Deconfinement and Duality of (super) Yang-Mills on Toroidially-Compactified Spacetimes for all Gauge Groups

by

Brett Teeple

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Physics
University of Toronto

© Copyright 2015 by Brett Teeple
Abstract

Deconfinement and Duality of (super) Yang-Mills on Toroidally-Compactified Spacetimes for all
Gauge Groups

Brett Teeple
Doctor of Philosophy
Graduate Department of Physics
University of Toronto
2015

I study gauge theories for any gauge group $G$, in particular Yang-Mills (YM) theories including super
Yang-Mills (SYM) and mass deformed super Yang-Mills (SYM*), on toroidally compactified spacetimes
$\mathbb{R}^{N-k} \times \prod_{i=1}^{k} S_{L_i}^{1}$ of $k$ compact dimensions of size $L_i$. Each compact direction introduces $r = \text{rank}(G)$
massive scalar fields into the theory and results in an effective potential added to the Lagrangian of the
theory. The mathematics of such potentials is interesting, however important applications begin with
special simple cases.

The first case studied here is finite temperature super Yang-Mills where a thermal circle of size $\beta = 1/T$
is introduced and the temperature dependence of phases of the theory can be studied including the
deconfinement phase transition. Further compactification on a spatial circle of size $L$ is useful. For
small such $L$ we are in a regime where semiclassical calculations can be performed at weak coupling.
The transition is found to be mediated by the competition between non-perturbative objects including
monopole-instantons and 'exotic' topological molecules: neutral and magnetic bions composed of BPS
and KK monopole constituents, with charges in the co-root lattice of the gauge group $G$, as well as
electrically charged W-bosons (and wino superpartners in the case of SYM) with charges in the root
lattice of $G$.

The second case is super Yang-Mills on $\mathbb{R}^3 \times S_L^1$, but with softly broken supersymmetry with a small
mass $m$ for the adjoint fermion. This is interesting as there is a conjectured continuity relating this
theory, and its quantum deconfining phase transition at some critical mass $m_c$ for the gluino, to pure
Yang-Mills with a thermal deconfinement transition at some critical $T_c$.

Furthermore, on $\mathbb{R}^2 \times S_L^1 \times S_{\beta}^1$, I determine a duality for all $G$ to a 2D Coulomb gas of bions of different
charges of their monopole constituents, and W-bosons of both scalar and electric charges. Aharonov-
Bohm interactions exist between magnetic bions and W-bosons. New scalar interactions of W-bosons
and neutral bions exist which attract like charges, as opposed to the magnetic and electric charges where
like charges repel. I then propose a dual lattice 'affine' XY-model with symmetry-breaking perturbations
coupled to the scalar field, for $SU(2)$, and show results of lattice studies to determine the nature of the
deconfinement phase transition and propose generalizations for other groups.
Dedication

I thank my advisor professor Erich Poppitz for providing me with interesting research projects and with help editing my research papers and giving advice for my seminar talks. The work we have done together was exciting and we all learned a lot, especially in Lie theory and new applications to gauge theory in general gauge group. Without his generous help and patience I would not have been able to co-author my first two papers and publish my latest paper as sole author.

Special thanks as well to postdoc Mohamed Anber for helping my research throughout my PhD. His advice was helpful especially in my work on the Callias index theorem on $\mathbb{R}^3 \times S^1_L$ and on our recent paper on deconfinement and continuity between super and thermal yang-mills theories for all gauge groups. I thank him as well for his time providing advice on my presentation for my oral qualifying exam and asking good questions helping me prepare for it.

Many thanks thanks as well to a lot of the staff and faculty at the University of Toronto. Professor Robin Marjoribanks for putting in a good word for me so I could continue my PhD., which began at Caltech, and having me able to work with my adviser Erich in a research area that very much interests me, and all in such short notice. I would also like to thank the graduate secretary Krystyna Biel and department heads for accepting me into the PhD program on such short notice and considering my application months after the deadline.

Furthermore I give thanks to the professors, including Sergei Gukov, Kip Thorne, and John Schwarz among others, and staff at Caltech who provided me with much useful education and knowledge for my research projects at the University of Toronto. I would like to thank my friends at Caltech as well, Himanshu Mishra, JD Bagert, and Agatha Hodsman for great times had and being there as support. I also thank course instructors I worked with at the University of Toronto including David Bailey for providing me with great teaching experience that taught me a lot along with my students. I really enjoyed my experience as a teaching assistant and lab demonstrator. I also would like to thank my friends in Toronto, Melissa Newton, Alysha Factley and William Archer for helping me in rough times of need and always being there, and for helping me with housing and in finding a place to stay when needed. Thanks as well to my friends in Calgary, where I completed my undergraduate degree, Kevin Alfano, Robert Parkinson and James Waugh for good times and sweet hikes.

And of course thanks to my family and my Mom for always being there through good and bad times and providing financial support when needed even though funds were lacking. Thanks so much to everyone!

I dedicate this work to my brother Clint, who passed away last year.
# Contents

1 Introduction and Outline 1  
1.1 Outline .......................................................... 8  

2 Yang-Mills Theory and Phases of Gauge Theory 11  
2.1 Duality in Gauge Theory ......................................... 12  
2.2 The Confining Phase .............................................. 14  

3 Deconfinement in $SU(2)$ Gauge Group 18  
3.1 Perturbative Theory of $SU(2)$ Yang-Mills on Toroidially Compactified Spacetimes .... 18  
3.1.1 Finite temperature perturbative dynamics of $SU(2)$ super Yang-Mills ............ 20  
3.2 Non-perturbative Theory of $SU(2)$ .................................. 26  
3.2.1 Monopole solutions for $SU(2)$ ..................................... 26  
3.2.2 Non-perturbative dynamics in $SU(2)$ gauge group ......................... 27  

4 Duality to Coulomb Gases and XY Spin Models of $SU(2)$ (Super) Yang-Mills 33  
4.1 Dual Double Coulomb Gas to $SU(2)$ (Super) Yang-Mills .......................... 33  
4.1.1 Results of lattice Monte Carlo simulations of dual Coulomb gas ................. 41  
4.2 Dual Affine XY Model to $SU(2)$ Double Coulomb Gas .......................... 44  
4.2.1 Results of lattice Monte Carlo simulations of dual affine XY model to dual Coulomb gas .......................... 46  

5 Lie Groups and Lie Algebras 49  
5.1 Notes on General Lie Theory .................................... 49  
5.2 The Roots and the Weights ...................................... 50  

6 Perturbative and Non-perturbative Theory for all Gauge Groups on Toroidially Compactified Spacetimes 59  
6.1 $T = 0$ Dynamics of Super Yang-Mills on $\mathbb{R}^3 \times S^1_\beta$ .......................... 59  
6.2 Finite Temperature Dynamics of SYM on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ and the Effective Potential .......................... 61  
6.2.1 Cases of $\mathbb{R}^3 \times S^1_\beta$ and general cases .......................... 64  
6.3 Non-perturbative Theory for General Gauge Group $G$ .............................. 66  
6.3.1 Monopole and Bion Solutions for General Gauge Group .......................... 66  
6.3.2 Non-perturbative dynamics in general gauge group .............................. 67
List of Tables

2.1 Phases of gauge theories and potentials of interaction in three dimensions. 11
3.1 Magnetic molecule vertices, charges, and amplitudes for different molecules. 29
4.1 Scalar, electric and magnetic charges of relevant Coulomb gas constituents. 41
5.1 (Dual) Kac labels and dual Coxeter numbers for all semi-simple Lie groups. 56
6.1 Magnetic molecule vertices, charges, and amplitudes for different molecules. 69
8.1 The critical mass, $c_{cr}$, for the spinor representation Polyakov loop and its correlator coefficients for the $Spin(2N)$ group. The string tensions $\hat{\sigma}$ and other coefficients measured by the two spinor representations are identical. 99
8.2 The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $Spin(6)$ group. 99
8.3 The critical mass, $c_{cr}$, the spinor representation Polyakov loop and its correlator coefficients for the $Spin(2N + 1)$ group. 100
8.4 The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $Spin(7)$ group. 100
8.5 The critical mass, $c_{cr}$, the fundamental representation Polyakov loop and the values of the coefficients of its two-point correlator for the $Sp(2N)$ group, including the dimensionless string tensions $\hat{\sigma}$. 101
8.6 The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $Sp(6)$ group. 101
8.7 The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $E_6$ group. 102
8.8 The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $G_2$ group. 104
8.9 The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $F_4$ group. 104
List of Figures

1.1 Confinement during the Big Bang until present day. ................................. 4
1.2 The running of the coupling of QCD and non-Abelian gauge theories. The blue line indicates the strong coupling scale $\Lambda$ where the gauge coupling becomes of order 1, and the red line indicates the leveling off of the coupling at a small value at energies less than $\approx 1/\sqrt{N_L}$. ................................................................. 5
1.3 Running of the QED coupling. ............................................................... 5
1.4 Torus compactified spacetime. Each point of $\mathbb{R}^2$ has a torus of cycle lengths $\beta$ and $L$. ...... 6
1.5 Confined quarks below the deconfinement phase transition temperature. ................. 7
1.6 Interaction potential between quark-anti-quark pairs .................................. 7

2.1 Centres of all simple Lie groups. ........................................................ 15
2.2 Symmetry breaking of gauge groups on $\mathbb{R}^3 \times S^1$. Note that most groups do not fully Abelianize after a single compactification in QCD(adj) with $n_f > 1$. The groups always Abelianize in the case of SYM. ......................................................... 17

3.1 Perturbative effective potential for torus of aspect ratio $L/\beta = 1$. Note the two minima at $(\phi, \psi) \in \{(0, \pi), (\pi, 0)\}$ with a bump in between them signaling a first order phase transition as $L \to \beta$. Note the $x, y$ axes are scaled to run from 0 to $2\pi$. ......................................................... 22
3.2 Perturbative effective potential for torus of aspect ratio $r = L/\beta = 3/2$. Note the global minimum is now at $(\phi, \psi) = (0, \pi)$ and is opposite for the case of $r < 1$. Note the $x, y$ axes are scaled to run from 0 to $2\pi$. ......................................................... 31
3.3 Effective potential for a torus of complex parameter $\tau = 1 + i$. The global minimum here is at $(\phi, \psi) = (\pi, 0)$ which is expected as here $r = L/\beta < 1$. Note again the $x, y = \phi, \psi$ axes are scaled to run from 0 to $2\pi$. ......................................................... 31
3.4 A magnetic bion. Note the approximate size and separation between bions in terms of $L$, the size of the compact spatial dimension, and the coupling $g$. ......................... 32

4.1 A Coulomb gas of W bosons and monopoles and bions. The lines represent the world lines of the W-bosons while the circles represent the monopole-instantons and bions, which are localized in space-time. even though $L \ll \beta$ the distance between gas constituents is much greater than the inverse temperature (the size of the vertical axis) and so the gas is effectively two dimensional. ......................................................... 35
4.2 Left: Magnetic and electric charge densities as a function of temperature. Right: Aharanov-Bohm phase contribution to the partition function of the Coulomb gas. ......................... 43
4.3 Histograms of distributions for $\phi$ at $N = 16$, for $T = 0.4, 2, 5$ from left to right. .......... 43
4.4 W-boson fugacity as a function of $\phi$ for $T = 0.4, 2, 5$, and $N = 16$. The result is the same for $N = 32$. .......... 44
4.5 XY-model 'magnetization' (left) and 'susceptibility' (right). See text for details. ........ 47
4.6 Distribution of $\theta$ angles at temperature $T = 0.4, 2, 5$ from left to right. Clearly the $Z_4$ symmetry breaking is observed as temperature is increased. ............. 47
4.7 The lattice showing the $\theta$ angles and positive and negative vortices at $T = 0.4, 2, 5$. See text for more discussion. ......................... 48
4.8 Histograms for $\phi$ for $N = 16$ and $T = 0.4, 2, 5$ from left to right. ........... 48

5.1 All simple Lie algebras and their affine Dynkin diagrams. The numbers inside the nodes denote the root, and the numbers beside each node are the Kac labels. The red circles are the lowest (affine) roots of the Lie algebra. ..................... 51
5.2 Gauge cells for Lie groups of rank 2. ......................... 58

7.1 A sketch of our dual 2D Coulomb gas of magnetic bions, monopoles and other particles. The monopole-instantons carry two fermionic zero modes and hence do not participate in determining the dynamics of the deconfining phase transition or the vacuum structure of the theory. One should add W bosons to the picture as well for the full vacuum. .... 78

8.1 The continuity conjecture between super Yang-Mills and thermal pure Yang-Mills. .... 84
8.2 Effective potential for the $b'$-field, showing a second order phase transition for lengths less than a critical length $L_c$ (with gaugino mass $m$ fixed) where the $b'$-field gets a non-zero VEV and centre symmetry is broken. Equivalently, with $L$ fixed, centre symmetry is broken for masses above a critical gaugino mass $m^*$. .......................... 95
8.3 Discontinuous jump in the string tension at the critical gaugino mass for $spin(7)$ group. 96
8.4 Theta dependence of critical gaugino mass (left) and jumps in the Wilson loop observable (right) at the transition for $spin(6)$ group. ......................... 96
8.5 Distribution of Wilson loop eigenvalues on the unit circle, below (left) and above (right) the critical gaugino mass for $Sp(12)$. Note the $Z_2$ centre symmetry is not preserved above the transition. ......................... 96
Chapter 1

Introduction and Outline

'Quarks have colour charge, electric charge, mass and spin...
And having colour charge means they exist solely in...
In side of other kinds of particles and cannot exist alone,
Which is why Quarks have never been studied on their own...
Oh ... UP DOWN STRANGE CHARM TOP BOTTOM...
If you don't know what a Quark is it don't MATTER you still got 'em!'
- Dr Avinash Sharma  Poem for Quarks!

Quantum Field Theory (QFT) describes the non-gravitational elementary particle interactions and the long-distance properties of various many-body systems. The physics of the electroweak interactions is well captured by perturbation theory, an expansion in a small coupling constant. The strong interactions governed by Quantum Chromodynamics (QCD) are different: observables involving short-distance physics can be studied via perturbation theory. However, QCD’s most interesting properties, including the structure of the ground state and low-lying excitations, are nonperturbative and new methods are needed to study such theories. The answer has been known from experiment for a long time, yet the reasons behind it, including the confining phase of quarks, still puzzle! The lattice is one theoretical tool to study QCD, but it alone yields little insight into the continuum dynamics of, say, confinement and chiral symmetry breaking. It is an immensely practical tool, whose modern use is a numerical experiment, a kind of ‘black box’ not immediately yielding to a physicist’s desire for intuitive qualitative understanding.

Thus, the development of any theoretical ideas that shed some dynamical insight into nonperturbative QCD is of interest. Furthermore, one should also consider other asymptotically free gauge theories, obtained by adding different numbers and representations of “quarks” and/or changing the gauge group. This excursion away from QCD is motivated: i.) due to our inherent theoretical curiosity and since exploring the “theory space” is likely to yield insights as to what makes the Standard Model so special and ii.) because non-QCD like dynamics has long been suggested as possibly relevant to models of physics Beyond the Standard Model.

The outstanding big questions, posed long ago and still waiting for a definitive answer, range from the more theoretical: “What is QFT?”, “What phases do gauge theories have?”, “Which theories are
Chapter 1. Introduction and Outline

conformal?” “When are quarks confined or not?”, “What happens upon ‘heating up’?”, “When and why does chiral symmetry break?”, ... to the more ‘practical’: “What is the mechanism behind the mass hierarchy in Nature?”.

Studying such questions for non-Abelian gauge theories, such as Yang-Mills theory and QCD-like theories in four dimensions, has been a challenge over the past few decades. In particular, the phases of such gauge theories have been proven difficult to study, because of strong coupling, and new techniques have been developed in order to better examine them. This includes the study of the confinement problem of quarks where one finds a deconfinement phase transition at a critical temperature where hadronic and mesonic matter dissociates into the quark-gluon plasma (QGP).

Studying finite temperature Yang-Mills theory is not only interesting for the purpose of studying such phases of gauge theories\(^1\) and understanding why quarks are not found free in Nature at low temperatures, but also has relevance in the study of early universe cosmology since the Big Bang. It is believed that the deconfinement temperature is around \(\approx 5 \times 10^{12}\) °C. This is the temperature of the early universe less than a nanosecond after the Big Bang. See Figure 1.1 for the whole picture of the universe since the Big Bang. During the first \(10^{-10}\) seconds after the Big Bang the universe cooled to the temperature of the confining phase transition and afterwards hadronic and mesonic matter formed and the earlier quark-gluon plasma of free Coulomb-interacting quarks (via gluon exchange) disappeared. Lattice studies\(^2\) and experiment give deconfinement temperatures of the order of \(\approx 100 - 110\) \(GeV\).\(^2\) This is the topic of this thesis. What is of interest is the generation of mass gap necessary for confinement and the underlying microscopic mechanisms underlying the transition. It is truly a 'million dollar question' to prove mass gap generation of Yang-Mills theory for gauge theories based on every semi-simple Lie algebra. Here I present a finite temperature study of confinement and the deconfinement phase transition for all gauge groups \(G\), using a setup allowing for analytically controlled studies.

Although direct analytic study of non-Abelian gauge theories such as (super) Yang-Mills in various dimensions is difficult, due mainly to their non-Abelian gauge group structure and the strongly coupled dynamics of the theory in the infrared, low energy (large distance) regime. This has forced studies of QCD and QCD-like theories (such as QCD(adj) with adjoint fermions, and (super) Yang-Mills) to be done numerically on the lattice. Several indirect approaches, other than numerical simulations, have been proposed recently, such as compactification, where a small compact extra dimension of size \(L\) is added to the theory to both Abelianize\(^3\) the theory making it simpler mathematically, but also to fix the gauge coupling \(g\) at a small value at energies well below the strong coupling scale \(\Lambda_{QCD} \equiv \Lambda\). This occurs whenever \(NLA \ll 1\) (for \(SU(N)\))\(^4\), where the mass of the lightest gauge boson (which I’ll call the W boson) is \(\approx 1/NL\). This mass also provides the scale at which the gauge coupling becomes fixed. In supersymmetric theories the theory always Abelianizes and there are no light charged matter fields at smaller mass to allow the coupling to run into stronger coupling. Figure 1.2 shows the running of

\(^{1}\)Think of a phase of a gauge theory in the way of condensed matter physics (like ice vs. water). More on the phases of gauge theories can have is discussed in Chapter 2.

\(^{2}\)Results may vary depending on the simulation or model used, but all agree qualitatively and give \(O(100 \, GeV)\) results for the deconfinement transition temperature.

\(^{3}\)A theory is Abelianized once its gauge symmetry is broken to an Abelian, i.e. commutative, gauge symmetry group.

\(^{4}\)For other gauge groups we replace \(N\) with \(c_2(G)\), the dual Coxeter number of the gauge group, which will be discussed later.
the QCD coupling. At higher energy and smaller length scales the theory is weakly coupled due to the asymptotic freedom of non-Abelian gauge theories such as QCD (compare this to the running of the coupling $g(\mu)$ in QED shown in Figure 1.3 where the strength of the coupling increases at higher energies), whereas at larger length scales and lower energy the theory becomes strongly coupled at the scale $\Lambda_{\text{QCD}}$ where the coupling diverges. This is shown by the blue line. If the size of the compact dimension is small the coupling will cease to grow at smaller energies than $O(1/NL)$, and is indicated by the red line where the coupling levels off. This is due to the Abelianization of the theory which will be discussed later, which implies the non-existence of light charged matter in the theory that can cause the coupling to continually increase at energies below the mass of the lightest W boson.

It has been found years ago [15], [21] that the deconfinement phase transition is mediated by the competition of exotic ‘molecules’ of monopole-instantons: neutral and magnetic bions, as well as W-bosons and their superpartners, the winos, in SUSY theories. More on the structure of such magnetic molecules is in [22] and will be described later. In this thesis the role of these perturbative and non-perturbative objects will come up again and again in determining the nature of the deconfinement transition and the underlying dynamics responsible for it. Recently in [1], for the case of SU(2), results of super Yang-Mills compactified on the torus $\mathbb{R}^2 \times S^1_L \times S^1_\beta$, where $L$ is a small compact dimension and $\beta$ is the inverse temperature, have shown the role of non-perturbative objects in the deconfinement phase transition, where now there is coupling to a scalar field $\phi$. Figure 1.4 shows a picture of such torus-compactified spacetimes. The resulting dual Coulomb gas of particles involves these perturbative and non-perturbative objects. This gauge theory was subjected to lattice studies in [1] both for the so-called double Coulomb gas and its related ‘affine’ XY-model with symmetry breaking perturbations and fugacities coupled to the scalar modulus $\phi$. It is this scalar field that breaks the electric-magnetic (Kramers-Vannier) duality enjoyed by the SU(2) theory [15] without supersymmetry on $\mathbb{R}^3 \times S^1_\beta$. In this thesis I generalize these results and dualities for general gauge group $G$ in Chapter 7 and suggest methods for future study of the phase transitions involved.

The study of SYM on toroidally compactified spacetimes, in particular $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ (illustrated in Figure 1.4), is interesting where a curious duality maps the theory onto an exotic Coulomb gas mentioned above. It is composed of magnetic monopole-instantons (Euclidean finite-action solutions of the gauge field equations of motion), doubly charged magnetic molecules, called ‘bions’, as well as W bosons and their wino superpartners. The interesting aspect of this gas is the variety of interactions it possesses. There are the usual Coulomb-Coulomb interactions between electric and magnetic charges (where like charges repel) as well as so-called scalar charges of neutral magnetic bions and W bosons that have the property that like charges attract! In addition there are the Aharanov-Bohm phases for interactions of electric and magnetic charges. This duality of the Yang-Mills theory to exotic Coulomb gases, with its perturbative and non-perturbative effective potential of W bosons and magnetic objects, respectively, which makes it related to cos-Gordon and cosh-Gordon theories\(^5\), is interesting and helpful in the study of the deconfinement phase transition. The transition is mediated by the competition between W-bosons, which dominate at higher temperatures, and neutral and magnetic bions which dominate at low temperatures, and studying the Coulomb gas can give an idea of the deconfinement phase transition. Furthermore, lattice studies can be done using Monte-Carlo simulations of the Coulomb gas with its

---

\(^5\)These are simply field theories with a cosine or cosh potential.
various particle constituents to study the nature of the phase transition numerically. I study the theory as in [1] with torus compactification, $\mathbb{R}^2 \times S^1_L \times S^1_\beta$. The result of Chapter 7 of this work is to generalize that for $SU(2)$ on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ for any gauge group $G$.

The usefulness of having a compact dimension to analytically study gauge theories in a theoretically-controlled manner at weak coupling for any gauge group has successfully been applied recently for super Yang-Mills on $\mathbb{R}^3 \times S^1_L$ with a mass for the gaugino. The deconfinement phase transition has been found to occur at a critical gaugino mass (depending on the theta angle$^7$ of the theory) and is shown to be first order for all gauge groups other than $SU(2)$ where it is second order [1]. It is conjectured that this zero temperature quantum phase transition is continuously related, as a function of gaugino mass $m$, to the thermal deconfinement transition of pure Yang-Mills as $m \to \infty$ at some critical deconfinement temperature $T_c$. Much supports this conjecture, including lattice studies where qualitative agreement to the zero temperature phase transition has been shown. The order of the transition, its universality class of centre-symmetry breaking, and dependence on theta-angle [17], agree with the ones in super Yang-Mills. I will discuss this conjecture in Chapter 8, along with a study of deconfinement on $\mathbb{R}^3 \times S^1_L$.

The method of study to determine the dynamics of the compactified non-Abelian gauge theory on

\begin{align*}
\text{k}_B T &\sim 100\text{MeV} \quad T \sim 10^{12}\text{K} \sim \wedge \\
(10^{-10}\text{s after big bang})
\end{align*}

Figure 1.1: Confinement during the Big Bang until present day.

$^6$I consider the covering group of the gauge group $\tilde{G}$, as it allows for all representations, such as the spinor representation, to be considered.

$^7$The topological theta angle of a gauge theory such as Yang-Mills modifies the Lagrangian of the theory to $\mathcal{L} = \frac{1}{2g^2} \text{Tr} F \wedge *F + \frac{\theta}{8\pi^2} \text{Tr} F \wedge F$. I will discuss more on this in Chapter 8 where non-zero theta angles are considered.
Figure 1.2: The running of the coupling of QCD and non-Abelian gauge theories. The blue line indicates the strong coupling scale $\Lambda$ where the gauge coupling becomes of order 1, and the red line indicates the leveling off of the coupling at a small value at energies less than $\approx 1/NL$.

Figure 1.3: Running of the QED coupling.
Figure 1.4: Torus compactified spacetime. Each point of $\mathbb{R}^2$ has a torus of cycle lengths $\beta$ and $L$.

$\mathbb{R}^2 \times S^1_L \times S^1_\beta$ and $\mathbb{R}^3 \times S^1_L$ involves computing the perturbative dynamics of the theory (the Gross-Pisarski-Yaffe (GPY) effective potential [39]), and the non-perturbative effective potential of the theory due to the presence of monopole-instantons and bions. This effective theory can be studied through lattice simulations of the Coulomb gas and its partition function, but another method of studying this Coulomb gas is to map it to a dual 'affine' XY spin model with symmetry-breaking perturbations as in [1] and [15]. The results of both simulation methods can be compared qualitatively, and are seen to agree. This lattice study will not be presented here for all gauge groups, but the method will be discussed for future lattice study. However, I will discuss the results of the $SU(2)$ case previously studied and give hints for determining the model for other gauge groups.

This 'affine' XY model with symmetry breaking perturbations and parameters depend on a scalar field arising from the compact spatial dimension. This lattice model can be studied using mean field theory and more accurate numerical results of the deconfinement phase transition can be deduced from lattice simulations as well [1]. This will be discussed for the case of $SU(2)$ in Chapter 4 and progress of determining dual spin models for other gauge groups will be mentioned in Chapter 7. Both looking at the dual Coulomb gas and spin models dual to SYM in general gauge group is not only interesting and somewhat elegant physically, but provides ways of numerically and analytically investigating the deconfinement phase transition of these compactified field theories.

Now what do these compactified theories tell us of phase transitions in 4D gauge theory? The idea is the concept of large-$N$ volume independence [19], [34] which states that expectation values of observables (order parameters, for example) in the 4D theory coincide to order $O(1/N^2)$ to those of the compactified theory (provided centre symmetry remains preserved in the limit). Hence, studying large rank gauge groups in compactified spaces can tell us of the nature of phase transitions in the non-Abelian 4D theory, which is of much interest and has been for decades. Studies of the compactified theory lead to interesting
exotic Coulomb gases as mentioned before, as well as 2d lattice XY-models, which can be studied by various methods [1], and generally 2D theories are easier and more understood than 4d ones. Hence there are many advantages of using compactified dimensions and the large-$N$ volume correspondence to study higher dimensional field theories. It is important to note, however, that the regime of interest for the large-$N$/volume correspondence requires $N_c$, or $c_2(G)$, to be large enough that $L\Lambda c_2(G) \gg 1$, and so to study higher dimensional theories one must forfeit the benefits of semi-classical calculability. Yet there may still be information one can get about the higher dimensional theory for 'intermediate' sized $N$ (with $L\Lambda$ sufficiently less than one) as the expectation values of observables are correct to $O(1/N^2)$ and so errors fall off quicker than $1/N \ll 1/L\Lambda$. This will not be discussed further in this work, however.

Now what is confinement exactly? In simple terms it is the linearly increasing potential of interaction between quark-anti-quark pairs. This results from the formation of a chromoelectric flux tube or 'string'
between the charges with a constant force (of about $10^6 N$) or energy density (giving a string tension between them). This linear potential persists as the quark pairs are separated until the energy of the flux tube becomes greater than the mass of two quarks and it becomes more energetically favourable to form a quark-anti-quark pair in between the two charges. When this happens it is called 'string-breaking' and the original pair of quarks is effectively non-interacting. Figure 1.5 illustrates string-breaking as the quarks become separated by a distance $2m_q/\sigma \approx 2m_q/\Lambda^2$, where $\sigma$ is the string tension which is constant as the quarks separate, yet is a function of temperature. For small quark-anti-quark separation the potential of their interaction becomes Coulombic due to the asymptotic freedom of the quarks at small distances ($< 1/\Lambda_{QCD}$) and higher energies (above $\Lambda_{QCD}$). These regimes of interaction are shown in Figure 1.6.

In summary, studying gauge theories, such as (super) Yang-Mills, on toroidially-compactified space-times of dimension $D$ with $d$ compact dimensions $\mathbb{R}^{D-d} \times \prod_{i=1}^{d} S^1_{L_i}$ is not only interesting mathematically and can be done analytically to some extent, but also has many benefits. Compactification allows analytically calculability as the theory Abelianizes and has the coupling fixed to small values. Furthermore, having a compact time-like dimension is a way to study theories at finite temperature and allows the study of thermal deconfinement. This motivates the study of super Yang-Mills on $\mathbb{R}^3 \times S^1_L$ and $\mathbb{R}^2 \times S^1_L \times S^1_\beta$, where $\beta = 1/T$ is the inverse temperature. These two spacetimes will be focused on in this thesis, and generalizations will be made on the general case of $\mathbb{R}^{D-d} \times \prod_{i=1}^{d} S^1_{L_i}$ where appropriate.

Let me now describe the outline of this work.

### 1.1 Outline

The purpose of this work is to study deconfinement of various theories on toroidially compactified space-times, in particular of Yang-Mills, super Yang-Mills, and QCD-adjoint, in any gauge group (although I will focus on SYM). Particularly of interest are such theories on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ and $\mathbb{R}^3 \times S^1_L$ where compact spatial dimensions allow for analytic control of the theory, and compact 'time' directions allow for study at finite temperature. In examining such theories it appears useful to study dualities to equivalent theories such as exotic Coulomb gases of electric and magnetic charges, and spin models, both of which can be subjected to lattice studies and simulations in order to study thermal deconfinement phase transitions when not analytically possible. The case of the quantum phase transition of SYM on $\mathbb{R}^3 \times S^1_L$ with mass for the gaugino is another case [2] where analytical study of the phase transition is possible, for any gauge group. This work includes research from published works [1], [2], [53] and additional work done in more general theories and special cases.

Chapter 2 introduces the theories that will be studied, in particular Yang-Mills, with or without supersymmetry, and QCD adjoint (with adjoint fermions). A discussion of phases of gauge theories will be included with detail on the confined phase. Dualities of gauge theories will be mentioned here as well. It also explores the confined phase of quark matter in more detail and the method of compactification to toroidal spacetimes will be discussed as a method of analytical calculation of such strongly coupled theories and not just as a curiosity. Through study of compactified theories one hopes to still learn much of the non-compactified, higher dimensional theory that may be less approachable or calculable.

Chapter 3 begins the study of (super) Yang-Mills for the initial case of $SU(2)$. Section 3.1 studies the perturbative dynamics of the theories on general compactified spacetimes of the form $\mathbb{R}^{N-d} \times \prod_{i=1}^{d} S^1_{L_i}$.
and specifically for $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ and $\mathbb{R}^3 \times S^1_L$. Particular attention will be spent on the case of $SU(2)$ super Yang-Mills on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ as in [1] where the supersymmetry adds additional complications to the theory but as well adds certain simplifications. The perturbative study leads to the Gross-Pisarski-Yaffe (GPY) effective potential of the compactified theory and the introduction of W-bosons. Section 3.2 describes the non-perturbative sector of the $SU(2)$ theory beginning with monopole solutions and structure of other non-perturbative objects: neutral and magnetic bions formed from BPS and KK monopole constituents. This discussion leads to the non-perturbative effective potential of the theory arising from such objects.

Chapter 4 continues the discussion of $SU(2)$ super Yang-Mills on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ and its duality to an exotic Coulomb gas of electric and magnetic charges: the W-bosons and magnetic monopoles and bions. This is done through a derivation of the partition function of the theory which gives the interactions of the various gas constituents. One remarkably finds, through the existence of the compact thermal direction, an additional scalar field which leads to a scalar 'charge' of W-bosons and neutral bions that attracts likely charged particles as opposed to oppositely charged ones. This dual double Coulomb gas was subjected to lattice Monte-Carlo simulations [1] leading to study of the deconfinement phase transition and dynamics of a gas of such particles at different temperatures. Section 4.2 looks at another dual to the Coulomb gas: the lattice version which is best described as an 'affine' XY model with symmetry-breaking perturbations and parameters that map onto particles of the Coulomb gas and couple to the scalar field. This was also subjected to lattice studies for $SU(2)$ and the second order deconfinement phase transition was examined. The results will be discussed here as well.

Chapter 5 provides the mathematical background of Lie groups, Lie algebras and representation theory necessary for the remainder of the work. Specific attention will be paid to the study of root and weight systems of Lie algebras and the notation for following chapters will be set up. Chapter 6 begins the case of theories in general gauge group $G = A_N, B_N, C_N, D_N, E_6, E_7, E_8, F_4, G_2$, starting with the perturbative theory as done for $SU(2)$ in section 4.1. Section 6.1 sets up the discussion of the matter in the theory in the case of any gauge group at zero temperature, followed by section 6.2 where the finite temperature theory on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ is examined in detail and the effective GPY potential derived. Section 6.3 considers the case of $\mathbb{R}^3 \times S^1_L$ which will be needed for Chapter 8 later, as well as some general details of general toroidally compactified spacetimes. The following sections discuss the non-perturbative aspects of theories of general gauge group including monopole and bion solutions, the presentation of the superpotential for super Yang-Mills, and of the non-perturbative effective potential and dynamics.

Chapter 7 returns to the discussion of deconfinement of SYM on $\mathbb{R}^3 \times S^1_L \times S^1_\beta$ for general gauge group $G$, by deriving the dual theories: the double Coulomb gas and the dual XY model. Although the question of finding a clear dual XY model in the case of general gauge group is not resolved the double Coulomb gas seems to be more approachable and suggestions for future studies of it including lattice Monte-Carlo simulations will be discussed. Only brief mention of general spin models for other gauge groups and the difficulties in deriving them will be mentioned here. however, the derivation of the double Coulomb gas partition function in general gauge group is a main result of this work and one hopes to soon do lattice simulations of this Coulomb gas as done in [1] for $SU(2)$.

Chapter 8 begins with a detailed discussion of super Yang-Mills in general gauge group on $\mathbb{R}^3 \times S^1_L$ where the gaugino is given a small mass $m$ (softly-broken supersymmetry). A quantum phase transition to a deconfined phase is observed at some critical mass for the gaugino and is analytically calculable and
determined to be of first order for all gauge groups, other than for $SU(2)$ where it was second order. It is interesting to note a continuity conjecture which states that this quantum deconfinement phase transition of SYM on $\mathbb{R}^3 \times S^1_L$ is continuously connected to the thermal deconfinement phase transition of pure Yang-Mills. This is discussed in section 9.1. Section 9.2 introduces the Callias index theorem, generalized to $\mathbb{R}^3 \times S^1_L$, which allows corrections to superpotentials and Kähler potentials due to non-cancellation of non-zero mode determinants in the background of monopoles. This allows the derivation of effective potentials for studying the deconfinement phase transition of this deformed SYM on $\mathbb{R}^3 \times S^1_L$ for any gauge group as well as for any theta angle. Section 9.3 presents the results from minimization of the effective potential of the theories and the critical gaugino mass for deconfinement, and begins with the case of $SU(2)$. It is found here that increasing theta angle decreases the critical mass of the transition.

Chapter 9 concludes the thesis and discusses areas of further study that can follow after this work, including future Coulomb gas simulations in general gauge group, spin model derivations in special cases, as well as other questions and treatments of theories that can be done in a future work.
Chapter 2

Yang-Mills Theory and Phases of Gauge Theory

'Things fall apart,
the centre cannot hold.'
- W. B. Yeats  The Second Coming

A gauge theory can come in many phases which include the confined phase, which is of interest in this work. In the confined phase there is a linear potential of interaction between particles of the theory (quarks) of the form \( V = \sigma(T)R \) for separation \( R \) and \( \sigma(T) \) is the (temperature-dependent) 'string' tension of the colour flux tubes stretched in between the quarks. Figure 1.5 of the Introduction section shows quarks in the confined phase. We see that once stretched apart by a distance \( R_c \approx 2m_q/\sigma(T) \) a quark-antiquark pair is formed and another meson is created. This is how below the deconfinement temperature quarks are not observed free in nature and a single quark has an infinite vacuum energy density. One can calculate the string tension of the quark-anti-quark pair, and determine the deconfinement transition temperature at the point when the string tension vanishes (the so-called dual photon\(^1\), to be introduced later, becomes massless and the theory loses its mass gap). This is one of many phases a gauge theory can have.

Table 2.1 shows some phases of gauge theories and the potentials of interactions in three dimensions. The Coulomb potential is the usual interaction potential between electrically (and magnetically) charged

\[ V(R) \approx \frac{1}{R} \]

\[ V(R) \approx \frac{1}{R} \log(R\Lambda) \]

\[ V(R) \approx \log(R\Lambda)/R \]

\[ V(R) \approx \text{constant} \]

\[ V(R) \approx \sigma(T)R \]

Table 2.1: Phases of gauge theories and potentials of interaction in three dimensions.

\(^1\)The dual photon \( \sigma \), a scalar in two dimensions, is related to the field tensor as \( \epsilon_{\mu\nu\lambda\sigma} \partial_\lambda \sigma = \frac{4\pi L}{\sigma^3} F_{\mu\nu} \), and will be discussed in the next Chapter.
particles due to photon exchange. In the free electric and magnetic cases the log(RΛ) factors take into account the electric charge screening at energy scale Λ (which decreases the apparent charge at higher energy scales) and the anti-screening of magnetic charges.\(^2\) These phases differ from the Higgs phase where there is no interaction force between particles, and the confined phase where they are attracted by a linear potential. It is this confining phase in these theories that I examine here.

Before discussing gauge theory in general let me discuss Yang-Mills theory. This is essentially a theory of the Standard Model with gauge group \(SU(3) \times SU(2) \times U(1)\). These gauge group factors correspond to the quark/gluon sector of the strong force, the lepton sector of the weak force with the W and Z bosons, and the electromagnetic sector of the photon, respectively. The full action of the Standard model, with all of its particles and their interactions is quite long [60], however the action of a basic pure (bosonic) Yang-Mills theory on a manifold \(M\) (taken here to be four dimensional, unless otherwise specified) can be written simply as

\[
S_{YM} = \int_M dVol \, \text{tr}\left[ \frac{1}{2g^2} F^{MN} F_{MN} \right]
\]

for the bosonic sector, where the field strength tensor is \(F_{MN} = F^a_{MN} T^a\) with \(F_{MN} = \partial_M A_N - \partial_N A_M + ig[A_M, A_N]\) and \(T^a\) are the generators of the Lie algebra of the gauge group. Capitalized indices \(M, N\) run over all coordinates of the manifold \(M\). In future cases I will add \(n_f\) (adjoint representation) Weyl fermions to the theory which adds a fermionic term \(\bar{\lambda} \sigma^M D_M \lambda\) to the Lagrangian in the case of four dimensions where \(\sigma^M\) become the usual Pauli matrices. In the case of adjoint representation fermions in the theory we call it \(QCD(adj)\). Supersymmetric versions with \(n_f = 1\) will also be considered here, with \(N = 1\) supersymmetries and will be called super Yang-Mills (SYM).

Some further definitions may be good to review. A **gauge field** \(A = A_\mu dx_\mu\) is essentially a Lie algebra-valued one-form on a spacetime manifold \(M\) and so \(A \in \Omega_1(M, g)\) where \(g = \text{Lie } G\) is the Lie algebra of the gauge group. A **(gauge) particle** is in general a quantization of a gauge field that may or may not have mass, as well as other properties such as charge or spin. These are the gauge bosons. **Charge** can be electric or magnetic and the particle inherits it from its Lie algebra’s roots and co-roots respectively. Other types of particles are the fermions (which constitute matter) and they can be thought of as quantizations of fields of half-odd integer spin, and so live on spin manifold equivalents of the manifolds the gauge fields live on. The fields with such spin would take on the spin representation of the gauge group, if not already a spin group. More on charges, roots, Lie algebras and representations will be covered in Chapter 5.

### 2.1 Duality in Gauge Theory

Before further discussion of confinement in the next sections I want to mention duality in gauge theories. A **duality** between two theories \(A\) and \(B\), say, is essentially a mapping between observables of the theories (these are gauge-invariant quantities of the theory such as expectation values of fields and their correlations. In particular of interest are the Wilson and Polyakov loops, important in the study

\(^2\)Note that the product of the screened/anti-screened charges is not dependent on the scale of the theory, due to Dirac quantization of electric and magnetic charges.
of phases of gauge theories, and will be mentioned in the next Section). Basically, although the two theories may be very different, knowing something about the one theory tells one something about the other theory. In some cases one theory may be strongly-coupled and difficult to study analytically, while the dual theory may be weakly-coupled and easily studied analytically.

Two common examples of dualities in gauge theory include S and T-duality. S-duality, also known as 'strong-weak' duality, matches theories of strong coupling $g$ (a generalized 'charge' one could say) to one of weak coupling strength $1/g$. This is important as studying strongly-coupled theories perturbatively and analytically is difficult, so duality to a more well studied weakly-coupled theory can tell one much about the dual strongly-coupled one.

T-duality, also known as 'target' duality, is in general a duality between theories of different target spaces, that is spaces where the fields live on. In the case of toroidally compactified theories this relates a theory with a compact dimension of size $R$ to one with size $1/R$ and the other dimensions fixed. This is important in the study of deconfinement of finite temperature theories in that it relates low and high-temperature limits of different theories, or of the same theory. At finite temperature one has a compact time-like direction of length $\beta = 1/T$. A high-temperature theory on this space can be related to a dual one at low-temperature with a compact thermal direction of length $T = 1/\beta$. This is interesting because the dynamics of a theory at low and high temperature can be very different. For example, in the case of SYM we find a low temperature theory is dominated by magnetic bions and is confining, whereas high temperature SYM is dominated by W bosons and is in the deconfined phase. T-duality is then somewhat related to electric-magnetic duality in this example and can be useful in studying the deconfinement phase transition from two different angles.

Besides these main 'theoretical dualities' there are also 'model dualities', as I will call them. These are different models of a theory, such as SYM, that can have entirely different pictures. The idea is that the different physical systems considered give rise to similar results of calculations of dual observables in the two pictures and each parameter or observable of one theory can be mapped onto one of the other model. For example, the sine-Gordon model of pure Yang-Mills can be shown to be dual to a Coulomb gas of electric and magnetic charges and has been well studied in the past [43], [47], and parameters of the former theory can be mapped onto constituents of the Coulomb gas. This dual picture can allow the study of the theory through renormalization group investigations as well as numerical simulations, as was done in [15] and [1], respectively, for the case of $SU(2) N = 1$ SYM on $\mathbb{R}^2 \times S_1 \times S_1 \beta$. This will be discussed more later on. Furthermore, each theory can be placed on a lattice and studied numerically. What one needs is to determine the right lattice spin model dual to the theory. What one can find is that particles of one theory or constituents of a Coulomb gas, can be mapped to topological excitations of the lattice theory, such as vortices, or to symmetry breaking perturbations. It was found in [1] for SYM and in [15] for $QCD(adj)$ that the W-bosons are dual to the vortices of the spin model (an XY spin model with symmetry-breaking perturbations) and the magnetic bions are related to the symmetry-breaking perturbations. This will, again, be discussed later.

After having discussed gauge theories in general, and their phases, let me return to the study of the confining phase of YM and SYM theories.
2.2 The Confining Phase

One way of studying Yang-Mills and QCD-like theories in four dimensions is to consider a compactified direction as a way of controlling the calculability of the theory and the validity of semi-classical treatment. The compactification on a small circle $S^1_L$ with $L_{QCD} \ll 1$, where $c_2(G)$ is the dual Coxeter number of the gauge group and equals $N$ for $SU(N)$, controls the strength of the coupling at a scale $\mu \approx 1/L_{QCD}$ where the coupling ceases to run into large values near $\Lambda_{QCD}$. Furthermore, the theory also Abelianizes, which simplifies the effective action. By further compactification on a cycle of length $\beta$ we can introduce a small temperature $T \ll 1/c_2 L$ to the theory to examine the nature of its thermal phase transitions. In order to study the deconfinement phase transition one uses as order parameter the expectation value of the so-called Wilson loop around a compact direction, which in any representation $\mathcal{R}$ is given by two loop observables along the spatial and thermal cycles respectively

$$\Omega_L = Pe^{i\oint_{x,x} dx_3 A_3}, \quad \Omega_\beta = Pe^{i\oint_{x,0} dx_0 A_0},$$

where $P$ denotes path ordering. A Wilson loop along a thermal direction is often called a Polyakov loop. I will use 0 to denote the thermal direction, and 3 for the compact $L$-direction. Greek indices $\mu, \nu$ will denote non-compact spatial directions 1,2. Latin indices $i, j$ will be used for the spatial directions 1,2,3 while uppercase Latin indices $M, N$ run over all directions 0,1,2,3. Latin indices $a, b$ run over the $r = \text{rank}(G)$ generators of the Cartan subalgebra of the Lie algebra. Note that it is the loop observable $\Omega_\beta$ that controls the thermal deconfinement phase transition, not $\Omega_L$, which controls the breaking of $Z(G)^{(L)}$, defined below, which occurs at temperatures beyond the deconfinement phase transition. The holonomies transform under $x$-dependent gauge transformations, $U_L(\vec{x}, x^0) \in G \times \mathbb{R}^3 \times S^1_L$, as $\Omega_L(\vec{x}, x^0) \rightarrow U_L^{-1}(\vec{x}, x^0)\Omega_L(\vec{x}, x^0)U_L(\vec{x}, x^0)$, and similarly for $\Omega_\beta(\vec{x}, x^3)$ with $x^0 \rightarrow x^3$. Hence the eigenvalues of the Wilson loops are gauge invariant. The (gauge-invariant) traces of these Polyakov or Wilson loops act as an order parameter for the deconfinement phase transition as they are zero in the confined phase (meaning their eigenvalues are spread out as $\text{tr}\Omega^k = 0$ for all $k = 1, \ldots, n = |Z(G)|$, and the $Z(G)^{(L)}_L, \beta$ centre symmetries are preserved, $Z(G)$ being the centre of the gauge group $G$) and non-zero elsewhere where the eigenvalues tend to clump together. The order parameters $\text{tr}\Omega_{L, \beta}$ are for the breaking of the centre symmetry of the gauge group along the respective compact circle, $Z(G)^{(L)}_{L, \beta}$, and the symmetries act on the traces of the Polyakov loops by multiplying by phases $e^{2\pi ik/n}$ in $Z(G)$, where $n = |Z(G)|$, $k = 0, \ldots, n$ (or in the exception of group $G = D_{2N}$, $n = 2$ for each of its $\mathbb{Z}_2$ factors).

Figure 2.1 shows all semi-simple Lie groups and their centres.\(^3\)

Let me describe these order parameters and their role in confinement in more detail as a reminder. Consider the formation of a quark-anti-quark pair, and then separating them by a distance $R$ followed by a propagation for a time $t$ after which they recombine and annihilate. Since the anti-quark can be thought of a quark traveling back in time we end up with a rectangular contour of area $tR$ (this is the so-called area law of confined phases), which is thought of as the Wilson loop in representation $\mathcal{R}$ of the quarks (taken to be infinitely massive in the fundamental representation to test for confinement) $\text{Tr}\Omega_{L, \mathcal{R}}(x^\mu)$. It is a Lie group-valued quantity whose trace is a gauge invariant observable of the theory, however at finite temperature it exhibits an area law in both confined and deconfined phases. Hence the

\(^3\)I consider the universal cover of the Lie group $G$, $\tilde{G}$, as it allows all representations, such as the spin representation, and is simply connected so $\pi_1(\tilde{G}/Z(\tilde{G})) \approx Z(\tilde{G})$. 

Figure 2.1: Centres of all simple Lie groups.

Wilson line is not an order parameter for confinement. However, at \( T > 0 \), one can insert the finite-T Polyakov loop \( \Omega^R_\beta(x^\mu) \) into the partition function of the theory and this acts as an order parameter of deconfinement. The finite-T potential of interaction between the quark-anti-quark pair can be found from the correlator of traces of two such loops

\[
\langle \text{Tr} \Omega^R_\beta(x^\mu) \text{Tr} \Omega^\dagger_{\beta}(0) \rangle = e^{-V(r,T)/T},
\]

where \( r = |x^\mu| \). Below the deconfinement transition temperature we have a linear potential \( V(r,T) = \sigma(T)r \) and so we have as \( r \to \infty \), \( \langle \text{Tr} \Omega^R_\beta(x^\mu) \text{Tr} \Omega^\dagger_{\beta}(0) \rangle \to 0 \). This means that necessarily \( \langle \text{Tr} \Omega^R_\beta \rangle = 0 \) everywhere in space in the confining phase.

At high temperatures above the deconfinement transition temperature, however, we find a Yukawa type potential which can be described generically by \( V(r,T) = v + be^{-m_e r/r} \), where \( m_e \) is some (electric) Debye screening mass. Now as \( r \to \infty \), \( \langle \text{Tr} \Omega^R_\beta(\infty) \text{Tr} \Omega^\dagger_{\beta}(0) \rangle \neq 0 \) and in general \( \langle \text{Tr} \Omega^R_\beta \rangle \neq 0 \). This qualitatively different behaviour of Polyakov loops allows them to serve as an order parameter of the deconfinement phase transition as has been done for several years in lattice studies of pure Yang Mills and other theories [47], [48].

Furthermore, since the Polyakov loop is unitary its eigenvalues lie on the unit circle \( e^{i\theta} \in S^1 = U(1) \) in the complex plane. At high temperatures the W bosons of the theory proliferate the vacuum and the perturbative Gross-Pisarski-Yaffe (GPY) potential (to be calculated in Chapter 3) dominates over the non-perturbative potential due to magnetic monopoles and bions (to be discussed in the next Chapter as well). This GPY potential always tends to clump eigenvalues together as it induces attractive forces between them. This ensures that \( \langle \text{Tr} \Omega^R_\beta \rangle \neq 0 \) and signals the deconfined phase. This 'clumping' of eigenvalues shows the breaking of the centre-symmetry in the vacuum of the theory at high \( T \), whereas if the eigenvalues were evenly spread out, the vacuum of the theory would preserve the centre symmetry. This is what occurs at lower temperatures where it is found that the non-perturbative contributions from bions (in particular, the neutral ones) dominate the perturbative contribution to the potential from W bosons which are thermally suppressed. This contribution to the potential which dominates at low \( T \) induces repulsive forces among eigenvalues of the Polyakov/Wilson loop and tends to preserve centre-symmetry. As all eigenvalues tend to spread out evenly along the \( S^1 \) we find that \( \langle \text{Tr} \Omega^k_{\beta} \rangle = \sum_{\theta \in Z(G)} \langle e^{ik\theta} \rangle = 0 \).

\(^4\)This is because, as the quark probes become infinitely separated they become uncorrelated (due to causality, say) and \( \langle \Omega_x \Omega_0 \rangle = \langle \Omega_x \rangle \langle \Omega_0 \rangle \).
and we notice the property of the confined phase at low $T$.

I can then summarize the confined phase of matter by the following properties:

(i) 'Area law': $\langle \text{Tr} \Omega^R_L (x^\mu) \text{Tr} \Omega^R_L (0) \rangle = e^{-V(c,T)}/T = e^{-\sigma(T)} \beta r$ for the spatial Wilson loop, where $A = \beta r$ is the area enclosed by the loop (this is different than the usual perimeter law for other phases of gauge theories).

(ii) Centre-symmetry preserved: vacuum $= Z(G)^{(\beta)}$ for the $\Omega^\beta$ Wilson loop.

(iii) Vanishing of thermal Polyakov loop observable: $\langle \text{Tr} R \Omega^\beta \rangle = 0$.

At finite temperature $T > 0$ condition (i) is not equivalent to the other two, as mentioned above, as Wilson lines exhibit the area law in both confined and deconfined phases as $\sigma(T) \neq 0$ identically in either phase.

One should note that for groups without centre there is no centre symmetry to break at the deconfinement phase transition. However, this does not prevent deconfinement as the eigenvalue dynamics of attraction and repulsion at different temperatures still holds, allowing eigenvalue clumping at higher temperatures, and spread out eigenvalues at lower $T$. Hence jumps in Polyakov loop traces are observed as in [2] but are not identically zero in the deconfined phase and so the thermal Polyakov loop still can serve as a probe for deconfinement but is not exactly an order parameter\(^5\) and the role of centre symmetry is no longer essential. One should also note that the 'Higgsing' of $G \to U(1)'$ to its maximal Abelian subgroup does not always occur from compactification on $S^1$ for all gauge groups in general theories like non-SUSY $QCD(adj)$. Figure 2.2 shows the symmetry breaking due to Higgsing from compactification [4]. The reason why the symmetry does not fully Abelianize is that the vacuum of the compactified theory has points lying on gauge cell boundaries, so not all particles get mass and there is just partial Higgsing. For more on gauge cells see the end of Chapter 5 as well as the appendix of [4]. However, in this work I discuss super Yang-Mills and the theories always fully Abelianize in this case.

\(^5\)The thermal Polyakov loop does change at the deconfinement phase transition but is not always strictly zero before the transition. It is then not always an order parameter in the strict sense that it is non-zero in one phase and strictly zero in another. We find in [2] that the deconfinement phase transition is first order for all groups other than $SU(2)$ (where it is second order), but is without order parameter for groups without centre. Most phase transitions often occur without order parameter.
Figure 2.2: Symmetry breaking of gauge groups on $\mathbb{R}^3 \times S^1$. Note that most groups do not fully Abelianize after a single compactification in QCD(adj) with $n_f > 1$. The groups always Abelianize in the case of SYM.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(N+1)$ $\simeq$ $A_N$ $\rightarrow$ $U(1)^N$ for $N \geq 1$</td>
<td></td>
</tr>
<tr>
<td>$SO(2N+1)$ $\simeq$ $B_N$ $\rightarrow$ $U(1)^{N-1} \times SO(3)$ for $N = 2, 3$ $\rightarrow$ $SO(4) \times U(1)^{N-3} \times SO(3)$ for $N \geq 4$</td>
<td></td>
</tr>
<tr>
<td>$Sp(2N)$ $\simeq$ $C_N$ $\rightarrow$ $U(1)^N$ for $N \geq 3$</td>
<td></td>
</tr>
<tr>
<td>$SO(2N)$ $\simeq$ $D_N$ $\rightarrow$ $SO(4) \times U(1)^{N-4} \times SO(4)$ for $N \geq 4$</td>
<td></td>
</tr>
<tr>
<td>$E_6$ $\rightarrow$ $SU(3) \times SU(3) \times SU(3)$</td>
<td></td>
</tr>
<tr>
<td>$E_7$ $\rightarrow$ $SU(2) \times SU(4) \times SU(4)$</td>
<td></td>
</tr>
<tr>
<td>$E_8$ $\rightarrow$ $SU(2) \times SU(3) \times SU(6)$</td>
<td></td>
</tr>
<tr>
<td>$F_4$ $\rightarrow$ $SU(3) \times SU(2) \times U(1)$</td>
<td></td>
</tr>
<tr>
<td>$G_2$ $\rightarrow$ $SU(2) \times U(1)$</td>
<td></td>
</tr>
</tbody>
</table>
Chapter 3

Deconfinement in $SU(2)$ Gauge Group

I begin the study of deconfinement of Yang-Mills and super Yang-Mills on toroidally compactified spacetimes with the case of the rank one gauge group $SU(2)$. The case of general compactifications on $\mathbb{R}^{D-d} \times \prod_i^d S_{L_i}$ of $d$ compact dimensions of sizes $L_i$ will be mentioned and the perturbative effective potential derived, with the end goal of focusing on the case of the torus $D = 4, d = 2, \mathbb{R}^2 \times S_{L_1} \times S_{L_2}$. The zero temperature case on $\mathbb{R}^3 \times S_{L_1}$ will be reviewed before looking at the finite temperature $T > 0$ case. This is done in Section 3.1. Section 3.2 reviews the non-perturbative sector of the theory in $SU(2)$ gauge group and presents monopole-instanton and bion solutions and derives the non-perturbative effective potential.

3.1 Perturbative Theory of $SU(2)$ Yang-Mills on Toroidally Compactified Spacetimes

Recall the action for our main theory of interest now: $\mathcal{N} = 1$ supersymmetric $SU(2)$ Yang-Mills theory on a toroidally compactified spacetime $M$, with focus on $M = \mathbb{R}^3 \times S_{L_1}$ or $\mathbb{R}^2 \times S_{L_1} \times S_{L_2}$, with a single adjoint Weyl fermion, the gaugino $\lambda = \lambda^a T^a$.

$$S = \int_M d^Dx \left[ \frac{1}{2g^2} F_{MN} F^{MN} + \frac{2i}{g^2} \bar{\lambda} \sigma^M D_M \lambda \right].$$

(3.1)

Here $\sigma_M$ are the $D$-dimensional Pauli matrices, which in four dimensions are taken to be $\sigma_M = (i, \vec{\tau})$ where $\vec{\tau}$ are the usual $SU(2)$ Pauli matrices. The action for (pure) Yang-Mills (YM) is the same as in (3.1) just without the second terms with $\lambda$’s in them. As before $F_{MN} = \partial_M A_N - \partial_N A_M + ig[A_M, A_N]$ is the gauge field strength tensor and the gauge covariant derivative is $D_M \lambda^a = \partial_M \lambda^a - gf_{abc} A^b_M \lambda^c$ with structure constants $f_{abc}$ (defined by $[T^a, T^b] = f^{abc} T^c$) for the gauge group in consideration. I will use indices $M, N$ to run over all dimensions of spacetime and lowercase $i, j$ to run over non-compact directions. Beginning at zero temperature, and in the regime $L_{A_{QCD}} << 1$ where accurate calculations integrating out heavy Kaluza-Klein modes along the compact cycles $S_{L_i}$ can be done, we can find the perturbative effective potential of the compactified theory, $V_{\text{eff pert.}}(\{\Omega_{L_i}\})$. Here, $\Omega_{L_i}$ are the Wilson
loops around the $i$-th compact cycle. In the case of supersymmetry this potential vanishes identically, as computing the two effective potentials, bosonic and fermionic, there is a sign difference due to fermionic Gaussian integrals having inverse determinants than the bosonic ones. Hence at zero temperature there is a whole moduli space of supersymmetric vacua. At finite temperature, however, I will find in the next subsection that these do not cancel entirely, as the fermions have periodic boundary conditions along the spatial cycle but antiperiodic around the thermal cycle. Nonetheless, as will be found in the next section, there are non-perturbative contributions $V_{\text{eff non-pert.}}(\Omega_L)$ from magnetic objects in the theory which, in the case of $SU(2)$ on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$, I find the minimum of the effective potential to be

$$\langle \Omega_L \rangle = \text{diag}(e^{i\pi/2}, e^{-i\pi/2}), \quad \langle A_3 \rangle = (A_3^3)^T = \frac{\pi}{L} T^3. \quad (3.2)$$

As mentioned in the Section on confined phases we have $\text{tr} \langle \Omega_L \rangle = 0$ and so the $\mathbb{Z}_2^L$ centre symmetry is preserved and we are in a confined phase at zero temperature. In this vacuum the $SU(2)$ symmetry is broken by the Higgs field $A_3$ to $U(1)$. Being in the adjoint representation it has three components, two of which become massive of mass $M_W = \pi/L$. These massive gauge field components are termed the W-bosons of the theory and these, once integrated out of the effective action, are responsible for the perturbative effective potential of the theory. The remaining component $A_3^3$ (the third colour component along the so-called Cartan subalgebra direction) gets an exponentially small mass $\approx e^{-8\pi^2/g^2}$ due to non-perturbative effects. The components of $\lambda^a$ that do not commute with $A_3$ also acquire mass $M_W = \pi/L$, and these are called the winos - the superpartner of the W bosons. In the case of supersymmetry the low temperature, low energy effective action reads

$$S_{\beta \to \infty} = \int_{\mathbb{R}^2 \times S^1_\beta} \frac{L}{g^2} \text{tr} [-\frac{1}{2} F_{\mu \nu} F^{\mu \nu} + (D_\mu A_3)^2 + 2i \bar{\lambda} \sigma^\mu D_\mu \lambda - i \bar{\sigma}_3 [A_3, \lambda]] \quad (3.3)$$

$$= \int_{\mathbb{R}^2 \times S^1_\beta} \frac{L}{2g^2} [\frac{1}{2} (D_\mu A_3)^2 + 2i \bar{\lambda} \sigma^\mu \partial_\mu \lambda],$$

where the last line occurs at distances $\gg L$ (Greek indices run over non-compact and thermal directions). Note that we are at relatively weak coupling in this regime as there is no coupling between the fermions and gauge fields so the coupling $g(\mu)$ ceases to run at energy scales $\mu < 1/L$. It is small due to asymptotic freedom and $L \Lambda_{\text{QCD}} \ll 1$ as discussed in the Introduction.

One further way to write our action is using Abelian duality

$$\epsilon_{\mu \nu \lambda} \partial_\lambda \sigma = \frac{4\pi L}{g^2} F^{3}_{\mu \nu}, \quad (3.4)$$

which maps the gauge field to a spin zero scalar field $\sigma$, called the dual photon, which is in fact a scalar in two dimensions and is compact with period $2\pi$. Defining another scalar field $\phi$

$$\phi \equiv \frac{4\pi L}{g^2} A_3^3 - \frac{4\pi^2}{g^2}, \quad (3.5)$$

(so $\phi = 0$ corresponds to centre symmetry being preserved with $\langle A_3^3 \rangle = \pi/L$ at the supersymmetric
vacuum of the theory), allows me to write the free bosonic part of the Lagrangian as
\[ L_{\text{free bosonic,low}} - T = \frac{1}{2} g^2 \left[ (\partial_{\mu} \sigma)^2 + (\partial_{\mu} \phi)^2 \right]. \] (3.6)

Let me pause here and turn now to the finite temperature case deriving the perturbative effective potential for \( SU(2) \) SYM.

### 3.1.1 Finite temperature perturbative dynamics of \( SU(2) \) super Yang-Mills

At finite temperature \( T > 0 \), with supersymmetry, the situation is different as the perturbative effective potential no longer vanishes identically. This is because, although both bosons and fermions have identical periodic boundary conditions along the spatial circle \( S^1 \), the fermions take anti-periodic boundary conditions along the thermal circle. This breaks the supersymmetry and the perturbative effective potential no longer vanishes. Let me derive it and its low temperature expansion.

This effective potential in general is just a sum over the eigenvalues of the Laplacian \( \Delta \) on \( \Sigma \) and so can be written as a zeta function and usual zeta function regularization techniques can be used.

\[ V_{\text{eff}} = -\ln \det(\Delta)/ \prod_{M=1}^{d} L_{M} = -\zeta'(0)/ \prod_{M=1}^{d} L_{M}. \] (3.7)

The effective potential is divided by the volume of the compact dimension \( \prod_{M=1}^{d} L_{M} \) so when the compact dimension is integrated out the effective action is integrated over the remaining non-compact space \( \mathbb{R}^{D-d} \). In the case of a flat torus we have the usual Laplacian \( -\Delta = \partial_{\mu} \partial^{\mu} \) but with Wilson lines added \( \Omega_{\gamma} = e^{\int_{\gamma} A_{\mu} dx_{\mu}} \) where the gauge field \( A_{\mu} \in \Omega^{1}(M, \text{Lie } G) \) is a Lie algebra 1-form on the manifold and \( [\gamma] \in H_{1}(M) \) is a homology cycle. Only non-contractible Wilson loops are considered, and the eigenvalues of these Wilson loop matrices we will denote by \( \theta_{i} \). The effect of the Wilson lines is that they add a phase to the gauge potential and hence the covariant derivative gets a phase term added. Thus the Laplacian becomes \[ -\Delta_{j} = \partial_{m} \gamma^{m} + \sum_{M}(\partial_{M} - i \theta_{M,j}/L_{M})^{2} \] where \( m \) denotes non compact space/space-time coordinates and \( M \) denotes coordinates of the torus, with the \( M \)-th cycle of length \( L_{M} \). The index \( j \) represents the gauge group’s indices and will be suppressed in the following formulae (I begin by considering the rank 1 Lie group \( SU(2) \) in this Chapter anyway and derive a formula for all gauge groups in Chapter 6). We can represent the zeta function by the following heat-kernel representation:

\[ \zeta(s) = \Gamma(s)^{-1} \int_{0}^{\infty} dt \ t^{s-1} \sum_{\lambda} e^{-\lambda t}, \] (3.8)

where the eigenvalues of the Laplacian \( \Delta \) are \( \{ \lambda = \sum_{M}(2\pi(n_{M} + \theta_{M}/2\pi)/L_{M})^{2} + k^{2} \} \). The formula is simplified using Poisson resummation, doing the \( k \)-integral and finally the \( t \)-integral to get the bosonic contribution to the perturbative one-loop effective potential:

\[ V_{\text{eff}} = 2^{D-d+1} \Gamma(D/2) \pi^{D/2-d} \sum_{\vec{n}_{M} \neq 0} \frac{e^{i \sum_{M} \theta_{M} n_{M}}}{(\sum_{M} L_{M}^{2} n_{M}^{2})^{D/2}} = 2^{D-d+2} \Gamma(D/2) \pi^{D/2-d} \sum_{\vec{n}_{M} \geq 0} \frac{\cos \sum_{M} \theta_{M} n_{M}}{(\sum_{M} L_{M}^{2} n_{M}^{2})^{D/2}}. \] (3.9)

Note that for fermions there would be a factor of \( -(-1)^{n} \times n_{f} \) in the sum over the modes in the \( \beta \)-
direction (periodic BCs in spatial direction, anti-periodic in time direction). We consider fermions in the adjoint representation, and for adjoint matter note that every time a Wilson line eigenphase appears in direction (periodic BCs in spatial direction, anti-periodic in time direction). We consider fermions in the

Note that this gives the correct result for studies on \( \mathbb{R}^3 \times S^1_L \) [4]. Let me derive (3.9).

I can represent the zeta function by the heat-kernel representation (3.8) to give

\[
\zeta(s) = \frac{V(S^{d-1})}{\Gamma(s)} \int_0^\infty dt t^{s-1} \int_{-\infty}^\infty dk k^{d-1} \times \sum_{\vec{n} \in \mathbb{Z}^d} \exp(-t[(2\pi(n_M + \theta_M/2\pi)/L_M)^2 + k^2]) \tag{3.10}
\]

where \( \vec{n} = (n_1, \ldots, n_d) \), the eigenvalues \( \{\lambda = \sum_M (2\pi(n_M + \theta_M/2\pi)/L_M)^2 + k^2\} \) of the Laplacian \( \Delta \) were put in (3.8), and the volume of the \((d-1)\)-sphere \( S^{d-1} \) of radius 1 is \( 2\pi^{d/2}/\Gamma(d/2) \).

I will simplify (3.10) using the following Poisson resummation formula (after doing the easy Gaussian \( k \)-integral)

\[
\sum_{n=-\infty}^\infty \exp(-t(2\pi(n+a)/L)^2) = \frac{L}{\sqrt{4\pi t}} \sum_{n'=-\infty}^\infty \exp(-L^2n'^2/4t + 2\pi in'a). \tag{3.11}
\]

Using this for each cycle of the torus and integrating over the momenta then gives the effective potential

\[
\zeta(s) = \frac{2\pi^{(D-d)/2}}{\Gamma(s)2^{d/2}\pi^{d/2}} \sum_{\vec{n} \in \mathbb{Z}^d} \int_0^\infty dt t^{s-D/2-1} \exp(-\sum_M (L_M^2n_M^2/4t + in_M\theta_M)).
\]

Remaining to do is the \( t \)-integral and we get our general effective potential formula for the torus after taking the derivative of the divergent factor \( \Gamma^{-1}(s) \) at \( s = 0 \):

\[
V_{\text{eff}} = 2^{D-d+1}\Gamma(D/2)\pi^{D/2-d} \sum_{\vec{n}_M \neq \vec{0}} \frac{e^{i\sum_{M=1}^d \theta_M n_M} \sum_{M=1}^d \theta_M n_M}{(\sum_M L_M^2 n_M^2)^{D/2}} = 2^{D-d+2}\Gamma(D/2)\pi^{D/2-d} \sum_{\vec{n}_M \neq \vec{0}} \frac{\cos \sum_{M=1}^d \theta_M n_M}{(\sum_M L_M^2 n_M^2)^{D/2}},
\]

where \( \vec{n}_M \neq \vec{0} \). \( \square \)

These functions I will plot, and will generalise further to arbitrary conformal parameter \( \tau \). Let me look at this in the case of \( D = 4 \) and a 2D torus \( d = 2 \) with cycle lengths \( L, \beta \), and I will mention methods to study other compact manifolds as well. For now I look analytically more into the \( V_{\text{eff}} \) and examine the low-temperature expansion of \( V_{\text{eff}} \) in the case of \( SU(2) \):

\[
V = \sum_{n=0}^\infty \sum_{m=1}^\infty \frac{(1 - (-1)^m n_f) \cos(n\phi L + m\psi \beta)}{(m^2\beta^2 + n^2 L^2)^2}.
\tag{3.12}
\]

Let me start by plotting some cases of the torus effective potential and finding their absolute minima. Looking at the first few figures, we see immediately that the local and global minima occur at \( \pi \) for the \( L \) cycle (\( \phi \) field) for aspect ratios \( L/\beta \) less than 1, and at 0 for aspect ratios greater than 1. This implies the existence of a phase transition at \( r_c = (L/\beta)_c = 1 \) and the plot Figure 3.1 for aspect ratio 1 shows that the phase transition is first order as there is a 'bump' in between two degenerate minima at the critical phases, adding a 'latent' heat to the transition. In our case of interest we have small \( L \ll \beta \) and

\footnotetext[1]{Only derivatives of divergent factors need to be computed as the rest become 0 as \( s \to 0 \).}
Chapter 3. Deconfinement in SU(2) Gauge Group

Figure 3.1: Perturbative effective potential for torus of aspect ratio $L/\beta = 1$. Note the two minima at $(\phi, \psi) \in \{(0, \pi), (\pi, 0)\}$ with a bump in between them signaling a first order phase transition as $L \to \beta$. Note the $x, y$ axes are scaled to run from 0 to $2\pi$.

so in all these cases we have the unique global minimum $(\phi, \psi) = (\pi, 0)$ and this will be important later.\(^2\)

Note also the symmetry between the regimes $L \ll \beta$ and $L \gg \beta$ about $L = \beta$, and this low-T-high-T duality can be thought of as T-duality mentioned in Chapter 2. See Figure 3.2 for the case $L/\beta = 3/2$ and note how the global minimum has shifted to $(\pi, 0)$.

Low temperature expansion:

Now, the scale in our problem is $L \ll \beta$, i.e. we have a low-temperature and a small compact spatial dimension (this will ensure large masses $\approx 1/L$ for KK excitations so we can integrate out massive modes of mass beyond our regime of consideration $LA \ll 1$). Keeping this in mind and writing $A_{nm}^2 \equiv 4\pi^2[(n/L + \phi/2\pi)^2 + (m/\beta + \psi/2\pi)^2]$ I get

$$
\zeta(s) = (8\pi L\beta \Gamma(s))^{-1} \int_0^\infty dt t^{s-1} \int_0^\infty dk^2 \sum_{n,m=-\infty}^\infty e^{-t(k^2 + A_{nm}^2)} = \sum_{n,m=-\infty}^\infty \int_0^\infty \frac{dk^2}{8\pi L\beta(k^2 + A_{nm}^2)} =
$$

$$
= \sum_{n,m=-\infty}^\infty \frac{\pi}{2L\beta^3(1-s)}[(\frac{\beta}{L})^2(n + \phi L/2\pi)^2 + (m + \psi \beta/2\pi)^2]^{1-s}.
\tag{3.13}
$$

In the last line I set some analytic factors in $s$ to $s = 0$, as in the end they will not be hit by the derivative with respect to $s$. The sum over $m$ can now be done using a useful series expansion from [24].

$$
\zeta(s) = \sum_{n=-\infty}^{\infty} \frac{\pi^{3/2}}{L\beta^3 \Gamma(s-1)} \frac{\beta}{L} (n + \phi L/2\pi)^{3-2s} \Gamma(s - 3/2) +
$$

\(^2\)Let us look at these minima more closely: for the case of the flat square torus $r = 1$ there are two minima $\phi_{min} \in \{0, \pi\}$ which are global and occur for Wilson lines in the thermal cycle to be respectively $\psi_{min} = \pi, 0$. There is a 'hump' in between these minima is and so have a 'latent heat' as in any first order phase transition. For the case of a rectangular torus (say $r = 3/2$) the only minimum is at $(\phi, \psi)_{min} = (0, \pi)$ and the opposite for the dual torus ($r < 1$). See Figure 3.2 for the case $L/\beta = 3/2$. 


Let me make some comments on the effective potential so far. In considering the deconfinement transition we are interested in temperatures much smaller than the W boson mass $m_W = \pi / L$, i.e. $LT \ll 1$, and so we consider only the $p = 1$ term in (3.17). This is because of the exponential (Boltzmann) suppression of the higher-$p$ terms $\approx e^{-2\pi p / LT}$. I then write (3.17) to good accuracy in the simpler form

$$V_{\text{eff. pert.}}(A_0^3, A_3^3) = -\frac{4}{\pi \beta^3 L} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi |n\beta/L + \beta A_3^3|/2\pi|}}{n^3 L^3} (1 + 2\pi |n\beta/L + \beta A_3^3/2\pi|) \cos(\beta A_0^3).$$

Let me make some comments on the effective potential so far.

The potentials (3.17) and (3.18) are doubly periodic with periods $2\pi / L$ and $2\pi / \beta$ for the fields $A_0^3$ and $A_3^3$.
and $A_{i}^{a}$, respectively, and the potential determines the breaking of the two centre-symmetries along the two compact directions: $Z_{2}^{(L)}$ and $Z_{2}^{(\beta)}$. At low temperatures the two centre symmetries, as found in simulations, are seen to be respected, with $Z_{2}^{(L)}$ preserved at temperatures even above the deconfinement transition temperature. However for temperatures greater than the W boson mass $T \approx M_{W} = \pi/L$ the potential (3.17) shows that the $Z_{2}^{(L)}$-symmetry breaks. T-duality can relate the temperature of the breaking of these two centre symmetries.

Also, at temperatures $T \ll M_{W}$, the W boson (and superpartner) one-loop contribution to the mass of the scalar modulus $\phi$ (given by (3.5)) is exponentially suppressed $\approx e^{-M_{W}/T}$, as is easily seen by looking at the exponential prefactors of (3.18). However, even in the $T = 0$ theory the scalar mass is not zero but acquires an exponentially small mass from non-perturbative contributions, $\approx e^{-4\pi^{2}/g^{2}}$, and dominates over the thermal $\approx e^{-1/LT}$ contribution at low-$T$. This allows the formation of mass gap and the theory is indeed confining at zero temperature. It is also important to mention that the analysis of the perturbative effective potential (3.18) is beyond that of simply looking at its minima alone as non-perturbative corrections to it must be considered, as will be done in the next Section. The dynamics of the deconfinement phase transition depends on the competition between both the perturbative and the non-perturbative sectors of magnetic charges (monopole-instantons) as well as non-self dual 'exotic' topological excitations: the neutral and magnetic bions. I will return to this in the next Section, and Chapter 4 will provide methods of analysing the deconfinement transition by investigating duality of the theory with effective potential (3.17) and non-perturbative contributions to a dual double Coulomb gas of electric W-bosons and magnetic and neutral bions. As well, a discrete 'affine' XY spin model dual to the Coulomb gas will be determined and studied.

I will end this section with some other derivations of effective potentials in more general cases, but which I will not return to.

**Special further case of torus and other effective potentials from compactification.**

Other generalizations of the effective potential due to toroidal compactification of $N = 1$ SYM or pure YM can be done, including the general case of complex $\tau$ parameter (a 'skew' torus) as well as for massive fields. I will also mention briefly the general case of the 'flat' Riemann surface compactification.

Before continuing on to the case of complex $\tau$ parameter, let me briefly describe the massive case. In the previous derivation I simply replace $\vec{k}^{2}$ with $\vec{k}^{2} + M^{2}$ and get, upon doing the $k$-integral,

$$V_{eff} = \frac{1}{2(2\pi)^{(d-1)/2}} \prod_{M} \frac{1}{L_{M}^{2}} \sum_{n_{M} = -\infty}^{\infty} \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} dt t^{-d/2-1} \exp(- \sum_{M} L_{M}^{2} n_{M}^{2} + i n_{M} \theta_{M} - tM^{2})). \quad (3.19)$$

Using the representation of the modified $K$-Bessel functions: $K_{\nu}(z) = \frac{1}{2}(\frac{z}{2})^{\nu} \int_{0}^{\infty} dt \exp(-t - \frac{2}{t}) t^{-\nu-1}$, I then get

$$\zeta(s) = \Gamma(s)^{-1} \sum_{\vec{n} \in \mathbb{Z}^{d}} \frac{4\pi^{(D-d)/2}}{M^{2}2^{d} \pi^{d/2}} \left( \frac{2/M}{\sqrt{\sum_{M} L_{M}^{2} n_{M}^{2}}} \right)^{s+D/2} K_{s+D/2}(M \sqrt{\sum_{M} L_{M}^{2} n_{M}^{2}}) e^{i \sum_{M} n_{M} \theta_{M}}.$$
which leads to the effective potential upon taking the derivative of the divergent part $\Gamma(s)^{-1}$ at $s = 0$, \[ V_{\text{eff}} = M^{D/2}2^{D/2-\frac{d}{2}}\pi^{D/2-d} \sum_{\tilde{m}, \tilde{n} \geq 0} K_{D/2}(M) \sqrt{\frac{\sum_{\tilde{m}} n_{\tilde{m}} L_{\tilde{m}}^2 \cos(\sum_{\tilde{m}} n_{\tilde{m}} \theta_{\tilde{m}})}{(\sum_{\tilde{m}} n_{\tilde{m}} L_{\tilde{m}}^2)^{D/4}}}. \] (3.20)

This gives the usual result (3.9) as $M \to 0$. This also gives the correct result on $\mathbb{R}^3 \times S^1_L$ [7] for fields massive along the $S^1_L$ by using the expansion for $K_2(z) \approx 2/z^2 - \frac{1}{2}$ for $z \to 0$ at small mass. This also agrees with Section 6.2.1, equation (6.23) for the gaugino having mass $M = m$, and for general gauge group.

The case of arbitrary complex parameter $\tau$ is similar. We need only consider the Eisenstein like series $\sum_{m,n} \frac{y^m x^n}{|m\tau+n|^4}$ which is plotted for the case $\tau = 1+i$ in Figure 3.3. We see there is just the one minimum at $\phi = \pi$, $\psi = 0$. It is the other way around for the cases of $|\tau| < 1$ (our case of interest is $|\tau| > 1$ now!). This is important then as we see that the minima (vacua) are independent of the conformal parameter as would be required/expected by gauge invariance. The proof of this formula goes as follows.

**Proof:**

To prove the formula in this case consider a skew (flat) torus with parameter $\tau = \tau_1 + i\tau_2$. The Laplacian in the complex coordinates is $\Delta = -2\partial_i \partial_\bar{j}$ which I will rewrite in the non-orthogonal coordinates $\xi_i$ which have $y = Im$ $z = \xi_2/\tau_2$ and satisfy $z = x + iy = \xi_1 + \tau \xi_2$ which can then give us $\xi_1 = i(\bar{\tau}z - \tau \bar{z})/2\tau_2$. The Laplacian in these coordinates becomes

$$
\Delta = \frac{-1}{2\tau_2^2} [||\tau|^2 \partial_1^2 - 2\tau_1 \partial_1 \partial_2 + \partial_2^2].
$$

Requiring the periodic boundary condition eigenfunctions to be of the form $\psi_{m,n} = \exp[2\pi i (n\xi_1 + m\xi_2)]$ and applying the Laplace operator to these gives the eigenvalues $\lambda_{n,m} = 2\pi^2|m + \tau n|^2/\tau_2^2$. Sticking these eigenvalues into the expression of the zeta function heat kernel as before gives us the result stated above for the effective potential with $|m + \tau n|^4$ in the denominator. □

I will also briefly mention a result for 'flat' Riemann surfaces (that is surfaces of genus $g$ which are $4g$-gons with opposite sides identified). This will not be considered in the sequel but in interesting in string theory applications and can help generalize work done in this paper to higher genus Riemann surfaces. Let me use an elliptic function identity for general zeta functions for the general case (tori and Riemann surfaces). This allows a very nice way to write the perturbative effective potential. For higher-genus cases on an arbitrary Riemann surface $\Sigma_g$ of genus $g$ we simply consider winding numbers $(n_1, n_2, \ldots, n_g, m_1, m_2, \ldots, m_g)$ along the $g$ pairs of cycles $\alpha_i, \beta_j \in H_1(\Sigma)$ and where the $n_i, m_i \in \mathbb{Z}/2$ indicating periodic or anti-periodic boundary conditions. I will also add general phases $\theta_i$ around the $a_i$ cycles, and $\phi_i$ around the $b_i$ cycles. We can decompose any differential form like $dx_{nm} = \sum_i \left[(n + \theta_i)\alpha_i + (m + \phi_i)\beta_i\right]$ where $\alpha, \beta$ are the forms (cocycles) $\in H^1(\Sigma)$ corresponding to the cycles. This allows me to write the basic action as simply the area law and the corresponding partition function:

$$
S_{n,m}[x] = 2\pi \int_{\Sigma} (dx \wedge *dx)_{m,n}, \quad Z = \left(\frac{det' - \nabla^2}{\sqrt{g}}\right)^{-1/2} \sum_{\tilde{m}, \tilde{n}} e^{-S_{\tilde{m}, \tilde{n}}}. \quad (3.21)
$$
I can also write the period matrix of Σ as \( \Omega = B^{-1}C_{ij} + iB^{-1} \) in the basis with \( A_{ij} = \int \alpha_i \wedge \alpha_j \), \( B_{ij} = \int \beta_i \wedge \beta_j \), and \( C_{ij} = \int \alpha_i \wedge \beta_j \). Now keeping \( \vec{n} \) fixed let us do a Poisson resummation over \( \vec{m} \) to give

\[
Z = \frac{1}{2g}(\int \sqrt{\det \text{Im} \Omega})^{-1/2} \sum_{\vec{m}, \vec{n}} e^{A_{mn} \bar{A}_{mn}},
\]

where \( A_{mn} = i\pi (\vec{n} + \vec{\theta} + \vec{m}/2) \cdot \vec{\phi} \) and \( \bar{A}_{mn} \) is the just the complex conjugate but also with \( \vec{m} \to -\vec{m} \). I then get

\[
Z = \frac{1}{2g}(\int \sqrt{\det \text{Im} \Omega})^{-1/2} \sum_{\vec{e}_1, \vec{e}_2} |\vec{e}_1 + i\vec{e}_2| (0, \Omega)^2 .
\]

(3.22)

Let me end this discussion and return to the case of toroidal compactification in the low-energy low temperature limit. What is missing to the effective action (3.15) is the aforementioned non-perturbative sector of the theory which contributes to the effective potential along with the GPY (Gross-Pisarki-Yaffe) potential (3.17). This non-perturbative sector arises from (anti) self-dual (Euclidean) solutions of finite action to the equations of motion of the SYM action (3.1) in 3 + 1 dimensions: the monopole-instantons, and non self-dual topological excitations formed from molecules of them (the neutral and magnetic bions). This I turn to now.

### 3.2 Non-perturbative Theory of SU(2)

This section reviews the non-perturbative objects and dynamics of SU(2) gauge theories in four dimensions and on compactified spacetimes \( \mathbb{R}^3 \times S^1 \) and \( \mathbb{R}^2 \times S^1 \times S^1 \). The non-perturbative effective potential from such objects, both magnetic monopoles, and so-called bions composed of pairs of monopoles, is derived and its minima are studied. The next Chapter combines results from this section into a universal Coulomb gas of magnetic molecules and monopoles, as well as electric W-bosons and superpartners and will allow us to study the deconfinement phase transition through other methods and numerical simulations.

#### 3.2.1 Monopole solutions for SU(2)

For SU(2) I present the monopole-instanton solution as given in [2] as the (anti) self dual Euclidean solutions of finite action to the equations of motion of action (3.23) below. We take as generators \( T^a = \tau^a/2 \) where \( \tau^a \) are the Pauli matrices, and satisfy \( tr[T^aT^b] = \delta^{ab}/2 \). The action is

\[
S = \frac{1}{4g^2} \int_{\mathbb{R}^3 \times S^1} F^a_{MN} F^{aMN} .
\]

(3.23)

In the hedgehog gauge the monopole solution is given by [2]

\[
A_\mu = A^\mu_\nu \tau^\nu = \epsilon_{\mu
u\varepsilon} \frac{x_\nu}{|x|^2} (1 - \frac{v|x|}{\sinh(v|x|)}) \tau^\varepsilon ,
\]

(3.24)

\[
A_3 = \Psi^\varepsilon \tau^\varepsilon = \frac{x_\varepsilon}{|x|^2} (v|x| \coth(v|x|) + 1) \tau^\varepsilon ,
\]

where \( v = \langle A_3 \rangle \). Putting these solutions (3.24) into the action (3.23) gives \( S = 4\pi vL/g^2 \) so that \( v = 2\pi/L \) gives the usual monopole instanton action \( 8\pi^2/g^2 \).
The magnetic field’s asymptotics are found to be (in the string/singular gauge)

\[ B_\mu = \frac{1}{2} \varepsilon_{\mu \nu \lambda} F_{\nu \lambda} \to |x| \to \infty - \frac{x_\mu}{2|x|^3} r^3. \]  

(3.25)

Let me now turn to the non-perturbative structures such as neutral and magnetic bions in the theory and their dynamics on \( \mathbb{R}^3 \times S^1_L \) and \( \mathbb{R}^2 \times S^1_L \times S^1_\beta \).

### 3.2.2 Non-perturbative dynamics in \( SU(2) \) gauge group

Let me begin with the zero temperature case and present the non-perturbative solutions of our theory on \( \mathbb{R}^3 \times S^1_L \). The action (3.23) admits monopole-instanton solutions as well as neutral and magnetic bions solutions, which are, respectively, self-dual and non self-dual non-perturbative Euclidean solutions to the gauge field’s equations of motion with finite action. In the path integral of our theory, which will give us the partition function of the theory with its W-boson constituents as well as the monopoles and bions, we must include both the perturbative and non-perturbative solutions. The latter I present here. The simplest of such non-perturbative objects are the BPS monopole-instantons on \( \mathbb{R}^3 \times S^1_L \). They arise from the non-trivial second homotopy group of the residual gauge symmetry

\[ \pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}, \]

where the first equality follows from the long exact sequence of homotopy groups:

\[ 0 \to U(1) \to SU(2) \to SU(2)/U(1) \to 0 \iff \cdots \to \pi_2(SU(2)) = 0 \to \pi_2(SU(2)/U(1)) \to \pi_1(U(1)) \to 0, \]

and the vanishing homotopy of the higher rank gauge groups. This implies the existence of one charge 'type' of monopole, and its anti self-dual anti-monopole, called BPS and \( \bar{\text{BPS}} \). Due to the compact \( x^3 \) direction there exists another solution, called the KK monopole and its anti self-dual KK solution. These are 'twisted' solutions of finite action, for \( L > 0 \), which 'flip' on traversing the compact spatial direction. Both solutions are (anti) self-dual particle like objects localized in space and time with long range magnetic fields due to the unbroken \( U(1) \) symmetry. The field of a BPS (\( \bar{\text{BPS}} \)) monopole-instanton at the origin \( x_0 = x_1 = x_2 = 0 \) in the singular/stringy gauge, ignoring internal structure in the dilute monopole-instanton gas approximation, is found to be

\[ A^{BPS,BPS}_0 = \mp \frac{x_1}{r(r + x_2)}, \]

\[ A^{BPS,BPS}_1 = \pm \frac{x_0}{r(r + x_2)}, \]

\[ A^{BPS,BPS}_2 = 0, \]

\[ A^{BPS,BPS}_3 = \pi \frac{1}{L} - \frac{1}{r}, \]

where \( r = \sqrt{x_0^2 + x_1^2 + x_2^2} \). Note the Dirac 'string' singularity at \( r = -x_2 \). These components of the gauge field give the correct asymptotics of the magnetic field at infinity as in the previous section. The field for the KK (\( \bar{K}K \)) monopole similarly reads

\[ A^{KK,KK}_0 = \pm \frac{x_1}{r(r + x_2)}. \]  

(3.26)
Chapter 3. Deconfinement in $SU(2)$ Gauge Group

$$A_1^{KK,KK} = \frac{x_0}{r(r^2 + x^2)},$$

$$A_2^{KK,KK} = 0,$$

$$A_3^{KK,KK} = \frac{\pi}{L} + \frac{1}{r}.$$ These monopole-instantons carry magnetic charge $Q_m$ from Gauss’ law

$$\int_{S^2} d^2 \Sigma B_\mu = 4\pi Q_m, \quad (3.28)$$

where $B_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda = Q_m \frac{x_\mu}{r^2}$ is the magnetic field. There is also a long-range scalar field (from the $A_3^3$ component of the gauge field) which can attract or repel these monopoles due to scalar charge interaction. There exists also a topological charge of these monopole instantons $Q_T$ defined by

$$Q_T = (32\pi)^{-1} \int_{\mathbb{R}^3 \times S^1} F_{MN}^a F^{aMN}. \quad (3.29)$$

Using the solutions (3.26) and (3.27) I find the charges $(Q_m, Q_T)$ for each monopole type, which for $SU(2)$ are:

$$BPS \ (+1, 1/2) \ \bar{BPS} \ (-1, -1/2) \ KK \ (-1, 1/2) \ \bar{KK} \ (+1, -1/2). \quad (3.30)$$

Note that in general, as will be seen in Chapter 8 for general gauge group, the values of the topological charge depend on the vacuum of the theory.

Due to the presence of fermions and supersymmetry (our gaugino), the Callias index theorem [8], [10] on $\mathbb{R}^3 \times S^1$ implies the existence of two adjoint fermionic zero modes attached to each monopole-instanton. Section 8.2 describes this index theorem in detail for general gauge group. Let me use the field $\phi$ instead of $A_3^3$ and $\sigma$ instead of $A_3^\mu$ and attach fermionic zero modes to get the ‘t Hooft vertices

$$\mathcal{M}_{BPS} = e^{-4\pi^2/\lambda^2} e^{-\phi + i\sigma} \bar{\lambda}, \quad \mathcal{M}_{KK} = e^{-4\pi^2/\lambda^2} e^{\phi - i\sigma} \bar{\lambda}. \quad (3.31)$$

The anti monopole vertices are just the complex conjugates of these. Inserting these into the partition function of the theory inserts the contribution of fermionic zero modes and the long range fields (the $e^{-\phi + i\sigma}$ factors). These in themselves do not alter the vacuum structure of the theory as they are attached to fermionic zero modes and do not generate a potential for the fields $\phi$ and $\sigma$, and no mass will be generated for the dual photon $\sigma$. I consider then the effect of the neutral and magnetic bions in the non-perturbative potential and their role in the deconfinement phase transition.

Let me now consider the non self-dual bion 'molecules' these monopole constituents can form. Their charges and amplitudes of the so-called magnetic and neutral bions that form are summarized below in Table 3.1.

These amplitudes can be summed up to give the total non-perturbative effective potential of the theory $V_{\text{eff, non-pert.}}(\phi, \sigma)$ which is proportional to $\cosh 2\phi - \cos 2\sigma$, which gives the theory a resemblance to cosh-Gordon and cos-Gordon theories.

Figure 3.4 shows the magnetic bion’s structure as well as its size, whose magnetic repulsion of monopole pairs is stabilized by the logarithmic attraction of the fermionic zero modes hopping between
Chapter 3. Deconfinement in $SU(2)$ Gauge Group

Table 3.1: Magnetic molecule vertices, charges, and amplitudes for different molecules.

<table>
<thead>
<tr>
<th>Molecule</th>
<th>Vertex</th>
<th>$(Q_m, Q_T)$</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>neutral BPS bion</td>
<td>$M_{BPS}M_{BPS}$</td>
<td>$(0, 0)$</td>
<td>$e^{-8\pi^2/g^2}e^{-2\phi}$</td>
</tr>
<tr>
<td>neutral KK bion</td>
<td>$M_{KK}M_{KK}$</td>
<td>$(0, 0)$</td>
<td>$e^{-8\pi^2/g^2}e^{2\phi}$</td>
</tr>
<tr>
<td>magnetic bion</td>
<td>$M_{BPS}M_{KK}$</td>
<td>$(2, 0)$</td>
<td>$e^{-8\pi^2/g^2}e^{2i\sigma}$</td>
</tr>
<tr>
<td>magnetic bion</td>
<td>$M_{KK}M_{BPS}$</td>
<td>$(-2, 0)$</td>
<td>$e^{-8\pi^2/g^2}e^{-2i\sigma}$</td>
</tr>
</tbody>
</table>

The monopoles. This gives the bion a stable size $r_*=4\pi L/g^2$ which is much smaller than the inter-bion distance as indicated on the figure. See [22] for more on bion structure. The neutral bions are rather different in how they form stable structures, but due to analytic continuation and certain methods such as the BZJ prescription [7] or a finite-size argument a stable equilibrium size can be obtained as the same size $r_*$ of the magnetic bion. ‘Resurgence’ theory comes into play here as well [4]. The long-range fields of the magnetic bions at distances $\gg r_*$ are given by summing up the contributions from each monopole:

$$A_{0 \text{ mag. bion}} = -2 \frac{x_1}{r(x_1 + x_2)},$$

$$A_{1 \text{ mag. bion}} = 2 \frac{x_0}{r(x_1 + x_2)},$$

$$A_{2 \text{ mag. bion}} = 0,$$

$$A_{3 \text{ mag. bion}} = \frac{2\pi}{L}.$$ (3.32)

Now the magnetic bions do not carry fermionic zero modes and so can affect the vacuum of the theory, giving a mass for the dual photon $\sigma$ allowing for confinement of electric charges. The neutral bions, on the other hand, source a long-range scalar field and stabilizes centre-symmetry allowing the repulsion of Wilson loop eigenphases. At zero temperature the perturbative effective potential vanishes and so only these neutral bions can allow for the centre symmetry to be preserved as required by the confined phase. The total non-perturbative potential is found by summing up the amplitudes of both bions and monopole-instantons and we arrive at the full Lagrangian at zero temperature:

$$L_{\beta \to \infty} = \frac{1}{2} \frac{g^2(L)}{(4\pi)^2L}[(\partial_\mu \sigma)^2 + (\partial_\mu \phi)^2] + i \frac{L}{g^2} \lambda \sigma_\mu \partial_\mu \lambda + \frac{\alpha}{g^2} e^{-4\pi^2/g^2(L)} [(e^{-\phi+i\sigma} + e^{\phi-i\sigma})\lambda \lambda + h.c.]$$ (3.33)

$$+ \frac{64\pi^2 e^{-8\pi^2/g^2}}{g^6 L^3} (\cosh 2\phi - \cos 2\sigma),$$

where $\alpha$ is a numerical factor that will not concern me as I am only interested in the effective potential of neutral and magnetic bions in studying the deconfinement phase transition. At finite temperature one would add the perturbative effective potential $V_{\text{eff, pert.}}(\phi, \sigma)$ to the Lagrangian above, which as can be seen from Section 3.1 will add a term $\propto \cosh \phi \cos \sigma$. At finite temperature all that changes is that the fermions acquire a thermal mass $\approx T$. These lighter gauginos do not participate in the deconfinement phase transition as they carry no electric or magnetic charge, however the heavier winos do and so behave just as the W-bosons and must be taken into consideration.

Now that we are equipped with the full set of ingredients that make up a gas of electric, magnetic and scalar charged particles, molecules and superpartners we can now turn to the study of this gas,
called the dual (universal) Coulomb gas to our cosh-cos-Gordon like theory.
Figure 3.2: Perturbative effective potential for torus of aspect ratio $r = L/\beta = 3/2$. Note the global minimum is now at $(\phi, \psi) = (0, \pi)$ and is opposite for the case of $r < 1$. Note the $x, y$ axes are scaled to run from 0 to $2\pi$.

Figure 3.3: Effective potential for a torus of complex parameter $\tau = 1 + i$. The global minimum here is at $(\phi, \psi) = (\pi, 0)$ which is expected as here $r = L/\beta < 1$. Note again the $x, y = \phi, \psi$ axes are scaled to run from 0 to $2\pi$. 
Figure 3.4: A magnetic bion. Note the approximate size and separation between bions in terms of $L$, the size of the compact spatial dimension, and the coupling $g$. 
Chapter 4

Duality to Coulomb Gases and XY Spin Models of $SU(2)$ (Super) Yang-Mills

One way to study the thermal deconfinement phase transition of Yang-Mills with or without supersymmetry is to consider dual Coulomb gases as studied in [15] and in [1], which can be studied by renormalization group methods as was done for QCD(adj) in [15] for $SU(2)$ and $SU(3)$ with quite success, as well as $SU(N)$ but with less known about the fixed points of the renormalization group flow in the general case. Alternatively, as in [1], they can be subjected to lattice Monte-Carlo simulations. Here I present the work of the dual Coulomb gas partition function for the case of $SU(2)$ super Yang-Mills on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ and present the results of simulations showing the existence of a second order phase transition for gauge group $SU(2)$.

One further way to study the deconfinement phase transition is to map the dual Coulomb gas to a spin model which turns out to be an 'affine' XY spin model with symmetry-breaking perturbations. This model was subjected to lattice simulations in [1] and I present the model for $SU(2)$ SYM on the lattice and the numerical results in Section 4.2. In Chapter 7 I will present the derivation of the double dual Coulomb gas in the case of any gauge group, and comment on the lack of a clear spin model dual in the general case.

4.1 Dual Double Coulomb Gas to $SU(2)$ (Super) Yang-Mills

Let me begin by ignoring the fermions in (3.33). The remaining terms involving $\sigma$ fields represent the magnetic bions and those with $\phi$ fields represent the neutral bions. As depicted in Figure 3.5 of the last Chapter, the distance between these bions is $\Delta L_{\text{bion}} \approx L e^{4\pi^2/3g^2}$, which is much larger than the inverse temperature of the gas. This means our gas is effectively two dimensional and the scalar field $\sigma$ can be treated as a compact 2D scalar. As in 2D we can decompose the field into two parts: the spin waves and the vortices. It is these vortices that we will take to be the W-bosons of the theory, as will be made clear when I present the dual XY spin model in the next Section. The 2D Coulomb gas of our model
will be derived following methods of the derivation for the finite temperature 3D Polyakov model in [6], where I take the original $U(1)$ field $F^{3}_{\mu\nu}$, instead of $\sigma$, and add the perturbative photon fluctuations to a background magnetic field of arbitrarily-placed magnetic bions (and their anti molecules). In Section 3.1 I evaluated the perturbative W-boson (plus wino) determinant for constant backgrounds. At distances $\gg L$ away from the bion cores the magnetic fields are small and I can use varying backgrounds in the constant field potential $V_{\text{eff. pert.}}(A_{0}^{3}, A_{3})$ from equation (3.18). Evaluating the partition function of the finite-T theory, at large distances, will show that it represents a dual Coulomb gas made up of electric and magnetic particles (W bosons and winos as well as magnetic and neutral bions) with fugacities coupled to the scalar field $\phi$.

As mentioned above I begin by splitting the fields into photon fluctuations and magnetic bion fields:

$$F_{\mu\nu}^{3} = F_{\mu\nu}^{3, \text{bion}} + F_{\mu\nu}^{3, \text{ph}}$$

$$A_{\mu}^{3} = A_{\mu}^{3, \text{bion}} + A_{\mu}^{3, \text{ph}}$$

where $A_{\mu}^{3, \text{bion}} = \sum_{i,q_{i}=\pm 1} q_{i} A_{\mu}^{3, \text{bion}}(x - x_{i})$ splits into a sum of an arbitrary numbers of bions and anti bions at positions $x_{i} \in \mathbb{R}^{3}$ with charges $q_{i}$, and $A_{\mu}^{3, \text{bion}}$ is from (3.32). In the partition function I will sum over arbitrary numbers of such bions and integrate over their positions. At finite temperature I have $\beta$ finite and so I must sum up an infinite number of image charges in the $0$-direction and so

$$A_{\mu}^{3, \text{bion}} = \sum_{q_{i}=\pm 1} \sum_{n \in \mathbb{Z}} q_{i} A_{\mu}^{3, \text{bion}}(\vec{x} - \vec{x}_{i}, x_{0} - x_{0,i} + n\beta).$$

The partition function of our Coulomb gas is the path integral of our field theory with path integrals over the gauge field $A_{\mu}^{3, \text{ph}}$ and the scalar field $\phi$, and sums over arbitrary numbers $N_{b_{\pm}}$ of magnetic bions, as well as $N_{W}$ W-bosons and their superpartners. Figure 4.1 shows a picture of the Coulomb gas.

Considering first the photon field in the background of magnetic bions, I simply replace the argument of the cosine of the potential (3.17) with the integral

$$\oint_{S_{3}^{1}} dx_{0} A_{0}^{3, \text{bion}} = \sum_{i,q_{i}=\pm 1} q_{i} \int_{0}^{\beta} \sum_{n \in \mathbb{Z}} A_{\mu}^{3, \text{bion}}(\vec{x} - \vec{x}_{i}, x_{0} - x_{0,i} + n\beta) = \sum_{i,q_{i}=\pm 1} q_{i} \int_{-\infty}^{\infty} A_{0}^{3, \text{bion}}(\vec{x} - \vec{x}_{i}, x_{0})$$

to get

$$V_{\text{eff. pert.}}(A_{0}^{3}, \phi) = -\frac{4}{\pi \beta^{3}} \sum_{n \in \mathbb{Z}} e^{-\beta(2n+1)\pi/L + g^{2}\phi/4\pi L} (1 + \beta(2n + 1)\pi/L + g^{2}\phi/4\pi L)$$

$$\times \cos(\oint_{S_{3}^{1}} dx_{0} A_{0}^{3}(x_{0}, \vec{x}))$$

in the bosonic action including the neutral bion potential

$$S_{\text{bosonic}} = \int_{\mathbb{R}^{2} \times S_{3}^{1}} \left[ \frac{L}{4g^{2}} (F_{\mu\nu}^{3})^{2} + \frac{g^{2}}{2(4\pi)^{2}L} (\partial_{\mu} \phi)^{2} + \frac{64\pi^{2} e^{-8\pi^{2}/g^{2}}}{g^{6}L^{3}} \cosh 2\phi + V_{\text{eff. pert.}}(A_{0}^{3}, \phi) \right].$$

In this approximation, instead of a constant holonomy background, we have a spatially varying one in
Chapter 4. Duality to Coulomb Gases and XY Spin Models of SU(2) (Super) Yang-Mills

Figure 4.1: A Coulomb gas of W bosons and monopoles and bions. The lines represent the world lines of the W-bosons while the circles represent the monopole-instantons and bions, which are localized in space-time. Even though $L \ll \beta$ the distance between gas constituents is much greater than the inverse temperature (the size of the vertical axis) and so the gas is effectively two dimensional.

The nonperturbative background of magnetic bions. Using equation (3.31) I find as in [1] the integral (4.3) above to be

$$\oint_{S^1} d\theta_0 A^{3,\text{bion}}_0 = 4 \sum_{i,q} q_i \Theta(\vec{x} - \vec{x}_i),$$

where

$$\Theta(\vec{x}) = -\text{sgn}(x_1)\pi/2 + \tan^{-1}\frac{x_2}{x_1}$$

is the angle in the $x_1$-$x_2$-plane between $\vec{x}$ and the magnetic bion at position $\vec{x}_i$. From here on $A, B$ denote the positions of the W-bosons, and $i, j$ the positions of the magnetic bions.

The grand partition function is then obtained as a path integral with action (4.5) over the perturbative fluctuations $A^{3,\text{ph}}_\mu$ and $\phi$, summing over all possible magnetic bion backgrounds with $N_{b+}$ magnetic bions and $N_{b-}$ anti bions, at positions $x_i = (x_{0,i}, \vec{x}_i)$:

$$Z_{\text{grand}} = \sum_{N_{b+},N_{b-}} \frac{\xi_{N_{b+}+N_{b-}}^{N_{b+}+N_{b-}}}{N_{b+}!N_{b-}!} \prod_i \int d^3x_i \int DA^{3,\text{ph}}_\mu \int D\phi$$

$$\times \exp\left[-\int_{S^1} d\theta_0 \left(\frac{L}{4g^2} (F^{3,\text{ph}}_{\mu\nu} + F^{3,\text{bion}}_{\mu\nu})^2 + \frac{1}{2} \frac{g^2}{(4\pi)^2 L} (\partial_\mu \phi)^2 + \frac{1}{g^6 L^3} \cosh 2\phi \right)

- 2T\xi_W(\phi) \cos\left(4 \sum_{i,q} q_i \Theta(\vec{x} - \vec{x}_i) + \oint_{S^1} d\theta_0 A^{3,\text{ph}}_0 \right)\right].$$
where the bion fugacity is written as \( \xi_b = e^{-8\pi^2/g^2/\beta L^3} \). The \( \phi \)-dependent W-boson fugacity is found from (4.4) to be

\[
\xi_W(\phi) = \frac{2}{\pi \beta^2} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2}((2n+1)\pi + g^2 \phi/4\pi)(1 + \frac{\beta}{L}(2n + 1)\pi + g^2 \phi/4\pi)}
\]

\[
= \frac{2}{\beta L \sinh(\beta \pi/\beta L)}[(\coth \frac{\beta \pi}{L} + \frac{L}{\pi \beta}) \cosh \frac{g^2 \phi}{4\pi L} - \frac{g^2 \phi}{4\pi^2} \sinh \frac{g^2 \phi}{4\pi L}],
\]

and in the second line it is assumed that the field \( \phi \) lies in the principal Weyl chamber \(-\pi/L < g^2 \phi/4\pi L \leq \pi/L\). This can be interpreted as the W-boson fugacity as, near \( \phi = 0 \) the dominant contribution to \( \xi_W(\phi) \) comes from the \( n = 0, -1 \) terms. With \( \phi = 0 \) I have \( \xi_W(0) = \xi_W \approx \frac{4}{\beta L} e^{-\beta \pi/\beta L} = \frac{4 M_W T}{\pi} e^{-M_W/T} \), where we recall \( \beta/L \approx M_W/T \gg 1 \). This is exactly 4 times that for a single W-boson, thermally produced using the Maxwell-Boltzmann distribution. This is because the zero and first excited Kaluza-Klein modes have the same mass as seen from (4.9), and the wino superpartners have the same mass as well.

One comment to make is that at the boundary of the Weyl chamber \(|g^2 \phi/4\pi L| = \pi/L\) a W boson becomes massless and the full \( SU(2) \) gauge symmetry is restored, and the Abelian description of the theory no longer applies. In our regime of semi-classics \( LT \ll 1 \) it is seen from the formula for \( \xi_W(\phi) \) that the W-boson fugacity increases as the Weyl chamber boundary is approached. However, there is nothing to worry about as the \( \cosh 2\phi \) term from the neutral bions strongly disfavour larger values of \( \phi \), and at low enough temperatures the neutral bion potential dominates over the perturbative contribution.

Now I can consider the electrically charged particles of the partition function (4.8) with fugacity (4.9) coupled to the magnetic bions. I can expand the cosine in (4.8) using the formula

\[
\exp[2 \xi \int dx \cos(f(x))] = \sum_{n_+ = 0}^{\infty} \sum_{n_- = 1}^{\infty} \xi_{n_+ + n_-}^{n_+ + n_-} (n_+)! (n_-)! \prod_{i=1}^{n_+ + n_-} \int dx_i e^{\sum_i i q_i f(x_i)}.
\]

The partition function can then be written as (summing over \( N_{W+} \) W-bosons of positive charge and \( N_{W-} \) negatively charged ones at positions \( x_A = (x_{0,A}, \vec{x}_A) \))

\[
Z_{\text{grand}} = \sum_{N_{b+},q_i = \pm} \xi_b^{N_{b+} N_{b-}} \prod_{i=1}^{N_{W+} + N_{W-}} \int d^3 x_i \sum_{N_{W+}, q_A = \pm} \frac{(T \xi_W(\phi))^{N_{W+} + N_{W-}}}{N_{W+}! N_{W-}!} \prod_A \int d^3 x_A \int D A_{\mu \nu}^{3,ph} \int D \phi
\]

\[
\times \exp\left[ - \int_{\mathbb{R}^2 \times S^3} \frac{L}{4g^2} (F_{\mu \nu}^{3,ph} + F_{\mu \nu}^{3,ph})^2 + \frac{1}{2 \langle 4\pi \rangle^2 L} (\partial_{\mu} \phi)^2 + \frac{64\pi^2 e^{-8\pi^2/\beta^2}}{g^6 L^3} \cosh 2\phi \right]
\]

\[-i \sum_A q_A A_{\mu \nu}^{3,ph}(\vec{x}, x_0) \delta^2(\vec{x} - \vec{x}_A) + 4i \sum_{i=1}^{N_{W+} + N_{W-}} q_i q_A \Theta(\vec{x}_A - \vec{x}_i),
\]

where the Aharanov-Bohm interaction term \( \Theta(\vec{x}_A - \vec{x}_i) \) was included.

Doing the path integral over the field \( A_{\mu \nu}^{3,ph} \) can be carried out using an Abelian duality transformation as in [6]. The equations of motion for the dual field can then be solved and put into the partition function. The final grand canonical partition function for our dual Coulomb gas, coupled to the scalar field \( \phi \) can
then be written as (including only the thermal zero mode of size \(\beta\) so \(\int d^3x = \beta \int d^2x\))

\[
Z_{\text{grand}} = \sum_{N_{b+},N_{b-}} \frac{\beta \xi_b}{N_{b+}!N_{b-}!} \prod_i \int d^2x_i \sum_{N_{W+},N_{W-}} \frac{(\xi_W(\phi))^{N_{W+}+N_{W-}}}{N_{W+}!N_{W-}!} \prod_A \int d^2x_A \int \mathcal{D}\phi
\]

\[\times \exp\left[\frac{32\pi LT}{g^2} \sum_{i>j} \log |\vec{x}_i - \vec{x}_j| + \frac{g^2}{2\pi LT} \sum_{A>B} \log |\vec{x}_A - \vec{x}_B| + 4iA \sum_{i,A} q_i q_A \Theta(\vec{x}_A - \vec{x}_i) + \frac{1}{g^2} \left(\frac{L}{4\pi^2 x^2} + \frac{64\pi^2 e^{-8\pi^2/g^2} \cosh 2\phi}{g^6 L^3}\right)\right].\]

I illustrate here the series of dualities used to arrive at the partition function (4.12). I start by giving one alternative way first to writing the partition function (4.8) ignoring the \(\phi\) field for now:

\[
Z_{\text{grand}} = \sum_{N_{b+},N_{b-}} \frac{\beta \xi_b}{N_{b+}!N_{b-}!} \prod_i \int d^3x_i \sum_{N_{W+},N_{W-}} \frac{(T\xi_W(\phi))^{N_{W+}+N_{W-}}}{N_{W+}!N_{W-}!} \prod_A \int d^3x_A \int \mathcal{D}\phi^3 \]

\[\times \exp\left[-\int_{\mathbb{R}^2 \times S^3} \frac{L}{4g^2} (F_{\mu\nu}^{3,ph} + F_{\mu\nu}^{3,bion})^2 + i\sum_{A} q_A A_0^{3,ph}(\vec{x},x_0)\right].\]

I introduce a scalar field \(\sigma\) as a Lagrange multiplier field by adding to \(S = \int d^3x F_{\mu\nu}^{2}/4g_3^2\) a term \(\delta S = \frac{1}{2} \int d^3x \sigma \epsilon_{\mu\nu\lambda} \partial_\mu F_{\nu\lambda}\) to enforce the Bianchi identity as the equation of motion for \(\sigma, \epsilon_{\mu\nu\lambda} \partial_\mu F_{\nu\lambda} = 0\) (except at the positions of magnetic charges). Varying \(S + \delta S\) with respect to \(F_{\mu\nu}\) gives \(F_{\mu\nu} = -g_3^2 \epsilon_{\mu\nu\lambda} \partial_\lambda \sigma\). Putting this back into the action gives \(S + \delta S = \int d^3x \frac{g_3^2}{2} (\partial_\mu \sigma)^2\) as an alternative way of writing the action in terms of a new scalar field along \(\phi\).

\[
\int \mathcal{D}\alpha \int \mathcal{D}\phi \int \mathcal{D}\sigma \exp[-\int d^3x (\frac{1}{2} g_3^2 (\partial_\mu \sigma)^2 + \frac{1}{2} \epsilon_{\mu\nu\lambda} \partial_\mu A_\nu \partial_\lambda \phi)]
\]

\[\equiv \int \mathcal{D}\alpha \int \mathcal{D}\phi \int \mathcal{D}\sigma \exp[-S_{U(1)^2}]\]

The \(U(1)^2\) symmetry acts on the fields by: \(B_{\mu} \rightarrow B_{\mu} + \partial_\mu \lambda\), and \(\sigma \rightarrow \sigma - \lambda\) for \(\lambda \in \mathbb{R}\), and the second \(U(1)\) acting as \(A_\mu \rightarrow A_\mu + \partial_\mu \sigma\) which leave the actions invariant. With \(\partial = 0\) (in unitary gauge) we can vary the action \(S_{U(1)^2}\) with respect to \(\Phi_{\mu}\) to get \(\Phi_{\mu} = -\frac{i}{g_3} \epsilon_{\mu\nu\lambda} \partial_\sigma A_\nu / g_3^2\). Placing this back into the action \(S_{U(1)^2}\) leads to \(S_{U(1)^2} = \int d^3x F_{\mu\nu}^{2}/4g_3^2\) proving the equality (4.14).

Using the new action \(S_{U(1)^2}\) allows me to work with \(S_{\text{aux}}\) given below where I substituted for \(F_{\mu\nu}^{2}/4g_3^2\):

\[
S_{\text{aux}} = \int d^3x \frac{1}{2g_3^2} (F_{\mu\nu}^{3,bion})^2 + \frac{1}{2g_3^2} F_{\mu\nu}^{3,ph} F_{\mu\nu}^{3,ph} + \frac{1}{2} g_3^2 \Phi^2_{\mu} + i\epsilon_{\mu\nu\lambda} \partial_\nu A_{\lambda} \Phi_{\mu} - i \sum_A q_A A_{\mu}^{3,ph}(\vec{x},x_0)\delta^2(\vec{x} - \vec{x}_A).\]

Varying this with respect to \(A_\mu^{3,ph}\) gives \(i\epsilon_{\mu\nu\lambda} \partial_\lambda \Phi_{\mu} + \partial_\mu F_{\mu\nu}^{3,bion} / g_3^2 = -i \sum_A q_A \Phi_{\mu}^2(\vec{x} - \vec{x}_A)\delta_{0\nu}\) which has solution \(\Phi_{\mu} = iB_{\mu} / g_3^2 + b_\mu\), where \(B_{\mu} = \epsilon_{\mu\nu\lambda} F_{\nu\lambda}^{3,bion} / 2\) is the magnetic field due to the background of magnetic bions and \(b_\mu\) splits into its divergence and curl free parts: \(b_\mu = \partial_\mu \sigma + \epsilon_{\mu\nu\lambda} \partial_\nu C_\lambda\) with
\( \partial_a C_\nu = 0 \). Putting \( b_\mu \) back in \( S_{\text{aux}} \) shows that \( \sigma \) drops out and the equation of motion for \( C_\mu \) is

\( \nabla^2 C_\mu = -\sum_A q_A \delta^2(\vec{x} - \vec{x}_A) \delta \rho_\mu \).

Introducing a Green’s function \( G(x) \) on \( \mathbb{R}^2 \times S_\beta^3 \) satisfying \( \nabla^2 G(\vec{x} - \vec{x}', x_0 - x_0') = -\delta^2(\vec{x} - \vec{x}') \delta(x_0 - x_0') \) with solution

\[
G(\vec{x} - \vec{x}', x_0 - x_0') = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{(\vec{x} - \vec{x}')^2 + (x_0 - x_0' + n\beta)^2}}
\]

gives the solution for \( C_\nu \),

\[
C_\mu = \int d^3 x' \sum_A q_A \delta_\rho_\mu \delta^2(\vec{x} - \vec{x}_A) G(\vec{x} - \vec{x}', x_0 - x_0') = -\frac{1}{4\pi} \sum_A q_A \log |\vec{x} - \vec{x}_A|.
\]

The solution for \( \Phi_\mu = iB_\mu/g_3^2 + b_\mu + K_\mu \) can also be found where a term \( K_\mu = -\frac{e_{\text{aux}}}{2\pi} \sum_A q_A \partial_\mu \log |\vec{x} - \vec{x}_A| \).

Substituting this into \( S_{\text{aux}} \) and integrating by parts yields

\[
S_{\text{aux}} = \int d^3 x \frac{1}{2} g_3^2 \left[ (\partial_\mu \sigma)^2 + K_\mu^2 - i\sigma \partial_\mu B_\mu + iB_\mu K_\mu - g_3^2 \sigma \partial_\mu K_\mu \right],
\]

where the last term is zero from the asymmetry of the \( \epsilon_{\mu\nu\beta} \) and \( \partial_\mu B_\mu = 4\pi \sum_A q_a \delta^3(x - x_a) \) from Gauss’ law. I can then write \( B_\mu = \sum_a q_a \frac{(x - x_a)}{|x - x_a|^2} \langle \rho \rangle \) where \( \langle \rho \rangle \) denotes the periodicity enforced along the \( S_\beta^3 \).

The next term \( iB_\mu K_\mu \) in \( S_{\text{aux}} \) can be evaluated as well and is seen to be zero:

\[
\int d^3 x B_\mu K_\mu = -\int d^3 x \sum_{aA} q_a q_A \int_0^\beta \sum_{n \in \mathbb{Z}} \frac{\epsilon_{ij}(x - x_a)_i(x - x_A)_j}{|\vec{x} - \vec{x}_A|^2 (|\vec{x} - \vec{x}_A|)^2 + (x_0 - x_0a + n\beta)^2} = 0
\]

by symmetric integration under asymmetric \( \epsilon_{ij} \).

What remains is the \( K^2 \) term in \( S_{\text{aux}} \), and we will see this gives rise to the Coulomb interactions of the W-bosons,

\[
\int d^3 x K_\mu^2 = \frac{1}{(2\pi)^2 T} \sum_{AB} q_A q_B \int d^3 x \partial_i \log |\vec{x} - \vec{x}_A| \partial_i \log |\vec{x} - \vec{x}_B| = \frac{1}{(2\pi)^2 T} \sum_{AB} q_A q_B \log (T|\vec{x}_A - \vec{x}_B|).
\]

Further, varying \( S_{\text{aux}} \) with respect to \( \sigma \) gives \( g_3^2 \nabla^2 \sigma = -4\pi i \sum_A q_a \delta^3(x - x_a) \), which has solution

\( \sigma = \frac{4\pi i}{g_3^2} \sum_A q_a G(\vec{x} - \vec{x}_a, x_0 - x_0a) \). Putting this back into \( S_{\text{aux}} \) gives \( S_{\text{aux}} = \frac{8\pi}{g_3^2} \sum_{ab} q_a q_b G(\vec{x}_b - \vec{x}_a, x_0b - x_0a) - \frac{g_3^2}{4\pi T} \sum_{AB} q_A q_B \log (T|\vec{x}_A - \vec{x}_B|) \). This gives us our final partition function for the double Coulomb gas as given in (5.111) once the \( \phi \) terms are restored (reminding ourselves that the W-boson fugacity depends on \( \phi \) once we turn it on). Putting everything together I get the final result

\[
Z_{\text{grand}} = \prod_{N_{\text{b}} \neq 0} \xi_{N_{\text{b}}+1/N_{\text{b}}-1} N_{\text{b}}+N_{\text{b}}-1 \prod_i \int d^3 x_i \sum_{N_{\pm}=\pm} \sum_{q_A=\pm} \frac{(T\xi_{W}(\phi))^{N_{\text{b}}+N_{\text{b}}-1} N_{\text{b}}+N_{\text{b}}-1}{N_{\text{b}}+N_{\text{b}}-1} \prod_A q_A \Theta(\vec{x}_A - \vec{x}_i) + 4i \sum q_i q_A \Theta(\vec{x}_A - \vec{x}_i) + \]

\[
\int d^3 x_A \int D\phi \exp \left[ \frac{32\pi L T}{g_2^2} \sum_{i>j} q_i q_j \log |\vec{x}_i - \vec{x}_j| + \frac{g_2^2}{2\pi L T} \sum_{A>B} q_A q_B \log |\vec{x}_A - \vec{x}_B| + 4i \sum_{i,A} q_i q_A \Theta(\vec{x}_A - \vec{x}_i) + \right]
\]
which is valid for all $T$ with $0 \leq T < M_W$, and I used $g_3 = g/L$ and the long-distance property of the Green’s function where it behaves like a logarithm. One could go one step further and expand the $\cosh 2\phi$ term using a similar version of equation (4.10) and solve the equation of motion for $\phi$ and find that the neutral bions (at positions $i,j$) and W-bosons interact also with a scalar charge where like charges attract! (the logarithm has opposite sign). This makes our general Coulomb gas (GCG) very interesting with different competing interactions which can be responsible for the deconfinement phase transition and determining its temperature.

Let me do this integration over $\phi$ and with neutral bions in the background as well to see how the scalar-scalar interactions work out. Starting with the effective action:

$$\Gamma_{eff} = -\int_0^\beta dx_0 \int d^2x \left[ \frac{1}{4g^2} (F_{\mu\nu}^A + F_{\mu\nu}^b)^2 + \frac{1}{2g^2} ((\partial_\mu \phi)^2 + 2\partial_\mu \phi \partial^\mu \phi) + \frac{T^2m_W}{\pi} \cos(\int_0^\beta (A_0^b + A_0)) \cosh(\phi/T = \varphi + \int_0^\beta dx_0 \phi')) \right],$$

(4.19)

let me derive the Coulomb gas with all of its interactions: electric, magnetic, and scalar.

Recall that for a certain singular gauge ansatz I can integrate $\int_0^\beta dx_0 A_0^b$ for periodic bions in the thermal direction starting at $r_n$ to give a term that equals (remember the bions are charge $\pm 2$) $4 \sum \phi_n \Theta(x - r_n)$. Similarly the effect of periodic neutral bions and scalars the term $\int_0^\beta dx_0 \phi_0'$ written as

$$\sum_q \sum_{-\infty}^\infty \int_0^\beta 2q_\alpha/ \sqrt{r^2 + (z + n\beta)^2} = -4 \sum \phi_\alpha \ln |r - r_\alpha|$$

(4.20)

gives a logarithm but with opposite sign! This gives a term $-4 \sum_{A,\alpha} q_\alpha q_A \ln |r_A - r_\alpha|$ to the action once this solution is substituted into (4.19) and integrated. (Similarly one could work out the propagator for the sinh-Gordon model and find that it is the same as the sin-Gordon model but with opposite sign. This means that the action is minimized by having like-charged particles closer together (like scalar charges attract!) while the electric/magnetic charges are attracted by opposites (a sort of pan-sexual gas)). These interaction terms add source terms to the Lagrangian and give rise to our Coulomb gas upon writing the partition function ($a$ denotes the index of the position of the magnetic bions, $\alpha$ the positions of the neutral centre-stabilizing bions, and $A$ the positions of the W-bosons):

$$Z = \frac{Z_0}{L^2 \beta} \sum_{N_b} \sum_{N_W} \sum_{N_{\phi}} \sum_{q_0} \frac{\xi^{N_b+N_W+N_{\phi}} (2\xi_{N_b})^{N_b} (2\xi_{N_W})^{N_W-N_b} (2\xi_{N_{\phi}})^{N_{\phi}-N_b-N_W} \xi_{N_{\phi}^+}^{N_{\phi}^+}}{N_b^+!N_b^-!N_W^+!N_W^-!N_{\phi}^+!N_{\phi}^-!} \times$$

$$\times \prod q^{N_b^+} q^{N_W^+} \prod_A \prod_{\alpha} \int d^{(2+1)}r_\alpha \int d^{(2+1)}r_A \int d^{(2+1)}r_\alpha \int DA_\mu \int D\varphi \times$$

$$\times \exp \left[ \int d^{(2+1)}x \left( \frac{1}{4g^2} (F_{\mu\nu}^A + F_{\mu\nu}^b)^2 + \frac{1}{2g^2} ((\partial_\mu \phi)^2 + 2\partial_\mu \phi \partial^\mu \phi') + i \sum_A q_A A_0(x) \delta(\vec{x} - \vec{r}_A) + \right) \right].$$

(4.21)
Putting in everything together I arrive at the partition function (note the overall minus sign again in $\phi$ cancel as well). This is seen by looking at the kinetic term piecewise as (after integrating by parts)

$$
\sum_A q_A \varphi(x) \delta(x - r_A)) - 4 \sum_{\alpha,A} q_A q_{\alpha} \ln |r_{\alpha} - r_A| + 4i \sum_{a,A} q_a q_A \Theta(\vec{r}_a - \vec{r}_A)].
$$

Note that actually the charges $q$ are triplet vectors $\vec{q}_X = (q_X^e, q_X^o, q_X^s)$ so that $q_W = (1, 0, 1)$ etc. I will also write in the sequel sometimes the shorthand $|r_{\alpha}| - |r_a - r_B|$, etc.

The next step is to vary the action with respect to $A_\mu$ and $\varphi$, solve the equations of motion, stick them back into the action, do the integral and voila we have the Coulomb gas! The solution for the vector potential involves some dualities and anzatze and was done in [76]. The result is just the Coulomb-Coulomb interactions between electric and magnetic charges. Now I can start with some calculations using

$$
\Box \ln |r| = -2\pi \delta^2(r) \quad (4.22)
$$

$$
\Box(\varphi_A' = \sum_\alpha q_\alpha \ln |\vec{x} - \vec{r}_\alpha| ) = +8\pi \sum_\alpha q_\alpha \delta(\vec{x} - \vec{r}_\alpha).
$$

The equation of motion for $\varphi$ gives way to the usual cosh-Gordon propagator with the minus sign:

$$
\frac{1}{g^2}(\Box \varphi + \Box \varphi') = \sum_A q_A \delta(x - r_A) \rightarrow \varphi = -\varphi' - \sum_\alpha q_\alpha \delta(\vec{x} - \vec{r}_\alpha) / 2\pi,
$$

and we can then find for the fluctuations $\varphi$ ('W' bosons)

$$
\Box \varphi = g^2 \sum_A q_A \delta(\vec{x} - \vec{r}_A) - 8\pi \sum_\alpha q_\alpha \delta(\vec{x} - \vec{r}_\alpha) \rightarrow
$$

$$
\varphi = 4 \sum_\alpha q_\alpha \ln |\vec{x} - \vec{r}_\alpha| - g^2 \sum_A q_A^2 \ln |\vec{x} - \vec{r}_A|\quad (4.23)
$$

Putting this into the action along with $\varphi'$ above (the kinetic term gives $+g^2/4\pi$ times the log, but the source term gives $-g^2/2\pi$ leading to an overall $-g^2/4\pi$ coefficient as expected, and other terms partially cancel as well). This is seen by looking at the kinetic term piecewise as (after integrating by parts)

$$
\Gamma^{kin}_{eff} = \int d^{(2+1)}x \frac{-1}{2g^2} [\varphi \Box \varphi + 2\varphi' \Box \varphi].
$$

Integrating and adding the other source terms finally gives the kinetic term of the effective action of the $\phi$ field to be

$$
\Gamma^{kin}_{\phi} = \frac{1}{2g^2} \sum_{A \neq B} q_A^2 q_B \ln |\vec{r}_A - \vec{r}_B| - 16g^2 \sum_{\alpha,A} q_\alpha^2 q_A \ln |\vec{r}_\alpha - \vec{r}_A| - 32\pi \sum_{\alpha \neq \beta} \ln |\vec{r}_\alpha - \vec{r}_\beta|\quad (4.24)
$$

Putting in everything together I arrive at the partition function (note the overall minus sign again in $Z = Z_0 \exp(-\beta H = \int d^2x \int d\phi H)$)

$$
Z = \frac{Z_0}{L^2 \beta} \sum_{N_b} \sum_{N_W} \sum_{N_b'} \sum_{q_X} (-\xi_{N_b}^{N_b_+} - \xi_{N_b}^{N_b_-}) \xi_{N_W}^{N_W_+} N_b^{N_b_+} N_W^{N_W_+} N_b'^{N_b'_+} N_W'^{N_W'_+} \times \quad (4.26)
$$
**Table 4.1:** Scalar, electric and magnetic charges of relevant Coulomb gas constituents.

<table>
<thead>
<tr>
<th>Coulomb gas constituent</th>
<th>$q_X = (q_{X,e}, q_{X,m}, q_{X,s})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>magnetic bions</td>
<td>$(0, \pm 2, 0)$</td>
</tr>
<tr>
<td>W-bosons</td>
<td>$(\pm 1, 0, \pm 1)$</td>
</tr>
<tr>
<td>neutral bions</td>
<td>$(0, 0, \pm 2)$</td>
</tr>
</tbody>
</table>

This result is reasonable, showing how scalar charges tend to attract like charges, whereas electric and magnetic charges tend to attract opposites: the action is minimized for like scalar charges being closer together and oppositely charged electric and magnetic charges closer together.

Let me make some comments on this partition function of electric and magnetic charges interacting together and coupled to a scalar field $\phi$, before going on to lattice simulation results. From the interaction terms in the partition function (4.18) we see that like scalar charges attract, whereas like electric and magnetic charges repel. This is due to the different sign in their interaction. The W-bosons have a double nature attracting opposite electrically charged W-bosons, and attracting like scalar charged W-bosons and neutral bions. These W-bosons can be written in the form

$$W^{\pm} = \phi \pm i\sigma,$$

giving them both scalar charges (under the scalar modulus $\phi$) and electric charges (from the photon $\sigma$). Let me summarize the components of the Coulomb gas and their charges, written as vectors $q_X = (q_{X,e}, q_{X,m}, q_{X,s})$. For $SU(2)$ there is only one root and co-root and so the charges are simply integers.

This Coulomb gas can be subjected to lattice study as in [1] for the case of $SU(2)$. See [1], [53], [54] for more on the Monte-Carlo simulations used in studying such Coulomb gases numerically. Another method of studying the deconfinement phase transition other than simulating the Coulomb gas is to map the Coulomb gas constituents to parameters of a dual spin model. The spin model that best suits the Coulomb gas at hand is an XY spin model with symmetry breaking perturbations and fugacities coupled to the scalar field $\phi$. This I turn to in the next Section. The next subsection recalls key results and methods of simulation done in [1] before turning to the results of the spin model simulations.

### 4.1.1 Results of lattice Monte Carlo simulations of dual Coulomb gas

The presence of the scalar field $\phi$ in the dual Coulomb gas partition function makes direct analytical study difficult compared to what was done previously in [5] and [15] for Yang-Mills without supersymmetry and for QCD(adj). There, renormalization group equations and other direct approaches to studying critical points and deconfinement were successful, but in the case of SYM this is not so easy.
Hence numerical approaches were done in [1] which I summarize now. First, a Monte Carlo study of the Coulomb gas partition function $Z_{\text{grand}}$ itself was done. It is an advantage that parameter values can be taken in the UV completion of the Coulomb gas (the 4D SYM theory) with the only drawback being the sign problem due to the Aharanov-Bohm phases. Secondly, in the next Section, a study of the dual 'affine' XY spin model with a symmetry-breaking perturbation (with coefficient dependent on the scalar $\phi$) will be described which does not have this sign problem. The only disadvantage of this approach is the magnetic bion fugacity is not a free parameter [1], but as seen in [5], [15] there is no real qualitative difference of the properties of the phase transitions determined from the dual Coulomb gas Monte Carlo approach. These two methods of studying the SYM $SU(2)$ theory will be found to agree qualitatively. One note is that in the simulations the values of the parameters in the UV completion of the theory are not in the regime of semi-classic calculability as in analytic approaches (small $L$, weak coupling $g$ and low temperature $L \ll \beta$), as the exponentially small fugacities make the simulation slow and impractical due to too few non-trivial excitations generated. However there is still agreement with analytics for fugacities of order $1/e$ being 'small' enough to agree even quantitatively with exponentially small 'analytic' studies [1]. However still a weak-coupling simulation should be done to be sure of the results of the phase transitions studied.

To study the Coulomb gas we need a discrete version of the action in the partition function of the Coulomb gas, with $L$ taken to be the lattice spacing of the simulation and will be set to unity. The discrete action reads

$$S = \sum_x \sum_\mu \left[ \frac{g^2}{32\pi^2 T} (\nabla_\mu \phi_x)^2 + \frac{64\pi^2 e^{-8\pi^2/g^2}}{T g^6} \cosh 2\phi_x \right] - \sum_A \log \xi_W(\phi(x_A)) - \frac{g^2}{2\pi T} \sum_{A < B} q_A q_B G(A, B) + \sum_a \frac{8\pi^2}{g^2} - \frac{32\pi T}{g^2} \log q_0 G(a, b) - 4i \sum_{a, A} q_a q_A \Theta(a, A).$$

(4.27)

The first two terms are for the action of the $\phi$ field. The next two terms are for the $W$ bosons and the two terms after are for the magnetic bions. The first term of each represents the fugacities as core energies, with the core energy of the $W$-boson in terms of the $\phi$ field at the position $A$ of the $W$-boson: $E(A) = -\log \xi_W(\phi(x_A))$. The electric $W$-bosons and the $\phi$ field live on the lattice at positions $A, B, \ldots$ and the magnetic bions live on the dual lattice at position $a, b, \ldots$. The functions $G$ and $\Theta$ are discrete versions of the continuum Green’s function and angle between the magnetic bion and $W$-boson, respectively. A Metropolis algorithm is used to simulate the grand partition function of action (4.18), as opposed to (4.26) where the $\phi$ field interactions lead to instabilities [1]. The full details of the algorithm will not be given here but are discussed in Appendix C of [1]. As mentioned above, in the semi-classical regime of weak coupling the algorithm will not generate magnetic bions and $W$ bosons in the regime of $LT \ll 1$, and so we use small fugacities (of $1/e$ for the magnetic bions, and core energies equal to unity) rather than exponentially small ones. The $W$-boson fugacities near $\phi = 0$ and with $T \approx g'/8\pi$ are then also around $\approx 1/e$. One other correction is to remove the $1/g^6$ factor of the neutral bion potential and at $g^2 = 8\pi^2$ the $W$-boson fugacity is also modified as there the fugacity becomes negative [1]. This would dramatically shift the transition point as negative $W$-boson core energies favour the liberation of $W$-bosons at lower temperatures. I then use a modified core energy $E_W = -\log \left[ \frac{\text{cosh}(g^2 \phi/4\pi T)}{2\sinh(\pi \beta)} \right]$, which amounts to removing pre-exponential factors of the $W$-boson fugacity (4.9). The idea is that these
Chapter 4. Duality to Coulomb Gases and XY Spin Models of SU(2) (Super) Yang-Mills

Figure 4.2: Left: Magnetic and electric charge densities as a function of temperature. Right: Aharanov-Bohm phase contribution to the partition function of the Coulomb gas.

Figure 4.3: Histograms of distributions for $\phi$ at $N = 16$, for $T = 0.4, 2, 5$ from left to right.

modifications, which are mainly to ease the simulations, will lead to qualitatively similar results to the weak coupling theory at small $L$.

In the simulations two volumes were used with lattice widths $N = 16, 32$. The temperatures are gradually increased with 10000 sweeps at each temperature (each with $N^2$ Metropolis iterations, and the first 500 removed for equilibration). At each step the action is calculated along with densities of magnetic bions and W-bosons and values of $\phi$ and $|\phi|$. Initially we begin with no W-bosons or magnetic bions and $\phi$ uniformly distributed in the range $[-4\pi^2/g^2, 4\pi^2/g^2]$. Next, neutral pair creation, annihilation or diffusion of magnetic bion or W-boson pairs is done with equal probability. Then the $\phi$ field at a random lattice site is changed to a random value in the range $[-4\pi^2/g^2, 4\pi^2/g^2]$ with probability $p = \min(1, e^{-\Delta S})$ so as to have Boltzmann statistics. See Appendix C of [1] for more details.

Figure 4.2 shows the electric and magnetic bion charge densities (on the left) at different temperatures and shows the characteristics of a deconfinement transition - electric charge proliferation at high temperature and magnetic bion charge abundance at lower temperatures. The non-zero electric charge density for $T > 3$ shows the breaking of the centre-symmetry in the deconfined phase. For most of the range outside the ‘critical’ temperature $T \approx 3$ the two gases are effectively decoupled from one another due to the lower density of one or the other gas component. Only near the transition point do they have significant interaction. On the left side of Figure 4.2 it is shown the average of the real and imaginary parts of the Aharanov-Bohm factor $e^{i\sum aA q_a q_A \Theta(a,A)}$. The bottom curve shows the re-weighting factor is real, as expected, and the upper curves show that as $N$ increases the Aharanov-Bohm interaction becomes more important near the transition.
The transition is seen to occur precisely at $T \approx \pi = g^2/8\pi$ for our value of $g^2$ chosen, where the electric and magnetic charge densities are equal. Studies of histograms reveal no doubly-peaked distributions of the action which would indicate a first-order phase transition and so a second order phase transition is expected for the $SU(2)$ Coulomb gas. This agrees with direct simulation of Yang-Mills on the lattice and analytically \[5,15\]. The fluctuations of $\phi$ do not seem to affect the transition much as well. To study this the averages of $\phi$, $|\phi|$ and its susceptibility were found:

$$\bar{\phi} = \frac{1}{N^2} \langle \sum_x \phi(x) \rangle \approx 0, \quad |\bar{\phi}| = \frac{1}{N^2} \langle |\sum_x \phi(x)| | \rangle$$

$$\chi(\phi) = \frac{1}{N^2} \langle (\sum_x \phi(x))^2 \rangle - \frac{1}{N^2} \langle (\sum_x \phi(x))^2 \rangle^2 \approx \sum_x \langle \phi(x)\phi(0) \rangle.$$ \[4.28\]

The average of $\phi$ is zero effectively at all temperatures. The second $\approx$ in the susceptibility shows that it is effectively the zero-momentum Green’s function of the field $\phi$ (the inverse mass squared in the continuum approximation). Figure 4.3 shows the results of histograms (normalized to 1) of the field $\phi$ for different temperatures. They are identical for different volumes. The distribution was also found to be effectively the same for simulations without electric and magnetic charges. Plots of W-boson fugacities at different temperatures as a function of $\phi$ are also given in \[1\] and Figure 4.4 and show they increase with temperature increasing and 'dip' for $\phi$ near zero. It seems that the W-bosons have higher fugacities near the boundary of the Weyl chambre, which would not allow Abelianization of the theory. However, the $\cosh 2\phi$ term of the neutral bion potential favours $\phi \approx 0$ and the theory remains Abelianized.

In summary, I find a transition from a magnetic dominated Coulomb gas to an electric dominated one as the temperature is increased. The transition agrees with previous results of YM and QCD(adj) in \[5,15\] and a second order transition is observed. In the next Section a dual spin model with symmetry-breaking perturbations with coefficients dependent of the scalar field $\phi$ is introduced, dual to the Coulomb gas studied here. Results of simulations of this spin model will be seen to qualitatively agree with those of the Coulomb gas.

### 4.2 Dual Affine XY Model to $SU(2)$ Double Coulomb Gas

The XY model description of the Coulomb gas partition function (4.18) has the form \[1\]

$$S_{XY} = \sum_{x,\hat{\mu}} -\frac{8T}{\pi \kappa} \cos \nabla_\mu \theta_x + \frac{\kappa}{16\pi T} (\nabla_\mu \phi_x)^2 + \ldots$$ \[4.29\]
\[
\sum_x \frac{8e^{-4\pi/\kappa}}{\pi T K^3} \cosh(2\phi_x) + 2\xi_W(\phi_x) \cos(4\theta_x),
\]
where \(\xi_W(\phi)\) is the \(\phi\)-dependent W-boson fugacity given by equation (4.9). The two fields \(\phi_x, \theta_x\) are represented by the symmetry perturbing \(\cos 4\phi\) of this field to be the magnetically charged objects: the magnetic bions. The electric particles (W-bosons and superpartners) are then represented by the symmetry perturbing \(\cos 4\phi\) term (or ’external field’ term), which interact by photon exchange and depend on \(\theta_x\) and the scalar \(\phi_x\). Note that for \(\phi = 0\) (4.29) reduces to that found in QCD(adj) for \(n_f > 1\) found in [5]. We can check the duality

Let me now look into the derivation of the Coulomb gas from the dual spin model, for any \(\mathbb{Z}_p\) symmetry breaking perturbation. Our partition function is given by (for a single field \(\theta\) for now)

\[
Z_{\mathbb{Z}_p}^D = \int d\theta e^{\sum_{<ij>} \cos(\theta_i - \theta_j) + g \sum_i \cos \theta_i},
\]
where \(<ij>\) denotes nearest lattice neighbours, and the action I will write as \(S = \int d^2x \left( \partial \theta \right)^2 - 2G \cos p \theta\) where \(2G = g/2a^2\) where \(a\) is the lattice spacing. We will use the following expansion for expanding the cosine in the action

\[
ed^2x 2G \cos p \theta = \sum_{k \geq 0} \frac{(2G)^k}{k!} \left( \int d^2x e^{ip\theta(x)} + e^{-ip\theta(x)} \right)^k.
\]

Doing the Gaussian integration over \(\theta\) implies that equal numbers of \(e^{ip\theta}\) and \(e^{-ip\theta}\) must contribute to make the action finite. This leads to the charge neutrality condition on the Coulomb gas. This leaves us with \(2k\) magnetic charges of charge \(pq_A\) at positions \(x_A\). Vortices need to also be included due to the periodicity of the \(\theta\) field with winding number \(wq_i\) at positions \(x_i\) (the was calculated before and it gives a \(\Theta(\vec{x} - \vec{x}_i)\) term) with \(\sum_i q_i = 0\) as well so as to keep the action finite. I then write for our \(\theta\)-field

\[
\theta(x) = -\frac{i p}{K} \sum_A q_A \ln |\vec{x} - \vec{x}_A| + \sum_i q_i \Theta(\vec{x} - \vec{x}_i) + \theta_0(x),
\]

where \(\theta_0(x)\) represents the spin-wave portion of the field. Substituting this \(\theta(x)\) into the action and integrating (I omit self-energies which are absorbed into the fugacities after renormalization) gives

\[
Z_{\mathbb{Z}_p}^D = Z_0 \sum_{(n,m) \in \mathbb{Z}^2} \sum_{q_i = \pm w} \sum_{q_A = \pm 1} \frac{G^{2n}}{n!^2} \frac{H^{2m}}{m!^2} \int d^{4(n+m)} r e^{\sum_{A>B} k \cdot q_A \cdot q_B \ln r_{AB} + \sum_{i>j} K q_i \cdot q_j \ln r_{ij} + ip \sum_{A,i} q_i \cdot q_A \Theta(r_{Ai})},
\]

where \(H\) represents the W-boson fugacity and I used the shorthand notation \(r_{iA} = |\vec{x}_i - \vec{x}_A|\), etc.

The mapping to the dual double Coulomb gas can be described by taking the dual photon field \(\sigma\) to represent the compact lattice scalar \(\theta_x\). In one dual version of the lattice model we can take the vortices of this field to be the magnetically charged objects: the magnetic bions. The electric particles (W-bosons and superpartners) are then represented by the symmetry perturbing \(\cos 4\phi\) term (or ’external field’ term), which interact by photon exchange and depend on \(\theta_x\) and the scalar \(\phi_x\). Note that for \(\phi = 0\) (4.29) reduces to that found in QCD(adj) for \(n_f > 1\) found in [5]. We can check the duality
by expanding the cosine term as in (4.10) and integrating over $\theta_x$ then gives the W-W boson Coulomb interactions as well as the W-boson-magnetic bion interactions by Aharanov-Bohm phases. Note also that an insertion of $e^{i\theta_x}$ in the XY-model partition function is interpreted as the insertion of an external particle of charge 1/4 that of the W-boson (hence the cos 4$\theta_x$ term, which expanded gives the original Coulomb gas partition function) and such probes do not exist in the SU(2) SYM theory. But insertion of a factor $e^{2i\theta_x}$ into the partition function represents an electric probe of half the charge of the W-boson, and is thought of as a fundamental non-dynamical quark probe for confinement. In the XY-model I will probe the $\mathbb{Z}_4^\beta$ symmetry realization with order parameter the magnetization, and the potentials from its susceptibility:

$$m = \frac{1}{N^2} \left| \sum \ e^{i\theta_x} \right| = \frac{\langle |M| \rangle}{N^2}, \quad \chi(m) = \frac{\langle |M|^2 \rangle - \langle |M| \rangle^2}{N^2} = \sum_x (e^{i\theta_x} e^{-i\theta_0})_{\text{conn.}}$$

(4.34)

### 4.2.1 Results of lattice Monte Carlo simulations of dual affine XY model to dual Coulomb gas

The simulation of the XY-model was done for the same values of the parameters given for the Coulomb gas simulation, but to get an idea of finite-size scaling lattice widths of $N = 16, 20, 24, 32$ were simulated. The simulation runs were done as in the Coulomb gas with 10000 iterations (the first 500 ignored for equilibration) and data such as action, magnetization, $\theta$-vortex density, and averages of $\phi$ and $|\phi|$ were computed after each sweep. Simulations started at low temperatures and with $\theta$ uniformly distributed in the interval $[-\pi, \pi]$ and those of $\phi$ uniformly distributed in the range $[-4\pi^2/g^2, 4\pi^2/g^2]$. Each Metropolis iteration changed a value of the $\theta$ field at a randomly chosen lattice site to one in the interval $[-\pi, \pi]$, and similarly for the $\phi$ field in its range. Changes are done with probabilities with a Boltzmann distribution as before with the Coulomb gas, $p = \min(1, e^{-\Delta S})$.

Figure 4.5 shows results of the magnetization and susceptibility as a function of temperature. There is a clear transition, at $T \approx 4$, that gets steeper with increasing $N$. To show more of the $\theta$ symmetry breaking due to the cos 4$\theta_x$ term Figure 4.6 shows the histograms of the $\theta$ angle distributions as the temperature is increased. A clear breaking of the symmetry down to a $\mathbb{Z}_4$-peaked distribution is seen as temperature increases. Figure 4.7 shows configurations and formation of vortices as a 2D lattice plot with $\theta$ directions given by arrows and positive and negative vortices indicated on the figure. It is clear that the vortices disappear above $T_c$ and become bound in a small number of dipole pairs, indicating the confinement of magnetic bions above $T_c$. It is also seen that at high $T$ the ordering of $\theta$ increases, whereas at low $T$ the magnetic plasma confines electric charges (it disorders the 'photon' $\theta_x$ and gives it a more finite correlation length).

Finally, Figure 4.8 shows the $\phi$ distributions at different temperatures, as in Figure 4.3 for the Coulomb gas. Qualitative agreement is once again seen between the two models. The only difference is that of a two-bump structure around $T = 2$. This can be attributed to the two small bumps in the W-boson fugacity of Figure 4.3 [1]. Furthermore, as in the dual-Coulomb gas simulations, the action histograms show no two coexisting peaks indicating a second order phase transition in agreement with previous results and theory. See [1] for more on extrapolating simulation results to weak-coupling, and how the results of simulation can be seen to be remarkably similar, even sometimes quantitatively, to
Chapter 4. Duality to Coulomb Gases and XY Spin Models of SU(2) (Super) Yang-Mills

Figure 4.5: XY-model 'magnetization' (left) and 'susceptibility' (right). See text for details.

Figure 4.6: Distribution of $\theta$ angles at temperature $T = 0, 2, 5$ from left to right. Clearly the $\mathbb{Z}_4$ symmetry breaking is observed as temperature is increased.

theoretical results at weak coupling.
Figure 4.7: The lattice showing the $\theta$ angles and positive and negative vortices at $T = 0.4, 2, 5$. See text for more discussion.

Figure 4.8: Histograms for $\phi$ for $N = 16$ and $T = 0.4, 2, 5$ from left to right.
Chapter 5

Lie Groups and Lie Algebras

For a sufficiently self-contained description of the mathematical constructs in our theory let me review Lie groups and Lie algebras. The familiar reader can skip to Section 5.2 on the subject of roots and weights.

5.1 Notes on General Lie Theory

Let me begin by defining a Lie algebra and give its properties.

A Lie algebra \( \mathfrak{g} \) is a vector space over a field \( F \) (which I take here to be either real, \( \mathbb{R} \), or complex, \( \mathbb{C} \)) with a binary operation (called the Lie bracket) \( \cdot \times \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \) satisfying the basic properties:

(i) bilinearity: \( \left[ ax+by, cz+dw \right] = ac[x, y] + ad[x, w] + bc[y, z] + bd[y, w], \forall a, b, c, d \in F \) and \( \forall x, y, z, w \in \mathfrak{g} \).

(ii) asymmetry: \( [x, y] = -[y, x], \forall x, y \in \mathfrak{g} \)

(iii) Jacobi identity: \( [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0, \forall x, y, z \in \mathfrak{g} \).

A Lie algebra is equipped with a basis of generators \( \{ T^a \}_{a=1}^r \) where \( r = \text{dim}(\mathfrak{g}) \), and these satisfy the same relations above. The generators, forming a basis, have commutators which are linear combinations of generators, \( [T^a, T^b] = f^{abc} T^c \), where the coefficients \( f^{abc} \) are the structure constants of the algebra.

In the fundamental representation this dimension \( r \) is minimal and equal to the rank of its corresponding Lie group.

A Lie algebra is called simple if it is non-Abelian and has no non-zero proper ideals, and is semi-simple if it is non-Abelian and has no non-zero proper Abelian ideals. Hence a semi-simple Lie algebra \( \mathfrak{g} \) can be written as a direct sum of simple Lie algebras \( \mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i \). I consider here just semi-simple Lie algebras.

A Lie algebra depends on its representation. A representation \( \mathcal{R} \) is given by a map \( \pi_\mathcal{R} : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \), where \( \mathfrak{gl}(V) \) is the enveloping associative Lie algebra of endomorphisms of a vector space \( V \). The dimension of the representation \( \text{dim}(\mathcal{R}) = \text{dim}(V) \) equals the dimension of the vector space \( V \), if it is finite. For example, the fundamental representation has \( \text{dim}(V) = \text{rank}(G) \). Also, in this paper, I use often the adjoint representation \( ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) \) where the action is \( ad(x)(y) = [x, y], \forall x, y \in \mathfrak{g} \).

A Lie group has a subgroup called the maximal torus \( T \subset G \), whose elements commute with all
other elements of the Lie group, and is topologically a torus \((S^1)^r\) where \(r = \text{dim}(G)\), the topological dimension of the group. Its Lie algebra \(t = \text{Lie}(T)\) is called the Cartan subalgebra of the Lie algebra and is of dimension \(r\). Its generators \(\{H^a\}_{a=1}^r\) with \([H^a, H^b] = 0\) form an \(r\)-dimensional subspace of \(\mathfrak{g}\) and can be chosen to satisfy the normalization \(\text{tr}(H^a H^b) = \delta^{ab}\) here.

The other generators of the Lie algebra can be represented by \(\text{dim}(G) - r\) raising and lowering operators, \(\{E_\alpha\} \text{ and } \{E_{-\alpha} = E^\dagger_\alpha\}\), which satisfy the relations

\[
[H^i, E_\alpha] = \alpha_i E_\alpha \tag{5.1}
\]

\[
[E_\alpha, E_{-\alpha}] = \alpha_i H^i
\]

\[
[E_\alpha, E_\beta] = N_{\alpha\beta\gamma} E_\gamma,
\]

where the constants \(N_{\alpha\beta\gamma}\) will not be needed later. The contravariant and covariant roots are related by the Cartan Killing form \(g^{ij} = \text{Tr}[H^i H^j]\).

A Lie group, as a reminder, is a group that is also a differentiable manifold, and hence has a differential structure or derivation (that satisfies the Leibnitz rule). In fact, the Lie algebra corresponding to it is the tangent space of the Lie group, specifically to its covering space \(\tilde{G}\). Figure 5.1 shows all possible simply-connected, semi-simple Lie algebras and their Dynkin diagrams. For more definitions and detailed theory see [9]. Dynkin diagrams will be briefly described in the next Section.

### 5.2 The Roots and the Weights

One way to define the roots of a Lie group \(G\) that will be useful later on is to consider it from the point of view of representations of its corresponding Lie algebra \(\mathfrak{g}\). In general I define the root \(\alpha_i\) as an eigenvalue. In fact it is a function valued on \(t = \text{Lie}T\), where \(T\) is the maximal torus of \(G\)

\[
\alpha_i : \mathbb{C}t \rightarrow \mathbb{C}
\]

with its eigenspace \(E_{\alpha_i} \in \mathbb{C}\mathfrak{g}\) defined by

\[
[H, E_{\alpha_i}] = \alpha_i(H)E_{\alpha_i}, \tag{5.2}
\]

where \(H \in \mathbb{C}t\), the Cartan subalgebra of \(\mathfrak{g}\).

We can see how this works for \(SU(N)\). Beginning with \(SU(2)\), we have Lie algebra \(\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})\). It is clear that \(t = \text{span}\{\lambda \sigma_3 = (\lambda \ 0 \ -\lambda)\}_{\lambda \in \mathbb{C}}\) and that there are two root spaces, one with root the negative of the other: \(E^+ = \text{span}\{(0 \ 1)\}, E^- = \text{span}\{(1 \ 0)\}\). It is easy to check that the roots satisfying equation (5.2) are \(\alpha_\pm(H(\lambda)) = \pm 2\lambda\).

This clearly generalizes to \(SU(N)\) with Lie algebra \(\mathfrak{sl}_N(\mathbb{C})\). The maximal torus is just \(T \approx \mathbb{T}^{N-1} \approx \{\text{diag}(e^{i\theta_j})_{j=1}^N \prod e^{i\theta_j} = 1\}\). The Cartan subalgebra is the set of matrices with complex numbers \(\lambda_j\) along the diagonal, accompanied by their negatives \(-\lambda_j\), so as to make the trace vanish (this is in fact
Figure 5.1: All simple Lie algebras and their affine Dynkin diagrams. The numbers inside the nodes denote the root, and the numbers beside each node are the Kac labels. The red circles are the lowest (affine) roots of the Lie algebra.
for the adjoint representation). The root spaces are just the span of each $E_{ij}$, the $N \times N$ matrix with a 1 in the $i, j$-th position and zeroes elsewhere. One easily checks that the roots obey $\alpha_{jk}(H(\lambda)) = \lambda_j - \lambda_k$ along with their negatives from

\[ [H(\lambda), E_{jk}] = (\lambda_j - \lambda_k)E_{jk}. \tag{5.3} \]

This is why in the adjoint representation, the roots take on values given by differences of Wilson line eigenphases $\theta_j$. This result will be used later on, and the roots for the other Lie algebras will be given soon.

In the next Section we will need to know the weights in the adjoint representation, which I prove are the roots (the full set) of the Lie algebra. Let me describe representation theory in general a bit first before proving this fact.

A representation of a Lie algebra is a homomorphism from the Lie algebra $\mathfrak{g}$ into the endomorphism group of a certain vector space $V$, 

$$ \phi : \mathfrak{g} \to \text{End}(V), $$

and preserves the Lie bracket. The dimension of the representation is the dimension of the vector space $V$ underlying the representation. The dimension of the Lie algebra itself is the number of independent generators of $\mathfrak{g}$. In the fundamental representation the dimension of $\mathfrak{g}$ equals the dimension of $V$ and hence of the dimension of the representation. The rank $r$, however, of a Lie algebra is the dimension of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The Cartan subalgebra has a set of Abelian generators in the Cartan-Weyl basis $\{H_i\}_{i=1}^r$ satisfying $[H^i, H^j] = 0$ and the roots, as mentioned before, satisfy eigenvalue-like expressions: $[H^i, E^\alpha] = \alpha_i E^\alpha$, and there are hence $r = rk(\mathfrak{g})$ simple positive roots of $\mathfrak{g}$. The Lie algebra then decomposes as

$$ \mathfrak{g} = \mathfrak{h} \oplus \alpha \in \Delta^+ \mathfrak{g}_\alpha, $$

where $\mathfrak{g}_\alpha$ is the eigenspace, spanned by $E^\alpha$. I can also prove what the weights (eigenvalues of the $H^i$) are in fact the roots in the adjoint representation. Indeed,

$$ \phi_{\text{adj}}(H^i)E^\alpha = \text{ad}_H E^\alpha \equiv [H^i, E^\alpha] = \alpha_i E^\alpha, \tag{5.4} $$

proving the claim. □

Below I will list the positive roots for each Lie algebra, but one more point to make is the role of the affine root in spaces with a compact direction. The Lie algebra with the affine root included is the Lie algebra of the loop group $LG$ of maps $\pi : S^1 \to G$, $\text{Lie}(LG) = L\mathfrak{g}$. Similarly on spaces with multiple compact directions there are more roots to be added and the resulting algebra is the toroidal Lie algebra. The affine roots are included below.

For all semi-simple Lie groups (described below) a choice of simple positive roots is given as follows:

$$ A_{N+1} \approx SU(N); $$

This is the group of rotations about the origin in $\mathbb{C}^N$. It preserves the lengths of vectors. The simple
roots are:
\[ \left\{ \alpha_i = e_i - e_{i+1} \right\}_{i=1}^N \] (5.5)

The affine root is \( \alpha_0 = -\sum_{i=1}^N \alpha_i = e_N - e_1 \).

\( B_N \approx Spin(2N + 1) \):

This is the double cover of the orthogonal group \( SO(2N + 1) \), the rotation group in \( \mathbb{R}^{2N+1} \). The simple roots are:
\[ \left\{ e_i - e_{i+1} \right\}_{1 < i < N-1} \cup \{ e_N \} \] (5.6)

The affine root is \( -\alpha_0 = e_1 + e_2 = \alpha_1 + 2 \sum_{i=2}^N \alpha_i \).

\( C_N \approx Sp(2N) \):

This is the group of \( 2N \times 2N \) matrices preserving the antisymmetric scalar product \( J = (0_{N \times N}, 1_N) \), so \( M^TJM = J \forall M \in Sp(2N) \). The simple roots are:
\[ \left\{ e_i - e_{i+1} \right\}_{1 < i < N-1} \cup \{ 2e_N \} \] (5.7)

The affine root is \( -\alpha_0 = 2e_1 = \sum_{i=1}^{N-1} 2\alpha_i + \alpha_N \).

\( D_N \approx Spin(4N), Spin(4N + 2) \) (N even, odd respectively):

These are the double covers of the orthogonal groups \( SO(4N) \) and \( SO(4N + 2) \) respectively. The simple roots are:
\[ \left\{ e_i - e_{i+1} \right\}_{1 < i < N-1} \cup \{ e_{N-1} + e_N \} \] (5.8)

The affine root is \( -\alpha_0 = e_1 + e_2 = \alpha_1 + 2 \sum_{i=2}^{N-2} \alpha_i + \alpha_{N-1} + \alpha_N \).

\( E_6 \):

This is the rank 6 exceptional Lie group of dimension 78. The simple roots are:
\[ \{ (1, -1, 0, 0, 0, 0) \} \] (5.9)

\( (0, 1, -1, 0, 0, 0) \)

\( (0, 0, 1, -1, 0, 0) \)

\( (0, 0, 0, 1, 1, 0) \)

\( -\frac{1}{2}(1, 1, 1, 1, 1, -\sqrt{3}) \)

\( (0, 0, 0, 1, -1, 0) \}

The affine root is \( -\alpha_0 = e_1 - e_8 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \). This is in the 8 dimensional basis and we note all vectors are orthogonal to \( \sum_{i=1}^8 e_i \) and to \( e_1 + e_8 \) and so gauge fields are constrained by
\[ \phi_1 + \phi_8 = \sum_{i=2}^{7} \phi_i = 0. \]

**E\(_7\):**

This is the rank 7 exceptional group of dimension 133. The simple roots are:

\[
\begin{align*}
(1, -1, 0, 0, 0, 0, 0) \\
(0, 1, -1, 0, 0, 0, 0) \\
(0, 0, 1, -1, 0, 0, 0) \\
(0, 0, 0, 1, -1, 0, 0) \\
(0, 0, 0, 1, 1, 0) \\
-\frac{1}{2} (1, 1, 1, 1, 1, -\sqrt{2}) \\
(0, 0, 0, 0, 0, -1, 0, 0)
\end{align*}
\]

The affine root is \(-\alpha_0 = e_2 - e_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7\). The fields are constrained to live on the plane orthogonal to \(\sum_{i=1}^{8} e_i = 0\) in the 8 dimensional basis.

**E\(_8\):**

This is the rank 8 exceptional group of dimension 248. The simple roots are:

\[
\begin{align*}
(1, -1, 0, 0, 0, 0, 0, 0) \\
(0, 1, -1, 0, 0, 0, 0, 0) \\
(0, 0, 1, -1, 0, 0, 0, 0) \\
(0, 0, 0, 1, -1, 0, 0, 0) \\
(0, 0, 0, 0, 1, -1, 0, 0) \\
(0, 0, 0, 0, 0, 1, 1, 0) \\
-\frac{1}{2} (1, 1, 1, 1, 1, 1, 1, 1) \\
(0, 0, 0, 0, 0, 0, -1, 0, 0)
\end{align*}
\]

The affine root is \(-\alpha_0 = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8\).

**F\(_4\):**

This rank 4 exceptional Lie group has dimension 52. The simple roots are given by:

\[
\{(0, 1, -1, 0)\}
\]
(0, 0, 1, -1)
(0, 0, 0, 1)
\(-\frac{1}{2}(-1, 1, 1, 1)\}\)

The affine root is \(-\alpha_0 = e_1 + e_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\).

\(G_2\): (embedded in 2D subspace of \(\mathbb{R}^3\), the plane perpendicular to line \(x + y + z = 0\))

This is the rank 2 exceptional Lie group of dimension 14. Its simple roots are:
\[
\{(0, 1, -1), (1, -2, 1)\} \quad (5.13)
\]
The affine root is \(-\alpha_0 = e_1 + e_2 - 2e_3 = 2\alpha_1 + 3\alpha_2\). All vectors are orthogonal to \(e_1 + e_2 + e_3 = 0\).

Note that the coefficients \(k_i\) in the definition of the affine root \(\vec{\alpha}_0 = -\sum_{i=1}^{r} k_i \vec{\alpha}_i\) are called the Kac labels of the Lie algebra. The Coxeter number of the Lie algebra is \(h(G) = \sum_{i=1}^{r} k_i + 1\).

We will also need co-roots later, defined as
\[
\vec{\alpha}^\lor = \frac{2}{\vec{\alpha}^2} \vec{\alpha} \in \Lambda_r^\lor, \quad (5.14)
\]
where they span the co-root lattice \(\Lambda_r^\lor\), and the \(\vec{\alpha}\)'s are the \(r = \text{rank}(G)\) simple roots given above span the root lattice \(\Lambda_r\).

The weights.

We also need to get to know the weight system, with lattice \(\Lambda_w\) and its co-weight lattice. The weight vectors \(\vec{w}_i\) for a set of simple roots \(\vec{\alpha}_i\) are defined via
\[
\vec{w}_i \cdot \vec{\alpha}^\lor = \delta_{ij}, \quad (5.15)
\]
and the co-weights are defined as were the co-roots:
\[
\vec{w}^\lor = \frac{2}{\vec{w}^2} \vec{w} \in \Lambda_w^\lor. \quad (5.16)
\]
Since I am dealing with affine Lie algebras I need to define the affine co-root in terms of simple co-roots (the affine root being as before, \(\vec{\alpha}_0 = -\sum_{j=1}^{r} k_j \vec{\alpha}_j\)):
\[
\vec{\alpha}_0^\lor = -\sum_{j}^{r} k_j \vec{\alpha}^\lor_j, \quad (5.17)
\]
and the dual Coxeter number is defined from the coefficients \(c_2 = \sum_{i=0}^{r} k_i^\lor\). For \(\mathfrak{g} = \mathfrak{su}(r+1)\), \(c_2 = r+1\) as all \(k^\lor\)'s are 1s, and all \(\alpha\)'s have norm \(\sqrt{2}\). We will also soon need these data for \(\mathfrak{g}_2\), where \(c_2 = 4\) for \(\{k_i^\lor\} = \{1, 2, 1\}\) and \(\{\alpha_i^2\} = \{2, 2, 2/3\}\). For data such as these for all semi-simple Lie groups see
Table 5.1: (Dual) Kac labels and dual Coxeter numbers for all semi-simple Lie groups.

<table>
<thead>
<tr>
<th>Group, G</th>
<th>$r = rk(G)$</th>
<th>$h$</th>
<th>$c_2(G)$</th>
<th>$[k_0^\vee, \ldots, k_r^\vee]$</th>
<th>$[k_0, \ldots, k_r]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(N+1)$</td>
<td>$N$</td>
<td>$N+1$</td>
<td>$N+1$</td>
<td>$[1,1,\ldots,1]$</td>
<td>$[1,1,\ldots,1]$</td>
</tr>
<tr>
<td>$SO(2N+1)$</td>
<td>$N$</td>
<td>$2N$</td>
<td>$2N-2$</td>
<td>$[1,1,1,2,\ldots,2]$</td>
<td>$[1,1,2,\ldots,2]$</td>
</tr>
<tr>
<td>$SO(2N)$</td>
<td>$N$</td>
<td>$2N$</td>
<td>$2N-2$</td>
<td>$[1,1,1,1,2,\ldots,2]$</td>
<td>$[1,2,\ldots,2,1]$</td>
</tr>
<tr>
<td>$Sp(2N)$</td>
<td>$N$</td>
<td>$2N-2$</td>
<td>$N+1$</td>
<td>$[1,1,\ldots,1]$</td>
<td>$[1,1,2,2,\ldots,2,1,1]$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>$[1,1,2]$</td>
<td>$[1,2,3]$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>4</td>
<td>12</td>
<td>9</td>
<td>$[1,1,2,3,2]$</td>
<td>$[1,2,3,4,2]$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>6</td>
<td>12</td>
<td>12</td>
<td>$[1,1,1,2,2,2,3]$</td>
<td>$[1,2,3,2,1,2]$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>7</td>
<td>18</td>
<td>18</td>
<td>$[1,1,2,2,2,3,3,4]$</td>
<td>$[1,2,3,4,3,2,1,2]$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>8</td>
<td>30</td>
<td>30</td>
<td>$[1,2,2,3,3,4,4,5,6]$</td>
<td>$[1,2,3,4,5,6,4,2,3]$</td>
</tr>
</tbody>
</table>

[11]. Table 5.1 shows some such data including the (dual) Kac labels and (dual) Coxeter numbers. It is interesting to note that the Coxeter number $h$ of a group is the number of roots divided by the rank of the group.

Figure 5.1 shows all semi-simple Lie algebras as Dynkin diagrams with the Kac labels included. As a reminder a Dynkin diagram is (for our purposes) a graph with single, double or triple lines connecting nodes, represented by simple roots. The multiplicity of the lines (edges) will not concern us, but are related to the length of roots represented by the nodes the edge connects. The affine Dynkin diagram contains the affine root $\alpha_0$ and is indicated in Figure 5.1 by red circles.

The weights of a Lie algebra in a given irrep $R$ represent the charges of particles possible for that irrep and hence are important Lie algebra data. The matrices $R(h)$ for any $h$ in the Cartan subalgebra can be simultaneously diagonalized giving vectors $\vec{w} \in \mathfrak{t}^*$ of eigenvalues so that $\vec{w} \cdot h$ is an eigenvalue of $R(h)$. These vectors $\vec{w}$ belong to the set of weights of $R$ $\Delta_w^R$ and their integral span $\mathbb{Z}[\Delta_w^R] = \Lambda_w^R$ is called the weight lattice of $R$. The group lattice $\Gamma_G = \cup_R \Lambda_w^R$ is the union of irrep weight lattices. At the level of Lie group, the eigenvalues of irrep $R$ of an element $g \in G$, the maximal torus of $G$, are $\exp(2\pi i \vec{w} \cdot h)$. The periodicity of the maximal torus are given by shifts in the lattice of those $h$ such that $\vec{w} \cdot h \in \mathbb{Z}$. The dual lattice of co-weights is defined by the lattice of such $h$, $\Lambda_w^R$. The smallest arising group lattice is called the root lattice $\Lambda_r$, whereas the largest is called the weight lattice $\Lambda_w$.

For completeness I now present the weights of the adjoint representation (which are in fact the set of ALL roots as was shown above) for each Lie algebra. The number of weights is equal to the dimension of the representation minus the rank of the Lie algebra (the rank is the number of null weights of eigenvalue zero for the action of the Cartan generators).

$A_{N+1} \approx SU(N)$:
There are $N^2 + N$ adjoint weights in all, $N(N+1)/2$ being positive. All are of length $\sqrt{2}$ with 1 in one entry $i$, -1 in position $j$, and zeros elsewhere. We denote them as $\alpha_{ij}^\pm$ where the superscript is positive if the root is. The positive roots are taken to be the ones with a +1 occurring in an earlier position than -1, i.e. $i < j$.

$B_N \approx Spin(2N+1)$:
There are $2N^2$ weights of two types: $\alpha_{ij}^\pm$ which are all integer vectors of length $\sqrt{2}$, and $\beta_i^{\pm,B}$ which
are all integer vectors of length 1. The positive weights are those with a +1 occurring before a -1 as usual.

\[ C_N \approx Sp(N) : \]
In all there are \(2N^2\) roots including the \(\alpha_{ij}^\pm\) above, and with \(\beta_i^\pm.c = \pm 2e_i\). Positive roots are as before.

\[ D_N \approx Spin(4N), Spin(N + 2) : \]
Here all roots are all integer vectors of length \(\sqrt{2}\). These include the \(\alpha_{ij}^\pm\) above, but also those with 2 entries both -1 or both +1, called \(\beta_{ij}^{\pm,D}\). There are \(2N(N - 1)\) in all.

\[ E_6 : \]
The adjoint weights include the \(4 \times \binom{5}{2}\) permutations of the entries of the vectors \((\pm1, \pm1, 0, 0, 0, 0)\) keeping a zero in the last entry, plus the vectors of the form \(\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm\sqrt{3})\) with an odd number of + signs. This gives a total of 72 weights.

\[ E_7 : \]
We have here \(4 \times \binom{6}{2}\) permutations of \((\pm1, \pm1, 0, 0, 0, 0, 0)\) keeping a zero in the last entry, plus the vectors of the form \(\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm\sqrt{2})\) with an even number of + signs, plus the two vectors \((\vec{0}, \pm\sqrt{2})\). This gives a total of 126 weights.

\[ E_8 : \]
We have 112 roots as permutations of \((\pm1, \pm1, 0, 0, 0, 0, 0, 0)\), plus the 128 vectors of the form \(\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm1)\) with an even number of - signs.

\[ F_4 : \]
We have here 48 roots: 24 as permutations of \((\pm1, \pm1, 0, 0, 0, 0)\) (call them type I), plus 8 roots as permutations of \((\pm1, \vec{0})\) (type J), and 16 roots of the form \((\pm1, \pm1, \pm1, \pm1)/2\) (type K).

\[ G_2 : \]
Here there are 12 adjoint weights:

\[(1, -1, 0), (2, -1, -1), (1, 0, -1), (1, -2, 1), (0, 1, -1), (1, 1, -2),\]
together with their negatives.

For more on Lie algebras, weights and representations, a great resource is [9].

**Gauge cells and Weyl chambers.**

The Weyl group \(W(g)\) is another group of gauge identifications on \(t\), that acts as a group of linear transformations on \(t^*\) that preserves the set of roots \(\Delta_r\) (permutes them). It includes a Weyl reflection for each simple root \(\alpha\) which acts on \(h \in t^*\) by \(\sigma_{\alpha}(h) = h - (h \cdot \alpha^\vee)\alpha\). It acts on \(\varphi \in t\) by \(\sigma_{\alpha}(h)[\varphi] = h \cdot \sigma_{\alpha}(\varphi)\) and so \(\sigma_{\alpha}(\varphi) = \varphi - 2(\alpha \cdot \varphi)\alpha^\vee\) and is a reflection about the plane with normal vector \(\alpha\) passing through the origin. Allowing translations of the co-root lattice, the group of transformations is the semi-direct product \(\hat{W}\) of \(W\) and \(\Lambda_r^\vee\). A fundamental domain or gauge cell (or affine Weyl chambre) \(\hat{t}\) for \(G\) is the
quotient $t/\hat{W}$. A choice often used for the affine Weyl chamber is

$$\hat{\mathfrak{t}} = \{ \varphi \in \mathfrak{t} | 0 \leq \alpha \cdot \varphi, \forall \alpha \in \Delta^s, \ -\alpha_0 \cdot \varphi \leq 1 \},$$

where $\Delta^s$ denotes the set of simple roots. This is the cell of interest here as it is the Cartan subalgebra modulo gauge equivalences. At points interior to $\hat{\mathfrak{t}}$ the unbroken gauge group is the maximal torus $U(1)^r$, while on the cell boundary, as mentioned previously, the gauge symmetry is enhanced due to elements being fixed by the gauge transformations $\hat{W}$, and the theory is no longer fully Abelianized. For explicit root systems and gauge cells see Appendix B of [4]. The gauge cells for Lie groups of rank 2 are shown in Figure 5.2.
Chapter 6

Perturbative and Non-perturbative Theory for all Gauge Groups on Toroidally Compactified Spacetimes

In this section I examine the perturbative dynamics of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ (and briefly for other toroidially compactified spacetimes) for any Lie group $G$. I begin with a review of the theory at zero temperature and move on to new results of the perturbative dynamics at finite temperature $T > 0$. The non-perturbative dynamics will be discussed in the next Section.

6.1 $T = 0$ Dynamics of Super Yang-Mills on $\mathbb{R}^3 \times S^1_L$

I consider $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with general gauge group $G$ with a single massless adjoint Weyl fermion (the gaugino/gluino). The action on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ is then

$$S = \int_{\mathbb{R}^2 \times S^1_L \times S^1_\beta} \text{tr} \left[ \frac{1}{2g^2} F^{MN} F_{MN} + \frac{2i}{g^2} \bar{\lambda} \sigma^M D_M \lambda \right],$$

(6.1)

where $F^{MN} = F^{MNa} T^a$, where $F_{MN} = \partial_M A_N - \partial_N A_M + ig[A_M, A_N]$ is the gauge field strength tensor, is written in the basis of the generators of the Lie group $G$, and similarly $\lambda = \lambda^a T^a$. $D_M$ is the covariant derivative $\partial_M + igA_M$, and $\sigma_M = (i, \vec{\tau})$ and $\bar{\sigma}_M = (-i, \vec{\tau})$ where $\vec{\tau}$ are the Pauli matrices. I write $\vec{x} \in \mathbb{R}^2$ for the non-compact (1-2) spatial directions and have $x_0 = x_0 + \beta$ and $x_3 = x_3 + L$ for the compact directions. Both the adjoint fermion and the gauge field (gluon) have periodic boundary conditions along the spatial circle $S^1_L$, but along the thermal cycle $S^1_\beta$ the former have anti-periodic boundary conditions while the latter have periodic boundary conditions. At zero temperature $\beta \to \infty$ and the differing boundary conditions do not matter; hence supersymmetry remains unbroken at zero temperature. Recall that I take $c_2 L A_{QCD} \ll 1$ so one-loop calculations can be easily done to integrate out Kaluza-Klein modes along the $S^1_L$, which calculates the Coleman-Weinberg (or Gross-Pisarski-Yaffe, GPY) effective potential $V_{\text{eff}}^{\text{pert}}(\Omega_L)$. Supersymmetry sets this potential to zero as the determinants of gluon and gluino cancel to all orders in perturbation theory. However, non-perturbative corrections in monopole backgrounds contribute a $V_{\text{eff}}^{\text{non-pert}}$ and will be found in the next Section.
At zero temperature I consider the action (6.1) in the vacuum \( \langle A_3 \rangle = \langle A_3^a \rangle H^a \) (as the action is minimized by gauge fields in the Cartan subalgebra where they commute), the group \( G \) gauge theory is broken by the Higgs field \( \langle A_3 \rangle \) spontaneously to some smaller group \( G \rightarrow H \times U(1)^{r-m} \) where \( H \) is a subgroup of rank \( m < r \). For example, \( SU(N) \) breaks down to \( U(1)^{N-1} \) and becomes fully Abelianized. Some other gauge groups do not fully Abelianize, however further compactification can fully Abelianize the theory. See Figure 2.2 for the symmetry breaking steps of each gauge group. This partial Abelianization occurs for cases where the number of flavors \( n_f > 1 \), but at zero temperature SUSY \( (n_f = 1) \) we always have full Abelianization and \( \Lambda_{\text{SU}(2)L} \) \( \ll 1 \). Examining the non-perturbative effective potential later we see that \( A_3 \)’s minima are located at the rank\( (G) + 1 = |Z(G)| \) roots of unity \( \langle A_3 \rangle = \langle A_3^a \rangle H^a \), where \( H^a \) are the Cartan generators of the Lie group \( G \) and the \( Z(G)(L) \) symmetry is preserved at \( T = 0 \). The action (6.1) then becomes:

\[
S_{\beta \rightarrow \infty} = \int_{\mathbb{R}^2 \times S_3^\beta} \frac{L}{g^2} \text{tr}
\left[ -\frac{1}{2} F_{\mu \nu} F_{\mu \nu} + (D_\mu A_3) \right] + i \lambda (\bar{\sigma} \partial^\mu D_\mu \lambda - i \bar{\sigma}_3 [A_3, \lambda])
\]

\[
= \frac{L}{g^2} \int_{\mathbb{R}^2 \times S_3^\beta} \left( F_{\mu \nu}^a H^a \right)^2 / 2 + (\partial_\mu A_3^a H^a)^2 / 2 + i \bar{\lambda} \sigma^\mu \partial_\mu \lambda^a,
\]

where in the second line all fields belong to the Cartan subalgebra, so \( A_3^a T^a \rightarrow A_3^a H^a \), and similarly for \( \bar{F}_{\mu \nu} \) and \( \bar{\lambda} \). One should note that all massive (non-Cartan) gauge fields have been integrated out in (6.2) above, leaving only massless fields \( \{X\} \) in the path integral \( \int \mathcal{D}\{X\} e^{-S[\{X\}]}. \)

Let me consider the matter present in this theory. The \( r \) components of the gauge field \( A_3 \) along the Cartan subalgebra direction are massless perturbatively, but acquire a small mass \( \approx e^{-8\pi^2/g^2} \) non-perturbatively. The remaining components acquire mass \( m_W = \pi / c_2(G) L \), where \( c_2(G) \) is the dual Coxeter number of the Lie group (Refer to Chapter 5 for Lie algebra conventions and data). These are the W-bosons of the theory. Similarly, the components of the gauginos that don’t commute with \( \langle A_3 \rangle \) acquire the same mass \( m_W \) and these constitute the superpartners of the W-bosons, the winos. We also have the remaining components \( A_3^a \) of the gauge field, or, equivalently, \( F_{\mu \nu}^a \). Let me assemble these components into a more compact notation. Define first \( r \) fields \( \sigma^a \) and combine them into a vector \( \vec{\sigma} \) (from now on the vector notation \( \vec{v} \) denotes \( r \)-dimensional vectors in the Cartan subalgebra), through Abelian duality:

\[
\epsilon_{\mu \nu \lambda} \partial_\lambda \vec{\sigma} \equiv \frac{4\pi L}{g^2} \vec{F}_{\mu \nu}.
\]

The components of \( \vec{\sigma} \) form \( r \) spin-zero dual photon fields in two dimensions. I also define a scalar field \( \vec{\phi} \) as

\[
\vec{\phi} \equiv \frac{4\pi L}{g^2} \vec{A}_3 - \frac{4\pi^2}{g^2} \vec{\rho},
\]

so that \( \vec{\phi} = 0 \) corresponds to \( Z(G)(L) \) center symmetry unbroken. Here, \( \vec{\rho} \) is the Weyl vector of the Lie algebra, \( \vec{\rho} = \sum_{i=1}^r \vec{\omega}_i \), where \( \vec{\omega}_i \) are the fundamental weights\(^1\) defined by \( \vec{\alpha}_i \cdot \vec{\omega}_j = \delta_{ij} \). It turns out that this can be written as a sum of positive roots, \( \vec{\rho} = \frac{1}{2} \sum_{\alpha \in \Delta^+} \vec{\alpha} \). The bosonic part of the Lagrangian from (6.2) can be written compactly as \( \mathcal{L}_{\text{free bosonic}, \beta \rightarrow \infty} = \frac{g^2}{2 (4\pi L)^2} (\partial_\mu \vec{\sigma})^2 + (\partial_\mu \vec{\phi})^2 \).\(^2\)

\(^1\)Do not confuse these with the weights of the fundamental representation. See Chapter 5 for more on weights.

\(^2\)Note that in supersymmetry we can obtain the kinetic terms for the gluon and gluino from the Kahler potential.
One further comment before proceeding is the computation of traces in different representations. In (6.2) I took the gauge fields to lie within the Cartan subalgebra of $g$. As outlined in Chapter 5, the weights of representation $\mathcal{R}$ are eigenvalue sets of the Cartan matrices $H^a$ in representation $\mathcal{R}$. Hence we have for any fields $X, Y$, 
\[
\text{tr}_{\mathcal{R}} XY = Tr(X^a H^a Y^b H^b) = \sum_{w \in \Delta_w} \bar{X} \cdot \bar{w} \bar{Y} \cdot \bar{w} = \bar{X} \cdot \bar{Y},
\]
where $\Delta_w$ is the set of weights of $\mathcal{R}$, which for the adjoint representation, as seen in Chapter 5, are the set of all roots of $g$. I use this result from now on in computing traces. I now turn to the finite temperature dynamics.

### 6.2 Finite Temperature Dynamics of SYM on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ and the Effective Potential

At finite temperature supersymmetry no longer holds as the anti-periodic boundary conditions for the gluino along the thermal cycle break supersymmetry, and the perturbative effective potential no longer vanishes due to non-cancelling of gauge and gaugino determinants. Hence, at finite temperature, our effective action takes the form (in the adjoint representation from now on)

\[
S = L \int d^2x \sum_{\bar{w} \in \Delta_{\text{adj}}} \left[ |F_{ij} F_{ij}|/4g^2 + |D_i \bar{\Lambda}_3 \cdot \bar{w}|^2/2g^2 + |D_i \bar{\Lambda}_0 \cdot \bar{w}|^2/2g^2 + V_{\text{GPY}}^{\text{eff}}(\bar{\Lambda}_0, \bar{\Lambda}_3) + V_{\text{non pert.}}(\bar{\Lambda}_0, \bar{\Lambda}_3) + \text{(fermions)} \right] \tag{6.5}
\]

where the non-perturbative potential $V_{\text{non pert.}}$ will be found in the next Section. In this section I compute the perturbative effective potential of the W-bosons and winos present from the Higgsing due to the compactification of the 4D theory.

On calculating the effective one-loop perturbative potential I integrate out the heavy Kaluza-Klein modes along both directions of the torus $T^2$, which is related to computing the determinant of the operator $\mathcal{O} = D_M^2$ on $S^1_L \times S^1_\beta$ for both gauge and gaugino fluctuations in the background of constant holonomies along the torus. This gives us $V_{\text{eff}}^{\text{pert}}(A_0^a, A_3^a)$. The other components of the gauge field not along the Cartan subalgebra are set to zero in the effective potential as it is minimized by commuting holonomies. I present a method of calculation using zeta functions.

Recall that the determinant of an operator $\mathcal{O}$ is given by the product of its eigenvalues

\[
\text{Det} \mathcal{O} = \prod_{\lambda} \lambda = \exp(\sum_{\lambda} \log \lambda), \tag{6.6}
\]

and using the zeta function

\[
\zeta_{\mathcal{O}}(s) = \sum_{\lambda(\mathcal{O})} \lambda^{-s}, \tag{6.7}
\]

we find that the determinant and effective potential are respectively,

\[
\text{Det} \mathcal{O} = \exp[-\zeta_{\mathcal{O}}'(0)], \quad V_{\text{bosonic}} = -\zeta_{\mathcal{O}}'(0)/L\beta \tag{6.8}
\]

$K = \frac{\tilde{g}^2}{2(4\pi)^2 L} B^\dagger B$, where $B$ is a dimensionless chiral superfield with lowest component $\bar{\phi} - i \tilde{\sigma} [2]$.

I considered here products of fields as $tr X^a H^a = 0$ for any single field.
for the fermionic operator on $\mathbb{R}^2 \times S^1_L \times S^3_B$. For the bosonic operator we need $[\text{det} \mathcal{O}]^{-1}$ (it is -1 not -1/2 as there are two degrees of freedom/polarization for the gauge field) where our operator of interest is $\mathcal{O} = D_M^2$, so $V_{\text{boson}} = -\zeta_D^2(0)/L\beta$. The eigenvalues of this operator are matrix-valued in the Lie group $G$

$$\lambda_{mn} = \vec{k}^2 + (\omega_n + A_3^a T^a)^2 + (\Omega_m + A_0^a T^a)^2,$$

(6.9)

where $\omega_n = 2\pi n/L$ and $\Omega_m = 2\pi m/\beta$ are the KK and Matsubara frequencies along the respective cycle of the torus $\mathbb{T}^2 = S^1_L \times S^3_B$. By further integrating out the massive non-Cartan components I am left with fields with colour components along the Cartan subalgebra $\text{span}\{H^a\}_{a=1}$. I also choose a gauge where the holonomies are constant (so $A_0, A_3$ are independent of $x_0$ and $x_3$). The Wilson loops are then (using the vector notation to represent r-dimensional vectors in the Cartan subalgebra (the maximal torus) of $G$, and $\vec{H} = (H^1, \ldots, H^r)$

$$\Omega_L = e^{iL\vec{A}_3 \cdot \vec{H}}, \quad \Omega_\beta = e^{i\beta \vec{A}_0 \cdot \vec{H}},$$

and I take the commutator of Wilson loops $[\Omega_L, \Omega_\beta] = 0$ on the flat torus as it minimizes the effective potential. Writing the zeta function I must calculate (with traces in the adjoint representation in consideration)

$$\zeta(s) = \int \frac{d^2 k}{(2\pi)^2} \sum_{(n,m) \in \mathbb{Z}^2} tr_{\text{adj}} [\vec{k}^2 + (2\pi n/L + A_3^a T^a)^2 + (2\pi m/\beta + A_0^a T^a)^2]^{-s}$$

(6.10)

In the second line I performed the integral over $d^2 k = \pi d(k^2)$ and used the fact that the eigenvalues of the Cartan matrices form weight vectors, and that these are simply the roots of the Lie algebra of the Lie group $g = \text{Lie}(G)$ for the adjoint representation.

I need to find a low temperature expansion of this expression ($\beta \gg L$) as done in Section 3.1 for the case of $SU(2)$, and so I use a useful identity (as used in [24] and Section 3.1)

$$\sum_{m \in \mathbb{Z}} \frac{1}{[(m+a)^2 + c^2]^s} = \frac{\sqrt{\pi}}{\Gamma(s)} |c|^{1-2s} \Gamma(s-1/2) + \frac{4}{\pi} \sum_{p=1}^{\infty} (\pi p |c|)^{s-1/2} \cos(2\pi p a) K_{s-1/2}(2\pi p |c|).$$

(6.11)

In our equation I have $a = \vec{A}_0 \cdot \vec{w} \beta / 2\pi$, $c = n\beta / L + \vec{A}_3 \cdot \vec{w} \beta / 2\pi$ and $s \to s - 1$ and get (doing the sum over $m$)

$$\zeta(s) = \frac{1}{4\pi(s-1)} \left( \frac{\beta}{2\pi} \right)^{2s-2} \frac{\sqrt{\pi}}{\Gamma(s)} \sum_{\vec{w} \in \Delta_\omega} \sum_{n \in \mathbb{Z}} |n\beta / L + \vec{A}_3 \cdot \vec{w} \beta / 2\pi|^{3-2s} \Gamma(s-3/2)$$

(6.12)

$$+ \frac{4}{\pi} \sum_{p=1}^{\infty} (\pi p |n\beta / L + \vec{A}_3 \cdot \vec{w} \beta / 2\pi|)^{s-3/2} \cos(2\pi p \vec{A}_0 \cdot \vec{w} \beta / 2\pi) K_{s-3/2}(2\pi p |n\beta / L + \vec{A}_3 \cdot \vec{w} \beta / 2\pi|).$$

To find the effective perturbative bosonic potential I need only take the derivative of the overall divergent factor $\Gamma(s-1)^{-1}$ as taking derivatives of other terms will give zero at $s = 0$. I note that $\frac{d}{ds}|_{s=0} \Gamma^{-1}(s-1) =$
\( \psi(-1)/\Gamma(-1) = +1 \) (\( \psi(z) \) is the logarithmic derivative of \( \Gamma(z) \)) and so I can safely set \( s = 0 \). I notice that the first term is a Hurwitz zeta function which is related to a Bernoulli polynomial \( B_4(z) = -\frac{3}{\pi^4} \sum_{k=1}^{\infty} \frac{\cos 2\pi k z}{k^4} \) (times \( (L/\beta)^{2s-3} \)). The Hurwitz zeta function is related to Bernoulli polynomials in the following way

\[
\zeta_H(s, z) = \sum_{n=0}^{\infty} \frac{1}{(n + z)^s} = B_{1-s}(z)/(s - 1) = -\frac{\Gamma(2-s)}{(2\pi i)^{1-s}} \sum_{k \neq 0} e^{2\pi i k z} k^{1-s}.
\] (6.13)

so that \( \zeta_H(-3, \bar{A}_3 \cdot \bar{w}L/2\pi) = -\frac{1}{2} B_4(\bar{A}_3 \cdot \bar{w}L/2\pi) \) and I note that the first term is exactly that obtained previously on \( \mathbb{R}^3 \times S^1_\beta \) as required at zero temperature \([1], [39]: -\frac{z^2}{12\pi^3} B_4(\bar{A}_3 \cdot \bar{w}L/2\pi)) \). This cancels the fermionic contribution so I am left with the remaining terms. The second term in (6.12) can be simplified using \( K_{-3/2}(z) = \sqrt{\frac{2}{\pi}}(1 + z^{-1})e^{-z} \). Collecting terms together I can get an exact expression using polylogarithms \( L_{i\beta}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\ln^k} \) and that \( \Gamma(-3/2) = 4\sqrt{\pi}/3 \)

\[
V_{eff}^{pert, \text{bosonic}} = -\sum_{w \in \Delta_w} \frac{2}{\pi^2 L^4} \sum_{p=1}^{\infty} \cos pL \bar{A}_3 \cdot \bar{w} \frac{p}{p^4} - \sum_{n \in \mathbb{Z}} \sum_{p=1}^{\infty} \frac{e^{-2\pi p |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi}}{\pi^2 L^3 p^3} (1 + 2\pi p |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi) \cos(p\beta \bar{A}_0 \cdot \bar{w}).
\] (6.14)

I can alternatively write for the bosonic contribution:

\[
V_{eff}^{pert, \text{fermionic}} = \sum_{w \in \Delta_w} \left[ -\frac{2}{\pi^2 L^4} \sum_{p=1}^{\infty} \cos pL \bar{A}_3 \cdot \bar{w} \frac{p}{p^4} + \sum_{n \in \mathbb{Z}} \sum_{p=1}^{\infty} (-1)^p \frac{e^{-2\pi p |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi}}{\pi^2 L^3 p^3} (1 + 2\pi p |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi) \cos(p\beta \bar{A}_0 \cdot \bar{w}) \right]
\] (6.16)

Combining everything together I get

\[
V_{eff}^{pert} = -2 \sum_{p=1}^{\infty} [1 - (-1)^p] \sum_{w \in \Delta_w} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi p |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi}}{\pi^2 L^3 p^3} (1 + 2\pi p |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi) \cos(p\beta \bar{A}_0 \cdot \bar{w}).
\] (6.17)

I now look at the low temperature contribution and consider just the \( p = 1 \) term as in Chapter 3 as other terms are suppressed by higher powers of the Boltzmann factor \( e^{-mW/T} = e^{-\beta/L} \). The result is

\[
V_{eff}^{pert, \text{lowT}}(\bar{A}_0, \bar{A}_3) \approx -4 \sum_{w \in \Delta_w} \sum_{n \in \mathbb{Z}} \frac{e^{-2\pi |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi}}{\pi^2 L} (1 + 2\pi |n\beta/L + \beta \bar{A}_3 \cdot \bar{w}|/2\pi) \cos(\beta \bar{A}_0 \cdot \bar{w}).
\] (6.18)
Let me make a few comments about this perturbative effective potential. The effective potential is doubly periodic in the fields $\hat{A}^3_0$ and $\hat{A}^3_3$, with respective periods $2\pi/\beta$ and $2\pi/L$. These fields encode the centre symmetries $Z^{(\beta)}$ and $Z^{(L)}$. Although the $Z^{(\beta)}$ centre symmetry breaks at the deconfinement temperature, the $Z^{(L)}$ centre symmetry remains unbroken until temperatures beyond the deconfinement phase transition, yet still breaks at a temperature of $T \approx M_W = 2\pi/c_2(G)L$. Also, (6.18) is valid for $c_2(G)LT \ll 1$ and the mass of the scalar fields $\phi^a$ (i.e. $\hat{A}^3_a$) is exponentially suppressed by the Boltzmann factor $e^{-2\pi/c_2(G)L}$. At $T = 0$ the scalar fields are not massless, however, as I will find later the field gets an exponentially small mass from non-perturbative contributions $\approx e^{-4\pi^2/g^2}$, and these dominate at very low temperature. Also, note that finding the vacuum of the theory is not found by simply minimizing the perturbative effective potential (6.18) alone. When the non-perturbative sector of the theory, discussed in Section 6.3, is taken into account the thermal electric charges couple to magnetic charges and the total effective potential gets a non-perturbative contribution.

6.2.1 Cases of $\mathbb{R}^3 \times S^1_L$ and general cases

Case of $\mathbb{R}^3 \times S^1_L$ with non-zero gaugino mass

For use later on in Chapter 8 on the case of deconfinement on $\mathbb{R}^3 \times S^1_L$ with softly broken supersymmetry for general gauge group SYM I present a derivation of the perturbative GPY potential on $\mathbb{R}^3 \times S^1_L$ with non-zero gaugino mass, $m$. The four dimensional action for the gaugino field is given by

$$L = \frac{2}{g^2} \int_0^{2\pi R} dx^3 \int d^3x \left[ i\lambda_3 \bar{\sigma}^m \lambda_3 + m\frac{2}{3} \lambda_3 + m\frac{2}{3} \bar{\lambda_3} \bar{\lambda_3} \right] ,$$

where $D_m \lambda_3 = \partial_m \lambda_3 + i [A_m, \lambda_3]$. The group generators are taken in the fundamental representation with normalization $\text{Tr} X^a X^b = \delta_{ab}$. Note that (6.19) is consistent with the normalization of the dimensionally reduced action $S = \frac{1}{g^2} \int d^3x \left[ -(\partial_\mu \bar{\phi})^2 / L^2 - \frac{1}{2} (\bar{F}_{\mu \nu})^2 - 2i \bar{\phi} \cdot \sigma^\mu \partial_\mu \bar{\lambda} \right]$ for the Cartan components. Next, I expand $\lambda_3$ in Fourier modes along the $x^4$ direction, using the decomposition of the generators from Chapter 5, as follows:

$$\lambda_3(x^\mu, x^3) = \sum_{\beta \in \Delta^+_L} \left[ \hat{\lambda}^\beta_3(x^\mu) \cdot \hat{H} + \sum_{\bar{w} \in \Delta^+_L} \lambda^\beta_{\alpha, w}(x^\mu) E_w + \sum_{\beta \in \Delta^+_L} \lambda^{\alpha, p}_{\beta, w}(x^\mu) E_{-w} \right] e^{i \beta \cdot \bar{w}},$$

where $\Delta^+_L$ denotes the set of positive roots in the adjoint representation. Remembering now that $\hat{A}_3 = \frac{\hat{\phi} \cdot \hat{H}}{\mathcal{F}}$, and $[H^i, E_{\pm \beta}] = \pm \beta^i E_{\pm \beta}$ ($\beta^i$ is the $i$-th component of the weight/root in the adjoint representation $\beta$, for general representation I stick to the notation $\bar{w}$), I find

$$[A_3, \lambda_3] = \frac{1}{L} \sum_{p \in \mathbb{Z}} \sum_{\bar{w} \in \Delta^+_L} \beta^i \phi \lambda^p_{\alpha, w} E_w e^{i \bar{w} \cdot \phi} - \frac{1}{L} \sum_{p \in \mathbb{Z}} \sum_{\bar{w} \in \Delta^+_L} \bar{w} \cdot \phi \lambda^{\alpha, p}_{\beta, w} E_{-w} e^{i \bar{w} \cdot \phi} ,$$

and using $\text{Tr} [E_w E_{-w}] = \delta_{ww'}$ and $\text{Tr} [E_w E_{w'}] = 0$, I find that the contribution of the derivative along the compact direction is

$$\int_0^{2\pi R} dx^3 \lambda_3 \bar{\sigma}^3 \lambda_3 D_3 \lambda_3 = \sum_{p \in \mathbb{Z}} \sum_{\bar{w} \in \Delta^+_L} \left[ \frac{p}{\mathcal{F}} + \frac{\bar{w} \cdot \phi}{L} \right] \lambda^{\alpha, p}_{\beta, w} \bar{\sigma}^3 \lambda^{\alpha, p}_{\beta, w} + \text{c.c.} .$$
Similarly, I find that the derivatives in the noncompact directions contribute

\[
\int_{0}^{2\pi R} dx^{3} \tilde{\lambda}_{\alpha} \tilde{\sigma}^{\mu} \tilde{\alpha} \tilde{\alpha} D_{\mu} \lambda_{\alpha} = \sum_{p \in \mathbb{Z}} \sum_{\vec{w} \in \Delta^{+}} \tilde{\lambda}_{\alpha \mu}^{p} \tilde{\alpha} \tilde{\sigma}^{\mu} \tilde{\alpha} \partial_{\mu} \lambda_{\alpha \mu}^{p} + c.c + \text{higher order corrections}, \tag{6.22}
\]

with a similar expression for the mass term. Since the Lagrangian is quadratic in \(\lambda\), I can calculate the determinant easily. Collecting the above expressions, the mass term, and also performing a Fourier transform in the \(x^{\mu}\) directions, I obtain the determinant of the gaugino in the holonomy background as a sum over Kaluza-Klein modes and positive roots:

\[
\log \text{Det}_{\text{gaugino}} = 2 \sum_{p \in \mathbb{Z}} \sum_{\vec{w} \in \Delta^{+}} \int d^{3} k (2\pi)^{3} \log \left[ m^{2} + k^{2} + \left( \frac{p}{R} + \frac{\vec{w} \cdot \vec{\phi}}{L} \right)^{2} \right]^{-s}.
\]

Performing the \(k\) integral, then performing the sum using the zeta function, as indicated in the second line above, we find the massive gaugino contribution to the Gross-Pisarski-Yaffe (GPY) effective potential

\[
V_{\text{GPY}}^{(\text{gaugino})} = \frac{m^{2}}{\pi^{3} R} \sum_{p=1}^{\infty} \sum_{\vec{w} \in \Delta^{+}} \frac{K_{2}(2\pi p m R)}{p^{2}} \cos(p \vec{w} \cdot \vec{\phi}). \tag{6.23}
\]

Further, we use \(K_{2}(x) \big|_{x \to 0} \sim \frac{4}{x^{2}} - \frac{1}{2}\) to find:

\[
V_{\text{GPY}}^{(\text{gaugino})} \big|_{m \to 0} \approx \frac{4}{\pi^{3} L^{4}} \sum_{p=1}^{\infty} \sum_{\vec{w} \in \Delta^{+}} \frac{2}{p^{4}} \cos(p \vec{w} \cdot \vec{\phi}) - \frac{m^{2}}{2 \pi^{3} R} \sum_{p=1}^{\infty} \sum_{\vec{w} \in \Delta^{+}} \frac{\cos(p \vec{w} \cdot \vec{\phi})}{p^{2}}. \tag{6.24}
\]

The bosonic contribution cancels the \(m = 0\) part of the gaugino contribution (the first term above, which was previously calculated for general gauge groups in [4], was also derived here in Section 3.1 for finite mass in the \(SU(2)\) case), leaving us with the \(O(m^{2})\) GPY potential:

\[
V_{\text{GPY}}^{O(m^{2})} = -\frac{m^{2}}{\pi^{2} L^{4}} \sum_{p=1}^{\infty} \sum_{\vec{\beta} \in \Delta^{+}} \frac{\cos(p \vec{\beta} \cdot \vec{\phi})}{p^{2}}. \tag{6.25}
\]

**General compactification case**

For a general toroidially compactified spacetime of \(d\) compact directions and \(D - d\) non-compact directions I can also give a result for general gauge group extending the results of Chapter 3 for \(SU(2)\), even though the applications are not immediate.

The usual GPY potential for fields \(A_{i}^{a}\), in the Cartan subalgebra, for \(i = 1, \ldots, d\) representing the compact directions follows just as from equation (3.9) to give

\[
V_{\text{eff. pert.}}(\{A_{i}^{a}\}) = \sum_{\vec{n} \in \mathbb{Z}^{d}} \text{tr}_{\text{adj}} \frac{\cos(\sum_{i=1}^{d} n_{i} L_{i} A_{i} \cdot \vec{H})}{(\sum_{i=1}^{d} n_{i}^{2} L_{i}^{2})^{2}}, \tag{6.26}
\]
where the compact dimensions are of size $L_i$. For thermal theories a low-$T$ expansion can be done as in Section 3.1 in this general case with the simple replacement of a phase $\theta_M$ with a set of phases $\theta_M = \vec{\theta}_M \cdot \vec{H}$.

### 6.3 Non-perturbative Theory for General Gauge Group $G$

As done for the case of $SU(2)$ in Chapter 3 I derive the non-perturbative solutions and objects in the theory for general gauge group $G$ and determine the non-perturbative effective potential in this general case. Deriving it will rely on supersymmetry and the Kähler potential and superpotential will be found to construct it.

#### 6.3.1 Monopole and Bion Solutions for General Gauge Group

Just as in section 3.2 I describe the non-perturbative solutions to the action of our SYM theory but here for any gauge group $G$, including the monopole-instantons, of which there are $r = \text{rank}(G)$ BPS solutions and one KK solution, and the magnetic and neutral bions. Bions form only from interacting pairs of magnetic bions and so, taking charges in the co-root lattice of $G$, they only form from combinations of either the same charge type (forming neutral bions) or of neighbouring charges on the Dynkin diagram of $G$ (forming magnetic bions). I begin by constructing the monopole solutions from the $SU(2)$ solution found in section 3.2.1.

To find the monopole solutions for general gauge group we can embed the $SU(2)$ solution into $G$, $SU(2) \subset G$, for each simple co-root $\vec{\alpha}_i$. The KK monopole solution will be given later to give $r + 1$ monopole solutions as is consistent with the symmetry breaking $G \rightarrow U(1)^r$ (for the case of full Abelianization), which presents $r$ BPS solutions as $\pi_2(G/U(1)^r) \approx \pi_1(U(1)^r) \approx \mathbb{Z}$ (since $\pi_2(G) \approx \mathbb{Z}$ for covering spaces of Lie groups $\tilde{G}$, which we consider here). The KK solution arises due to the compact direction and will be given later associated to the affine co-root $\vec{\alpha}_0$.

The $SU(2)$ embedding into $G$ for each simple root $\vec{\alpha}_i$ is given by

$$t^1 = \frac{1}{\sqrt{2\vec{\alpha}_i^2}}(E_{\alpha_i} + E_{-\alpha_i}), \quad t^2 = \frac{1}{\sqrt{2\vec{\alpha}_i^2}}(E_{\alpha_i} - E_{-\alpha_i}), \quad t^3 = \frac{1}{2} \vec{\alpha}_i^\vee \cdot \vec{H},$$

(6.27)

which obey the $SU(2)$ algebra commutation relations $[t^a, t^b] = i\epsilon^{abc}t^c$. The solutions for the gauge field are the same as (3.23) but with

$$A_3 = \Psi^c \tau^c + (\vec{\phi} - \frac{1}{2} \vec{\alpha}_i^\vee v) \cdot \vec{H},$$

(6.28)

where $\vec{\phi}$ determines the asymptotics of the gauge field $v = \vec{A}_3 \cdot \vec{\alpha}_i = \vec{\alpha}_i \cdot \vec{\phi}/L$. The solution $A_3$ is as is to guarantee these asymptotics since (in string gauge)

$$\Psi^c \tau^c |_{|x| \rightarrow \infty} = \frac{x^c}{|x|} t^c \frac{\vec{\alpha}_i \cdot \vec{\phi}}{L} = t^3 \frac{\vec{\alpha}_i \cdot \vec{\phi}}{L} = \frac{1}{2} \frac{\vec{\alpha}_i \cdot \vec{\phi}}{L} \vec{\alpha}_i^\vee \cdot \vec{H}.$$
The BPS magnetic monopole’s magnetic field’s asymptotics are given by

\[ B^\alpha_\mu = -\frac{x_\mu}{|x|^3} \frac{\vec{\alpha}_i^\vee \cdot \vec{H}}{2}, \quad i = 1, \ldots, r. \]  

Its action and instanton number are given by

\[ S^{\alpha_i} = \frac{4\pi}{g^2} \vec{\alpha}_i^\vee \cdot \vec{\phi}, \quad K^{\alpha_i} = \frac{\vec{\alpha}_i^\vee \cdot \vec{\phi}}{2\pi}, \]  

respectively. Note that in the supersymmetric \( Z(G) \)-symmetric vacuum \( S^\alpha = \frac{8\pi^2}{c_2(G)g^2} \).

The other solution mentioned before, the KK monopole, can be found by a Weyl reflection as in [14]. Its asymptotic magnetic field is

\[ B^\alpha_\mu = -\frac{x_\mu}{|x|^3} \frac{\vec{\alpha}_0^\vee \cdot \vec{H}}{2}. \]  

Note that it has negative magnetic charge. (Also, since \( \vec{\alpha}_0^\vee = -\sum_{i=1}^{r} k_i^\vee \vec{\alpha}_i^\vee \) an instanton can be formed from the collection of \( 2c_2(G) \) monopoles.) Its action and monopole number are found to be [2]

\[ S^{\alpha_0} = \frac{4\pi}{g^2} (2\pi + \vec{\alpha}_0^\vee \cdot \vec{\phi}), \quad K^{\alpha_0} = \frac{2\pi + \vec{\alpha}_0^\vee \cdot \vec{\phi}}{2\pi}. \]  

### 6.3.2 Non-perturbative dynamics in general gauge group

We now are ready to examine the non-perturbative sector of the theory, its constituents and its dynamics, and derive the non-perturbative effective potential for any gauge group \( G \). This includes the effects of magnetic monopole-instantons ((anti) self-dual objects) which are charged in the co-root lattice \( \Lambda'_\vee \) of the Lie algebra \( \mathfrak{g} \), as well as exotic topological ‘molecules’: the neutral bions and the the magnetic bions (non self-dual objects). These enter into the path integral with action (6.2), and hence into the partition function of the theory. I begin by describing the zero temperature dynamics of such particles.

Due to the topology of gauge groups I find that for a gauge group \( G \) fully abelianized to \( U(1)^r \) there are \( r \) BPS monopole solutions as \( \pi_2(G/U(1)^r) \approx \pi_1(U(1)^r) \approx \mathbb{Z}^r \). Also, due to the compactness of the \( x^3 \)-coordinate there is another solution, as mentioned in Chapter 3, called the twisted or KK monopole. These solutions are (anti) self-dual objects localized in space and time, and I treat the dilute monopole gas case so we can ignore their internal structure and examine their long-range fields. The field of a single BPS (\( B\bar{P}S \)) monopole of type \( j, \quad j = 1, \ldots, r \), localized at the origin is, in the stringy gauge, [2] (Here \( A_\mu = A_\mu^a H^a \))

\[ A^\alpha_{0,BPS,\bar{B}PS} = \mp \frac{x_1}{r(r+x_2)} \vec{\alpha}_j^\vee \cdot \vec{H}, \]  

\[ A^\alpha_{1,BPS,\bar{B}PS} = \pm \frac{x_0}{r(r+x_2)} \vec{\alpha}_j^\vee \cdot \vec{H}, \]  

\[ A^\alpha_{2,BPS,\bar{B}PS} = 0, \]  

\[ A^\alpha_{3,BPS,\bar{B}PS} = (\frac{\pi}{L} - \frac{1}{r}) \vec{\alpha}_j^\vee \cdot \vec{H}, \]  

where \( r = \sqrt{x_0^2 + x_1^2 + x_2^2} \). These gauge field components are charged under the co-root lattice of the Lie
algebra and so are multiplied by $\vec{\alpha}_0^\vee$. These components of the gauge field give the correct asymptotics of the magnetic field at infinity as in (6.28).

The field for the KK ($\bar{K}K$) monopole similarly reads

$$A_{0,KK,\bar{K}K}^0 = \pm \frac{x_1}{r(r + x_2)} \alpha_0^\vee \cdot \vec{H},$$
$$A_{0,KK,\bar{K}K}^1 = \mp \frac{x_0}{r(r + x_2)} \alpha_0^\vee \cdot \vec{H},$$
$$A_{0,KK,\bar{K}K}^2 = 0,$$
$$A_{0,KK,\bar{K}K}^3 = \left(\frac{\pi}{L} + \frac{1}{r}\right) \alpha_0^\vee \cdot \vec{H}.$$  

These gauge fields have an additional charge factor $\alpha_0^\vee$, the affine co-root of $\mathfrak{g}$. See Section 6.3.1 for more on monopole-instanton solutions. These monopole-instantons carry magnetic charge $Q^a_m$ from Gauss’ law

$$\int_{S^2} d^2 \Sigma_{\mu} B^a_{\mu} = 4\pi Q^a_m, \quad a = 0, \ldots, r,$$  

where $B^a_{\mu} = \epsilon_{\mu\nu\lambda} \partial_{\nu} A^a_{\lambda} = Q^a_m \frac{\vec{x}_\mu}{r^2}$ is the magnetic field. I will write $Q^a_m = q^a_m \vec{\alpha}_0^\vee$ as the monopole charges belong to the co-root lattice of the Lie algebra of the gauge group, $\Lambda^\vee$. Monopoles of charge type $a$ and $b$ only interact when $\vec{\alpha}_a \cdot \vec{\alpha}_b \neq 0$, or, in other words, $a = b$ or they are nearby neighbours on the Dynkin diagram of $\mathfrak{g}$ (i.e. they correspond to non-zero elements of the Cartan matrix of the Lie algebra). See Chapter 5 for more on the Dynkin diagrams for each (affine) Lie algebra. As mentioned in the introduction, there is also a long-range scalar field (from the $A_3^a$ component of the gauge field) which can attract or repel these monopoles due to scalar charge interaction. There is further a topological charge of these monopole instantons $Q_T$ defined by

$$Q^{(i)}_T = (32\pi)^{-1} \int_{\mathbb{R}^3 \times S^1} F^{(i)}_{MN} F^{a(i)MN},$$

where $F^{a(i)}_{MN}$ is the field strength tensor due to the presence of a monopole of charge type $(i)$. Using the solutions (6.33) and (6.34) we can find the charges $(Q_m, Q_T)$ for each monopole type, which for $SU(2)$ were:

$$BPS (+1,1/2) \quad B\bar{P}S (-1,-1/2) \quad KK (-1,1/2) \quad K\bar{K} (+1,-1/2).$$

In general, as can be seen from Section 8.2, the values of the topological charge depend on the vacuum of the theory. There it is derived that the topological and magnetic charges, for monopoles of type $i$, are

$$Q^{(i)}_T = \frac{L}{2\pi} \sum_{w \in \Delta^{adj}_w} (\vec{w} \cdot \vec{\phi}_0)(\vec{w} \cdot \vec{\alpha}_i^\vee), \quad \vec{Q}^{(i)}_m = \vec{\alpha}_i^\vee.$$  

The topological charge clearly depends on the vacuum $\vec{\phi}_0$ of the theory, and usually gives fractional charges.

Due to the presence of fermions and supersymmetry (our gaugino), the Callias index theorem [8], [10] on $\mathbb{R}^3 \times S^1$ implies the existence of two adjoint fermionic zero modes attached to each monopole-
instanton. Section 8.2 describes this index theorem in detail for general gauge group. Let me use the fields \( \vec{\phi} \) instead of \( \vec{A}_3 \) and \( \vec{\sigma} \) instead of \( \vec{A}_1 \) and attach fermionic zero modes to get the ’t Hooft vertices
(The field \( \vec{z} = \vec{\phi} + i\vec{\sigma} \) is the lowest component of the chiral superfield \( \vec{X} \).)

\[
\mathcal{M}_{BPS,j} = e^{-4\pi^2/g^4} e^{\vec{z} \cdot \vec{\sigma} \cdot \vec{\lambda}} \
\mathcal{M}_{KK} = e^{-4\pi^2/g^4} e^{\vec{z} \cdot \vec{\sigma} \cdot \vec{\lambda}} 
\]

The indices of the theory inserts the contribution of fermionic zero modes and the long range fields (the \( e^{-\phi^\alpha + i\sigma^a} \) factors). These in themselves due not alter the vacuum structure of the theory as they are attached to fermionic zero modes and do not generate a potential for the fields \( \vec{\phi} \) and \( \vec{\sigma} \). Hence no mass will be generated for the dual photons \( \sigma^a \) from the monopole-instantons themselves, however the bions can contribute to the vacuum structure and indeed generate mass gap. I consider then the effect of the neutral and magnetic bions in the non-perturbative potential and their role in the deconfinement phase transition.

Let me now consider these non self-dual 'molecules' formed from monopole-instanton constituents. Their charges and amplitudes of the so-called magnetic and neutral bions that form are summarized below in Table 6.1. In all cases the index \( a \in \{0, \ldots, r\} \). A neutral KK bion would have \( a = 0 \) and is formed from a KK-anti-KK monopole pair. In the case of magnetic bions they are formed from two (BPS or KK) monopoles of charge types \( a \neq b \) as long as \( \vec{\alpha}_a \cdot \vec{\alpha}_b \neq 0 \), that is the monopoles are Dynkin neighbours on the Dynkin diagram of \( G \).

Note for the above cases we can have \( a \) or \( b = 0 \) allowing for molecules containing KK monopoles. The occurrences (when they occur) of \( \phi^0, \sigma^0 \) can be written in terms of the \( \phi^a, \sigma^a \) as the linear combination \( \sum_{a=0}^{r} k^a_a \vec{\alpha}^a = 0 \), with \( k^a_0 = 1 \), requires that \( \phi^0 = - \sum_{a=1}^{r} k^a_a \phi^a \), where \( k^a_0 \) are the dual Kac labels of the roots of the Lie algebra \( g \). The anti-bions are just the complex conjugates of these amplitudes, and have the negative of the charges of the bion. Since the magnetic bion carries no fermionic zero modes, it generates a potential for the fields \( \vec{\phi} \) and \( \vec{\sigma} \) and gives a mass to the dual photon fields \( \sigma^a \) and the theory can confine electric charges. These magnetic bions are stabilized by the attractive force due to exchange of adjoint fermionic zero modes giving them an effective radius \( r_* = 4\pi L/g^2 \). See [22] for more on bion structure. The fields of the magnetic bions, in the long distance approximation, can be found simply by adding the fields (6.33) and (6.34). For example, for the magnetic bion,

\[
A_{0}^{ab,\text{bion}} = -2 \frac{x_1}{r(r + x_2)} (\alpha_a^\vee - \alpha_b^\vee) \cdot \vec{H},
\]

\[
A_{1}^{ab,\text{bion}} = 2 \frac{x_0}{r(r + x_2)} (\alpha_a^\vee - \alpha_b^\vee) \cdot \vec{H},
\]

\[
A_{2}^{ab,\text{bion}} = 0,
\]

Table 6.1: Magnetic molecule vertices, charges, and amplitudes for different molecules.
where the Kähler potential is found from [2], to one loop quantum corrections, the effective action is then found from (see [31] for more on superpotentials and supersymmetry) this work. I shall again ignore the quantum corrections and so will not present the details here. After

\begin{equation}
A^{ab,\text{bion}}_3 = \frac{2\pi}{T} (\alpha'^b - \alpha'^a) \cdot \vec{H}.
\end{equation}

These gauge fields are accompanied by the differences of charges \( \vec{\alpha}'_a - \vec{\alpha}'_b \). Neutral bions involve combinations of monopoles and anti-monopoles and generate no long-range magnetic fields. They do however generate long range scalar fields and have imaginary charge \( 2i\vec{\alpha}'_a \) instead of the \( (\alpha'^a - \alpha'^b) \) charges for the magnetic bions above.

The neutral bions, however, are a little more tricky to see how they form. Usually an analytic continuation is required (a so-called BZJ prescription) to control the attractive forces of the monopole constituents, or a finite volume argument to make sure they are stable with finite size [4], [22], 'Resurgence' theory applies here as well [4]. Supersymmetry can also be invoked [2] to allow for their stability. Nonetheless, these objects are stable and generate a centre-stabilizing potential for confinement. Since the perturbative potential vanishes at \( T = 0 \), only the neutral bion-induced potential can lead to centre-stabilization.

The total effective potential of the non-perturbative contributions \( V_{\text{eff, pert.}}(\vec{\phi}, \vec{\sigma}) \) is found by adding the amplitudes in Table 6.1. As in [2] we can obtain the effective potential from the superpotential in terms of the chiral superfield \( \vec{X} \),

\begin{equation}
\mathcal{W} = \frac{\kappa L}{g^2} \mu^3 \left( \frac{2}{\bar{\alpha}'_j} e^{\bar{\alpha}'_j} \vec{X} + \frac{2}{\bar{\alpha}'_0} e^{\bar{\alpha}'_0} \vec{X} + 2\pi i\tau \right),
\end{equation}

where \( \kappa \) is a numerical factor that will not matter to us, \( \vec{X} \) is the chiral superfield with lowest component \( \vec{\phi} - i\vec{\sigma} \) and \( \tau \equiv i \frac{3\pi^2}{16} + \theta/2\pi \) will be taken with \( \theta = 0 \) here. In [2] it is shown that quantum corrections to the superpotential change the scale of the coupling to be not \( \Lambda_{PV} \) but rather \( \mu = 2/R \), with \( L = 2\pi R \). The effective action is then found from (see [31] for more on superpotentials and supersymmetry)

\begin{equation}
S(\vec{\phi}, \vec{\sigma}) = \int d^3x [K_{ij} \partial_{\mu} X^i \partial^\mu X^j + K^{ij} \partial X^i \partial X^j],
\end{equation}

where the Kähler potential is found from [2], to one loop quantum corrections, \( K^{ij} = \frac{16\pi^2 L}{g^2} [\delta_{ij} - \frac{3\pi^2}{16\pi^2} \sum_{w \in \Delta_{\mu+}} u_i w_j \psi(\bar{w} \cdot \vec{\phi}/2\pi) + \psi(1 - \bar{w} \cdot \vec{\phi}/2\pi)] \), where \( \psi(z) = \Gamma'(z)/\Gamma(z) \). The inverse Kähler metric has the overall coefficient inverted and the second term becomes negative. I shall ignore the one-loop corrections for now in this thesis. The non-perturbative effective potential due to bions is from the second term of (6.40) and so I get

\begin{equation}
V_{\text{bion}} = \frac{16\pi^2 L}{g^2} \delta^{ij} \frac{\partial \mathcal{W}}{\partial X^i} \frac{\partial \mathcal{W}}{\partial X^j} = 64\pi^2 \kappa^2 \left( \frac{2\pi R}{g^2} \right)^3 \left( \frac{2}{R} \right)^6 \left[ \sum_{i,j=1}^{r} \frac{\bar{\alpha}'_i \cdot \bar{\alpha}'_j}{\bar{\alpha}'_0} e^{\bar{\alpha}'_0} \vec{X} + \bar{\alpha}'_i \cdot \bar{\alpha}'_j \vec{X} + 2\pi i\tau \right],
\end{equation}

where \( \vec{X} = i(\tau \vec{\phi} + \vec{\sigma}) - \frac{3}{2} \sum_{w \in \Delta_{\mu+}} (\bar{w} \log \frac{\Gamma(\bar{w} \cdot \vec{\phi}/2\pi)}{\Gamma(1 - \bar{w} \cdot \vec{\phi}/2\pi)}) \) is the one-loop correction to the superfield as found in [2] and can be derived from results in its Appendix as well from the results in Section 8.2 of this work. I shall again ignore the quantum corrections and so will not present the details here. After
some algebra the non-perturbative effective potential becomes

\[ V_{\text{bion}} = V_{\text{bion}}^0 \sum_{i,j=0}^r k_i^{\vee} k_j^{\vee} \alpha_i^{\vee} \cdot \alpha_j^{\vee} e^{-\left(\alpha_i^{\vee} + \alpha_j^{\vee}\right) \cdot \vec{b}} \cos\left(\alpha_i^{\vee} - \alpha_j^{\vee}\right) \cdot \vec{\sigma}', \tag{6.43} \]

where \( \vec{\sigma}' \) and \( \vec{b} \) are the fluctuations about the supersymmetric vacuum, \( \vec{\phi}_0, \vec{\sigma}_0 \), and

\[ V_{\text{bion}}^0 = 16\pi^2 \kappa^2 \left( \frac{512\pi^3}{g^6 R^3 |v|^2} \right) e^{-16\pi^2/g^2 c_2(G)}, \tag{6.44} \]

with \( |v| = \left| \prod_{i=0}^r \left( \frac{k_i^{\vee} \alpha_i^{\vee} \vec{k}_i^{\vee}}{2} \right) ^{1/c_2(G)} \right| \). Note that this gives the \( SU(2) \) result with \( \alpha_0 = -\alpha_1 \) leading to \( \cosh 2\phi - \cos 2\sigma \) terms as found in the non-perturbative effective potential in [1].

The monopole terms carry two fermionic zero modes (the \( \lambda \)'s in (6.37)), and will not be considered here as there is no mass gap generation from such terms. However, with a finite mass for the gaugino these zero modes are lifted and the monopole-instantons participate in the vacuum structure of the theory and contribute to the effective potential. This will be considered in Chapter 8. With zero gaugino mass, however, I consider only the W-boson/photon and magnetic and neutral bion contributions to the effective potential.

The idea at zero temperature is that the \( Z(G)_L \) centre symmetry is unbroken as \( \langle \vec{\phi} \rangle = 0 \) minimizes the potential (6.42) (at zero temperature, and low \( T \) in general, we can ignore the perturbative, deconfining, effective potential (6.18) due to Boltzmann suppression of the W-bosons). There is also a mass gap for the dual photon, the \( \phi^a \)'s and \( \lambda^a \)'s have equal masses and the electric charges are confined.

In studying the deconfinement phase transition one could attempt to minimize the total effective potential \( V_{\text{eff}}^{\text{pert.}} + V_{\text{bion}} \) where \( V_{\text{eff}}^{\text{pert.}} \) is the perturbative effective potential of the W-bosons (6.18) and \( V_{\text{bion}} \) is (6.42). By varying the temperature one would look for a sudden change in the global minimum of the total effective potential and thus determine the deconfinement transition temperature and the order of the phase transition, as was done in [2]. It is harder in this case though as in [2] we were able to look only at the monopole and bion potentials and ignore the W-bosons at zero temperature. The perturbative effective potential here is much more complicated so this analysis may not be easy, and even using low-\( T \) approximations like (6.18) or others may not be valid near the deconfinement temperature. However, this still could be an exercise for future work. Let us consider what happens now at finite temperature.

**Monopole-instantons and bion structure at finite temperature**

From [1] I recall that the dual photon and \( \vec{\phi} \) fields have masses \( m_{\phi^a} = m_{\phi^a} \approx \frac{e^{-4\pi^2/g^2}}{L} \), and the fermions acquire a thermal mass \( \approx T \). The light gauginos do not participate in the deconfinement phase transition as they carry no electric, magnetic or scalar charge. However they do allow the formation of the bions and so have an indirect role in the transition. The heavy fermions (the winos) do participate, however, in a way similar to the W-bosons. The deconfinement transition temperature is found to be of the order \( T_c \approx \frac{g^2}{8\pi L} \) from both simulations and from setting the W-boson and magnetic bion fugacities to the same order [15]. This temperature is smaller then the inverse bion radius and so we
need not worry about the dissolution of the bions before the deconfinement transition and our gas of such particles persists beyond the deconfinement transition temperature. See [1] for more details. The best way to study the finite temperature dynamics of our gas of all these particles is to map it to a double Coulomb gas and examine its partition function. This I derive in the next Chapter.
Chapter 7

Deconfinement on $\mathbb{R}^2 \times S^1_\beta \times S^1_L$

In this Chapter I derive dualities useful for studying the deconfinement transition of SYM on $\mathbb{R}^2 \times S^1_\beta \times S^1_L$. These include the dual double Coulomb gas as well as the ‘affine’ XY model with symmetry breaking perturbations. This Chapter extends the results of Chapter 4 done for $SU(2)$. Although spin models are hard to come by for groups other than $SU(2)$ and $SU(3)$, I will still provide hints for finding them and studying them numerically on the lattice as future pursuits. Nonetheless, the duality to the Coulomb gas appears more possible to perform Monte Carlo simulations and it is one main result of this thesis. I would hope one may do the simulations of this dual double Coulomb gas and learn much of the nature of its deconfining phase transition. These results could then be compared to the quantum phase transition results of the next Chapter, and provide support for the continuity conjecture to be discussed in Section 8.1.

7.1 Dual Double Coulomb Gas to (Super) Yang-Mills with General Gauge Group

To derive our electric-magnetic Coulomb gas dual to our theory I will follow a derivation done in [6] for the finite temperature 3D Polyakov model. I go back to our field $F^a_{\mu
u}$ instead of $\sigma^a$ and add to the perturbative photon fluctuations the contribution of the magnetic field of our magnetic bions (and anti-magnetic bions). This involves calculating the electric W-boson (and wino) determinant in the multi-instanton/anti-instanton background. I can similarly find the W-boson determinant in the background of the neutral bions (both the W’s and neutral bions carry scalar charge due to their coupling to the scalar fields $\phi^a$). In the end I find a duality of our model to a dual double electric-magnetic Coulomb gas with also scalar charges which couple to the fields.

Hence I begin by splitting the fields into photon fluctuation and magnetic field components

$$F^a_{\mu\nu} = F^a_{\mu\nu}^{\text{bion}} + F^a_{\mu\nu}^{\text{ph}}$$

$$A^a_{\mu} = A^a_{\mu}^{\text{bion}} + A^a_{\mu}^{\text{ph}},$$

where $A^a_{\mu}^{\text{bion}} = \sum_{i,q_i=\pm 1} q_i A^a_{\mu}^{\text{bion}}(x-x_i)$ splits into a sum of an arbitrary numbers of bions and anti-
bions at positions \( x_i \in \mathbb{R}^3 \), and \( A^{a,\text{bion}}_\mu \) is from (6.39). At finite temperature we have \( \beta \) finite and so we must sum up an infinite number of image charges in the 0-direction and so

\[
A^{a,\text{bion}}_\mu = \sum_{a,q_a=\pm 1} \sum_{n \in \mathbb{Z}} q_a A^{a,\text{bion}}_\mu (\bar{x} - \bar{x}_a, x_0 - x_{0,a} + n \beta).
\]

(7.2)

The partition function of our Coulomb gas is the path integral of our field theory with path integrals over the gauge fields \( A^{a,\text{ph}}_\mu \) and the scalar fields \( \phi^a \), and sums over arbitrary numbers \( N_{b\pm} \) of magnetic bions, as well as \( N_W \) W-bosons and their superpartners. There is also a sum over 'colours', that is the sum over components of the Cartan subalgebra of \( g \). In the action I integrate the photon fields in the background of magnetic bions. To do this I simply replace the argument of the cosine of the potential (6.18) with the integral

\[
\oint_{S^1_B} \sum_{a,bion} q_i \int_0^\beta A^{a,\text{bion}}_\mu (\bar{x} - \bar{x}_a, x_0 - x_{0,a} + n \beta) = \sum_{a,q_a=\pm 1} q_a \int_{-\infty}^{\infty} A^{a,\text{bion}}_0 (\bar{x} - \bar{x}_a, x_0),
\]

(7.3)

which, using equation (3.8), I find as in [1] the integral above to be

\[
\oint_{S^1_B} \sum_{a,bion,ij} q_i \int_0^\beta \sum_{i,j=0}^r (\alpha_i^\vee - \alpha_j^\vee) q_a \Theta(\bar{x} - \bar{x}_a),
\]

(7.4)

where \( \Theta(\bar{x}) = -\text{sgn}(x_1)\pi/2 + \tan^{-1} \frac{x_2}{x_1} \) is the angle in the \( x_1-x_2 \)-plane between \( \bar{x} \) and the magnetic bion of type \( ij \) at position \( \bar{x}_a \). From here on \( A, B \) denote the positions of the W-bosons, and \( a, b \) the positions of the magnetic bions.

Using this term in the cosine in the action with GPY potential (6.18) I can write the action of the partition function in the grand canonical ensemble for bions at positions \( a \) as

\[
S = \int_{S^1_B \times \mathbb{R}^2 \times S^1_L} \frac{L}{2g^2} (\partial_{\mu} \tilde{\phi})^2 - \frac{L}{4g^2} (\tilde{F}^{ph}_{\mu \nu} + \tilde{F}^{\text{bion}}_{\mu \nu})^2 - \sum_{a} 2T_W (\tilde{\phi}) \sum_{i,j,k} (\alpha_i^\vee - \alpha_j^\vee) \cdot \tilde{A}_{k \mu} q_\alpha \Theta(\bar{x} - \bar{x}_a) + \oint_{S^1_B} dx_0 A^{a,\text{ph}}_\mu + V_{\text{neutral bion}}(\bar{x}_a),
\]

where \( V_{\text{neutral bion}}(\bar{x}_a) \) is given by (6.42) for neutral bions \( (a = b) \) with the fields at \( \bar{x}_a \). Note the dependence of the W-boson fugacity, \( \xi_W \), on \( \tilde{\phi} \). The grand canonical partition function \( Z_{\text{grand}} = \int D\tilde{\phi} \int D\tilde{A}_\mu^{a,\text{ph}} e^{-S} \) can be expanded using the equation

\[
\exp[2\xi \int dx \cos(f(x))] = \sum_{n_+,n_-=0}^{\infty} \frac{2\pi}{(n_+)! (n_-)!} \prod_{i=1}^{n_+ + n_-} \int dx_i e^{x_i f(x_i)},
\]

(7.6)

and putting in \( V_{\text{neutral bion}} \) I rewrite the partition function as

\[
Z_{\text{grand}} = \sum_{N_{b\pm}} \sum_{N_W} \prod_{a} \int d^3x_a \prod_{A} \int d^3x_A \int D\tilde{\phi} \int DA^{a,ph}_\mu \frac{\xi_W(\phi^a)_{N_{b\pm}}}{N_{b\pm}! N_{b\pm}!} \times
\]

(7.7)
\[ \frac{(T\xi_{W}(\phi^{a}))^{N_{W}+N_{W}}}{N_{W}!N_{W}!} \exp\left[ \int_{\mathbb{R}^{3} \times S^{3}} \frac{L}{2g^{2}} (\partial \phi^{a})^{2} + 2i \sum_{i,j,q_{a}=\pm} (\alpha_{i}^{\gamma} - \alpha_{j}^{\gamma}) q_{a} \Theta(\mathbf{x} - \mathbf{x}_{a}) - \int_{\mathbb{R}^{3} \times S^{3}} \frac{L}{4g^{2}} (\bar{F}_{\mu \nu} + F_{\mu \nu}^{bion})^{2} \right] \]

\[-i \sum_{A} \sum_{a=0}^{r} q_{A} \bar{\alpha}_{a} \cdot \bar{A}_{A}^{bion}(\mathbf{x}, x_{0}) \delta(\mathbf{x} - \mathbf{x}_{A}) + V_{bion}^{0} \sum_{a=0}^{r} (k_{a}^{\gamma} \alpha_{a}^{\gamma})^{2} \exp\left( - \frac{8\pi^{2}}{g^{2}} \bar{\alpha}_{a}^{\gamma} \phi^{a} \right) \].

It is clear that by writing a path integral \( \int \mathcal{D} \phi \) (or \( \int \mathcal{D} A_{\mu}^{a,ph} \)) is equivalent to integrating over \( \bar{b} \) introduced earlier. Also I wrote the bion fugacity as \( \xi_{b}(\phi^{a}) = V_{bion}^{0} = 16\pi^{2} \kappa^{2}(\frac{3125pi^{3}}{g^{2}R^{3}v^{2}}) e^{-16\pi^{2}/(g^{2}c_{2}(G))} \), and the fugacity of the W-bosons is from the Boltzmann distribution \( \xi_{W} = 2 \int \frac{d^{2}p}{(2\pi)^{2}} e^{-m_{W}/T - p^{2}/2m_{W}T} = 2 \frac{m_{W}}{2\pi} e^{-m_{W}/T} \), and \( m_{W} = 1/c_{2}(G)L \) is the mass of the lightest W-bosons. This W-boson fugacity actually depends on \( \phi^{a} \) in general, as found from the calculation (6.18) of the W-boson determinant. Recall

\[ \xi_{W}(\phi^{a}) = \frac{2}{g^{2}} \pi^{2} \sum_{n \in \mathbb{Z}} \sum_{w \in \Delta_{\mathbb{R}}} e^{-\frac{g^{2}}{4}(2n+1)^{2} + g^{2} \phi \cdot \bar{w}/4\pi} (1 + \frac{\beta}{L} \frac{g^{2} \phi \cdot \bar{w}}{4\pi})(7.8) \]

\[ = \sum_{w \in \Delta_{\mathbb{R}}} \frac{2}{\beta L \sinh(\beta \pi / L)} [(\coth \frac{\beta}{L} + \frac{L}{\beta}) \cosh \frac{g^{2} \phi \cdot \bar{w}}{4\pi L} - \frac{g^{2} \phi \cdot \bar{w}}{4\pi^{2}} \sinh \frac{g^{2} \phi \cdot \bar{w}}{4\pi L}] \]

where I assume that \( \phi^{a} \) lies in the Weyl chambr \( \alpha_{i} \cdot \phi \geq 0 \) for each \( \alpha_{i} \) simple, and \( -\alpha_{0} \cdot \phi \leq 1 \). For small values of \( \phi^{a} \approx 0 \) near its minimum I find the contribution to \( \xi_{W}(\phi) \) is dominated by the \( n = 0, -1 \) terms and we are left with (at the minimum \( \phi^{a} = 0 \)) \( \xi_{W} \approx \frac{4}{\beta L} e^{-\beta \pi / L} \frac{4c_{2}(G)m_{W}T}{\pi} e^{-c_{2}(G)m_{W}T} / \beta L \), as expected, but I get \( 2c_{2}(G) \) times the usual Boltzmann factor due to the fact that there are \( c_{2}(G) \) lightest Kaluza-Klein modes with the same mass \( m_{W} = \pi / c_{2}(G)L \) (due to the unbroken centre symmetry), and the wino superpartners have the same mass as well and contribute equally. I also represent the charges as vectors \( q_{X,i} = \alpha_{i}^{(v)} \) for electric and magnetic charges, respectively, for charge type (‘colour’) \( i \). The last term in the exponent of (7.7) is the contribution from the neutral bions introduced in the previous Chapter.

Next, a duality transformation can be done as in [6] for the gauge field \( A_{\mu}^{a,ph} \) (and consider only the zero mode of its \( \beta \)-component in the low-\( T \) approximation) to exchange the gauge fields \( F_{\mu \nu} \) for scalar fields \( \sigma^{a} \). This goes as follows:

\[ S_{cl} = \frac{L}{g^{2}} \int d^{3}x \frac{1}{L^{2}} (\partial_{\mu} \phi^{a})^{2} - \frac{1}{2} (\bar{F}_{\mu \nu})^{2} + 2i \lambda \cdot \bar{\sigma}_{\mu} D_{\mu} \bar{\lambda} - i \frac{\theta}{8\pi^{2}} \int d^{3}x \epsilon_{\mu \nu \rho} \partial_{\mu} \phi^{a} \cdot \bar{F}_{\nu \rho} \]
now reads
\[
S_{\text{free-bosonic}} = \frac{1}{L} \int d^3x \left[ \frac{1}{g^2} (\partial_\mu \tilde{\phi})^2 + \frac{g^2}{16\pi^2} (\partial_\mu \tilde{\sigma} + \frac{\theta}{2\pi} \partial_\mu \tilde{\phi})^2 \right] = (4\pi L)^{-1} \int d^3x \partial_\mu \tilde{z}^a \cdot \partial_\mu \tilde{z}^a / \Im \tau, \quad (7.10)
\]
where \( \tilde{z} = i(\tau \tilde{\phi} + \tilde{\sigma}) \) is the lowest component of the superfield \( \tilde{X} \) with fermionic component \( \tilde{X} \). I take \( \theta = 0 \) in the sequel. Using this duality, the free bosonic part of our Lagrangian becomes:
\[
S_{\text{free-bosonic}} = \beta \int d^2x \frac{L}{2g^2} [(\partial_\mu \tilde{\phi})^2 + (\partial_\mu \tilde{\sigma})^2]. \quad (7.11)
\]

To evaluate the path integral over the gauge fields I can go one step further, as done for the case of \( SU(2) \) in Section 4.1, and enhance the \( U(1)^r \) symmetry of the fields \( \sigma^a \) to \( U(1)^{2r} \) by introducing another set of scalar fields \( \lambda^a \) and new fields \( \Phi^a_\mu \) and \( \theta^a [6] \), which transform under the two \( U(1)^r \) symmetries as
\[
\partial^a \rightarrow \partial^a - \lambda^a, \quad \Phi^a_\mu \rightarrow \Phi^a_\mu + \partial_\mu \lambda^a, \quad A^a_\mu \rightarrow A^a_\mu + \partial_\mu \sigma^a.
\]
The path integral over the gauge fields becomes
\[
\int D\tilde{A}_\mu e^{-\int d^2x (\tilde{F}^\mu_\nu)^2 / 4g_3^2} = \int D\tilde{A}_\mu D\tilde{\Phi}_\mu D\tilde{\sigma} \exp[-\int d^2x (g_3^2 (\partial_\mu \tilde{\phi} + \tilde{\Phi}_\mu)^2 / 2 + i \sum_{\mu \nu \rho} \epsilon_{\mu\nu\rho} \partial_\mu \tilde{A}_\nu \cdot \tilde{w} \tilde{F}_\rho \cdot \tilde{w})], \quad (7.12)
\]
where \( g_3 \) is the effective three dimensional coupling. Taking \( \tilde{\sigma} = 0 \) (unitary gauge) and varying the new action with respect to \( \tilde{\Phi}_\mu \) gives \( \tilde{\Phi}_\mu = -i \epsilon_{\mu\nu\rho} \partial_\nu \tilde{A}_\rho / g_3^2 \), which once substituted in the Lagrangian in (7.12) yields the original Lagrangian \( (\tilde{F}^\mu_\nu)^2 / 4g_3^2 \). I now evaluate (I used \( \sum_{\mu \nu \rho} \tilde{X} \cdot \tilde{w} \tilde{Y} \cdot \tilde{w} = \tilde{X} \cdot \tilde{Y} \) as done before)
\[
\int D\tilde{A}_\mu \int D\tilde{\Phi}_\mu \exp[-\int d^2x \left[ \frac{1}{4g_3^2} (\tilde{F}^\mu_\nu)^2 + \frac{1}{2g_3^2} \tilde{F}^\mu_\nu \cdot \tilde{F}_\mu^{\text{bion}} + \frac{g_3^2}{2} \tilde{F}^2 + i \epsilon_{\mu\nu\rho} \partial_\mu \tilde{A}_\nu \cdot \tilde{F}_\rho \right]] \quad (7.13)
\]
\[
\times \exp[-\int d^2x \left[ \sum_{a=0}^r \sum_{A} q^a_\lambda \tilde{A}_a^{\text{ph}}(\tilde{x}, x_0) \delta(\tilde{x} - \tilde{x}_A) \right]].
\]

Using the new action in (7.12) allows me to work with \( S_{\text{aux}} \) given below where i substituted for \( \tilde{F}^\mu_\nu / 4g_3^2 \):
\[
S_{\text{aux}} = \int d^3x \left[ \frac{1}{4g_3^2} (\tilde{F}^\mu_\nu)^2 + \frac{1}{2g_3^2} \tilde{F}^\mu_\nu \cdot \tilde{F}_\mu^{\text{ph}} + \frac{1}{2} g_3^2 \tilde{F}^2 + i \epsilon_{\mu\nu\rho} \partial_\mu \tilde{A}_\nu \cdot \tilde{F}_\rho - i \sum_{a=0}^r \sum_{A} q^a_\lambda \tilde{A}_a^{\text{ph}}(\tilde{x}, x_0) \delta(\tilde{x} - \tilde{x}_A) \right]. \quad (7.14)
\]
Varying this with respect to \( \tilde{A}_a^{\text{ph}} \) gives \( i \epsilon_{\mu\nu\rho} \partial_\nu \tilde{F}_\rho^{\text{ph}} / g_3^2 = -i \sum_{a=0}^r \sum_{A} q^a_\lambda \tilde{A}_a^{\text{ph}}(\tilde{x} - \tilde{x}_A) \partial_\nu \), which has solution \( \tilde{\Phi}_\mu = i \tilde{B}_\mu / g_3^2 + \tilde{b}_\mu \), where \( \tilde{B}_\mu = \epsilon_{\mu\nu\rho} \tilde{F}_\rho^{\text{ph}} / 2 \) is the magnetic field due to the background of magnetic bions and \( \tilde{b}_\mu \) splits into its divergence and curl free parts: \( \tilde{b}_\mu = \tilde{\partial}_\mu \tilde{\phi} + \epsilon_{\mu\nu\rho} \partial_\nu \tilde{C}_\rho \), with \( \partial_\nu \tilde{C}_\rho = 0 \). Putting \( \tilde{b}_\mu \) back in \( S_{\text{aux}} \) shows that \( \tilde{\sigma} \) drops out and the equation of motion for \( \tilde{C}_\mu = \nabla^2 \tilde{C}_\mu = -\sum_{a=0}^r \sum_{A} q^a_\lambda \tilde{A}_a^{\text{ph}}(\tilde{x} - \tilde{x}_A) \partial_\mu \), introducing a Green’s function \( G(x) \) on \( \mathbb{R}^2 \times S_3 \) satisfying \( \nabla^2 G(\tilde{x} - \tilde{x}', x_0 - x'_0) = -\delta^2(\tilde{x} - \tilde{x}') \delta(x_0 - x'_0) \) with solution \( G(\tilde{x} - \tilde{x}', x_0 - x'_0) = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{(\tilde{x} - \tilde{x}')^2 + (x_0 - x'_0 + n\beta)^2}} \)
gives the solution for $\vec{C}_\mu$,

$$
\vec{C}_\mu = \sum_{a=0}^{r} \int d^3x' \sum_A q_A^a \sigma_a \partial_0 \delta^2 (\vec{x}' - \vec{x}_A') G(\vec{x} - \vec{x}_A', x_0 - x_0') \tag{7.15}
$$

$$
= \frac{1}{4\pi} \sum_{a=0}^{r} \sum_A q_A^a \sigma_a \partial_0 \int_{S^3_B} d\vec{x}_0 \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{(\vec{x} - \vec{x}_A)^2 + (x_0 - x_0^a + n\beta)^2}} = -\delta_{\mu 0} \sum_{a=0}^{r} \sum_A q_A^a \sigma_a \partial_0 \log |\vec{x} - \vec{x}_A|.
$$

The solution for $\vec{\Phi}_\mu = i\vec{B}_\mu / g_3 + \vec{b}_\mu + \vec{K}_\mu$ can also be found where a term $\vec{K}_\mu = -\frac{e_{\mu\nu}}{2\pi} \sum_{a=0}^{r} \sum_A q_A^a \sigma_a \partial_\mu \log |\vec{x} - \vec{x}_A|$.

Substituting this into $S_{\text{aux}}$ and integrating by parts yields

$$
S_{\text{aux}} = \int d^3x \frac{1}{2} q_3^2 |(\partial_\mu \vec{\sigma})^2 + \vec{K}_\mu^2| - i\vec{\sigma} \cdot \partial_\mu \vec{B}_\mu + i\vec{B}_\mu \cdot \vec{K}_\mu - g_3^2 \vec{\sigma} \cdot \partial_\mu \vec{K}_\mu, \tag{7.16}
$$

where the last term is zero from the asymmetry of the $\epsilon_{\mu\nu\lambda\rho}$ and $\partial_\mu \vec{B}_\mu = 4\pi \sum_{a=0}^{r} \sum_i q_i^a \sigma_i^a \delta^2 (x - x_i^a)$ from Gauss' law. I can then write $\vec{B}_\mu = \sum_{a=0}^{r} \sum_i q_i^a \sigma_i^a \delta^2 \frac{(x - x_i^a)}{|x - x_i^a|^2}$ where $(p)$ denotes the periodicity enforced along the $S^1_B$. The next term is $\vec{B}_\mu \cdot \vec{K}_\mu$ in $S_{\text{aux}}$ can be evaluated as well and is seen to be zero:

$$
\int d^3x \vec{B}_\mu \cdot \vec{K}_\mu = - \int d^3x \sum_{a,b=0}^{r} \sum_{iA} q_i^a q_i^b \sigma_i^a \cdot \sigma_b \int_{S^3_B} d\vec{x}_0 \sum_{n \in \mathbb{Z}} \frac{\epsilon_{kl}(x - x_i^a)k(x - x_i^b)}{|\vec{x} - \vec{x}_A|^2 |(x - x_i^a)^2 + (x_0 - x_0^a + n\beta)^2|^{3/2}}
$$

$$
= -2 \int d^3x \sum_{a,b=0}^{r} \sum_{iA} q_i^a q_i^b \sigma_i^a \cdot \sigma_b \epsilon_{kl}(x - x_i^a)k(x - x_i^b)\frac{|\vec{x} - \vec{x}_A|^2 |(x - x_i^a)^2 + (x_0 - x_0^a + n\beta)^2|^{3/2}}{|\vec{x} - \vec{x}_A|^2 |(x - x_i^b)^2|} = 0
$$

by symmetric integration under asymmetric $\epsilon_{ij}$.

What remains is the $\vec{K}_\mu$ term in $S_{\text{aux}}$, and I will see this gives rise to the Coulomb interactions of the W-bosons.

$$
\int d^3x \vec{K}_\mu^2 = \frac{1}{(2\pi)^2 T} \sum_{a,b=0}^{r} \sum_{AB} q_A^a q_B^b \sigma_a \cdot \sigma_b \int d^3x \partial_i \log |\vec{x} - \vec{x}_A| \partial_i \log |\vec{x} - \vec{x}_B| \tag{7.17}
$$

$$
= \frac{1}{(2\pi)^2 T} \sum_{a,b=0}^{r} \sum_{AB} q_A^a q_B^b \sigma_a \cdot \sigma_b \int d^3x \frac{(x - x_A)_l (x - x_B)_l}{|x - x_A|^2 |x - x_B|^2} = -\frac{1}{2\pi T} \sum_{a,b=0}^{r} \sum_{AB} q_A^a q_B^b \sigma_a \cdot \sigma_b \log (T|x_A^a - x_B^b|).
$$

Further, varying $S_{\text{aux}}$ with respect to $\vec{\sigma}$ gives $g_3^2 \nabla^2 \vec{\sigma} = -4\pi i \sum_{a=0}^{r} \sum_i q_i^a \delta^2 (x - x_i^a)$, which has solution $\vec{\sigma} = \frac{4\pi i}{g_3^2} \sum_{a=0}^{r} \sum_i q_i^a \sigma_i^a G(\vec{x} - \vec{x}_i^a, x_0 - x_0^a)$. Putting this back into $S_{\text{aux}}$ gives $S_{\text{aux}} = \frac{8\pi}{g_3^2} \sum_{a,b=0}^{r} \sum_{ij} q_i^a q_j^b \sigma_i^a \cdot \sigma_j^b \sigma_i^a \sigma_j^b G(\vec{x}_i^a - \vec{x}_j^b, x_0^a - x_0^b) - \frac{4\pi i}{g_3^2} \sum_{a,b=0}^{r} \sum_{AB} q_A^a q_B^b \sigma_a \cdot \sigma_b \log (T|x_A^a - x_B^b|)$. This gives us our final partition function for the double Coulomb gas once the $\vec{\phi}$ terms are restored (reminding ourselves that the W-boson fugacity depends on $\phi$ once we turn it on). Putting everything together I get the final result

$$
Z_{\text{grand}} = \sum_{N_{A+}, N_{A-}, N_{B+}, N_{B-}} \frac{\xi_{N_{A+} + N_{A-}}}{N_{B+} + N_{B-}!} \prod_i \int d^3x_i \frac{(T\xi_W(\phi))^{N_W+ + N_W-}}{N_W+! N_W-!} \prod_A \tag{7.17}
$$

$$
\int d^3x_A \int D\vec{\sigma} \exp \left( \frac{32\pi LT}{g^2} \sum_{a,b=0}^{r} \sum_{ij} q_i^a q_j^b \sigma_i^a \cdot \sigma_j^b \log |x_i^a - x_j^b| + \frac{g_3^2}{2\pi LT} \sum_{A>B} q_A^a q_B^b \sigma_a \cdot \sigma_b \log |x_A^a - x_B^b| + \right)
$$
Figure 7.1: A sketch of our dual 2D Coulomb gas of magnetic bions, monopoles and other particles. The monopole-instantons carry two fermionic zero modes and hence do not participate in determining the dynamics of the deconfining phase transition or the vacuum structure of the theory. One should add W bosons to the picture as well for the full vacuum.

\[ 4i \sum_{i,A} q_i^a q_A^b \bar{\alpha}_a \cdot \bar{\alpha}_b \Theta(x_A^b - x_i^m) + \int_{\mathbb{R}^2} \left[ \frac{1}{2} \frac{g^2}{(4\pi)^2 L} (\partial_\mu \bar{\phi})^2 + V_{\text{bion}}^0 \sum_{a=0}^{r} (k^a_{\lambda} \bar{\alpha}_{a} \lambda)^2 \exp\left(-\frac{8\pi}{g^2} \bar{\alpha}_a \cdot \bar{\phi}\right) \right], \]

which is valid for all \( T \) with \( 0 \leq T < M_W \), and I used \( g_3 = g/L \) and the long-distance property of the Green’s function where it behaves like a logarithm.

Let me make a few comments about the results so far. Note already the usual Coulomb-Coulomb interactions between electric W-bosons and magnetic bions, as well as the Aharonov-Bohm interaction given by the \( \Theta \) term as in [1]. One further point to consider is the dependency of the W-boson fugacity on the fields \( \phi^a \). In the case of circle compactification where the \( \bar{\phi} \) field is absent (as in the zero temperature limit) there is, as in the \( SU(2) \) case, a Kramers-Wannier duality \( 32\pi LT/g^2 \rightarrow g^2/2\pi LT \) as in the sine-Gordon model which is the zero-temperature limit of our theory. Figure 7.1 shows such a gas of magnetic bions and monopole-instantons with the fermionic zero modes attached - the non-perturbative vacuum of the theory. The magnetic monopoles are not present in the partition function of our Coulomb gas, as these I ignore as they do not contribute to the dynamics of the deconfinement phase transition and the vacuum structure of the theory. The magnetic monopoles still interact with a potential similar to the W-bosons: \( V_{m-m} = \frac{4\pi}{LTg^2} \sum_{i,j=1}^{N_{m}} \sum_{q, q_1 = \pm}^{r} \sum_{a, b=0}^{r} q_i^a q_j^b \bar{\alpha}_a \cdot \bar{\alpha}_b \ln |x_i^m - x_j^m| \). Note also the hierarchy of scales in the effective 2D Coulomb gas: \( r_m \approx M \ll r_b \approx L/g^2 \ll d_{m-m} \approx L e^{2\pi^2/g^2} \ll d_{b-b} \approx L e^{4\pi^2/g^2} \) of monopole size, bion size, monopole-monopole separation distance, and bion-bion separation distance respectively. This holds at weak coupling and shows that the vacuum partition function is truly that of an effective 2D dilute Coulomb gas of monopoles and bions. Note that the hierarchy fails at strong coupling and the Coulomb gas ‘collapses’, showing the importance of weak coupling to our duality.
The $r$ W-bosons of the theory can be written in terms of the scalar and photon fields:

$$\vec{W}^\pm = \vec{\phi}^\pm i\vec{\sigma}.$$ 

These particles can be thought of as having two charges: scalar and electric. The electric charges belong as usual to the root lattice $\mathbb{Z}[\{\vec{\alpha}_i\}] = \Lambda_r$, whereas the scalar charges belong to the imaginary root lattice $i\mathbb{Z}[\{\vec{\alpha}_i\}] = i\Lambda_r$ (the magnetic charges belong to the co-root lattice of $G$, $\Lambda_r^\vee$).

To explain this, one could go one step further and evaluate the path integral over the $\vec{\phi}$ field, but this proves difficult for general gauge group. This was done in Chapter 4 for the case of $SU(2)$ however. The difference here is in trying to expand the neutral bion potential into source terms of the $\vec{\phi}$ fields, which is not simply done. However, one can still use the methods of simulating the Coulomb gas (4.18) for $SU(2)$ as was detailed in Section 4.1.1. Compared to the $SU(2)$ case I note many similarities of the partition function to that of general gauge group. Firstly, we still have a gas composed of W-bosons, of which there are $\text{rank}(G)$ different electric charges belonging to the root lattice of $G$, $\Lambda_r$, as well as magnetic bions of magnetic charge 2 (which can be found from formula (6.35) for example with Dynkin-neighbouring charges $\vec{\alpha}_a^\vee - \vec{\alpha}_b^\vee$ in the co-root lattice $\Lambda_r^\vee$ of $G$). The neutral bions can still be thought of having scalar charge 2 from the $e^{-2\vec{\alpha}_a^\vee \cdot \vec{\phi}}$ amplitude from the scalar fields $\phi^a$, and where like scalar charges attract. This is unlike the magnetic bions where a cosine type potential leads to dynamics where opposite charges attract. Note though how the picture of neutral and magnetic bions is not as crystal clear and simple as in the $SU(2)$ case.

This Coulomb gas can be subjected to lattice study as in [1] for the case of $SU(2)$, but for other gauge groups. Perhaps extending first the results to $SU(3)$ and $SU(N)$ would be a start in future research. See [1], [53], [54] for more on the Monte-Carlo simulations used in studying such Coulomb gases numerically. Another method of studying the deconfinement phase transition other than simulating the Coulomb gas is to map the Coulomb gas constituents to parameters of a dual spin model. The spin model that best suits the Coulomb gas at hand is a multiple component XY spin model with symmetry breaking perturbations and fugacities coupled to the scalar field $\vec{\phi}$. This I turn to now in the next Section.

### 7.2 Dual Affine XY Model to Double Coulomb Gas with General Gauge Group

Before commenting on spin models in general gauge group, first let me consider the $SU(N)$ cases and the issues that arise, beginning with $SU(2)/\mathbb{Z}_2$. Recall from [1], and Section 4.2, that the spin model dual to our action (6.5), with perturbative and non-perturbative potentials added, for the $SU(2)$ case is given by

$$S_{XY} = \sum_{x,\hat{\mu}} -\frac{8T}{\pi\kappa} \cos \nabla_{\hat{\mu}} \theta_x + \frac{\kappa}{16\pi T} (\nabla_{\hat{\mu}} \phi_x)^2 + \sum_{x} \frac{8e^{-4\pi/\kappa}}{\pi T \kappa^3} \cosh(2\phi_x) + 2\xi_W(\phi_x) \cos(4\theta_x),$$

(7.18)
where \(\xi_W(\phi)\) was given by (7.9) in the case of a single colour. This action is a spin theoretical model of the action (6.5) with unit lattice spacing and \(N\) sites along each direction. The fields \(\theta_x\) are the compact scalars satisfying \(\theta_x = 2\pi + \theta_x\) and live on the same lattice as the scalar fields \(\phi^a_x\). \(\nabla_\mu\) represents the forward lattice derivative along direction \(\mu = 1, 2\). The partition function is a path integral over these fields at their positions, \(Z = \int D\phi_x \int D\theta_x e^{-S_{xy}}\). The fields \(\theta_x\) are thought of as the dual photon fields \(\sigma_x\). The vortices of these fields are then the magnetically charged objects: the magnetic bions. The \(\cos 4\theta_x\) terms represent the electrically charged objects: the W-bosons and their (wino) superpartners. These interact with the scalar fields \(\phi_x\) as their fugacity couples to them. Although, before in [15], we considered the vortices as W bosons and the symmetry breaking terms as related to magnetic bions, here we shall take the alternate view as it holds better for general gauge group with scalar field \(\phi_x\). Let me first remind ourselves of [15] for the \(SU(N)\) cases without the field \(\phi\).

For \(SU(2)/\mathbb{Z}_2\) XY spin model with symmetry breaking perturbations \((U(1) \to \mathbb{Z}_2)\) the spin model Hamiltonian is

\[-\beta H = \sum_{x, \mu} \frac{\kappa}{2\pi} \cos \nabla_\mu \theta_x + \sum_x \tilde{y} \cos 4\theta_x.\]  

(7.19)

Here the magnetic bion fugacity is represented by \(\tilde{y}\) and so the symmetry breaking perturbations are associated with the magnetic bions, and the vortices of this model are considered to be the W-bosons of the Coulomb gas representation, and \(\kappa = g^2_4(L)/2\pi LT\) denotes the strength of their Coulomb interaction (\(\kappa^{-1}\) would be proportional to the magnetic bion Coulomb interaction (actually \(16/\kappa\))). When the strengths of these interactions are of the same order, \(\kappa \approx 16/\kappa\), we have a phase transition as the vortices begin to proliferate and break centre symmetry at \(\kappa_c = 4 \to T_c \approx g^2_4(L)/8\pi L\). The critical exponent \(\nu\) given by \(\zeta \approx |T - T_c|^{-\nu} = |T - T_c|^{-1/16\pi \sqrt{30}}\) is continuously varying as in the BKT transition of the XY model without symmetry breaking terms \((y_0\) denotes the W-boson fugacity) and the photon and its dual fields \(\sigma, \tilde{\sigma} \to 0\) as \(1/\zeta\) as \(T \to T_c^\pm\) (from the left for \(\sigma\) and from the right for \(\tilde{\sigma}\)). Hence this model enjoys an electric-magnetic duality \(\kappa \iff 16/\kappa\) as described in [1], [15]. Note for future reference that the \(\mathbb{Z}_4 \to \mathbb{Z}_2^{sp} \oplus \mathbb{Z}_2^{op}\) symmetry breaks into two parts. The first represent the discrete chiral symmetry and the second is the topological symmetry from \(\pi_1(SU(2)/\mathbb{Z}_2) \approx \mathbb{Z}_2\) (as our Lie groups are taken to be simply-connected (they satisfy \(\pi_1(G) = 0\)) we have topological symmetry \(\mathbb{Z}_c \approx \pi_1(G/Z(G))\) where \(c = |Z(G)|\).

For the \(SU(3)/\mathbb{Z}_3\) case [5], [15] we have two colours of photon field \(\vec{\theta}_x = (\theta^1_x, \theta^2_x)\) which are (doubly) periodic along the root lattice of \(SU(3)\), \(\vec{\theta}_x = \vec{\theta}_x + 2\pi \vec{\alpha}_1 = \vec{\theta}_x + 2\pi \vec{\alpha}_2\), where we can take the simple roots to be \(\vec{\alpha}_1 = (1, 0)\) and \(\vec{\alpha}_2 = (\frac{1}{2}, -1, \sqrt{3})\). In representation \(\mathcal{R}\) the spin model Hamiltonian becomes

\[-\beta H = \sum_{x, \mu} \sum_{i=0}^2 \frac{\kappa}{4\pi} \cos 2\vec{w}_i \cdot \nabla_\mu \vec{\theta}_x + \sum_x \sum_{i=0}^2 \tilde{y} \cos 2(\vec{\alpha}_i^c - \vec{\alpha}_{i+1}^c) \cdot \vec{\theta}_x,\]  

(7.20)

where the weights \(\vec{w}_i\) are taken in representation \(\mathcal{R}\) and the roots (co-roots) include the affine root (co-root). Note that I used co-roots \(\vec{\alpha}_i^c\) in the symmetry breaking term so as to highlight the interpretation of the symmetry breaking terms as due to magnetic bions. The interpretation of the symmetry breaking term is that they represent vortices which, in this model of now two fields \(\theta^i\), are interpreted as dislocations of a 2d crystal (Burgers’ vectors) which are responsible for the study of melting of 2d crystals [42]. There is no clear physical interpretation of such symmetry breaking terms for higher rank
gauge groups. Here the global symmetry $\theta \rightarrow \theta + \tilde{\theta}$, $U(1)^2 \rightarrow \mathbb{Z}_2$, is broken again into its discrete chiral and topological components, both of which are $\mathbb{Z}_2$. This model is found [15] to also have an electric-magnetic symmetry $\kappa \iff 12/\kappa$ with a critical transition temperature at $T_c \approx g^2/4\sqrt{3}\pi L$. It would be a future project to generalise (7.20) to the case of SYM with scalar field $\phi_a^x$. The main difficulty to finding the spin model dual to the SYM Coulomb gas for cases other than $SU(2)$ is that both kinetic terms depend on the scalar $\phi_a^x$ and there is no clear duality map to the Coulomb gas that works easily. One could, however, find a simpler dual model that works for small values of the scalars $\phi_a^x$.

Before going on to general group, let me look at the $SU(N_c)/\mathbb{Z}_{N_c}$ case given in [15]. The result (7.21) below is actually not correct although published in [15]. The issue is that it does not give the correct dual Coulomb gas when the duality is performed. I will still, however, present the details that must hold from such a spin model in the $SU(N)$ case and give ideas of properties of spin models in general gauge group. Here our compact field $\hat{\theta} \in \mathbb{R}^{N_c-1}$ has periodicity determined by the root lattice $\tilde{\theta}_x = \tilde{\theta}_0 + 2\pi a_i$. The global symmetry $U(1)^{N_c-1} \rightarrow \mathbb{Z}_{N_c}$ breaks down into discrete chiral symmetry and topological symmetry parts. The spin model Hamiltonian one might suspect is just as in (7.20) with a simple generalization

$$-\beta H = \sum_{x,\tilde{\theta}} \sum_{i=0}^{N_c-1} \frac{\kappa}{4\pi} \cos 2\tilde{\theta}_i \cdot \nabla_\mu \tilde{\theta}_x + \sum_x \sum_{i=0}^{N_c-1} \tilde{y} \cos 2(\tilde{\alpha}_i - \tilde{\alpha}_{i-1}) \cdot \tilde{\theta}_x. \quad (7.21)$$

The W-bosons represent the vortices again and interact with Coulomb interactions of strength $\kappa \times (\tilde{\alpha}_i \cdot \tilde{\alpha}_j)$ and hence are proportional to the Cartan matrix of the Lie algebra. The magnetic bions represented by the symmetry breaking terms interact with a Coulomb interaction $\kappa^{-1} \times (\tilde{Q}_i \cdot \tilde{Q}_j)$. The charges of the magnetic bions are $\tilde{Q}_i = \tilde{\alpha}_i - \tilde{\alpha}_{i-1}$ which generalize to any gauge group by taking the charges to be $\tilde{Q}_{ij} = \tilde{\alpha}_i - \tilde{\alpha}_j$ whenever node $i$ is connected to node $j$ directly by a line in the Dynkin diagram of $g$ (so $\tilde{\alpha}_i \cdot \tilde{\alpha}_{i-1} \neq 0$).\footnote{For $SU(N)$ we have $\tilde{\alpha}_i \cdot \tilde{\alpha}_j = \delta_{ij} - \frac{1}{2} \delta_{i,j+1} - \frac{1}{2} \delta_{i,j-1}$, so that $\tilde{Q}_i \cdot \tilde{Q}_j = 3\delta_{ij} - 2\delta_{i,j+1} - 2\delta_{i,j-1} + \frac{1}{2} \delta_{i,j+2} + \frac{1}{2} \delta_{i,j-2}$. Note that this $= 3\tilde{\alpha}_i \cdot \tilde{\alpha}_j$ for the case of $SU(3)$}. I will mention that the topological symmetry arises by shifts long the weight lattice: $\mathbb{Z}_{N_c}^{top} : \tilde{\theta} \rightarrow \tilde{\theta} - 2\pi k \tilde{u}_i$ for each $i$ and $k = 0, \ldots, N - 1$. This changes the phase of the W-boson vertex $e^{2\pi i \tilde{\mu}_i \cdot \tilde{\theta}} \rightarrow e^{2\pi i k/N \tilde{\mu}_i \cdot \tilde{\theta}}$, and also holds for the bions $e^{2\pi i (\tilde{\alpha}_i - \tilde{\alpha}_j) \cdot \tilde{\theta}}$. The (discrete) chiral symmetry acts rather by shifts of fundamental weights $\tilde{\mu}_i$, which satisfy in our case $\tilde{\alpha}_i \cdot \tilde{\mu}_j = \frac{1}{2} \delta_{ij}$, and so $\mathbb{Z}_{N_c}^N : e^{2\pi i \tilde{\mu}_i \cdot \tilde{\theta}} \rightarrow e^{2\pi i \tilde{\mu}_i \cdot \tilde{\theta}}$. Taking the Weyl vector $\tilde{\rho} = \sum_{i=1}^{N-1} \tilde{\mu}_i$ (so $\tilde{\rho} \cdot \tilde{\alpha}_i = 1/2$) the discrete chiral symmetry acts as $\mathbb{Z}_{N_c}^N : \tilde{\theta} \rightarrow \tilde{\theta} + \frac{2\pi \tilde{\rho}}{N}$. One might guess that (7.22) is valid for any gauge group $G$ replacing $N - 1$ with $r$ and $\sum_{u_i}$ with $\sum_{\bar{w} \in \Delta^R}$ for general representation $R$ (taking to be the adjoint representation here where the weights are the set of all roots), yet, as for the $SU(N_c)$ case it does not yield the correct dual Coulomb gas. This work of deriving the dual spin model for SYM for groups other than $SU(2)$ and $SU(3)$, with coupling to the scalars $\phi_a^x$, is a project of future research.

It is also interesting to note the competition between ferromagnetic and antiferromagnetic interactions present here. I find that our XY-model with symmetry-breaking perturbations gets 'frustrated' due to competing ferromagnetic interactions of scalar charges (W’s and neutral bions) and antiferromagnetic interactions of electric and magnetic charges (W’s and magnetic bions), as seen more easily in the case of $SU(2)$. At different temperatures the free energy of the theory will be dominated by one or the other of these interactions. In future work one could investigate the role of this 'frustration' for different gauge
groups, mentioned in studies [49], [50].

Let me comment about the behaviour a correct spin model of SYM in general gauge group must have. The difference from [15] is the inclusion of the $\phi^a$ fields which makes the W-boson fugacity $\phi$-dependent. The fugacity of the magnetic bions is not a free parameter now as it depends on the coefficient of the kinetic term for the $\theta^a$’s (as well as the lattice spacing). In general I still get a low-$T$ prevalence of the magnetic bions (XY vortices) due to less suppression of the $\theta^a$ field fluctuations than at higher temperatures. These (external magnetic field) $\cos 2(\alpha_i^Y - \alpha_j^Y) \cdot \theta_x$ terms, as in the case of $SU(3)/Z_3$ without supersymmetry, break the $U(1)$ symmetry of each field component to $Z_2$, which acts by shifts $\theta_x \rightarrow \theta_x + 2\pi/c$. Its $Z_2^{(\beta)}$ subgroup represents the centre-symmetry of the fields along the thermal cycle. The factor of 2 is present as the insertion of $e^{2i\theta^a_x}$ represents an electric probe which can be thought of a fundamental quark-like insertion, having half the charge of the W-bosons and can be used to probe confinement. In simulations for other gauge groups that may be done in the future, and which were done for $SU(2)$ in [1], I probe the $Z_2^{(\beta)}$ symmetry breaking using as order parameter the XY ’magnetization’ $m^a$, and susceptibility $\chi^a(m)$ as in [1]

$$m^a = \frac{1}{N^2} \left\langle \sum_x e^{i\theta^a_x} \right\rangle \equiv \left\langle |M^a| \right\rangle / N^2,$$

and

$$\chi^a(m) = \frac{\left\langle (|M^a|^2) - \left\langle |M^a| \right\rangle^2 \right\rangle}{N^2} = \sum_x \left\langle e^{i\theta^a_x} e^{-i\theta^a_0} \right\rangle.$$

In future work one may do further lattice simulations to generalize the results of [1] as presented in Section 4.2 to arbitrary gauge group, and hope to find a first order phase transition for groups other than $SU(2)$ (recall that in [1] a second order phase transition is found for $SU(2)$ in agreement with lattice results). This may help shed light on the continuity conjecture described in [2],[7] and is a project for future research.

Let me now turn from the case of torus compactification and dual Coulomb gas and spin models to a case that can be studied analytically: the case of mass deformed SYM (with a mass for the gaugino) on $R^3 \times S^1_L$. Here calculations can be performed such as the phase transition at some critical mass for the gaugino, as well as properties of the Wilson loop observables, their correlators, and the potentials of interactions of quark probes near the deconfinement transition.
Chapter 8

Deconfinement of SYM on $\mathbb{R}^3 \times S^1_L$

' So I parted with the scientist,  
he continued on his journey  
of discovering gravitinos, selectrons, and squarks,  
and so far finding instead instrument blips, hiccups, and quirks;  
yet there is no dearth of questions to pursue  
if pursuing this theory answers no questions...  
after all, he is not wedded to any theories,  
he just wants to find things out about nature -  
least those things that are just within reach of our senses,  
though not under the lamp posts, but in the dark corners...'  
- Numi Who  

Supersymmetric Illuminations

In this Chapter I look at the case of a single compact dimension $\mathbb{R}^3 \times S^1_L$ and the (zero temperature) quantum phase transition of deformed super Yang-Mills (SYM*). Here I add a mass for the gaugino and determine a deconfinement phase transition at some critical gaugino mass. Through a continuity conjecture, discussed in the next Section, it is believed that this transition is related to the thermal deconfinement transition in pure Yang-Mills and so the study of quantum phase transitions in SYM* can teach us about the order and nature of the thermal phase transition of pure Yang-Mills, which, being strongly coupled, is harder to study analytically and only some lattice studies for few gauge groups have been done thus far. I begin by describing the continuity conjecture and then go on to the analysis of the quantum phase transition of SYM* for all gauge groups.

8.1 Softly-Broken Supersymmetry and the Continuity Conjecture

The continuity conjecture can be briefly described by Figure 8.1. It was studied in [2] how broken/deformed super Yang-Mills (SYM*) on $\mathbb{R}^3 \times S^1_L$ with a non-zero mass for the gaugino, $m$, exhibits deconfinement and has its quantum zero temperature phase transition continuously connected to the thermal deconfinement phase transition of pure Yang-Mills on $\mathbb{R}^3 \times S^1_L$ with $\beta_c = L_c$. This would imply that the nature of the phase transitions are similar, in that their order and universality class, as well as
other properties agree with simulations done thus far. It is found in [2], that the deconfinement phase transition is first order for all gauge groups except for $SU(2)$, where it is second order. I suspect that this 'continuity conjecture' is valid as the underlying mechanism of the phase transition was seen to be due to competition between non self-dual topological molecules/bions and the perturbative W-bosons, even though there are no magnetic bions in the pure Yang-Mills theory (however there are still neutral bions). Even so, comparison to lattice studies of thermal Yang-Mills with various gauge groups has been done [47], [48], and there is further support for this conjecture. See [2] and [7] for more on this continuity conjecture and support for it. I show here how also the dependence on theta angle on discontinuous jumps of Wilson loop observables and the critical transition mass is also in qualitative agreement with lattice studies of pure Yang-Mills for gauge groups studied thus far ($Sp(4)$, $SU(N)$, and $G_2$). It is hoped that further lattice studies can be done in thermal Yang-Mills and of the thermal theory discussed here on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ for general gauge group to further verify or support this conjecture. It is also worth noting that at $m = 0$ the theory is supersymmetric and the partition function is the Witten index, hence there is no transition point here as $L$ is varied.

8.2 Callias Index on $\mathbb{R}^3 \times S^1_L$ and Corrections to Monopole Determinants and Effective Potentials

I present here a derivation of the Callias index (an index for Dirac operators on non-compact, open, odd-dimensional spaces [57] generalizing the Atiyah-Singer index which is defined only on compact spaces and is zero when the dimension is odd) on $\mathbb{R}^3 \times S^1_L$ for all gauge groups. I find that in the adjoint representation the index is two for any gauge group, showing that monopoles always carry two fermionic zero modes each. Furthermore, an 'index function' allows one to find corrections to the superpotential and effective potential of the theory due to non-cancelling of non-zero mode determinants in monopole backgrounds. I present here a derivation of these corrections as well.

8.2.1 Callias index on $\mathbb{R}^3 \times S^1_L$

I describe the index theorem which determines the number of fermionic zero modes attached to each monopole (BPS and KK) of Yang-Mills theory, with more detail than in [2], and for arbitrary gauge group generalizing the results of [8]. To compute the index of our Dirac operator in the background of
BPS and KK monopoles, I shall use a convenient definition for the number of zero modes of $\Delta_0 = D\ast D$ minus the number of zero modes of $\Delta_0 = D\ast D$, with $D_\mu = \partial_\mu + i A_\mu^a T^a$. It can be written as in [8], [10]
for representation $\mathcal{R}$ (which I will take to be the adjoint representation)

$$I_R = \lim_{M^2 \to 0} \left[ (T R \frac{M^2}{D\ast D + M^2} - T R \frac{M^2}{D\ast D + M^2}) = M T R \gamma_5 (-\hat{D} + M)^{-1}), \right.$$  \hspace{1cm} (8.1)

where $\gamma_5 = \prod_i^4 \gamma_i$ and $D\ast D = -D_\mu D_\mu$ and $D\ast D = -D_\mu D_\mu + 2\sigma^m B^m$. The last equality follows from cyclicity of the trace and how $\{\gamma_5, D\}$ = 0, where $\hat{D} = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & D \end{array}\right)$ = $\gamma_5 D_\mu$. Using now $\langle x|(-\hat{D} - M)^{-1}|y\rangle = \langle \psi(x)\bar{\psi}(y)\rangle$ so that $I_R(M^2) = M \int d^3 x d y < \bar{\psi}\gamma_5 \psi >$, where $y \in S^1_L$. I can then write

$$\langle J^a_k \rangle = tr\langle x|\gamma^a_{5}(-\hat{D} + M)|x\rangle = tr\langle x|\gamma^a_{5}\hat{D}/(-\hat{D}^2 + M^2)|x\rangle.\] Using integration by parts and the fact that $\partial_\mu(J^a_k = \bar{\psi}\gamma_5 \psi) = -2M \bar{\psi}\gamma_5 \psi - T(R)G^a_{\mu\nu}G^a_{\mu\nu}/8\pi^2$,

$$I_R(M^2) = - \int_{S^2} d^2 \Sigma^\mu \partial_j (J^a_k)/2 - T(R)G^a_{\mu\nu}G^a_{\mu\nu}/16\pi^2. \hspace{1cm} (8.2)$$

These two terms are the surface term and topological charge contributions respectively. I then get for the index in general representation $\mathcal{R}$ (with both surface and charge contributions)

$$I_R(M^2) \equiv I_1 + I_2(M^2) = \frac{T(R)}{16\pi^2} \int dy \int d^3 x G^a_{\mu\nu}G^a_{\mu\nu} - \frac{1}{2} \int dy \int_{S^2} d^2 \sigma^k (J^a_k). \hspace{1cm} (8.3)$$

Let me look first at the surface contribution of the Chern-Simons current, $I_2(M^2)$.

$$I_2 = \lim_{M^2 \to 0} Tr(\gamma_5 \frac{M^2}{D\ast D + M^2}) = \int dy \int d^3 x \partial_j J_j = \int dy \int_{S^2} d^2 \sigma J_0, \hspace{1cm} (8.4)$$

with $J_0 = -\frac{1}{2} Tr\langle x|\gamma_5 D|x\rangle$ and the denominator can be expanded as a power series in $G$ to first order as $G \approx 1/r^2$ as seen in [8]. This gives two terms: (i) $Tr(\gamma_5 \gamma_5 \psi)$ which is zero, and (ii) $Tr(\gamma_5 \gamma_5 \gamma_5 G)$ which is zero for $\mu \neq 4$ where I get $\propto \epsilon_{i j k} \hat{r}_i \hat{r}_j$. The remaining term is then (where the factor of 2 comes from the traces of the Pauli matrices and using $g_0, \phi_0 \in t$, where $\phi_0 = diag(\{\phi_i\})$ and $g_0$ are the charges in the Cartan subalgebra $\sum_{a=0}^\ast \alpha_a^* n_a \cdot \hat{H}$)

$$J_i \approx 2 \frac{\hat{r}_i}{r} \sum_{n \geq 0} Tr_{adj}(x, y) \langle i D_4 g_0 \cdot \hat{H}/(-D_j^2 + (\phi_0 + 2\pi n/L)^2 + M^2)^2 \rangle |x, y \rangle \approx \hspace{1cm} (8.5)$$

$$2 \frac{\hat{r}_i}{r} \sum_{n \geq 0} \sum_{m \geq 0} \langle \bar{\psi} \cdot \phi_0 + 2\pi n/L \rangle \langle \bar{\psi} \cdot g_0 \rangle Tr\langle x, y | (1/(-D_j^2 + (\phi_0 + 2\pi n/L)^2 + M^2)^2) |x, y \rangle,$$

where $-i D_4 = 2\pi n/L + A_0$ and the trace was done in representation $\mathcal{R}$ with weights in $\mathcal{R}$. The KK terms are added in series with the Wilson phases $\phi_0$ as it shifts the winding number on the spatial circle. For $SU(N)_{adj}$ the weights (positive) have 1 in the $i$th position and -1 in the $j$th position with $i < j$. Integrating over the $S^2$ gives just 4$\pi$, and over the circle gives 2$\pi L$. Doing the expectation value in momentum space is easy using

$$< x, y | (-D_j^2 + M^2 + (p/L + \bar{\phi} \cdot \hat{H}/2\pi L)^2)^{-1/2} |x, y > = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} k^2 + M^2 + (p/L + \bar{\phi} \cdot \hat{H}/2\pi L)^2 =$$
where I used \( f_\infty \frac{k^2 d_k}{(k^2 + A^2)^2} = \frac{\pi}{4A} \). The index due to this surface contribution is then put together:

\[
I_2(0) = \lim_{M^2 \to 0} \sum_{\substack{n \geq 0 \ w \in \Delta^G \ Z}} \sum_{\substack{i \in \mathbb{Z}}} \frac{(\vec{w} \cdot \phi_0 + 2\pi n/L)(\vec{w} \cdot g_0)}{(\vec{w} \cdot \phi_0 + 2\pi n/L)^2 + M^2} = \sum_{\vec{w}} \sum_{\phi_0 \neq 0 \ n \in \mathbb{Z}} sgn(\vec{w} \cdot \phi_0 + 2\pi n/L)(\vec{w} \cdot g_0)/2. \tag{8.6}
\]

Note that the following property can be used (from \( \text{tr} \) \( \vec{H} \cdot \vec{H} \)) in terms of asymptotics is

\[
\text{tr}_\mathcal{R}(\vec{a} \cdot \vec{H}_0 \cdot \vec{H}) = T(\mathcal{R})\vec{a} \cdot \vec{b}. \tag{8.7}
\]

Replacing \( \vec{H} \) with the weights is as in the adjoint representation. Another way to write the mass dependent index (for a BPS monopole type \( i \)) in terms of asymptotics is

\[
I_2(M^2) = \frac{1}{2} \text{tr}_\mathcal{R} \sum_{p \in \mathbb{Z}} \frac{(\vec{w} \cdot \phi_0 + 2\pi k/L)(\vec{w} \cdot g_0)}{M^2 + (p/L + \phi_0 \cdot \vec{H}/2\pi L)^2}. \tag{8.8}
\]

In the case of \( SU(N) \) we have \( \phi_0 = \text{diag}(\phi_1, \ldots, \phi_N) \) so that \( g_0 = \text{diag}(n_1, n_2 - n_1, \ldots, -n_{N-1}) \) with \( n_j \in \mathbb{Z}^+ \). The first term \( (n = 0) \) gives the usual result on \( \mathbb{R}^3 \) given in [8], [10]: \( I = N_{\text{adj}} = \sum_{j \geq 1} ((n_{j+1} - n_j) - (n_{j+1} - n_j)) = 2 \sum_{j} n_j \). Evaluating the second term I can generalize my previous result on \( \mathbb{R}^3 \times S^1 \) for \( SU(2) \) and higher ranks to arbitrary gauge group.

I can write the surface contribution to the index alternatively using \( \sum_{p \in \mathbb{Z}} sgn(p/L + \phi_0 \cdot \vec{H}/2\pi L) = 1 - 2L\phi_0 \cdot \vec{H}/2\pi L + 2|L\phi_0 \cdot \vec{H}/2\pi L| \)

\[
\sum_{\vec{w} \in \mathcal{R}} \sum_{k \in \mathbb{Z}} sgn(\vec{w} \cdot \phi_0 + 2\pi k/L)(\vec{w} \cdot g_0) = (1 - 2\frac{L}{2\pi}(\vec{w} \cdot \phi_0) + 2\frac{L}{2\pi}(\vec{w} \cdot \phi_0)))(\vec{w} \cdot g_0). \tag{8.9}
\]

This is also from taking the sum over \( n \) by analytic continuation of the eta function \( (v_j = w_j \cdot \vec{v}, \text{here, and } \lambda \text{ are the eigenvalues of } i \frac{d}{dy} + A_0|_{\infty}) \):

\[
\eta[v_j, s] = \sum_{\lambda \neq 0} sgn(\lambda) |\lambda|^s = \zeta(v_j L/2\pi - |v_j L/2\pi|, s) - \zeta(1 - v_j L/2\pi + v_j L/2\pi, s), \tag{8.10}
\]

where \( \zeta(x, s) = \sum_n (n + x)^{-s} \) is the Hurwitz zeta function. Using \( \zeta(x, 0) = 1/2 - x \) I get the result.\(^1\)

What is missing now is the topological charge contribution. I have from [7] that this contribution to the index is\(^2\)

\[
I^{\text{top}}_\mathcal{R} = -2T(\mathcal{R})Q, \tag{8.11}
\]

where

\[
Q = \frac{1}{32\pi^2} \int \int \int F^{\alpha \nu} G^{\alpha \mu} = \frac{L}{4\pi^2} \int_{S^1} d^2 \Sigma^k \text{tr} A_d B^k. \tag{8.12}
\]

\(^1\)Recall that in the adjoint representation we have \( T(\mathcal{R}) = 2c_2(G) \), twice the dual Coxeter number, and for the fundamental it is \( 1/2 \) (in some conventions/normalizations of generators it is 1).

\(^2\)I will need the normalization choice in any representation \( \text{tr}_\mathcal{R} H^i H^j = T(\mathcal{R})\delta^{ij} \) so that \( \text{tr}_\mathcal{R}(\vec{a} \cdot \vec{H}_0 \cdot \vec{H}) = T(\mathcal{R})\vec{a} \cdot \vec{b} \). I also have \( \text{tr}_\mathcal{R} T^a T^b = T(\mathcal{R})\delta^{ab} \) and \( \text{tr}_\mathcal{R} E^a E^b = T(\mathcal{R})\delta^{ab} \).
The factor $T(R)$ comes from $8tr(A_4B^k)|_\infty = -8\frac{e}{2\pi r^4} r^c \tau^b T^b T^c$. Also as $A_4 |_\infty \rightarrow \phi_0$ and $B^k = \frac{\pi}{r} g_0 + O(1/r^2)$, the last equality of the previous equation becomes

$$Q = -\sum_{\bar{w}} \frac{L}{2\pi} (\phi_0 \cdot \bar{w})(\phi_0 \cdot \bar{w}),$$

(8.13)

as taking the trace over the representation gives (as $H^i|w_i) = w_i|w_i)$ $tr_R(g_0 \cdot \bar{H} \phi_0 \cdot \bar{H}) = \sum w_i \sum w_j \langle w_i|g_0|w_i\rangle\langle w_j|\phi_0|w_j\rangle (\bar{w} \cdot g_0)(\bar{w} \cdot \phi_0)$.

Combining this topological charge contribution to the surface contribution earlier cancels the non-integer part to the index leaving us with an integer as required:

$$I^\text{tot} R(0) = -\sum_{\bar{w} \in \mathcal{R}} (\bar{w} \cdot g_0)(1 + 2\frac{L}{2\pi}(\bar{w} \cdot \phi_0)),$$

(8.14)

as the constant term 1 from the surface contribution cancels as weights are summed over both positive and negative weights. Now using $[-x] = -1 - [x]$ I get the final expression for the Callias index on $\mathbb{R}^3 \times S^1$:

$$I_R(0) = \sum_{\bar{w}^+ \in \Delta_+^R} (\bar{w} \cdot g_0)(1 + 2\frac{L}{2\pi}(\bar{w} \cdot \phi_0)),$$

(8.15)

This means for $(\bar{w} \cdot \phi_0)L \in [0, 2\pi)$, which we can always take to be in the main Weyl chambre, the Callias index is simply given by

$$I_{BPS} = \sum_{\bar{w}^+ \in \Delta_+^R} (\bar{w} \cdot g_0).$$

(8.16)

I evaluated this for all semi-simple Lie algebras for each monopole $g_0 = \bar{\alpha}^\vee_i$ and showed it is always 2 in the adjoint representation, as required, showing that each monopole carries 2 fermionic zero modes.

For the KK monopole I simply note that the magnetic charge is negative, and use a Weyl reflection. This affine monopole, in the case of $\mathbb{R}^3 \times S^1$, comes about by a Weyl reflection (reflection about a plane orthogonal to a given root $\bar{\alpha}$ passing through the origin) followed by translations to higher charges in the co-root lattice ($\sigma_\alpha$ representing the Weyl reflection about root $\alpha$):

$$\langle \bar{\phi} \rangle \rightarrow \sigma_\alpha(\langle \bar{\phi} \rangle) + \pi n \bar{\alpha}^\vee,$$

(8.17)

and the Weyl reflection is easily seen to be

$$\sigma_\alpha(\bar{\phi}) = \bar{\phi} - \frac{2\bar{\alpha} \cdot \bar{\phi}}{|ar{\alpha}|^2} \bar{\alpha}.$$

The resulting $\nu' = -\bar{\alpha} \cdot \langle \bar{\phi} \rangle / 2\pi L + n/L > 0$ is positive if to remain in the new Weyl chambre. The new solution is then (with $\bar{\tau}^c = \exp(inx_0\bar{\alpha}^\vee \cdot \bar{H}/2L)\sigma_\alpha^c \tau^c \sigma_\alpha^c \exp(-inx_0\bar{\alpha}^\vee \cdot \bar{H}/2L)$)

$$A_0(x_\mu) = \Phi^c(n/L - v, x_\mu)\bar{\tau}^c + (2\pi L)^{-1}((\bar{\phi} - \frac{1}{2}(\bar{\phi} - \bar{\alpha} + 2\pi n)\bar{\alpha}^\vee) \cdot \bar{H}.$$  

(8.18)

The asymptotics of the simple KK monopole solution are obtained by replacing $v \rightarrow 1/L - v$, which in the notation of this thesis means $\nu L/2\pi \rightarrow 1 - \nu L/2\pi$. Carrying on as in my calculation of the BPS
monopole I find the index is

$$I^{KK} = 2n_0 T(R) - \sum_{w \in \Delta_w} \left( \frac{L \phi \cdot w}{2 \pi} \right) \alpha_0^\vee \cdot w. \quad (8.19)$$

This is easily seen to be equal to 2 also as $\sum_{i=0}^r k_i^\vee \alpha_i^\vee = 0$, and so the KK monopoles carry 2 fermionic zero modes as well.

I can also add the BPS solutions and gather a final formula for any number of BPS and KK monopoles.

$$2n_0 T(R) - \sum_{w \in \Delta_w} \left( \frac{L \phi \cdot w}{2 \pi} \right) \alpha_0^\vee \cdot w + 2 \sum_{\bar{w} \in \Delta_+^w} \sum_{i=1}^r \left( \bar{w} \cdot n_i \alpha_i^\vee \right) \left[ \frac{L}{2 \pi} (\bar{w} \cdot \phi_0) \right]. \quad (8.20)$$

Writing the sums just over positive roots and with fields in the principal Weyl chambre, the integer part terms vanish and I am left with the total index

$$I^{KK+BPS} = 2n_0 + \sum_{i=1}^r \sum_{\bar{w} \in \Delta_+^w} n_i \alpha_i^\vee \cdot w = 2 \sum_{i=0}^r n_i. \quad (8.21)$$

The last equality above is from $\sum_{\bar{w} \in \Delta_+^w} n_i \alpha_i^\vee \cdot \bar{w} = 2$ for each $i \in \{1, \ldots, r\}$. This was computed and verified for all gauge groups in the adjoint representation by a weight generating code for each co-root (each monopole charge type) I wrote. Hence each monopole, BPS or KK, gets two fermionic zero modes attached to it in the case of the adjoint representation. Note also that whenever a field $\phi^a$ crosses a Weyl chambre boundary, the index increases by 4, as is seen by the floor functions in (8.20).

### 8.2.2 Corrections to monopole determinants and effective potentials

Now I shall use the index theorem to find the quantum corrections in the monopole background. I use the mass-dependent 'index' given by (8.23) below to calculate the determinant. In this Section, I also will need the expressions for $I^{\alpha}_{1\text{adj}}(M^2)$ and $I^{\alpha}_{2\text{adj}}(M^2)$ in (8.22). Using the identity $\text{Tr}_{\text{adj}}[H^i H^j] = \delta^{ij} T(\text{adj})$, I find

$$I^{\alpha}_{1\text{adj}}(M^2) = T(\text{adj}) \frac{\bar{\alpha}^\vee \cdot \bar{\phi}}{2 \pi},$$

$$I^{\alpha}_{2\text{adj}}(M^2) = \frac{1}{2} \sum_{p \in \mathbb{Z}} \sum_{\bar{w}} \left[ \left( \frac{p}{\bar{w}} + \frac{\bar{\phi} \cdot \bar{w}}{2 \pi} \right) \alpha^\vee \cdot w \right].$$

where $T(\text{adj}) = 2c_2$, and $c_2 = \sum_{a=0}^r k_a^\vee$ is the dual Coxeter number.

I begin with

$$\int_{\mu^2}^{\Lambda^2} dM^2 \frac{dM^2}{M^2} I^{\alpha}(M^2) = \text{tr} \log \left[ \Delta_- + M^2 \right]^{\Lambda^2_{1PV}} \mu^2 - \text{tr} \log \left[ \Delta_+ + M^2 \right]^{\Lambda^2_{1PV}} \mu^2$$

$$= \text{tr} \log \left[ \frac{\Delta_- + \Lambda^2_{1PV}}{\Delta_- + \mu^2} \right] - \text{tr} \log \left[ \frac{\Delta_+ + \Lambda^2_{1PV}}{\Delta_+ + \mu^2} \right]. \quad (8.23)$$
Using the identity \( \text{tr log } A = \log \text{Det } A \), it is found that

\[
\int_{\mu^2}^{\Lambda^2} \frac{dM^2}{M^2} I^a(M^2) = \log \text{Det} \left[ \frac{(\Delta_+ + \mu^2)(\Delta_- + \Lambda_{P\bar{V}}^2)}{(\Delta_- + \mu^2)(\Delta_+ + \Lambda_{P\bar{V}}^2)} \right].
\]  

(8.24)

The usefulness of the expression (8.24) is that it determines the one-loop corrections around the supersymmetric monopole-instanton background. The contribution of the nonzero modes of the fluctuations of the fermions and gauge fields in the \( \bar{\alpha}V \)-monopole-instanton background is given by the ratio \( R_{n\bar{V}} \) of determinants defined as

\[
R_{n\bar{V}} = \lim_{\mu \to 0} \left( \frac{\mu^{I_{\bar{R}}(0)}(\Delta_+ + \mu^2)(\Delta_- + \Lambda_{P\bar{V}}^2)}{(\Delta_- + \mu^2)(\Delta_+ + \Lambda_{P\bar{V}}^2)} \right)^{3/4},
\]

(8.25)

where \( I_{\bar{R}}(0) \) is the number of zero modes of the operator \( \Delta_- \). Hence, I find

\[
\left( e^{-S_0^\alpha} R \right)_{\bar{V}} = e^{-S_0^\alpha} \lim_{\mu \to 0} \left( \mu^{I_{\bar{R}}(0)} e^{I_{\bar{a}}(\text{adj}) \log \frac{\Lambda_{P\bar{V}}^2 \mu^2}{4\pi}} \times e^{I_{\alpha}^2 \bar{M}^2 \frac{I_{n\text{adj}}(M^2)}{2\bar{M}^2}} \right)^{3/4},
\]

(8.26)

where

\[
S_0^\alpha = \frac{4\pi}{g^2(\Lambda_{P\bar{V}})} \bar{\alpha}V \cdot \vec{\phi}
\]

(8.27)

is the bare monopole action defined at the scale \( \Lambda_{P\bar{V}} \). Using the SYM running coupling (\( \Lambda \) is the strong-coupling scale to one loop order)

\[
\frac{4\pi}{g^2(\mu)} = \frac{3}{8\pi} T(\text{adj}) \log \frac{\mu^2}{\Lambda^2},
\]

(8.28)

I find

\[
-S_0^\bar{\alpha} + \frac{3}{4} I_{\text{adj}} \log \frac{\Lambda_{P\bar{V}}^2}{\mu^2} = -\frac{4\pi}{g^2(\Lambda_{P\bar{V}})} \bar{\alpha}V \cdot \vec{\phi} + \frac{3}{8\pi} T(\text{adj}) \bar{\alpha}V \cdot \vec{\phi} \log(2\pi R \Lambda_{P\bar{V}})^2
\]

\[
-\frac{3}{8\pi} T(\text{adj}) \bar{\alpha}V \cdot \vec{\phi} \log(\mu 2\pi R)^2
\]

\[
= -\frac{4\pi}{g^2(1/2\pi R)} \bar{\alpha}V \cdot \vec{\phi} - \frac{3}{4\pi} T(\text{adj}) \bar{\alpha}V \cdot \vec{\phi} \log(2\pi R \mu).
\]

(8.29)

Next, I have to calculate \( \int_{\mu^2}^{\Lambda^2} \frac{dM^2}{M^2} I_{2,\text{adj}}^a(M^2) \). Given the integral

\[
\int_0^\infty \frac{dM}{M \sqrt{A^2 + M^2}} = \frac{1}{|A|} \sinh^{-1} \left[ \frac{|A|}{\mu} \right] = \frac{1}{|A|} \log \left[ \frac{2|A|}{\mu} \right] + O(\mu^2),
\]

(8.30)

I find

\[
\int_{\mu^2}^{\Lambda^2} \frac{dM^2}{M^2} I_{2,\text{adj}}^a(M^2) = \sum_{p \in \mathbb{Z}} \sum_{\vec{w}} \left[ \text{sign} \left( \frac{\vec{\phi} \cdot \vec{w}}{2\pi R} + \frac{p}{R} \right) \bar{\alpha}V \cdot \vec{w} \log \frac{2|A|}{\mu} \right].
\]
Using $\sum_{p \in \mathbb{Z}} \text{sign} \left( a + \frac{p}{R} \right) = 1 - 2aR + 2|aR|$, and the index theorem in the adjoint representation, I get

$$\sum_{p \in \mathbb{Z}} \sum_{\vec{w} \in \Delta_{\text{adj}}} \left[ \text{sign} \left( \frac{\vec{\phi} \cdot \vec{w}}{2\pi R} + \frac{p}{R} \right) \vec{\alpha} \cdot \vec{w} \log \frac{2}{\mu R} \right] = (2I_{\text{adj}}^a(0) - 2I_{1\text{adj}}^a) \log \frac{2}{\mu R}. \quad (8.32)$$

Collecting everything I obtain:

$$\log(e^{-S_0^a} R^{-a}) = -\frac{4\pi}{g^2(1/2\pi R)} \vec{\alpha} \cdot \vec{\phi} - \frac{3}{4\pi} T(\text{adj}) \vec{\alpha} \cdot \vec{\phi} \log(4\pi) + \frac{3}{2} I_{\text{adj}}^a(0) \log \frac{2}{R}$$

$$+ \frac{3}{4} \sum_{p \in \mathbb{Z}} \sum_{\vec{w} \in \Delta_{\text{adj}}} \left[ \text{sign} \left( p + \frac{\vec{\phi} \cdot \vec{w}}{2\pi} \right) \vec{\alpha} \cdot \vec{w} \log \left| p + \frac{\vec{\phi} \cdot \vec{w}}{2\pi} \right| \right]$$

$$= -\frac{4\pi}{g^2(1/2\pi R)} \vec{\alpha} \cdot \vec{\phi} + \frac{3}{2} I_{\text{adj}}^a(0) \log \frac{2}{R}$$

$$+ \frac{3}{4} \sum_{p \in \mathbb{Z}} \sum_{\vec{w} \in \Delta_{\text{adj}}} \left[ \text{sign} \left( p + \frac{\vec{\phi} \cdot \vec{w}}{2\pi} \right) \vec{\alpha} \cdot \vec{w} \log \left| p + \frac{\vec{\phi} \cdot \vec{w}}{2\pi} \right| \right], \quad (8.33)$$

where I have used (8.28) to simplify the first two terms in the first line.

Further, to calculate the last term I note that

$$\sum_{p \in \mathbb{Z}} \text{sign}(p + a) \log |p + a| = -\frac{d}{ds} [\xi(s, a) - \xi(s, 1 - a)]|_{s=0}, \quad (8.34)$$

where $0 < a < 1$, and $\xi(s, a) = \sum_{p \geq 0} \frac{1}{|p + a|^s}$ is the incomplete gamma function. Similarly we have

$$\sum_{p \in \mathbb{Z}} \text{sign}(p - a) \log |p - a| = -\frac{d}{ds} [\xi(s, 1 - a) - \xi(s, a)]|_{s=0}. \quad (8.35)$$

Then, using $\xi'(0, x) = \log \Gamma(x) - \frac{1}{2} \log(2\pi)$, $I_{\text{adj}}^a(0) = 2$, and taking into account that $\vec{\phi} \cdot \vec{w}$ can be positive or negative, I finally arrive at

$$(e^{-S_0} R)^a = \left( \frac{2}{R} \right)^a e^{-\vec{\alpha} \cdot \vec{\phi}} \sum_{s_+} \left( \vec{w} \log \frac{r(\frac{\vec{w}}{2\pi})}{\Gamma(1 - \frac{s_+}{2\pi})} \right)^a. \quad (8.36)$$

This gives an effective correction to the value of $\vec{\phi}$ that can be used in previous formulae such as the superpotential and thus used as a further one-loop correction to the effective potentials. However, I will not do this here but state one-loop corrected results sparingly. I am satisfied with using the non quantum corrected formulae in the study of the effective non-perturbative effective potentials and in studying the deconfinement phase transition in SYM*.
8.3 Deconfinement on $\mathbb{R}^3 \times S^1_L$ and Theta Dependence for all Gauge Groups

Now with the monopole corrections to the superpotential at hand it is time to study the deconfinement of SYM* as the gaugino mass $m$ is varied. I begin by deriving the full effective potential from monopole-instantons and bion contributions, as well as the GPY perturbative effective potential (which is $1/\log \Lambda L$ suppressed so will be later ignored). From this, one can find the critical mass of the gaugino $m^*$ at the quantum phase transition and determine it to be a first order phase transition for all gauge groups other than $SU(2)$, a case I will consider first where the transition is second order. Furthermore, I compute traces of Wilson loop observables near the phase transition, their eigenvalues, and their discontinuous jumps at $m^*$, and further find their dependence on the theta angle of the theory. One finds qualitative agreement of these results with lattice simulations of pure thermal Yang-Mills for groups studied to date. Furthermore, by computing correlators of Wilson loops the effective potential of interaction between quark-anti-quark pairs can be found from

$$\langle \text{Tr} \Omega(x^\mu) \text{Tr} \Omega^\dagger(y) \rangle = e^{-V(r,T)/T}, \quad (8.37)$$

where $r = |x - y|$. The general form of the correlator can be expressed for groups with non trivial centre as

$$\langle \text{Tr} \Omega(x) \text{Tr} \Omega^\dagger(y) \rangle = \begin{cases} \frac{g^2 R}{4r} A_0 \ e^{-\hat{\sigma}_0 m_0 r}, & c_m = 0 \\ \frac{g^2 R}{4r} A_r \ e^{-\hat{\sigma}_r m_0 r}, & c_m = c_{cr} \\ \left(\frac{g^2}{4\pi^2}\right)^2 D_{cr+} + \frac{g^2 R}{4r} C \ e^{-\lambda c m_0 r}, & c_m = c_{cr+}, \end{cases} \quad (8.38)$$

where $r = |x - y|$. The $\hat{\sigma}$ parameters can be thought of as dimensionless string tensions. Note that above the transition the constant term $D_{cr+}$ dominates the exponential and the string tension vanishes. Appendix D of [2] derives the formula (8.38) and the scale $m_0$ in the exponents is given by $m_0^2 = \frac{16\pi^2}{g^2 \Lambda L} V_{bion}^0 = 18\sqrt{2} \kappa g^4 \lambda L^3 \log \frac{m_0 R}{\Lambda}$. The parameters above and below the transition will be computed for all gauge groups. For now, I begin with a review of the $SU(2)$ case examined in [7]. This has easy analytical calculations so let us see how it works.

The $SU(2)$ case was discussed in [1], however the Polyakov correlator was not considered there. The trace in the fundamental representation of the Polyakov loop operator, with $\phi_0^{(1)} = 0$ is $\text{Tr} \Omega(x) = i(e^{i g^2 b(x)/4r} - e^{-i g^2 b(x)/4r})$, and clearly vanishes in the supersymmetric vacuum $\langle b \rangle = 0$. In this vacuum (below the transition) the two-point correlator becomes

$$\langle \text{Tr} \Omega(x) \text{Tr} \Omega^\dagger(0) \rangle = 2 \sinh \frac{g^2 R}{4r} e^{-\hat{\sigma} m_0 r}, \quad (8.39)$$

At scales between $R$ and $m_0^{-1}$ the exponential factor is negligible and the correlator behaves as $1/r$. Yet, at scales larger than this Debye screening length the Polyakov loop correlator is dominated by the exponential term and we have a non-zero string tension $\sigma = \hat{\sigma} m_0 / R$ and are in the confined phase. Above the transition the constant $D$-terms dominate the exponential and lead to a zero string tension at $c_m > c_{cr}$. Let me describe the $SU(2)$ case in more detail and its deconfinement phase transition.
8.3.1 SU(2) case

For completeness I mention that the one-loop order perturbative effective potential of the Wilson line for SU(2) with periodic boundary conditions on both the gauge field and adjoint Weyl fermion is [7]

\[ V_{\text{pert.}}^{\text{SYM}}(\Omega, m) = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \left[ -1 + \frac{1}{2} (nmL)^2 K_2(nLm) \right] \frac{|\text{tr} \Omega^n|^2}{n^4}. \]  

(8.40)

This can be found from the result for general gauge group given by (6.23) taking the weights/roots to be \( \vec{\alpha}_1 = \frac{1}{\sqrt{2}} (1, -1) = -\vec{\alpha}_0 \) and \( \vec{\phi} = (\phi, -\phi) / \sqrt{2} \), for example and using \( K_2(z \ll 1) \approx 2z^2 - \frac{1}{2} + O(z^2) \).

This potential vanishes in the limit \( m \to 0 \) where the contributions from the gauge field cancels that of the fermion (we note that \( K_2(z \to \infty) \approx \sqrt{\pi/2} e^{-z} \)) and this holds to any order in perturbation theory. Non-perturbative effects, however, give the Wilson line a non-zero value \( \Omega = \text{diag} (e^{i\Delta\theta/2}, e^{-i\Delta\theta/2}) \), where \( \Delta\theta \) is the separation of eigenvalues of the Wilson line and equals \( \pi \) in the centre-symmetric SYM vacuum at \( m = 0 \).

With a small gaugino mass \( m \ll \Lambda \), corresponding to soft supersymmetry breaking, the mass independent term of the effective potential vanished as seen in Section 6.2.1 for general gauge group and we are left with a \( O(m^2) \) contribution

\[ V_{\text{pert.}}^{\text{SYM}}(\Omega) = -\frac{m^2}{2\pi^2 L^2} \sum_{n=1}^{\infty} \frac{|\text{tr} \Omega^n|^2}{n^2} = -\frac{m^2}{L^2} B_2 \left( \frac{\Delta\theta}{2\pi} \right), \]  

(8.41)

where \( B_2(x) = x^2 - x + 1/6 \) is the second Bernoulli polynomial valid for \( \Delta\theta \in [0, 2\pi] \) and with periodic continuation outside that interval. Note that (8.41) has minima at \( \Delta\theta = 0, 2\pi \) and so the \( n_f = 1 \) theory has broken centre-symmetry, however exponentially small non-perturbative corrections can lift the Wilson lines to a centre symmetric phase.

I return now to a discussion of the non-perturbative effects at \( m = 0 \) (similar to that done in Section 3.2.2). Here the SU(2) theory is broken to an Abelian U(1) theory through the ‘Higgsing’ of the Wilson line \( \Omega \). As before, Abelian duality \( \epsilon_{\mu\nu\lambda} \partial_\lambda \sigma = \frac{4\pi L}{g^2} F_{\mu\nu} \) maps the gauge field onto a spin-zero dual photon \( \sigma \). I also define the exponent of gauge holonomy \( b \equiv \frac{4\pi}{g^2} \Delta\theta \). The kinetic term is that obtained previously

\[ L = \frac{1}{2} \frac{g^2}{(4\pi)^2 L} [(\partial_\rho \sigma)^2 + (\partial_\rho b)^2], \]

which corresponds to a Kähler potential that is given by \( K = \frac{1}{2} \frac{g^2}{(4\pi)^2 L} B^1 B \) in terms of the chiral superfield \( B \) whose lowest component is \( b - i\sigma \) and has fermionic component \( \lambda \) being the massless gluino field.

On \( \mathbb{R}^3 \times S^1_L \) for SU(2) there are the two types of monopole-instantons: the BPS and KK monopole-instantons, with amplitudes given in Section 3.2.2 as

\[ \mathcal{M}_{\text{BPS}} = e^{-b+i\sigma} \lambda \lambda, \quad \mathcal{M}_{\text{KK}} = e^{-8\pi^2/g^2} e^{b-i\sigma} \lambda \lambda, \]  

(8.42)

respectively. The anti-monopole-instantons are the complex conjugates of these \( \mathcal{M}_{\text{BPS,KK}} \). The topological theta angle is set to zero for now, and they each carry two fermionic zero modes due to the Callias
index theorem on $\mathbb{R}^3 \times S^1_L$. The superpotential in terms of $B$ is given by

$$W_{\mathbb{R}^3 \times S^1_L} = \frac{M_{PV}^3 L}{g^2} (e^{-B} + e^{-8\pi^2/g^2} e^B) = 2 \frac{M_{PV}^3 L}{g^2} e^{-4\pi^2/g^2} \cosh(B - \frac{4\pi^2}{g^2}), \quad (8.43)$$

normalized at the cutoff scale $M_{PV}$. The Lagrangian is given by [31]

$$\mathcal{L} = \int d^4\theta K + \left( \int d^2\theta W + \text{h.c.} \right), \quad (8.44)$$

leading to the scalar potential

$$V(b, \sigma) \equiv \left( \frac{\partial B}{\partial B} \right)^{-1} \left| \frac{\partial W}{\partial B} \right|^2 = \frac{64\pi^2 M_{PV}^6 L^3 e^{-8\pi^2/g^2}}{g^2} (\cosh 2b - \cos 2\sigma), \quad (8.45)$$

where I used the shifted field $b' \equiv b - 4\pi^2/g^2$ and $\Delta \theta = \frac{4\pi}{4\pi} b' + \pi$. The potential can also be written in terms of the strong coupling scale $\Lambda$ [7]

$$\Lambda^3 = \frac{M_{PV}^3}{g^2} e^{-4\pi^2/g^2}, \quad 4\pi^2/g^2 \approx \log \frac{1}{\Lambda L}. \quad (8.46)$$

The expression (8.45) can be rewritten to give the contribution of the bions to the effective potential

$$V_{\text{bion}}(b', \sigma) = 48 L^3 \Lambda^6 \log \frac{1}{\Lambda L} (\cosh 2b' - \cos 2\sigma). \quad (8.47)$$

This potential has minimum at $\langle b' \rangle = 0$ which corresponds to a repulsive potential separating the eigenvalues as $\Delta \theta = \pi$ at the minimum (the sigma field has two minima $\langle \sigma \rangle = \{0, \pi\}$ corresponding to discrete chiral symmetry breaking). Hence the bions allow for centre-stabilization of the theory. Furthermore they generate mass gap for the fields (gauge fluctuations $\sigma, b'$) respectively by the operators $e^{\pm 2i\sigma}$ and $e^{\pm 2ib'}$.

Putting everything together I get the effective Lagrangian of the monopole-bion gas to be:

$$\mathcal{L} = \frac{1}{2} \frac{g^2}{(4\pi)^2 L} \left[ (\partial_{\mu} \sigma)^2 + (\partial_{\mu} b)^2 \right] + i \frac{L}{g^2} \lambda \sigma_{\mu} \partial_{\mu} \lambda + \alpha e^{-4\pi^2/g^2(L)} \left[ (e^{-b' + i\sigma} + e^{b' - i\sigma}) \lambda \lambda + \text{h.c.} \right]$$

$$+ \beta \frac{e^{-8\pi^2/g^2(L)}}{L^3} (\cosh 2b' - \cos 2\sigma), \quad (8.48)$$

where the prefactors contain $\log(1/\Lambda L)$ factors. The monopole terms carry two fermionic zero modes and so do not contribute to the vacuum structure of the theory when $m = 0$, however they do at non zero $m$ which will be discussed next.

Let me now add a small mass $m << \Lambda$ to the gaugino breaking the $N = 1$ supersymmetry. This gives a mass perturbation to the Lagrangian of $\Delta \mathcal{L}_m = \frac{m}{g^2} \text{tr} \lambda \lambda + \text{h.c.}$ and lifts the zero modes of the monopole-instanton amplitudes (8.42) allowing the monopole-instantons to contribute to the vacuum structure and effective potential of the theory. The result of the contribution of monopole instantons,
to leading order in $m$ with $mL << 1$ is

$$V_{\text{mono.}} = 24mL^3 \Lambda^3 \cos \sigma \left( \log \frac{1}{\Lambda L} \cosh b' - \frac{b'}{3} \sinh b' \right). \quad (8.49)$$

The details of arriving at this result will not be carried out here as they are done in the Appendix of [7], but I briefly describe them now. Firstly, the integral over the collective coordinates for a BPS monopole in SYM on $\mathbb{R}^3 \times S^1_L$ for $SU(2)$ is

$$\int d\mu_{\text{BPS}} = \frac{M_{PV}^3 L}{2\pi g^2} \int d^3\tilde{a} e^{-\tilde{a}^2 \xi} e^{-b + i\sigma},$$

where $\tilde{a}$ is the coordinate of the centre of the monopole, $\phi$ is the $U(1)$ angular collective coordinate, and $\xi$ are the two Grassmannian collective coordinates for the two fermionic zero modes attached to the monopole. The prefactor is the product of all collective coordinate Jacobians. Then, when adding a fermion mass term to the Lagrangian $L_m = \frac{mL}{g^2} \text{tr} \left[ \lambda \lambda \right]$ one must integrate over the fermion zero modes at position $\tilde{a}$, $\lambda^0(x - a)$:

$$\int d\mu_{\text{BPS}} e^{-\int d^4x L_m} = \frac{M_{PV}^3 L}{g^2} \int d^3\tilde{a} e^{-\tilde{a}^2 \xi} \text{tr} \left[ \lambda^0(x - a) \right]^2. \quad (8.50)$$

It is calculated in [58] that the last integral $\frac{mL}{g^2} \text{tr} \left[ \lambda^0(x - a) \right]^2 = 2mb$ and so the contribution to the effective potential of a single BPS monopole is $\Delta V_{\text{BPS}} = \frac{2mL^3}{g^2} \Lambda^3 \cos \sigma \left( \log \frac{1}{\Lambda L} \cosh b' - \frac{b'}{3} \sinh b' \right)$. Similarly for a single KK monopole the contribution is $\Delta V_{\text{KK}} = \frac{2mL^3}{g^2} \Lambda^3 \cos \sigma \left( \log \frac{1}{\Lambda L} \cosh b' - \frac{b'}{3} \sinh b' \right)$. To get the total contribution of the monopoles to the effective potential in SYM for $SU(2)$ I add the contributions from all monopole types and their anti-monopoles to get:

$$V_{\text{mono.}} = \frac{32\pi^2 LM_{PV}^3}{g^4} e^{-4\pi^2/g^2} \cos \sigma \left( \cosh b' - \frac{g^2}{4\pi^2} b' \sinh b' \right), \quad (8.51)$$

which is equivalent to (8.49). The case for general gauge group can be carried out in a similar way but will not be presented here but is briefly discussed in [2].

One should note that even with a mass lifting the fermionic zero modes the bions still can form provided the gaugino mass is smaller than the inverse size of the bions, where $r_{\text{bion}} = 4\pi L/g^2$. The full scalar effective potential finally reads

$$V_{\text{total}}^{\text{eff.}} = 48L^3 \Lambda^6 \log \frac{1}{\Lambda L} \left( \cosh 2b' - \cos 2\sigma \right)$$

$$+ 24mL^3 \Lambda^3 \cos \sigma \left( \log \frac{1}{\Lambda L} \cosh b' - \frac{b'}{3} \sinh b' \right) - \frac{m^2}{36L \log \frac{1}{\Lambda L}} (b')^2, \quad (8.52)$$

where the last term comes from expanding the perturbative GPY potential about its minimum. This term is $\log \frac{1}{\Lambda L}$ suppressed and so we can ignore it when examining the deconfinement phase transition. Let me examine the symmetry breaking of the potential (8.52) and the nature of the transition.

In the regime $c_m = \frac{m}{L^2 \Lambda^6} << 1$ the bion centre-stabilizing term $\cosh b'$ dominates over the monopole and GPY contributions, which favour centre symmetry breaking. If I expand (8.52) about the minimum
Chapter 8. Deconfinement of SYM on $\mathbb{R}^3 \times S^1_L$

Figure 8.2: Effective potential for the $b'$-field, showing a second order phase transition for lengths less than a critical length $L_c$ (with gaugino mass $m$ fixed) where the $b'$-field gets a non-zero VEV and centre symmetry is broken. Equivalently, with $L$ fixed, centre symmetry is broken for masses above a critical gaugino mass $m^*$. 

\[ \langle \sigma \rangle = \pi, \langle b' \rangle = 0, \text{ to second order in fluctuations } \delta \sigma, \delta b', \text{ I get (ignoring a constant term and overall factor) [7]}: \]

\[ V' = (1 - \frac{c_m}{8} \left[ 1 + \frac{2}{3 \log \Lambda L} - \frac{c_m}{432 \log^3 \Lambda L} \right]) (\delta b')^2 + (1 + \frac{c_m}{8}) (\delta \sigma)^2. \]  

(8.53)

It is clear in the regime $c_m < \approx 8$ that we are in a centre-symmetric phase and the system is confined. I then obtain the approximate shape of the transition line in Figure 8.1: $L \approx \sqrt{m/8\Lambda^3}$. This can also give us an estimate of the thermal deconfinement transition temperature by rough extrapolation of the curve. For $m >> \Lambda$ the curve should no longer depend on $m$. For rough values of the fermion mass where the transition approaches that of a pure Yang-Mills theory, say $m \approx (5 - 10)\Lambda$, we expect the thermal deconfinement transition to occur at $T_c = 1/L_c \approx (0.8 - 1.3)\Lambda$, in agreement with lattice results. See [7] and references cited there. Outside this regime the $b'$ field takes on a non-zero VEV and the centre symmetry is broken and the system becomes deconfined.

Figure 8.2 shows the (in fact) second order phase transition where below a certain critical length $L_c$ the $\mathbb{Z}_2$ centre symmetry is broken and the $b'$-field gets a non-zero VEV (equivalently with $L$ fixed one observes a second order phase transition to a deconfined centre-broken phase at masses above a critical gaugino mass $m^*$). I will now go on to general gauge group.

8.3.2 Case of general gauge group

Before going on to general gauge group let me first present some qualitative features of deconfinement of SYM* such as string tensions, centre symmetry breaking, Wilson line eigenvalues, and theta dependence for some sample gauge groups.

Figure 8.3 shows how the string tension depends on the gaugino mass for an example spin(7) gauge group. It jumps discontinuously to zero for masses above a critical gaugino mass. This is qualitatively the same for all groups with non-trivial centre, and is also observed in lattice simulations.
Chapter 8. Deconfinement of SYM on $\mathbb{R}^3 \times S^1_L$

Figure 8.3: Discontinuous jump in the string tension at the critical gaugino mass for spin(7) group.

Figure 8.4: Theta dependence of critical gaugino mass (left) and jumps in the Wilson loop observable (right) at the transition for spin(6) group.

Figure 8.5: Distribution of Wilson loop eigenvalues on the unit circle, below (left) and above (right) the critical gaugino mass for Sp(12). Note the $\mathbb{Z}_2$ centre symmetry is not preserved above the transition.
Figure 8.4 shows the theta dependence of the critical gaugino mass and the discontinuous jump in the Wilson loop trace at the critical point for spin(6) gauge group. This behaviour (of the jump increasing with increasing theta and the critical gaugino mass decreasing with increasing theta angle) is universal for all gauge groups and agrees qualitatively for all gauge groups with lattice simulations done thus far.

Figure 8.5 shows how centre-symmetry breaks, for the example symplectic gauge group $sp(12)$, as the gaugino mass (or equivalently the temperature in a simulation of pure Yang-Mills) is increased. Above the critical gaugino mass the $\mathbb{Z}_2$ centre symmetry ($\Omega \rightarrow -\Omega$) clearly is broken and the eigenvalues tend to clump together, whereas they remain equally spaced apart so $\langle \text{Tr} \Omega^k \rangle = 0 \forall k$ in the confined phase below $m^*$. This behaviour is also universal for all groups with centre $Z(G)$ and is qualitatively similar to studies done so far on the lattice of (pure) Yang-Mills.

To begin the study of deconfinement in general gauge group of SYM I need to derive the full effective potential. The bion term was found from the superpotential and is $V_{\text{bion}}(\vec{b}, \vec{\sigma}) \equiv (\partial_B, \partial_B K)^{-1} \left| \frac{\partial W}{\partial B} \right|^2$ as found in Section 6.3.2, equation (6.42). The monopole contribution with a mass for the gaugino is found similarly as described in the Appendix in [7] and in [2]. The idea is that I integrate over the moduli space of the monopole-instanton, integrate over the mass insertion term $\propto \int \text{Tr} \vec{X} \cdot \vec{\lambda}$, and the rest of the factors come from the superpotential (6.39). The result for any gauge group $G$ and the $\vec{\alpha}_a$-th monopole is [2]

$$V_{\text{mono.}}^{\alpha_a} = \kappa \frac{R m}{g^2} \frac{2}{\alpha_2^2} \frac{2}{R} (\frac{8 \pi^2}{g^2} \delta_{a\alpha} - \vec{\alpha}_a \cdot \vec{\sigma}) e^{2 \pi i \delta_{a\alpha} + \vec{\alpha}_a \cdot \vec{X}}. \quad (8.54)$$

I then sum over all monopole (and anti-monopole) types to get the total monopole effective potential, which after using

$$V_m = -V_m^0 \sum_{a=0}^{r} k_a^\nu \nu e^{-\vec{\alpha}_a^\nu \vec{\delta}} \cos(\vec{\alpha}_a^\nu \cdot \vec{\sigma}) + \frac{\theta + 2 \pi u}{c_2(G)}, \quad (8.55)$$

where $V_m^0 = \frac{m \nu}{R} \frac{128 \pi^2}{c_2(G) |\nu| g^4} e^{-8 \pi^2 / c_2(G)(G)}$ with $|\nu| = \prod_{i=0}^{r} (\frac{k_i^\nu \delta_i^\nu}{2})^{1/c_2(G)}$, and $u = 1, \ldots, c_2(G)$ label the $c_2(G)$ different vacua of the theory.

Putting everything together I obtain a (rather elegant) total effective potential from the monopole-instanton and bion contributions:

$$\frac{V_T}{V_{\text{bion}}} = \sum_{a,b=0}^{r} k_a^\nu k_b^\nu \alpha_a^\nu \cdot \alpha_b^\nu e^{-(\vec{\alpha}_a^\nu + \vec{\alpha}_b^\nu) \cdot \vec{\delta}} \cos(\vec{\alpha}_a^\nu \cdot \vec{\sigma}) - c_m \sum_{a=0}^{r} k_a^\nu e^{-\vec{\alpha}_a^\nu \vec{\delta}} \cos(\vec{\alpha}_a^\nu \cdot \vec{\sigma} - \frac{\theta + 2 \pi u}{c_2(G)}), \quad (8.56)$$

where $c_m = V_m^0 / V_{\text{bion}}^0$. The total bosonic part of the Lagrangian then takes the form

$$L_{\text{bosonic}} = \frac{1}{2} \frac{g^2 (m_W)^2}{(4 \pi)^2 R} (\partial_\mu \vec{b})^2 + (\partial_\mu \vec{\sigma})^2 + V_T + V_{\text{GPY}}. \quad (8.57)$$

Once again, I ignore the perturbative GPY corrections found in Section 6.2.1 as they are $\log 1/c_2 \Delta L$ suppressed as in the case of $SU(2)$. The goal now is to minimize the potential (8.56) and look for degenerate minima at some critical gaugino mass and for different values of theta. I also pick the value of $u$ which gives the global minimum of the total potential. The occurrence, at some critical gaugino mass, of
Chapter 8. Deconfinement of SYM on $\mathbb{R}^3 \times S^1_L$

degenerate minima determines the phase transition to be first order, shifting from the supersymmetric minimum of $\vec{b} = \vec{\sigma}' = 0$ to one where $\vec{b} \neq 0$. Furthermore, calculations of Polyakov loops and their correlators can be done to determine the parameters of the potential of interaction of quark-anti-quark pairs, such as strings tensions, as well as the discontinuous jumps of Polyakov loop traces at the transition for different values of theta.

To determine the discontinuous nature of the phase transition and the breaking of centre symmetry the traces of Polyakov loops expectation values need to be calculated in the appropriate representation. To leading order in $g^2$ we have the approximation;

$$\text{Tr} \langle \Omega \rangle = \sum_{\vec{w} \in \Delta^g} e^{i\vec{w} \cdot \langle \vec{\phi} \rangle} \approx \sum_{\vec{w} \in \Delta^g} e^{i\vec{w} \cdot (\vec{\phi}_0^{(0)})} (1 + i \frac{g^2}{4\pi} \vec{w} \cdot \vec{\phi}_0^{(1)} + i \frac{g^2}{4\pi} \vec{w} \cdot \langle \vec{b} \rangle). \quad (8.58)$$

Here $\vec{w}$ are the weights of the corresponding representation and $\langle \vec{b} \rangle$ is the expectation value above the transition. The magnitude of the discontinuity of the Polyakov loop can then be written as

$$\frac{4\pi}{g^2 c_2(G)} |\text{Tr} \langle \Delta \Omega \rangle| = \frac{1}{c_2(G)} \sum_{\vec{w} \in \Delta^g} e^{i\vec{w} \cdot (\vec{\phi}_0^{(0)})} \vec{w} \cdot \langle \vec{b} \rangle. \quad (8.59)$$

For symplectic groups $Sp(2N)$ I use the fundamental representation and for $Spin(2N + 1)$ theories I use the spinor representation while for $Spin(2N)$ I use the two chiral spinor representation, as probes of centre symmetry breaking. For $E_6,7$ I did not calculate traces of $\Omega$ in any representation and instead centre symmetry breaking was found by explicit application of the $Z_2$ ($Z_3$), respectively, on $\vec{b}$ to determine if the symmetry is broken with the required number of degenerate vacua above the critical gaugino mass $c_{cr}$ [2].

Let me now describe these results for all gauge groups both with or without centre.

8.3.3 Groups with centre

Beginning with groups that have centre I summarize the results of the Polyakov loop correlators and theta dependence of discontinuous jumps of Polyakov loop traces and of critical gaugino mass. Beginning with the cases of the orthogonal and symplectic group I will make some comments on the literature: $Sp(4)$ has been the first non-$SU(N)$ thermal gauge theory studied on the lattice. Despite the fact that a $Z_2$ centre-symmetry allows for a continuous transition in the 3d Ising universality class, as $SU(2)$, it is found that the transition is first order. This indicates that, not only do the results here support this, assuming they are continuously connected to the thermal theory, but also show that centre symmetry and universality class do not alone determine the nature of the phase transition. I begin a discussion and presentation of results for the symplectic and orthogonal groups (with universal cover the spin groups) and exceptional groups with centre. I will then compare these to the cases of groups without centre.

$spin(2N)$ case:

This group is the double (universal) cover of the $SO(2N)$ special orthogonal group. In calculating
Chapter 8. Deconfinement of SYM on $\mathbb{R}^3 \times S_L^1$

Table 8.1: The critical mass, $c_{cr}$, for the spinor representation Polyakov loop and its correlator coefficients for the $Spin(2N)$ group. The string tensions $\hat{\sigma}$ and other coefficients measured by the two spinor representations are identical.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>$\frac{\pi}{12}$</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{cr}$</td>
<td>2.953</td>
<td>2.951</td>
<td>2.946</td>
<td>2.937</td>
<td>2.925</td>
<td>2.908</td>
<td>2.886</td>
<td>2.859</td>
<td>2.825</td>
</tr>
<tr>
<td>$</td>
<td>\text{Tr}(\Delta\Omega)</td>
<td>$</td>
<td>2.277</td>
<td>2.279</td>
<td>2.287</td>
<td>2.299</td>
<td>2.318</td>
<td>2.342</td>
<td>2.371</td>
</tr>
</tbody>
</table>

Table 8.2: The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $Spin(6)$ group.

Polyakov loops I use weights in the two spinor representations that are of the form

$$\{w\} = \{\pm \frac{1}{2}, \pm \frac{1}{2}, \ldots, \pm \frac{1}{2}\}$$

where the two representations are whether there are an even or odd number of minus signs. The $spin(2N)$ groups have a $\mathbb{Z}_4$ or a $\mathbb{Z}_2^2$ centre when $N$ is odd or even, respectively. It acts on the scalar field $\phi$ in the respective cases of $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$ as:

$$\phi_i \rightarrow \begin{cases} 
\pi + \phi_N, & \text{for } i=1, \\
\pi - \phi_{i+1}, & \text{for } i>1, 
\end{cases}$$

$$\phi_i \rightarrow \pi \pm \phi_N (i = 1), \pi - \phi_{N+1} (i > 1), \mp \pi \pm \phi_i (i = N).$$

Table 8.1 shows the critical mass $c_{cr}$, Polyakov loop discontinuity and the parameters of the Polyakov loop correlator in (8.38). A first order phase transition is found for all $N \geq 3$. Both for even and odd $N$, the $spin(N)$ groups both have dimensionless string tensions equal to $2\sqrt{2}$ at $c_{m} = 0$. There is a slight decrease in the dimensionless string tensions for odd $N$ at $c_{m} = c_{cr}$, and is larger for even $N$.

It is also found that, as seen in Figure 8.4, that the value of $c_{cr}$ decreases with increasing $\theta$ angle, while the discontinuity of the Polyakov loop increases with increasing $\theta$. This is qualitatively the same for all gauge groups. The $\theta$ dependence of an example group $spin(6)$ is shown in Table 8.2 illustrating this dependence.

$spin(2N+1)$ case:

Here this is the double (universal) cover of the $SO(2N+1)$ special orthogonal group. The only difference here to that of $spin(2N)$ is that there is just one spinor representation, and the centre symmetry is $\mathbb{Z}_2$ and acts on $\phi$ as

$$\phi_i \rightarrow \begin{cases} 
1 - \phi_i, & \text{for } i=1, \\
\phi_i, & \text{for } i>1. 
\end{cases}$$
Table 8.3: The critical mass, $c_{cr}$, the spinor representation Polyakov loop and its correlator coefficients for the $Spin(2N + 1)$ group.

| Spin | $c_{cr}$ | $\frac{4\pi}{g^2} \times |\Gamma(\Delta \Omega)|$ | $A_0$ | $\hat{A}_0$ | $A_{cr}$ | $\hat{A}_{cr}$ | $\lambda_{cr}$ | $D_{cr}$ |
|------|---------|---------------------------------|------|---------|---------|---------|--------|--------|
| (5)  | 3.8     | 1.503                           | 3    | $2\sqrt{2}$ | 3        | 0.593   | 2.756  | 0.878  | 2.26   |
| (7)  | 2.21    | 4.69                            | 5    | $2\sqrt{2}$ | 5        | 1.89    | 3.57   | 1.23   | 22     |
| (9)  | 1.324   | 7.55                            | 7    | $2\sqrt{2}$ | 7        | 2.3     | 4      | 0.86   | 57     |
| (11) | 0.871   | 10.46                           | 9    | $2\sqrt{2}$ | 9        | 2.48    | 4.42   | 0.595  | 109.4  |
| (13) | 0.615   | 13.58                           | 11   | $2\sqrt{2}$ | 11       | 2.6     | 4.65   | 0.42   | 182    |
| (15) | 0.457   | 16.7                            | 13   | $2\sqrt{2}$ | 13       | 2.66    | 4.8    | 0.32   | 278    |

Table 8.4: The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $Spin(7)$ group.

Table 8.3 shows the results for the odd dimensional $Spin(N)$ groups including critical masses and parameters of the Polyakov loop correlator. A first order transition is again found for all $N \geq 3$. Table 8.4 gives the $\theta$ dependence of the critical masses and the Polyakov loop jumps for an example $Spin(7)$ group. Again, a qualitatively universal behaviour of increasing $\theta$ causing a decrease in $c_{cr}$ and an increase in $|\langle \Delta \Omega(\theta) \rangle|$. is observed.

**Symplectic groups Sp(2N):**

This is the symplectic group which is defined as the set of $2N \times 2N$ unitary matrices $\mathcal{M}$ which preserve the anti-symmetric scalar product

$$\mathcal{M}^T J \mathcal{M} = J,$$

where $J = \left( \begin{array}{cc} 0 & I_N \\ -I_N & 0 \end{array} \right)$. (8.63)

For computing traces of Polyakov loop observables I use fundamental representation probes, and the weights of the fundamental representation are

$$\{ \vec{u} \} = \{ \vec{e}_i / \sqrt{2}, \text{ for } i = 1, \ldots, N \\ -\vec{e}_i / \sqrt{2}, \text{ for } i = N + 1, \ldots, 2N \}. \quad \text{(8.64)}$$

The centre for all $N$ is $\mathbb{Z}_2$ and acts on the field $\tilde{\phi}$ by

$$\phi_i \mapsto \pi - \phi_{N+1-i},$$

for $i = 1, \ldots, N$. By studying the minima of the full potential $V_T = V_{bion} + c_m V_m$ as $c_m$ is varied a first order phase transition is found for all $N \geq 2$. Table 8.5 shows the results of critical masses, Polyakov loop jumps and parameters of the Polyakov loop correlator. As $N$ is increased we see a decrease in the value of $c_{cr}$, and the discontinuity of the trace of the Polyakov loop, normalized to unity can be fitted as follows [2]

$$\frac{1}{2N} |Tr \langle \Delta \Omega \rangle| \approx (0.217 - \frac{0.178}{N} - \frac{0.005}{N^2}) \pi \hat{g}^2 c_2 \frac{4}{4\pi},$$

(8.65)
acts on the field $\vec{\phi}$. The results are in qualitative agreement with that of the spin and orthogonal groups shown previously, with $c_{\text{cr}}$ decreasing and the Polyakov loop jump increasing with increasing $\theta$. The transition is still first order and has the same qualitative behaviour as the $SU(N)$ cases in [61] and agree to studies performed there on the lattice.

### Case of $E_6$.

This is one of the exceptional groups with centre symmetry, and $Z(E_6) = Z_3$ is its centre. The center acts on the field $\vec{\phi}$ as:

$$
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_5 \\
\phi_6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\pi + \frac{1}{2} \phi_2 - \frac{1}{2} \phi_3 \\
-\frac{\pi}{3} + \frac{1}{2} \phi_2 + \frac{1}{2} \phi_4 + \sum_{i=4}^{6} \phi_i \\
-\frac{\pi}{3} - \frac{1}{2} \phi_2 - \frac{1}{2} \phi_4 - \phi_6 \\
-\frac{\pi}{3} - \frac{1}{2} \phi_2 - \frac{1}{2} \phi_3 - \phi_5 \\
-\frac{\pi}{3} - \frac{1}{2} \phi_2 - \frac{1}{2} \phi_3 - \phi_4 \\
-\frac{\pi}{3} + \frac{1}{2} \phi_2 + \frac{1}{2} \phi_3 + \phi_1
\end{pmatrix}.
$$

(8.67)
It is easy to check that the supersymmetric minimum
\[
\tilde{\phi}_0 = \tilde{\phi}^{(0)} + \frac{g^2}{4\pi} \tilde{\phi}^{(1)} = (2.879, -1.31, -0.78, -0.262, 0.262, 0.78, 1.31, -2.879)
+ \frac{g^2}{4\pi} (-0.155, 0.155, -0.155, -0.119, 0.119, 0.155, -0.155, -0.155),
\]
is invariant under the $Z_3$ symmetry.

As we increase $c_m$, the theory experiences a first order transition at $c_{cr} = 0.360$. There are three broken minima which are related via the $Z_3$ symmetry. Table 8.7 shows the results of the $\theta$-angle dependence on the critical mass and Polyakov loop jumps. The results are qualitatively similar to the other groups.

**Case of $E_7$.**

This is the rank 7 special group with dimension 133 and $Z_2$ centre symmetry.

It is easy to check that the supersymmetric minimum
\[
\tilde{\phi}_0 = \tilde{\phi}^{(0)} + \frac{g^2}{4\pi} \tilde{\phi}^{(1)}
= (-4.276, 1.658, 1.309, 0.960, 0.611, 0.262, -0.0873, -0.436)
+ \frac{g^2}{4\pi} (-0.342, 0.111, 0.217, 0.120, -0.120, -0.217, -0.111, 0.342),
\]
is invariant under the $Z_2$ symmetry.

As $c_m$ is increased, the group experiences a first order transition at $c_{cr} = 0.360$. There are two broken minima which are related via the $Z_2$ symmetry.

**8.3.4 Groups without centre**

The case of groups without centre differs from those with centre in that there is a phase transition without order parameter: there is no centre-symmetry to break and the Polyakov loop traces do not vanish identically in the confined phase. Yet there is still a first order phase transition with a jump in the Polyakov loop observable at the critical gaugino mass. Let me begin by revisiting the case of $G_2$ considered previously in [7], [30].

The general Polyakov loop correlator in this case looks as follows:

\[
\langle \text{Tr} \Omega(x) \text{Tr} \Omega^\dagger(y) \rangle = \left\{ \begin{array}{ll}
\left( \frac{g^2}{4\pi} \right)^2 D_0 + \frac{g^2 R}{4r} A_0 \ e^{-\sigma_{m0r}}, & c_m = 0 \\
\left( \frac{g^2}{4\pi} \right)^2 D_{c_{cr}} + \frac{g^2 R}{4r} A_{c_{cr}} \ e^{-\sigma_{c_{cr}m0r}}, & c_m = c_{cr} \\
\left( \frac{g^2}{4\pi} \right)^2 D_{c_{cr}+} + \frac{g^2 R}{4r} C \ e^{-\lambda c_{cr}m0r}, & c_m = c_{cr+},
\end{array} \right.
\]
where there are extra constant terms below the transition point, which makes it differ from the case of groups with centre.

**$G_2$ revisited:**

This group has rank 2 and dimension 14. The Cartan generators of the 7 dimensional fundamental representation (or equivalently, the weights of the fundamental representation) are:

\[ H_1 = \text{diag} \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right), H_2 = \text{diag} \left( \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{\sqrt{2}}{3}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{3}, 0 \right). \]

The supersymmetric minimum for $G_2$ has $g^2$ dependence:

\[
\vec{\phi}_0 = \vec{\phi}^{(0)} + g^2 \frac{4\pi}{9} \vec{\phi}^{(1)} = (3.33, 0.64) + g^2 \frac{4\pi}{9} (0.025, -0.2389).
\] (8.71)

The $g^2$ dependence gets contributions from both tree-level and one-loop corrections. The trace of the Polyakov loop in the fundamental representation at the supersymmetric minimum reads:

\[
\langle \text{Tr} \Omega \rangle = \text{Tr}_f \left[ e^{i\vec{\phi}_0 \cdot \vec{H}} \right] \cong \text{Tr}_f \left[ e^{i\vec{\phi}_0^{(0)} \cdot \vec{H}} \left( 1 + i g^2 \frac{4\pi}{9} \vec{\phi}_0 \cdot \vec{H} \right) \right] = g^2 \frac{4\pi}{9} 0.0746. \] (8.72)

By studying the total potential, we find that the $G_2$ group experiences a first order phase transition at the critical value $c_{cr} = 3.174$. The trace of the Polyakov loop in the fundamental representation at the broken minimum is

\[
\langle \text{Tr} \Omega \rangle = \text{Tr}_f \left[ e^{i(\vec{\phi}_0 + \frac{2g}{4\pi} \vec{b}) \cdot \vec{H}} \right] \cong \text{Tr}_f \left[ e^{i\vec{\phi}_0 \cdot \vec{H}} \left( 1 + i g^2 \frac{4\pi}{9} \vec{\phi}^{(1)} \cdot \vec{H} \right) \right] = g^2 \frac{4\pi}{9} 3.437. \] (8.73)

The Polyakov loop correlator $\langle \text{Tr} \Omega(x) \text{Tr} \Omega(y) \rangle$ is now,

\[
\langle \text{Tr} \Omega(x) \text{Tr} \Omega(y) \rangle = \begin{cases} 
0.056 \left( \frac{g^2}{4\pi} \right)^2 + \frac{g^2 R}{4\pi} \times 4 e^{3.586 m_0 r}, & c_m = 0 \\
0.056 \left( \frac{g^2}{4\pi} \right)^2 + \frac{g^2 R}{4\pi} \times 4 e^{-2.194 m_0 r}, & c_m = c_{cr-} \\
11.3 \left( \frac{g^2}{4\pi} \right)^2 + \frac{g^2 R}{4\pi} \times 3.98 e^{-1.49 m_0 r}, & c_m = c_{cr+}
\end{cases} \] (8.74)

Naturally, as expected in a theory without center symmetry, the above correlators show that there is no linear confinement of fundamental charges, but rather “string breaking”.

The lattice studies of thermal $G_2$ pure YM theory [30] found that the trace of the Polyakov loop changes from a small (close to zero) value below $T_c$ to a large positive value above $T_c$, “exactly” as a look at equation (8.74) would indicate. Table 8.8 gives the $\theta$-angle dependence of $c_{cr}$ and Polyakov loop jumps for $G_2$.

**$F_4$ group:**

This is one of the five special groups, it has rank 4 and dimension 52 and has no centre. It is the isometry group of a 16 dimensional Riemannian manifold known as the octonionic projective plane.
Chapter 8. Deconfinement of SYM on $\mathbb{R}^3 \times S^1_L$

Table 8.8: The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $G_2$ group.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>$\frac{\pi}{12}$</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{9\pi}{4}$</th>
<th>$\frac{5\pi}{3}$</th>
<th>$\frac{3\pi}{2}$</th>
</tr>
</thead>
</table>

Table 8.9: The $\theta$-dependence of the critical transition mass and the Polyakov loop discontinuity for the $F_4$ group.

Examine the full potential reveals a first order phase transition at $c_{cr} = 0.922$ not associated with symmetry breaking. Table 8.9 shows the $\theta$-angle dependence of $c_{cr}$ and the jumps of the Polyakov loop traces in this case.

$E_8$ group:

This is the last of the five special groups and has no centre. It is of rank 8 and dimension 248.

The supersymmetric minimum is found at

$$\vec{\phi}_0 = \vec{\phi}_0^{(0)} + \frac{g^2}{4\pi} \vec{\phi}_0^{(1)} = (4.817, 1.257, 1.047, 0.838, 0.628, 0.419, 0.209, 0)$$

$$+ \frac{g^2}{4\pi} (0.669, -0.014, 0.293, 0.398, 0.359, 0.2087, -0.0329, -0.0719).$$

The trace of the Polyakov loop in the fundamental (for $E_8$, the same as the adjoint) representation at the supersymmetric minimum reads

$$\langle \text{Tr} \Omega \rangle = \text{Tr} \left[ e^{i\vec{\phi}_0} \cdot \hat{H} \right] \cong \text{Tr} \left[ e^{i\vec{\phi}_0^{(0)} \cdot \hat{H}} \left( 1 + i \frac{g^2}{4\pi} \vec{\phi}_0^{(1)} \cdot \hat{H} \right) \right] = -1 - \frac{g^2}{4\pi} \times 4.47.$$  \hspace{1cm} (8.76)

As we increase $c_m$, the theory experiences a first order transition at $c_{cr} = 0.1936$. The trace of the Polyakov loop at the broken minimum is

$$\langle \text{Tr} \Omega \rangle \cong -1 + \frac{g^2}{4\pi} \times 80.06,$$  \hspace{1cm} (8.77)

and the discontinuity at the transition is $|\text{Tr}(\Delta \Omega)| \cong 84.53 \frac{g^2}{4\pi} [2]$.

It has been found here that for all gauge groups there is a first order quantum zero temperature phase transition of deformed SYM at some critical gaugino mass that decreases with increasing theta angle. Furthermore, the jumps in Polyakov loop observables increased with increasing theta angle. All this agrees with simulations of pure thermal Yang-Mills theory done to date and supports the continuity conjecture discussed in Section 8.1. It is hoped that future study and lattice simulations on other parts of the phase diagram Figure 8.1 can be reached and other gauge groups studied as well to provide further support for this conjecture which would give us much insight into pure thermal Yang-Mills theory at strong coupling itself.
Chapter 9

Conclusion

Studying (super) Yang-Mills theory on compactified spacetimes allows the study of gauge theories in general gauge group by Abelianizing the theory and fixing the coupling at small values, allowing analytical control of the theory: a full playground at weak coupling where semiclassical calculations can be performed with relative ease in the regime $c_2(G)L\Lambda \ll 1$. This is what having a single compact spatial dimension can do. Furthermore, having a compact time/thermal direction allows study at finite temperature which permits a study of the thermal deconfinement phase transition in (super) Yang-Mills theory. In the regime $L \ll \beta = 1/T$ at low temperature the perturbative effective potential was found for all gauge groups on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$. The non-perturbative sector was also described in the language of general gauge group.

Furthermore, studies of (mass deformed) super Yang-Mills (SYM*) on $\mathbb{R}^3 \times S^1_L$ with a finite mass for the gaugino have shown that there is a first order quantum (zero temperature) deconfinement phase transition at some critical gaugino mass for all gauge groups other than $SU(2)$, where the transition is second order. It is also believed, yet so far not proven, that this quantum phase transition is continuously related, as a function of the gaugino mass, to the thermal deconfinement phase transition of pure Yang-Mills as the gaugino mass tends to infinity. From lattice studies done for some gauge groups there is qualitative agreement with the phase transitions including the order of the transition, universality class, and centre-symmetry breaking. Dependence on theta angle, jumps of Wilson loop expectation values and string tensions at the critical transition temperature/mass (with transition mass/temperature decreasing with increasing theta and jumps in Wilson loops increasing with increasing theta angle) are also seen to agree qualitatively with lattice simulations. This dependence was calculated for all gauge groups in SYM* in this work and in [2].

It was also found that $\mathcal{N} = 1$ super Yang-Mills on $\mathbb{R}^2 \times S^1_L \times S^1_\beta$ has a dual description as a double Coulomb gas of various particles: W-bosons and their wino superpartners, monopole-instantons and neutral and magnetic bions and their anti-particles. The partition function was computed as well as the duality maps to the Coulomb gas of $r = \text{rank}(G)$ such types of electric and magnetic charges, and several types of magnetic and neutral bions formed from combinations of BPS and KK monopoles (and their anti-monopoles) that interact with each other (that is, they are Dynkin neighbours). The electric charges are charged under the root lattice of the gauge group $G$, $\Lambda_r$, in the adjoint representation, and
the magnetic charges are charged under the co-root lattice, $\Lambda^\vee_r$. The elementary charges are the simple roots (co-roots), and their negatives. The interesting feature of this 'universal' Coulomb gas is that it presents a gas of particles of three charges: electric, magnetic and scalar. The first two interact with Coulomb-Coulomb interactions with particles of same charge type, or charge containing a root nearby on the Dynkin diagram. The scalar charges make the Coulomb gas unique as they interact such that like charges attract, and this introduces instability and exotic behaviour of the gas at different temperatures.

A dual spin model was also found to be of XY-model type with discrete symmetry breaking perturbations, and fugacities coupled to the scalar fields $\phi^a$. The model has two coupled lattices (one for the W’s which couples to a lattice of neutral and magnetic bions on one lattice) and the presence of the scalar fields $\phi^a$, which are ferromagnetic in nature as opposed to antiferromagnetic electric and magnetic charges, introduces competition between lattices the W’s couple to. Hence the model can be viewed as a 'frustrated' XY-model with symmetry-breaking perturbations [49]. The W-bosons here are interpreted as the symmetry-breaking perturbations with strength proportional to the W-boson fugacity, which depends on the fields $\phi^a$. The magnetic bions represent the vortices of the XY model instead of the W-bosons which were interpreted as vortices in [15]. The neutral bions introduce the 'frustration' and electromagnetic - scalar competition. As found in previous works [21], [23] the magnetic bions lead to mass gap for the dual photon fields $\sigma^a$, allowing for confinement of electric charges, and the neutral bions lead to a centre-stabilizing potential which dominates at low-$T$. The magnetic monopole-instantons do not lead to a mass gap as they contain fermionic zero modes and so were not considered in the study of the phases of the theory. It is noted that in studying the supersymmetric theory on a torus is that the theory is not as simple as the non-supersymmetric version, due to the presence of the scalar fields $\phi^a$, even though the GPY potential vanishes at zero temperature and partially cancels at $T > 0$. Nonetheless, the dualities derived here are interesting and have led us to new phenomena and new ways of studying Yang-Mills theory at finite temperature.

Future directions of study include the following pursuits

1. Lattice studies, as done in [1], can be done in the general gauge group setting, even if for particular gauge group such as $SU(3)$ or $G_2$, in both the dual Coulomb gas model or the XY-spin model, in order to gain better understanding of the phase transition as found in [2]. A first order phase transition is expected as opposed to the second order transition in the $SU(2)$ theory [1]. This can also lead to further study of the continuity conjecture as mentioned in [2], [7] by comparing phase transitions in pure thermal Yang-Mills to the quantum phase transitions in mass deformed super Yang-Mills. Comparison can be made to previous lattice studies and new studies in general gauge group may be possible as well, at least for the Coulomb gas as the XY models have not been yet found for general gauge group.

2. It has also been of recent interest to consider finite density QCD-like theories, in particular super Yang-Mills, and their phase transitions. There is a known sign problem due to finite chemical potential and so imaginary chemical potentials have been studied instead [3], [35]. This leads to a theory with twisted boundary conditions for the adjoint fermions along the compact direction. Computing the Callias index as a function of the ‘twist’ angle leads to a twist-dependent index, which equals the usual answer, 2, at the centre-symmetric and supersymmetric vacuum. This recent work can be generalized
to general gauge group and dependence on the boundary conditions is quite interesting.

3. Mean field theory methods can be used for the XY spin models considered here, as well as related spin models in special cases. Although not exact, mean field theory can tell us about phase transitions and their orders, although at transition temperatures that are not always correct, but at least are within an order of magnitude. Studies of XY-models with symmetry breaking perturbations have been studied [44], [45] for different values of $p$ in the $\cos p\theta$-term and phase diagrams mapped out. It would be curious to implement a mean field theory that takes into account vortices and can verify known results, and produce new ones for other gauge groups not studied before. This would be interesting even for the case of zero scalar fields $\phi^a = 0$. Cases with $\vec{\phi} \neq 0$ can be done as well in the mean field method. These cases are related to 'frustrated' XY models in the case that, on some lattices, bonds are ferromagnetic (like the scalar charged W’s and neutral bions), while on others they are antiferromagnetic (like electric W’s and magnetic bions). These competing interactions lead to 'frustration', that is a ground state that is degenerate and not at minimum possible energy without frustration. Models with competing F and AF interactions were studied in [44] and [46].

4. Renormalization group equations and flow can be determined from the partition function (4.18). Special cases for $SU(2)$ and $SU(3)$ have been done with good success in [15] leading to known results of deconfinement, transition temperatures, and scaling parameters and critical exponents. In the cases of higher rank it was found that no fixed points appeared to exist for the RGEs and that the electric-magnetic duality no longer holds. It would be interesting to continue investigating the $SU(N)$ RGE cases and other groups of higher rank to see (possibly by going to higher order in the expansion) if there are fixed points and to find the nature of the critical points.

5. General compactifications on toroidal spaces such as $\mathbb{R}^D \rightarrow \mathbb{R}^d \times (S^1)^{D-d}$ can be done in this framework, although the applications or interests may not be immediate.

6. Further lattice studies of pure Yang-Mills theory can be done for other gauge groups not yet studied to show more support of the continuity conjecture (only $Sp(4)$, $SU(N)$ and $G_2$ have been studied to date). Also more areas of the phase diagram can be reached with simulations at weaker coupling, although they are less efficient to do at weak coupling. Performing these weaker-coupled simulations would still be of importance in justifying the continuation of simulation results into regimes of weak coupling.

7. Pure Yang-Mills theory at stronger coupling itself can be studied by monopole-instanton liquid models (MILMs) considered in [59]. These studies can also help verify the continuity conjecture.

It is hoped that the work presented in this thesis is novel and interesting and has provided a framework for further research. I would expect in future work one can do simulations of the dual double Coulomb gas in the general gauge group, and perhaps dual spin models for gauge groups other than $SU(2)$ and $SU(3)$ can be proposed soon to allow simulations of them as well. I hope this work is fairly self-contained and explained the background necessary for future research. Learning and applying the new material and discovering novel Coulomb gases with interesting interactions, as well as being able to study deformed
super Yang-Mills analytically in all general cases, was an exciting and interesting task. However, there is always more work to be done!
Bibliography


[29] N. Seiberg, E. Witten. Gauge dynamics and compactification to three dimensions. [hep-th/9607163]


[40] K. Lee. Instantons and magnetic monopoles on $R^3 \times S^1$ with arbitrary simple gauge groups. hep-th/9802012 (1998)


[63] Teeple, Brett. *Deconfinement on $\mathbb{R}^2 \times S_1^1 \times S_3^1$ and duality to dual double Coulomb gas for all gauge groups*. arXiv:hep-th/1506.02110 (2015)