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1 **Analysis of consolidation with constant rate of displacement**

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7 **Abstract**

8 The ‘constant rate of strain’ consolidation test has been widely used for several
9 decades to provide a continuous load:displacement response. An exact theoret-
10 ical solution can be obtained for this problem. Two other solution techniques
11 have pedagogic advantage in their simplicity, their ability to clarify the phases
12 of response of the consolidating sample, and their ability to be extended to in-
13 corporate additional details of material response or testing configuration. The
14 parabolic isochrone technique imposes a particular mode shape on the isochrones
15 at all times. Treating the sample as a single system, and applying the boundary
16 conditions at the system level, the governing equation becomes an ordinary dif-
17 ferential equation. For more elaborate soil properties or experimental procedures
18 a finite difference description of the problem is readily programmed and solved.

19 These solution techniques are used to illustrate the importance of distinguish-
20 ing between the observed response of the system - the soil specimen under test
21 - and the behaviour of the soil elements that make up this system. Examples
22 are given of erroneous conclusions that might be drawn if this distinction is not
23 recognised.

24 *Keywords:* consolidation, analysis, isochrones, constant rate of strain, seepage

25 **1 Introduction**

26 Different configurations of consolidation test are used for the determination of
27 flow and deformation properties of the soil. The standard incremental loading test
28 produces a well-known analytical solution, but this mode of loading generates
29 high gradients of pore pressure which may not realistically develop in prototype
30 applications (Stolle and Stolle, 2011).

31 The constant rate of strain - or more precisely constant rate of displacement
32 (CRD) - consolidation test is valuable because it is performed as a continuous
33 process and not as a series of discrete stages, and that consequently, for example,
34 a more reliable estimate of preconsolidation pressure is possible (Pu et al., 2013).
35 The constant rate of displacement consolidation test has been in widespread use
36 since the 1960s (Wissa et al., 1971; Smith and Wahls, 1969; Sällfors, 1975; Lee
37 et al., 1993). The present paper revisits the analysis of this test using different
38 solution techniques, distinguishing between an initial transient phase and a sub-
39 sequent long term phase. Identification of appropriate dimensionless quantities
40 broadens the applicability of the analytical results.

41 Whereas the individual increments of an incremental loading oedometer test
42 might reasonably each be described with a constant set of soil parameters (grouped
43 into a coefficient of consolidation c_v), the constant rate of displacement test is de-
44 liberately run without intermediate pauses and may produce significant changes
45 in geometry (sample thickness) and associated changes in soil properties. Analy-
46 ses by Lee (1981) show, not surprisingly, that the inadequacy of the small strain

analysis becomes more apparent as the applied displacement rate increases.

More elaborate large strain analyses - incorporating more extensive variation of material properties - have been described by Stolle and Stolle (2011) and Pu et al. (2013). As the realism (and complexity) of the problem being analysed increases so the need to adopt sophisticated numerical analysis solution procedures becomes more inevitable. The results of analyses have been used to establish limits to displacement rates that should be adopted for constant rate of displacement testing (Ozer et al., 2012; Feng, 2010). However, as Lee (1981) shows - and as will become evident from the dimensionless analyses reported here - it is not displacement rate on its own but rather a grouping of displacement rate, coefficient of consolidation and specimen size which limits the success of the small strain approach.

Solution of the governing equation for the boundary conditions of this CRD test is thus not new. The intention here, however, is to present some insights into the constant rate of displacement test obtained from simple analyses. The analyses may be deficient in their neglect of changes in geometry but the insights are expected to be relevant also to large strain analyses.

There are two aims for this paper: first, to show that there are rather straightforward routes by which solutions may be obtained to the analysis of the application of the diffusion equation to the constant rate of displacement consolidation test, and to emphasise the importance of identifying relevant dimensionless groups for formulation of the problem (Palmer, 2008); and second to use these solutions to illustrate the importance of distinguishing between the behaviour of individual soil elements and the behaviour of the system that is the CRD specimen under test.

(i) Solutions: Three solution techniques for the governing diffusion equation will

72 be briefly described. A theoretical solution can be directly applied to our consoli-
73 dation problem in certain restricted circumstances.

74 Alternatively, assuming a particular mode shape for the variation of pore pres-
75 sure within the sample, the problem can be analysed at the system level ensuring
76 that the conservation of volume of soil particles and water is maintained for the
77 specimen as a whole but not for every location or element within the specimen.
78 The mathematics of such mode shape analyses are simpler but the technique is re-
79 markably powerful and instructive. For linear problems, some effects of interest -
80 such as the analysis of non-monotonic loading - can be included by superposition
81 of separate loading episodes using either of these solutions.

82 As a third possibility, the governing equation can be written in a finite differ-
83 ence form for which the solution procedure is rather simple. Independent variation
84 of stiffness and hydraulic conductivity during consolidation can then be readily
85 included in the solution.

86 The grouping of these solution techniques, and the illustration of the potential
87 to apply them to realistic extensions of the simplest problem, may be of interest
88 both to practising engineers and to students who may discover features of this and
89 other consolidation situations which are accessible through straightforward analy-
90 sis. The computational challenge of implementing these three solution techniques
91 is not great.

92 (ii) *System and element*: The diffusion equation which governs the consolidation
93 of soils applies the volume conservation requirements at every point within the
94 soil. The response of the soil specimen subjected to constant rate of displacement
95 testing is usually presented in terms of the externally observed *system* response
96 - the load and displacement - which are integrated effects of variations of effec-

Fig. 1: Sample height H subjected to constant rate of boundary displacement $\rho_0 = Ct$.

97 tive stresses and strains for each infinitesimal *element* within the specimen. It is
 98 then tempting to forget these variations and to assume that the *system* response
 99 is actually the response of a single soil *element*. The various solution techniques
 100 will be used to obtain analytical results which explore the effects of jumps in im-
 101 posed displacement rates, of cycles of reversal of displacement direction, and of
 102 systematic variations in stiffness and hydraulic conductivity. If the distinction be-
 103 tween *element* and *system* is not recognised then quite erroneous conclusions can
 104 be drawn concerning various aspects of the behaviour of the soil.

105 **2 One-dimensional consolidation: governing equation**

106 The consolidation is driven by a constant rate of displacement $C = d\rho_0/dt$ at
 107 the drainage boundary ($z = 0$, Fig 1). Assuming that the soil particles and pore
 108 fluid are essentially incompressible, conservation of mass is equivalent to conser-
 109 vation of volume. The rate of displacement is equal to the rate of outflow of water,
 110 controlled by the exit hydraulic gradient. With compressive stresses and strains

assumed positive, the governing equation for a material with one-dimensional stiffness E_o , hydraulic conductivity k , pore fluid unit weight γ_w (coefficient of consolidation $c_v = kE_o/\gamma_w$) can be written:

$$\frac{k}{\gamma_w} \frac{\partial^2 u}{\partial z^2} = -\frac{k}{\gamma_w} \frac{\partial^2 \sigma'}{\partial z^2} = -c_v \frac{\partial^2 \varepsilon}{\partial z^2} = -\frac{\partial \varepsilon}{\partial t} = -\frac{1}{E_o} \frac{\partial \sigma'}{\partial t} \quad (1)$$

since the total stress is constant throughout the sample at any time.

The governing equation can be written in terms of effective stresses or strains:

$$c_v \frac{\partial^2 \sigma'}{\partial z^2} = \frac{\partial \sigma'}{\partial t} \quad \text{or} \quad c_v \frac{\partial^2 \varepsilon}{\partial z^2} = \frac{\partial \varepsilon}{\partial t} \quad (2)$$

3 Theoretical solution

Many of the examples of solutions to the diffusion equation for flow of heat analysed by Carslaw and Jaeger (1959) are also applicable to the analysis of consolidation of soils. Temperature is equivalent to effective stress or pore pressure and heat flow is equivalent to flow of pore water. Thermal conductivity is equivalent to permeability and specific heat is equivalent to soil stiffness. They present solutions to two relevant problems.

There is a transient stage during which the perturbation at $z = 0$ is progressively felt further into the sample. The sample can be treated as though it has infinite thickness: the actual thickness of the sample H can play no role. This corresponds to the semi-infinite solid with a constant flux of heat $F_0 =$ per unit time and unit area, at $z = 0$ (Carslaw and Jaeger (1959) §2.9).

The boundary conditions are:

- 129 • for $z = \infty$, $\partial\varepsilon/\partial z = 0$, $\partial\sigma'/\partial z = 0$ for $t > 0$ (impermeable boundary at
130 $z = \infty$)
- 131 • for $z = 0$, $u_0 = 0$, $\sigma = \sigma'_0 = E_o\varepsilon_0$; $c_v\partial\varepsilon/\partial z = C$, $(c_v/E_o)\partial\sigma'/\partial z = C$ for
132 $t > 0$
- 133 • for $t = 0$, $\varepsilon = \sigma' = 0$ for $0 < z < 1$

134 The displacement of the loaded boundary is calculated from strain:

$$\rho_0 = \int_{\infty}^0 \frac{\sigma'}{E_o} dz \quad (3)$$

135 with $\rho_{\infty} = 0$. The imposed constant rate of displacement implies that $\rho_0 = Ct$.
136 Subscripts $_0$ and $_{\infty}$ indicate values at the drainage boundary $z = 0$ and at the
137 infinitely distant undrained boundary $z = \infty$ respectively.

138 This has solution:

$$\begin{aligned} \sigma' &= \frac{CE_o}{c_v} \int_z^{\infty} \operatorname{erfc} \frac{z}{2\sqrt{c_v t}} dz \\ &= \frac{2CE_o}{c_v} \left(\frac{c_v t}{\pi} \right)^{1/2} \{ \exp[-\theta^2] - \sqrt{\pi} \theta \operatorname{erfc}(\theta) \} \end{aligned} \quad (4)$$

139 At $z = 0$ the effective stress varies with square root of time according to:

$$\sigma'_0 = \frac{CE_o}{c_v} \left(\frac{4c_v t}{\pi} \right)^{1/2} \quad (5)$$

140 The variation of effective stress (or pore pressure) through the sample relative to

141 σ'_0 is a function of the combined variable $\theta = z/2\sqrt{c_v t}$

$$\frac{\sigma'}{\sigma'_0} = \{ \exp[-\theta^2] - \sqrt{\pi} \theta \operatorname{erfc}(\theta) \} \quad (6)$$

142 Plotted in dimensionless terms, as the ratio σ'/σ'_0 , the isochrone is an unchanging
143 function of θ .

144 We can broaden the applicability of solutions by describing the problem in
145 terms of normalised dimensionless quantities. As noted, the actual thickness of
146 the sample H can have no significance during this transient phase. However, we
147 note that the ratio c_v/C has dimensions of length and can be seen as a ratio of
148 a *property* of the soil (coefficient of consolidation c_v) and the chosen *constraint*
149 of the testing configuration (imposed rate of displacement C). Appropriate nor-
150 malised quantities are therefore: location $\hat{z} = zC/c_v$; time $\hat{t} = C^2t/c_v$; dis-
151 placement $\hat{\rho} = \rho C/c_v$; pore pressure $\hat{u} = u/E_o$; effective stress $\hat{\sigma}' = \sigma'/E_o$;
152 one-dimensional strain $\hat{\varepsilon} = \varepsilon$; $\hat{\theta} = \hat{z}/2\sqrt{\hat{t}} = \theta$.

$$\hat{\sigma}' = \left(\frac{4\hat{t}}{\pi}\right)^{1/2} \left\{ \exp[-\hat{\theta}^2] - \sqrt{\pi}\hat{\theta}\text{erfc}(\hat{\theta}) \right\} \quad (7)$$

153 The longer term solution corresponds to a finite slab with prescribed (constant)
154 flux of heat at one surface and no flow of heat at the opposite surface $z = H$ (§3.8).
155 We redefine the normalised variables: location $\tilde{z} = z/H$; time $\tilde{t} = c_v t/H^2$.
156 There is a dimensionless group of variables which controls and characterises the
157 problem: $\phi = CH/c_v$. This group combines external demand (displacement
158 rate C), material property (coefficient of consolidation c_v), and specific problem
159 dimension (sample thickness H). The solution for effective stress or pore pressure
160 turns out to be linear in ϕ so that it is convenient to define the normalised stress
161 quantities as: pore pressure $\tilde{u} = u/\phi E_o$; effective stress $\tilde{\sigma}' = \sigma'/\phi E_o$; one-
162 dimensional strain $\tilde{\varepsilon} = \varepsilon/\phi$; displacement $\tilde{\rho} = \rho/\phi H$. It is suggested by Lee
163 (1981) that typical values of ϕ might lie in the range 0.01-1 but that the small strain

analysis breaks down for values of ϕ greater than 0.1-0.2 and that it is the value of this characteristic group rather than displacement rate alone that is important.

$$\tilde{\sigma}' = \tilde{t} + \frac{1}{3} + \frac{\tilde{z}^2}{2} - \tilde{z} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp[-n^2 \pi^2 \tilde{t}] \cos n\pi \tilde{z} \quad (8)$$

The total stress follows the effective stress at $\tilde{z} = 0$. Theoretical isochrones are shown in Fig 2a and variation of total stress with time in Fig 2d. The results shown here have all used 20 terms. However, the summation in (8) converges very rapidly: $(2/\pi^2) \exp[-\pi^2 \tilde{t}] \sim 0.01$ for $\tilde{t} \sim 0.3$ and the transient stage can be regarded as complete by $\tilde{t} \sim 0.5$ (Lee, 1981; Stolle and Stolle, 2011). For $\tilde{t} \rightarrow \infty$, $\tilde{\sigma} = \tilde{\sigma}'_0 \rightarrow \tilde{t} + 1/3$; for $\tilde{t} > \sim 0.5$, the pore pressure isochrone is

$$\tilde{u} = \tilde{z} - \frac{\tilde{z}^2}{2} \quad (9)$$

with $\tilde{u}_1 = 1/2$.

Returning to the analysis of the initial progress of consolidation, for $\theta \geq 2$, $\sigma'/\sigma'_0 \sim 0$ (6) so that for a finite sample of length H we expect the 'infinite slab' analysis to break down for $t \geq H^2/16c_v$ or $\tilde{t} \geq 1/16$ and this is confirmed by the deviation of the short term solution from the complete theoretical solution in Fig 2d for $\tilde{t} > \sim 0.0625$.

4 Parabolic isochrones

If a mode shape for the pore pressure isochrone is assumed (Schofield and Wroth, 1968; Terzaghi and Fröhlich, 1936), then the problem can be analysed as a *system*: the mathematical description and solution are simplified. Any plausible function

Fig. 2: (a), (b), (c) Isochrones of pore pressure $0 < \tilde{t} < 0.5$: (a) theoretical solution; (b) parabolic isochrone solution; (c) finite difference solution of theoretical equations; (d) comparison of time variation of total stresses (solid line: theoretical solution; dashed line: parabolic isochrones; dotted line: theoretical solution for small time).

Fig. 3: Parabolic isochrone.

could be adopted for the mode shape (Muir Wood, 2004) but a parabola provides a precisely correct result at large times (see (9)) (Stolle and Stolle, 2011), and also has helpfully simple geometrical characteristics.

The imposition of a constant rate of displacement implies:

$$\frac{d\rho_0}{dt} = C \quad (10)$$

The pore pressure $u_0 = 0$.

There are two phases of the analysis corresponding to the two problems considered by Carslaw and Jaeger (1959). In the transient phase, the effect of the imposed constant rate of displacement is progressively felt further into the sample which might as well be of infinite length. The pore pressure rises parabolically to u_1 at depth h and remains constant at $u = u_1$ for $h < z < H$ (Fig 3a). The development of effective stress requires deformation of the soil; deformation of the soil requires pore water movement; pore water movement requires a gradient of pore pressure. Until the effect of the drainage boundary can be felt the pore pressure cannot change and there can be no development of effective stress. Thus in this first phase, the total stress $\sigma = u_1$. The effective stress is non-zero only in the region $0 < z < h$ and the value of the effective stress for any location within this region is the difference between the pore pressure (the parabolic isochrone) and the total stress ($\sigma' = u_1 - u$) (Fig 3a).

We can deduce the rate of displacement of the boundary of the sample from the gradient of the parabolic isochrone at $z = 0$.

$$C = \frac{d\rho_0}{dt} = \frac{k}{\gamma_w} \frac{du}{dz} = 2 \frac{c_v u_1}{E_o h} \quad (11)$$

202 The depth of experience of boundary drainage (h) is directly linked to the pore
 203 pressure u_1 for $h < H$ through the imposed rate of boundary displacement. Re-
 204 calling our previous choice of dimensionless quantities: $u_1/E_o = hC/2c_v$ or
 205 $\hat{u}_1 = \hat{h}/2$

206 The area above the isochrone indicates the current average effective stress in
 207 the soil $\bar{\sigma}'$. The displacement is calculated from the average strain:

$$\rho_0 = h \frac{\bar{\sigma}'}{E_o} = \frac{hu_1}{3E_o} = \frac{h^2C}{6c_v} \quad (12)$$

208 and the rate of displacement

$$\frac{d\rho_0}{dt} = C = \frac{2hC}{6c_v} \frac{dh}{dt} \rightarrow h^2 = 6c_v t \quad \text{or} \quad \frac{Ch}{c_v} = \hat{h} = \sqrt{6\hat{t}} \quad (13)$$

209 and

$$\frac{u_1}{E_o} = \frac{\sigma'_0}{E_o} = C \sqrt{\frac{3t}{2c_v}} \rightarrow \hat{u}_1 = \hat{\sigma}'_0 = \sqrt{\frac{3}{2}\hat{t}} \quad (14)$$

210 During this first phase the typical parabolic isochrone is:

$$\hat{u} = \hat{z} - \frac{\hat{z}^2}{2\hat{h}} = \hat{z} - \frac{\hat{z}^2}{2\sqrt{6\hat{t}}} \quad (15)$$

211 and the variation of effective stress:

$$\frac{\hat{\sigma}'}{\hat{\sigma}'_0} = \left[1 - \theta \sqrt{\frac{8}{3}} + \frac{2}{3}\theta^2 \right] \quad (16)$$

212 where $\theta = \hat{z}/2\sqrt{\hat{t}}$ as before. The consolidation front reaches the base of the layer,

213 $\hat{h} = \hat{H} = \phi$, at $\hat{t} = \phi^2/6$.

214 The second stage begins at time $\tilde{t} = c_v t / H^2 = 1/6$ or $(\hat{t} = \phi^2 \tilde{t})$. The
 215 isochrone in the second stage is (Fig 3b):

$$\tilde{u} = \tilde{z} - \frac{\tilde{z}^2}{2} \quad (17)$$

216 which does not change with time because a constant outflow gradient is required
 217 to match the imposed constant rate of displacement. This isochrone is exactly
 218 the same as the theoretically determined limiting isochrone (8) for time $\tilde{t} > \sim$
 219 0.5 (Lee, 1981; Stolle and Stolle, 2011). The pore pressure at the impermeable
 220 boundary $\tilde{z} = 1$ is $\tilde{u}_1 = 1/2$ for all $\tilde{t} > 1/6$.

221 The displacement

$$\frac{\rho_0}{H} = \frac{\sigma - 2u_1/3}{E_o} \quad (18)$$

$$\frac{d\rho_0}{dt} = \frac{H}{E_o} \frac{d\sigma}{dt} = C \rightarrow \frac{\sigma}{E_o} = \frac{Ct}{H} \quad \text{or} \quad \tilde{\sigma} = \tilde{t} - \frac{1}{6} \quad (19)$$

223 The *average* effective stress continues to increase linearly with time (it is di-
 224 rectly proportional to the steadily increasing displacement).

$$\tilde{\sigma}' = \tilde{t} - \frac{1}{6} \quad (20)$$

225 so that the total stress

$$\tilde{\sigma} = \frac{1}{2} + \tilde{t} - \frac{1}{6} \quad (21)$$

226 and the effective stress

$$\tilde{\sigma}' = \tilde{\sigma} - \tilde{u} = \frac{1}{3} + \tilde{t} - \tilde{z} + \tilde{z}^2/2 \quad (22)$$

227 The parabolic isochrones through the two phases are shown in Fig 2b and the
 228 variation of total stress with time is shown in Fig 2d. The values obtained from
 229 this simple approximate solution are remarkably close to the precise theoretical
 230 values. During the first stage the total stress $\tilde{\sigma} = \sqrt{3\tilde{t}/2} = 1.225\sqrt{\tilde{t}}$ compared
 231 with the theoretical value of total stress (5), $\tilde{\sigma} = \sqrt{4\tilde{t}/\pi} = 1.128\sqrt{\tilde{t}}$.

232 This analysis suggests a constant isochrone (17) for $\tilde{t} > 1/6$; in the theoretical
 233 analysis this limiting isochrone is effectively reached at $\tilde{t} \sim 0.5$. The CRD test
 234 is usually performed with pore pressure measurement at the impermeable base of
 235 the sample. For sufficiently large time from the start it might be assumed that
 236 the second stage of consolidation has been reached and that an average effective
 237 stress can be calculated from the parabolic isochrone spanning the entire sample.
 238 A requirement that the pore pressure at the base should not exceed 10% of the
 239 total stress (Ozer et al., 2012) implies

$$\tilde{t} + \frac{1}{3} > 5 \quad \rightarrow \quad \tilde{t} > \frac{14}{3} \gg 0.5 \quad (23)$$

240 and the transient terms have certainly become negligible. However, for small
 241 times the isochrone is quite different (Fig 2b).

242 Sheahan and Watters (1997) present data from CRD tests on Boston Blue Clay
 243 in a Rowe cell with internal pore pressure probes to produce information concern-
 244 ing the distribution of pore pressure through the sample. Their data for tests with
 245 a displacement rate of 3%/hr have been reprocessed and plotted as isochrones of
 246 u/u_1 ($u = u_1$ for $\tilde{z} = 1$). With increasing displacement the isochrones spread pro-
 247 gressively as expected during the transient phase. The pore pressure is monitored
 248 at only three internal points in the sample so there is not great detail concerning

the actual shape of the isochrones. However, parabolic isochrone analysis for the transient stage with $c_v = 10^{-7} \text{m}^2/\text{s}$ (Fig 4a) or $5 \times 10^{-8} \text{m}^2/\text{s}$ (Fig 4b) provides a reasonable match up to 5% or 10% displacement respectively. These values lie in the range of values of coefficient of consolidation reported for these tests by Sheahan and Watters (1997). (However, some aspects of the results reported by Sheahan and Watters (1997) are counterintuitive: as displacement continues further the pore pressure u_1 falls sharply.)

5 Finite difference solution

The one-dimensional problem lends itself to a central finite difference formulation in space and time. With time step $\Delta \tilde{t}$ and position step $\Delta \tilde{z}$, and with subscript i referring to position and superscript j referring to time, the diffusion equation becomes for the one-dimensional problem:

$$\frac{1}{2} \left[\frac{\tilde{\varepsilon}_{n-1}^{j+1} - 2\tilde{\varepsilon}_n^{j+1} + \tilde{\varepsilon}_{n+1}^{j+1}}{\Delta \tilde{z}^2} + \frac{\tilde{\varepsilon}_{n-1}^j - 2\tilde{\varepsilon}_n^j + \tilde{\varepsilon}_{n+1}^j}{\Delta \tilde{z}^2} \right] - \frac{\tilde{\varepsilon}_n^{j+1} - \tilde{\varepsilon}_n^j}{\Delta \tilde{t}} = 0 \quad (24)$$

This is equivalent to calculating the values of the variable at time $\tilde{t}^j + \Delta \tilde{t}/2$. This leads to a set of simultaneous equations for $\tilde{\varepsilon}_i^{j+1}$:

$$\begin{aligned} -\frac{\beta}{2}\tilde{\varepsilon}_{n-1}^{j+1} + (1 + \beta)\tilde{\varepsilon}_n^{j+1} - \frac{\beta}{2}\tilde{\varepsilon}_{n+1}^{j+1} \\ = \frac{\beta}{2}(\tilde{\varepsilon}_{n-1}^j + \tilde{\varepsilon}_{n+1}^j) + (1 - \beta)\tilde{\varepsilon}_n^j \end{aligned} \quad (25)$$

where $\beta = \Delta \tilde{t}/\Delta \tilde{z}^2$. The resulting tridiagonal set of equations can be solved quite efficiently by the Thomas substitution method (Thomas, 1949; Wikipedia, 2013).

Analysis using a spatial discretisation $\Delta \tilde{z} = 0.001$ and a time increment

Fig. 4: Isochrones of pore pressure from constant rate of displacement consolidation test on Boston Blue clay (points calculated from data in Sheahan and Watters (1997)); imposed strains indicated. Comparison with parabolic isochrone analysis for (a) coefficient of consolidation $c_v = 10^{-7} \text{ m/s}^2$ and (b) coefficient of consolidation $c_v = 5 \times 10^{-8} \text{ m/s}^2$. Staggered plotting to separate isochrones.

$\Delta\tilde{t} = 0.0001$ produces results indistinguishable from the theoretical analysis. The actual programming of this solution is relatively straightforward: it is instructive to observe the stages of the solution process so that they can be adapted to other boundary conditions.

6 Application 1: variation in imposed displacement rate

The system being analysed is a linear system. We can superpose separate solutions in order to calculate the response in a test in which the rate of displacement is changed. A displacement rate C_1 is imposed at time t_1 ; then a displacement rate C_2 at time t_2 . Immediately the rate of displacement is changed the pore pressure gradient at $\tilde{z} = 0$ must change in order to provide an outflow rate equal to C_2 . For $t > t_2$ we add a solution with displacement rate $C_2 - C_1$ and continue our time description with the new time origin t_2 . This process of superposition of analyses can be applied using either the full theoretical analysis or the parabolic isochrone analysis.

Figures 5 show isochrones of pore pressure for History 1 (Table 1). Each time the rate of displacement changes the isochrones combine by superposition of the pore pressures continuing from the first loading and the pore pressures from subsequent loadings. There are modest divergences between the isochrones for the theoretical solution and the solution using parabolic isochrones, but the relationships between total stress and displacement in Fig 5f are remarkably close. Even though a purely elastic constitutive model is being used, hysteresis is seen in the cycles of unloading and reloading; this will be explained later.

Histories 2 and 3 (Table 1) cover the same displacements but at different rates

Fig. 5: Application 1: non-monotonic history of applied displacements (History 1 (Table 1)): (a)-(e) isochrones of pore pressure (finite difference solution) (a) $0 < \tilde{t} < 0.45$; (b) $0.45 < \tilde{t} < 0.58$; (c) $0.58 < \tilde{t} < 0.98$; (d) $0.98 < \tilde{t} < 1.35$; (e) $1.35 < \tilde{t} < 1.85$; (f) total stress and displacement: comparison of theoretical results (solid line) and parabolic isochrone results (dashed line).

Fig. 6: Application 1: non-monotonic histories of applied displacement (Histories 2 and 3 (Table 1)): step change in displacement rate at $\tilde{\rho}_0 = 1$ (heavy lines: rate=2; thin lines: rate = 2 then 1 or rate = 1 then 2; solid lines: theoretical solution; dashed/dotted lines: parabolic isochrones).

289 $C = 1$ and $C = 2$. An ‘isotache’ effect is observed (Šuklje, 1957) with the load-
 290 displacement response switching between curves associated with the two rates
 291 (Fig 6).

292 Using the finite difference solution, the response to changing displacement
 293 rates is simply treated by changing the outflow boundary condition at required
 294 times. Such a procedure gives results identical to those obtained using successive
 295 superposition of theoretical solutions.

296 **7 Application 2: stiffness variation with overconsolidation history**

297 We first adopt a simple ‘quasi-elasto-plastic’ framework in which the stiffness is
 298 low (E_o) if $\tilde{\sigma}' = \tilde{\sigma}_{max}$ but is high (κE_o - with $\kappa = 5$) if $\tilde{\sigma}' < \tilde{\sigma}_{max}$. The governing

Table 1: Histories of constant rate of displacement loading and unloading

history 1		history 2		history 3		history 4		history 5	
end $\tilde{t}$	rate	end $\tilde{t}$	rate	end $\tilde{t}$	rate	end $\tilde{t}$	rate	end $\tilde{t}$	rate
0.45	1	1	1	0.5	2	0.45	1	0.45	1
0.58	-1	1.5	2	1.5	1	0.48	-1	0.48	-1
0.98	1	-	-	-	-	0.88	1	0.88	2
1.35	-1	-	-	-	-	0.98	-1	0.98	-1
1.85	1	-	-	-	-	1.5	1	1.5	1

equation does not change.

The boundary condition on outflow at $\tilde{z} = 0$ involves hydraulic conductivity but not stiffness. Incremental strain is $\delta\tilde{\epsilon} = \delta\tilde{\sigma}'/\kappa$ and ρ_0 is calculated from incremental strain: the operational stiffness will in general vary with time and location.

$$\delta\tilde{\rho}_0 = \int_1^0 \delta\tilde{\epsilon} d\tilde{z} = \int_1^0 \frac{\delta\tilde{\sigma}'}{\kappa} d\tilde{z} \quad (26)$$

with $\tilde{\rho}_1 = 0$. As usual with elastic-plastic modelling, trial and error is used to choose the ‘elastic’ or ‘plastic’ stiffness to be used in the calculation.. History 4 (Table 1) again shows a hysteretic total stress: displacement relationship (Fig 7). For given rates of unloading and reloading the stiffer soil obviously unloads more rapidly than the constant stiffness soil.

8 Application 3: stiffness proportional to stress

Next, we incorporate a stiffness which varies continuously with current effective stress and retain the jump in stiffness on overconsolidation.

$$E_o = E_o^* \frac{\tilde{\sigma}'}{\tilde{\sigma}'_r} \quad \text{or} \quad \frac{E_o}{E_o^*} = \frac{\tilde{\sigma}'}{\tilde{\sigma}'_r} = \check{\sigma} = \exp\tilde{\epsilon} \quad (27)$$

Fig. 7: Application 2: total stress and displacement: History 4 (Table 1): stiffness constant (solid); stiffness increased by factor 5 when $\tilde{\sigma}' < \tilde{\sigma}'_{max}$ (dashed)

where $\tilde{\sigma}'_r$ is a reference stress. Analysis begins with an effective stress $\tilde{\sigma}'/\tilde{\sigma}'_r =$
 $\tilde{\sigma} = 1$, and $E_o = E_o^*$. The value of stiffness for each element is updated after each
calculation step using the calculated value of effective stress.

The coefficient of consolidation is defined from initial values:

$$c_v = \frac{kE_o^*}{\gamma_w} \quad (28)$$

and the governing equation:

$$\frac{\partial^2 \tilde{\sigma}'}{\partial \tilde{z}^2} = \frac{1}{\kappa \tilde{\sigma}} \frac{\partial \tilde{\sigma}'}{\partial \tilde{t}} \quad (29)$$

Changes in effective stress are calculated from the finite difference solution to
(29); these increments are used to compute the displacements and pore pressures.
The total stress $\tilde{\sigma} = \tilde{\sigma}'_0$. The total of initial effective stress plus consolidation

320 increments controls the current stiffness of the soil.

321 The total stress:displacement relationships are presented in Figs 8 for Histories
322 4 and 5 (Table 1). The dependency of stiffness on stress implies a somewhat linear
323 relationship in logarithmic plotting (Fig 8b). History 5 includes an episode of
324 double displacement rate loading which, once again, produces an ‘isotache’-like
325 effect (Šuklje, 1957), returning to the reference curve for the base displacement
326 rate when the rate reduces again.

327 **9 Application 4: hydraulic conductivity dependent on strain**

328 Finally we allow the hydraulic conductivity to vary with void ratio. If the initial
329 specific volume is $v_o = 1 + e_o$ then a strain ε corresponds to a specific volume v :

$$\varepsilon = \ln(v_o/v) \rightarrow v = v_o \exp[-\varepsilon] \rightarrow e = (1 + e_o) \exp[-\varepsilon] - 1 \quad (30)$$

330 A simple rule might be $k \propto e^3/(1 + e)$ (Taylor, 1948; Carrier, 2003) or:

$$\zeta = \frac{k}{k_o} = \frac{1 + e_o}{1 + e} \left[\frac{e}{e_o} \right]^3 = \frac{1 + e_o}{1 + e} \{ ((1 + e_o) \exp[-\tilde{\varepsilon}/\xi] - 1) / e_o \}^3 \quad (31)$$

331 and we define the coefficient of consolidation:

$$c_v = \frac{k_o E_o^*}{\gamma_w} \quad (32)$$

332 The parameter ξ is included in (31) in order to give some control on the rate of
333 change of hydraulic conductivity with strain: a value $\xi = 5$ has been used.

Fig. 8: Application 3: stiffness proportional to effective stress and dependent on overconsolidation: total stress and displacement: (a) linear stress axis; (b) logarithmic stress axis (dashed line: History 4 (Table 1); solid line: History 5 (Table 1))

Fig. 9: Application 4: non-monotonic history of applied displacements (History 1 (Table 1)): (a)-(e) isochrones of pore pressure (finite difference solution) (a) $0 < \tilde{t} < 0.45$; (b) $0.45 < \tilde{t} < 0.58$; (c) $0.58 < \tilde{t} < 0.98$; (d) $0.98 < \tilde{t} < 1.35$; (e) $1.35 < \tilde{t} < 1.85$; (f) total stress and displacement

$$\frac{\zeta k_o}{\gamma_w} \frac{\partial^2 u}{\partial z^2} = -\frac{\partial \varepsilon}{\partial t} = -\frac{1}{E_o} \frac{\partial \sigma'}{\partial t} \quad \text{or} \quad \zeta c_v \frac{\partial^2 u}{\partial z^2} = -\frac{1}{\kappa \hat{\sigma}'} \frac{\partial \sigma'}{\partial t} \quad (33)$$

334 The governing equation is:

$$\frac{\partial^2 \tilde{\sigma}'}{\partial \tilde{z}^2} = -\frac{1}{\zeta \kappa \hat{\sigma}'} \frac{\partial \tilde{\sigma}'}{\partial \tilde{t}} \quad (34)$$

335 The flow boundary condition now becomes:

$$\left. \frac{\partial \tilde{\sigma}'}{\partial \tilde{z}} \right|_{\tilde{z}=0} = \frac{1}{\zeta_0} \quad (35)$$

336 If the hydraulic conductivity reduces, then the pore pressure gradient must in-
 337 crease to maintain the constant flow rate. A set of isochrones for the several
 338 successive stages of History 1 (Table 1) is shown in Fig 9.

339 **10 Interpretation**

340 (i) *System and element response*: All elements within the sample are governed
 341 by the same effective stress model for determination of stiffness, and by the same
 342 model for variation of hydraulic conductivity. Figure 10a shows the stress:strain
 343 paths for Application 4 for 5 points through the sample: $\tilde{z} = 0, 0.25, 0.5, 0.75, 1$.
 344 Similarly, Figure 10b shows the combinations of hydraulic conductivity and strain
 345 for the same locations. The staggered plots are intended to ease the separation of
 346 the response for each of the 5 elements. The element at $\tilde{z} = 0$ always has zero pore
 347 pressure. This element is always ahead of the rest of the sample - consolidation to
 348 the highest stress, the greatest reduction in hydraulic conductivity, and the greatest

Fig. 10: Application 4: (a) Effective stress and strain paths followed at $\tilde{z} = 0, 0.25, 0.5, 0.75, 1$; (b) hydraulic conductivity and strain paths for $\tilde{z} = 0, 0.25, 0.5, 0.75, 1$ (simulation shown in Fig 9). Individual paths staggered to show variations through the sample, though the response of each element is controlled by the same constitutive model.

Fig. 11: Application 1: time histories of total stress (solid line: theoretical; dashed line: parabolic isochrones) and effective stress at mid point (dotted line) (History 1 (Table 1))

349 degree of unloading on reversal. The overall behaviour of the soil sample is an
 350 integrated effect of an infinite number of different stress:strain responses.

351 The periods of unloading for History 1 (Table 1) ensure that the total stress
 352 does not become tensile. (It is a practical problem to maintain contact between
 353 loading ram and sample during displacement controlled unloading.) The varia-
 354 tions of total stress and of effective stress at the mid-height of the sample $\tilde{z} \sim 0.5$
 355 are shown in Fig 11 for the first reversal of applied displacement. The pore water
 356 continues to flow down any pore pressure gradient. It takes time after the reversal
 357 for the effect of the changed boundary condition to be felt inside the sample (Fig
 358 11). The effective stress at mid-depth continues to rise a little before beginning to
 359 fall and shows a very different response from the sharp fall in external total stress.
 360 The internal response is heavily damped. The change to normally consolidated
 361 stiffness does not occur simultaneously through the sample but the general effect

Fig. 12: Application 4: (a) Variation of hydraulic conductivity, stiffness and coefficient of consolidation with strain; (b) hydraulic conductivity at $\tilde{z} = 0$ and pore pressure at $\tilde{z} = 1$ (simulation shown in Fig 9)

of reaching the previous maximum stress is evident in the total stress:displacement relationship in Figs 7, 8.

(ii) *Variation of hydraulic conductivity:* The particular expressions that have been introduced in Application 4 to describe the variation of stiffness with strain (27) and hydraulic conductivity with strain (31) have been plotted in Fig 12a for $e_o = 2$ and $\xi = 5$. The value of coefficient of consolidation that is applied at each calculation step is $c_v = kE_o/\gamma_w$. This is also plotted in Fig 12a. Coincidentally, the changes in stiffness and hydraulic conductivity more or less compensate so that the coefficient of consolidation is somewhat constant with increasing strain. However, the internal variations of pore pressure and displacement are certainly controlled by the individual values of stiffness and conductivity and an analysis with constant E_o and constant k (and thus constant c_v) would give quite different results (Stolle and Stolle, 2011). The finite difference solution can handle these more general variations.

376 If the hydraulic conductivity is falling the exit gradient has to increase. The
 377 rate of outflow must be compatible with the imposed displacement rate. The lim-
 378 iting isochrone thus requires a much higher pore pressure at $\tilde{z} = 1$: compare
 379 Figs 9e and 5e. Ignoring variations through the sample, the theoretical limiting
 380 isochrone at infinite time is still somewhat parabolic. If the hydraulic conductiv-
 381 ity has fallen by a factor ζ then the gradient must increase by a factor $1/\zeta$ and the
 382 pore pressure at the undrained boundary, which for the simple case is $\tilde{u}_1 = 1/2$,
 383 should be approximately $\tilde{u} = \zeta/2$. The hydraulic conductivity in Fig 12e falls by
 384 a factor of about 5; the pore pressure $\tilde{u}_1$ is approximately $\tilde{u}_1 \sim 2.5$ (Fig 9b).

385 (iii) *Isotaches*: If successive displacement stages are long enough $\Delta\tilde{t} \gg 1/6$,
 386 the transient effects at the start of each stage have been forgotten and the isochrone
 387 is essentially the limit isochrone for the current displacement rate: compare Fig
 388 5b ($\Delta\tilde{t} = 0.13$) with Figs 5a, c, d, e. The limiting isochrone reaches $\tilde{u}_1 = \pm 0.5$
 389 depending on the sign of the displacement rate.

390 With reference displacement rate C_o , the long term development of total stress
 391 and displacement for other displacement rates can be scaled. Thus:

$$\tilde{\sigma} \rightarrow \frac{C}{C_o} \left[\tilde{t} + \frac{1}{3} \right] \quad \text{and} \quad \tilde{\rho}_0 = \frac{C}{C_o} \tilde{t} \quad (36)$$

392 A displacement history which involves doubling or halving of displacement rate
 393 (compare Histories 2 and 3 (Table 1)) produces a transient shift in the total stress:
 394 displacement relationship $\Delta\tilde{\sigma} = \pm 1/3$ respectively (Fig 6). The response pro-
 395 duces an ‘isotache’ effect (Šuklje, 1957) linking stress:displacement response with
 396 displacement rate even though there is no viscous component in the constitutive
 397 description of the soil.

(iv) *Hysteretic response: energy balance:* The soil skeleton has no dissipative mechanisms of response (Fig 10a). However, cycles of reversal of direction of displacement produce energy dissipation in terms of imposed displacement and total stress (Figs 5f, 7, 8, 9f). We use the parabolic isochrone approximation as a mathematically convenient way of investigating energy dissipation and storage for the first loading stage of History 1 (Fig 5).

Work ΔW is done by the external stress displacing the soil boundary. Elastic energy ΔU is stored in the soil. Energy is dissipated internally as the soil is displaced against the seepage force provided by the flowing water.

We define $\lambda = h/H$ for convenience. The soil is initially unloaded; the parabolic isochrone steadily penetrates over time $0 < \tilde{t} < 1/6$. The typical isochrone is:

$$\tilde{u} = \tilde{z} - \frac{\tilde{z}^2}{2\lambda} \quad (37)$$

The seepage stress is

$$\tilde{\sigma}_s = \frac{\partial \tilde{u}}{\partial \tilde{z}} = 1 - \frac{\tilde{z}}{\lambda} \quad (38)$$

The total stress is

$$\tilde{\sigma} = \frac{\lambda}{2} = \sqrt{\frac{3\tilde{t}}{2}} \quad (39)$$

The normalised effective stress (and normalised strain)

$$\tilde{\varepsilon} = \tilde{\sigma}' = \frac{\lambda}{2} - \tilde{z} + \frac{\tilde{z}^2}{2\lambda} \quad (40)$$

The displacement is:

$$-\tilde{\rho} = \int_{\lambda}^{\tilde{z}} \left[\frac{\lambda}{2} - \tilde{z} + \frac{\tilde{z}^2}{2\lambda} \right] d\tilde{z} = \left[\frac{\lambda\tilde{z}}{2} - \frac{\tilde{z}^2}{2} + \frac{\tilde{z}^3}{6\lambda} - \frac{\lambda^2}{6} \right] \quad (41)$$

414 The displacement and effective stress are zero for $\tilde{z} > \lambda$. At $\tilde{z} = 0$, $\tilde{\rho}_0 = \lambda^2/6$
 415 and $\tilde{\rho}_0 = \tilde{t}$.

416 The incremental displacement in time $\delta\tilde{t}$

$$\delta\tilde{\rho} = \frac{d\tilde{\rho}}{d\lambda} \frac{d\lambda}{d\tilde{t}} \delta\tilde{t} = - \left[\frac{\tilde{z}}{2} - \frac{\lambda}{3} - \frac{\tilde{z}^3}{6\lambda^2} \right] \sqrt{\frac{3}{2\tilde{t}}} \delta\tilde{t} \quad (42)$$

417 The work done against the seepage force in this time increment

$$\delta V = - \int_0^\lambda \left[1 - \frac{\tilde{z}}{\lambda} \right] \left[\frac{\tilde{z}}{2} - \frac{\lambda}{3} - \frac{\tilde{z}^3}{6\lambda^2} \right] \sqrt{\frac{3}{2\tilde{t}}} d\tilde{z} \delta\tilde{t} \quad (43)$$

418 The total energy absorbed over time $0 < \tilde{t} < 1/6$ is

$$\Delta V = - \int_0^{1/6} \int_0^\lambda \left[1 - \frac{\tilde{z}}{\lambda} \right] \left[\frac{\tilde{z}}{2} - \frac{\lambda}{3} - \frac{\tilde{z}^3}{6\lambda^2} \right] \sqrt{\frac{3}{2\tilde{t}}} d\tilde{z} d\tilde{t} \quad (44)$$

419 or, since $d\tilde{t} = (2\lambda/3)d\lambda$

$$\Delta V = - \int_0^1 \int_0^\lambda \left[1 - \frac{\tilde{z}}{\lambda} \right] \left[\frac{\tilde{z}}{2} - \frac{\lambda}{3} - \frac{\tilde{z}^3}{6\lambda^2} \right] d\tilde{z} d\lambda = \frac{11}{360} \quad (45)$$

420 Energy is supplied by the applied total stress which steadily compresses the
 421 sample. The displacement rate $d\tilde{\rho}/d\tilde{t} = 1$; the total stress is $\tilde{\sigma} = \sqrt{3\tilde{t}/2}$. The
 422 work done by the external stress is:

$$\Delta W = \int_0^{1/6} \sqrt{\frac{3\tilde{t}}{2}} d\tilde{t} = \frac{20}{360} \quad (46)$$

423 There will be stored elastic energy $\delta U = \tilde{\sigma}'^2/2$ per unit volume. The isochrone
 424 has just reached the undrained boundary $\lambda = 1$ for $\tilde{t} = 1/6$. The effective stress

425 is

$$\tilde{\sigma}' = \frac{1}{2} - \tilde{z} + \frac{\tilde{z}^2}{2} \quad (47)$$

426

$$\Delta U = \frac{1}{2} \int_0^1 \left[\frac{1}{2} - \tilde{z} + \frac{\tilde{z}^2}{2} \right]^2 d\tilde{z} = \frac{9}{360} \quad (48)$$

427 The work done by the external load ΔW is equal to the elastic energy stored
428 ΔU plus the work dissipated as the soil is pushed against the seepage force ΔV .

429 The same procedure can be followed for a later stage of unloading: see the
430 Appendix for the detailed calculation. The direction of displacement is reversed
431 at time $\tilde{t} = 1$ until time $\tilde{t} = 1 + 1/6$, sufficient to produce the fully penetrating
432 isochrone for the reversed displacement.

433 The elastic strain energy released is $\Delta U = -55/360$. The energy dissipated
434 by the movement of the soil against the seepage force $\Delta V = 10/360$. The work
435 done in pushing against the boundary total stress is $\Delta W = -45/360$ and $\Delta W -$
436 $\Delta V - \Delta U = 0$.

437 **11 Conclusions**

438 Constant rate of displacement (CRD) consolidation testing has the advantage over
439 incremental loading consolidation testing that a continuous record of response is
440 produced from which the deformation behaviour of a soil can be deduced. The
441 basic diffusion equation governing the consolidation process remains unchanged
442 but the boundary conditions are different. We have shown three possible solution
443 techniques.

444 1. Theoretical solutions can be obtained for many diffusion problems. The
445 same governing equation is found in many areas of science/engineering.

446 The compendious volume of Carslaw and Jaeger (1959) provides a source
447 of relevant solutions.

448 2. Introducing an assumed mode shape for the isochrone describing the varia-
449 tion of pore pressure through the soil, the governing equations can be solved
450 at the system level instead of the element level. An assumed parabolic form
451 is mathematically simple and, in the long term, matches the theoretical re-
452 sult.

453 3. Finite difference solution of the governing equation and boundary condi-
454 tions can be simply programmed and applied to more general problems.
455 Examples have shown stiffness varying with normal/overconsolidation in
456 an elastic-plastic manner, stiffness varying with effective stress, and hy-
457 draulic conductivity varying with strain. Such variations cannot readily be
458 incorporated in theoretical or system level analyses.

459 These latter two techniques are particularly useful in providing insights into the re-
460 sponse of consolidation samples subject to constant rate of displacement loading.
461 All three of these techniques can be applied in the form of 'back-of-the-envelope'
462 calculations in order to provide benchmark results against which more elaborate
463 numerical analyses can be checked. Such analyses can also provide qualitative
464 predictions of response in advance of the deployment of heavier computing power.

465 A consolidation test is a system not an element test. The theoretical and finite
466 difference solutions apply the diffusion equation at each discretised point or ele-
467 ment within the specimen but the results of oedometer tests are usually displayed
468 in terms of system quantities - total stress, overall specimen displacement - be-
469 cause there is no possibility of observing internal variations. Pore pressure and

470 effective stress vary through the sample so that estimation of an average effective
471 stress requires an assumed internal pore pressure variation. Because the degree of
472 drainage will differ for each element within the sample, each element will follow
473 a different effective stress path even though they are all governed by the same
474 model of effective stress:strain response. As the constitutive model for the soil
475 becomes more realistic so the challenge of deconstructing the response from the
476 system-level total stress observation becomes more challenging.

477 Having presented the results of CRD tests in terms of system quantities it is
478 easy to forget that these are outputs of the response of a boundary value problem
479 and to interpret them as though they are describing the response of a single soil
480 element. The system-level response - formally, the external relationship between
481 total stress and displacement - may show features which appear to indicate the
482 presence of viscous mechanisms of behaviour. It is then tempting to interpret rate-
483 dependent system response as rate-dependent material response. Real soils may
484 indeed possess such constitutive characteristics but, even with inviscid material
485 response, apparent viscous behaviour in the form of 'isotaches' may be observed:
486 with somewhat parallel relationships between applied load and displacement cor-
487 responding to different imposed rates of displacement.

488 Similarly, the observation of hysteretic energy dissipation in cycles of unload-
489 ing and reloading can naturally be explained in terms of conservation of energy of
490 the system. During any stage of deformation, elastic energy is stored in the sam-
491 ple. The changing distribution of effective stresses through the sample leads to
492 internal displacements which imply work done against the internal seepage forces
493 associated with the flow and consolidation of the sample. Work is done by the
494 external load generating external displacement of the sample. The work done by

495 the external load balances the elastic energy stored and the work done against
496 the seepage force. It is the latter term which provides the dissipative mechanism
497 for the system which could be misinterpreted as an apparent hysteretic material
498 response.

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548
Notation

c_v	coefficient of consolidation
C	rate of displacement
e	void ratio
E_o	one-dimensional stiffness
E_o^*	reference stiffness
h	$= \lambda H$; location of consolidation front
H	sample thickness/reference dimension
k	hydraulic conductivity
n	term in series summation
t	time
u	pore pressure
U	strain energy
v	specific volume $v = 1 + e$
V	work done against seepage force
W	work done by external stresses
z	position
β	finite difference ratio
γ_w	unit weight of pore water
ε	strain
ζ	hydraulic conductivity multiplier
θ	$= z/2\sqrt{c_v t}$ combined location and time variable
κ	stiffness multiplier
λ	location of consolidation front $= h/H$
ξ	parameter controlling influence of void ratio on hydraulic conductivity
ρ	displacement
σ	total stress
σ_s	seepage stress
σ'	effective stress
σ'_r	reference stress for stiffness

550 Appendix: energy balance for displacement reversals

551 On reversal, the pore pressure has two components: one continuing isochrone
 552 from the first stage and the superimposed isochrone from the second stage. The
 553 displacement rate in reversal is just the opposite of that of the original loading so
 554 we superimpose a constant rate of displacement of double the original magnitude.
 555 We divide the sample into two regions: region A ($\lambda < \tilde{z} < 1$) and region B ($0 <$
 556 $\tilde{z} < \lambda$); the descriptions of the pore pressure isochrone are different in these two
 557 regions. During the transient stage, while the effect of the boundary perturbation
 558 is gradually spreading through the soil with current location characterised by the
 559 variable λ : $1 < \tilde{t} < 7/6$, $\tilde{t} = 1 + \lambda^2/6$ or $\lambda^2 = 6(\tilde{t} - 1)$.

560 The isochrones in the two regions are:

$$\begin{aligned} \lambda < \tilde{z} < 1 : \quad \tilde{u}_A &= \tilde{z} - \frac{\tilde{z}^2}{2} - \lambda \\ 0 < \tilde{z} < \lambda : \quad \tilde{u}_B &= \tilde{z} - \frac{\tilde{z}^2}{2} - 2\tilde{z} + \frac{\tilde{z}^2}{\lambda} = -\tilde{z} - \frac{\tilde{z}^2}{2} + \frac{\tilde{z}^2}{\lambda} \end{aligned} \quad (49)$$

561 These equations ensure continuity of value of pore pressure at $\tilde{z} = \lambda$.

562 The seepage stress is

$$\begin{aligned} \lambda < \tilde{z} < 1 : \quad \tilde{\sigma}_{sA} &= 1 - \tilde{z} \\ 0 < \tilde{z} < \lambda : \quad \tilde{\sigma}_{sB} &= -1 - \tilde{z} + \frac{2\tilde{z}}{\lambda} \end{aligned} \quad (50)$$

563 The total stress must be constant throughout the sample:

$$\tilde{\sigma} = \frac{1}{2} + \tilde{t} - \frac{1}{6} - \lambda = \frac{4}{3} - \lambda + \frac{\lambda^2}{6} \quad (51)$$

564 The effective stress is

$$\begin{aligned} \lambda < \tilde{z} < 1 : \quad \tilde{\sigma}'_A &= \frac{4}{3} - \lambda + \frac{\lambda^2}{6} - \tilde{z} + \frac{\tilde{z}^2}{2} + \lambda \\ 0 < \tilde{z} < \lambda : \quad \tilde{\sigma}'_B &= \frac{4}{3} - \lambda + \frac{\lambda^2}{6} + \tilde{z} + \frac{\tilde{z}^2}{2} - \frac{\tilde{z}^2}{\lambda} \end{aligned} \quad (52)$$

565 and this is equivalent to the strain.

566 The displacement is obtained by integrating the strain in the two regions, with
567 continuity of displacement at $\tilde{z} = \lambda$:

$$\begin{aligned} \lambda < \tilde{z} < 1 : \quad \tilde{\rho}_A &= \int_1^{\tilde{z}} \left[\frac{4}{3} + \frac{\lambda^2}{6} - \tilde{z} + \frac{\tilde{z}^2}{2} \right] d\tilde{z} \\ &= \frac{4\tilde{z}}{3} + \frac{\lambda^2\tilde{z}}{6} - \frac{\tilde{z}^2}{2} + \frac{\tilde{z}^3}{6} - \frac{\lambda^2}{6} - 1 \end{aligned} \quad (53)$$

$$\tilde{\rho}_A|_{\tilde{z}=\lambda} = -1 + \frac{4\lambda}{3} - \frac{2\lambda^2}{3} + \frac{\lambda^3}{3} \quad (54)$$

$$\begin{aligned} 0 < \tilde{z} < \lambda : \quad \tilde{\rho}_B &= \tilde{\rho}_A|_{\tilde{z}=\lambda} + \int_{\lambda}^{\tilde{z}} \left[\frac{4}{3} - \lambda + \frac{\lambda^2}{6} + \tilde{z} + \frac{\tilde{z}^2}{2} - \frac{\tilde{z}^2}{\lambda} \right] d\tilde{z} \\ &= -1 + \frac{\lambda^2}{6} + \frac{4\tilde{z}}{3} - \lambda\tilde{z} + \frac{\lambda^2\tilde{z}}{6} + \frac{\tilde{z}^2}{2} + \frac{\tilde{z}^3}{6} - \frac{\tilde{z}^3}{3\lambda} \end{aligned} \quad (55)$$

568 The incremental displacement in time $\delta\tilde{t}$ - or equivalent $\delta\lambda$

$$\begin{aligned} \lambda < \tilde{z} < 1 : \quad \frac{d\tilde{\rho}_A}{d\lambda} &= -\frac{\lambda}{3} + \frac{\lambda\tilde{z}}{3} \\ 0 < \tilde{z} < \lambda : \quad \frac{d\tilde{\rho}_B}{d\lambda} &= +\frac{\lambda}{3} + \frac{\lambda\tilde{z}}{3} + \frac{\tilde{z}^3}{3\lambda^2} - \tilde{z} \end{aligned} \quad (56)$$

569 The work done by the seepage force in this time increment $\delta\lambda$

$$\delta V = \int_0^\lambda \tilde{\sigma}_{sB} \delta \tilde{\rho}_B d\tilde{z} + \int_\lambda^1 \tilde{\sigma}_{sA} \delta \tilde{\rho}_A d\tilde{z} \quad (57)$$

570 and the seepage energy dissipation for the complete reversal stage is:

$$\Delta V = \int_0^1 \left\{ \int_0^\lambda \tilde{\sigma}_{sB} \delta \tilde{\rho}_B d\tilde{z} + \int_\lambda^1 \tilde{\sigma}_{sA} \delta \tilde{\rho}_A d\tilde{z} \right\} d\lambda = -\frac{10}{360} \quad (58)$$

571 At time $\tilde{t}_r = 1$, just before the reversal of the loading direction, the effective
572 stress is

$$\tilde{\sigma}' = \frac{4}{3} - \tilde{z} + \frac{1}{2}\tilde{z}^2 \quad (59)$$

573 We calculate the strain energy as before.

$$U_1 = \int_0^1 \frac{1}{2} \left[\frac{1}{3} + \tilde{t}_r - \tilde{z} + \frac{1}{2}\tilde{z}^2 \right]^2 d\tilde{z} = \frac{184}{360} \quad (60)$$

574 After reversal has proceeded far enough for the isochrone to have penetrated
575 through the sample: $\lambda = 1$, $\tilde{t} = \tilde{t}_r + 1/6$ and the effective stress is

$$\tilde{\sigma}' = \frac{1}{2} + \tilde{z} - \frac{1}{2}\tilde{z}^2 \quad (61)$$

576 and we calculate the strain energy as before.

$$U_2 = \int_0^1 \frac{1}{2} \left[\frac{1}{2} + \tilde{z} - \frac{1}{2}\tilde{z}^2 \right]^2 d\tilde{z} = \frac{129}{360} \quad (62)$$

577 During the reversal process, the total stress is

$$\tilde{\sigma} = \frac{1}{2} + \tilde{t} - \frac{1}{6} - \lambda = \frac{4}{3} - \lambda + \frac{\lambda^2}{6} \quad (63)$$

578 and the displacement is $\tilde{\rho}_r - \tilde{t} = \tilde{t}_r - \tilde{t}$. We have $\lambda = \sqrt{6(\tilde{t} - \tilde{t}_r)}$; $2\lambda\delta\lambda = 6\delta\tilde{t}$

$$\begin{aligned} \Delta W &= - \int_1^{1+1/6} \left[\frac{4}{3} - \sqrt{6(\tilde{t} - 1)} + (\tilde{t} - 1) \right] \delta\tilde{t} \\ &= -\frac{45}{360} \end{aligned} \quad (64)$$

579 The elastic strain energy released is $\Delta U = -(184 - 129)/360 = -55/360$.

580 The energy dissipated by the seepage flow through the soil is $\Delta V = 10/360$. The

581 work done in pushing against the boundary total stress is $\Delta W = -45/360$ and

582 $\Delta W - \Delta V - \Delta U = 0$. The change in stored energy and the energy dissipated

583 through the seepage flow match the work done against the total stress.