrium belief of \( \delta_t = 1 \) and with the bid-ask spread < 1, a limit order with the new price \( \text{bid}^{**}_{t+1} \) outbids any limit buy order that yields investors positive expected profits.

The beliefs upon an out-of-equilibrium sell order are symmetric. The above out-of-equilibrium beliefs ensure that no investor deviates from his equilibrium strategy. I emphasize that these beliefs and actions do not materialize in equilibrium. Instead, they can be loosely thought of as a “threat” to ensure that investors do not deviate from their prescribed equilibrium strategies.
Figure 2.5: Entry and Order Submission Timeline

This figure illustrates the timing of events upon the arrival of an investor at an arbitrary period, \( t \), until their departure from the market. Value \( y_t \) is the private valuation of the period \( t \) investor and \( \delta_t \) is the innovation to the security’s fundamental value in period \( t \).

Professional liquidity providers post limit orders to empty side(s) of the visible book, and limit orders to the dark market with probability \( \Pr(DLS_{t-1}) \)

<table>
<thead>
<tr>
<th>Period</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>Period ( t ) investor enters market, learns ( y_t ) and ( \delta_t )</td>
</tr>
<tr>
<td>( t )</td>
<td>Period ( t ) investor submits order (if any)</td>
</tr>
<tr>
<td>( t )</td>
<td>Period ( t ) dark market trades are reported, Period ( t - 1 ) investor leaves market (if still present)</td>
</tr>
<tr>
<td>( t + 1 )</td>
<td>Period ( t + 1 ) investor enters market, learns ( y_{t+1} ) and ( \delta_{t+1} )</td>
</tr>
<tr>
<td>( t + 1 )</td>
<td>Period ( t + 1 ) investor submits order (if any)</td>
</tr>
<tr>
<td>( t + 1 )</td>
<td>Period ( t + 1 ) dark market trades are reported, Period ( t ) investor leaves market (if still present)</td>
</tr>
<tr>
<td>( t + 1 )</td>
<td>Period ( t + 1 ) limit orders either trade against the period ( t + 1 ) market order or get cancelled</td>
</tr>
</tbody>
</table>

<table>
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<tr>
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<th>XXX</th>
<th>ABC</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>456</td>
<td>789</td>
</tr>
</tbody>
</table>

Chapter 2: Should Dark Pools Improve Upon Visible Quotes? The Impact of Trade-at Rules
The panels below depict equilibrium valuation thresholds $z^M$, $z^L$ and $z^D$ as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks $\lambda^*$; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of $\mu$ are qualitatively similar.
Figure 2.7: Volume, Market Participation

The panels below depict volume (visible market and total), and market participation as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks $\lambda^*$; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of $\mu$ are qualitatively similar.
Figure 2.8: Quoted Half-Spread

The panels below depict the quoted half-spread (also price impact) as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks $\lambda^*$; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of $\mu$ are qualitatively similar.
Figure 2.9: Informational Efficiency

The panels below depict informational efficiency as a function of the trade-at rule (\( \lambda < \lambda^* \) on the left, \( \lambda > \lambda^* \) on the right). Higher values than the benchmark are less efficient. A vertical dashed line marks \( \lambda^* \); the horizontal dashed line indicates the visible market only benchmark value. Parameter \( \mu = 0.5 \). Results for other values of \( \mu \) are qualitatively similar.

Price Efficiency \((\lambda < \hat{\lambda})\)

Price Efficiency \((\lambda > \hat{\lambda})\)
Figure 2.10: Total Expected Welfare

The panels below depict total expected welfare as a function of the trade-at rule ($\lambda < \lambda^*$ on the left, $\lambda > \lambda^*$ on the right). A vertical dashed line marks $\lambda^*$; the horizontal dashed line indicates the visible market only benchmark value. Parameter $\mu = 0.5$. Results for other values of $\mu$ are qualitatively similar.
Chapter 3

Broker Fees in the Maker-Taker Fee System
3.1 Introduction

The most recent decade has brought dramatic changes to the organization of equity markets worldwide. Exchanges maintain limit order books, where orders to trade pre-specified quantities at pre-specified prices are arranged in a queue, according to a set of priority rules. To improve the level liquidity provided to their limit order books, many exchanges provide cash incentives for filled limit orders. Exchanges that pay cash rebates to limit order traders that provide or “make” liquidity typically levy higher fees to remove or “take” liquidity on submitters of market orders. This practice is referred to as “maker-taker” pricing,\(^1\) and it has been a contentious issue in regulatory and policy debates on market structure.

These incentives, however, do not find their way to all investors equally. While investors who access the exchange directly may receive maker rebates for filled limit orders (and pay taker fees for market orders), many investors access the exchange through a broker, who pays fees and receives rebates on their behalf. As a practical matter, many long-term investors do not pay taker fees; instead, they pay a flat fee per trade to their broker.\(^2\) Proponents maintain that the new trading environment benefits all market participants through increased competition. Opponents argue that the increased competition for liquidity provision lowers the execution probability of long-term investors’ limit orders and compels these investors to trade with the more expensive market orders.\(^3\)

The industry debates notwithstanding, the academic literature has questioned whether maker-taker pricing per se can play an economically meaningful role. Angel, Harris, and Spatt (2011) and Colliard and Foucault (2012) argue that maker-taker fees affect trading only through the total fee that is retained by the exchange, and that in the absence of regulatory and market frictions, the split of this fee into a maker rebate and a taker fee is irrelevant. If a maker rebate is introduced in competitive markets, the bid-ask spread will decline by (twice) the maker rebate. Provided that the exchange finances the maker rebate by an increase in the taker fee, the takers’ cum-fee trading costs, i.e. the bid-ask spread plus (twice) the taker fee, remain unaffected. Building on these insights, we investigate the impact of maker-taker pricing when some investors pay only a flat (average) maker-taker fee.

In this paper, we build upon the model of Foucault (1999) to study the role that cash incentives for liquidity provision play in current equity markets when investors may face different fee schedules. Traders (who we refer to as “investors”) enter the market sequentially to trade a single unit of a risky security, motivated private values. These investors trade using market or limit orders; if using a limit


order, they must specify a price at which the order must be filled—the ‘limit price’. A subset of investors access the market directly through the exchange, and pay maker-taker fees on a per-trade basis. The remainder of investors trade through a broker who interacts with the exchange on their behalf; the broker charges a per-trade fee—independent of order type—that is equal to the average trading fees that the broker remits to the exchange. This generates a “two-tiered” trading fee system.

In equilibrium, investors choose the limit price of their limit orders to ‘target’ (a subset of) investors that arrive in the subsequent period. Because the flat fee paid by the subset of investors who trade through a broker is an average of the maker and taker fees that the broker collects, the flat fee that they pay to trade with market orders is less than the taker fee paid by investors that access the exchange directly. As a result, flat-fee investors are willing to trade against limit orders with worse limit prices than maker-taker investors. This implies that there are two equilibrium prices: a more favourable price that targets both types of investors, and a less favourable price, that targets only flat-fee investors. Moreover, the more favourable fee maker-takers face for submitting limit orders leads them to always submit limit orders at (weakly) more favourable prices than flat-fee investors.

In this two-tiered system, the relative size of the subset of flat-fee investors has an impact on market quality. When the measure of flat-fee investors is sufficiently large, flat-fee investors submit limit orders at less favourable prices than maker-taker investors, leading maker-taker investors to take on a more prominent role in liquidity provision. In fact, if the subset of flat-fee investors is large enough, maker-taker investors become de-facto market makers.

By placing limit orders at less favourable prices, flat-fee investors then face a lower fill rate for their limit orders than they would at a more favourable price, as maker-taker investors “pass over” the less favourable limit orders, in favour of submitting their own limit order. The result is lower overall trading volume in two-tiered system. We also examine investor welfare by following the work of Bessembinder, Hao, and Lemmon (2012) to define a measure that reflects allocative efficiency. We find that social welfare co-moves with volume, and hence, a two-tiered system reduces welfare. Finally, maker-taker investors earn greater profits, on average.

We find that the split between the maker and taker fee, too, has an economically meaningful impact on market quality and welfare in a two-tiered system. Consistent with the previous literature (see Angel, Harris, and Spatt (2011) and Colliard and Foucault (2012)), when all investors face the same fee schedule—either maker-taker or flat-fee—the split does not play an economically meaningful role in our model; any increase in the maker rebate is passed to the takers through a narrower bid-ask spread, exactly offsetting an increase in the taker fee. In a two-tiered system, the split has a meaningful—

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4When discussing our results, we focus on the prevalent industry practice of a negative maker fee, or a rebate, but our
and potentially positive—impact. When flat-fee investors submit less favourably-priced limit orders, in equilibrium, a reduction in the taker fee (maker rebate) lowers the relative cost of submitting limit orders that target all investors, thus improving the fill rates of their limit orders. A large enough decrease maximizes trading volume and welfare; the profit differential between investor groups also declines to zero.

If maker-taker investors place more favourably-priced limit orders than flat-fee investors, in equilibrium, an increase in the maker-taker split can also improve volume and welfare. By increasing the maker-taker fee split in this scenario, maker-taker investors find the cost of submitting less favourable limit orders increases at a slower rate, compared to the cost of submitting limit orders that target all investors. This induces all investors to prefer less favourable limit orders; in fact, maker-taker investors take on a purely market-making role, as taking liquidity becomes too costly, relative to liquidity provision that targets only flat-fee investors. By doing so, flat-fee investors find fewer empty limit order books, as maker-taker investors who enter the market ahead of them do not remove liquidity from the book. The end result is an improvement in trading volume and welfare.

Our paper is most closely related to Colliard and Foucault (2012) and Foucault, Kadan, and Kandel (2013), who theoretically analyze the impact of maker-taker fees. Colliard and Foucault (2012) study trader behavior in a model where symmetrically informed traders choose between limit and market orders. They show that, absent frictions, the split between maker and taker fees has no economic impact, and they focus on the impact of the total fee charged by an exchange. Foucault, Kadan, and Kandel (2013) argue that in the presence of a minimum tick size, limit order book prices may not adjust sufficiently to compensate traders for changes in the split between maker and taker fees. They then show that exchanges may use maker-taker pricing to balance supply and demand of liquidity, when traders exogenously act as makers or takers. Subsequent to our work, O’Donoghue (2015) models the impact of a broker who charges a flat commission to investors irrespective of order type, also finding that the split between the maker and taker fees has an economically meaningful impact on market quality.


Our work is also closely linked to Degryse, van Achter, and Wuyts (2012), who study the impact of the post-trade clearing and settlement fees. In their model, the clearing house may set a flat fee for all trades or impose different fees, depending on whether a trade was internalized. They find that the fee

analysis extends to the case of a positive maker fee.
schedule affects the welfare of market participants, and that the optimal schedule depends on the size of the clearing fee.

The maker-taker pricing model is related to the payment for order flow model, see, e.g., Kandel and Marx (1999), Battalio and Holden (2001), or Parlour and Rajan (2003), in the sense that both systems aim to incentivize order flow; Battalio, Shkilko, and Van Ness (2012) and Anand, McCormick, and Serban (2013) empirically compare market quality under maker-taker pricing and payment for order flow.

3.2 The Model

We model an infinite-horizon financial market where risk-neutral investors enter the market sequentially to trade a single risky security in the sense of Foucault (1999) and Degryse, van Achter, and Wuyts (2012). Trading is conducted on a single exchange via limit order book. Investors choose between posting a limit order to trade at a pre-specified price, and submitting a market order to trade immediately with a previously posted limit order. The exchange charges fees for each filled order that depend on the order type. Some investors may access the exchange through a broker, who charges a flat fee per filled order, regardless of order type.

**Investors.** At each period $t$, a risk-neutral, expected utility maximizing investor enters the market willing to trade a single unit of the risky security. Investors are buyers or sellers with equal probability, and have a private valuation for the security that is equal to $V > 0$ for buyers, and $-V$ for sellers.

**Limit Order Book.** Trading for the single security conducts via limit order book. An investor may place an order to buy or sell the security using market orders or limit orders. An investor that enters the market in period $t$ may trade immediately using a market order only if there is an available limit order at the exchange on the correct side of the market (e.g., if the investor wants to buy, there is a limit sell order.) If no desirable limit orders are available, an investor may place a limit order to trade in period $t+1$. A trader who wishes to buy places a limit order at price $\text{bid}_{t+1}$, whereas a seller places a limit order at price $\text{ask}_{t+1}$. We assume that investors who are indifferent between market orders and limit orders choose to submit market orders. Limit orders that are not filled by the subsequent investor are cancelled (as in Foucault (1999)).

**Exchange Fees.** The limit order book is maintained by an exchange that charges time-invariant fees for executing orders. Our paper focuses on maker-taker fees, which depend on the order type (market or limit). The fee for market orders, or the “taker fee”, is denoted $f_T$. Similarly, the fee for limit orders, or the “maker fee” is denoted $f_M$. We refer to the sum of maker and taker fees as the “total fee”, (i.e.,
Moreover, we assume that the exchange collects a total fee per transaction equal to \( f_{\text{total}} = 2c \). The total fee is then split into a taker fee, written as \( f_T = c + f \), and a maker fee, \( f_M = c - f \). While we focus on the prevalent practice where a taker pays a fee, and a maker receives a rebate (i.e., \( f_T > 0 > f_M \)), the intuition for most of our results extends to the reverse scenario where market order submitters receive a rebate and makers pay a positive fee for filled limit orders.\(^5\) Finally, so that investors prefer trading to not trading, we assume that the total gains from trade exceed the per-trade costs imposed by the exchange, \( 2V > 2c \).

We study a setting with two types of investors, \( \theta_i \in \{\theta_{MT}, \theta_{FF}\} \), each of whom accesses the exchange differently: proportion \( \lambda \in [0, 1] \) of investors are of type \( \theta_{FF} \), who access the exchange through a broker and pay a flat-fee \( \bar{f} \) for any filled order; the remaining fraction \( (1 - \lambda) \) are of type \( \theta_{MT} \), and access the exchange directly, paying maker fees on filled limit orders, and taker fees on filled market orders. We assume that investors who access the exchange through a broker do so through a single competitive broker that sets their fee equal to the average of the maker and taker fees they remit to the exchange. These flat fees reflect a common practice in the industry: long-term investors typically access exchanges via brokers, who pay the exchange maker-taker fees but levy a flat fee per transaction on their customers.\(^6\)

**Investor Payoffs.** The payoff to an investor who buys the security in period \( t \) is given by the difference between the investor’s private valuation, and the price that the investor pays for the unit, plus the fee incurred; similarly for a sell decision. We normalize the payoff of a non-executed order to zero. The period \( t \) investor has the following expected payoffs to submitting, respectively, a market buy order to trade immediately at the prevailing ask price \( \text{ask}_t \), and a limit buy order at price \( \text{bid}_{t+1} \):

\[
\begin{align*}
\pi_{t}^{\text{MB}} &= V - \text{ask}_t - \text{fee for market order} \quad (3.1) \\
\pi_{t}^{\text{LB}} &= \Pr(\text{fill} | \text{bid}_{t+1}) \times (V - \text{bid}_{t+1} - \text{fee for limit order}) \quad (3.2)
\end{align*}
\]

where \( \Pr(\text{fill} | \text{bid}_{t+1}) \) is the probability that an investor’s period \( t \) limit order is filled in period \( t + 1 \) (by a market sell order) given the order’s price, \( \text{bid}_{t+1} \). Payoffs to a market sell order at the prevailing bid price \( \text{bid}_t \), and a limit sell order at the price \( \text{ask}_{t+1} \) are analogously defined. In what follows, for an order submitted in period \( t \), we denote a market buy as \( \text{MB}_t \), a market sell as \( \text{MS}_t \), a limit buy as \( \text{LB}_t \), and a limit sell as \( \text{LS}_t \).

\(^5\)This “inverted” pricing is often referred to by industry participants as “taker-maker pricing”, as it is utilized, for instance, by NASDAQ OMX BX.

3.3 Equilibrium

The exchange operates the common maker-taker fee system where investors that submit market orders pay a fee, (i.e., \( f_T > 0 \)), and those who submit limit orders receive a rebate (i.e., \( f_M < 0 \)). A proportion \( \lambda \) of investors are of type \( \theta_{FF} \), accessing the market through a broker who charges them a flat fee per trade. The remaining \( (1 - \lambda) \) proportion of investors are of type \( \theta_{MT} \), and access the exchange directly where they pay maker-taker fees on a per-trade basis.

3.3.1 Single Investor Type

We begin by considering two extreme cases, where investors either all pay maker-taker fees \( (\lambda = 0) \), or all pay flat fees \( (\lambda = 1) \). Using the solution approach of Degryse, van Achter, and Wuyts (2012), we search for stationary, symmetric equilibria, where the prices of buy and sell limit orders, \( \text{bid}^* \) and \( \text{ask}^* \) respectively, are set such that an incoming investor is indifferent between submitting a market order or a limit order. Because the model is symmetric across buyers and sellers, we focus on the decisions of buyers, without loss of generality. Hence, buyers who submit limit orders do so with the intention to trade with an incoming seller.

An investor who enters the market chooses both what type of order they submit (market or limit), and if they are to submit a limit order, at which price they submit it. Investors who submit limit orders set a limit price such that an incoming seller just prefers to trade with them (using a market order) over submitting their own limit order. This occurs when the price is such that an investor is indifferent between market and limit orders, because, when indifferent, we assume investors choose market orders. Such price setting is optimal, as any price that is marginally more favourable to the incoming investor is unnecessary, as sellers that would trade at the indifference price, would submit market orders at the lower price anyway (i.e., the limit order fill rate is unchanged). A price that is marginally less favourable would lead the incoming seller to prefer submitting their own limit order over a market order, thus reducing the fill rate of the limit order by the positive measure of investors who would sell at the “indifference price” (and subsequently, reduce the expected payoff of the order).

Let \( \lambda = 0 \), the setting where all investors pay maker-taker fees. For investors are indifferent between market and limit orders, \( \text{ask}^* \) and \( \text{bid}^* \) solve the system,

\[
V - \text{ask}^* - f_T = \Pr(\text{fill} | \text{bid}^*) \times (V - \text{bid}^* - f_M) \tag{3.3}
\]
\[
\text{bid}^* + V - f_T = \Pr(\text{fill} | \text{ask}^*) \times (\text{ask}^* + V - f_M) \tag{3.4}
\]
Because buyers and sellers choose market orders when faced with a limit order priced at ask\(^*\) or bid\(^*\), respectively, a limit order’s execution probability is equal to the probability that an investor from the opposite side of the market will arrive next period, \(\Pr(\text{fill} | \text{ask}^*) = \Pr(\text{fill} | \text{bid}^*) = 0.5\). We can then solve the system of equations (3.3)-(3.4) for equilibrium prices ask\(^*\) and bid\(^*\):

\[
\begin{align*}
\text{ask}^* &= \frac{V}{3} - \frac{(2f_T - f_M)}{3} \\
\text{bid}^* &= -\frac{V}{3} + \frac{2f_T - f_M}{3}
\end{align*}
\]  

Substituting the equilibrium prices from (3.5)-(3.6) into the payoff functions yields:

\[
\pi_{\text{MB}}|_{\lambda=0} = \frac{2V}{3} - \frac{f_T + f_M}{3} = \pi_{\text{LB}}|_{\lambda=0}
\]  

Note that \(f_T + f_M = f_{\text{total}} = 2c\), which is independent of the split between maker and taker fees, \(f\). The equilibrium payoffs in (3.7) share a similar property with those in Colliard and Foucault (2012), in that an investor’s equilibrium order placement decision depends only on the total fee, \(f_{\text{total}}\), when all investors face the same fee schedule. Finally, as \(V > c\), investors prefer trading to not trading.

Suppose instead that all investors pay a flat fee \(\bar{f}\) per transaction (i.e., \(\lambda = 1\)). We model the flat fee as the average fee per investor that the broker remits to the exchange:

\[
\bar{f} = \frac{2 \cdot (\Pr(\text{MB}_t^* | \theta_{FF}) \times f_T + \Pr(\text{LB}_t^* | \theta_{FF}) \times \Pr(\text{MS}_{t+1}^*) \times f_M)}{2 \cdot (\Pr(\text{MB}_t^* | \theta_{FF}) + \Pr(\text{LB}_t^* | \theta_{FF}) \times \Pr(\text{MS}_{t+1}^*))} \\
= \frac{\Pr(\text{MB}_t^* | \theta_{FF}) - \Pr(\text{LB}_t^* | \theta_{FF}) \times \Pr(\text{MS}_{t+1}^*)}{\Pr(\text{MB}_t^* | \theta_{FF}) + \Pr(\text{LB}_t^* | \theta_{FF}) \times \Pr(\text{MS}_{t+1}^*)} \times f + c
\]  

where \(\Pr(\text{MB}_t^* | \theta_i)\) and \(\Pr(\text{LB}_t^* | \theta_i)\) denote the probability that a period \(t\) investor of type \(\theta_i\) places a market buy order or limit buy order, respectively. Equation (3.8) then simplifies to equation (3.9) by substituting \(f_T = c + f\), and \(f_M = c - f\). On average, the broker remits the fixed cost of the order, \(c\), plus the average redistribution cost, \(f\). It follows immediately from (3.9) that \(f_M < \bar{f} \leq f_T\).

In a stationary equilibrium, \(\Pr(\text{MB}_t^*) = \Pr(\text{MB}_{t+1}^*) = \Pr(\text{MB}^*)\). Then, the probability that an investor submits a market buy order in period \(t\) must be equal to the probability that an investor is induced to submit a market buy order next period; that is, the probability a sell limit order is placed in period \(t\), multiplied by the probability that the investor that enters in period \(t+1\) is a buyer. Therefore, the solution in the case of \(\lambda = 0\) or \(\lambda = 1\) is \(\Pr(\text{MB}^*) = 1/3\), which simplifies the average fee to \(\bar{f} = c\). Hence, investors’ strategies are also independent of \(f\) when all investors pay a flat fee. This discussion yields the following result.
Proposition 1 (Independence of the Maker-Taker Split) For $\lambda \in \{0, 1\}$, investors’ equilibrium strategies and payoffs are independent of $f$.

Though the maker-taker split $f$ does not impact investor decisions, it does impact the bid-ask spread, which we define as:

$$\text{spread} \equiv \text{ask}^* - \text{bid}^* = 2 \times \left( \frac{V - c}{3} - f \right)$$

(3.10)

where we use the fact that $f_T + f_M = 2c$. Because $f_T = c + f$, the fee split reduces the quoted spread by a factor of 2. The split does not, however, impact the cum-fee spread. The cum-fee spread represents a more accurate measure of transaction costs, as it takes into account the fees paid by an investor for a round-trip transaction. The cum-fee spread is defined as the total cost of using market orders to buy and then sell the security (a round-trip):

$$\text{cum-fee spread} \equiv \text{ask}^* - \text{bid}^* + 2f_T = 2\left( \frac{V + 2c}{3} \right)$$

(3.11)

By accounting for round-trip transaction costs in the spread measure, the impact of the fee split is exactly offset. We summarize these results in the following proposition:

Proposition 2 (Spreads and the Maker-Taker Split) An increase in $f$ by $\Delta > 0$ reduces the quoted bid-ask spread by $2\Delta$, but the cum-fee spread is unchanged.

Proposition 2 illustrates how investors’ order placement decisions are independent of any fee redistribution from taker to maker: the increase (or decrease) in $f$ is exactly compensated for by a reduction in the quoted spread. When comparing an environment where all investors pay maker-taker fees ($\lambda = 0$) to one where all pay flat fees ($\lambda = 1$), Proposition 2 implies the following corollary.

Corollary 1 (Maker-Taker Fees vs. Flat Fees) Given $c$, the quoted spread is narrower in a maker-taker fee system ($f > 0$) than in a system that charges a flat fee per trade; conversely, the quoted spread in a taker-maker system ($f < 0$) is wider. The cum-fee spread is unchanged.

We note that our model admits identical results on quoted and cum-fee spreads (up to a scaling of the fees) to Degryse, van Achter, and Wuyts (2012). When all investors remit a flat fee per trade to the broker, the environment is similar to investors submitting a uniform post-trade fee to a clearing and settlement agent. Whereas Degryse, van Achter, and Wuyts (2012) departs from this baseline by considering trade-specific fees (i.e., trades that are within brokerage versus inter-brokerage), we examine the impact of charging investors different order-specific fees, depending on method of market access (direct-to-exchange or via broker).
3.3.2 Two Investor Types

When all investors face the same fee schedule, an investor’s order placement decisions are invariant to the redistribution of the total fee between takers and makers. If, however, both type $\theta_{MT}$ and $\theta_{FF}$ investors participate in the market, investors’ order placement decisions are not neutral to the fee split \( f \). We show this by first characterizing investors’ equilibrium strategies in terms of the measure of flat-fee investors \( \lambda \); we describe how \( f \) impacts these strategies in Section 3.4.

An investor’s order placement strategy depends on the state of the book upon their arrival at the market. A buyer can expect one of three possible (on-the-equilibrium-path) states of the sell side of the book: no available orders, a sell limit order optimally placed by a flat-fee investor, or a sell limit order optimally placed by a maker-taker investor. Hence, an investor’s order placement strategy must include a best response to each of these three states of the book.

We now show how an investor’s order placement strategy is driven by their best response to an empty limit order book (i.e., their limit order placement strategy). When an investor arrives at the market and the book is empty, they will submit a limit order if the payoff from doing so is non-negative, a criterion easily satisfied by pricing the limit buy (sell) order below (above) the difference between the investors valuation and the fee they pay for their limit order. Conditional on satisfying this constraint, the investor must consider how their choice of limit price impacts both their terms of trade for a filled limit order, and the likelihood that investors of each type will fill the order in the next period. In this way, the limit price can ‘target’ groups of investors to maximize their expected payoff: if an investor from the targeted group enters the market and finds an available limit order at such a price, they will submit a market order against it.

Similarly to the case with only one type of investor, the investor placing the limit order chooses the limit price such that all targeted investors (weakly) prefer submitting a market order against it, to posting a new limit order, with at least some subset of the targeted group indifferent. As before, such price setting is optimal, as any price that is marginally more favourable to a targeted incoming investor is unnecessary, as all such investors will trade against the limit order. A price that is marginally less favourable would lead the subset of targeted investors who were indifferent to market orders and limit orders at the more favourable limit price to prefer submitting a new limit order instead, thus reducing the fill rate of the limit order (and its expected payoff).

When all investors face the same fee schedule, Corollary 1 implies that $\text{bid}^*|_{\lambda=0} \geq \text{bid}^*|_{\lambda=1}$ and $\text{ask}^*|_{\lambda=0} \leq \text{ask}^*|_{\lambda=1}$. In an environment where investors of both types participate, Corollary 1 provides an intuition that type $\theta_{MT}$ investors would never willingly trade against limit orders priced at $\text{bid}^*|_{\lambda=1}$.
or $\text{ask}^*|_{\lambda=1}$. We denote these bid and ask prices that attract only type $\theta_{FF}$ investors as $\text{bid}_{FF}$ and $\text{ask}_{FF}$, respectively. On the other hand, because an available limit order priced at $\text{ask}^*|_{\lambda=0}$ or $\text{bid}^*|_{\lambda=0}$ is desirable to a type $\theta_{MT}$ investor, so would it be to a type $\theta_{FF}$ investor. We denote these bid and ask prices that attract investors of all types as $\text{bid}_{all}$ and $\text{ask}_{all}$, respectively.

**Lemma 1 (Limit Order Fill Rates)** In any equilibrium, a (weakly) larger measure of investors will submit a buy (sell) market order when faced with sell (buy) limit order at $\text{ask}^*_{all}$ ($\text{bid}^*_{all}$), than when faced with sell (buy) limit order at $\text{ask}^*_{FF}$ ($\text{bid}^*_{FF}$).

Lemma 1 illustrates the trade-off between limit order prices and fill rates: more favourable limit prices improve the likelihood that an incoming investor will fill the order.

Continuing our focus on buyers, investors choose the limit price of their limit order by analyzing the difference in payoff from submitting a limit order at price $\text{bid}^*_{all}$ and a limit order at price $\text{bid}^*_{FF}$. Let $g(f_i; \lambda)$ be the difference in expected payoffs to submitting a limit buy order with price $\text{bid}^*_{all}$ and a limit buy order with price $\text{bid}^*_{FF}$, for an investor that pays a fee $f_i$ for a filled limit order.

$$g(f_i; \lambda) = \pi^{LB}(\text{bid}^*_{FF}) - \pi^{LB}(\text{bid}^*_{all}) = \frac{\lambda}{2} \times (V - \text{bid}^*_{FF} - f_i) - \frac{1}{2} \times (V - \text{bid}^*_{all} - f_i)$$

The interpretation of $g(f_i; \lambda)$ is that, if negative, the investor prices any limit orders they submit at $\text{bid}^*_{all}$, whereas $g(f_i; \lambda) > 0$ implies that the investor prices all limit orders at $\text{bid}^*_{FF}$. If $g(f_i; \lambda) = 0$, the investor adopts a mixed strategy over the two limit prices. We summarize the choice of limit price by a probability function $\delta_i(\lambda)$, where an investor of type $\theta_i$ chooses limit price $\text{bid}^*_{all}$ with probability $\delta_i(\lambda)$, and limit price $\text{bid}^*_{FF}$ otherwise.

$$\delta_{MT} = \begin{cases} 1 & \text{if } g(f_M; \lambda) < 0 \\ \delta_{MT} & \text{if } g(f_M; \lambda) = 0 \\ 0 & \text{if } g(f_M; \lambda) > 0 \end{cases}$$

$$\delta_{FF} = \begin{cases} 1 & \text{if } g(\tilde{f}; \lambda) < 0 \\ \delta_{MT} & \text{if } g(\tilde{f}; \lambda) = 0 \\ 0 & \text{if } g(\tilde{f}; \lambda) > 0 \end{cases}$$

Because the flat fee is defined as the weighted average of the maker rebate and taker fee, $f_M \leq \tilde{f} \leq f_T$, a flat-fee investor always earns less than a maker-taker investor when trading with an identically-priced limit order. Hence, $g(f_i; \lambda)$ is increasing in $f_i$, for all $\lambda$, which leads us to the following useful corollary.
Corollary 2 (Limit Prices) In equilibrium, if \( f_M < \bar{f} \), then \( g(f_M; \lambda) < g(\bar{f}; \lambda) \).

From Corollary 2, it follows that if a flat-fee investor submits a limit order priced at \( \text{bid}_{all}^* \) with positive probability, then a maker-taker investor strictly prefers limit orders priced at \( \text{bid}_{all}^* \) (i.e., \( \delta_{FF} > 0 \Rightarrow \delta_{MT} = 1 \)). The reverse is true if maker-taker investors submit limit orders priced at \( \text{bid}_{FF}^* \) with some probability (i.e., \( \delta_{MT} < 1 \Rightarrow \delta_{FF} = 0 \)). Thus, an equilibrium in our environment has a feature that those who pay lower fees for limit orders (receive rebates) will place limit orders at (weakly) more favourable prices.

Using Corollary 2, we can combine (3.13) and (3.14) into the following joint function,

\[
(\delta_{FF}, \delta_{MT}) = \begin{cases} 
(1, 1) & \text{if } g(f_M; \lambda) < 0 \\
(\delta_{FF}, 1) & \text{if } g(f_M; \lambda) = 0 \\
(0, 1) & \text{if } g(\bar{f}; \lambda) < 0 < g(f_M; \lambda) \\
(0, \delta_{MT}) & \text{if } g(\bar{f}; \lambda) = 0 \\
(0, 0) & \text{if } g(\bar{f}; \lambda) > 0 
\end{cases}
\] \hspace{1cm} (3.15)

We can simplify the function in (3.15) one step further by employing the following Lemma about the relationship between \( g(f_i; \lambda) \) and \( \lambda \).

Lemma 2 (Limit Order Thresholds) Given a pair \( (\delta_{FF}, \delta_{MT}) \) that satisfies function 3.15, \( g(f_i; \lambda) \) is increasing in \( \lambda \).

Applying Lemma 2 to the function in (3.15), we can rewrite it as:

\[
(\delta_{FF}, \delta_{MT}) = \begin{cases} 
(1, 1) & \text{if } \lambda \leq \Delta_{FF} \\
(\delta_{FF}, 1) & \text{if } \Delta_{FF} < \lambda < \bar{\lambda}_{FF} \\
(0, 1) & \text{if } \bar{\lambda}_{FF} \leq \lambda \leq \Delta_{MT} \\
(0, \delta_{MT}) & \text{if } \Delta_{MT} < \lambda < \bar{\lambda}_{MT} \\
(0, 0) & \text{if } \lambda \geq \bar{\lambda}_{FF} 
\end{cases}
\] \hspace{1cm} (3.16)

To obtain the above thresholds, we solve the following equilibrium conditions for \( \lambda \):

1. \( \Delta_{FF} = \{ \lambda \mid g(\bar{f}; \delta_{MT} = \delta_{FF} = 1) = 0 \} \)
2. \( \bar{\lambda}_{FF} = \{ \lambda \mid g(\bar{f}; \delta_{MT} = 1, \delta_{FF} = 0) = 0 \} \)
3. \( \Delta_{MT} = \{ \lambda \mid g(f_M; \delta_{MT} = 1, \delta_{FF} = 0) = 0 \} \)
4. \( \bar{\lambda}_{MT} = \{ \lambda \mid g(f_M; \delta_{MT} = \delta_{FF} = 0) = 0 \} \)

Thus, equilibrium limit order placement decisions hinge on the existence of (unique) thresholds in \( \lambda \).
Given investors’ best responses to an empty book as defined by function \(3.16\), we can now show that this function characterizes investors’ decisions for all three possible equilibrium states of the limit order book. Consider an investor who enters the market to buy. When the book is empty, a maker-taker investor who places a limit buy order at \(\text{bid}_{all}^*\) must weakly prefer this limit order over a limit order priced at \(\text{bid}_{FF}^*\). If, instead, a limit sell order priced at \(\text{ask}_{all}^*\) is available, this investor would trade against it, as they are indifferent between a market order at \(\text{ask}_{all}^*\), and a limit order at \(\text{bid}_{all}^*\). Then, given the availability of a limit sell order at \(\text{ask}_{FF}^*\), a maker-taker investor strictly prefers trading against a limit order at \(\text{ask}_{all}^*\) over limit orders priced at \(\text{ask}_{FF}^*\), and thus, a maker-taker investor strictly prefers placing a limit order at \(\text{bid}_{all}^*\) over trading against the limit order priced at \(\text{ask}_{FF}^*\). If, in equilibrium, the maker-taker investor submits a limit order at \(\text{bid}_{FF}^*\), a small modification to the above argument leads us to conclude that the investor prefers placing limit orders priced at \(\text{bid}_{FF}^*\) to any other buy orders, in any state of the book.

A flat-fee buyer’s strategy is similar, except where an (on-the-equilibrium-path) limit order at \(\text{bid}_{FF}^*\) or \(\text{ask}_{FF}^*\) is available. Because a flat-fee buyer is indifferent to trading against a limit order at \(\text{ask}_{FF}^*\), and placing a limit buy order at \(\text{bid}_{FF}^*\), the investor would trade against any on-the-equilibrium-path limit order at \(\text{ask}_{FF}^*\) or \(\text{bid}_{FF}^*\). We can now state the following existence theorem. The equilibrium is illustrated in Figure 3.1.

**Theorem 1 (Existence and Uniqueness)** Given \(V > c\) and \(\lambda \in (0,1)\), there exist unique values \(0 < \lambda_{FF} < \lambda_{FF} < \lambda_{MT} < \lambda_{MT} < 1\) that constitute an equilibrium in threshold strategies, where investors choose limit prices as defined in the function \((3.16)\). Investors’ sell decisions are symmetric to buy decisions.

Theorem 1 outlines the five possible equilibrium types. For a given \(\lambda\), Theorem 1 dictates that the model admits a single equilibrium type, characterized by investors’ limit order pricing decisions \((\delta_{FF}, \delta_{MT})\) outlined in \((3.16)\). For example, the model admits a Type 1 equilibrium when \(\lambda < \lambda_{FF}\), where all investors submit limit orders that attract all investors; the model admits a Type 5 equilibrium when \(\lambda > \lambda_{MT}\), in which case all investors submit limit orders that attract only flat-fee investors.

From Corollary 2, the rebate paid to maker-taker investors incentivizes them to submit limit orders priced more favourably to incoming investors than flat-fee investors. Indeed, we see from figure 3.1 that for all \(\lambda\), maker-taker investors take on a more prominent liquidity-providing role than do flat-fee investors, always placing limit orders at (weakly) better quotes. Moreover, when the measure of flat-fee investors that participate in the market is relatively large, maker-taker investors become de-facto market makers. In this case, it is more profitable to forego potential trades with maker-taker investors.
by quoting wider spreads to target flat-fee investors alone. What is perhaps a surprising corollary to Theorem 1, in our environment, maker-taker investors adopt a market-making role without possessing a speed advantage over other investors. Instead, maker-taker investors need only be provided with an advantage to providing better quotes—via maker rebates—to gain a market-making advantage over other investors.

### 3.3.3 Impact of $\lambda$ on Market Quality and Welfare

To measure trading volume in our environment, we examine the probability that a trade is initiated by the incoming investor in any period $t$. Because a trade occurs only when an incoming investor submits a market order, our measure of trading volume is equivalent to the probability of an incoming investor submitting a market order at period $t$, $Pr(\text{market order}_t)$. In a symmetric and stationary equilibrium, this simplifies to $Pr(\text{MB})$. To compute this measure, we decompose volume into two components: the probability that a maker-taker investor submits a market order $Pr(\text{MB} \mid \theta_{MT})$, and the probability that a flat-fee investor submits a market order $Pr(\text{MB} \mid \theta_{FF})$.

\[
Pr(\text{MB} \mid \theta_{MT}) = \lambda \times \frac{1 - Pr(\text{MB} \mid \theta_{FF})}{2} \times \delta_{FF} + (1 - \lambda) \times \frac{1 - Pr(\text{MB} \mid \theta_{MT})}{2} \times \delta_{MT}
\]  
(3.17)

\[
Pr(\text{MB} \mid \theta_{FF}) = \lambda \times \frac{(1 - Pr(\text{MB} \mid \theta_{FF}))}{2} + (1 - \lambda) \times \frac{(1 - Pr(\text{MB} \mid \theta_{MT}))}{2}
\]  
(3.18)
A trade occurs in the current period when the previous period investor was not able to trade, and they post a limit order on the correct side of the market with an on-the-equilibrium-path limit price. Weighting these components by the proportion of investors in each respective class, we write the total volume function as:

\[
\text{Trading Volume} = \lambda \times \text{Pr}(\text{MB} \mid \theta_{FF}) + (1 - \lambda) \times \text{Pr}(\text{MB} \mid \theta_{MT})
\]  

(3.19)

We examine total volume along the \( \lambda \) dimension, comparing it to the environment where all investors are subject to the same fee schedule (i.e., \( \lambda \in \{0, 1\} \)).

**Proposition 3 (Trading Volume)** Trading volume is maximized for all \( \lambda < \Delta_{FF} \) (Type 1 Equilibrium). Moreover, trading volume in a Type 1 Equilibrium is equal to the volume of a single investor type environment.

When all investors place limit orders priced at \( \text{ask}^*_\text{all} \) or \( \text{bid}^*_\text{all} \), or face the same fee schedule, their limit orders attract all investors that enter the market on the correct side of the book. However, when some investors choose a limit price of \( \text{ask}^*_{FF} \) or \( \text{bid}^*_{FF} \), thereby only attracting a subset of investors, which sacrifices a higher fill rate for a higher return, conditional on the order being filled. Then, because limit buy orders with prices \( \text{bid}_{FF} \) fill with a lower probability (maker-taker investors who wish to sell will pass over this limit order), fewer limit orders are filled in equilibrium, leading to lower total volume. Hence, the maker-taker investors impose a negative externality on flat-fee investors: flat-fee investors either submit limit orders with a limit price, \( \text{bid}_{\text{all}} \), worse than what makes them indifferent to market orders, or, face a reduced fill rate for limit orders submitted at limit price \( \text{bid}_{FF} \).

Our result on trading volume is similar to that of Degryse, van Achter, and Wuyts (2012), who also find, in their analysis of clearing and settlement costs, that an equilibrium where investors submit quotes that target all investors has the highest rate of trade. Intuitively, trading volume is highest when limit buy (sell) orders are not passed over by sellers (buyers).

Our measure of investor welfare computes the per-investor expected gains from trade. Because each investor in our setting has a private valuation for the security, we can follow Bessembinder, Hao, and Lemmon (2012) to define a social welfare measure that reflects allocative efficiency. If a transaction occurs in period \( t \), then the welfare gain is given by the valuation of a buyer, net of the fees paid by the buyer, minus the valuation of a seller, net of the fees paid by the seller. Let \( \text{LO}(\theta_{MT}) \) and \( \text{LO}(\theta_{FF}) \) denote a type \( \theta_{MT} \) or \( \theta_{FF} \) investor who submit (equilibrium) limit orders, respectively. Similarly, let \( \text{MO}(\theta_{MT}) \) and \( \text{MO}(\theta_{FF}) \) denote a type \( \theta_{MT} \) or \( \theta_{FF} \) investor who submit market orders, respectively.
Then, $W$ can be written as the probability that each possible pair of taker and maker meet to trade:

$$W = \Pr(\text{trade} \mid \text{MO}(\theta_{MT}) \cap \text{LO}(\theta_{MT})) \times (V - \text{ask} - f_T + (\text{ask} - (-V) - f_M))$$

$$+ \Pr(\text{trade} \mid \text{MO}(\theta_{MT}) \cap \text{LO}(\theta_{FF})) \times (V - \text{ask} - f_T + (\text{ask} - (-V) - \bar{f}))$$

$$+ \Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{MT})) \times (V - \text{ask} - \bar{f} + (\text{ask} - (-V) - f_M))$$

$$+ \Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{FF})) \times (V - \text{ask} - \bar{f} + (\text{ask} - (-V) - \bar{f}))$$

(3.20)

In every trade, one investor pays for the security and the other receives that payment, thus, the price at which trade occurs does not impact welfare. Similarly for the fees charged, the direct impact of the fees on welfare is the taker fee, $f_T$, for the market order, and the maker fee, $f_M$ for the limit order. It does not matter on average whether the trading parties are type $\theta_{MT}$ or type $\theta_{FF}$ investors because the exchange itself always receives $f_T$ for the filled market order and $f_M$ for the filled limit order. The fact that some investors pay $\bar{f}$ is irrelevant, because the $\bar{f}$ is set by the broker in such a way that the broker breaks even on average. That is, the broker pays fees to the exchange equal to the per-leg cost of filling an order, $c$. Thus, the total expected welfare is equal to the total gains from trade, minus the total fee remitted to the exchange:

$$W = 2 \times (V - c) \times \Pr(\text{trade})$$

(3.21)

A welfare measure that captures allocative efficiency, in this context, thus reflects the likelihood that a trade occurs. Or more simply, how likely is it that an order is not cancelled—that is, how likely is it that the incoming investor does not “pass over” an available limit order. We summarize this in the following proposition.

**Proposition 4 (Social Welfare)** Social welfare comoves with trading volume.

The notion that unfilled limit orders leads to welfare loss is documented empirically by Hollifield, Miller, Sandás, and Slive (2006), where they find that non-executed limit orders account for 70% of all welfare loss in their sample. In general, investors pass over limit orders in one of two cases: one, when an investor arrives on the wrong side of the book, and; two, when an investor finds that the limit order on the right side of the book is posted at an unfavourable price. Because investors arrive to the market on the wrong side of the book with probability $1/2$, the probability that an order goes unfilled in this case is independent of investors’ strategies. In the case where an investor passes over an order on the right side of the book, this is determined by an investor’s type (i.e., which fee schedule imposed on them), the price of the limit order available to them, and the relative split $\lambda$ between the two investor types.

In equilibrium, flat-fee investors will trade against available orders at any equilibrium price, so their strategies do not directly impact welfare. Maker-taker investors, however, will pass over any order that is
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priced with $\text{ask}_{FF}$ or $\text{bid}_{FF}$. Moreover, maker-taker investors will pass over any equilibrium limit order if, in equilibrium, their strategy implies that they submit limit orders with prices $\text{ask}_{FF}$ or $\text{bid}_{FF}$. Because social welfare is proportional to trading volume, welfare is maximized in the cases where investors do not pass over available limit orders on the correct side of the book (in favour of submitting their own).

We also have an interest in the impact of two-tiered systems on trading costs. As noted in the case of a single investor type, the quoted spread is an insufficient measure of total trading costs, as it does not take into account the fees paid by an investor. In this environment, the cum-fee spread is also misleading, as the round-trip cost of using only market orders does not account for the occasions when an investor finds the limit order book empty, and then is forced to make the market in order to trade; a case where floor-fee investors are at a relative disadvantage, as $\bar{f} > \bar{f}_M$.

A better measure to compare the trading costs of maker-taker and floor-fee investors in a two-tiered system is to examine average (per period) profits. We focus on the relative expected profits of floor-fee investors versus maker-taker investors, as we are additionally interested in whether, were $\lambda$ endogenous (i.e., if investors could choose their fee schedule ex-ante) what would be a stable partition (or partitions) of investors into maker-taker and floor-fee. In equilibrium, if an investor can choose whether to adopt the maker-taker or floor-fee schedule, any stable partition(s) (value(s) of $\lambda$) would be where the difference in profits between the two investor groups is zero.

Let the expected profit differential between maker-taker and floor-fee investors be written as $\Delta \Pi = \pi_{\theta_MT} - \pi_{\theta_FF}$.

**Proposition 5 (Profit Differential)** Given $\lambda$, $\Delta \Pi \geq 0$, and $\Delta \Pi_{\lambda} \geq 0$. Moreover, $\Delta \Pi = 0, \forall \lambda \leq \lambda_{FF}$ (Type 1 Equilibrium).

In an environment where some investors pay a floor-fee per trade, investors who face a maker-taker fee schedule are at an advantage. Because a maker-taker investor pays lower fees for limit orders than a floor-fee investor, the maker-taker investor earns greater profits when they submit a limit order. Then, when an investor faces an empty book, a maker-taker investors always earns higher expected profits than floor-fee investors. When an appropriate limit order is available, the only case where a floor-fee investor earns higher profit is if they can submit a market order against a limit order priced at $\text{ask}_{all}^*$. However, on average, the benefit to floor-fee investors from trading against buy orders placed at $\text{ask}_{all}^*$ (the narrower cum-fee spread) is outweighed by the cost of being forced to make the market when the book is empty.

The implication of Proposition 5 is that, in a market where some investors pay maker-taker fees, floor-fee investors would prefer to switch to a maker-taker fee structure, even when their broker sets the floor fee competitively.
Consider the extension where investors who know their private valuation arrive at the market, and subsequently choose whether to trade through the broker, or the exchange directly (with the associated fee schedule). We assume they choose through whom to submit their order before they see the state of the limit order book. Then, Proposition 5 implies the following corollary.

**Corollary 3 (Endogenous Fee Schedules)** Before arriving at the market, let an investor of type \( \theta_i \) choose whether to access the market directly, or through the broker. Then an equilibrium only exists for \( \lambda < \Delta_{EF} \) (Type 1 Equilibrium); moreover, the (mixed strategy) equilibrium is unique, and maximizes total volume and investor welfare.

Corollary 3 suggests that the broker’s redistribution of maker-taker fees creates a disparity of profits between maker-taker and flat-fee investors, privileging the former.

### 3.4 Impact of Fees on Market Quality and Welfare

In this section, we investigate the impact of the split between the maker rebate and the taker fee \( f \), on aspects of market quality, and the welfare of investors, when both \( \theta_{MT} \) and \( \theta_{FF} \) type investors participate in the market. In what follows, we assume that parameters \( V, c \) and \( f \) are such that the quoted spread is positive; that is, \( V - c - 3f > 0 \). For a study of the total fee \( c \), we refer the reader to Colliard and Foucault (2012).

When \( \lambda > 0 \), we look at the impact of the maker-taker fee split, \( f \), on our market quality measures from Section 3.3. In the case of a single fee schedule (i.e., a single investor type), Colliard and Foucault (2012) find that the impact of maker-taker fees is independent of \( f \); Proposition 1 concurs with these findings. When both maker-taker and flat-fee investors participate in the market, however, we find that \( f \) impacts \( g(f_i; \lambda) \), which in turn impacts market quality.

By examining \( g(f_i; \lambda) \) for each type of investor, we find that increasing \( f \) impacts investors’ limit order placement decisions, governed by \( \delta_{MT} \) and \( \delta_{FF} \). Differentiating \( g(f_i; \lambda) \), we find the following:

**Lemma 3 (Fee Split and Limit Prices)** For any \( f \), \( g_f(f; \lambda) < g_f(f_M; \lambda) < 0 \).

Lemma 3 demonstrates that for an increase in the maker-taker fee split, investors will prefer limit buy orders priced at bid\(_{FF}\) for lower values of \( \lambda \). The result is a leftward shift of the investor limit price probability functions \( \delta_{MT} \) and \( \delta_{FF} \) for each \( \lambda \). Lemma 3 then generates the following corollaries, in relation to our results from Section 3.3.
**Corollary 4 (Fee Split and Welfare)** Let $(\lambda, f) = (\hat{\lambda}, \hat{f})$ be such that the equilibrium is of Type 2-5. Then there exists a unique $f^* < \hat{f}$ such that $(\hat{\lambda}, f < f^*)$ is an equilibrium Type 1. Moreover, any $f < f^*$ maximizes total volume and welfare, and minimizes $\Delta \Pi$.

The intuition of Corollary 4 is that if there is a sufficiently large enough measure of flat-fee investors (i.e., $\lambda > \overline{\lambda}_{FF}$) such that flat-fee investors only place limit orders that attract other flat-fee investors, we can improve volume, welfare, and reduce the profit differential between maker-taker investors and flat-fee investors by reducing $f$.

A reduction in $f$ decreases the difference in the fee schedules of maker-taker and flat-fee investors, and flat-fee investors prefer to submit limit orders priced at $\text{ask}_{all}^*$ and $\text{bid}_{all}^*$ for higher values of $\lambda$. At these limit prices, maker-taker investors do not pass over limit orders, submitted by flat-fee investors, and hence the fill rates limit orders placed by flat-fee investors increase, improving volume and investor welfare. Ultimately, investors who must trade through a broker prefer to narrow the gap between the fees they pay for limit orders, and those paid by maker-taker investors.

We note here that, in the case of a monopolistic, profit-maximizing exchange, the interests of investors and the exchange align. In our environment, the exchange pays no costs, and collects a maker fee and a taker fee for each completed trade. Thus, the maximum (average) profit exchange makes per period is the probability that a trade occurs, plus the fees they collect:

$$E[\pi_{\text{exchange}}] = \Pr(\text{trade}) \times (f_T + f_M) = 2 \times \Pr(\text{MB}) \times 2c$$

(3.22)

Which is equal to a rescaling of trading volume by $2c$. Hence, a monopolistic exchange would prefer to eliminate the maker-taker pricing scheme (set $f = 0$) in the presence of a broker who charges a flat-fee $\overline{f}$ for orders they route to the exchange.

Consider the more restrictive setting where $\lambda$ is such that the model admits a Type 3 Equilibrium (i.e., $\overline{\lambda}_{FF} < \lambda < \overline{\lambda}_{MT}$). In this case, flat-fee investors submit limit orders that target other flat fee investors, and maker-taker investors submit limit orders that attract all investors. Suppose further that the exchange is constrained from decreasing the fee split, $f$. Then, we can improve volume and investor welfare (but not maximize it) by increasing $f / f^{**}$ such that $\overline{\lambda}_{MT}(f^{**}) = \lambda$. By increasing $f$, in this way, maker-taker investors find limit orders that target all investors to be relatively too costly of a trade-off the benefit of a higher fill rate. Instead, they target only flat-fee investors, submitting limit orders priced at $\text{ask}_{FF}^*$ and $\text{bid}_{FF}^*$. The result is a Type 5 Equilibrium. Here, while maker-taker investors still pass over limit orders placed by flat-fee investors, flat-fee investors who arrive to the market face fewer empty limit order books (type $\theta_{MT}$ investors never take liquidity), and thus, submit fewer limit orders that go unfilled. Thus when maker-taker investors are induced to act as de-facto market makers,
total volume and overall investor welfare improves.

**Corollary 5 (Maker-Taker Investors as Liquidity Providers)** Given \((\lambda, \hat{f})\) that implies a Type 3 Equilibrium, there exists a unique \(f^{**} > \hat{f}\) such that \((\lambda, f^{**})\) yields a Type 5 Equilibrium. Moreover, increasing \(\hat{f} > f^{**}\) improves total volume and investor welfare.

Lastly, in a Type 5 Equilibrium, all investors always submit limit orders that attract only flat-fee investors. As such, \(g(\hat{f}; \lambda) < g(f_{M}; \lambda) < 0\), implying by equation (3.15) that increasing \(f\) beyond \(f^{**}\) will not change the behavior of either type of investor, and thus cannot further improve welfare or trading volume.

### 3.5 Conclusion

We apply a similar model to Foucault (1999) to study the impact of maker-taker fees in an environment with a two-tiered fee system. When all investors face the same fee schedule, investor behavior is affected only through the total fee charged by the exchange (the taker fee minus the maker rebate), consistent with Colliard and Foucault (2012). When, however, some investors access the exchange directly (and pay maker-taker fees), while others pay flat fees through a broker (e.g., Fidelity cites a flat fee of $7.95 per trade on their website), the split of the total exchange fee into the maker fee and the taker fee plays a meaningful role (even when the maker-taker fees are passed through, on average, to those investors paying a flat-fee).

In a two-tiered system, investors who pay flat fees always submit limit orders that are (weakly) less favourable than ones submitted by maker-taker investors. Our model predicts that trading volume is lower than in a single-tier system (either pure maker-taker or flat-fee system). We also examine investor welfare, using a measure that reflects allocative efficiency. In our model, the measure simplifies to a scaling of expected trading volume, and as a result, investor welfare is lower in a two-tiered fee system. We also find that maker-taker investors earn greater profits on average, compared to flat-fee investors.

When the measure of flat-fee investors is large enough such that they submit less favourably-priced limit orders than maker-taker investors, the split between the maker and taker fee can have a positive impact on market quality and investor welfare. By reducing the split between the maker and taker fee, the order cost differential between the two investor groups declines, and flat-fee investors improve the competitiveness of their limit orders, leading to an improvement in volume and welfare; a large enough reduction can improve trading volume and welfare to the levels of a single-tier system.

Market quality and welfare may improve with a fee split increase, too, if the increase is large enough to induce maker-taker investors to become de-facto market makers. By taking on the role of liquidity
providers, maker-taker investors reduce the occasion that flat-fee investors find the limit order book empty. We find that this reduction leads to an improvement in trading volume and welfare.

Our predictions have important policy implications. First, we find that in markets with a two-tiered fee system, the levels of maker and taker fees have an economic effect beyond that of the total exchange fee. Our results show, in particular, that when the fee is passed through to some investors only on average, through a flat commission, those investors’ trading incentives are different to those who pay taker fees and receive maker rebates for each executed trade. The flat fee reduces volume and worsens welfare by inducing flat-fee investors to post limit orders with less favourable prices, reducing the likelihood of their order being filled. Second, we contribute to the debate on incentives for liquidity provision in limit order markets. Our model suggests that providing maker-taker pricing to a subset of investors induces them to take on a more prominent market-making role—a result that is independent of the relative speed of investors. By endowing some investors with rebates for supplying liquidity, these investors use this advantage to quote more favourably, and at times, pricing investors who pay flat fees for their limit orders out of the market.

Finally, we reiterate the importance of accounting for the exchange trading fees (see, e.g., Angel, Harris, and Spatt (2011), Colliard and Foucault (2012), or Battalio, Shkilko, and Van Ness (2012)), as incentives for liquidity provision offered to one subset of investors may not pass down liquidity improvements to all.
3.6 Appendix

Here we detail all proofs not given in-text.

3.6.1 Proofs in Section 3.3

Preliminaries. In the proofs that follow, we use a simplification of \( \bar{f} \) that introduces the notation \( \varphi(\lambda) \):

\[
\bar{f} = \frac{\Pr(MB^* \mid \theta_{FF}) - \Pr(LB^* \mid \theta_{FF}) \times \Pr(MS^*)}{\Pr(MB^* \mid \theta_{FF}) + \Pr(LB^* \mid \theta_{FF}) \times \Pr(MS^*)} \times f + c = \varphi(\lambda) \times f + c \tag{3.23}
\]

where \( \varphi(\lambda) \) represents the weighted average of the redistribution, \( f \), paid by the broker to the exchange. As a function of \( \delta_{MT} \), \( \delta_{FF} \), and \( \lambda \), we can write explicitly \( \varphi(\lambda) \):

\[
\varphi(\lambda) = \frac{(2 - \lambda - \lambda^2) - (1 + \lambda - 2\lambda^2) \cdot \delta_{FF} - (1 - 2\lambda + \lambda^2) \cdot \delta_{MT} \times \delta_{FF}}{(2 + \lambda + \lambda^2) + (1 - \lambda) \cdot \delta_{FF} + 2(\lambda - \lambda^2) \cdot \delta_{MT} + (1 - 2\lambda + \lambda^2) \cdot \delta_{MT} \times \delta_{FF}} \tag{3.24}
\]

Proof (Proposition 1). From the payoff functions in (3.7), we see that maker and taker fees only enter the investor’s order placement decision as the sum \( f_T + f_M = f_{total} \). Because \( f_{total} = 2c \), investor’s order placement decisions are independent of \( f \). ■

Proof (Proposition 2). The quoted spread, \( \text{ask} - \text{bid} \), is equal to:

\[
S = \frac{2V}{3} - \frac{4f_T - 2f_M}{3} = \frac{2(V - c)}{3} - 2f
\]

Immediately, we see that by increasing \( f \) to \( f + \Delta \), \( S \) increases by \( 2\Delta \). The cum-fee spread is \( S + 2f_T \), which simplifies to \( \frac{2(V + 2c)}{3} \), and is independent of \( f \). ■

Proof (Lemma 1). Consider an investor who wishes to buy. Investors that are subject to flat fees are indifferent between market buy orders \( \text{ask}^*_{FF} \) and limit orders at limit price \( \text{bid}^*_{FF} \). Because \( \text{ask}^*_{all} < \text{ask}^*_{FF} \), market orders at \( \text{ask}^*_{all} \) are preferred to market orders at \( \text{ask}^*_{FF} \), and thus flat fee investors would submit a market order against limit orders with either prices \( \text{ask}^*_{all} \) or \( \text{ask}^*_{FF} \), if available. Investors that are subject to maker-taker fees are indifferent between market buy orders at \( \text{ask}^*_{all} \) and limit orders at limit price \( \text{bid}^*_{all} \). However, because \( \text{ask}^*_{all} < \text{ask}^*_{FF} \), maker-taker investors prefer submitting a limit order at price \( \text{bid}^*_{all} \) over submitting a market order against a limit order with price \( \text{ask}^*_{FF} \). Hence, maker-taker investors only submit market orders against limit orders with price \( \text{ask}^*_{all} \). Then, all investors will submit market buy orders against a limit sell order with price \( \text{ask}^*_{all} \), but only a fraction \( \lambda \) of investors will submit market buy orders against a limit sell order with price \( \text{ask}^*_{FF} \). The argument is similar for investors who sell. ■
Proof (Lemma 2).

I prove this lemma in steps: first, I analyze the behaviour of \( \varphi(\lambda) \) in \( \lambda \), for the given possible (equilibrium) pairs of \((\delta_{FF}, \delta_{MT})\). Then, I use this information to show that \( g(f_M; \lambda) \) and \( g(\check{f}; \lambda) \) are both decreasing in \( \lambda \), given \((\delta_{FF}, \delta_{MT})\).

**Step 1:** From Corollary 2, it must be true that for all \( \delta_{MT} < 1 \), that \( \delta_{FF} = 0 \). Then, taking the first derivative of \( \varphi(\lambda) \), evaluated at \( \delta_{FF} = 0 \):

\[
\frac{\partial \varphi(\lambda)}{\partial \lambda} \mid_{\delta_{FF}=0} = \frac{-2 + 6\delta_{MT} - 8\lambda \delta_{MT} - (7 - 2\lambda(1 + \delta_{MT}))(1 - 4\lambda \delta_{MT})^3}{(2 + \lambda + \lambda^2 + 2\lambda(1 - \lambda)\delta_{MT})^2} \leq 0 \quad (3.25)
\]

Then, again from Corollary 2, it must be true that for all \( \delta_{FF} < 0 \), that \( \delta_{MT} = 1 \). Taking the first derivative of \( \varphi(\lambda) \), evaluated at \( \delta_{MT} = 0 \):

\[
\frac{\partial \varphi(\lambda)}{\partial \lambda} \mid_{\delta_{MT}=1} = \frac{-4(1 - \lambda)(1 - \delta_{FF}^2) - (1 - \delta_{FF})^2(3 - 2\lambda - \lambda^2)}{(2 + 3\lambda - \lambda^2 + (2 - 3\lambda + \lambda)\delta_{FF})^2} \leq 0 \quad (3.26)
\]

Thus, at any possible equilibrium pair \((\delta_{FF}, \delta_{MT})\), \( \varphi'(\lambda) \leq 0 \).

**Step 2:** In this step, I show that \( g(f_M; \lambda) \) is increasing in \( \lambda \). For any pair \((\delta_{FF}, \delta_{MT})\) that satisfies (3.15), take the first derivative of \( g(f_M; \lambda) \) with respect to \( \lambda \). We have,

\[
\frac{\partial g(f_M; \lambda)}{\partial \lambda} = \frac{8(V - c) + (4 - 4\lambda - \lambda^2)(1 - \varphi(\lambda))f - \lambda(4 - \lambda^2)\varphi'(\lambda)f}{(2 + \lambda)^2} \quad (3.27)
\]

Recall that we assume that the bid-ask spread must be positive, implying that \( V - c - 3f > 0 \). Thus, \( \frac{\partial g(f_M; \lambda)}{\partial \lambda} \) is positive by the fact that,

\[
8(V - c) + (4(1 - \lambda) - \lambda^2)(1 - \varphi(\lambda))f > V - c - 3f \quad (3.28)
\]

Hence, \( g(f_M; \lambda) \) is increasing in \( \lambda \).

**Step 3:** Finally, I show that \( g(\check{f}; \lambda) \) is increasing in \( \lambda \), in similar fashion to step 2. For any pair \((\delta_{FF}, \delta_{MT})\) that satisfies (3.15), take the first derivative of \( g(\check{f}; \lambda) \) with respect to \( \lambda \). We have,

\[
\frac{\partial g(\check{f}; \lambda)}{\partial \lambda} = \frac{8(V - c) - (8\varphi(\lambda) - (4(1 - \lambda) - 3\lambda^2)\varphi'(\lambda))f}{(2 + \lambda)^2} \quad (3.29)
\]

Then, to show that the numerator is positive, we evaluate it at \( \lambda \) and \( \delta_{MT} \) such that the negative term in the numerator is at its upper-bound. Doing so yields:

\[
\text{numerator} \left( \frac{\partial g(\check{f}; \lambda)}{\partial \lambda} \right) = \begin{cases} 
8(V - c) - 15f & \text{if } (\delta_{FF}, 1) \\
8(V - c) - (8 + \frac{10\delta_{MT}}{4})f & \text{if } (0, \delta_{MT})
\end{cases} \quad (3.30)
\]
By inspection, numerator \( \frac{\partial g(\bar{f}; \lambda)}{\partial \lambda} \) is always greater than \( 8(V - c - 3f) \), a scaling of the positive bid-ask spread constraint. Thus, \( g(\bar{f}; \lambda) \) is increasing in \( \lambda \).

**Proof (Theorem 1).** The equilibrium existence proof proceeds in three steps: i) we prove the existence of the three pure strategy regions under an empty book, ii) we show the existence of the 2 mixed strategy regions under an empty book, and iii) we show that these strategies hold under the states of the book when either a limit order priced at \( S_{MT} \) or \( S_{FF} \) is available. The argument is made for buy-side investors, but the argument for sellers follows analogously by symmetry.

Expression (3.15) reduces the potential set of pure and mixed-strategy Bayesian equilibria to the set \( \{(1, 1), (\delta_{FF}, 1), (0, 1), (0, \delta_{MT}), (0, 0)\} \).

**Part 1a)** When an investor arrives to buy and finds the limit order book empty, they decide whether to submit a buy limit order with price \( bid_{all} \) or \( bid_{FF} \). The pure strategy \( (\delta_{FF}, \delta_{MT}) = (1, 1) \) is a Bayesian equilibrium of this game if \( \lambda \) is such that expression \( g(\bar{f}; \lambda) \leq 0 \). We need only to consider those investors who pay flat fees in this case, because in any equilibrium where \( \delta_{MT} = 1 \), it automatically follows that \( \delta_{FF} = 1 \). Evaluating \( \lambda \) at 0 and 1, we have,

\[
\begin{align*}
\lambda = 0 & \Rightarrow g(\bar{f}; \lambda) = - (V - bid_{all} - c) < 0 \\
\lambda = 1 & \Rightarrow g(\bar{f}; \lambda) = bid_{FF} - bid_{all} > 0
\end{align*}
\]

where the last line holds by Corollary 1.

Then, by the intermediate value theorem, there exists a \( \Delta_{FF} \) such that for all \( \lambda < \Delta_{FF} \), \( (\delta_{FF}, \delta_{MT}) = (1, 1) \) is an equilibrium. By Lemma 2, we have that \( \frac{\partial g(f_M; \lambda)}{\partial \lambda} > 0 \), implying that \( \lambda_{MT} \) is unique.

**Part 1b)** We now show that \( (\delta_{FF}, \delta_{MT}) = (0, 1) \) is an equilibrium for \( \lambda \in (\overline{\lambda}_{FF}, \lambda_{MT}) \), where \( \overline{\lambda}_{FF} \) and \( \lambda_{MT} \) are unique, and \( \Delta_{FF} < \overline{\lambda}_{FF} < \lambda_{MT} \).

To show that flat fee investors choose \( \delta_{FF} = 0 \), evaluate \( g(\bar{f}; \lambda) \) at \( (\delta_{FF}, \delta_{MT}) = (0, 1) \). Checking the endpoints of \( \lambda \), we have,

\[
\begin{align*}
\lambda = 0 & \Rightarrow g(\bar{f}; \lambda) = \left( - \left( \frac{V - c}{3} - f \times (1 + \varphi(\lambda)) \right) \right) < 0 \\
\lambda = 1 & \Rightarrow g(\bar{f}; \lambda) = bid_{all} - bid_{FF} > 0
\end{align*}
\]

where the last line holds by Corollary 1. Moreover, \( \overline{\lambda}_{FF} \) is unique by the fact that \( \frac{\partial g(f_M; \lambda)}{\partial \lambda} > 0 \) when evaluated at \( (\delta_{FF}, \delta_{MT}) = (0, 1) \), which follows from Lemma 2.

We need now to show that \( \lambda_{FF} < \overline{\lambda}_{FF} \). This follows immediately from comparing \( g(\bar{f}; \lambda) \) when evaluated at \( (\delta_{FF}, \delta_{MT}) = (1, 1) \) and \( (\delta_{FF}, \delta_{MT}) = (0, 1) \). The only difference is that \( \phi(\lambda | \delta_{FF} = 1) = \phi(\lambda | \delta_{FF} = 0) \).
0, \delta_{MT} = 1) > \phi(\lambda \mid \delta_{FF} = 1, \delta_{MT} = 1), which implies that bid_{FF} is less under \((\delta_{FF}, \delta_{MT}) = (0, 1)\), and thus \(g(f_M; \lambda)\) is higher. Therefore, \(g(f_M \mid \delta_{FF} = \delta_{MT} = 1) > g(f_M \mid \delta_{FF} = 0, \delta_{MT} = 1)\) when evaluated at the same \(\lambda\). Hence, \(\Delta_{FF} < \lambda_{FF}\).

Next, we show that there exists a unique \(\Delta_{MT} > \lambda_{FF}\). Because \(f > \bar{f}\), it must be true that \(g(f_M \mid \delta_{FF} = 0, \delta_{MT} = 1) < g(\bar{f} \mid \delta_{FF} = 0, \delta_{MT} = 1)\). Therefore, if there exists a \(\Delta_{MT}\), then \(\Delta_{MT} > \lambda_{FF}\) necessarily. Evaluating \(g(\bar{f}; \lambda)\) at \(\lambda = 0\) and \(\lambda = 1\) when \((\delta_{MT}, \delta_{FF}) = (1, 0)\):

\[
\begin{align*}
\lambda &= 0 \Rightarrow g(f_M; \lambda) = \frac{2}{3} \times (V - bid_{all} - c) < 0 \\
\lambda &= 1 \Rightarrow g(f_M; \lambda) = bid_{all} - bid_{FF} > 0
\end{align*}
\]

which shows that \(\Delta_{MT}\) exists, and by Lemma 2, it is unique. Thus, there exists a unique \(\Delta_{MT} > \lambda_{FF}\) such that \((\delta_{FF}, \delta_{MT}) = (0, 1)\) is an equilibrium for all \(\lambda \in (\lambda_{FF}, \lambda_{MT})\).

**Part 1c** We now show that \((\delta_{FF}, \delta_{MT}) = (0, 0)\) is an equilibrium for \(\lambda \in (\lambda_{MT}, 1)\), where \(\lambda_{MT}\) is unique, and \(\Delta_{MT} < \lambda_{MT}\).

Because the expressions in (3.33) still hold, we need only show that \(\frac{\partial g(f_M; \lambda)}{\partial \lambda} > 0\) when evaluated at \((\delta_{FF}, \delta_{MT}) = (0, 0)\). As in previous parts, we achieve this through Lemma 2. Thus, there exists a unique \(\lambda_{MT} \in (0, 1)\) such that \((\delta_{FF}, \delta_{MT}) = (0, 0)\). Lastly, to see that \(\lambda_{MT} > \Delta_{MT}\), note that \(\hat{f}(\delta_{MT} = 1, \delta_{FF} = 0) > \hat{f}(\delta_{MT} = \delta_{FF} = 0)\) for all \(\lambda\). Then, \(\lambda_{MT} > \Delta_{MT}\) follows by the similar argument of \(\lambda_{FF} > \Delta_{FF}\) in part 1b.

**Part 2a** We show here that for each \(\lambda \in (\Delta_{FF}, \lambda_{FF})\), there exists a unique mixed-strategy equilibrium, \((\delta_{FF}, 1), \delta_{FF} \in (0, 1)\). Let \(A = (\Delta_{FF}, \lambda_{FF})\). A mixed-strategy equilibrium exists if there exists some \(\delta_{FF}\) where \(\lambda \in A\) such that \(g(\bar{f}; \lambda) = 0\). We know that, for all \(\lambda \in A\), \(g(\bar{f} \mid \delta_{FF} = \delta_{MT} = 1) < 0\) and \(g(\bar{f} \mid \delta_{FF} = 0, \delta_{MT} = 1) > 0\). Hence, we search for a value of \(\delta_{FF}\) for each \(\lambda\), such that \(g(\bar{f}; \lambda) = 0\).

If we differentiate \(g(\hat{f}; \lambda)\) with respect to \(\delta_{FF}\), holding \(\delta_{MT} = 1\), we find that,

\[
\frac{\partial g(\hat{f}; \lambda)}{\partial \delta_{FF}} = \frac{-(1 - 2\lambda + \lambda^2) \times (4 + \lambda)}{(3 + 3\lambda - \lambda^2) + (1 - 2\lambda + \lambda^2) \cdot \delta_{FF})^2} < 0
\]

Because \(g(\hat{f}; \lambda)\) is continuous and decreasing in \(\delta_{FF}\) at each \(\lambda \in A\), there must exist a unique \(\delta_{FF}\) for each \(\lambda\) such that \(g(\hat{f}; \lambda) = 0\).

**Part 2b** Similarly, we show that for each \(\lambda \in (\Delta_{MT}, \lambda_{MT})\), there exists a unique mixed-strategy equilibrium, \((0, \delta_{MT}), \delta_{MT} \in (0, 1)\). Let \(B = (\Delta_{MT}, \lambda_{MT})\). A mixed-strategy equilibrium exists if there exists some \(\delta_{MT}\) where \(\lambda \in B\) such that \(g(f_M; \lambda) = 0\). We know that, for all \(\lambda \in B\), \(g(f_M \mid \delta_{FF} = 0, \delta_{MT} = 1) < 0\) and \(g(f_M \mid \delta_{FF} = \delta_{MT} = 0) > 0\). Hence, we search for a value of \(\delta_{MT}\) for each \(\lambda\), such that \(g(f_M; \lambda) = 0\).
If we differentiate \( g(f_M; \lambda) \) with respect to \( \delta_{MT} \), holding \( \delta_{FF} = 0 \), we find that,

\[
\frac{\partial g(f_M; \lambda)}{\partial \delta_{MT}} = \frac{-(2 - \lambda - \lambda^2) \times (4 + \lambda)}{((2 + 3\lambda - \lambda^2) + (2 - 2\lambda + \lambda^2) \cdot \delta_{MT})^2} < 0 \tag{3.35}
\]

Because \( g(f_M; \lambda) \) is continuous and decreasing in \( \delta_{MT} \) at each \( \lambda \in B \), there must exist a unique \( \delta_{MT} \) for each \( \lambda \) such that \( g(f_M; \lambda) = 0 \).

**Part 3a** Now, we verify that all equilibrium strategies that hold for an empty book also hold when investors arrives to a book that has a limit order priced at \( \text{ask}_{all} \) when they wish to buy. In this case, all investors should submit market

For maker-taker investors, if their strategy is such that \( \text{LB(bid}_{all}) \succ \text{LB(bid}_{FF}) \), then \( \text{MB(ask}_{all}) \sim \text{LB(bid}_{all}) \) implies that they will submit a market order against the order at \( \text{ask}_{all} \) (by assumption), as is consistent with their strategy. If their strategy dictates that \( \text{LB(bid}_{FF}) \succ \text{LB(bid}_{all}) \), then because \( \text{LB(bid}_{all}) \sim \text{MB(ask}_{all}) \), the investor will still submit a limit order at \( \text{bid}_{FF} \), consistent with their equilibrium strategy.

For flat-fee investors, if their strategy is such that \( \text{LB(bid}_{all}) \succ \text{LB(bid}_{FF}) \), then \( \text{MB(ask}_{all}) \succ \text{LB(bid}_{all}) \) implies that they will submit a market order against the order at \( \text{ask}_{all} \), as is consistent with their strategy. If their strategy dictates that \( \text{LB(bid}_{FF}) \succ \text{LB(bid}_{all}) \), then because \( \text{ask}_{all} \prec \text{ask}_{FF} \), \( \text{MB(ask}_{all}) \succ \text{MB(ask}_{FF}) \sim \text{LB(bid}_{FF}) \succ \text{LB(bid}_{all}) \), the investor will submit a market order, consistent with their equilibrium strategy.

**Part 3b** Now, we verify that all equilibrium strategies that hold for an empty book also hold when investors arrives to a book that has a limit order priced at \( \text{ask}_{FF} \) when they wish to buy. In this case, all investors should submit market

For maker-taker investors, if their strategy is such that \( \text{LB(bid}_{all}) \succ \text{LB(bid}_{FF}) \), then \( \text{MB(ask}_{all}) \sim \text{LB(bid}_{all}) \succ \text{MB(ask}_{FF}) \) implies that they will submit \( \text{LB(bid}_{all}) \), as is consistent with their strategy. If their strategy dictates that \( \text{LB(bid}_{FF}) \succ \text{LB(bid}_{all}) \), then because \( \text{LB(bid}_{all}) \sim \text{MB(ask}_{all}) \succ \text{MB(ask}_{FF}) \), the investor will still submit a limit order at \( \text{bid}_{FF} \), consistent with their equilibrium strategy.

For flat-fee investors, if their strategy is such that \( \text{LB(bid}_{all}) \succ \text{LB(bid}_{FF}) \), then \( \text{MB(ask}_{FF}) \sim \text{LB(bid}_{FF}) \) implies that they will submit a limit order at \( \text{bid}_{all} \), as is consistent with their strategy. If their strategy dictates that \( \text{LB(bid}_{FF}) \succ \text{LB(bid}_{all}) \), then because \( \text{ask}_{all} \prec \text{ask}_{FF} \), \( \text{MB(ask}_{all}) \succ \text{MB(ask}_{FF}) \sim \text{LB(bid}_{FF}) \succ \text{LB(bid}_{all}) \), the investor will submit a market order, consistent with their equilibrium strategy.

**Proof (Proposition 3).** First, we solve the system (3.17)-(3.18) in terms of \( \delta_{MT}, \delta_{FF} \) and \( \lambda \), and
substitute into the volume expression in (3.19).

$$\text{Volume} = \frac{2\lambda + \delta_{MT} \cdot (2 - 3\lambda + \lambda^2) + \delta_{FF} \cdot (\lambda - \lambda^2)}{2(2 + \lambda) + \delta_{MT} \cdot (2 - \lambda - \lambda^2) - \delta_{FF} \cdot (\lambda - \lambda^2)}$$

(3.36)

By setting $\lambda = 0$ (which, from equation (3.15), implies that $\delta_{MT} = \delta_{FF} = 1$) we see that the left limit of the volume equation (3.36) is equal to 1/3. Similarly, setting $\lambda = 1$ generates the same result. The proposition then follows by showing that the volume equation in (3.36) is less than 1/3, for all $\lambda \in (\Delta_{FF}, 1)$, and equal to 1/3 on $\lambda \in (0, \Delta_{FF}]$.

$$\begin{align*}
\text{Volume} & = \frac{2\lambda + \delta_{MT} \cdot (2 - 3\lambda + \lambda^2) + \delta_{FF} \cdot (\lambda - \lambda^2)}{2(2 + \lambda) + \delta_{MT} \cdot (2 - \lambda - \lambda^2) - \delta_{FF} \cdot (\lambda - \lambda^2)} < \frac{1}{3} \\
\iff -4(1 - \lambda) \times (1 - \delta_{MT}(1 - \lambda) - \delta_{FF}\lambda) & < 0
\end{align*}$$

(3.37) (3.38)

Now, let $\lambda \in [0, \Delta_{FF})$. Then, we have $(\delta_{MT}, \delta_{FF}) = (1, 1)$. The left-hand side of equation (3.38) is then equal to zero, showing that on $[0, \Delta_{FF})$, volume is equal to 1/3. Thus, trading volume is always lower when $\lambda \in (\Delta_{FF}, 1)$ than on $\lambda \in [0, \Delta_{FF}] \cup \{1\}$. ■

**Proof (Proposition 4).**

Total expected welfare, $W$, is equal to the probability-weighted gains from trade for each pair of counterparties (i.e., for each maker and taker combination of $\theta_{MT}$ and $\theta_{FF}$ investors.) Recall the $W$ function:

$$W = \Pr(\text{trade} \mid \text{MO}(\theta_{MT}) \cap \text{LO}(\theta_{MT})) \times (V - \text{ask} - f_T + (\text{ask} - (-V) - f_M)) + \Pr(\text{trade} \mid \text{MO}(\theta_{MT}) \cap \text{LO}(\theta_{FF})) \times (V - \text{ask} - f_T + (\text{ask} - (-V) - \bar{f})) + \Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{MT})) \times (V - \text{ask} - \bar{f} + (\text{ask} - (-V) - f_M)) + \Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{FF})) \times (V - \text{ask} - \bar{f} + (\text{ask} - (-V) - \bar{f}))$$

(3.39)

Substituting $f_T = f + c$, $f_M = c - f$, and $\bar{f} = \varphi(\lambda) \cdot f + c$, we simplify equation (3.39) to:

$$W = 2 \times \Pr(\text{trade}) \times (V - c) - 2 \times \varphi(\lambda) \times \Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{FF})) \times f + \left(\Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{MT})) - \Pr(\text{trade} \mid \text{MO}(\theta_{MT}) \cap \text{LO}(\theta_{FF}))\right) \times f - \varphi(\lambda) \times \left(\Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{MT})) + \Pr(\text{trade} \mid \text{MO}(\theta_{MT}) \cap \text{LO}(\theta_{FF}))\right) \times f$$

(3.40)

But we can rewrite $\varphi(\lambda) = \frac{\Pr(\text{trade} \mid \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{FF})) - \Pr(\text{trade} \mid \text{LO}(\theta_{FF}))\Pr(\text{trade} \mid \theta_{FF})}{\Pr(\text{trade} \mid \theta_{FF})}$, which leads the denominator of $\varphi(\lambda)$ in
equation (3.40) to cancel out. $W$ then simplifies to:

$$W = 2 \cdot \Pr(\text{trade}) \times (V - c) - (\Pr(\text{trade} | \text{MO}(\theta_{FF})) - \Pr(\text{trade} | \text{LO}(\theta_{FF}))) \times f$$

$$+ (\Pr(\text{trade} | \text{MO}(\theta_{FF}) \cap \text{LO}(\theta_{MT})) - \Pr(\text{trade} | \text{MO}(\theta_{MT}) \cap \text{LO}(\theta_{FF}))) \times f$$

$$= 2 \cdot \Pr(\text{trade}) \times (V - c)$$

But $\Pr(\text{trade}) = \Pr(\text{MB})$, which is, by definition, our measure of volume. Hence, social welfare is simply our volume measure, scaled by the gains from trade, $2 \times (V - c)$. ■

**Proof (Proposition 5).**

We prove this proposition in 4 steps, according to the equilibrium limit order placement probabilities, $\delta_{MT}$ and $\delta_{FF}$. In each, we show that, for a given subset of $(\delta_{MT}, \delta_{FF}) \in (0, 1)^2$, $\Delta \Pi \geq 0$. As we have done throughout, the argument focuses on investors that enter the market to buy.

**Step 1.** Consider $(\delta_{MT}, \delta_{FF}) = (0, 0)$, which is the unique equilibrium on $[0, \Delta_{FF}]$. In this equilibrium, investors only trade with market orders that trade against quotes at $\text{ask}^*_\text{all}$, or with limit orders placed at $\text{bid}^*_\text{all}$. Thus, we have $\Delta \Pi$ equal to:

$$\pi^{MB}_{\theta_{MT}}(\text{ask}^*_\text{all}) \geq \Pr(\text{MB} | \theta_{FF}) \times \pi^{MB}_{\theta_{FF}}(\text{ask}^*_\text{all})$$

$$+ (1 - \Pr(\text{MB} | \theta_{FF})) \times \pi^{LB}_{\theta_{FF}}(\text{bid}^*_\text{all})$$

$$V - \text{ask}^*_\text{all} - f_T \geq \Pr(\text{MB} | \theta_{FF}) \times (V - \text{ask}^*_\text{all} - \bar{f})$$

$$+ (1 - \Pr(\text{MB} | \theta_{FF})) \times \frac{V - \text{bid}^*_\text{all} - \bar{f}}{2}$$

where the left-hand side of the system (maker-taker profit) is equal to $\pi^{MB}_{\theta_{MT}}(\text{ask}^*_\text{all})$, because in equilibrium, $\pi^{MB}_{\theta_{MT}}(\text{ask}^*_\text{all}) = \pi^{LB}_{\theta_{MT}}(\text{bid}^*_\text{all})$. Then, applying symmetry, and solving the system of equations (3.17)-(3.18) for $\Pr(\text{MB} | \theta_{FF})$, allows us to simplify (3.42):

$$V - \text{ask}^*_\text{all} - f - c \geq \frac{1}{3} \times (V - \text{ask}^*_\text{all} - c) + \frac{1}{3} \times (V + \text{ask}^*_\text{all} - c)$$

$$\iff \frac{2(V - c)}{3} \geq \frac{2(V - c)}{3}$$

Thus, for all $\lambda \in [0, \Delta_{FF}], \Delta \Pi \geq 0$.

**Step 2.** Consider $(\delta_{MT}, \delta_{FF}) = (0, \delta_{FF})$, which is the unique equilibrium on $(\Delta_{FF}, \overline{\lambda}_{FF})$. In this equilibrium, flat-fee investors may trade with market orders that trade against quotes at $\text{ask}^*_\text{all}$ or $\text{ask}^*_F F$, or with limit orders placed at $\text{bid}^*_\text{all}$ or $\text{bid}^*_F F$. Here, because $\pi^{LB}_{\theta_{FF}}(\text{bid}^*_\text{all}) = \pi^{LB}_{\theta_{FF}}(\text{bid}^*_F F) = \pi^{MB}_{\theta_{FF}}(\text{ask}^*_F F)$, in equilibrium, we separate the flat-fee investor’s profit functions into $\Pr(\text{MB} \text{ at } \text{ask}^*_\text{all} | \theta_{FF})$, the probability that a flat-fee investor submits a market order against a limit order with price $\text{ask}^*_\text{all}$, and the
probability of any other outcome. $\Delta \Pi$ is then equal to:

$$
\pi_{\theta_{MT}}^{MB}(ask_{all}^{*}) \geq \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF}) \times \pi_{\theta_{FF}}^{MB}(ask_{all}^{*}) \\
+ (1 - \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})) \times \pi_{\theta_{FF}}^{LB}(bid_{all}^{*}) \\
V - ask_{all}^{*} - f_T \geq \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF}) \times (V - ask_{all}^{*} - \bar{f}) \\
+ (1 - \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})) \times \left(\frac{V - bid_{all}^{*} - \bar{f}}{2}\right)
$$

Then, applying symmetry once more, and solving the system of equations (3.17)-(3.18) which generates $\Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})$, we arrive at:

$$
\frac{2(V - c)}{3} \geq \frac{2(V - c) - 1 + \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})}{2} \times f \\
- \frac{(1 - 3 \cdot \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})) \cdot \varphi(\lambda)}{2} \times f
$$

Which follows from the fact that $\Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF}) < \Pr(\text{MB \mid \theta_{FF}}) \leq 1/3$. Thus, for all $\lambda \in (\Delta_{FF}, \lambda_{FF})$, $\Delta \Pi \geq 0$.

**Step 3.** Consider $(\delta_{MT}, \delta_{FF}) = (\delta_{MT}, 1)$, which is the unique equilibrium on $[\lambda_{FF}, \lambda_{MT}]$. In this equilibrium, flat-fee investors may trade with market orders that trade against quotes at $ask_{all}^{*}$ or $ask_{FF}^{*}$, or with limit orders placed at $bid_{FF}^{*}$. Again, because $\pi_{\theta_{FF}}^{LB}(bid_{FF}^{*}) = \pi_{\theta_{FF}}^{MB}(ask_{FF}^{*})$, in equilibrium, we separate the flat-fee investor’s profit functions into $\Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})$, the probability that a flat-fee investor submits a market order against a limit order with price $ask_{all}^{*}$, and the probability of any other outcome. Moreover, the left-hand side of the system (the maker-taker profit) only shows $\pi_{\theta_{MT}}^{MB}(ask_{all}^{*})$, because $\pi_{\theta_{MT}}^{MB}(ask_{all}^{*}) = \pi_{\theta_{MT}}^{LB}(bid_{all}^{*}) = \pi_{\theta_{MT}}^{LB}(bid_{FF}^{*})$. $\Delta \Pi$ is then equal to:

$$
\pi_{\theta_{MT}}^{MB}(ask_{all}^{*}) \geq \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF}) \times \pi_{\theta_{FF}}^{MB}(ask_{all}^{*}) \\
+ (1 - \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})) \times \pi_{\theta_{FF}}^{MB}(ask_{FF}^{*}) \\
V - ask_{all}^{*} - f_T \geq \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF}) \times (V - ask_{all}^{*} - \bar{f}) \\
+ (1 - \Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF})) \times \left(\frac{V - ask_{FF}^{*} - \bar{f}}{2}\right)
$$

To show that $\Delta \Pi \geq 0$ in this case, we endeavour to show the opposite, by examining an upper bound for $\pi_{\theta_{FF}}$, and showing that, even at that upper bound, $\Delta \Pi \geq 0$. We first assume the upper-bound for $\Pr(\text{MB at } ask_{all}^{*} \mid \theta_{FF}) = 1/3$, as a market order priced at $ask_{all}^{*}$ is the most profitable order for a type $\theta_{FF}$ investor. We then also assume that $\varphi(\lambda) = 0$, as the cost of a market order at $ask_{FF}^{*}$, $ask_{FF}^{*} + \bar{f}$ is
minimized at \( \varphi(\lambda) = 0 \). Then, we arrive at:

\[
V - \text{ask}^*_M - f_T \geq V - \frac{(2\text{ask}^*_F + \text{ask}^*_M)}{3} - \bar{f}
\]

\[
\frac{2}{3} \times \left( \frac{V - c}{3} - f \right) + f < \frac{2}{3} \times \frac{2 - \lambda}{2 + \lambda} \times (V - c)
\]

Then, by our assumption that the bid-ask spread is positive \((V - c > 3f)\), we substitute \( \frac{V - c}{3} = f \), to arrive at:

\[
\frac{2}{9} \times (V - c) < \frac{2 - \lambda}{2 + \lambda} \times (V - c)
\]

Which holds for all \( \lambda \). Thus, for all \( \lambda \in (\lambda_F, \lambda_F) \), \( \Delta \Pi \geq 0 \).

**Step 4.** As a final step, we examine \((\delta_{MT}, \delta_{FF}) = (1, 1)\), which is the unique equilibrium on \([\lambda_{MT}, 1]\). Here, flat-fee investors only trade with market orders that trade against quotes at \( \text{ask}^*_F \), or with limit orders placed at \( \text{bid}^*_F \). Maker-taker investors only submit limit orders priced at \( \text{bid}^*_F \), and never use market orders. We arrive at the result that \( \Delta \Pi \geq 0 \) through corollary 2, which implies that \( \pi_{0,MT}^{\text{LB}}(\text{bid}^*_F) > \pi_{0,FF}^{\text{LB}}(\text{bid}^*_F) \), for all \( \lambda \). Thus, for all \( \lambda \in [0, \lambda_F] \), \( \Delta \Pi \geq 0 \).

This concludes the proof that for all \( \lambda \in [0, 1] \), \( \Delta \Pi \geq 0 \).

**Proof (Corollary 3).** From Proposition 5, we have that for all \( \lambda \in (\lambda_F, 1] \), a maker-taker investor earns positive profits in equilibrium. Let an investor enter the market to trade. Because they choose whether to trade through the broker or through the exchange, before knowing the state of the limit order book, any investor would prefer to pay fees to the exchange directly, as the expected profit from trading through the exchange is always greater. Thus, \( \lambda \in (\lambda_F, 1] \) cannot be an equilibrium when fee schedules are endogenous.

For any \( \lambda \in [0, \lambda_F] \), Proposition 5 yields that all investors earn zero profits. Thus, the unique mixed strategy equilibrium for each \( \lambda \in [0, \lambda_F] \) is where an investor chooses to send their order to the broker with probability \( \lambda \), and directly to the exchange with probability \((1 - \lambda)\). It follows from Propositions 3 and 4 that the resulting equilibrium maximizes volume and welfare.

### 3.6.2 Proofs in Section 3.4

**Proof (Lemma 3).** First, we show that \( g(f_i; \lambda) \) is decreasing in \( f \) for all \( f_i \in \{f_M, \bar{f}\} \). Evaluating \( f_i \) at \( f_M \) and differentiating by \( f \),

\[
g_f(f_M) = -\lambda \times \frac{2(1 - \varphi(\lambda)) + \lambda \times (1 + \varphi(\lambda))}{2 + \lambda} < 0
\]
Then, evaluating $f_i$ at $\bar{f}$ and differentiating by $f$,

$$g_f(\bar{f}) = -\frac{2(1 + \varphi(\lambda)) + \lambda - 3\varphi(\lambda) \times \lambda}{2 + \lambda} < 0$$

Then, to show that $g_f(\bar{f}) < g_f(f_M)$, we compare the above.

$$-\frac{2(1 + \varphi(\lambda)) + \lambda - 3\varphi(\lambda) \times \lambda}{2 + \lambda} < -\lambda \times \frac{2(1 - \varphi(\lambda)) + \lambda \times (1 + \varphi(\lambda))}{2 + \lambda}$$

$$\iff (1 + \varphi(\lambda)) \times (1 - \lambda) > 0$$

which holds for all $\lambda \in (0, 1)$.

Proof (Corollary 4). It follows from Lemma 3 that, because $g(\bar{f}; \lambda)$ increases at a slower rate in $f$ (decreases at a faster rate) than $g(f_M; \lambda)$, we need not worry about $g(f_M; \lambda)$ crossing $g(\bar{f}; \lambda)$.

Then, $\hat{\lambda} = \Delta_{FP}(0) = 1$, there must be an $f^* \in (0, \bar{f})$ such that $\lambda_{FP}(f^*) = \hat{\lambda}$.

Finally, $\hat{\lambda} = \Delta_{FP}$ implies that $(\delta_{MT}, \delta_{FF}) = (0, 0)$. It follows by Propositions 3, 4 and 5 that volume and welfare are maximized, and $\Delta \Pi$ is minimized.

Proof (Corollary 5). First, we know that by (3.15), $\delta_{FF} = 0$ for all $\lambda \in [\lambda_{FF}, 1]$. Then, the volume expression in (3.36) simplifies to:

$$\text{Volume} = \frac{2\lambda + \delta_{MT} - (2 - 3\lambda + \lambda^2)}{2 \times (2 + \lambda) + \delta_{MT} \times (2 - \lambda - \lambda^2)}$$

Further, we know that $\frac{\partial \delta_{MT}}{\partial f} > 0$ by Lemma 3, as an increase in $f$ moves the thresholds in $\lambda$ that define $\delta_{MT}$ leftward. Thus, to show that trading volume increases in $f$ for all $\lambda \in [\lambda_{FF}, \lambda_{MT}]$, we differentiate (3.57) by $f$ to get:

$$\frac{\partial \text{Volume}}{\partial f} = \frac{2 - 3\lambda + \lambda^3}{(4 + 2\delta_{MT} + 2\lambda - \delta_{MT}(\lambda - \lambda^2))^2} \times \frac{\partial \delta_{MT}}{\partial f}$$

Then, the numerator of $\frac{\partial \text{Volume}}{\partial \delta_{MT}}$ is positive by the fact that evaluating $\lambda = 0$, the numerator is equal to 2, and evaluating at $\lambda = 1$, the numerator is equal to zero. Then, by differentiating by $\lambda$, we get $-3 + 3\lambda$, which is negative for all $\lambda \in [0, 1]$. Thus, the numerator of $\frac{\partial \text{Volume}}{\partial \delta_{MT}} > 0$ for all $\lambda \in [\lambda_{FF}, \lambda_{MT}]$, implying that $\frac{\partial \text{Volume}}{\partial f} > 0$ on $\lambda \in [\lambda_{FF}, \lambda_{MT}]$. Finally, it follows from Proposition 4 that welfare also increases on $\lambda \in [\lambda_{FF}, \lambda_{MT}]$.
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