THREE ESSAYS ON MACROECONOMIC LABOUR SEARCH

by

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Abstract

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This thesis examines heterogeneous worker productivity in labour search models.

In Chapter 1 I examine how involuntary part-time employment varies over the business cycle. It is known that the proportion of workers claiming to be involuntarily part-time employed increases during recessions and decreases during normal times and expansions, but the mechanism through which this occurs has been generally under-studied. A model incorporating heterogeneous worker productivity, on the job search and firm deciding whether to operate jobs part-time or full-time. The model is similar to Menzio and Shi (2010), with the novel inclusions of worker specific productivity and part-time jobs. There are three primary goals: 1) provide a framework for future research in non-standard employment contracts 2) determine the relative importance of worker specific and match specific productivity in determining involuntary part-time employment and 3) determine the impact
of involuntary part-time employment on the unemployment rate and the level of welfare. It is found that while match-specific productivity is important in explaining the fluctuations of involuntary part-time employment over the business cycle, such employment is concentrated among workers with fixed low productivity. The welfare effects of involuntary part-time employment are found to be modest for the economy as a whole but relatively important for the low fixed productivity workers.

In Chapter 2, I examine the relationship between wage-tenure contracts and heterogeneous workers with fixed productivity. As a proxy for productivity, I use educational achievement. I then investigate the relationship between the wage profile for workers with less than high school, high school and college education, respectively. I consider a directed search model with on the job search similar to Shi (2009) with workers of heterogeneous fixed productivity. Following the model, I estimate a baseline wage profile which is the implied relationship between the current wage and future wage growth for a worker. I find that more educated workers do indeed enjoy a steeper wage-tenure profile. The model explains the steeper profile solely based on wage-tenure contracts, demonstrating that higher learning rates for more educated workers is not necessary to explain their steeper wage profile. The calibration exercise establishes this qualitatively, but not quantitatively, leading to the conclusion that differential learning rates are an important part of the steeper wage profiles.

In Chapter 3, I examine the interaction of wage-tenure contracts and on the job learning. In combining these two effects I am able to deduce the relative importance of each of them in explaining tenure wage growth. The model is a directed search model with on the job search and wage-tenure contracts similar to Shi (2009), but workers learn on the job.
A natural extension of this study is combining it with the second Chapter, in order to better identify the relative importance of learning on the job among workers with differing educational achievement.
Dedication

To my parents, brother and nieces.
I wish to thank Miquel Faig, Gueorgui Kambourov, Shouyong Shi and Ronald Wolthoff for excellent supervision. I would also like to thank Grigoris Spanos for his encouragement and many insights, thoughtful comments and suggestions. This thesis also benefited from discussions with Elton Dusha, Kevin Fawcett, Florian Hoffman, Kinda Hachem, Li Li and Kunio Tsuyuhara, as well as with seminar participants at the University of Toronto and at the Canadian Economic Association Meetings in Ottawa in 2011 and in Toronto 2015, respectively.
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Chapter 1

Involuntary Part-time Employment and the Business Cycle
1.1 Introduction

It is well known that output decreases during a recession mainly due to an increase in the unemployment rate. It is also known but less studied, that other measures of work intensity also change during a recession. The participation rate, for example, also decreases and, more importantly from the point of view of this study, the number of workers employed in part-time jobs increases. A worker is defined by the CPS to be involuntarily part-time employed if he is employed part-time but states that he wishes to be employed full time. Involuntary part-time (IPT) employment, like unemployment, is counter-cyclical; this means that it is an additional source of lost output during recessions. Furthermore, like unemployment, it is much more prevalent during recessions. For example, while the proportion of people searching for full-time jobs that report as being involuntarily part-time employed has averaged approximately 4 percent since 1980, it reached a peak of approximately 10 percent during the 2009 recession.

There are several important facts concerning IPT employment. Firstly as stated above, it is important mostly during recessions. The proportion of workers reporting themselves as being under-employed is only about 2 percent during expansions but it increases to approximately 5 percent during recessions. Furthermore, since full-time employment drops during recessions, the ratio of IPT workers to full-time workers increases by more than 5 percent during recessions, so that roughly 9 percent of all normally full-time workers report being involuntarily part-time employed. Secondly, the majority of the increase in IPT workers comes from cut backs in hours worked and during recoveries, IPT employment declines before unemployment does, as worker’s hours are increased. Thirdly, the average wage of involuntarily part-time employed workers is substantially less than that of full-time workers suggesting that worker productivity plays an important role. Finally, involuntary part-time employees are 29 percent more likely to leave their current job for a new one than are regular full-time employees. Figure 1 shows the evolution of involuntary part-time employment over the business cycle.

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1 The data comes from the United States Bureau of Labour.
2 During the 2009 recession, 3.8 percent out the 5.2 percent of involuntary part-time workers was due to cut backs in work hours (versus workers finding new part-time employment).
3 See Appendices for some additional graphs.
The aforementioned facts provide a framework for understanding IPT employment. Since wages are lower for IPT jobs, these tend to be of low quality. A simple opportunity cost argument, therefore, may explain why IPT employment grows during recessions. During recessions the opportunity cost of leisure is greater and therefore less productive jobs are best operated less intensively (e.g. part-time). The increased labour mobility of IPT workers suggests two possibilities. The first is that, due to having more time to search for a job, IPT workers search for new jobs more intensively and therefore transition to new jobs at a higher rate as compared with full-time workers. The second possibility is that IPT workers are in worse jobs than full-time employees and therefore search more intensively.

Given that involuntary part-time jobs are less productive ones, it is important to determine the cause for this low productivity. There are two potential causes. The first is that IPT employees may be low ability workers, while the second is that they may generally be employed in a relatively bad matches. In the first case, worker mobility will have no impact on an individual’s productivity, since the worker has a low productivity regardless of the job he works at; in the second case, however, worker mobility may have a large impact on a worker’s productivity since it is possible that a worker is more productive at a new job. Therefore, only the latter type of worker heterogeneity will affect a worker’s on the job search decision. The rate at which IPT employed workers find new employment therefore confers important information regarding the nature IPT workers’ low productivity.

The purpose of this study is to provide a model linking unemployment, involuntary part-time employment, full-time employment and the business cycle as well as quantifying
both the nature and importance of this type of employment contract. The two most important contributions of this study is providing a framework in which IPT employment is counter cyclical as well as determining the role of a worker’s search technology and idiosyncratic productivity in explaining IPT employment. The idiosyncratic productivity can be decomposed into two types: An ex-ante observable component, which does not influence the job to job transition rate of workers, and an ex-post component that is job specific and affects the search intensity of a worker. It is therefore possible to determine the relative importance of each type of heterogeneity in explaining the existence of IPT employment.

More specifically, this study presents a directed search model with heterogeneous workers and on the job search, in which jobs can be operated full-time, operated part-time or destroyed. Worker heterogeneity is present in two forms: workers have a different, observable and permanent ex-ante productivity and workers have an on the job productivity reflecting the quality of the match that the worker is currently in. In addition to idiosyncratic productivity, which is either specific to a job or a worker, there is an aggregate productivity component common to all jobs in the economy. Firms must decide, given a job’s productivity, whether to operate the match and whether to operate the match part-time of full-time. Workers, meanwhile, decide whether to search for a job and subsequently what submarket to search in, where different submarkets offer workers different wages and different probabilities of finding a new job. This leads to the existence of a Block Recursive Equilibrium that separates workers by their ex-ante observable characteristics, thus greatly simplifying the computation of an equilibrium.

There are a few theoretical results that are worth noting. As long as the search technology of part-time workers is as good as that of full-time workers, conditional on a worker’s ex-ante productivity, IPT employment is concentrated among jobs with low match specific productivity. This allows for one to consider the effect on IPT employment from a change in the relative returns to unemployment, part-time employment, and full-time employment, for example. An increase in unemployment benefits, for example, not only increases the unemployment rate but also reduces the proportion of IPT employed workers in the economy. Another somewhat related result is that in order for there to be an increase in IPT employment due to an increase in the returns of part-time labour, it must be the case that

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Meisenheimer and Ilg (2000) state that 14.2 of Involuntary part-time workers searched for a new job in the previous month versus 4.9 percent for full-time workers.

these returns rise both relative to the returns to unemployment and the returns to full-time employment.  

Whether IPT employment is primarily due to worker or match-specific productivity is an important issue. These two sources of productivity heterogeneity have different policy implications. If IPT employment jobs are occupied by ex-ante low ability workers, for example, the best way to eliminate these types of jobs is to Fa-train these workers thereby increasing their productivity. If, on the other hand, these jobs represent mainly a bad match between workers and firms, it may be optimal to subsidize on the job search so that these otherwise productive workers can transition into better jobs. Moreover, the existence of IPT employment is more troubling in the former case than in the latter case. In the former case, a worker in an IPT employment job cannot transition to a good job since he is a low quality worker. In the latter case, on the other hand, the IPT employment job is transitory and the worker can transition into a good job by searching for a new one. A calibration exercise reveals the importance of these two types of productivity heterogeneity in determining IPT employment.

The calibration exercise determines the extent to which IPT employment is due to ex-post job specific productivity as compared with ex-ante worker specific productivity. It is found that ex-ante worker productivity heterogeneity is primarily responsible for the existence of IPT employment. In fact, the calibration exercise determines that 95 percent of IPT employment exists in workers with a fixed productivity in the bottom 10 percent. Match specific heterogeneity is, however, important in explaining the cyclicality of IPT employment. This occurs since even jobs involving workers with a low ex-ante productivity are operated full time if the match specific productivity is high enough. The fluctuations in IPT employment observed during the business cycle are therefore driven mainly by the match-specific productivity cutoff below which all jobs are operated part-time, for the bottom 10 percent workers, decreasing.

There are also two experiments conducted each answering an important question. The first experiment demonstrates the importance of ex-ante heterogeneity by considering a restricted version of the model without ex-ante worker productivity heterogeneity and drop-

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5There are numerous examples of policies that may affect the relative returns to part-time employment relative to full-time employment; one example may be the mandating of increased perks and benefits for full-time employees such as dental care, health care etc...
ping the monthly job to job transition rate of IPT workers. It is found that the implied
monthly job to job transition rate implied by such a model is a factor of 10 greater than
that observed in the data. This essentially shows that if IPT employment workers could
obtain a highly productive job, one which would in all likelihood be operated full-time, there
would be a large incentive to search for such a job. The second experiment investigated
the positive effects of IPT employment on the economy by considering a restricted version
of the model which does not allow for a job to be operated part-time. It is found that
while the welfare effects of IPT employment are modest, the effect of IPT employment on
the unemployment rate is significant. Without IPT employment the average unemployment
rate of the economy on the whole would rises by approximately 8 percent (not percentage
points). These effects are, however, larger for low productivity workers. The elimination of
IPT employment causes the average unemployment rate to rise by 38 percent among the
lowest productivity workers.

The rest of this study consists of five sections. The following section will present some
of the relevant literature. This study will be placed in the context of two literatures: that of
non-standard employment contracts and that of directed search with on the job search. The
third section will present the model. The fourth section will discuss the main theoretical
results of the model. These results include the existence of an equilibrium as well as a
discussion of the effects of changes to return of unemployment, part-time employment and
full-time employment. The fifth section presents a calibration of the model. While most
of the calibration is superficially described, more attention is given the novel features in
the model. These features include the ex-ante heterogeneity in the productivity of workers
as well as the existence of the option of operating jobs part-time. In the sixth section
some numerical exercises are undertaken. First a restricted version of the model, one with
no ex-ante heterogeneity, is calibrated in order to highlight the importance of this ex-ante
heterogeneity in explaining the job to job monthly transition rate of IPTW. Secondly, a
counterfactual exercise is conducting which eliminates the option of operating a job part-
time. The final section concludes with a summary of the main results as well as a brief
discussion of possible avenues for future research.
1.1.1 Comparison with the Literature

It is well known that during recessions non-standard employment relationships are formed. Kalleberg (2000) documents some forms of these relationships. During recessions firms may hire temporary workers, subcontract to outside companies, or employ part-time workers (by hiring or cutting back work hours). Of these non-standard relationships, part-time work is by far the most important in North America and consequently the most studied. Tilly (1996) is one such study. It highlights the importance of involuntary part-time employment, as it is estimated to represent nearly a quarter of part-time workers. Sightler and Adams (1999) is another such study. They document the historical fact that involuntary part-time employment grows during recessions and declines during expansions. My study contributes to the literature on non-standard contracts by considering involuntary part-time employment in a search setting. The framework provided can also be used to analyze other sorts of non-standard employment contracts such as seasonal employment or temporary work contracts for example.

Search models were pioneered in the seminal papers of Pissarides (1979), Diamond (1982), and Mortensen (1982); they model the importance of search and recruiting costs in explaining the existence of unemployment.⁶ In these models, firms post vacancies while unemployed workers search for jobs. A matching function links the two sides, meaning that not every vacancy and/or unemployed worker matches in a given period. This gives rise to equilibrium unemployment and offers insights into the evolution of vacancies and unemployment over the business cycle. At the same time, these models ignore the non-negligible monthly job-to-job transitions observed in the data.

The potential importance of on the job search was first explored in Mortensen and Pissarides (1994). Subsequently other authors have also explored this importance including Nagypal (2007), Ramey (2007) and Menzio and Shi (2011). Models with wage heterogeneity among workers struggled with the problem that value functions were not independent of the distribution of employees. For example, the probability of a worker accepting a firm’s offer is dependent on the wage at his current job. In order to determine the equilibrium, therefore, one would generally have to make an unrealistic assumption such as the option value of unemployment being a worker’s outside option when negotiating with a new match.

⁶For a brief review of the literature until 1998, see Mortensen and Pissarides 1999.
Chapter 1: Involuntary Part-time Employment and the Business Cycle

Delacroix and Shi (2006) seem to construct the first Directed Search model that incorporates on the job search, which alleviates the aforementioned problem. Directed search ensures that a worker of a given type searches in his own preferred submarket. This in turn ensures a Block Recursive equilibrium, which is essentially one in which the value function is not dependent on the distribution of workers. Menzio and Shi (2011) incorporates on the job search into a business cycle framework. To their framework this study adds the possibility of operating a job part-time as well as the existence of ex-ante productivity heterogeneity amongst workers. The possibility of introducing ex-ante productivity heterogeneity into a directed search model with on the job search is discussed in Menzio and Shi (2010). This model therefore contributes to this line of literature by allowing for different hours of worked and ex-ante worker productivity heterogeneity. It is worth noting that such an exercise would be more difficult in a random search literature, this study therefore underlines the benefits of directed search environments.

There are a few studies examining non-standard work contracts using similar search models. Cao, Shao and Silos (2011) is one such recent study. They examine the existence of fixed-term and permanent employment contracts and claim that such work contracts are primarily chosen due to firing costs. Their study differs significantly from the present one in that it lacks a business cycle. They obtain, however, the similar result of there being a productivity cutoff rule that determines how jobs are operated. Cahuc and Postel-Vinay (2002) construct a similar model but in their model the government decides whether to operate the job on a permanent or temporary basis, whereas in Cao, Shao and Silos (2011) the firm-worker pair makes that decision. While this study is clearly different in the sense that involuntary part-time employment is a different form of a non-standard work contract, some of the points studied in the aforementioned studies may be readily transferable to the current one. For example, involuntary part-time employment may also be affected by government policies such as the mandating of firing costs for full-time employees.

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7 Shi (2009) formalizes the notion of block recursivity.
8 Andolfatto (1994) and Merz (2005) analyze a leisure/production tradeoff in a different search context.
1.2 Model

This section presents the model and the contractual environment. While the model presented is similar to that of Menzio and Shi (2011) there are a few important differences. The two most important differences are that workers are ex-ante heterogeneous and jobs can be operated part-time.\(^9\) Matches are operated part-time for two reasons: the opportunity cost of leisure is bigger at low productivity levels and workers in part-time jobs have more time to dedicate to searching (which is more important in lower productivity matches). Another minor difference is that matches do not instantly produce but rather remain idle for 1 period before producing. This assumption simplifies the value functions without significantly altering the equilibria.

There is a mass 1 of ex-ante observably idiosyncratic workers following a distribution \(G_h\) with support \(\{h_1, h_2... h_N\}\) where \(h_1 < h_2... < h_N\). The mass of workers of type \(h_i\) is therefore \(g(h_i) \sim G_h\). It is important to note that a worker’s productivity type is observable to the firm and that it is fixed. In addition to workers, there is also an infinite measure of potential firms. These firms post contracts \(a\) that stipulate the terms of employment that a worker can either accept or reject. The contract can be conditional on a worker’s type and there is perfect commitment on the part of the firm.

Each worker has a utility function of the following form: given a consumption allocation \(\bar{c} = \{c(t)\}_{t=0}^{\infty}\), \(u(\bar{c}) = \sum_{t=0}^{\infty} \beta^t c_t\) where \(c_t \in \mathbb{R}\) represents a worker’s consumption in period \(t\) and \(\beta \in (0, 1)\) represents the discount factor. Similarly, \(\pi_t \in \mathbb{R}\) represents a firm’s profit in period \(t\) and therefore \(\sum_{t=0}^{\infty} \beta^t \pi_t\) gives a firm’s lifetime profit.

Time, in the model, is discrete and each period consists of the following stages: (1) Realization of the aggregate shock (2) Separation (exogenous and endogenous) and job type operation decision (full-time or part-time) (3) Search and matching (4) Production. During the first stage the aggregate state of the economy is revealed as the aggregate productivity parameter \(y_a\) is realized. This aggregate shock is common to all ex-ante submarkets. In the second stage every match faces some probability of destruction, \(\delta\), and matches have the option of shutting down. During this stage, firms also decide whether to operate a job part-time or full-time. In the Search and Matching stage unemployed and employed...
workers select a submarket to visit and search for a job, while firms post vacancies. Lastly, in the final stage, all jobs that were active at the beginning of the period produce regardless of whether the job was terminated or if the worker found a new job. This means that workers only begin producing at their new job at the beginning of the next period. Figure 2 illustrates the timing.

Figure 2

The aggregate productivity parameter, $y_a$, determines the overall economy’s productivity, with larger values of $y_a$ indicating an increase in production for both full-time and part-time jobs. At the beginning of each period a $y_a \equiv \{y_{a1}, y_{a2}..y_{Ny}\}$, where $Ny \geq 2$, is realized according to a markov process, $y_a(t) \sim f(y_a(t)|y_a(t - 1))$. The aggregate component is common between all worker types $h_i$ so that it represents the state of the entire economy. Every match has, in addition to the worker’s ex-ante productivity and the current aggregate components, a match-specific productivity component $z \in [z_1, z_2,..z(Nz)]$. This productivity parameter captures how well suited to a particular job a worker is. Production therefore occurs as follows: If the aggregate realization is $y_a$, a worker of type $h_i$ in a job with a match-quality parameter of $z$ produces $Y_f^i(y_a, z)$ in a full-time job and $Y_p^i(y_a, z)$ in a part-time job. It is worth noting that for a given $h_i, y_a$ and $z$, full-time production and part-time production differ for a fundamental reason: a relatively smaller share of full-time production comes from leisure/home production as compared with part-time production. $Y_{f(p)}^i(y_a, z)$ captures all production in including home production and leisure.\(^\text{10}\)

The state of the economy can be summarized by the ex-ante productivity $g_h$ and $\psi \in \Psi$, where $\psi \equiv (y_a, u^i, g^i)$. The first element is the aggregate productivity of the economy in the

\(^{10}\)Throughout this study a superscript i denotes that the variable is a specific to a worker with productivity $h_i$. 

current period. The second element, \( u^i \), denotes the measure of unemployed workers of type i, and therefore \( u^i \in [0, 1] \). The last element, \( g^i \) is a function \( g_z : Z \to [0, 1] \) representing the measure of workers of type i in productivity z matches. Since all workers of type i are either unemployed or working at some job \( z, u^i + \sum_j g^i_z(z_j) = g_h(h_i) \). 11

Once the aggregate shock has been observed, the labour contract specifies how every existing match is to be operated. Since at best a match faces an exogenous separation probability of \( \delta \), \( \tau \in [\delta, 1] \) gives the probability that a separation occurs. The variable \( \zeta \in [0, 1] \) gives the probability that it will operated full-time this period. Whether a job is operated or not depends on whether the contract specifies the job is to be operated full-time or part-time.

In the third stage, workers who did not separate from a job in the preceding stage, have an opportunity of searching for a new job. Whether a worker is in a full-time job, part-time job or unemployed, affects their probability of being allowed to search. Specifically, full-time workers are allowed to search for a job with probability \( \lambda_f \in [0, 1] \), part-time workers are allowed to search with probability \( \lambda_p \in [0, 1] \) and unemployed workers are allowed to search with probability \( \lambda_u \in [0, 1] \). This reflects the fact that workers with more non-production time at their disposal can be expected to search more intensively and consequently enjoy a higher matching rate. Workers who lost a job in the second stage, that is earlier this period, do not search for a job.

Workers that are allowed to search next decide which submarket, \( x^i \), to visit where \( x \in \mathbb{R} \) represents the expected utility from offers in that submarket. Simultaneously, firms decide whether to post a vacancy and where to locate it. The cost of posting a vacancy is \( k > 0 \). Since both workers and firms are too small to influence the tightness, \( \theta^i(x) \), of a submarket offering value x to workers of type i, they take it as given. 12 The variable \( \theta^i(x) \) therefore acts as a price and for this reason directed search is sometimes referred to as competitive search.

A matching function \( M(q) \) relates the probability of a firm filling a vacancy, \( q(\theta^i(x)) \), with the probability of a worker matching with a firm in that submarket, \( p(\theta^i(x)) = M(q(\theta^i(x))) \). The matching function \( M(.) \) has the following features: it is strictly decreasing, continuously

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11 Since search is directed, the value functions will depend on the state variables only through \( y_a \). This is referred to as Block-Recursivity and will be elaborated on in the text.

12 A worker of type i refers to a worker with ex-ante productivity \( h_i \).
differentiable, and concave in $q$. Furthermore $M(.) : [0, 1] \rightarrow [0, 1]$, so that if $q \in [0, 1]$ then $p \in [0, 1]$. These assumptions ensure that the matching function is well-behaved in the sense that a worker’s search decision has a unique solution and that in submarkets where it is easy for firms to attract workers it is hard for workers to find jobs (and vice-versa).

Once a worker and a vacancy match, the worker is offered an employment contract by the firm. If the worker rejects the offer, he continues working in his old job. If, however, he accepts the offer he obtains the expected utility of searching in submarket $x^i$, which is simply equal to $x$.\textsuperscript{13} A worker accepting an offer continues to work at his current job for the remainder of the period and begins working at the new firm at the beginning of the next period. At this point a match specific component of the match, $z$, is drawn according to the cumulative density function $F(z) : Z \rightarrow [0, 1]$ which has an associated probability function $f(z)$. The parameter, $z$, represents how good the worker is at the job he has found and therefore remains fixed for the remainder of the match.

Finally, in the final stage production occurs. Workers who were employed in a full-time job produce $Y^f_i(y_a, z)$, while those employed in a part-time job produce $Y^p_i(y_a, z)$ units of production. Workers that were unemployed at the beginning of the period produce $b^i_u$ units of production. Units of production include any leisure utility enjoyed by workers. The parameter $b^i_u$ can therefore be thought of as the value of unemployment benefits plus the additional value of leisure that an unemployed worker enjoys with respect to a full-time worker and $Y^p_i(y_a, z)$ can be thought of as any labour market production plus the additional value of leisure that part-time workers enjoy over full-time workers.

Following Menzio and Shi (2011), I focus on an efficient equilibrium, which can be viewed as the result of firms offering complete contracts.\textsuperscript{14} In other words, I focus on the allocation that maximizes the joint surplus or a worker and a firm. In the context of the current study non-efficient equilibria may be interesting. An important driver of involuntary unemployment, for example, may be a constant or sticky wage. During bad times it could be the case that a normally productive job cannot cover the relatively high constant wage guaranteed to the worker by the contract. Alternatively, government mandated fringe benefits for full-time workers may cause firms to operate certain jobs part-time. While such considerations may

\textsuperscript{13}The notation may seem a bit confusing but essentially I am labeling the sub-market a worker visits using the value the worker expects to get from finding a job there. The superscript $i$ represents the fact that different contracts are offered to different ex-ante workers.

\textsuperscript{14}See the working paper version of Menzio and Shi 2011 for details.
be interesting a complete contracts environment offers added tractability and serves as a benchmark for future studies.

A worker’s history can be summarized by a vector \( \{h, z; y_a\} \). The first component, \( h \), is the worker’s permanent productivity parameter, while the second component, \( z \), is the worker’s productivity parameter for the duration of a match. The key difference between these two parameters is that the former is permanently associated with a worker, while the latter is match specific. The last component, \( y_f = \{y_{a1}, y_{a2}, ..., y_{at}\} \), is the history of aggregate realizations since the beginning of the match. Since I am focusing on the efficient allocation, the current aggregate state \( y_{at} \) is sufficient to characterize a firm’s optimal decisions and henceforth I will refer to this simply as \( y_a \).

1.3 Theoretical results

1.3.1 Decentralized Market

This section develops the decentralized problem facing workers and firms of a given productivity \( h_i \). It is assumed that firms make offers that are specific to a type \( i \) worker and that, similarly, workers only respond to offers made to their type.\(^{15}\) It is therefore convenient to consider an arbitrary worker, with productivity \( h_i \) and characterize the optimal solution to this worker’s problem. A key feature of the equilibrium is that it is block recursive. This means that the equilibrium market tightness in any submarket \( x^i, \theta(x^i, y^a) \equiv (y_a, u^i, g^i) \), is dependent on the state of the economy only through \( y_a \) and not through the distributions \( u^i \) and \( g^i \). Block-recursivity therefore makes the problem tractable, since otherwise equilibrium market tightness, and consequently the value functions, would be functions of the distribution of workers over different employment states. An added advantage of the block-recursive nature of the equilibrium is that it renders numerical computation relatively straightforward.

Consider the search decision facing any worker with current expected utility of \( v \). A submarket that offers a worker an expected value of \( x \) has a tightness \( \theta(x, y_a) \). Consequently, if the worker is allowed to search and decides to search in a submarket \( x^i \), he will successfully match, and obtain an expected value of \( x \), with probability \( p(\theta_i(x, y_a)) \) and will fail to find

\(^{15}\) The subsequent section will prove the existence of such an equilibrium.
a successful match with probability $1 - p(\theta^i(x, y_a))$\footnote{I write $\theta^i(x, y_a)$ instead of $\theta(x^i, y_a)$ in order to discuss the worker’s search problem as choosing the submarket that offers the value $x$. See eqn (3.1).}. If the worker succeeds he begins employment next period at a job offering the value $x$, while if he fails he continues working at his current job and receives the value $v$. A worker therefore solves the following search problem:

$$D^i(v; y_a) = \max_x p(\theta^i(x, y_a))(x - v)$$

let $m^i(v; y_a)$ represent the solution to the problem, $x$

During the search stage an unemployed worker is allowed to search with probability $\lambda_u$. If allowed to search, he will visit the submarket that maximizes his expected value. The unemployed worker therefore visits submarket $m^i(U(y_a); y_a)$ and receives an expected value of $D^i(U^i; y_a)$ . If the worker matches with a vacancy, he draws a $z$ from the idiosyncratic distribution $f$. At the end of the period, regardless of his search outcome, he produces and consumes $b^i_u$ units of output. At the beginning of the next period he either begins employment at the new job, with idiosyncratic component $z$, if his search was successful or continues as an unemployed worker otherwise. The unemployed worker’s value function at the beginning of the period can be written as:

$$U^i(y_a) = b^i_u + \beta \mathbb{E}[U^i(\hat{y}_a) + \lambda_u D^i(U^i(\hat{y}_a); \hat{y}_a)]$$

where $D(U^i; y_a) = \max_x p(\theta(x^i, y_a))((x^i - U^i(y_a))$

For the remainder of the paper $\mathbb{E}$ will denote the conditional expectation of $\hat{y}_a$ given the distribution $\phi(\hat{y}_a|y_a)$.

Since I am considering the efficient solution, it can be assumed that the firm makes all the relevant job decisions. That is firms choose $\{w, \zeta, \tau, n\}$ where $w$ is the current wage, $\zeta$ is the probability that a job is operated full-time, $\tau$ is the probability that a job is shut down. If the match is destroyed the worker will finish the period at his current job but begin the next period as an unemployed worker. If the match is not destroyed, with probability $\lambda_f^i$ he will be allowed to search for a job if the job is operated full-time and with with probability $\lambda_p^i$ if the job is operated part-time. If allowed to search, a worker searches in submarket $n$. If the worker finds a new job, he produces at his current job this period and then begins the next period at the new job. At the beginning of the period, the lifetime utility, $W^i(z; y_a)$,
of a worker matched with a firm is therefore:

\[ W^i(z; y_a) = w + (1 - \tau)\zeta(\lambda_f p(\theta^i(n(y_a), y_a))n(y_a) + (1 - \lambda_f p(\theta^i(n(y_a), y_a)))\beta\mathbb{E}[W^i(\tilde{y}_a)]) \\
+ (1 - \tau)(1 - \zeta)[\lambda_p p(\theta^i(n(y_a), y_a))n(y_a) + (1 - \lambda_p p(\theta^i(n(y_a), y_a)))\beta\mathbb{E}[W^i(\tilde{y}_a)] \\
+ \tau\beta\mathbb{E}[U^i(\tilde{y}_a)] \]

(1.3)

The firm receives the output net of the payment to the worker plus the continuation value of the match. This means that if the match is operated full-time the firm receives \( Y^i_j(z, y_a) - w \), while if it is operated part-time it receives \( Y^i_p(z, y_a) - w \). During the separation stage, the match separates with probability \( \tau \), if separation occurs the firm receives its share of the surplus this period as outlined above and consequently receives a continuation value of 0. Finally, the submarket \( x^i \) in which the worker searches affects the continuation value through the worker’s job matching rate in that submarket. The firm’s expected lifetime value, at the beginning of the period, is therefore:

\[ J^i(z, y_a) = \zeta Y^i_j(z, y_a) + (1 - \zeta)Y^i_p(z, y_a) - w \\
+ (1 - \tau)\zeta[1 - \lambda_f p(\theta^i(n(y_a), y_a))\beta\mathbb{E}(J^i(z, \tilde{y}_a))] \\
+ (1 - \tau)(1 - \zeta)[1 - \lambda_p p(\theta^i(n(y_a), y_a))\beta\mathbb{E}(J^i(z, \tilde{y}_a))] \]

(1.4)

Assuming that utility/production is transferable between firms and workers, the joint surplus can be written as:

\[ V^i(z, y_a) = \zeta Y^i_j(z, y_a) + (1 - \zeta)Y^i_p(z, y_a) + \tau\mathbb{E}[U^i(\tilde{y}_a)] \\
+ (1 - \tau)[\zeta\lambda_f p(\theta^i(n(y_a), y_a)n(y_a) + \zeta[1 - \lambda_f p(\theta^i(n(y_a), y_a))]\beta\mathbb{E}(V^i(z, \tilde{y}_a))] \\
+ (1 - \zeta)\lambda_p p(\theta^i(n(y_a), y_a)n(y_a) + (1 - \zeta)[1 - \lambda_p p(\theta^i(n(y_a), y_a))]\beta\mathbb{E}(V^i(z, \tilde{y}_a))] \]

(1.5)

The following proposition summarizes the optimal allocation \( w^*, \zeta^*, \tau^*, \) and \( n^* \). The optimal allocation specifies that a match should be operated full-time if it yields a higher expected value when operated full-time instead of part-time. This expected value includes the value, to the worker, of finding a new job. This means that if \( \lambda_p \) is significantly larger than

---

17I suppress \( w, \zeta, \tau, \) and \( n^* \)’s dependence on \( z \).
it is possible that a job will be operated part-time even when \( Y^i(h_i, z, y_a) > Y^j(h_i, z, y_a) \).
Similarly, it specifies that a match is terminated if and only if destroying the match yields a higher expected payoff to operating it either full-time or part-time. Finally it specifies that the submarket, \( n^*(y_a) \) that a worker searches in maximizes the expected joint return to the match. Since both parties are risk neutral, the optimal allocation can specify any \( w^* \).

**Proposition 1.1** The optimal allocation prescribes the following: a) \( n^i(z; y_a) = m(V^i(z, y_a)) \)

b) \( \tau^i(z; y_a) = 1 \) if \( E[U^i(y_a)] > E[V^i(z, y_a)] + \zeta^i(z, y_a)\lambda_p + (1 - \zeta^i(z, y_a))\lambda_f)p(\theta^i(n(y_a), y_a)) - E[V^i(z, y_a)]; \) otherwise \( \tau^i(z; y_a) = \delta \)

c) \( \zeta^i(z, y_a) = 1 \) if \( Y^i(z, y_a) - Y^j(z, y_a) > (1 - \tau^i(z; y_a))(\lambda_p - \lambda_f)n^i(y_a) - E[V^i(z, y_a)]; \) otherwise \( \zeta^i(z, y_a) = 0 \)

All contracts offered in the equilibrium I consider implement the above allocation. This will be proven after a discussion of the equilibrium vacancy-posting decision of firms.

### 1.3.2 Equilibrium tightness

In addition to deciding on how to operate a job, the firm must decide on the optimal number of vacancies to post in any submarket \( x^i \). The firm will fill such a vacancy with probability \( q(\theta^i(x, y_a)) \) and receive the expected value of a match, \( \sum_j V^i(z_j, y_a)f(z_j) \), minus the portion promised to the worker \( x \). There is an incentive for a firm to post vacancies in submarket \( x \) as long as the expected return from posting it surpasses its cost of \( k \). The following equations therefore guides the vacancy posting process:

\[
\begin{align*}
\frac{k}{q(\theta^i)} & \geq \frac{\sum_j \beta E[V^i(z_j, y_a)f(z_j)]}{1 - x} \\
\theta^i(x; y_a) & \geq 0 \text{ with complementary slackness}
\end{align*}
\]

(1.6)

The above equation guarantees that, due to the properties of the matching function discussed previously, there is a meaningful tradeoff from between the probability of finding a job and the value offered by the job. Consider an offer made in equilibrium, in other words one for which \( \theta^i(x; y_a) > 0 \). A higher offer \( x \) implies a lower firm matching rate of \( q(\theta^i(x; y_a)) > 0 \), since the overall expected benefit must equal \( k \). Since the matching function, \( M(.) \), is decreasing in \( q \), this in turn implies a lower job finding rate \( p(x) \).

\(^{18}\)All proofs are included in the Appendices.
1.3.3 Law of Motion

The final thing to discuss before defining the equilibrium is the law of motion faced by a worker of type i. The block recursive nature of the equilibrium allows one to easily calculate these using the optimal policy functions. Consider arbitrary functions \( d^i(z, y_a), e^i(z, y_a), m^i(z, y_a), \theta^i(z, y_a) \), representing the job-destruction policy, job-operation policy, search policy function and equilibrium tightness function. First consider an unemployed worker searching for a job in the search stage. Since \( m^i(U^i(y_a); y_a) \) denotes the expected value of finding a job for the worker, \( \theta^i_u(y_a) = \theta^i(m^i(U^i(y_a); y_a)) \) gives the tightness of the submarket the worker visits through ?? . The probability that the worker is still unemployed, at the end of the period, is \( 1 - \lambda u p(\theta^i_u(y_a)) \) and the probability that the worker is employed at a job \( \tilde{z} \) is \( \lambda u p(\theta^i_u(y_a)) f(\tilde{z}) \). Similarly we can consider an employed worker just before the search stage and during the search stage. The worker loses his job with probability \( d^i(z, y_a) \) and is consequently not allowed to search for a job this period. Letting \( \theta^i_z \) represent \( \theta^i(m^i(U^i(y_a); y_a)) \), the probability that a worker that survived the destruction stage does not find a new job \( \tilde{z} \) is \( (1 - (e^i(z, y_a) \lambda_p + (1 - e^i)(z, y_a) \lambda_p))p(\theta^i_z(y_a))g_z(\tilde{z}) \) while the probability that he does find a new job \( \tilde{z} \) is \( (e^i(z, y_a) \lambda_p + (1 - e^i(z, y_a)) \lambda_p)p(\theta^i_z(y_a))g_z(\tilde{z}) \). Letting \( \hat{x} \) denote the level of a variable at the beginning of the next period, the following laws of motion can be written as follows \(^{19}\):

\[
\hat{u}^i = u^i(1 - \lambda u p(\theta^i_u(y_a))) + \sum_j d^i(z, y_a)g^i(z_j)
\]

\[
\hat{g}^i(z) = s^i(\Psi)f(z) + (1 - d^i(z, y_a))(1 - (e^i(z, y_a) \lambda_p + (1 - e^i)(z, y_a) \lambda_p))p(\theta^i_z(y_a))g^i(z)
\]

\[
s^i(\Psi) = \lambda u \lambda u p(\theta^i_u(y_a))u^i + \sum_j (1 - d^i(z, y_a))(e^i(z, y_a) \lambda_p + (1 - e^i(z, y_a)) \lambda_p)p(\theta^i_z(y_a))g^i(z_j)
\]

Determining the proportion of workers of a type i working full-time, \( fte^i \), and part-time, \( pte^i \), is straight-forward. One takes the distribution of workers over jobs \( z \) and multiply by the probability that that job is operated full-time and full-time respectively. Given the above distribution, this gives:

\[^{19}\text{It is worth noting that production in this model takes place according to the beginning of period distributions See Model section.}\]
\[fte^i(y_a) = \sum_j e(z_j, y_a)g^i(z_j);\]
\[pte^i(y_a) = \sum_j e(z_j, y_a)g^i(z_j);\]

1.3.4 Definition of Equilibrium

The preceding discussion allows one to define a block recursive equilibrium under the assumption that workers are segmented by type.\(^{20}\)

**Definition 1.1** *A Block Recursive Equilibrium (BRE) consists of a collection of functions \(\forall i \in [1, 2, ..., N_h]\) as follows: market tightness functions \(\theta^{i^*} : \mathbb{R} \times Y_a \to \mathbb{R}_+\); search value functions \(D^{i^*} : \mathbb{R} \times Y_a \to \mathbb{R}\) along with their associated policy function \(m^{i^*} : \mathbb{R} \times Y \to \mathbb{R}\); Unemployment value function \(U^{i^*} : Y_a \to \mathbb{R}\) and a match Value functions \(V^{i^*} : Z \times Y_a \to \mathbb{R}\) along with its policy functions \(d^{i^*} : Z \times Y_a \to [\delta, 1]\) and \(e^{i^*} : Z \times Y_a \to [0, 1]\); and laws of motion \(\hat{u}^{i^*} : \Psi \to [0, 1]\) and \(\hat{g}^{i^*} : Z \times \Psi \to [0, 1]\). The above functions satisfy the following conditions:

(i) For all \(x \in \mathbb{R}\) and all \(\psi^i \in \Psi, \theta^{i^*}\) satisfies (1.6)  
(ii) For all \(V \in \mathbb{R}\) and all \(\psi^i \in \Psi, D^{i^*}\) satisfies (1.1) and \(m^{i^*}\) is its associated policy function.  
(iii) For all \(\psi \in \Psi, U^{i^*}\) satisfies (1.2).  
(iv) For all \(z \in Z \psi^i \in \Psi, V^{i^*}\) satisfies (1.5) and \(d^*, e^*\) are its associated policy functions.  
(v) For all \(\psi^i \in \Psi^i, \hat{u}^*\) and \(\hat{g}^*\) satisfy (1.7).*

1.3.5 Existence and Properties of Equilibrium

An important feature of the equilibrium presented in the previous subsection is that it is block-recursive. In this study’s context this means that value functions and their associated policy functions are dependent on the state of the economy, \(\psi^i\), only through the aggregate productivity parameter \(y_a\). This simplicity allows one to compute the equilibrium without reference to distribution of workers, thus making the model tractable. A key reason for the existence of such an equilibrium is that, fixing worker type i, with directed search workers sort themselves into different submarkets depending on their current value. Each submarket is therefore visited by a worker with a different outside option. A firm that posts a vacancy in a submarket \((x, i)\) knows that only workers of type i such that \(x = m(V_i; y_a)\)

\(^{20}\)This definition closely follows Menzio and Shi (2011).
will apply. They therefore do not need to take into account the actions of workers at other job (including unemployed workers). A further consequence of this is that the solution to the social planner’s problem is identical to that of the competitive economy.

**Theorem 1.1** A BRE exists. Furthermore, in this equilibrium firms operate jobs optimally according to $\zeta^*, \tau^*, n^*$ and the equilibrium separates workers by type.

The intuition for the existence of an equilibrium that separates types is straightforward. Suppose that two different types of workers, $h_1$ and $h_2$, apply for the same contract $x'$. Additionally, assume without loss of generality that the current value of the match of worker 1, $V_1$, is less than that of worker 2, $V_2$. Then the firm is making strictly larger profits when matching with an $h_2$ worker since an $h_2$ worker is more productive than an $h_1$ worker. Since in equilibrium 1.6 holds, the tightness prevailing in submarket $x'$, $\theta'(x^3, y_a)$ is larger than the tightness prevailing in submarket $x^1(V_1)$. Firms can create a new contract that gives a value 0 to the $h_2$ workers and the same value $x$ to the $h_1$ workers. Since the equilibrium tightness will be $x^1(V_1)$, workers of type 1 will prefer this contract.

Part-time workers differ fundamentally from full-time workers in that they, by definition, spend relatively more time at home. It is therefore natural to assume that workers obtain some utility from these hours. This means that a relatively larger portion of part-time production, $Y^p(z, y_a)$, is due to non-market home production/leisure. An increase in productivity, either due to higher aggregate productivity in the economy, a larger $y_a$, or due to a larger match specific productivity, a larger $z$, will have a disproportionately larger impact on full-time production as compared to part-time production. In the context of this model, this means that it is natural to assume that $\frac{Y^f(z,y_a)}{\partial y_a(z)} > \frac{Y^p(z,y_a)}{\partial y_a(z)}$. This assumption implies that in economic expansions, when $y_a$ is relatively high, the opportunity cost of part-time employment is greater. This helps explain the fact that involuntary part-time employment is counter-cyclical.

In addition to the above, part-time workers spending more time at home implies that they have access to a search technology that is at least as efficient as that of full-time workers. In the context of the model this means that $\lambda_p \geq \lambda_f$. This assumption, unlike the previous assumption, does not necessarily imply that in economics expansions the opportunity cost of full-time employment increases. The reason for this is that during economic expansions having a good match, a high match specific $z$, can become relatively more important. The
better search technology of part-time workers, which allows movement into these better jobs at a higher rate, may actually reduce the opportunity cost of full-time employment.\footnote{This was actually found to be the case during a Calibration exercise in an earlier draft of this paper.}

Under the above assumptions, and the assumption that all three types of jobs are present amongst workers of type i, there is a straightforward breakdown of a job, in ex-ante sub-market i, according to the idiosyncratic draw of the match. The least productive jobs, those with a productivity z less than \( z_u \), are closed down. Intermediate productive jobs, those with a productivity z between \( z_u \) and \( z_p \), are operated part-time, and high productivity jobs, those with a productivity z higher than \( z_p \) are operated full-time. It is clear that higher a higher z makes it more profitable to operate a job full-time as opposed to part-time and more profitable to operate part-time as opposed to closing the position. The search technology assumption, \( \lambda_u \geq \lambda_p \geq \lambda_f \), reinforces this effect. Due to the fact that contracts are complete, the search decision maximizes a match’s value. Since high z jobs have more joint value, the optimal contract stipulates less intensive search.

**Proposition 1.2** Under the above assumptions, there is a natural breakdown of jobs according to \( Z \) for a given worker type i. Supposing that some jobs are operated, \( \exists z_u \in Z \) such that if \( z < z^i_u \) \( d^i(z,y_a) = 1 \) while if \( z < z^i_u \) \( d^i(z,y_a) = \delta \). If a job is operated part-time, \( \exists z^i_p \in Z \) with \( z^i_p > z_u \) such that if \( z < z^i_p \) \( \zeta^i(z,y_a) = 0 \) while if \( z > z^i_p \) \( \zeta^i(z,y_a) = 1 \).

The above proposition highlights the relationship between job quality and how a job is operated. Conditional on permanent ability \( h_i \), involuntary part-time employment jobs are those of low quality but still good enough to not be shutdown. The next section will investigate some of the consequences of this breakdown.

### 1.3.6 Heterogeneous Equilibrium

A larger \( h_i \) represents a worker that is more productive at every job and therefore relatively less productive at home. For high \( h_i \) workers, therefore, it is optimal to operate jobs with a relatively low match specific component (i.e low z jobs) as full-time jobs. Therefore the productivity cutoff above which a job is operated full-time, \( z_p \), is decreasing in \( h_i \). The above argument ignores the fact that if z and h are complements, search may be more important for high z matches. Since \( \lambda_p \geq \lambda_f \), this would make it more profitable to operate jobs...
part-time. The assumption that \( \frac{\partial Y(h,z_i,y_a)}{\partial h \partial z_i(y_a)} = \frac{\partial Y(h,z_j,y_a)}{\partial h \partial z_i(y_a)} \forall z_i, z_j \in Z \) is sufficient to guarantee that this is the case. A similar argument holds for \( z_u \), the cutoff \( z \) below which matches are destroyed. Since both \( z_u \) and \( z_p \) are decreasing in \( h \), it is unclear theoretically whether there is more or less involuntary part-time employment in higher \( h \) submarkets.

Even in the case where the part-time operating region (between \( z_u \) and \( z_p \)) is larger in lower \( h \) ex-ante submarkets, it is not necessarily true these submarkets consist of the majority of involuntary part-time employees. Low productivity submarkets have a higher unemployment rate, since all jobs are relatively less productive, and therefore the proportion of workers employed may be low. In other words, although most workers in the submarket are part-time employed, there are not many employed workers in the ex-ante submarket. When considering the mass of all involuntary part-time employees, this ex-ante submarket may represent a relatively low proportion of such workers.

There is an important difference between a job that is of low productivity owing to its worker’s low permanent component, \( h_i \), and due to a low match specific component \( z \). In the former case, changing job will have a modest impact on productivity since the worker’s permanent productivity does not change upon finding a new employer. In the latter case, changing employer may have a more significant impact on a worker’s productivity since the worker will draw a new \( z \) upon beginning work at the new employer. There is therefore a link between involuntary part-time employees’ job mobility rate and the extent to which involuntary part-time employment is caused by a bad temporary match in the form of a low \( z \). This is the central insight of this study and will be discussed further after calibrating the model. The analysis of the calibration exercise will demonstrate which type of productivity is the primary driver of IPT employment.

**Proposition 1.3**  
1) The unemployment cutoff, \( z_u^i \), and the part-time cutoff \( z_p^i \) is decreasing in the ex-ante productivity of workers \( h_i \). 2) Consider \( Y_p^i(z,y_a) = Y_p^j(z',y_a) \) \( j > i \) \((z' < z)\). If \( \zeta^i(z,y_a) = \zeta^j(z',y_a) = 0 \), the monthly job-to-job transition rate of worker \( j \) is greater than that of worker \( i \).

### 1.3.7 Theoretical Policy Implications

There are several interesting policy implications of the model. This subsection will consider policies that increase the instantaneous value of unemployment and policies that increase
the relative value of part-time to full-time labour. Policies that affect the relative returns
of one type of job vis-a-vis another job will have an effect on the composition of jobs in
the economy. If the return to unemployment increases, for example, then for any given $h_i$, the
proportion of jobs not operated will increase and consequently total unemployment will
increase. While this result is standard in the search literature, here there is an additional
implication.

If $b_u^i$ increases, due to an increase in unemployment benefits, then the productivity cutoff
$z_u^i$ increases and therefore a smaller proportion of jobs are operated part-time. Therefore an
increase in unemployment benefits has the effect of decreasing the proportion of involuntary
part-time to full-time employees. The reason for this is clear, operating a bad match (one
with low $z$ ) has a higher opportunity cost relative to a good match. Therefore when
unemployment becomes more attractive, these are the first jobs to be shut down. These
low productivity jobs are incidentally ones where part-time employment is most attractive
since a portion of the value of part-time employment comes in the form of increased leisure,
which is independent of the match’s quality.

Policies affecting the relative value of part-time work have similar effects. Consider a
policy that increases the value of part-time employment relative to both full-time employ-
ment and unemployment. In the model this would be an increase in $Y^i(z, y_a) \forall i, z, y_a$. By
increasing the relative return of part-time employment to full-time employment, this would
represent an increase in the cutoff above which jobs are operated full-time, $z_p$. Additionally,
by increasing the value of the relatively low $z$ jobs, the cutoff above which jobs are operated,
$z_u$, simultaneously decreases. There is therefore an unambiguous increase in the proportion
of workers being involuntarily part-time employed. Some of the increase, however, is due to
a decrease in full-time employment while some of the increase is due to an increase in the
proportion of jobs that are operated instead of shutdown.

Policies that affect the return to part-time employment in the manner of the preceding
paragraph are less common than policies such as employment insurance. I therefore provide
an example. Since the majority of IPT employment comes from matches switching from
being operated full-time to part-time, a policy offering IPT employment insurance could
be instituted. This program could provide additional funds, for a fixed period of time, to
workers whose employers have decided to operate a previously full-time position part-time.
As noted above, such a policy would have two effects: a decrease in jobs shutdown and an increase in jobs operated part-time instead of full-time. Whether such a policy is beneficial, therefore, is an empirical question since if employment comes predominantly from the former it will generally be welfare improving while if it comes from the latter it will not be. An important advantage in pursuing such a policy during a recession, however, is that the loss of full-time jobs to part-time status is temporary while the increase in employment is permanent. Once the economy recovers, workers in IPT jobs will already be matched with a firm and a share of them will see their job updated to full-time status.

Another type of policy is one that decreases the relative values of both IPT employment and unemployment towards full-time employment but increases the value of unemployment towards IPT employment. In the model this could be a decrease in $Y_{ip}(z, y_{a}) \forall i, z, y_{a}$ and a smaller decrease in $b_{u}$. The decrease in the relative return to part-time employment towards full-time employment would decrease $z_{p}$, while the decrease towards unemployment would increase $z_{u}$. The proportion of IPTE is ambiguous since it is decreasing due to more matches being operated full-time but is increasing due to less jobs being shutdown. An example of a policy inducing such a change is the introduction of subsidized daycare. This makes full-time employment more attractive with respect to both part-time work and unemployment, while it also makes part-time work attractive with respect to unemployment.

1.4 Quantitative Analysis

1.4.1 Calibration

One of the main purposes of the study is to determine the relative importance of match specific productivity as compared with permanent observable productivity in explaining involuntary part-time employment. As discussed previously, these types of productivity differ from one another in the sense that in the first case workers have an incentive to find a new job while in the second case they do not. The monthly job-to-job transition rate of involuntary part-time employment therefore reveals information on how much the worker’s low productivity is due to the former and how much is due to the latter. By targeting the IPT job-to-job transition rate, the calibration answers this question.

There are other interesting insights that a calibration and subsequent counter-factual
analysis allows. The overall impact of the involuntary part-time sector, in terms of production or welfare, can be determined. Additionally one can determine the leisure value of an involuntary-part time job as well as the effects of changes to this leisure value by, perhaps, government policy. These issues and other are discussed in the subsequent section.

In order to calibrate the model, a few additional assumptions have to be made. Following the literature, a period is defined as being one month long. Consequently, the discount parameter $\beta$ is set to 0.996 in order to match the annual interest rate of 5 percent. The job-finding probability is assumed to be the same as in Menzio and Shi (2010). It is therefore of the form $p(\theta) = \min\{1, \theta^\gamma\}$ with $\gamma \in (0, 1)$ and the parameter values are taken from their study.

It is assumed that full-time production is linear in the aggregate productivity parameter, $y_a$, and the idiosyncratic productivity parameters $z$ and $h_i$. This means that $Y_{f}^i(z, y_a) = h_i + z + y_a$. Part-time production can be broken down into two components: a work related component and a leisure based component. It is assumed that a part-time job is as efficient as a full-time job, with the only difference being that a part-time job is operated for less hours. The part-time component, $b_{pt}^i$, captures the fact that a worker enjoys additional leisure time as compared with a full-time worker. Specifically, $Y_{p}^i(h_i, z, y_a) = (h_i + z + y_a)hr_p + b_{pt}^i$, where $hr_p$ represents the ratio of the average part-time work week to the average full-time one.

In order to replicate a business cycle, it is assumed that the aggregate productivity can take on three possible values $y_1$, $y_2$ and $y_3$, representing a period of economic recession, normal economic activity and economic expansion, respectively. Furthermore, it is assumed that $y_1$ and $y_3$ are equidistant from $y_2$. In this way the booms and recessions are modeled as being symmetric. Without loss of generality, $y_1$ is set equal to 1. Having specified the aggregate productivity set $Y$, the transition process is as follows: the probability of remaining in a give state i is $\rho$, while the probability of moving to an adjacent state is given by the $(1 - \rho)/N_{adj}$, where $N_{adj}$ is the number of adjacent states; the probability of moving to a non-adjacent state is 0. The lower (and therefore upper) productivity bound, $y_1(y_3)$, is chosen such that the average standard deviation of productivity is .020. The productivity persistence parameter, $\rho$ is chosen such that the autocorrelation of production is 0.878.\footnote{These values are taken from Shimer (2005).}
For simplicity it is assumed that there are three essential types of workers with different permanent productivities. A worker may therefore have a low, average, or high permanent productivity associated with him. The support of the set $H$ is therefore $[h_1, h_2, h_3]$. Since in the data involuntary part-time employment is found to represent at most 10 percent of all employees who seek full-time employment, it is assumed that the probability distribution over the types is $[.1, .8, .1]$. While at first glance, this assumption might seem unsatisfactory, it allows for the full mass of involuntary part-time workers to be of low type. It is also assumed that $h_1$ and $h_3$ are equidistant from $h_2$, and $h_2$ is set to 0. In order to identify $h_1$ (and therefore $h_3$), the average ratio of Involuntary part-time workers to Full-time workers, 0.044, is used. A decrease in $h_1$, from $h_1 = 0$ has the effect of first increasing the proportion of involuntary part-time workers, as jobs in that ex-ante submarket remain active and convert to being operated part-time, and then decreasing it as jobs are too unproductive to operate.\footnote{This is just a decrease in $z_p$ and increase in $z_u$ from the previous section.}

The match-specific productivity distribution, $g(z)$ is assumed to follow a Weibull distribution with scale parameter $Sc$ and shape parameter $Sh$.\footnote{In the actual calibration the Weibull distribution is approximated using a state space consisting of 10,000 $z$'s.} With the scale parameter, one is able to target the EU rate. A small (large) scale parameter makes the distribution of jobs more similar (different) therefore reducing (raising) the number of job types, $z$, that are destroyed. The shape parameter, along with the exogenous probability of a match being destroyed $\delta$, are targeted using tenure data. This is possible since these two parameters affect the tenure distribution in different ways. The key difference is that the shape parameter affects the hazard rates at jobs in the middle of the tenure distribution differently form jobs at the beginning and end of the tenure distribution, while $\delta$ increases the hazard rate at all jobs. Using data from the 2003 CPS Tenure supplement, therefore, it is possible to target $Sh$ using the complete tenure distribution and target $\delta$ using the average tenure length.\footnote{See Menzio and Shi (2011) for details.}

There are three parameters, $\lambda_u, \lambda_p, \lambda_f$, that differentiate the search technology between unemployed, part-time and full-time workers, respectively. Since these parameters reflect differences in the search technology, $\lambda_u$ can be set to 1. The parameters $\lambda_p and \lambda_f$ then reflect the relative decline in the probability of a part-time and full-time worker finding a job, given the same market tightness $\theta$. In order to determine $\lambda_f$, the average monthly job-to-job
Chapter 1: Involuntary Part-time Employment and the Business Cycle

Table 1.1: Parameter Values

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Identifier</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.46</td>
<td>Vacancy/unemployment data</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.452</td>
<td>Job finding rate</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.46</td>
<td>Vacancy/unemployment data</td>
</tr>
<tr>
<td>$\lambda_f$</td>
<td>0.46</td>
<td>Employment-Employment transition rate</td>
</tr>
<tr>
<td>$b_u$</td>
<td>[.24 0.89 1.5]</td>
<td>$b_u / \text{Average productivity} = 0.71$ for each $h_i$</td>
</tr>
<tr>
<td>$b_{pt}$</td>
<td>0.147</td>
<td>Average IPTE monthly transition rate $\text{Average } H_{pt}/\text{Average } H_{ft}$</td>
</tr>
<tr>
<td>$hr_p$</td>
<td>25/40</td>
<td>$\text{Average } H_{pt}/\text{Average } H_{ft}$</td>
</tr>
<tr>
<td>$k$</td>
<td>1.4</td>
<td>UE rate</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.00956</td>
<td>Average tenure length</td>
</tr>
<tr>
<td>$H$</td>
<td>[-0.851 0 0.851]</td>
<td>Average FTE/IPTE</td>
</tr>
<tr>
<td>$Sc$</td>
<td>0.078</td>
<td>Tenure distribution</td>
</tr>
<tr>
<td>$Sh$</td>
<td>4</td>
<td>EU rate</td>
</tr>
<tr>
<td>$y_1$</td>
<td>[0.968 1 1.32]</td>
<td>Std of quarterly hourly productivity</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.969</td>
<td>Quarterly autocorrelation of hourly productivity</td>
</tr>
</tbody>
</table>

transition rate of 0.0303. In this section it is assumed that part-time and full-time search technologies are identical, $\lambda_p = \lambda_f$.

The parameters $b_p$, $b'_u$, and $k$ are not directly observable in the data. The value for $b_p$, which represents the non-production value of part-time work, is determined by the average job-to-job transition rate of involuntary part-time workers, 0.0394. To see why consider an increase in $b_p$. This increase will make involuntary part-time employment more attractive in higher ex-ante submarkets than $h_1$. The jobs that were previously operated full-time but are consequently operated part-time will be low $z$ jobs. These low $z$ jobs have a high job mobility rate as workers in these jobs wish to move to better ones. The parameter $b'_u$, which represents the instantaneous value of being unemployed, is set equal .71 of the average productivity of a worker of productivity $h_i$. This choice of $b'_u$ reflects the idea that high productivity workers are more productive, and therefore enjoy higher unemployment leisure/production, than less productive workers. The parameter $k$, which represents the cost of posting a vacancy, targets the average unemployment rate of 0.0567. A second way to target the part-time benefits $b_{pt}$, would be to calibrate the implied wage ratio of full-time to part-time workers to the real world data. The parameters are summarized in table 1.

The Calibration itself reveals the magnitude of some important parameters. Firstly, there is a large dispersion in ex-ante productivity. The distance between the lowest and the highest ability worker, for a fixed match-specific and aggregate productivity parameter, is
1.702 production units. This large dispersion in ex-ante productivity is necessary in order to match the job to job transition rate of IPT workers. In order to see how dispersion in ex-ante productivity affects the transition rate, recall the earlier discussion about the difference between the two types of idiosyncratic productivity. The worker working part-time due to the former has a large incentive to move jobs while the latter worker has a smaller incentive to move jobs since he is relatively unproductive at every job.

Furthermore, the standard deviation of ex-ante production over the standard deviation of match specific productivity is 4.4614. This essentially means that workers differ in their ex-ante productivity a lot more than they differ in their match specific productivity. This suggests that policies seeking to eliminate IPT employment should target improving a worker’s intrinsic productivity rather than improving his on the job productivity.\textsuperscript{26}

A second important quantitative result concerns the value of part-time off the job production. The ratio of part-time production to unemployment production for the $h_1$ productivity worker, $b_{pt}/b_{u}^{1}$, is found to be 0.6125 for the workers with the lowest ex-ante productivity. Since the majority of IPT workers are of this type, this essentially represents the relative value of part-time to unemployment leisure for Involuntary part-time workers in the economy. Additionally, one can decompose the average value of IPT employment into the average on-the-job production and the off-the-job production of a IPT job. This decomposition shows that approximately 0.61 percent of the value of an involuntary part-time job comes from part-time leisure.

1.4.2 Quantitative Results

Table 2 presents some model correlations and standard deviations.

The above correlations and standard deviations provide evidence for the veracity of the model since none of the above quantities were targeted in the calibration. If the calibrated model is a good approximation of the real world one would expect that it produces similar second moments.

The first part of the table presents the implied model autocorrelations of the unemployment rate, the v/u ratio and the FTE/PTE. All of the autocorrelations are of the right sign\footnote{This could be interpreted as a greater incentive for investment in General rather than on Specific Human Capital.}
Table 1.2: Calibration results

<table>
<thead>
<tr>
<th>Variable</th>
<th>Autocorrelation model</th>
<th>Autocorrelation data</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>0.9252</td>
<td>0.936</td>
</tr>
<tr>
<td>v/u</td>
<td>0.8634</td>
<td>0.941</td>
</tr>
<tr>
<td>PTE/FTE</td>
<td>0.8805</td>
<td>0.8130</td>
</tr>
<tr>
<td>PTE/FTE,y</td>
<td>-0.9974</td>
<td>-</td>
</tr>
<tr>
<td>PTE/FTE,u</td>
<td>0.988</td>
<td>+</td>
</tr>
<tr>
<td>FTE/FTE</td>
<td>0.0098</td>
<td>0.0127</td>
</tr>
</tbody>
</table>

and all quantities in the data are reasonably approximated by the model. It is worth noting that the implied autocorrelation of PTE/FTE is a bit higher in the model than in the data, 0.8805 to 0.8130, while both the implied autocorrelation of u and of v/u are slightly lower, 0.9252 to 0.936 and 0.8634 to 0.941 respectively.

The second part of the table presents the correlation of IPTE/FTE and y and the correlation of IPTE/FTE and u. One of the important facts concerning IPTE/FTE, which was discussed earlier, is that this quantity is procyclical. The calibrated model indeed implies that IPTE/FTE is heavily anti-cyclical with the correlation between IPTE/FTE and y being -0.9974 (and the correlation between IPTE/FTE and u being .988). The magnitude of this relationship is partially driven by the assumption that $\lambda_p = \lambda_f$. With $\lambda_p > \lambda_f$ it is possible that the correlation between output and IPTE/FTE weakens, since IPTE then allows for a superior search technology and the returns to moving from a bad job to a good job are higher in a boom than in a recession.

The final part of the table presents the standard deviations of u and IPTE/FTE. While the standard deviation of u implied by the model is relatively smaller than that of the data, the standard deviation of IPTE/FTE implied by the model is similar to that in the data. This is an important result since the existence of Involuntary part-time employment and its anti-cyclical behaviour is an important concern of this study. While the standard deviation is approximately matched, IPT employment is too symmetric in the model. In the model, IPTE/FTE attains a maximum of 0.0592 and a minimum of 0.0301, while in the data it attains a maximum of approximately 10 percent and a minimum of 2.2 percent since 1980. The asymmetry is an issue that may be studied in a future study.

\footnote{See Menzio and Shi (2010) for details.}
Table 1.3: Experiment 1: No ex-ante Heterogeneity

<table>
<thead>
<tr>
<th>Variable</th>
<th>New Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_u$</td>
<td>0.90</td>
</tr>
<tr>
<td>$b_p$</td>
<td>0.379</td>
</tr>
<tr>
<td>$\lambda_f$</td>
<td>0.602</td>
</tr>
<tr>
<td>IPTE EE</td>
<td>0.4150</td>
</tr>
</tbody>
</table>

The final, and perhaps most important, result from the calibration concerns the importance of ex-ante as compared with ex-post job specific productivity. The main fact is that almost all involuntary part-time employment is found among ex-ante low productivity workers, that is workers with productivity $h_1$.\(^{28}\) As per the theoretical results, however, among workers with a low ex-ante productivity, it is workers with a low match specific shock that are IPT workers. Approximately 49 percent of $h_1$ workers are in IPT jobs while the remainder are in regular full-time jobs. Furthermore, since all IPT employment is concentrated among $h_1$ workers, the fluctuations in IPT over the business cycle are driven completely by the match-specific productivity rather than the ex-ante productivity. This means that while the phenomenon of IPT employment is driven by a worker’s ex-ante productivity, the cyclical nature of IPT employment is driven primarily by ex-post on the job productivity.

1.5 Simulation and Numerical Policy Analysis

In this section several altered forms of the model are calibrated. Firstly, a restricted version of the model with no ex-ante heterogeneity is considered. Secondly, the value of the involuntary part-time sector to the economy is estimated by considering a restricted model with no involuntary part-time employment.

For the first experiment I use the same calibration strategy as in the first except that I eliminate the ex-ante heterogeneity of workers and therefore worker productivity heterogeneity comes solely from the match-specific component. Since there is one less parameter, namely $h_1$, to estimate, the IPT monthly job transition rate is dropped as a target. The non-production component of IPT employment, $b_p$, is now used to target the ratio of IPT to FT workers. Table 3 presents the important parameters from the calibration.

The main result above is that this restricted form of the model cannot match the IPT

\(^{28}\)Less than 5 percent of IPT workers are of productivity $h_2$ and 0 percent are of productivity $h_3$.\)
monthly job to job transition rate. The rate implied by the model, 0.4150, is more than a magnitude of 10 larger than the actual rate of 0.03904. In order to match the actual rate, therefore, $\lambda_p$ needs to be significantly smaller than $\lambda_f$. Furthermore, the difference has to be relatively large and is consequently difficult to take seriously. The reason for the above failure is that without, ex-ante heterogeneity, IPT jobs are solely operated as such due to a low match specific productivity. There is, therefore, a large incentive to re-allocate these workers into better jobs and consequently the $\theta_z$, or the vacancy - searching worker ratio, is relatively large for these jobs. With worker heterogeneity, however, IPT workers tend to be workers with low ex-ante productivity and therefore they cannot transition into ”good” jobs even with a high matching rate (large $\theta$ implies a high matching rate).

The experiment also shows that in the absence of low ex-ante heterogeneity, the implied return to non-production part-time, $b_p$, labour is relatively higher: 0.379 vs 0.147. The reason for this result is that in the absence of a low ex-ante productivity, part-time employment must have relatively large returns relative to full-time employment. Since a larger proportion of a full-time job’s value comes from workplace production, a low $h_i$ has a negative impact on full-time employment than on part-time employment and therefore a relatively lower $b_p$ will induce jobs to be operated part-time.

The second experiment investigates the impact of IPT employment by considering a world where it is not possible to operate jobs in such a manner. Some interesting questions include whether full-time employment would increase, how much of a welfare cost would the economy endure and to what extent would unemployment increase. The counter-factual analysis uses the same calibration as the previous section with $\zeta = 1$ (all jobs are operated full-time).

Since a job is operated as an IPT job only if it is more productive to do so, clearly the elimination of IPT employment will generally have negative consequences. Furthermore, since IPT employment is found mostly among workers with a low ex-ante productivity, one would expect the impact to be greater among such workers. The following table summarizes the important features of experiment 2.

The most important thing to note is that the total welfare in the economy, meaning the sum of total production and home (leisure) production, is slightly increased by IPT employment. This increase, however, is relatively modest, 1.2285 vs 1.2276. Since IPT
employment is concentrated among workers with low ex-ante productivity, the gains to workers of type $h_1$ are relatively larger, 0.393 vs 0.384. It is therefore the welfare of these workers that is most affected by IPT employment. IPT employment, however, decreases the proportion of workers working full-time from 0.9510 to 0.9125. Since part-time workers produce less than full-time workers, total production of goods in the economy is decreased by the existence of IPT employment. The difference between the latter result and the former follows from the fact that a portion of part-time welfare comes in the form of home production/leisure rather than from market production at a job.

While the effects on welfare were relatively modest, IPT employment has an important impact on the unemployment rate. IPT employment lowers both the average unemployment rate and the maximum unemployment rate, which is experienced during a recession. IPT employment allows the average unemployment rate to fall from 0.0490 to 0.0455 and the maximum unemployment rate to fall from 0.0597 to 0.0548. Similarly to the case of welfare, the impact is greatest amongst $h_1$ workers. IPT employment allows their average and maximum unemployment rates to fall from 0.159 to 0.115 and from 0.190 to 0.136 respectively. Without IPT employment, therefore, the unemployment rates of the lowest productivity workers would be significantly higher.

### 1.6 Conclusion

This study has provided a theoretical framework linking involuntary part-time employment, full-time employment and unemployment. The model utilized is a directed search model with on the job search based on Menzio and Shi (2011); the key novel features are that jobs may be operated part-time or full-time and workers may be ex-ante heterogeneous.
The central purpose of the study is to determine what kind of worker heterogeneity (e.g. ex-ante versus ex-post) is primarily responsible for the existence of IPT employment and the observed increase in IPT employment during recessions. Identification is possible due to the differential effect that the two types of heterogeneity have on a worker’s on the job search behaviour. A low ex-ante productivity does not affect a worker’s search decision, while a low ex-post productivity does. The differential in the job to job mobility rate of IPT employment workers (0.0394 vs .0303) is used as a target in the calibration exercise.

There are several other important theoretical observations in the study. Conditional on a worker’s ex-ante productivity, IPT employment is found amongst workers with low match specific productivities. Indeed, for each ex-ante productivity level $h_i$, there exist two cutoffs, a cutoff $z_u$ above which a job is operated and a cutoff $z_p$ below which all jobs are operated part-time. This means that IPT employment exists in all jobs with a match specific productivity between $z_u$ and $z_p$. This characterization of IPT employment allows one to consider the effects of certain policies on the level of IPT employment. An increase in unemployment benefits, for example, will raise $z_u$ and therefore reduce the proportion of jobs operated part-time in addition to increasing the proportion of workers unemployed.

The calibration exercise revealed that in ex-ante worker productivity is primarily important in the existence of IPT employment, but ex-post match specific worker productivity accounts for the anti-cyclicality of IPT employment. More specifically, the model is calibrated such that the bottom and top 10 percent of workers have a different fixed productivity. The required productivity dispersion is targeted using the job to job mobility rate of IPT employment workers. It is found that the average low productivity worker is 85.1 percent less productive in full-time employment than an average productivity worker. Additionally, it is found that on average 38.94 percent of low productivity workers are in a part-time job.

Finally two experiments are conducted investigating certain features of IPT employment. The first experiment considers a restricted version of the model which does not possess ex-ante worker heterogeneity. It demonstrates that the implied monthly job to job transition rates for IPT workers would be 0.4150 which is more than a factor of 10 greater than the actual rate. The implied returns to part-time leisure, in the restricted model, is also greater 0.39 as opposed to 0.147. The second experiment investigates the importance of IPT employment by considering a restricted version in which IPT employment is not allowed.
The experiment finds that while the welfare gains of IPT employment are modest, they are greatest among the lowest productivity workers. Furthermore, the unemployment rate would rise from 0.0490 to 0.0455 and from 0.159 to 0.115 among the lowest productivity workers. In this way, therefore, IPT is important in reducing unemployment especially amongst the worst workers. The existence of IPT employment does, however, reduce total production in the economy since a portion of the returns to part-time employment is leisure based.

In a broader context this study is one of the first to study involuntary part-time employment. It can therefore serve as a benchmark for future research, which may consider alternative contract structures. Contracts specifying different wage setting rules may be particularly interesting as well as those considering differential hiring/firing costs for workers in part-time positions. One particularly interesting possible alternative contract setting is one where wages are sticky. Sticky wages may explain part of the prevalence of involuntary part-time employment since under such a wage structure a firm may not be able to lower an individual’s wage but may more easily reduce their work hours.
# Appendix A

## Figure 1.1: Involuntary Part-Time Employment Facts

### Table 2. Involuntary part-time employment, by age, class of worker, and industry, selected quarterly data, not seasonally adjusted

(Numbers in thousands)

<table>
<thead>
<tr>
<th>Age, industry, and class of worker</th>
<th>2006 Q3</th>
<th>2008 Q3</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total</td>
<td>Percent</td>
<td>Total</td>
</tr>
<tr>
<td></td>
<td>distribution</td>
<td></td>
<td>distribution</td>
</tr>
<tr>
<td><strong>Total, 16 years and over, both sexes</strong></td>
<td>4,096 100.0</td>
<td>5,830 100.0</td>
<td>1,734 100.0</td>
</tr>
<tr>
<td>16 to 19 years</td>
<td>406 9.9</td>
<td>477 8.2</td>
<td>71 4.1</td>
</tr>
<tr>
<td>20 to 24 years</td>
<td>759 18.5</td>
<td>958 16.4</td>
<td>199 11.5</td>
</tr>
<tr>
<td>25 years and over</td>
<td>2,932 71.6</td>
<td>4,385 75.4</td>
<td>1,463 84.4</td>
</tr>
<tr>
<td>25 to 34 years</td>
<td>895 21.9</td>
<td>1,346 23.1</td>
<td>451 26.0</td>
</tr>
<tr>
<td>35 to 44 years</td>
<td>785 19.2</td>
<td>1,120 19.2</td>
<td>335 19.3</td>
</tr>
<tr>
<td>45 to 54 years</td>
<td>704 17.2</td>
<td>1,077 18.5</td>
<td>373 21.5</td>
</tr>
<tr>
<td>55 years and over</td>
<td>547 13.4</td>
<td>852 14.6</td>
<td>305 17.6</td>
</tr>
</tbody>
</table>

### Industry and Class of Worker

<table>
<thead>
<tr>
<th>Total, nonagricultural industries</th>
<th>4,007 100.0</th>
<th>5,739 100.0</th>
<th>1,732 100.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Self-employed workers</td>
<td>478 11.9</td>
<td>726 12.7</td>
<td>248 14.3</td>
</tr>
<tr>
<td>Unpaid family workers</td>
<td>3 0.1</td>
<td>5 0.1</td>
<td>2 0.1</td>
</tr>
<tr>
<td>Wage and salary workers</td>
<td>3,526 88.0</td>
<td>5,008 87.3</td>
<td>1,482 85.6</td>
</tr>
<tr>
<td>Mining</td>
<td>5 0.1</td>
<td>11 0.2</td>
<td>6 0.3</td>
</tr>
<tr>
<td>Construction</td>
<td>399 10.0</td>
<td>602 10.5</td>
<td>203 11.7</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>221 5.5</td>
<td>314 5.5</td>
<td>93 5.4</td>
</tr>
<tr>
<td>Wholesale and retail trade</td>
<td>619 15.4</td>
<td>970 16.9</td>
<td>351 20.3</td>
</tr>
<tr>
<td>Wholesale trade</td>
<td>55 1.4</td>
<td>80 1.4</td>
<td>25 1.4</td>
</tr>
<tr>
<td>Retail trade</td>
<td>565 14.1</td>
<td>890 15.5</td>
<td>325 18.8</td>
</tr>
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<td>144 3.6</td>
<td>237 4.1</td>
<td>93 5.4</td>
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<tr>
<td>Transportation and warehousing</td>
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<td>229 4.0</td>
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<td>303 5.3</td>
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<td>Public administration</td>
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<td>56 1.0</td>
<td>23 1.3</td>
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Figure 1.2: Full-Time Average Wage
Figure 1.3: Involuntary Part-Time Average Wage

- v3 (original data)
- v3 (smoothed)

Cyclical component of v3
Appendix B: Proofs

Proposition 1

From (1.5): \( V^i(z, y_a) = \zeta Y^i_j(z, y_a)+(1-\zeta)Y^i_k(z, y_a)+\tau E[U^i_j(y_a)]+(1-\tau)[\zeta \lambda_f p(\theta^i(n(y_a), y_a)n(y_a)+\zeta[1-\lambda_f p(\theta^i(n(y_a), y_a))]\beta E(V^i(z, \hat{y}_a))] + (1-\zeta)\lambda_p p(\theta^i(n(y_a), y_a)n(y_a)+(1-\zeta)[1-\lambda_p p(\theta^i(n(y_a), y_a))]\beta E(V^i(z, \hat{y}_a))]. \)

Consider a match with value \( v. \)

a) If \( \tau(z, y_a) = 1 \) there is no search this period and therefore any \( n \ast (y_a) \) is optimal. If \( \tau(z, y_a) = \delta, \) the optimal contract maximizes \( \max_x p(\theta_x^i)(x - v) \equiv m^i(v(z, y_a)) \) independent on whether \( \zeta(z, y_a) = 1 \) or \( \zeta(z, y_a) = 0. \)

b) Since \( V^i(z, y_a) \) is linear in \( \zeta, y_a, \) \( \tau(z, y_a) = 1 \) if \( \beta E(U^i_j(y_a)) > \zeta \lambda_f p(\theta^i(n(y_a), y_a)n(y_a)+\zeta[1-\lambda_f p(\theta^i(n(y_a), y_a))]\beta E(V^i(z, \hat{y}_a)] + (1-\zeta)\lambda_p p(\theta^i(n(y_a), y_a)n(y_a)+(1-\zeta)[1-\lambda_p p(\theta^i(n(y_a), y_a))]\beta E(V^i(z, \hat{y}_a)). \)

c) Since \( V^i(z, y_a) \) is linear in \( \zeta, y_a, \) and the decision to operate is independent of whether a match is operated, \( \zeta, y_a = 1 \) if \( \lambda_f p(\theta^i(n(y_a), y_a)n(y_a)+\zeta[1-\lambda_f p(\theta^i(n(y_a), y_a))]\beta E(V^i(z, \hat{y}_a)] \geq \lambda_p p(\theta^i(n(y_a), y_a)n(y_a)+(1-\zeta)[1-\lambda_p p(\theta^i(n(y_a), y_a))]\beta E(V^i(z, \hat{y}_a))]. \)

Theorem 1

The proof has two steps. The first step establishes the existence of an equilibrium to a related social planner’s problem. The second step verifies that a Block-Recursive equilibrium can be constructed using the value and policy functions that solve the social planner’s problem.

First the social planner’s problem must be developed. Let \( S(\psi) \) represent the social planner’s value function where \( \psi \) is the state of the economy as in the main text. The social planner decides how many vacancies to post for both unemployed workers and employed workers. Given a current distribution of workers this amounts to determining the market tightness \( \theta^{is}(y_a) \), \( d^{is}, \zeta^{is}(z, y_a) \) where the superscript \( i \) refers to the ex-ante type of worker and \( s \) denotes that the function solves the social planner’s problem. An unemployed worker produces \( b_u^i \) units and finds a job with probability \( \lambda_u p^i(\theta^{is}(y_a)) \). A worker in job \( z \) produces \( \zeta^{is}(z, y_a)\lambda_f p^i(\theta^{is}(y_a)) \) and finds a job with probability \( \zeta^{is}(z, y_a)\lambda_p p^i(\theta^{is}(y_a)). \) Creating vacancies costs \( \lambda_u \theta^{is}(z, y_a)ku^i, \lambda_f \theta^{is}(z, y_a)kg^i(z), \lambda_p \theta^{is}(z, y_a)kg^i(z). \)

Let \( F(d^i, \zeta^i, \theta^i_u, \theta^i_z) \) denotes that the function solves the social planner’s problem. An
(1 - zeta(z_j, y_a)) Y_p(z_j, y_a) g(z_j) + b^i_u u^i and ψ^i = (u^i, g^i(z), y_a), then the social planner problem can be written as:

\[ S^i(Ψ) = \max_{d, ζ, θ_z, θ_u} F(d^i, ζ^i, θ_z^i, θ_u^i) \]

s.t 1) \[ \hat{u}^i = u^i[1 - λ_u p^i(θ_u)] + \sum_j d^i(z_j) g(z_i) \]

2) \[ \hat{g}^i = h(Ψ) f(z) + [1 - d(z, y_a)] [ζ(z, y_a)(1 - λ_f p(θ_z(z, y_a)))] + (1 - ζ(z, y_a))(1 - λ_p p(θ_z(z, y_a))) \]

3) \[ h^i(Ψ^i) = λ_f p(θ_z) u + \sum_i [1 - d(z_i, y_a)] [ζ(z, y_a) λ_f p(θ_z(z, y_a))] \]

\[ S^0 = F(d, ζ, θ_z, θ_u) + β E[S^0(Ψ)] \]

Let Ψ denote the set \( Y \times [0, 1]^{N_i+1} \) (\([0, 1]^{N_i+1}\) is the measure of workers of each type).

Consider an economy with a single type i worker (I therefore drop the i superscript, although it is implied throughout). Let \( C(Ψ) \) denote the set of bounded continuous functions defined over Ψ and consider \( r \in C(Ψ) \) define an operator \( T : C(Ψ) \rightarrow C(Ψ) \) as follows:

\[ (Tr)(Ψ) = \max_{ζ, d, θ_z, θ_u} F(d, θ_u, θ_z, Ψ) + β E[r(Ψ)] \]

s.t 1) \[ \hat{u}^i = u^i[1 - λ_u p^i(θ_u)] + \sum_j d^i(z_j) g(z_i) \]

2) \[ \hat{g}^i = h(Ψ) f(z) + [1 - d(z, y_a)] [ζ(z, y_a)(1 - λ_f p(θ_z(z, y_a)))] + (1 - ζ(z, y_a))(1 - λ_p p(θ_z(z, y_a))) \]

3) \[ h^i(Ψ^i) = λ_f p(θ_z) u + \sum_i [1 - d(z_i, y_a)] [ζ(z, y_a) λ_f p(θ_z(z, y_a))] \]

\[ ζ : Z \rightarrow [0, 1], d : Z : [δ, 1], θ_u ∈ [0, 0], θ \]

For every \( r \in C(Ψ) \) the problem above is to maximize a continuous function over a compact set. Hence the maximum is attained and the argmax is non-empty. Since both \( F \) and \( r \) are bounded, \( Tr \) is also bounded. It follows from the theory of the maximum that \( Tr \) is also continuous hence, \( T : C(Ψ) \rightarrow C(Ψ) \).

Furthermore, \( T \) satisfies the required discounting and monotonicity properties of Blackwell’s sufficiency conditions. \( T \) therefore has a unique fixed point in \( C(Ψ) \). This establishes that there exists a solution \( S ∈ C(Ψ) \) to the Social Planner’s problem. Since, \( \lim_{t \rightarrow ∞} β S(Ψ) = 0 ∀ψ ∈ Ψ \) it follows that \( S = S^0 \).

Now obtaining the optimality conditions. First establishing that we can decompose the Central Planner’s Problem. First establishing that we can decompose the Central Planner’s Problem so that it only depends on \( Ψ \) through \( y_a \). \( S^0(Ψ) = S^0_u(y_a) u + S^0_z(y_a) g(z) \).

Consider \( r^0 \in R : r^0(Ψ) = r^0_u(y_a) u + r^0_z(y_a) g(z) \). The above mapping \( T \) takes \( r \in R \rightarrow R \).
Let $L(\Psi)$ denote the set of bounded continuous functions that are linear in $u$ and $g(.)$. Recall $F(\zeta, d, \theta_u, \theta_z) = -k\lambda_u u\theta_u + [\sum_j \int (1-d(z_j))][\zeta(z, y_a)\lambda_f + (1-\zeta(z, y_a))]g(z_j)\theta_z(z_j) + bu + \sum_j [\zeta(z, y_a)Y_f(z, y_a) + (1-\zeta(z, y_a))Y_p(z, y_a)]$. Take F.O.C wrt $\theta_u$.

$$-k\lambda_u u + \beta p'(\theta_u)E[-\lambda_u ur_u(y_a) + \sum_j r_z(y_a)\lambda_u uf(z_j)] \leq 0 \text{ with c.s.}$$

$$k \geq p'(\theta_u)\beta E[\sum_j (r_z(y_a)f(z_i)) - r_u(y_a)]$$

$p'(.)$ is strictly decreasing in $\theta$ therefore there is a unique solution $\theta_u$. $\hat{\theta}_u : y_a \rightarrow [0, \delta]$ (Since $\theta_u(.)$ only depends on $\Psi$ through $y_a$)

Similarly for $\theta_z$: $k \geq p'(\theta_z(z))\beta E[\sum_i (r_{zi}(y_a) - r_{zi}(\hat{y}_a))]$

$$\hat{\theta}_z : Z \times Y_a \rightarrow [0, \overline{\theta}]$$

Given $\zeta(\cdot), \hat{\theta}_z$ for each $\psi \in \Psi d$ is given optimally by $d(z) = 1$ if $E[U(y_a)] > E[V^i(z, y_a)]$ or in terms of the social planner’s value function $E_{r_u(y_a)} > E[r_z(y_a)] + [\zeta(y_a)\lambda_f + (1 - \zeta(y_a))]\lambda_p]\{p(\hat{\theta}_z(y_a))[E[S] - E[r_z(y_a)]\}]$ where $S$ is the value from search. $d = \delta$ otherwise. We see that $d(y_a)Z \times Y_a \rightarrow [0, \overline{\theta}]$.

$$\zeta = 1 \text{ if } Y_f(z, y_a)+(1-d(z, y_a))\lambda_f p(\theta_z)[ES-E[r_z(y_a)]] \geq Y_p(z, y_a)+(1-d(z, y_a))\lambda_p p(\theta_z)[ES-E[r_z(y_a)]] \text{. Otherwise } \zeta = 0$$

We see that $\zeta(z, y_a) : Z \times Y_a \rightarrow [0, 1]$.

Define $\tilde{r}_u(y_a) = -k\lambda_u \theta_u + b + [1-\lambda_u p(\theta_u(y_a))]\beta E[r_u(y_a)] + \lambda_u p(\theta_u(y_a))\beta E[\sum_i r_{zi}(y_a)f(z_i)]$ and similarly $\tilde{r}_z(y_a) = -[\lambda_f \zeta(z, y_a) + \lambda_p (1-\zeta(z, y_a))]$

We have shown that the solution to the social planner’s problem can be written in the above form. The last thing to establish is that this function is non-decreasing in $z$. Let $M(\Psi)$ denote the set of functions $r : \Psi \rightarrow \mathbb{R}$ such that $r \in L(\Psi)$ and $r_z : Z \times Y \rightarrow \mathbb{R}$ is non-decreasing in $z$. We need to show that the mapping defined above, $T$, maps $r(\Psi) \in M(\Psi) \rightarrow M(\Psi)$.

Consider $z_2, z_1 \in Z : z_2 > z_1$

In order to prove that a BRE exists I take the solution to the central planner’s problem and demonstrate that such an equilibrium can be constructing using this solution. First I consider a single worker type $i$, then I show that an equilibrium exists that separates the workers. In other words I demonstrate the existence of functions $\{D^{is}, m^{is}, U^{is}, V^{is}, d^{is}, \zeta^{is}, \theta^{is}\}$ that satisfy the conditions of definition 1 built upon the central planner’s solution above along with its associated policy functions. A superscript $s$ will designate a variable belonging to this set.

Consider the fixed point value function of the central planner, $s(y_a)$. Let $s_i^*(y_a)$ be the
derivative of \( s(y_a) \) with respect to \( g_h \) and \( u_i \) and \( s'_i(y_a) \) be the derivative of \( s(y_a) \) with respect to \( g_h \) and \( g_z \). Define \( U^{is} = b^i_u + \beta s'_u(y_a) \) and \( V^i(z, y_a) = \zeta^i(z, y_a)Y_f^i + (1 - \zeta^i(z, y_a))Y_p + \beta E s'_z(y_a) \).

The market tightness function, \( \theta^{is}(x, y_a) \), is set equal to \( q^{-1} \sum_{i=1}^{k} \{\theta^{is}(z, y_a)\} \) for all \( x \leq \bar{x}(y_a) \) and equal to 0 otherwise. The search variables \( D^{is}(z, y_a) \) and \( m^{is}(z, y_a) \) are set equal to the maximizer of \( p^i(\theta^{is}(x, y_a))(x - v) \).

By construction the first condition \((i)\) of the definition is satisfied and \( D^{is}(z, y_a) \) and \( m^{is}(z, y_a) \) satisfy condition \((ii)\). It must be shown that \( D^{is} \) and \( m^{is} \) are equivalent to those of the central planner \( D^{is} \) and \( m^{is} \). First an important feature of the optimal contract must be discussed. Since the firm offers a wage \( w \) such that the contract offers a worker searching for such a job \( x \), the firm acts as a residual claimant on the value of operating a job. An optimal contract therefore stipulates policy functions \( \{D^{is}, m^{is}, d^{is}, \zeta^{is}\} \) that maximize the joint value of a match \( V^{is}(z, y_a) \). Using (3.6), and \( p^i(\theta^{is}(x, y_a))(x - v) \) and noting that any submarket solves the search problem if \( \theta^{is} = 0 \), the search problem can be rewritten \( m^{is} = \arg \max_x \{ -k \theta^{is}(x; y_a) + p^i(\theta^{is}(x, y_a)) \sum_j E V^{is}(z_j) f_j(z_j) - v \} \). This is equivalent to the central planner’s problem and therefore \( D^{is}(z, y_a) = D^{is}(z, y_a) \) and \( m^{is}(z, y_a) = m^{is}(z, y_a) \) and the equilibrium tightness functions \( \theta^{is}_u \) and \( \theta^{is}_z \) are equal to \( \theta^{is}_u \) and \( \theta^{is}_z \) respectively.

The next step is to show that the third \((iii)\) and fourth \((iv)\) parts of the definition hold. Writing out the derivative of the central planner’s problem with respect to \( u \) gives \( s'_u = -k \lambda_u \theta^{is}_u(y_a) + b_u + [1 - \lambda_u p^i(\theta^{is}_u(y_a))] \beta E [s'_{z_u}(y_a)] + \lambda_u p^i(\theta^{is}_u(y_a)) \beta E [\sum_j s'_{z_j}(z_j, \hat{y}_a)] \) and with respect to \( g(z) \), \( s'_z = \zeta^{is}(z, y_a) Y_f^i(z, y_a) - k \lambda \theta^{is}_z(y_a) + (1 - \zeta^{is})(z, y_a) Y_f^i(z, y_a) - k \lambda \theta^{is}_z(y_a) \) + \( (1 - \theta^{is}(z, y_a)) \zeta^{is}(z, y_a) \lambda p^i(\theta^{is}_z(y_a)) E [s'_{z_u}(y_a)] \) and \( d^{is}(z, y_a) \beta E [s'_{z_u}(y_a)] \). Replacing \( U^i(y_a) = s'_u \) and \( V^i(z, y_a) = s'_z(z, y_a) \) and using the previous result \( D^{is}(v, y_a) = D^{is}(v, y_a) \), it is clear that \((iii)\) and \((iv)\) hold. Furthermore, since the maximization problem for the competitive market is identical to that of the central planner, the policy functions \( \{d^{is}, \zeta^{is}\} \) are equal to \( \{d^{is}, \zeta^{is}\} \).

The last part \((v)\) of the definition requires that the value functions and policy functions are independent of the distribution of workers. Since the functions above are independent of \( u, g(z) \), this condition is satisfied.

The above equilibrium treated every worker type separately. In the case of observable ex-ante heterogeneity such an equilibrium exists as shown by Menzio and Shi (2010).
Chapter 2

Heterogeneous Productivity and the Wage-tenure Profile


2.1 Introduction

The wage profile of workers is a key area of interest. It is well established that workers generally experience an increase in wages through the early part of their work history until reaching a plateau towards the end part of their career. The wage gains come through two potential channels. The first channel is through increases of their wage at their current employer while the second channel is through an increase in wages from finding a better employment offer at another firm. Furthermore, due to worker heterogeneity, the process of obtaining higher wages may differ across workers. More productive workers, for example, may experience faster increases in wages through either or both of the above channels.

One way of interpreting wage increases at the current firm is via productivity growth. Since Becker (1975) and Mincer (1974), this explanation has become quite standard. Workers over time become more proficient at their jobs and consequently receive higher wages as a consequence of a competitive market environment. The fact that much earnings growth comes from workers changing firms, however, suggests that the labour market might not be perfectly competitive. With non-competitive labour markets it is no longer clear whether, and to what degree, wage increases are due to increases in worker productivity.

Wage-tenure contracts relate a workers increasing wage profile to a need by firms to reduce worker turnover while promising workers some initial value. The possibility of firms back-loading contracts was investigated in Stevens (2004). In her framework, firms optimally backload contracts for in order to reduce tenure in a random search environment. Since workers are assumed to be risk neutral, the optimal contract is a step-wage contract. Burdett and Coles (2003) extend this framework by assuming that workers are risk-averse. The consequence of this assumption is that a workers wage rises continuously in tenure. Since workers receive offers exogenously, the equilibrium wage-tenure contract is dependent on the distribution of workers, which makes introducing additional heterogeneity difficult. Shi (2009) considers instead a directed search environment. This assumption makes the equilibrium wage-tenure contracts independent of the distribution of workers, or in other words the equilibrium is made Block-Recursive. Theoretically, this study closely follows the framework of Shi (2009). The inclusion of workers with heterogeneous productivity being the major difference. In order to test the model there is also the inclusion of exogenous
In directed search models, heterogeneous workers search in different submarkets. A free-entry condition for vacancy postings then ensures that in equilibrium all submarkets offer a firm the same expected profit. Peters (1991) and Montgomery (1991) provide early treatments. Moen (1997) constructed an equilibrium which was pareto optimal and could be attained if employers publicly announced their wages. The equilibrium he constructed has the feature that all submarkets visited actively by workers give the same expected profit to the firm.\footnote{For a related study using a directed search framework, See Hoffman and Shi (2010)} This study benefits from search being directed since, in addition to the tractability afforded by the resulting Block-Recursive nature of the equilibrium, the mechanism through which more productive workers obtain steeper wage-tenure profiles is strengthened. If search is random, all workers benefit from the increased number of job postings due to the presence of high types in the labour market. The only rationale for steeper wage-tenure profiles in that environment is that firms make higher profits off of the high types and therefore raise wages faster to retain them. In the directed search framework, high types are not only more profitable to the firms but also search in submarkets that offer a higher matching rate. There is therefore an additional incentive to backload wages.

The main goal of the paper is to demonstrate that even in the absence of differential rates of learning, directed search implies that more productive workers experience faster increases in their wages through tenure for the simple reason that they are better able to find a successful match when searching on the job. There are several other interesting insights. Assuming labour market frictions and on the job directed search, the average difference between a workers marginal benefit to the firm and the wage he receives is decreasing. A calibration exercise is used to determine the model’s effectiveness in explaining the tenure profiles of high educated workers vis-a-vis medium and low educated workers. A baseline wage-tenure profile is first estimated in order to determine whether more educated workers have a steeper baseline wage-tenure profile. The results show that the unconditional returns to tenure in log wages are 0.034, 0.04 and 0.06 for low, medium and highly educated workers. Interpreting education as begin a proxy for productivity, more productive workers therefore enjoy steeper wage profiles. Furthermore, more productive workers are initially more mobile but, since their wages increase faster than less productive workers, eventually
less productive workers are more mobile. Finally, the fact that workers in simulation reach their maximum wage in 4 years provides some evidence that learning on the job is also an important.

2.2 Model

Consider an environment with $N$ different types of heterogeneous workers each type being of measure $M_j$ for $j \in \{1, 2, \ldots, N\}$. Time is infinite and continuous with workers and firms discounting at a constant rate $\rho$. Workers preferences are given by a utility function with the standard properties: $0 < u'(w) < \infty$ and $-\infty < u''(w) < 0$ for all $w \in (0, \infty)$, and finally $u'(0) = \infty$. An employed worker produces output that depends on his productivity type. A worker of type $j$ produces $h_j > 0$ units of production when employed at a firm, and $b_j$ units when unemployed. Workers of all types face the same exogenous rate of death $\delta_d$, and are replaced by a worker of the same ability upon death. Finally workers employed at a firm also face an exogenous rate of separation $\delta_s$.

Firms are identical and risk-neutral. They make 2 decisions: First, they decide what value and to what kind of worker to make an offer, then they decide on the optimal wage path over time for that offer $\{w(t)\}_{t=0}^\infty$. The wage path, called the wage-tenure contract, determines the workers compensation after being at the firm for $t$ units of time. These contracts are full-commitment ones from the point of view of the firms but not from the point of view of the workers. This means that while a worker can leave a job at any time for a better offer, firms must pay workers according to the terms agreed upon at the beginning of the contract. Furthermore, firms are not allowed to respond to outside options. Since, as stated earlier, both workers and firms discount at the rate $\rho$, and matches exogenously end at rate $\delta_d + \delta_s$, the effective discount rate is $r = \rho + \delta_d + \delta_s$.

As stated above, wage-tenure contracts offer a particular value to a particular type of worker. $f^j(V(t))f \in \{J, \frac{dp}{dV}, \frac{d^2p}{dV^2}, \ldots\}$ will henceforth refer to a function for type $j$ workers with contracts with current value $V$. An offer to worker of type $j$ refers to the value $V(0)$.

$^2$This is a standard assumption in the literature since otherwise firms would bid and counter bid until the worker received the full value of production.

$^3$The existence of a separating equilibrium of this type is proven in subsequent sections.

$^4$It is important to note that $t$ here refers not to time but rather to tenure at a particular firm. Also, the dependence of $V$ on $t$ will be dropped occasionally.
For completeness, denote an unemployed worker’s tenure as $\emptyset$. An unemployed worker of type j’s current value is therefore $V_u^j = V(\emptyset)$ and his wage $b_u = w(\emptyset)$.

The lowest value that a worker can receive is clearly $V = u(b)/r$, since he can reject all offers and enjoy the unemployment benefits in perpetuity. Similarly, for a given j-type of worker there is a highest wage $\bar{w}^j$ which firms will not be able to profit above, therefore the highest conceivable offer is $\bar{V}^j$. Consequently every offer is contained in the set $[V, \bar{V}^j]$, where explicitly

$$V = u(b_u)/r, \bar{V}^j = \bar{w}^j/r$$

(2.1)

Unemployed as well as employed workers of every type can search for a new offer. It is assumed that they can direct their search so that they target a specific submarket $x^j$, depending on their current offer and the ease with which they expect to find employment at a firm offering a value of $x$ to workers of ability j. The ease at which they find such employment is determined by the “tightness” $\theta(x^j)$, which is the ratio of j applicants searching for the value $x$ to vacancies $v^j$ offering it. The total number of matches in a given submarket is given by a linearly homogenous matching function $M(N(x^j), v(x^j))/\theta(x^j)$, where $N(x^j)$ is the number of applicants of type j to an offer $x$. This implies that the Poisson matching rate is $p(x^j) = M(1, 1/\theta(x^j))$ for workers and $q(x^j) = M(\theta(x^j), 1)$ for firms. The function $p(.)$ can therefore be referred to as the employment rate and $q(.)$ as the hiring rate.

In equilibrium it must be the case that $p(.)$ is decreasing and $q(.)$ is increasing. A decreasing employment rate introduces a tradeoff between workers searching for high values and finding employment at that value. Workers with low values will therefore target relatively easier to obtain jobs, while workers with already high values will target harder to obtain jobs. The same reasoning applies in reverse with firms offering low values receiving relatively fewer applicants than those offering high values. The search process is therefore competitive with the respective matching rates acting as hedonic prices.

The competitive nature of search gives a glimpse of what to expect in equilibrium. Since all else equal more productive workers give firms a higher level of profit, a contract offering the same value $x$ to workers i and j, with $h^j > h^i$ must be easier to obtain for worker j.

---

5Note that $\bar{V}^j$ and $\bar{w}^j$ is written as a function of j. This is since the more productive a worker the higher the maximum wage for that worker is.

6In equilibrium firms must receive the same value regardless of which submarket they post an offer in
than worker i. This means that more productive workers should search in relatively higher value submarkets. Furthermore, the maximum wage a firm can offer without losing money is higher. This means that the upper-bound on offers is increasing in productivity, $V^i < V^j$ for $i < j$. The above relates to wage tenure contracts in that the maximum wage offered more productive is not only higher, as would be expected in most models, but due to the higher turnover, firms have an incentive to increase wages faster.

2.3 Optimal Decisions

2.3.1 Worker’s Search Decision

This section details the firm’s and worker’s optimal decisions. It provides the first details leading towards the theoretical section. First worker’s decisions are considered and then the firm’s. A worker j’s decision is based on the matching rate function $p(\cdot)$. The equilibrium considered here satisfies the following:

Assumption 2.1 (i) $p(V^j) = 0$ (ii) $p(x^j)$ is bounded, continuous and concave for all $x$; (iii) $p(x^j)$ is strictly decreasing and continuously differentiable for all $x^j \in \bigcup_{j=1}^{N_h} [V^j, \bar{V}^j]$

The first part ensures that the highest valued offer for a worker is never attained. The second part is useful in proving an existence of the equilibrium. The third part ensures that there is a tradeoff, from the point of view of the worker, between obtaining a high value job and a high matching rate, while the function’s differentiability makes the analysis easier. With the above assumption it is possible to characterize the worker’s search problem. A worker with a current value of $V$ searches for the submarket $x^j$ that gives him the optimal tradeoff between surplus over the current value, $(x-V)$, and probability of finding a match at that submarket, $p(x^j)$. The optimal search decision is therefore:

$$S^j(V) \equiv \max_{x^j \in [V^j, \bar{V}^j]} p(x^j)(x - V) \quad (2.2)$$

7Unlike other functions, $p$ is written solely as a function of the submarket since $x^j$ makes it clear that it is implicitly a function of a worker’s productivity.

8The super-script $j$ is used to highlight that the workers are heterogeneous. A worker searches for value $x$, for example, but if I want to discuss the tightness of a submarket offering value $x$, I will write $\theta(x^j)$ since the tightness will be lower for more productive workers.
Denoting the solution \( x^* = F^j(V) \), the following lemma holds:

**Lemma 2.1**  \( F^j(V^j) = V^j \) for all \( V < V^j \) the following holds: i) \( F^j(V) \) is unique for each \( V \), and continuous in \( V \); ii) \( S^j(V) \) is differentiable, with \( S''^j(V) = -p(F^j(V)) \); iii) \( F^j(V_2) - F^j(V_1) \leq (V_2 - V_1)/2 \) for all \( V_2 \geq V_1 \); iv) if \( p'(.) \) exists, then \( F^j(V) \) and \( S''^j(V) \) exist, with \( 0 < F^j(V) \leq 1/2 \); v) \( V = F^j(V) + p(F^j(V))/p'(V) \)

The above lemma provides several useful insights. The uniqueness of \( F^j(V) \) follows from the above discussion of the tradeoff between the matching rate and the value of a match. Furthermore, the higher the value the more willing a worker is willing to tradeoff a high matching rate for a higher value. This separation of workers into different states makes the equilibrium more tractable. Finally, the expected gain, \( S^j(V) \), and actual gain, \( F^j(V) - V/V \), from search is decreasing in \( V \) and the expected gain \( S^j(V) \) is diminishing at a decreasing rate since \( S''^j(V) > 0 \).

### 2.3.2 Value Functions

Since wage-tenure contracts imply changing match-surplus division between workers and firms, both the worker and firm value functions are evolving over time. Consider a worker of type \( j \) with current value \( V \). The worker’s value function has four components. The worker receives a wage \( w(t) \), value from search \( S^j(V(t)) \), changes in value due to moving along the wage profile \( V^j(t) \), and he becomes unemployed at a rate \( \delta_s \) receiving a flow benefit of \( -\delta_s U^j \). Using \( r = \delta_d + \delta_s + \rho \) as the effective discount rate, the worker’s value function is given by:

\[
rV^j(t) = u(w(t)) + S(V^j(t)) + \dot{V}^j(t) - \delta_s U^j \tag{2.3}
\]

As mentioned above, the non-stationarity of \( V^j(t) \) is due to a worker moving along the optimal wage profile determined by the firm. A worker receiving a constant wage would therefore experience no change in his value over time, \( \dot{V}^j(t) = 0 \). Since unemployed workers receive the same benefits \( b_u \) irrespective of time, \( \dot{V}^j_u(t) = 0 \) for all \( j \). Unemployed workers also face a different effective discount rate than employed workers since, clearly, there is no exogenous separation from unemployment.
0 = (\rho + \delta d) V^j_u(t) - u(b_u) - S(V^j_u) \tag{2.4}

It should be noted that since \( p(.) \) depends on a worker’s productivity, however, the wages and wage-profile facing a worker depends on his type. Furthermore, since \( S(.) > 0 \) for all \( V^j(t) \), the value of unemployment is clearly larger than \( V \). This means that if \( \overline{w}^j > b_u \), then \( V_u \in [V, \overline{V}] \). This is assumed to hold for all \( j \).

For the firm side, consider a worker with value \( V(t) \) and productivity \( h^j \). The value of a wage contract for a firm consists of the following: the productivity of the worker, the transition rate of the worker (which depends on the wage profile and the contract offers), and the wage profile directly.

\[
\frac{dJ(t)}{dt} = [r + p(F^j(V(t)))^j]J(t) - h^j + w^j(t) \tag{2.5}
\]

Consider the discount rate facing a firm. In addition to the effective discount rate, a worker’s endogenous separation rate also affects the firm’s discounting. The firm increases wages, and consequently the value to the worker, in order to reduce this transition rate. For an arbitrary \( t_o \in [o, t] \) define:

\[
\gamma^j(t, t_o) \equiv \exp(- \int_{t_o}^t [r + p(F^j(V(j))))]d\tau) \tag{2.6}
\]

Integrating the above differential condition and using the boundary condition \( \lim_{t \to \infty} \gamma^j(t, t_o) = 0 \) gives the following:

\[
J^j(t_o) = \int_{t_o}^\infty [h^j - w(t)]\gamma^j(t, t_o)dt \tag{2.7}
\]

### 2.3.3 Optimal Recruiting and Contracts

The firm has two decisions to make. The first is to choose an offer \( x^j \) that maximizes \( q^j(x)J^j(0) \), taking \( q(.) \) competitively. The second is to choose the optimal wage profile, \( \{w^j(t)\}_{t=0}^\infty \) that delivers the promised value \( x^j \). This subsection examines the first decision and subsequently the second part.

The decision on what offers to make is straightforward. For a fixed firm value, \( J^j(0) \), Firms
maximize the expected value of the offer \( q(x^j)J^j(0) \). Since search is directed, firms will post vacancies for an offer until the cost of posting that offer is equal its benefit. This means that \( q(x^j)J^j(0) = k \) for all offers that are made in equilibrium and, for completeness, \( q(x^j)J^j(0) < k \) for all offers not made in equilibrium. In the context of the model, this means that firms are indifferent between all values offered to workers as well as between workers. This is driven by search being directed and firms being perfectly committed to all offers made. All else equal firms will make greater profits if matched with workers of high productivity and low promised value. The probability of a firm matching with such a worker is, however, lower. Therefore in expectation a firm is indifferent between posting contracts targeting any type of worker in the economy.

As stated previously, the highest offer possible offer \( \overline{V}^j \), is determined by a contract offering the highest possible wage for all \( t \), \( \{w^j(t) = \overline{w}\}^\infty_0 \). This wage is solved for by considering the maximum wage that a firm can offer without making a loss. The search value, \( S(\overline{V}^j) = 0 \) since \( \overline{V}^j \) is the highest offer a \( j \)-type worker can obtain. Furthermore, due to the stationarity of the wage, \( \dot{\overline{V}}^j = 0 \). The stationarity of wages and the fact that \( p(\overline{V}^j) = 0 \) also implies that \( \dot{J}^j = 0 \). Consequently the lowest profit a firm makes, \( J^j \), is given by \( (h^j - \overline{w}^j)/r \). Recalling the maximum \( \overline{w} \), the maximum wage that a firm can offer without making a loss is \( \overline{q}_J J^j = k \). Solving for \( \overline{w} \) gives: \( \overline{w} = h^j - rk/\overline{q} \).

The contract part of the decision is choosing the wage profile that maximizes the starting value from the point of view of the firm subject to offering the worker the promised value \( x \).

\[
\max_{w^j(t)} \{w^j(t) = \overline{w}\}^\infty_0 J^j(0), \text{ subject to } V^j(0) = x
\] (2.8)

**Theorem 2.1** The optimal contract has the following features: (i) \( 0 < w^j(t) \geq \overline{w} \) for all \( t \); (ii) \( \frac{dw^j(t)}{dt} > 0 \) for all \( t, w(t) \to \overline{w} \to \infty \) and \( \frac{dw^j(t)}{dt} = \frac{u'(w^j(t))}{[u'(w^j(t))]^2} J^j(t)[\frac{dp(F(V^j(0)))}{dV}] \), all \( t \); (iii) \( V^j(t) > 0 \) and \( \frac{dJ^j(t)}{dt} < 0 \) for all \( t < \infty \), with \( V^j(t) \to \overline{V}^j \) and \( J^j(t) \downarrow \overline{J} \) as \( t \to \infty \).

The optimal contracts convey several features of an equilibrium. The most important features is that wages are strictly increasing continuously throughout tenure. This result comes from the firm’s incentives to reduce worker turnover by back-loading wages. Firms,

\[^9\] If \( q'(\overline{V}) < \overline{q} \) a firm can obtain a higher value by offering a slightly higher wage. See Shi(2009).
would ideally want to backload the wages completely but, due to worker risk aversion, that would be suboptimal. The tension between a firm backloading and worker’s being risk averse, leads to the continuous profile. The varying worker productivity affect the wage profile through \( J^j(t) \) and through \( \frac{dp(F(V^j(t)))}{dV} \), since more productive workers may face better offers.

Two other important features is that the contract induces efficient sharing of the surplus and there is a baseline contract associated with each worker type \( j \). To see the former that

\[
-\frac{dJ^j(t)}{dt} = \frac{V^j(t)}{w'(w^j(t))}.
\]

This means that the marginal cost to the firm of the rising wage profile is equivalent to the marginal benefit to the worker. The baseline contract means that every contract can be written as a truncation of a special contract. \( \{w^j(t)\}_{t=0}^{\infty} : w^j(t) = w^j(t + t_a), t_a \in [0, \infty] \). In other words for a given \( j \) productivity worker, any contract can be written as the latter part of the baseline contract. For the computation of equilibrium this fact allows one to solve for a single baseline contract for each type of worker.

### 2.4 Equilibrium

The equilibrium will make extensive use of the Block Recursivity that directed search affords. With random search, a firm would need to consider the distribution of workers over both productivity types, \( h_j \), and values \( V^j \) when making its offer decision. The block recursive nature of the equilibrium breaks this dependence of value functions on the distribution of workers. This occurs since firms post contracts, \( x^j \), that target only a particular type of worker, namely a worker of productivity \( j \) searching for a the value \( x \). A free entry condition then ensures that firms post vacancies until the equilibrium tightness is achieved where it is optimal for only 1 type of worker to search in that submarket, namely the worker that is targeted. This makes the equilibrium tractable and simplifies the numerical computation of the equilibrium.

#### 2.4.1 Definition of Equilibrium

The equilibrium consists of the following: A set of offers \( V = \{V^j(t)\} \), an application strategy \( F(.) \) along with hiring and job finding rate functions \( q^i(.) \) and \( p^j(.) \), Firm and Worker value functions \( J^j, V^j \) and finally a wage function \( w^j(.) \) that satisfy the following:
1) Given \( p^j(.) \), \( F^j(.) \) solves the workers problem (2.2) 

2) Given \( F^j(.) \) and \( p^j(.) \), the value of unemployment is given by 

3) Given \( F^j(.) \) and \( p^j(.) \), each offer \( x^j \) is given by a contract \( \{w(t)\}_{t=0}^{\infty} \) that solves (2.8) . Furthermore, \( J^j(.) \) is the consequent value function. 

3) Given \( J^j(.) \), \( q^j(.) \), free entry ensures that \( k = q(x^j)J(x^j) \) for all \( x \in [V, \bar{V}] \) and \( q(x)J(x) < k \) otherwise, where \( q(x) \) is as presented previously. 

Additionally there is a function \( G^j(.) \) that gives the distribution of workers of type \( j \) over \( V \), and a function \( n^j \), that gives the fraction of type \( j \) workers employed. 4) \( G \) and \( n \) are stationary 

As alluded to previously, parts 1) - 3) are independent of the distribution of workers due to the equilibrium’s block-recursive nature. Therefore the functions in 1) - 3) only affect the distribution and employment functions \( G^j \) and \( n^j \), where \( G^j(.) \) gives the measure of workers of type-j employed at values less than \( V \) and \( n^j \) gives the measure of employed workers. There is no feedback, unlike random search models. Most of the above definition is obvious. Examining 3), one concludes that since \( J^j(t) \) is decreasing in \( V \), there is a meaningful tradeoff between an offer, \( x^j \)’s value and the probability of a worker finding a match in that submarket. Specifically, \( q(x^j) = \frac{k}{J(x^j)} \) and therefore \( q^j(.) \) is decreasing in the offer \( x^j \). Since \( p(x^j) = M(q) \) and \( M(.) \) is a decreasing function, \( p(.) \) is indeed decreasing in \( x^j \). A further note concerning 3) is that it applies to all submarkets in \([V, \bar{V}]\) not just equilibrium submarkets \([v1, \bar{V}]\), where \( v1 = F(V_j^1) \) the submarket the unemployed workers search in.

Specifically outlining the Block-Recursive nature of the equilibrium. Given \( q^j(.) \), the matching function \( p^j(.) \) can be obtained. Since \( q^j(.) \) is independent of the distribution of workers, so too is \( p^j(.) \). The matching rate function \( p^j(.) \) is sufficient for workers to choose their preferred submarket \( F^j(.) \) and consequently gives the firm the transition rate of a worker over values \( V^j \). This transition rate is sufficient for the firm to choose the wage-profile, \( w^j(.) \) and consequently the optimal contract.

### 2.4.2 Existence of an Equilibrium

The equilibrium is constructed in two steps. The first step establishes an equilibrium for an economy consisting only of \( j \)-type workers, while the second establishes the existence of
an equilibrium consisting of all type workers. The key insight is that directed search allows firms to contract a particular $V^j$ worker so that the distribution of worker productivity is irrelevant. Note that this subsection only solves for the functions $p^j(\cdot)$, $q^j(\cdot)$, $w^j(\cdot)$, $F^j(\cdot)$, and $J^j(\cdot)$. The equilibrium distributions $G^j(\cdot)$ and $n^j$ will be considered in a subsequent section.\footnote{Recall that this is possible due to Block-Recursivity. (e.g Value functions are independent of the worker distribution functions)}

The strategy for the first step is develop a mapping of all functions onto $w^j(t)$ and prove the existence of a fixed point. Take an arbitrary wage profile $w$. Let the subscript $w$ on variables to denote that they are a function of the wage path $w$ under consideration. Integrating (iii) from theorem 2.3 and noting that $J^j(V^j) = k/\bar{q}^j$, gives equation (2.9) below.

The other equations have been established in the text:

\begin{equation}
J^j_w(V) = \frac{k}{q^j} + \int_V^V \frac{1}{u'(w^j(z))}dz
\tag{2.9}
\end{equation}

\begin{equation}
p^j_w(V) = M(\frac{k}{J^j_w(V)})
\tag{2.10}
\end{equation}

\begin{equation}
S^j_w(V) = \max_x \{p_w(x)(x - V)\}, F^j_w(V) = \arg\max_x \{p_w(x)(x - V)\}
\tag{2.11}
\end{equation}

The equations for $\frac{dV}{dt}(V)$ and $\frac{dJ^j}{dt}(V)$ are used to obtain $\frac{dV}{dt^w}$ and $\frac{dJ^j}{dt^w}$. Using the above, the optimality conditions and requiring $V^j_w(t) \geq 0$, one can write the following function $w^j(V) = \psi w^j(V)$ where:

\begin{equation}
\psi w^j(V) = h^j - [r + p^j_w(F^j_w(V))J^j_w - \max \{0, rV - S^j_w - u(w^j(V))\}] / u'(w^j(V))
\tag{2.12}
\end{equation}

Using the preceding relationship between $w^j$ and the functions $q^j_w$, $p^j_w$, $J^j_w$, $F^j_w$, and $S^j_w$, a fixed point in $w$ implies a fixed point in the latter.

In order to prove the existence of the fixed point, the following assumption is useful.

\textbf{Assumption 2.2} (i) $M(q)$ is continuous and $q(V) \in [\underline{q}, \bar{q}]$, where $q$ will be specified later; (ii) $M^j(q) < 0$ and $M(\bar{q}) = 0$; (iii) $M(q)$ is twice differentiable for all $q \in [\underline{q}, \bar{q}]$, where $|M'| \leq m_1$ and $|M''| \leq m_2$ for some finite constants $m_1$ and $m_2$; (iv) $qM''(q) + 2M'(q) \leq 0$. \footnote{Recall that this is possible due to Block-Recursivity. (e.g Value functions are independent of the worker distribution functions)}
The assumption makes it possible to ensure the application of a fixed-point theorem. The first part ensures continuity of $p^j$ when $J^j$ is continuous. Furthermore, the boundedness of $q$ is required to ensure the boundedness of $w^j$. The second part ensures that as firms find it easier to fill a vacancy, workers find it more difficult to find a job in that submarket. The third part and fourth parts simplify the proof of existence and help establish the concavity, and thus uniqueness for a given type j worker, of the matching rate function $p$.

Before presenting assumption 3, some additional definitions are needed. Let $J^j ≡ kq^j_j$, $J ≡ J^j w_j(V)$, $q^j ≡ kJ^j_j$, $p^j ≡ M(q^j)$, $S^j ≡ S^j w_j(V_j)$. Furthermore, let $w_j$ be a wage lower bound close to 0. 11

Assumption 2.3 Assume that $b^j_u, V^j, w^j$ satisfy the following:

i) $(0 <) b^j_u < W^j = h^j - \frac{rk}{V_j}$

ii) $h^j - [r + p_w(F_w(V^j))]J^j ≥ w^j + \frac{[u(b^j_u) - S^j w_j(V^j) - u(w^j)]}{u'(w^j)}$

iii) $1 + \frac{u''(w^j)}{u'(w^j)}[u(w^j) - u(w^j)] ≥ 0, \text{ all } w^j ∈ [w^j, W^j]$

The above assumptions must hold for all j-type workers. It is also worth noting that the above can be derived from exogenous objects of the model.

Theorem 2.2 Under assumption 1 and 2, the mapping $ψ$ has a fixed point $w^j_\ast$. The equilibrium for each j-type worker also has the following features: (i) $J^j_w(V)$ is strictly positive, bounded in $[J^j, J^j]$, strictly decreasing, strictly concave and continuously differentiable for all $V$, with $J^j_w(V) = J^j$; (ii) $p^j_w(V)$ has all the properties in (2.2) and is strictly concave for all $V < V^j$; (iii) $V^j_w > 0$ and $dJ^j_w dt < 0$ for all $V < V^j$.

The above equilibrium holds for an economy made up of only j-type workers. The next theorem establishes that there exists a joint equilibrium where workers behave exactly as in the above equilibrium although the economy is made up of heterogeneous workers. The idea of Block-Recursivity is once again central to this as workers separate into different submarkets based on both their current job’s value, $V$, and their productivity type-j. A worker of ability $h^j$ will therefore only consider jobs targeting him, and thus solely be concerned with the matching rate function $p^j$ that would result from an equilibrium consisting of solely similar type workers.

11Since $J^j_w, p^j_w$ and $S^j_w$ are decreasing in $V$, and $q_w(V)$ is increasing in $V$, all functions are contained within their respective bounds.
Theorem 2.3 An equilibrium consisting of \( \{h_1, h_2, ..., h_n\} \) productivity workers, with an associated distribution function \( g_h \) exists. Furthermore the equilibrium separates workers by type so that each worker type \( j \) behaves as if the economy consists of only \( j \)-type workers.

2.5 Equilibrium Distribution of workers

This section develops the distribution of workers over values \( V \). Due to the fact that worker types separate, in much the same way as the description of equilibrium, one can derive the equilibrium distribution over \( V \) for a fixed \( j \) worker. The aggregate distributions are then obtained by multiplying the measure of workers of type \( j \), \( M^j \), with the distribution over values of that worker \( g^j(V) \).

Let \( G^j \) be the cumulative distribution function of \( j \)-type workers over values \( V \in [v^j_1, V^j] \) and \( g^j \) the corresponding density function. In order to develop the intuition for the stationary distributions, consider an arbitrary value \( V \) and a small interval of time \( dt \) and examine the inflows and outflows into this group. The measure of the group is \( n^j M^j G^j(V) \), that is the number of type \( j \) workers employed multiplied by the measure of workers of that type multiplied by the measure of workers of that type in \( [v^j_1, V] \). Examining \( n^j G^j(V) \), there is a unique inflow into this group, that of unemployed workers of productivity type \( j \). These workers find matches at \( v^j_1 \) at the rate \( p(v^j_1) \). The inflow is therefore \((1 - n^j)p(v^j_1)dt\).

There are three outflows. Worker death generates an outflow, \( \delta n^j G^j(V)dt \). The increasing value of contracts causes workers in \((V - \dot{V}^j, V]\) to obtain a value greater than \( V \). Finally some workers find a better job and quit their current job; these workers are employed in \((Finv^j(V), V]\) if \( Finv^j(V) \geq v^j_1 \) where \( Finv^j \) is the inverse of the submarket search function for a worker of type \( j \). Quitting therefore generates the flow:

\[
(dt) n^j \int_{max\{v^j_1, Finv^j(V)\}}^{V} p^j(F^j(z))dG^j(z)
\]

Solving for the stationary distribution means equating the inflows and the outflows. Doing this and taking the limit \( dt \to 0 \) gives:

\[
\lim_{dt \to 0} \frac{\text{G}^j(V) - \text{G}^j(V - \dot{V}dt)}{dt} = \frac{1 - n^j}{n^j} p(v^j_1) - \delta G^j(V) - \int_{max\{v^j_1, Finv^j(V)\}}^{V} p^j(F^j(z))dG^j(z)
\]

(2.13)
Denote $v^j_k = F^{j,k}$, $k = 1, 2, \ldots$, where $F^{j,k} = v^j_0 \equiv V^j_u$ and $F^{j,k}(v^j_0) = F^j(F^{j,k-1})$. Then, $G^j(V)$ is continuous for all $V$, with $G^j(v^j_1) = 0$ The density function, $g^j(V)$, is continuous for all $V$, and differentiable except for $V = v^j_2$. The measure of employment for a given $j$-type is

$$n^j = \frac{p^j(v^j_1)}{\delta + p^j(v^j_1)}$$

(2.14)

Additionally, the following holds:

$$g^j(V)\dot{V}^j = \delta[1 - G^j(V)] - \int_{\max v^j_1, Finv^j(V)}^{V} p^j(F^j(z))dG^j(z)$$

(2.15)

The above theorem documents a few interesting features. The first is that the contract is continuous without any jumps. The second is that there is no mass points in the support of the distribution. This may be surprising since the mass of workers searching in submarket $v^j_1$ is positive. The reason for this is that workers move out of $v^j_1$ instantly due to increase in the value of their contract, death, or finding better employment at $v^j_2$ where $v^j_2$ is the submarket that people employed at $v^j_1$ search in.

The previous discussion and the characterization of the stationary distribution for $g^j(V)$ and $G^j(V)$ allows one to construct the stationary distribution for the economy. With the assumption that $g^j(V) = 0$ for all $V > V^j$, the following equation for the aggregate probability density of a value $V$ holds:

$$g^V = \sum M_j g^j(V)$$

(2.16)

$$n = \sum_{j=1}^{N_j} M_j n^j$$

(2.17)

---

\footnote{An additional feature of the stationary distribution not discussed here is that the distribution can be solved piece-wise. See Appendix.}
2.6 Empirical Analysis and Calibration

2.6.1 Data

This section evaluates the model presented in the study. A regression determines the actual returns to tenure for all education groups. This is then used to plot wage profiles for each education group which is then compared with the profile implied by the Calibrated model.

In order to evaluate the model, data from the National Longitudinal of Youth 1979 (NLSY79) is used. The dataset contains a sample of American youth born between the years 1957 and 1964 and begins in 1979. The original cohort consisted of 12,686 people but after two sub-samples were dropped, 9,964 remained. The participants in the project were interviewed yearly from 1979-1994 and bi-annually from 1994-2010. Each interview consisted of numerous questions varying from basic labour force questions to personal lifestyle questions. For the purposes of this study, only work related variables are necessary. Specifically, the average hourly wage, an industry dummy variable, age, tenure and highest grade completed are used. The empirical analysis is conducted with the full sample, which contains an oversampling of underprivileged groups, and with a sub-sample restricted to white males.

This dataset is useful in this study since the panel aspect allows one to track wage growth over tenure at a specific firm. This, along with the highest grade completed variable, allows one to track the wage-tenure profile for workers with varying observable human capital.

A particular feature of the model is that, controlling for worker heterogeneity, the starting wage at the job is sufficient for summarizing the wage-tenure path at a job. Starting for the year 1989, the starting wage for each new job is tracked for all workers in the sample. The wage-tenure profile is then estimated by allowing for a non-starting-wage tenure component and a component that interacts the starting wage with tenure. The reason for this is that the model implies the existence of a baseline contract that is dependent on the starting wage. A worker with a particular tenure, 5 years for example, will experience different returns to tenure than another one if their starting wages were different. The returns to tenure are decreasing in the starting wage since according to the model a worker is moving on the wage-profile towards his productivity.
The above figure shows a hypothetical baseline contract for an arbitrary worker. The wage associated with time 0 is the initial wage in the submarket unemployed workers visit. The wage $w(t_1)$ refers to the wage that a worker would receive if he reached tenure $t_1$ from starting at the wage $w(0)$. A worker may, however, find a job paying $w(t_2)$ from searching. He would then follow the baseline contract from time $t_2$ onwards. This highlights the dependence of the model’s implied wage growth on the starting wage.

In order to determine the effect of human capital on the wage-tenure profile the sample is separated into different sub-samples with differing levels of education. The following sub-samples are considered: 1) High school non-graduate 2) High school graduate and/or some college 3) College graduate and/or some Post-graduate studies. The regression is therefore:

$$
\ln(w(t)) = X_i + \ln(w_{st}) + Ten(t) + Ten^2(t) + \ln(w_{st})Ten(t) + \ln(w_{st})Ten^2(t) $$  \hspace{1cm} (2.18)

In order to control for some of the heterogeneity the following controls are included in $X_i$: A worker’s AFQT score, the occupation and industry of the current job, and the highest grade completed. The starting wage, $\ln(w_{st})$ is meant to capture a worker’s previous search history since according to the model it is a sufficient statistic. A worker’s current tenure,

---

13In addition to this specification, a regression is run with highest grade completed as a variable. The results are qualitatively similar, See Appendix
Table 2.1: Tenure Regression by Education Group

<table>
<thead>
<tr>
<th></th>
<th>Low Education</th>
<th>Middle Education</th>
<th>High Education</th>
</tr>
</thead>
<tbody>
<tr>
<td>starting wage</td>
<td>1.059***</td>
<td>1.070***</td>
<td>1.183***</td>
</tr>
<tr>
<td></td>
<td>(42.26)</td>
<td>(70.33)</td>
<td>(45.69)</td>
</tr>
<tr>
<td>tenure</td>
<td>0.034***</td>
<td>0.040***</td>
<td>0.066***</td>
</tr>
<tr>
<td></td>
<td>(7.09)</td>
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</tr>
<tr>
<td>tenure squared</td>
<td>-0.000</td>
<td>-0.000***</td>
<td>-0.000***</td>
</tr>
<tr>
<td></td>
<td>(-1.67)</td>
<td>(-8.18)</td>
<td>(-8.44)</td>
</tr>
<tr>
<td>starting wage x tenure</td>
<td>-0.005***</td>
<td>-0.006***</td>
<td>-0.010***</td>
</tr>
<tr>
<td></td>
<td>(-7.20)</td>
<td>(-16.22)</td>
<td>(-17.43)</td>
</tr>
<tr>
<td>starting wage x tenure squared</td>
<td>0.000</td>
<td>0.000***</td>
<td>0.000***</td>
</tr>
<tr>
<td></td>
<td>(1.19)</td>
<td>(10.11)</td>
<td>(13.38)</td>
</tr>
<tr>
<td>(R^2)</td>
<td>0.790</td>
<td>0.723</td>
<td>0.740</td>
</tr>
<tr>
<td>Sample size</td>
<td>1248</td>
<td>5380</td>
<td>2080</td>
</tr>
</tbody>
</table>

Notes: *** \(p < 0.01\), ** \(p < 0.05\), * \(p < 0.1\). t-statistics are reported in parentheses. OLS regression results for equation 1. Each column displays the estimate from a separate regression. starting wage refers to an employee’s wage at the beginning of the spell. tenure refers to time spent at the current employer, while tenure squared is the square of this number. starting wage x tenure and starting wage x tenure squared refers to a variable obtained by multiplying the above.

Ten\((t)\), denotes how long a worker has been continuously employed at a particular firm.

The important variables from the regression are the starting wage of a match and tenure. The starting wage captures a level effect on the current wage. Tenure enters the regression both directly and through an interaction term with wages. The first two terms, tenure and tenure squared, relate to the tenure returns to a worker starting at a log wage of 0. For the low, medium and high types the coefficients are positive for the tenure variable and negative for the tenure squared variable (although not significant for low education workers). There is a clear ranking of the tenure variable, high productivity workers enjoy an increase in log wages of 0.066, medium education workers an increase in 0.04 and low education workers an increase in 0.034. The tenure squared variable has the opposite ranking among the workers but is relatively small for each type. These two results imply that wages are increasing in tenure at a decreasing rate and, for low tenures, the effect is larger for more productive
workers. This means that, outside of any starting wage effects, high productivity workers have a steeper wage-profile.

The interaction term of starting wage and tenure is included in order to capture the fact that in the model, as a worker’s wage reaches his productivity, the returns to tenure decrease. The result shows that this effect is greater for higher educated workers since the effect is -0.01 for high education workers, -0.006 for medium educated workers and -0.005 for low educated workers. The interaction term of wages and tenure squared are all negligible suggesting that the starting wage-tenure effect is linear.

The results from the direct tenure and tenure squared are consistent with the model. Due to better search conditions, high education workers enjoy faster wage growth as firms attempt to retain them. Therefore the non-conditional returns to tenure are greater for these types of workers. The results from the starting wage and tenure interaction terms is a bit problematic as the effect from having a higher wage should have a less negative impact on the returns to tenure for more productive workers. The magnitude of the first effect dominates the latter, however, and therefore the data supports the hypothesis that more productive workers have a steeper wage profile.

2.6.2 Calibration

The previous sections served to develop a heterogeneous productivity wage-tenure model. I will now outline the calibration method that will be made in order to quantitatively evaluate the model and in order to further investigate the conclusions of the model. Most of the parameters necessary for this exercise are derived in a standard way except for the productivity ones. The section makes use of the NLYS79 and the Current Population Survey (CPS) data sets as well as statistics from other sources.

Instead of discretizing the grid and solving a related discrete version of the continuous model, a different approach is taken for the calibration. All value functions are approximated using Chebyshev polynomials. The coefficients for this approximation that minimize the deviation from optimality, for a pre-set number of collocation points, are then solved for. The resulting function is therefore continuous. The parameters calibrated for below therefore relate to a continuous time model. \(^{14}\)

\(^{14}\)See appendix for details on the method
There are two functional form assumptions that have to be made: the form of the utility function and the matching function. The utility function utilized is the standard log-utility one.\textsuperscript{15} For the search problem, the Cobb-Douglas matching function satisfies most of the assumptions required for the existence of an equilibrium. Therefore, the matching function used is $M(u, v) = \beta u^\alpha v^{1-\alpha}$.

In order to match the model to the data, worker productivity must be broken down into a finite number of type, $N_h$. For the purposes of this study I consider 3 productivity types relating to levels of educational achievement. $h_1$ refers to workers with less than a high school degree, $h_2$ to those with a high school degree and/or some college, $h_3$, to those with an undergraduate degree or more. The measure of the workers in each type is simply determined by their proportion in the dataset for the year 1989. The proportion of type 1 workers, $N_1$ is 0.1582, $N_2$ is 0.6590 and $N_3$ workers is 0.1882. In order to determine the level of productivity of each type, $h_1$ is normalized to 1. The other productivity levels are determined in comparison to $h_1$. I take the 90th percentile of wages for each of the education types and take the ratio of it to the 90th percentile of $h_1$ workers (e.g 90th percentile of wages for those without a high school degree). According to the model, a worker’s wage approaches his productivity over time, the 90th percentile wage should therefore approximate a worker’s productivity.

The rate of discount, $\rho$, is set to the standard value using the interest rate. The death rate, $\delta_d$, is identified by targeting the average years a worker spends in the labour force, while $\delta_s$ is targeted using the rate at which workers transition from employment to unemployment (EU rate). Since there is no endogenous job destruction or exogenous job separation, a second Calibration uses a higher $\delta$ in order to better match the economy’s employment to unemployment rate (EU rate).\textsuperscript{16} The level of unemployment benefits is set by targeting the rate at which unemployed workers find employment (UE rate).

The final parameters that need to be identified relate to the job-finding rates. In order to account for different matching probabilities between unemployed and employed workers. The parameter $\lambda_u$ and $\lambda_f$, measure the relative productivity of the search technologies of unemployed and employed workers. $\lambda_u$ is normalized to 1, while $\lambda_f$ is determined by

\textsuperscript{15}A different possibility is to calibrate a utility function from the exponential family of equations as was done in Sim (2006).
\textsuperscript{16}The results of this exercise are omitted since they are similar to the first calibration.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_d$</td>
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<td>Years in Labour Force</td>
</tr>
<tr>
<td>$\delta_s$</td>
<td>.1425</td>
<td>Employment-Unemployment Transition Rate</td>
</tr>
<tr>
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<td>Discount Rate</td>
</tr>
<tr>
<td>$h_1$</td>
<td>1</td>
<td>Normalized to 1</td>
</tr>
<tr>
<td>$h_2$</td>
<td>1.06</td>
<td>99th Wage Percentile High School Graduate</td>
</tr>
<tr>
<td>$h_3$</td>
<td>1.16</td>
<td>99th Wage Percentile College+</td>
</tr>
<tr>
<td>$N_1$</td>
<td>0.195</td>
<td>Proportion Less than High School</td>
</tr>
<tr>
<td>$N_2$</td>
<td>0.64</td>
<td>Proportion High School Graduate</td>
</tr>
<tr>
<td>$N_3$</td>
<td>0.165</td>
<td>Proportion College +</td>
</tr>
<tr>
<td>$\lambda_u$</td>
<td>1</td>
<td>Normalized to 1</td>
</tr>
<tr>
<td>$\lambda_f$</td>
<td>0.033</td>
<td>Employment-Employment Transition Rate</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.72</td>
<td>Elasticity of Matching Function</td>
</tr>
<tr>
<td>$\mu$</td>
<td>1.13</td>
<td>Normalizing the v/u Rate to 1</td>
</tr>
<tr>
<td>$b_u$</td>
<td>[0.71,0.75,0.8236]</td>
<td>Set to $.71 * average productivity of employed workers</td>
</tr>
<tr>
<td>$k$</td>
<td>0.53</td>
<td>Unemployment-Employment Transition Rate</td>
</tr>
</tbody>
</table>

targeting the rate at which employed workers transition to new jobs (EE rate), .033 monthly rate. The $\zeta$ parameter is targeted by setting the elasticity of the matching function equal to .28. ¹⁷ The other match parameter, $\mu$, is set by targeting an average vacancy-unemployment rate of 1. This is without loss of generality since the level of unemployment.

### 2.6.3 Simulation

This section analyzes the results of the Calibration. The time-evolution of wages and mobility suggested by the Calibration exercise is investigated.

There are two effects relating to heterogeneous productivity and wage tenure contracts. The first is that since more productive workers produce more, firms can offer workers higher utility by raising wages on portions of the wage-tenure contract that currently offer lower wages; this acts to smooth the wage tenure profile. The second effect is that higher productivity workers have an easier time finding other jobs; this acts to increase backloading. Whether more productive workers experience a steeper wage-profile therefore depends on which of these two effects dominate. It is worth noting the role that directed search plays in the second effect. More productive workers enjoy a higher matching rate because they visit different submarkets than less productive workers. If search was random, higher abil-

¹⁷This elasticity is the same one used in Shimer (2005)
ity workers would increase the expected value of a match and induce firms to post more vacancies at any particular wage. Less productive workers would therefore also enjoy a higher matching rate and there would be no extra incentive to backload wages for high type workers. The figure below plots out the baseline wage-tenure profiles for the three types of workers over four years.

The resulting wage-tenure profiles clearly produce a ranking with more productive workers earning more for the same amount of tenure. Furthermore, high productivity workers experience a steeper baseline profile and attain their highest wage before middle productivity workers who in turn attain theirs before low productivity workers. This means that the second effect dominates the first and consequently the increased turnover potential of high productivity workers allows them to extract a greater share of the surplus faster than medium and low productivity workers.

The figure also demonstrates that the empirical facts are qualitatively matched. More productive workers receive higher wages for any baseline tenure and the slope of the high types profile is steeper. The fact that workers attain their maximum wage (e.g capture the full surplus of the match) within the first four years suggests that on the job learning may play an important role in determining actual wage-tenure profiles. That said the results introduce a caveat for empirical studies interpreting all wage growth as on the job learning. A study ignoring endogenous that a portion of wage growth may be aimed at reducing turnover will overestimate the difference in learning between high types and low types.

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18 It is consequently imperative that the ability of workers be observable. In determining the ability of workers I therefore choose educational achievement since this is easily observable to firms.

19 Recall that each worker has a baseline wage-tenure contract. Time 0 here refers to a worker starting at the minimum wage accepted by his respective type.
The second determinant of life-cycle earnings is wage growth that occurs due to job mobility. The job mobility rate across worker types is therefore equally important. It might be presumed that since high ability workers enjoy better job seeking opportunities, they should have a higher job mobility rate throughout the baseline wage-tenure profile. This is, however, not necessarily true since job mobility is an endogenous outcome of the model. Recall that firms backload wages precisely to reduce worker tenure. There may therefore be regions of the baseline contract where higher ability workers enjoy a higher transition rate and regions where they do not. The following figure plots out the transition rates enjoyed by the three types of workers over their respective baseline wage-tenure profile over four years.

The figure clearly shows a ranking at the beginning of the baseline wage-tenure contracts with high productivity workers enjoying the highest mobility, middle productivity a middling mobility and low productivity workers the lowest. At some tenure point, however, the transition rate of high type workers falls below the middle type and eventually the lowest type. Similarly, eventually the middle type falls below the low type. This means that wage gains from job search are relatively more important for high types early on their baseline contract and relatively less important later on their baseline contracts. Furthermore, low-types continue searching for new jobs until later in their tenure. The fact that the respective mobilities are similar for all workers is partially due to the fact that in the calibration unemployment benefits were set as a proportion of productivity. High-type workers therefore search in relatively riskier submarkets when unemployed. If unemployment benefits were fixed at a similar level for all workers the early mobility gap would be wider.
Chapter 2: Heterogeneous Productivity and the Wage-tenure Profile

Noting that employed young workers are most likely to be at the beginning of the baseline wage-tenure contract, there are some life cycle implications. Lower types enjoy both smaller wage increases and smaller gains from search at the beginning of their careers. Since high-types attain their maximal wage earlier, eventually low-types continue to experience gains while high types no longer do. Although the time gap between the cessation of wage growth is not large for the respective workers, noting that an exogenous separation resets the workers position on the profile one can conclude that these differences in the time required to attain their maximum wage may have a significant impact on inequality between types. Furthermore, the impact may be exacerbated if a larger share of job separations are exogenous for low type workers.

A further interesting implication is that it is possible for a low type worker to currently have a higher wage than a high type worker but this is due to them being further along the baseline contract either through good luck searching or through being at a firm longer. The high type worker will, however, enjoy higher wage increases both through higher job mobility and through larger raises at their current job.

2.7 Conclusion

This study analyzes the relationship between wage-tenure contracts and observable worker heterogeneity. The model incorporates directed search, worker risk-aversion and the allowance of firms posting wage-tenure contracts. The optimal contract induces firms to backlog wages in order to efficiently reduce worker turnover. An important feature of the optimal contract is that any offered contract is a truncation of a single baseline contract. Since more productive workers are more valuable to firms and they enjoy a higher rate of matching, it is postulated that in equilibrium more productive workers will experience a steeper baseline wage-tenure profile.

In order to establish the relationship between worker productivity and a baseline wage tenure contract, workers are separated into three groups. These groups are workers with no high school degree, workers with a high school degree and finally workers with a college degree. It is worth noting that it is important that the workers are observably more productive since otherwise the assumption that firms can target the different types of workers does not
hold and a random search environment would be better suited. A regression of log wages on worker characteristics, starting wage, tenure, tenure squared and some interaction terms of tenure, tenure-squared with starting wages establishes that higher educated workers do possess a steeper baseline wage-tenure profile.

A calibration exercise is then conducted to establish whether wage-tenure contracts do indeed give rise to the above phenomenon. It is found that qualitatively it does although the length of time required for a worker to reach his maximum wage is only 4 years of continuous employment at a firm. An interesting result is that job mobility is relatively more important in increasing the wages for high productivity workers at low levels of the baseline contract but less important at high levels. The reason for this is that firms increase the wages of high types quickly in order to reduce their probability of finding another job. This increase in wages offsets the fact that the submarkets high types search in offer a higher matching rate.

Although this study ignores the role of human capital accumulation, it does not discount its importance. The goal was to establish that even in the absence of any additional learning advantage, directed search implies that more productive workers enjoy faster wage growth as a result of a higher matching rate. A model including both on the job learning and wage-tenure contracts could illuminate the relative importance of each mechanism.

Appendix

Chebyshev Method

The simulation part of this paper uses Chebyshev Projection methods to approximate the value and policy functions. The method uses Chebyshev polynomials constructed recursively using the rule $T_0(x) = 1$, $T_1(x) = x$ and $T_{n+1} = 2xT_n(x) - T_{n-1}(x)$. These polynomials prove extremely useful due to certain orthogonality properties at the zero points of the function. This improves the accuracy of the projection compared with other projection methods. The method uses an approximation of the firm value, $J(V)$, over the domain of $V$. All other functions are derived in terms of $J$ through the optimality conditions. The optimality conditions are then written in terms of the Chebyshev polynomials and a minimizing routine is used to solve for the optimal coefficients.
The first step is to determine the interval over which the function is to approximated over. Due to the block recursive nature of the equilibrium, the only state variable is V. The program then must be run for each level of human capital h. The choice at this point is to choose Vmin, the minimum value to the worker, and Vmax, the maximum value possible. Since Chebyshev polynomials can be written as \( T_n(x) = \cos(n \cdot \arccos(x)) \), the domain of the polynomial is [-1,1]. The domain in the function space, \([Vmin, Vmax]\), must therefore be translated into [-1,1] space. The translation is given by \( zv(x) = Vmin + z(Vmax - Vmin) \) where \( z = \frac{x - Vmin}{Vmax - Vmin} \).

The second step is therefore to determine the polynomial size \( \text{nv} \), which has \( \text{nv} \) zeros associated with it, that is points at which \( T_n(z) = 0 \). The zeros are given by: \( x_k = \cos\left(\frac{k\pi}{\text{nv}+1}\right), k = 1, \ldots, \text{nv} \) These zeros points will serve as the collocation points, which are the points over which the function is approximated. The approximated \( J(.) \) function, \( J_{\text{cheb}}(V) \) will be represented by a polynomial with a degree of \( \text{nv} \); therefore \( J_{\text{cheb}}(V) = \sum_{i=0}^{\text{nv}} \theta_i T_i(V) \).

The third step is to write all other functions in terms of \( J \), using the following relationships:

1. Free entry: \( q = \frac{k}{J_{\text{cheb}}} \). From this one solves the worker’s search problem \( S(V) = \max_{\tilde{V}} p(\tilde{V})(\tilde{V} - V) \).
2. Using the above function, \( \tilde{p}(V) \), one derives \( \tilde{p}_V(V) \).
3. Using \( J_{\text{cheb}} \), obtain its derivative in terms of \( V \), \( \frac{dJ}{dV} = J'_V \)
4. Using \( w'_V(V) = J'_V \)
5. Using \( \frac{dV}{dt} = rV - u(w(V)) - S(V) \) and \( \frac{df}{dt} = rJ(V) + w(V) - h \)
6. The above derivatives and functions can be used to obtain \( \frac{df(V)}{dt} \) for \( f \in \{w, p, \frac{dp}{dt}\} \)
7. For the Collocation points \( \{z_i\}^{\text{nv}}_1 \) one can obtain the deviation from the optimality conditions (and the boundary point \( J(z_{nv}) = 0 \)) for a given set of Chebyshev coefficients \( \theta \).

a) \( \frac{dw}{dt}(z_i) = w(z_i)J(z_i)\frac{df}{dt}(z_i) \)

b) \( \frac{df}{dt}(z_i) = \frac{dV}{dt}(z_i)w(z_i) \)

c) \( J(z_{nv}) = 0 \)
Once all of the Chebyshev coefficients are solved for, the time evolution of any function can be determined. Since all of the functions are based on a baseline contract, it is sufficient to calculate the baseline function evolution. That is, the functions time evolution from time 0.

1. Calculate the second derivatives using the best approximation obtained above. Obtain
\[ \frac{d^2 f(V)}{dt^2} = \frac{d^2 f(V)}{dV^2} \frac{dV}{dt} + \frac{df}{dV} \frac{d^2 V(V)}{dt^2} \] for \( f \in \{w, p, \frac{dp}{dt}\} \)

2. Choose an interval length \( dt \). For given \( t \), define \( N dt = t/dt \).

3. \( f(t) = f(0) + \sum_{0}^{N dt} \frac{d^2 f(V)}{dt^2} dt + \sum_{0}^{N dt} \frac{d^2 f(V)}{dt^2} dt^2 \)

The final step is to calculate the stationary distribution \( g(V) \). The method is similar to the above. It is easier to use the piece-wise formulation of \( g \) and therefore each \( g_j \) is approximated.

1. Since no contract reaches \( \bar{V} \) in finite time, one must choose a deviation from \( \bar{V}, dgv \).

2. Determine the value an unemployment worker achieves, \( V_u \). In order to do this one determines the value \( V_u \) that sets \( 0 = rV_u + u(b_u) + S(V_u) \) (since \( \frac{dV_u}{dt} = 0 \)). Using the Chebyshev representation for the probability of finding a job \( p(V) \) one solves for \( F(V_u) = v_1 \), the submarket that an unemployed worker searches in. Similarly, one solves for \( F(v_j) = v_{j+1} \) as long as \( v_j < \bar{V} - dgv \). A different function \( g_i \) is approximated for each of these values.

3. Similarly to the beginning step above, one chooses the degree of the Chebyshev polyno-mial and and obtains the collocation points.

4. Solve \( g_u(z_i)z_i \dot{z} = \delta \Gamma (z, v_1) \)

5. Solve \( g_j(z_i)z_i \dot{z} = g_j(z_j)z_j \Gamma (z, v_1) = \int_{v_j}^{V} \Gamma (z, s)p(s)g_{j-1}(F^{-1}(s)dF^{-1}(s)) \)
Table 2.2: Tenure Regression with education: Full Sample

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Each column displays the estimate from a separate regression. The controls common to all regressions are gender, hispanic race, black race, afqt (test score). The first regression includes experience, occupation and industry effects. The second regression excludes experience while the third and fourth exclude industry and occupation respectively. The last regression excludes experience, industry and occupational effects. Experience refers to age minus highest grade completed minus 5. Experience squared is the squared term of experience. Highest grade completed refers to the highest grade completed. of starting wage refers to an employees wage at the beginning of the spell. tenure refers to time spent at the current employer, while tenure squared is the square of this number. starting wage x tenure and starting wage x tenure squared refers to a variable obtained by multiplying the above.
### Table 2.3: Tenure Regression with education: White Males

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R^2: 0.188 0.182 0.187 0.178 0.181

Sample Size: 19586 19586 28197 19586 28197

Notes: *** p < 0.01, ** p < 0.05, * p < 0.1. t-statistics are reported in parentheses. OLS regression results for equation 1. Each column displays the estimate from a separate regression. The control common to all regressions is afqt (test score). The first regression includes experience, occupation and industry effects. The second regression excludes experience while the third and fourth exclude industry and occupation respectively. The last regression excludes experience, industry and occupational effects. Experience refers to age minus highest grade completed minus 5. Experience squared is the squared term of experience. Highest grade completed refers to the highest grade completed. of starting wage refers to an employee’s wage at the beginning of the spell. tenure refers to time spent at the current employer, while tenure squared is the square of this number. starting wage x tenure and starting wage x tenure squared refers to a variable obtained by multiplying the above.
Chapter 3

The Determinants of Tenure-Wage Growth
3.1 Introduction

Wage growth over the life-cycle is a robust relationship that has been well documented.\(^1\) It is especially understood that wages increase the longer a worker's tenure at a firm. The most natural explanation is that workers learn on the job and therefore experience an increase in human capital over time. This human capital can be attained either through on the job training or through on the job learning. An alternative explanation is, however, unrelated to any human capital acquisition. With workers searching on the job the optimal contract, called the wage-tenure contract, backloads worker wages in order to reduce worker turnover. This study considers a model incorporating both of the aforementioned potential sources of an increasing wage-tenure profile. Specifically, workers learn on the job exogenously and firms offer wage-tenure contracts that take such learning into account.

Since this study considers wage-tenure contracts as well as exogenous learning by doing, it is possible to investigate the importance of each in determining a worker’s wage growth over his tenure at a firm. This is in contrast to models that attribute a worker wage growth entirely to marginal productivity increases or to backloaded wage-tenure contracts. This decomposition illuminates not only the relative importance of each explanation but also any meaningful interactions between them.\(^2\)

The model considers a framework where workers search on the job. As in Shi (2009), search is directed meaning that workers can direct their search to submarkets offering a particular offer. Additionally, while employed workers learn at an exogenous rate. When a worker learns, he moves up an discrete worker productivity profile. This learning is public information and workers cannot therefore exploit this knowledge when searching for new jobs. The equilibrium is therefore segmented by worker types, with firms offering contracts that specify a contingent wage for each potential productivity level.

As in Shi (2009), increases to the minimum wage do not change the optimal contract structure of wages but solely reduce the number of active submarkets. In this study, however, increased minimum wages reduce employment and consequently on-the-job learning. The mechanism is different from most other papers considering such an effect since jobs are not

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\(^1\)See Topel and Ward (1992)

\(^2\)See Chapter 2 for a study investigating differences in wage growth of workers with different observable productivity.
destroyed due to a worker’s marginal productivity falling below the new minimum wage, but rather due to the fact that jobs offering low wages are easier to find and therefore preferred by unemployed workers.

The study finds that workers jump up on the knowledge ladder at a constant rate of 0.287. Furthermore, in the absence of wage-tenure contracts workers would attain the maximal wage in the economy after just 17.2 years and the median wage in the economy after just 10. The fact that a portion of early low wages comes from workers receiving a low share of the surplus accounts for this.

3.1.1 Literature Review

The literature on attributing wage growth to human capital acquisition is extensive. The literature investigating environments with both frictions and human capital accumulation is more limited. Burdett, Carillo-Tudela, and Coles (2009) extend the original wage-posting with on the job search model, Burdett and Mortensen (1998). Workers are offered contracts that promise a piece rate and therefore see their wages rise as they acquire more human capital through an exogenous on the job learning process. Furthermore, on the job search allows workers a higher share of output. There is no clear reason, however why offering such a contract is optimal from the point of view of firms.

Wage-tenure contracts relate a workers increasing wage profile to a need by firms to reduce worker turnover while promising workers some initial value. The possibility of firms back-loading contracts was investigated in Stevens (2004). In her framework, firms optimally backload contracts for in order to reduce tenure in a random search environment. Since workers are assumed to be risk neutral, the optimal contract is a step-wage contract. Burdett and Coles (2003) extend this framework by assuming that workers are risk-averse. The consequence of this assumption is that a worker’s wage rises continuously in tenure. Following a similar view of the Burdett, Carillo-Tudela, and Coles (2009) model as above, Sim (2013) constructs a model with on the job learning based on Burdett and Coles (2003). The wage contract in his model, therefore, is an endogenous outcome of firms being allowed to pre-empt worker turnover. This study differs from that of Sim (2013) by considering a directed search environment rather than the random search environment considered in the

\[3\] See Mincer (1974) and Becker (1975) for two early examples.
former. Directed search allows for a more tractable equilibrium which leads to simpler computation of the equilibrium. Additionally, the wage increases are dependent on the current value of the employee unlike in random search models where workers with low offers are equally likely to find employment at the highest offers. The limited wage jumps implied by a directed search model are therefore more realistic.

Two early examples of directed search are Peters (1989) and Moen (1997). In directed search models workers target submarkets offering a particular value. Firms, therefore, do not have to consider the complete distribution of workers over values since they know that contracts posted in a particular submarket will only be visited by workers that will accept that offer. This is unlike the models based on Burdett and Mortensen (1998), where firms must consider the reservation wage policy of all workers searching for a job since workers are randomly matched with any offer. Shi (2009) is similar to Burdett and Coles (2003), in that firms offer wage-tenure contracts in order to optimally solve a worker retention problem, but differs in that search is directed rather than random. This study extends this model by allowing for workers to learn on the job via an exogenous learning by doing process.

The rest of the study is organized as follows: In section 2 the model is developed. In section 3 theoretical results are presented. This includes establishing the existence of an equilibrium as well as characterizing its structure. Section 3 describes the data used as well as the Calibration procedure followed. Finally, the last section provides a brief conclusion as well as a discussion of possible extensions to the model.

### 3.2 Model

Consider an environment with \( N_h \) different types of heterogeneous workers indexed such that worker \( i \) is more productive than worker \( j \) if \( j > i \). In other words \( \{h_1, h_2, \ldots, h_{N_h}\} \) is organized from lowest to highest level of human capital. Employed workers face a constant rate of learning on the job, \( \zeta_f \). Specifically, a worker of type \( j \) becomes a worker of type \( j + 1 \) at a rate \( \zeta_f \). It is further assumed that a worker of type \( N_h \) cannot learn further.\(^4\)

\(^4\)This is equivalent to assuming that a worker of type \( N_h \) gets promoted to a worker of type \( N_h \).
or separation shock as described below.

Time is both continuous and infinite with both workers and firms discounting at a constant rate \( \rho \). It is also assumed that workers retire at a constant rate \( \delta_d \) and separate from their current firm at a rate of \( \delta_s \). Furthermore, workers preferences are given by a utility function with the standard properties: \( 0 < u'(w) < \infty, -\infty < u''(w) < 0 \) for all \( w \in (0, \infty) \), and \( u'(0) = \infty \). These assumptions, as outlined in chapter 2, ensure an interior solution for wages at all tenure lengths.

Production occurs in active matches. Specifically, an employed worker produces output according to his current level of human capital; a worker of type \( j \) produces \( h_j > 0 \) units of production when employed at a firm, and \( b_u \) units when unemployed. The mass of workers receiving a retirement shock are replaced by new workers of ability \( n_1 \) upon death, while those receiving a separation shock enter into the pool of unemployed workers with their current level of human capital.

Firms are identical and risk-neutral. They face 2 decisions: First, they decide what value and to what kind of worker to make an offer, then they decide on the optimal wage path over time for that offer \( \{w(t)\}_{t=0}^{\infty} \). The wage path, called the wage-tenure contract, determines the workers compensation after being at the firm for \( t \) units of time. These contracts are full-commitment ones from the point of view of the firms but not from the point of view of the workers. This means that while a worker can leave a job at any time for a better offer, firms must pay workers according to the terms agreed upon at the beginning of the contract. Furthermore, firms are not allowed to respond to outside offers.\(^5\) Since, as stated earlier, both workers and firms discount at the rate \( \rho \), and matches exogenously terminate at rate \( \delta_d + \delta_s \), the effective discount rate is \( r = \rho + \delta_d + \delta_s \). As in chapter 2, in addition to the effective discount rate, from the point of view of the firm the endogenous job finding rate from the worker adds to discounting. It is as a consequence of firms controlling this rate that a worker’s wages grow over time while employed at the same firm.

Wage-tenure contracts offer a particular value to a particular type of worker depending on his current productivity. A function \( f_j(V(t)) \{ J, \frac{dp}{dv}, \frac{dJ}{dv} \ldots \} \) therefore refers to a function for type \( j \) workers with contracts offering value \( V \).\(^6\) An offer to worker of type \( j \)

\(^5\)This is a standard assumption in the literature since otherwise firms would bid and counter bid until the worker received the full value of production.

\(^6\)It is important to note that \( t \) here refers not to time but rather to tenure at a particular firm. Also, the dependence of \( V \) on \( t \) will be dropped occasionally
refers to the value $V^j(0)$. For completeness, denote an unemployed worker’s tenure as $\emptyset$. An unemployed worker of type j’s current value is therefore $V^j_u = V^j(\emptyset)$ and his wage $b_u = w(\emptyset)$.

The lowest value that a worker can receive is clearly $V = u(b)/r$, since he can reject all offers and enjoy the unemployment benefits in perpetuity. Similarly, for a given j-type of worker there is a highest wage $\bar{w}^j$ which firms will not be able to profit above, therefore the highest conceivable offer is $\bar{V}^j$. Consequently every offer is bounded by the set $[V, \bar{V}^j]$, where explicitly:

$$V = u(b)/r, \bar{V}^j = \bar{w}^j/r$$

(3.1)

The highest wage offered, $\bar{w}^j/r$ is dependent on the optimal contract, since firms benefit when workers learn and therefore the usual maximum wage, as in Shi (2009), where firms offer workers their full productivity minus the cost of recruiting does not hold.

Workers search both on and off the job.\(^7\) This study assumes that search is directed, meaning that workers target a specific submarket $x^j$. This optimal submarket depends on the value they receive at their current job and the ease with which they expect to find employment at other firms offering contracts to their type of worker. The rate at which they find employment at other firms is determined by the “tightness” of that submarket, $\theta(x^j)$, or the ratio of applicants searching for that value $x^j$ to firms offering it. The total number of matches in a given submarket is given by a linearly homogenous matching function $M(N(x^j), v(x^j))/\theta(x^j)$, where $N(x^j)$ is the number of applicants of type j to an offer $x^j$ and $v^j$ is the number of vacancies posted by firms. This implies that the Poisson matching rate is $p(x^j) = M(1, 1/\theta(x^j))$ for workers and $q(x^j) = M(\theta(x^j), 1)$ for firms. The function $p(.)$ can therefore be referred to as the employment rate and $q(.)$ as the hiring rate. In order to better match the data in the numerical section, it is assumed that unemployed workers are better able to match than employed ones. This is modeled as the matching rate of an employed worker visiting the same submarket as an unemployed worker, for example $x^j$, matching at the rate $\lambda_f p(x^j)$ instead of $p(x^j)$.\(^8\)

In equilibrium it must be the case that $p(.)$ is decreasing and $q(.)$ is increasing. A

---

\(^7\)Note that $\bar{V}^j$ and $\bar{w}^j$ is written as a function of j. This is since the more productive a worker the higher the maximum wage for that worker is.

\(^8\)The parameter $\lambda_f$ is generally smaller than 1 implying that search on the job is more difficult than unemployed search.
decreasing employment rate introduces a tradeoff between workers searching for high values and finding employment at that value. Workers with low values will therefore target relatively easier to obtain jobs, while workers with already high values will target harder to obtain jobs. The same reasoning applies in reverse with firms offering low values receiving relatively fewer applicants than those offering high values. The search process is therefore competitive with the respective matching rates acting as hedonic prices. Workers who have previously learned on the job, and consequently are of a higher type, will visit higher value submarkets given the same value as a less productive worker. This is a consequence of firms creating more vacancies for more productive workers.

3.3 Optimal Decisions

This section details the firm’s and worker’s optimal decisions. First the worker’s decisions are considered and then the firm’s decisions. Finally, the optimal contract structure will be discussed and consequently the recursive solution procedure for the numerical section will be made clear.

3.3.1 Worker’s Search Decision

A worker j’s decision is based on the matching rate function \( p(\cdot) \). The equilibrium considered here satisfies the following:

**Assumption 3.1** (i) \( p(V_j = 0) \) (ii) \( p(x^j) \) is bounded, continuous and concave for all \( x \); (iii) \( p(x^j) \) is strictly decreasing and continuously differentiable for all \( x^j \in \bigcup_{j=1}^{Nh} [V_j^j, V_j^j] \)

The first part ensures that the highest valued offer for a worker is never attained. The second part is useful in proving an existence of the equilibrium. The third part ensures that there is a tradeoff, from the point of view of the worker, between obtaining a high value job and a high matching rate as well and the function being differentiable makes the analysis easier.

---

9 In equilibrium firms must receive the same value regardless of which submarket they post an offer in

10 Unlike other functions, \( p \) is written solely as a function of the submarket since \( x^j \) makes it clear that it is implicitly a function of a worker’s current productivity.
With the above assumption it is possible to characterize the worker’s search problem. A worker of type j with a current value of V searches for the submarket $x^j$ that gives him the optimal tradeoff between $(x - V)$ and the job finding rate, $p(x^j)$, of that submarket.\footnote{The super-script j is used to highlight that the workers are heterogeneous. A worker searches for value x, for example, but if I want to discuss the tightness of a submarket offering value x, $θ$ I will write $θ(x^j)$ since the tightness will be lower for more productive workers.} The optimal search decision, $x^j$, is therefore:

$$S^j(V) ≡ \max_{x^j ∈ [V; V]} λ_f p(x^j)(x - V)$$

(3.2)

Denoting the solution $x = F^j(V)$, the following lemma holds:

**Lemma 3.1** $F^j(V) = V^j$ for all $V < V^j$ the following holds: i) $F^j(V)$ is unique for each V, and continuous in V; ii) $S^j(V)$ is differentiable, with $S^j(V) = -p(F^j(V))$ iii) $F^j(V_2) - F^j(V_1) ≤ (V_2 - V_1)/2$ for all $(V_2 ≥ (V_1$ iv) If $p''(.)$ exists, then $F''(V)$ and $S''(V)$ exist, with $0 < F''(V) ≤ 1/2$ v) $V = F^j(V) + p(F^j(V))/p'(V)$

The above lemma provides several useful insights. The uniqueness of $F^j(V)$ follows from the above discussion of the tradeoff between the matching rate and the value of a match. Furthermore, the higher the value the more willing a worker is willing to tradeoff a high matching rate for a higher value. This separation of workers into different states makes the equilibrium more tractable. Finally the expected gain, $S^j(V)$, and actual gain $F^j(V) - V/V$ from search is decreasing in V. Additionally, the expected gain $S^j(V)$ is diminishing at a decreasing rate since $S''(V) > 0$.\footnote{These proofs are similar to Shi (2009) and are therefore omitted for brevity.}

### 3.3.2 Value Functions

Since wage-tenure contracts imply changing match-surplus division between workers and firms, both the worker and firm value functions are evolving over time. Consider a worker of type j with current value V. The worker’s value function has five components. The worker receives a wage $w^j(t)$, value from search $S^j(V(t))$ changes in value due to moving along the wage profile $V^j(t)$, he becomes unemployed at a rate $δ_s$, thus receiving a flow benefit of $-δ_s U^j$, and he learns at rate $μ$, thus receiving a flow benefit of $μ V^j(t)$ where
\( \tilde{V}^j(t) \) represents the jump in his utility from learning. Using \( r = \delta_d + \delta_s + \rho + \mu \) as the effective discount rate, the worker’s value function is given by:

\[
rV^j(t) = u(w^j(t)) + + S(V^j(t)) + \dot{V}^j(t) - \delta s U^j + \mu \tilde{V}^j(t) \tag{3.3}
\]

The non-stationarity of \( V^j(t) \) is due to a worker moving along his wage profile. A worker receiving a constant wage would therefore experience no change in his value over time, \( V^j(t) = 0 \). Since unemployed workers receive the same benefits \( b_u \) irrespective of time, \( V_u^j(t) = 0 \) for all \( j \). Unemployed workers also face a different effective discount rate than employed workers since, clearly, there is no exogenous separation from unemployment.\(^{13}\)

\[
0 = (\rho + \delta d)V_u^j(t) - u(b_u) - S(V_u^j) \tag{3.4}
\]

For the firm side, consider a worker with value \( V(t) \) and productivity \( h^j \). The value of a wage contract for a firm consists of the following: the productivity of the worker, the transition rate of the worker (which depends on the wage profile and the contract offers), the likelihood of the worker learning and the wage profile directly.

\[
\frac{dJ(t)}{dt} = [r + p(F^j(V(t)))]J(t) - h^j - \mu \tilde{J} + w^j(t) \tag{3.5}
\]

The discount rate facing a firm is similar to that of a worker except that it also faces the probability of a worker quitting for another job. It is the tradeoff between decreasing this rate and maintaining the wage that gives the increasing wage profile. For an arbitrary \( t_a \in [0, t] \) define:

\[
\gamma^j(t, t_a) \equiv \exp(-\int_{t_a}^{t} [r + \mu + p(F(V^j(t)))]d\tau) \tag{3.6}
\]

Integrating the above differential condition and using the boundary condition \( \lim_{t \to \infty} \gamma^j(t, t_a) = \) gives the following:

\[
J^j(t_a) = \int_{t_a}^{\infty} [\tilde{J}^j(t) + h^j - w(t)]\gamma(t, t_a)dt \tag{3.7}
\]

The value of a contract to a type-j worker, from the firm’s point of view, therefore also

\(^{13}\)It is worth noting that there is no loss of knowledge in the current version of the model. Allowing for human capital depreciation would introduce a flow of moving into a new unemployment type.
depends on how the value changes as a worker learns on the job.

### 3.3.3 Optimal Recruiting and Contracts

The firm has two decisions to make. The first is to choose an offer $x^j$ that maximizes $q^j(x)^J(0)$, taking $q(.)$ competitively. The second is to choose the optimal wage profile, \{\{w^j(t)\}_{t=0}^{\infty}\} that delivers the promised value $x^j$. This subsection examines the first decision and subsequently the second part.

The decision on what offers to make is straightforward. For a fixed firm value, $J^j(0)$, firms maximize the expected value of the offer $q(x^j).J^j(0)$. Since search is directed, firms will post vacancies for an offer until the cost of posting that offer is equal its benefit. This means that $q(x^j).J^j(0) = k$ for all offers that are made in equilibrium and for completeness, $q(x^j).J^j(0) < k$ for all offers not made in equilibrium. In the context of the model, this means that firms are indifferent between all values offered to workers as well as between workers. This is driven by search being directed and firms being perfectly committed to all offers made. All else equal firms will make greater profits if matched with workers of high productivity and low promised value. The probability of a firm matching with such a worker is, however, lower. Therefore in expectation a firm is indifferent between posting contracts targeting any type of worker in the economy.

The highest possible wage for a type-j, $w^j(t)$, can be characterized recursively. For the highest productivity type N, this wage is simply one that offers the entire surplus to the worker accounting for the firm’s cost of recruiting. Denoting the value to the worker as $V^N$: The search value, $S(V^N) = 0$ since $V^N$ is the highest offer a worker can obtain and consequently there is no incentive to further search. Due to the stationarity of the wage, the time derivative, $\dot{V}^N = 0$. Similar reasoning on the part of the firm value function gives $J^N(t) = 0$. Since the highest value to the worker represents the lowest value to the firm, $J^N = (h^N - w^N)/r$.\(^\text{14}\) Recalling the maximum $\bar{q}$, the maximum wage that a firm can offer without making a loss is $\bar{q}J^N = k$. Solving for $\bar{w}^N$ gives:\(^\text{15}\)

\[
\bar{V}^N = h^N - rk/\bar{q}
\]

\(^\text{14}\) Note that no further learning occurs once a worker reaches the level of N.

\(^\text{15}\) If $q^j(V) < \bar{q}$ a firm can obtain a higher value by offering a slightly higher wage. See Shi(2009).
For lower productivity types the maximum wage is similar except that workers may also learn on the job. The equation for the minimum value the firm receives changes to 

\[(r + \mu)\bar{J}^j(t) = h^j - (\bar{w}^j(t)) + \mu \bar{J}^j(t)\].

This gives the maximum wage:

\[
\bar{w}^j(t) = h^j - (r + \mu)k/\bar{q} + \mu \bar{J}^j(\bar{V}^j) \quad (3.9)
\]

The second decision firms face is offering the optimal wage profile once matched with a worker. This contract stipulates the full wage profile for a worker conditional on the productivity shocks that the worker experiences. The contract therefore stipulates \(\{w(t)|h(t)\}_{t=0}^{\infty}\) where \(h(t)\) is the workers productivity at time \(t\). It is first useful to characterize the optimal contract for the highest ability worker (type \(N_h\)), who experiences no further learning.

Consider the optimal contract offered to a worker of the highest ability. Such a worker will experience no changes to his level of human capital.\(^\text{16}\) The value of a contract to a worker therefore consists of the instantaneous utility from his current wage \(u(w_N(t))\), the value from searching on the job \(S_N(V(t))\), the change in value of the contract \(dV_N(t)/dt\) and the value from suffering a job separation \(\delta_s U_N(t)\). An unemployed worker enjoys the instantaneous utility from unemployment benefits \(b_u\) and the value from searching while unemployed. The value functions of the highest productivity type is therefore:

\[
rV^N(t) = u(w^N(t)) + S^N(V(t)) + dV^N(t)/dt + \delta_s U^N(t) \quad (3.10)
\]

\[
rU^N(t) = u(b_u) + S^N(U(t)) \quad (3.11)
\]

Since it is counter-intuitive for workers to take wage cuts after learning, it is assumed that when workers experience learning they receive the same current wage but on the higher human capital baseline contract.\(^\text{17}\) Since the consequent wage growth is faster, the effect of learning is positive. It is worth noting that although the wage grows faster upon learning, workers do not capture the entire marginal productivity of their knowledge immediately like in the competitive models. Let \(\hat{V}_j\) denote the value received by a type-\(j\) worker that learns

\(^{16}\text{Since a firm matched with a highest ability worker does not have to take further learning into account, the optimal contract for this type of worker is identical to Shi (2009) with the addition of exogenous retirement.}

\(^{17}\text{See Stevens(2004) for an early presentation of the wage-tenure baseline contract. The second chapter also discusses this concept in some detail.}
Assumption 3.2 Upon learning, workers receive the same wage, but move onto the higher type baseline contract. The new value of their contract is therefore: 
\[
\tilde{V}_j \equiv \min \{ V_k \text{s.t } k = j + 1 \text{ and } w_k = w_j \}
\]

The above assumption leads directly to the following Value function for workers of type-j:
\[
1)(r + \mu)V_N(t) = u(w_N(t)) + S_N(V(t)) + dV_N(t)/dt + \delta_u U_N(t) + \mu \tilde{V}_j \tag{3.12}
\]

The optimal contract part of the decision is choosing the wage profile that maximizes the starting value from the point of view of the firm subject to offering the worker the promised value x.
\[
\max_{w^j(t)_{t=0}} J^j(0), \text{ subject to } V^j(0) = x \tag{3.13}
\]

This leads to the following theorem. Assuming that the firm does not make strategic decisions in determining the optimal wage for type N, the optimal contract is the same as in Shi (2009), with the new worker value functions described above:

**Theorem 3.1** The optimal contract has the following features: i) \(0 < w^j(t) \geq \bar{w} \) for all t; ii) \( \frac{dw(t)}{dt} > 0 \) for all t, \( w(t) \rightarrow \bar{w} \) as \( t \rightarrow \infty \) and \( \frac{dw(t)}{dt} = \left[ \frac{u'(w(t))}{u'(w(t))} \right]^2 J^j(t) \left[ \frac{d\phi(F(V^j(t)))}{dV} \right], \text{ all } t; \) iii) \( \dot{V}^j(t) > 0 \) and \( \frac{dV^j(t)}{dt} < 0 \) for all \( t < \infty \), with \( V^j(t) \rightarrow \bar{V}^j \) and \( J^j(t) \downarrow \bar{J} \) as \( t \rightarrow \infty \).
\[
\frac{dJ^j(t)}{dt} = -\frac{\dot{V}^j(t)}{[u'(w^j(t))]}, \text{ all } t
\]

The optimal contracts convey several features of an equilibrium. The most important features is that wages are strictly increasing continuously throughout tenure. This result comes from the firm’s incentives to reduce worker turnover by back-loading wages. Firms, naturally, would ideally want to backload the wages completely but, due to worker risk aversion, that would be suboptimal. The tension between a firm backloading and worker’s being risk averse, leads to the continuous profile. The varying worker productivity affect the wage profile through \( J^j(t) \) and through \( \frac{d\phi(F(V^j(t)))}{dV} \), since more productive workers encounter better offers.
3.4 Theoretical Results

3.4.1 Equilibrium

An equilibrium in this economy consists of the following:

A set of offers $V = \{V^j(t)\}$, an application strategy $F(.)$ along with hiring and job finding rate functions $q^j(.)$ and $p^j(.)$, Firm and Worker value functions $J^j, \hat{V}^j$ and finally a wage function $w^j(.)$ that satisfy the following:

1) Given $p^j(.)$, $F^j(.)$ solves the worker’s problem.
2) Given $F^j(.)$ and $p^j(.)$, the value of unemployment as above.
3) Given $F^j(.)$ and $p^j(.)$, each offer $x^j$ is given by a contract $\{w(t)\}_{t=0}^{\infty}$ that satisfies the optimal contract characterization and $\hat{V}^j$ is the resulting value-jump function for type $j$ workers. Furthermore, $J^j(.)$ is the consequent value function for the firm and the .
4) Given $J^j(.)$, $q^j(.)$, free entry ensures that $k = q(x^j)J(x^j)$ for all $x \in [V, \hat{V}]$ and $q(x)J(x) < k$ otherwise, where $q(x)$ is as presented previously.

Additionally there is a function $G^j(.)$ that summarizes the distribution of workers of type $j$ over $V$, and a function $n^j$, that gives the fraction of type $j$ workers employed. (The associated probability density function of $G^j(.)$ is $g^j(.)$.)

5) $G^j$ and $n^j$ are stationary.

In order to prove the existence of the equilibrium, the following assumption for the matching function is useful. The assumption basically consists of some regularity conditions that ensure that the respective matching rates are bounded and well-behaved.

Assumption 3.3

(i) $M(q)$ is continuous and $q(V) \in [\underline{q}, \bar{q}]$, where $\underline{q}$ will be specified later;
(ii) $M'(q) < 0$ and $M(\bar{q}) = 0$; (iii) $M(q)$ is twice differentiable for all $q \in [\underline{q}, \bar{q}]$, where $|M'| \leq m_1$ and $|M''| \leq m_2$ for some finite constants $m_1$ and $m_2$; (iv) $qM''(q) + 2M'(q) \leq 0$.

In addition to the previous definitions, the following one is needed to ensure existence.

The following definitions are useful in its statement: $J^j = \frac{k}{q^j}, \bar{J} = J_{w^j}(V), \bar{q}^j = \frac{k}{\bar{J}}, \bar{p}^j \equiv M(\bar{q}^j), \bar{S}^j \equiv S_{w^j}(\bar{V}^j)$. Furthermore, let $w^j$ be a wage lower bound close to 0.

Assumption 3.4 Assume that $b_u, V^j, \text{ and } w^j$ satisfy the following:

18Since $J^j, \rho^j$, and $S^j$ are decreasing in $V$, and $q_{w}(V)$ is increasing in $V$, all functions are contained within their respective bounds.
i) \(0 < \beta < \bar{w}^j = h^j - \frac{r^j}{q^j}\)

ii) \(h^j - [r + p_{w^j}(F_{w^j}(V^j))]J^j \geq w^j + \frac{\left[ u(b^j_h - S_{w^j}(V^j)) - u(w^j) \right]}{u'(w^j)}\)

iii) \(1 + \frac{u''(w^j)}{u'(w^j)^2}[u(\bar{w}^j) - u(w^j)] \geq 0, \text{ all } w^j \in [w^j, \bar{w}^j]\)

With the above assumptions holding an equilibrium can be shown to exist. The proof follows from Shi (2009) except for (iv). The important fact that the value functions are independent from the distribution of workers is called Block-Recursivity and is discussed below.

**Theorem 3.2** Under assumptions 1, 2, 3 and 4, the mapping \(\psi\) has a fixed point \(w^j^*\). The equilibrium for each \(j\)-type worker also has the following features: (i) \(J^j_{w^j^*}(V)\) is strictly positive, bounded in \([J^j, \bar{J}^j]\), strictly decreasing, strictly concave and continuously differentiable for all \(V\), with \(J^j_{w^j^*}(\bar{V}^j) = \bar{J}^j\); (ii) \(p_{w^j^*}(V)\) has all the properties in (2.2) and is strictly concave for all \(V < \bar{V}^j\); (iii) \(\dot{V}^j_{w^j^*} > 0\) and \(\frac{dJ^j_{w^j^*}}{dt} < 0\) for all \(V < \bar{V}^j\) (iv) The condition \(\dot{V}^j(t) > V^j(t)\) holds for all \(j < N\).

The Block-Recursive nature of the equilibrium means that the Value and policy functions are independent of the distribution of workers over jobs. This is a direct result of the fact that search is directed in this model. In a random search environment, a firm making an offer must take into account the distribution of workers over all values when calculating the probability of forming a successful match. With directed search this problem is alleviated since only workers of a single type visit any given submarket. The Block Recursivity thus greatly simplifies both the characterization of the equilibrium and its numerical computation.

### 3.4.2 Stationary Probability Distributions

The Block-Recursive nature of the equilibrium greatly simplifies the stationary job distribution. The policy functions impact the probability distributions but the probability distributions do not feed back into the policy functions. In order to characterize the probability distribution, the inflow and outflow of workers over both type and employment status (including the position on the wage-tenure contract) is set equal to 0 so that the mass is stationary over time. There are three stationary distribution types to consider. The first is
the human capital stationary distribution $G_h$, the second is the stationary distribution of employment $G_n$ and the third is the stationary distribution over contract values $G_V$. This section characterizes each of these stationary distributions.

Consider first the stationary distribution of human capital $G_h$. In order to determine the measure of each type of human capital, I describe the inflow and outflow for each type of human capital. Setting the inflow into a human capital type equal to its outflow characterizes the stationary distribution. The inflow into the lowest human capital level, $h_1$, comes from the birth of new workers replacing those that retired at higher levels of human capital. This occurs at a rate $\delta_d(1 - h_1)dt$. The outflow for this group is simply the workers that learn on the job. This occurs at a rate $\mu n_1 dt$, where $n_1$ refers to the proportion of type 1 workers employed.\footnote{The notation for other human capital types is similar with $h_j$, $n_j$ referring to the proportion of type $j$ workers and workers of that type employed respectively.}

The inflow for workers at a higher type, $j$, comes from workers at human capital level $j - 1$ learning on the job. This occurs at a rate of $\mu h_{j-1} n_{j-1} dt$. The outflow comes from two sources. The first is from worker retirement and the second comes from workers learning on the job. The outflow rate is therefore $\mu h_j n_j + \delta_d h_j$ (The highest human capital type, $N$, has a single outflow coming from retirement since no further learning occurs for workers of this type).

Similarly one can consider the inflows and outflows into unemployment. Consider first the unemployment measure of workers with the lowest human capital level denoted $1 - n_1$. The inflows into this group are new workers entering the labour force and workers with the lowest human capital level that have experienced a job separation. Since the distributions are stationary the measure of new workers is equivalent to that of old retired workers. The rate of inflow into unemployment is therefore: $n_1(1 - h_1)\delta_d dt + n_1(\delta_s + \delta_d)n_1 h_1 dt$. The outflow comes from unemployed workers that successfully find a job. The outflow rate is therefore $(1 - n_1)p(V_u)h_1 dt$. The inflow into employment for other types, besides the highest type $N_h$ is similar except that new workers are not born into unemployment and workers that successfully learn leave. This gives a new inflow into unemployment rate of $(n_j\delta_s h_j + n_j \mu h_j)dt$. The outflow out of unemployment is the same and therefore the rate is $(1 - n_j)p(V_u^j)h_j dt$.

Finally, using the above distributions, it is possible to characterize the distribution of workers over the current value of employment $V$. These distributions are considered sepa-
rately for each human capital type but a simple addition across types obtains the unconditional distribution over values. Consider first the lowest human capital type and the changes to the cumulative probability distribution (cdf) \( G_1(V) \) for arbitrary value \( V \). The inflow into this distribution comes from workers successfully finding a job. This occurs at a rate of \( p(v_1^1)(1 - n_1)h_1dt \). The outflows come from four sources: Workers that experience a job separation or retirement, workers that find employment at a higher value than \( V \), workers that experience wage growth that gives them a higher value than \( V \) and workers that learn on the job and become a higher human capital type worker. The rate of outflow is therefore \( (\delta_s + \delta_d)G_j(V) n_1h_1dt + \int_{\max\{v_1^1,F-1(V)\}}^{V} p(s)dG_j(s)dt + (G_j(V(t+dt)) - G_j(V(t))) + \mu G_j(V(t)) \).

For workers with higher human capital than 1, there is an additional inflow due to lower ability workers learning. This changes the inflow to \( p(v_1^1)(1 - n_1)h_1dt + \mu G_{j-1}(V) \). The outflow is identical except for the highest type, \( N_h \), since these workers can no longer learn.

Taking the above inflows and outflows into consideration and equating them to 0 gives the following characterization:

**Theorem 3.3** Letting \( G_h \), \( G_V \) denote the Cumulative Distribution functions of Human Capital and Employment respectively, and \( g_h \), \( g_V \) denote their Probability Distributions. The following equations therefore characterize the stationary distributions of the economy:

1) \( 0 = \delta_d[1 - \overline{Gh}(j)] + \mu n_{j-1}h_{j-1} - \mu n_jh_j \) where \( \overline{Gh}(j) = \sum_{i=0}^{j} gh(i) \)

2a) \( n_1 = (\delta_d(h_1 - 1) + p(v_1^1))/ (\delta_s + \delta_d + p(v_1^1)) \)

2b) \( n_j = (\delta_s - \delta_d)/(\delta_s - \delta_d + p(v_1^1)) \)

3) \( 0 = p(v_1^1)(1 - n_j)/(n_j) + \mu h_{j-1}n_{j-1}G_{j-1}(V)/h_jn_j - (\delta_s + \delta_d)G_j(V) - \int_{\max\{v_1^1,F-1(V)\}}^{V} p(s)dG(s) - g_j(V)\dot{V}_j - \mu G_j(V) \)

An important theoretical insight following from the block recursive nature of the equilibrium is that the effect of an increase in the minimum wage, \( w_{min} \), is straightforward. The increase in \( w_{min} \), like in Shi (2009), eliminates some of the lower valued submarkets as firms cannot offer the associated wages. The optimal contract, in contrast, is unaffected. This means that jobs with a wage \( w > w_{min} \) are operated in exactly the same way as before. In the original model, the only impact on workers came through a reduction in submarkets that they could visit. While this is clearly an important source of welfare loss, the more important loss from such a policy may come from reduced learning. As in other models with some form of learning on the job, an increase in the minimum wage decreases the amount
of learning. The usual mechanism is through the inability of a firm to offer wage-cuts in compensation for providing worker training. Here the mechanism is a bit different. Noticing that the stationarity condition for employment is increasing in the worker inflow from unemployment and recalling that the probability of finding a job is inversely related to the value it offers; it is obvious that the elimination of low value jobs from the economy reduces the amount of learning in the economy.

**Proposition 3.1** An increase in the minimum wage, \( w_{\text{min}} \), leads to a decrease in learning in the economy.

### 3.5 Empirical Analysis

#### 3.5.1 Data

The data used in this study is the 1979 National Longitudinal Survey of Youth (NLSY79). The fact that it contains weekly work records from 1978 to 2010 makes it particularly well suited to the needs of the study. The model implies that workers’ wages vary due to increases in human capital (unobserved) and increases in tenure. In order to identify the parameters of the model, a complete work history must be constructed for each worker.

The survey initially consists of a sample of individuals aged 14-22 years old in 1979. In order to eliminate as much worker heterogeneity as possible, some restrictions are made to the data. The largest demographic is white male high school graduates, therefore these workers are extracted from the initial sample. In order to eliminate individuals with unusual histories, only those receiving a high school diploma or GED between the ages of 17 and 20 are kept. Under similar reasoning, all individuals that entered military service are dropped. Finally, all workers with work experience before 1979 are dropped since their entire work history is therefore unobservable. These restrictions leave a sample of 534 workers.

The study defines full-time employment to mean an individual reporting more than 30 hours of work a week at a particular job. For each job the average number of hours worked per week is taken in order to determine whether the job is considered full-time or not. Since

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20 This study therefore focuses solely on the learning and tenure effects of low-skilled workers. The second chapter investigates the differences in this relationship among workers with differing levels of education.

21 See appendix for details concerning the construction of the data.
the model only allows for 2 employment states, full-time employment and unemployment, part-time jobs as well as any period of non-employment following entry into the labour force are recoded as unemployment. Entry to the labour force is defined similarly to Farber and Gibbons (1996). An individual employed full-time more than half a year during three consecutive years is coded as having entered the labour force. Additionally, workers over the age of 20 are assumed to have made the transition into the labour force since workers in the sub-sample do not possess high school degrees.

A key concept of this study is job tenure. This refers to the length of time a worker is continuously employed at the same firm without any transitions into non-employment as defined above. It should be noted that transitions to different positions within a firm do not count as breaks in tenure. Employment tenure is defined as a continuous period of employment. It is worth noting that these two concepts differ in that some workers experience a period of non-employment between employment at different jobs while others transition from one job to another without experiencing an interceding period of non-employment. In order to accommodate instances where an intervening period of non-employment between two jobs is probably due to scheduling (e.g. the job was found while the worker was employed at the previous firm), a job-to-job transition is recorded when the time elapsed since the end of the previous job and the start of the new one is 3 weeks or less.

The model assumes that human capital is accumulated solely on the job. It is therefore necessary to distinguish worker and market experience. Worker experience is defined as the sum of a worker’s employment spells, while market experience is defined simply as the worker’s current age minus the age of labour market entry. In the dataset, wages are reported on the interview date and when a job’s end date. Starting from the 1985 survey, however, the interviewer asked for the wage at the beginning of a job. For jobs starting before 1985, the wage reported in that survey year is used instead due to the fact that workers might misreport the starting wage for jobs started before 1985. Additionally there are coding errors for some wage entries. The top and bottom 2.5% of wages are therefore dropped in order to eliminate this problem.
3.5.2 Simulation and Estimation

In order to calibrate the model to the best degree possible, I create a simulated sample of 10000 workers and obtain the required transition probabilities and regression parameters. Time is discretized such that one period in the simulations corresponds to one month in the data.

There are a few functional form assumptions that have to be made. The first is the form of the utility function. The standard log-utility function is chosen as it satisfies the requirements of the model.\(^{22}\) The Cobb-Douglas matching function with parameters \(\alpha\) and \(\beta\), \(M(u, v) = \beta u^\alpha v^{1-\alpha}\), is utilized for the search problem.

While most of the required parameters target data from the data set described in the previous section, a few of them come from external sources. The search elasticity parameter \(\alpha\) is set following Shimer (2005). The unemployment benefits are also set as is standard in the literature, so that they are equal to 0.73 of the average productivity of the economy. Finally, since the data-set follows a cohort that has not entirely reached the retirement age, the exogenous retirement rate, \(\delta_d\), targets an average work career of 40 years.

Since the goal of this paper is to explore the interactions between wage-tenure contracts and learning, the key variables are the learning rate, \(\mu\), along with the levels of human capital \(h_j\). The support of worker probability is assumed to be a discrete grid with 5 possible human capital states; that is \(G_h \equiv [h_1, h_2...h_5]\). The distance between each of these levels of human capital is additionally assumed to be constant. The choice of \(h_1\) is arbitrarily set to 5 in the simulation.\(^{23}\) Disentangling \(\mu\) from the distribution of productivity types is made possible since worker’s who lose their job maintain their level of human capital and do not learn while unemployed.

The table below summarizes the variables and how they are identified in the data:

The NLYS79 data, with the restrictions discussed in the previous section, is used in determining the rest of the simulation targets.\(^{24}\) The restrictions were made in order to build a relatively homogeneous pool of workers and while this study focuses on low-skilled workers it is straightforward to extend it to consider heterogeneous workers.

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\(^{22}\)A different possibility is to calibrate a utility function from the exponential family of equations as was done in Sim (2006).

\(^{23}\)With a different utility structure it may be possible to target this \(h_1\) along with a preference parameter.

\(^{24}\)See the appendix for additional notes on the data.
Table 3.1: Parameter Values

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Identifier</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>The learning rate</td>
<td>The re-employment wage after unemployment.</td>
</tr>
<tr>
<td>$\delta_s$</td>
<td>The job separation rate</td>
<td>The employment-unemployment transition rate: EU.</td>
</tr>
<tr>
<td>$\delta_d$</td>
<td>The retirement probability</td>
<td>Average lifetime career equal to 40 years.</td>
</tr>
<tr>
<td>$k$</td>
<td>The cost of posting a vacancy</td>
<td>The unemployment-employment transition rate: UE.</td>
</tr>
<tr>
<td>$b_u$</td>
<td>The unemployment benefits</td>
<td>Set to .73 of the average productivity in the model.</td>
</tr>
<tr>
<td>$\lambda_f$</td>
<td>Employment search technology parameter</td>
<td>The employment-employment transition rate: EE.</td>
</tr>
<tr>
<td>$h_1$</td>
<td>The lowest level of productivity</td>
<td>Set equal to 5.</td>
</tr>
<tr>
<td>$h_j$</td>
<td>Higher productivity levels</td>
<td>The ratio of earnings.</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Matching function parameter</td>
<td>Elasticity of Matching Function.</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Matching function parameter</td>
<td>Normalizing the v/u Rate to 1.</td>
</tr>
</tbody>
</table>

The identifiers for most of the parameters is evident. The support of human capital is identified primarily through the shape of the wage-tenure profile. Specifically, the ratios of the average wage over three time periods and the first period is considered. The ratios used are: the average wage after 5, 10 and 15 years over the average wage in the first year of labour force participation ($w_5/w_1$, $w_{10}/w_1$ and $w_{15}/w_1$, respectively). Distinguishing between the learning rate, $\mu$, and the support of human capital is made possible only by considering the wage experience of workers that have lost their job. The variable of interest is therefore the re-employment wage of workers that have lost a job. Using the complete worker history I constructed, I take the 2nd and 3rd quartile of the re-employment wage after 5 years and take their ratio with the first quartile ($\tilde{w}_2/\tilde{w}_1$, $\tilde{w}_3/\tilde{w}_1$ and $w_{15}/w_1$).

The flow rates of workers identifies the final set of parameters. There are two methods used in matching the data to the model, only the first of which is presented here. Since the model only has 2 states of worker status, employed or unemployed, a decision regarding part-time employment must be made. In the first method part-time employment is considered as employment while in the second it is considered unemployment. The rate at which workers transition from one job to another job, the employment-employment rate (EE rate), is calculated under the assumption that a job starting within 21 days of another ending is actually a movement from one job to another and therefore the brief unemployment spell is ignored. Furthermore, if a worker begins employment at another job and quits employment at his previous job within 14 days, it is similarly considered to be a job to job transition. The rate at which workers transition into unemployment (EU rate) and from unemployment into employment (UE rate) are constructed using the same definition. An unemployment

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\[^{25}\]See appendix for the other method's implied flow rates

\[^{26}\]This explains the relatively higher EE rate found in this study as compared with similar studies.
### Table 3.2: Parameter Values

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data</th>
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<tr>
<td>EU rate</td>
<td>0.01375</td>
</tr>
<tr>
<td>EE rate</td>
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<tr>
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<td>$w_{5}/w_1$</td>
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<td>$w_{10}/w_1$</td>
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<tr>
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<tr>
<td>$\bar{w}_2/\bar{w}_2$</td>
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<tr>
<td>$\bar{w}_3/\bar{w}_2$</td>
<td>1.6289</td>
</tr>
</tbody>
</table>

spell is therefore at least 21 days long. This assumption is necessary due to the survey nature of the data and consequent inexactness of job start and end dates as well due to the fact that in the case of a job to job transition it is likely the new job does not begin immediately after the worker ceases working at his current job.

The following table presents the rates and wages obtained from the data-set.\(^\text{27}\):

#### 3.5.3 Numerical and Counter-Factual Analysis

There are two main purposes of this study. The first is to provide a model for analyzing the impact of learning while taking into account that firms optimally back-load wages due to retention considerations, while the second is to study the interactions of learning and wage-tenure contracts.

An interesting observation regarding learning is that every time a worker learns there is a delay between the time that the learning takes place and the time it takes a worker to extract the entire benefit of the acquired knowledge. This creates a staggered wage profile where a worker’s wages grow slowly after he has been employed at a firm for a long time without learning and then accelerate rapidly after he learns and the firm finds it optimal to increase his wages since he is both more productive, and hence profitable to the firm, and better able to find employment at other firms.

In the context of the current study the first question relates to the support of the human capital and the rate of learning $\mu$.\(^\text{28}\) The simulation suggests that the highest productivity a worker achieves is roughly 37% higher than his initial level. The fact that this number

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\(^{27}\)The table shows the monthly transition rates

\(^{28}\)The simulation is preliminary at this point.
is smaller than the implied wage difference between 15 and 1 year is to be expected since workers begin employment at really disadvantageous jobs and work their way up through the wage-profile. Furthermore the rate of learning, $\mu$, is found to be 0.287. This rate of learning implies that the average worker takes approximately 17 years to attain his highest level of human capital. This means that in the absence of wage-tenure contracts, it would take an average worker 10.2 years to reach 50% of their maximal wage and 11.8 years to reach 75% of their wage.

A final experiment is conducted investigating the effect of increasing the minimum wage on learning. It is found that a 10 percent increase in the minimum wage reduces learning in the economy by approximately 0.89 percent. The effect is modest as a consequence of the mechanism. Recall that increasing the minimum wage simply eliminates the existence of certain submarkets, namely the ones that offer the worse terms to workers. This has the effect of decreasing the job finding of unemployed workers and consequently decreasing employment. The finding rate of jobs, however, is sufficiently concave so that this effect is not large.

### 3.6 Conclusion and discussion

This study considers a model with wage-tenure contracts as well as exogenous learning by doing. This is in contrast to models that attribute a worker wage growth entirely to marginal productivity increases or to backloaded wage-tenure contracts only. Using a modified version of the directed search model presented in Shi (2009), the model accounts for the joint effects of the firm’s incentive to backload wages and workers learning on the job. One of the central attractions for using such a model is that the directed nature of search allows for the construction of a Block-Recursive equilibrium. Such an equilibrium makes the characterization of equilibrium much simpler and allows for a straightforward calibra-
tion/simulation of the model. Most models considering human capital accumulation with wage-tenure contracts rely on a random search framework which greatly complicates both the former and the latter.

The study makes use of the National Longitudinal Survey of Youth (NLYS79) data-set in order to build a wage-tenure profile and obtain the required employment transition rates. In order to accomplish this complete worker histories, including all employment, unemployment, and out of labour force spells, are constructed. In order to ensure relative homogeneity of the workers, many restrictions are made. The sample is restricted to white-males that did not complete a high school and have not provided any military service. Furthermore, only workers whose complete histories are completed in order to minimize any potential biases. While restricting the sample in such a way may seem limiting, it is done under the assumption that such workers are more likely to be homogeneous when compared to workers with a college degree or technical training for example.

The first direct results relate to the estimated learning rate and the support of human capital. The simulation suggests that the highest productivity a worker achieves is roughly 37% higher than his initial level. Furthermore, workers are found to learn at the rate 0.287 meaning that it would take 17.2 years for them to attain the highest productivity level.

The study also proposes a mechanism by which an increase in the minimum wage leads to less learning. Instead of the usual mechanism, that low paying jobs provide learning experience, here it is that low paying jobs are easy to find. An increase in the minimum wage stops firms from making such low offers and therefore unemployed workers are forced to look for higher paying jobs that offer lower matching rates. A simulation exercise determined that this effect is probably modest, due in part to the concavity of the matching rate function.

In addition to any direct insights provided by the study, the demonstration of how to incorporate wage-tenure contracts in a directed search setting is useful. One specific application is introducing such learning into the model presented in Chapter 2. There insights could be obtained regarding the differential learning rates of different workers; those of high ability and those of low ability. An interesting question, in that the high ability workers already have steeper wage growth due to a greater threat of moving to another firm which in turn forces firms to raise wages faster, is by how much wage-tenure contracts increase income inequality in the early parts of workers careers. As demonstrated
in the second chapter of this thesis, high ability workers enjoy a faster transition rate and a steeper wage-profile early in their careers. This means that such workers would be able to take advantage of a high learning rate better than low ability workers. The degree to which the mechanism presented in the second Chapter is important can therefore be ascertained.\textsuperscript{29}

3.7 Appendix

3.7.1 Data Notes

The following are a few additional notes and details concerning the data used for this study.

1. The NLYS79 dataset contains a lot of missing entries for the age of individuals. I consequently construct my own age variable using the date of the interview and an individual’s birth date.\textsuperscript{30}

2. In order to construct the full employment history of each worker I manually track the beginning and end of each new job using the dataset’s job start date and job end date variables.

3. In order to utilize as many individual’s history as possible, I define employment spells as the length of time between the start date of a job and the end date of a job. Similarly, I define unemployment spells as the length of time between the end date of a job and the start date of another job. Employment is further segmented into 2 categories: Full-time and Part-time. The type of job is determined by coding the average number of hours worked at a job through its entire duration.

4. Missing start and end date problems are dealt with by ignoring employment spells containing such missing entries. While this might introduce a bias it is not clear which direction this bias moves the estimates.

5. Since the model encompasses only one type of employment, I create transition rates and estimate the regressions for 2 formulations. In the first formulation I treat part-time employment as unemployment, while in the second I treat it as full-time employment. The results do not differ significantly between the two formulations and therefore I present only the first in the paper.

\textsuperscript{29}See appendix for the regression of wages on tenure and highest grade completed.

\textsuperscript{30}This age variable and the original age variable are nearly identical for most individuals. They never disagree by more than a year.
Chapter 3: The Determinants of Tenure-Wage Growth

6. Wages are adjusted for inflation using the CPI, with the base year being 2000.
Table 3.4: Parameter Values

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### 3.7.2 Alternate rates

The following table gives the EE,EU,UE rates under the alternative formulation of unemployment.
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Notes: *** p < 0.01, ** p < 0.05, * p < 0.1. t-statistics reported in parentheses. OLS regression results for equation 1. Each column displays the estimate from a separate regression. The control common to all regressions is afqt (test score). The first regression includes experience, occupation and industry effects. The second regression excludes experience while the third and fourth exclude industry and occupation respectively. The last regression excludes experience, industry and occupational effects. Experience refers to age minus highest grade completed minus 5. Experience squared is the squared term of experience. Highest grade completed refers to the highest grade completed. of starting wage refers to an employees wage at the beginning of the spell. tenure refers to time spent at the current employer, while tenure squared is the square of this number. starting wage x tenure and starting wage x tenure squared refers to a variable obtained by multiplying the above.
Bibliography


Bibliography


