Betti Tables of Maximal Cohen-Macaulay Modules over the Cones of Elliptic Normal Curves

by

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Abstract

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Graded Betti numbers are classical invariants of finitely generated modules describing the shape of a minimal free resolution. We show that for maximal Cohen-Macaulay modules over a homogeneous coordinate rings of smooth Calabi-Yau varieties $X$ computation of Betti numbers can be reduced to computations of dimensions of certain Hom groups in the bounded derived category $D^b(X)$.

In the simplest case of a smooth elliptic curve $E$ we use our formula to get explicit answers for Betti numbers. Description of the automorphism group of the derived category $D^b(E)$ in terms of the spherical twist functors of Seidel and Thomas plays a major role in our approach. We study the homogeneous coordinate rings of the embeddings of the elliptic curve $E$ into projective spaces by a complete linear system of degree $n > 0$.

Case $n = 3$ is the simplest case of a smooth plane cubic. Here we show that there are only four possible shapes of the Betti tables up to a shifts in internal degree ($\bullet$), and two possible shapes up to a shift in internal degree and taking syzygies.

For elliptic normal curves of degree $n > 3$ recursive formulae for the Betti numbers are given, and possible shapes of the Betti tables are described. Results on the Betti numbers are applied to study Koszul and Ulrich modules over the homogeneous coordinate ring.

For $n = 1, 2$ the elliptic curve $E$ is embedded as a hypersurface into a weighted projective space. Homogeneous coordinate rings are known as minimal elliptic singularities $\widetilde{E}_7$ (for $n = 2$) and $\widetilde{E}_8$ (for $n = 1$). We show that the same approach to Betti numbers works. In fact formulae for the Betti numbers in these cases are even simpler than in the plane cubic case.
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Chapter 1

Introduction

Elliptic curves is a vast and old topic in algebraic geometry, mostly famous by its arithmetical facet. This thesis concentrated on another aspect, namely on the vector bundles on elliptic curves and more generally on the bounded derived category of coherent sheaves over an elliptic curve. Before going into technical details and precise results we briefly remain reader history of the subject and provide an elementary formulation of the main question motivating this thesis.

We fix an elliptic curve $E$ over an algebraically closed field $k$ of characteristic zero. Indecomposable objects in $\operatorname{Coh}(E)$ were classified by M. Atiyah (1957). Every coherent sheaf $\mathcal{F} \in \operatorname{Coh}(E)$ is isomorphic to the direct sum of a torsion sheaf and a vector bundle $\mathcal{F}'$:

$$\mathcal{F} \cong \operatorname{tors}(\mathcal{F}) \oplus \mathcal{F}'.$$

Isomorphism classes of indecomposable vector bundles with fixed charge $Z(\mathcal{F}) = \begin{pmatrix} r \\ d \end{pmatrix}$, are in bijection with $\operatorname{Pic}^0(E) \cong E$. Here we use the charge $Z(\mathcal{F})$ as an ordered pair of rank $r = \operatorname{rk}(\mathcal{F})$ and degree $d = \deg(\mathcal{F})$ of $\mathcal{F}$. Observing that category $\operatorname{Coh}(E)$ is hereditary we conclude that objects in the derived category $\mathbb{D}^b(E) = \mathbb{D}^b(\operatorname{Coh}(E))$ are formal, that is every complex of coherent sheaves is isomorphic to the direct sum of its cohomology. Thus, indecomposable objects of $\mathbb{D}^b(E)$ are vector bundles and skyscraper sheaves possibly translated by $[k]$.

Suppose that the elliptic curve $E$ is given as a cubic in projective plane $i : E \to \mathbb{P}^2$. Let $\mathcal{F}$ be a vector bundle over $E$, an elementary preliminary version of the question that we study in this thesis can formulated as follows. What is the structure of minimal free resolution

$$0 \leftarrow i_* \mathcal{F} \leftarrow P_0 \leftarrow^A P_1 \leftarrow 0$$

of $i_* \mathcal{F}$ over $\mathbb{P}^2$? This question has two parts: what are $P_0$ and $P_1$ and what is the map $A$? We have a complete answer to the first part of the question, but the second is much harder. Note that as a free $\mathcal{O}_{\mathbb{P}^2}$-modules $P_0$ and $P_1$ have the form

$$P_0 \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(-j)^{\beta_{0,i}}, \text{ and } P_1 \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(-j)^{\beta_{1,i}},$$

where positive exponents $\beta_{i,j}$ for $i = 0, 1$ a called Betti numbers. The Betti numbers are the main objects.
Chapter 1. Introduction

of study in this thesis. If we choose a basis in $P_0$ and in $P_1$ then differential $A$ of the resolution can be treated as a matrix with homogeneous polynomial entries. The Betti numbers give us information about degrees of entries of this matrix. Note also that matrix $A$ is the first matrix of a matrix factorization associated to $F$, we going to return to matrix factorizations later. In this considerations we have not defined what is a minimal resolution. It is well known that notion of a minimal resolution is more natural for modules over homogeneous graded coordinate ring of $E$. By definition homogeneous coordinate ring is

$$R_E = \bigoplus_{i \geq 0} H^0(E, \mathcal{L}^\otimes i),$$

where $\mathcal{L}$ is a very ample line bundle, sections of $\mathcal{L}$ provide the embedding of $E$ into $\mathbb{P}^2$. Serre’s functor $M = \Gamma_*(\mathcal{F}) = \bigoplus_{k \geq 0} H^0(E, \mathcal{F}(k))$ associate graded module $M$ over $R_E$. It is known classically that $M$ is maximal Cohen-Macaulay module for a vector bundle over a curve, in higher dimensions vector bundles with this property a called arithmetically maximal Cohen-Macaulay modules. Therefore, our original formulation of the main question naturally leads us to the study of maximal Cohen-Macaulay modules over $R_E$. We are going to abbreviate maximal Cohen-Macaulay as MCM.

We call $\text{Spec}(R_E)$ the cone over the elliptic curve $E$. It is a singular affine surface, with an isolated singularity at the irrelevant ideal of $R_E$. We started with a smooth curve $E$, but the cone over $E$ is a singular space, vector bundles and more generally derived category of coherent sheaves over $E$ are naturally related to MCM modules over the cone.

Category of MCM modules over a hypersurface $S/(f)$ is equivalent to the category of matrix factorizations. Object of the latter category are ordered pairs of matrices $\{A, B\}$ over $S$ such that $AB = BA = f \text{Id}$, and MCM module corresponding to $\{A, B\}$ is $M = \text{coker} \ A$. For details and references see corresponding section in the next chapter.

We want to return to the history of the topic of MCM modules over smooth cubics and name several papers. Matrix factorizations of MCM modules of rank 1 over Fermat cubic $x_0^3 + x_1^3 + x_2^3 = 0$ were completely classified in [25]. Classification over the general Hesse cubic $x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0$ follows immediately from our results, where the only non-trivial case is given by Moore’s matrices. Analogous classification for rank 1 MCM modules over Weierstrass cubics $y^2 z - x^3 - axz - bx^3 = 0$ here obtained more recently in [18].

Note that if one of the matrices say $A$ of a matrix factorization $\{A, B\}$ has linear entries and size equal to the degree of the hypersurface $f$ then $\det(A) = f$ and $B = \text{adj}(A)$, where $\text{adj}$ is operation of taking adjugate matrix (transpose of the cofactor matrix). Find such special matrix factorizations is equivalent to finding determinantal presentations of $f$. Some relevant results about determinantal presentations and Betti numbers of line bundles over plane curves could be found in [7].

Determinantal presentations of smooth cubics were also studied in [12]. Where authors obtained an explicit solution for the Legendre form of the cubic.

We study the Betti numbers in more general context of elliptic normal curves. If $\mathcal{L}$ is a line bundle of degree $n \geq 3$ on an elliptic curve $E$, then $\mathcal{L}$ is very ample and provides an embedding of $E$ into $\mathbb{P}^{n-1}$. Such embedded curves are called elliptic normal curves of degree $n$. The homogeneous coordinate ring of an elliptic normal curve is

$$R_E = \bigoplus_{i \geq 0} H^0(E, \mathcal{L}^\otimes i).$$

For example, for $n = 3$ we get a cubic in $\mathbb{P}^3$, while for $n = 4$ a normal elliptic curve is a complete
intersection of two quadrics in $\mathbb{P}^4$.

Key ingredient of our treatment of Betti numbers of graded modules over $R_E$ is an equivalence of triangulated categories

$$\Phi : D^b(E) \to \text{MCM}_{gr}(R_E)$$

proved by D. Orlov in [26]. In this paper we call this equivalence of categories Orlov’s equivalence. We explain notations and provide some details in the next section. This equivalence is used to formulate questions about graded MCM modules over $R_E$ in terms of coherent sheaves on $E$.

The first result (see theorem [3.0.6 below] in this direction is immediate.

**Theorem 1.0.1.** The graded Betti numbers of an MCM module $M$ are given by

$$\beta_{i,j}(M) = \dim \text{Hom}_{D^b(E)}(\Phi^{-1}(M), \sigma^{-j}(\Phi^{-1}(k^{st}))[i]).$$

For a description of the functor $\sigma$ see the next section. Despite the fact that the formula for Betti numbers above looks more complicated than the standard one

$$\beta_{i,j}(M) = \dim \text{Ext}^i_{R_E}(M, k(-j)),$$

it reduces computations of Betti numbers to computations in the bounded derived category $D^b(E)$ of coherent sheaves on $E$.

The derived category $D^b(E)$ of the elliptic curve is a relatively simple mathematical object. In particular, computations of dimensions $\dim \text{Hom}_{D^b(E)}(\Phi^{-1}(M), \sigma^{-j}(\Phi^{-1}(k^{st}))[i])$ can be done explicitly.

The case of $n = 3$ (smooth plane cubic) is the most elementary case because here we have (quasi-) periodicity of complete resolutions that allow us the reduce computations to $\beta_{0,j}$ and $\beta_{1,j}$.

**Theorem 1.0.2.** The Betti diagrams of complete resolution of an indecomposable MCM module over the homogeneous coordinate ring $R_E$, up to shift of internal degree and taking syzygies, has one of the following two forms.

1. The discrete family $F_r$, where $r$ is a positive integer.

```
... ← 3r ← 3r ← 1
     ↑     ↑     ↑
1 ← 3r ← 3r ← 1
     ↑     ↑     ↑
1 ← 3r ← 3r ← ...  
```

2. The continuous family $G_\lambda(r,d)$. Elements in the family $G_\lambda(r,d)$ are parameterized by a pair of
integers \((r, d)\), satisfying conditions \(r > 0, d \geq 0, 3r - 2d > 0\), and a point \(\lambda \in E\).

\[
\begin{array}{cccccc}
\ldots &\leftarrow &d &\leftarrow &3r - 2d &\leftarrow &3r - d &\leftarrow &d &\ldots \\
\end{array}
\]

Note that in this theorem \(F_r\), and \(G_\lambda(r, d)\) stand for vector bundles on \(E\), the diagrams represent the complete resolutions of the MCM modules that correspond to these vector bundles under Orlov’s equivalence.

In commutative algebra various numerical invariants can be associated with a module. Most important invariants include: dimension, depth, multiplicity, rank, minimal number of generators etc. For a graded module over a graded ring values of many invariants can be extracted from the Hilbert polynomial and Hilbert polynomial is easy to compute if we know the Betti numbers.

Numerical invariants - multiplicity \(e(M)\), rank \(rk(M)\) and minimal number of generators \(\mu(M)\) of an MCM module \(M = \Phi(F)\) - can be expressed in terms of the rank \(r\) and the degree \(d\) of \(F\). The results are summarized in table 4.3.

For Betti numbers of MCM modules over the cone over elliptic normal curves of degree \(n > 3\) we have answers in form of recursive sequences. These results are applied to give a criterion of (Co-)Koszulity. We also describe when \(M = \Phi(F)\) is maximally generated in terms of properties of \(F\).

In the last two sections we apply our methods to study the cases \(n = 1, 2\). Geometrically these correspond to the embedding of an elliptic curve into a weighted projective space. The singularities of the corresponding cones are called minimal elliptic. They were studied by K.Saito [29], where he introduced the notation \(\tilde{E}_8\) for \(n = 1\), \(\tilde{E}_7\) for \(n = 2\) and \(\tilde{E}_6\) for the cone over a smooth cubic, that is, for the case \(n = 3\). For the singularities \(\tilde{E}_7\) and \(\tilde{E}_8\) we obtain analogous formulae for the Betti numbers and the numerical invariants of MCM modules.
Chapter 2

Preliminaries

In this chapter we summarize some well known results that we need in this thesis. All rings and algebras we consider are Noetherian.

2.1 Maximal Cohen-Macaulay modules, complete resolutions and matrix factorizations

In this subsection $R$ is a commutative (local or graded) Gorenstein ring. If $M$ is a finitely generated module over $R$, then depth of $M$ (with respect to the unique maximal ideal in the local case, or the irrelevant ideal in the graded case) is bounded by the dimension of $M$, namely

$$\text{depth}(M) \leq \dim(M) \leq \dim(R).$$

If depth$(M) = \dim(M)$, then the module $M$ is called Cohen-Macaulay, and if depth$(M) = \dim(R)$ the module $M$ is called a maximal Cohen-Macaulay (MCM). In the following we are going to use the stable category of MCM modules. It means that we consider the full subcategory of MCM modules in the category of finitely generated modules, and in this subcategory morphisms of MCM modules that can be factored through a projective module are considered to be trivial. In other words, in the stable category we have

$$\text{Hom}(M, N) = \text{Hom}(M, N)/\{\text{morphisms that factor through a projective}\}.$$ 

In particular, all projective modules (and projective summands) in the stable category are identified with the zero module. Note that the operation of taking syzygies is functorial on the stable category, we denote this functor $\text{syz}$. The functor $\text{syz}$ is an autoequivalence of the stable category of MCM modules, the inverse of this functor we denote $\text{cosyz}$. Moreover, the stable category of MCM modules admits a structure of triangulated category with autoequivalence $\text{syz}$. The details of the construction of the stable category can be found in [11]. The stable category of MCM modules is denoted $\text{MCM}(R)$ in the local case and $\text{MCM}_{gr}(R)$ in the graded case. The following theorem also can be found in [4].

**Theorem 2.1.1.** A finitely generated module $M$ over a Gorenstein ring $R$ is MCM if and only if the following three conditions hold.

1. $\text{Ext}_{R}^{i}(M, R) \cong 0$, for $i \neq 0$, 

\[ \text{depth}(M) \leq \dim(M) \leq \dim(R). \]
2. $\text{Ext}^i_R(M^*, R) \cong 0$, for $i \neq 0$, where $M^* = \text{Hom}_R(M, R)$ is the dual module.

3. $M$ is reflexive, this means that the natural map $M \rightarrow M^{**}$ is an isomorphism.

These three conditions are completely homological and can be used to define an MCM module over a not necessarily commutative (but still Gorenstein) ring $R$. From now on we assume that all rings that we consider are Gorenstein.

The above criterion allows us to give a simple construction of a complete projective resolution of an MCM module $M$. In this thesis we use only complete projective resolutions by finitely generated projectives and we refer to them as complete resolutions. Let $P^* \rightarrow M$ be a projective resolution of $M$ and $Q^* \rightarrow M^*$ a projective resolution of $M^*$. Then we have a projective coresolution of $M^{**} \rightarrow (Q^*)^* = \text{Hom}(Q^*, R)$. The projective resolution $P^*$ and the coresolution $(Q^*)^*$ can be spliced together

$$
\begin{array}{c}
P^* & \longrightarrow & M & \longrightarrow & 0 \\
\vert & \cong & \vert & & \\
0 & \longrightarrow & M^{**} & \longrightarrow & (Q^*)^*
\end{array}
$$

We call the map of complexes $P^* \rightarrow (Q^*)^*$ the norm map and we obtain a complete resolution of $M$ as cone of the norm map:

$$\text{CR}(M) = \text{cone}(P^* \rightarrow (Q^*)^*)[-1].$$

The complex CR($M$) is an unbounded acyclic complex of finitely generated projective $R$-modules. On the other hand, if $K$ is an acyclic unbounded complex of finitely generated projectives then coker($K_1 \rightarrow K_0$) is an MCM module. This yields an equivalence of categories

**Theorem 2.1.2.**

$$\text{MCM}(R) \cong K^{ac}(\text{proj}(R)),$$

where $K^{ac}(\text{proj}(R))$ stands for the homotopy category of acyclic unbounded complexes of finitely generated projective modules.

More generally, if $X^* \in D^b(R)$ is an object of the bounded derived category of $R$, and $r : P^* \rightarrow X^*$ is a projective resolution we take $s : Q^* \rightarrow (P^*)^*$ to be a projective resolution of $(P^*)^*$. As before, we get a norm map

$$N : P^* \rightarrow (P^*)^{**} \rightarrow (Q^*)^*.$$

A complete resolution of $X^*$ is given by the cone of the norm map

$$\text{CR}(X^*) = \text{cone}(N)[-1].$$

The complete resolution of $X^*$ is unique up to homotopy.

Just as projective resolutions are used to compute extension groups $\text{Ext}(-, -)$, complete resolutions are used to compute stable extension groups $\text{Ext}(-, -)$

$$\text{Ext}^i(X^*_1, X^*_2) = H^i\text{Hom}(\text{CR}(X^*_1), X^*_2).$$

The next simple lemma is a crucial step in the derivation of our main formula for graded Betti numbers.
Lemma 2.1.3. If $M$ is an MCM module over $R$, and $N$ is a finitely generated $R$-module, then

$$\text{Ext}^i(M, N) \cong \text{Ext}^i(M, N),$$

for $i > 0$. Moreover, if $N = k$ the above isomorphism is also true for $i = 0$,

$$\text{Ext}^i(M, k) \cong \text{Ext}^i(M, k),$$

for $i \geq 0$.

Proof. The complete resolution of $M$ coincides with a projective resolution of $M$ in positive degrees. In the case $N = k$ if we choose a minimal resolution of $M$ the result is immediate.

We also need the following result about stable extension groups.

Lemma 2.1.4. For any finitely generated module $N$ over $R$ there is a finitely generated MCM module $M$ such that there is a short exact sequence

$$0 \to F \to M \to N \to 0,$$

where the module $F$ has finite projective dimension. For any $L \in \text{MCM}(R)$ there is an isomorphism of stable extension groups

$$\text{Ext}^i(L, M) \cong \text{Ext}^i(L, N),$$

for $i \in \mathbb{Z}$.


The module $M$ in the lemma above is called a maximal Cohen-Macaulay approximation or stabilization of $N$. We use the latter name and return to this construction in the next section in the context of bounded derived categories. For other properties of stable extension groups see [11], details on the construction of maximal Cohen-Macaulay approximations can be found in [5].

Now suppose that $R$ is a hypersurface singularity ring $R = S/(w)$, where $S$ is a regular ring, and $w \in S$ defines a hypersurface and $w \neq 0$. In this case for an MCM module $M$ any complete resolution is homotopic to a 2–periodic complete resolution. Moreover, such 2–periodic (ordinary and complete) resolutions can be constructed using matrix factorizations of $w$ introduced in [15]. We give a brief outline of the construction.

Let $M$ be an MCM module over $R = S/(w)$. If $M$ is considered as an $S$-module, then by Auslander-Buchsbaum formula

$$\text{pd}_S(M) = \text{depth}(S) - \text{depth}(M),$$

and so the projective dimension of $M$ as $S$ module is 1. Hence a projective resolution of $M$ over $S$ has the following form

$$0 \to P_1 \xrightarrow{A} P_0 \to M \to 0.$$
Since multiplication by \( w \) annihilates \( M \), there is a homotopy \( B \) such that the diagram

\[
\begin{array}{ccc}
P^1 & \xrightarrow{A} & P^0 \\
\downarrow{w} & & \downarrow{w} \\
P^1 & \xrightarrow{B} & P^0 \\
\end{array}
\]

commutes. One can check that the following is true.

\[ AB = BA = wid. \]

Ordered pairs of maps satisfying the above property are called matrix factorizations of \( w \). Note that our assumption that \( R \) is local or graded implies that the modules \( P^0 \) and \( P^1 \) are free, and if we choose a basis in \( P^0 \) and \( P^1 \), we can present \( A \) and \( B \) as matrices. Morphisms in the category of matrix factorizations are defined in a natural way (for details see below), the category of matrix factorizations is denoted \( MF(w) \). It is easy to see that the module \( M \) can be recovered from a matrix factorization \( \{A, B\} \) as

\[ M \cong \text{coker } A, \]

while the cokernel of the second matrix recovers the first syzygy of \( M \) over \( R \):

\[ \text{syz}(M) \cong \text{coker } B. \]

In this thesis we are interested in graded rings, modules and matrix factorizations, thus from now on all objects are graded. In particular, we assume that \( w \) is homogeneous of degree \( n \). The category of graded matrix factorizations is denoted \( MF_{gr}(w) \).

A matrix factorization alternatively can be interpreted as a (quasi) 2-periodic chain \( P \) of modules and maps between them

\[
\begin{array}{c}
\ldots \rightarrow P^0(-n) \xrightarrow{p^0} P^1(-n) \xrightarrow{p^1} P^0 \rightarrow P^1 \rightarrow \ldots
\end{array}
\]

such that composition of two consecutive maps is not zero (as in the case of chain complexes) but is equal to the multiplication by \( w \):

\[ p^0 \circ p^1 = p^1(n) \circ p^0 = w. \]

Abusing notation we still call the maps \( p^0 \) and \( p^1 \) differentials. Note that we assume that \( p^0 \) is of degree 0, and \( p^1 \) is of degree \( n \). A morphism of matrix factorizations \( f : P \rightarrow Q \) is an ordered pair \((f^0, f^1)\) of maps of graded free S-modules of degree zero commuting with the differentials

\[
\begin{array}{c}
\ldots \rightarrow P^0(-n) \xrightarrow{f^0(-n)} P^1(-n) \xrightarrow{f^1} P^0 \rightarrow P^1 \rightarrow \ldots \\
\ldots \rightarrow Q^0(-n) \xrightarrow{q^0(-n)} Q^1(-n) \xrightarrow{q^1} Q^0 \rightarrow Q^1 \rightarrow \ldots
\end{array}
\]
It is more convenient to consider periodic infinite chains

\[
\cdots \longrightarrow P^{-1} \xrightarrow{p^{-1}} P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^2 \longrightarrow \cdots
\]

of morphisms of free graded modules such that composition \(p^{i+1} p^i\) of any two consecutive maps is equal to multiplication by \(w\). Periodicity in the graded cases means that \(P^i[2] = P^i(n)\) and \(p^i[2] = p^i(n)\), where translation \([1]\) is defined in the same way as for complexes: \(P^i[1] = P^{i+1}\) and \(p^i[1] = -p^i\). Morphisms of matrix factorizations also satisfy the periodicity conditions \(f^{i+2} = f^i(n)\).

The category of matrix factorizations \(MF(w)\) satisfies properties similar to the properties of categories of complexes, here we only review some basic definitions and facts, details can be found for example in [26].

For example, the definition of homotopy of morphisms of matrix factorizations is completely analogous to the case of complexes. A morphism \(f : P \rightarrow Q\) is called null-homotopic if there is a morphism \(h : P \rightarrow Q[-1]\) such that \(f^i = q^{i-1} h^i + h^{i+1} p^i\). The category of matrix factorizations with morphisms modulo null-homotopic morphisms is called the stable (or derived) category of matrix factorizations and is denoted \(MF_{gr}(w)\).

For any morphism \(f : P \rightarrow Q\) from the category \(MF_{gr}(w)\) we define a mapping cone \(C(f)\) as an object

\[
\cdots \longrightarrow Q^i \oplus P^{i+1} \xrightarrow{c^i} Q^{i+1} \oplus P^{i+2} \xrightarrow{c^{i+1}} Q^{i+2} \oplus P^{i+3} \longrightarrow \cdots
\]

such that

\[
c^i = \begin{pmatrix} q^i & f^{i+1} \\ 0 & -p^{i+1} \end{pmatrix}.
\]

There are maps \(g : Q \rightarrow C(f)\), \(g = (id, 0)\) and \(h : C(f) \rightarrow P[1]\), \(h = (0, -id)\).

We define standard triangles in the category \(MF_{gr}(w)\) as triangles of the form

\[
P \xrightarrow{f} Q \xrightarrow{g} C(f) \xrightarrow{h} P[1].
\]

A triangle is called distinguished if it is isomorphic to a standard triangle.

**Theorem 2.1.5.** The category \(MF_{gr}(w)\) endowed with the translation functor \([1]\) and the above class of distinguished triangles is a triangulated category.

Reducing a matrix factorization modulo \(w\) and extending it 2-periodically to the left we get a projective resolution of \(M\) over \(R\)

\[
\cdots \longrightarrow P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0 \longrightarrow M \longrightarrow 0.
\]

Extension of this resolution to the right produces a 2-periodic complete resolution of \(M\) over \(R\)

\[
\cdots \longrightarrow P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0 \xrightarrow{p^0} P^1 \longrightarrow \cdots
\]

The next theorem follows from the previous discussion and an extension of Eisenbud’s result to the graded case.
Theorem 2.1.6. For a hypersurface ring \( R = S/(w) \) there is an equivalence of triangulated categories

\[
MF\text{}_{\text{gr}}(w) \cong \text{MCM}(R) \cong K\text{ac}(\text{proj}(R)).
\]

2.2 The singularity category and Orlov’s equivalence

Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded algebra over a field \( k \). We assume that \( R_0 = k \), such algebras are called connected. Moreover, we assume that \( R \) is Gorenstein, which means that \( R \) has finite injective dimension \( n \) and that there is an isomorphism

\[
\text{RHom}_R(k, R) \cong k(a)[-n],
\]

where the parameter \( a \in \mathbb{Z} \) is called the Gorenstein parameter of \( R \). In [26] D. Orlov described precise relations between the triangulated category \( D^b(\text{qgr}(R)) \) and the graded singularity category \( D_{\text{Sg}}\text{gr}(R) \). We only use his result for the case \( a = 0 \), and \( R \) a homogeneous coordinate ring of some smooth projective Calabi-Yau variety \( X \).

The abelian category of coherent sheaves \( \text{Coh}(X) \) is equivalent, by Serre’s theorem, to the quotient category \( \text{qgr}(R) \)

\[
\text{Coh}(X) \cong \text{qgr}(R),
\]

here the abelian category \( \text{qgr}(R) \) is defined as a quotient of the abelian category of finitely generated graded \( R \)-modules by the Serre subcategory of torsion modules

\[
\text{qgr}(R) = \text{mod}_{\text{gr}}(R)/\text{tors}_{\text{gr}}(R),
\]

where \( \text{tors}_{\text{gr}}(R) \) is the category of graded modules that are finite dimensional over \( k \). Alternative notation for this category is \( \text{Proj}(R) = \text{qgr}(R) \), this notation emphasizes relation of this category with the geometry of the projective spectrum of \( R \) for commutative rings, and replaces such geometry for general not necessarily commutative \( R \). Details can be found, for example, in [2].

The bounded derived category of the abelian category of finitely generated graded \( R \)-modules \( D^b(\text{gr}-R) = D^b(\text{mod}_{\text{gr}}(R)) \) has a full triangulated subcategory consisting of objects that are isomorphic to bounded complexes of projectives. The latter subcategory can also be described as the derived category of the exact category of graded projective modules, we denote it \( D^b(\text{grproj}-R) \). The triangulated category of singularities of \( R \) is defined as the Verdier quotient

\[
D^\text{gr}_{\text{Sg}}(R) = D^b(\text{gr}-R)/D^b(\text{grproj}-R).
\]

In fact, the triangulated category of singularities is one of many facets of the stable category of MCM modules. It is well known (see [11]) that the stable category of MCM modules is equivalent as triangulated category to the singularity category \( D^\text{gr}_{\text{Sg}}(R) \)

\[
\text{MCM}_{\text{gr}}(R) \cong D^\text{gr}_{\text{Sg}}(R).
\]
We have the following diagram of functors

\[ D^b(X) \xrightarrow{a} D^b(\text{gr}^R \geq i) \xleftarrow{\Phi} D^b(\text{gr}^R \geq i) \]

In this diagram \( D^b(\text{gr}^R \geq i) \) denotes the derived category of the abelian category of finitely generated modules concentrated in degrees \( i \) and higher. We call the parameter \( i \in \mathbb{Z} \) a cutoff. The functor \( R\Gamma \geq i : D^b(X) \rightarrow D^b(\text{gr}^R \geq i) \) is given by the following direct sum:

\[ R\Gamma \geq i(M) = \bigoplus_{l \geq i} R\text{Hom}(\mathcal{O}_X, M(l)). \]

The functor \( \text{st} : D^b(\text{gr}^R \geq i) \rightarrow D^\text{gr}_S^R(R) \) is the stabilization functor extended to the bounded derived category \( D^b(\text{gr}^R \geq i) \). Its construction for any parameter \( i \) was given by Orlov. The functor \( a \) is an extension of the sheafification functor to the derived category \( D^b(\text{gr}^R \geq i) \). It is well known and easy to check that the functor \( a \) is left adjoint to the functor \( R\Gamma \geq i \). Finally, the functor \( b \) is a left adjoint functor to the stabilization functor. It can be easily described explicitly if we go to the complete resolutions of MCM modules using the equivalence of triangulated categories

\[ D^\text{gr}_S^R(R) \cong K^{\text{ac}}(\text{proj}(R)). \]

We choose a complete resolution of an MCM module and then leave in this complex only free modules generated in degrees \( d \geq i \). The result is an unbounded complex, but it has bounded cohomology, and thus gives an element in \( D^b(\text{gr}^R \geq i) \).

The special case of Orlov’s theorem (see Theorem 2.5 in [26]) that we need in this thesis can be formulated as follows.

**Theorem 2.2.1.** Let \( X \) be a smooth projective Calabi-Yau variety, \( R \) its homogeneous coordinate ring (corresponding to some very ample invertible sheaf). Then for any parameter \( i \in \mathbb{Z} \) the composition of functors

\[ \Phi_i = \text{st} \circ R\Gamma \geq i : D^b(X) \rightarrow D^\text{gr}_S^R(R) \]

is an equivalence of categories.

The subtle point is that we get a family of equivalences \( \Phi_i \) parameterized by integer numbers. We choose the cutoff parameter \( i = 0 \) and denote \( \Phi = \Phi_0 \).

Abusing notation, we denote the composition of Orlov’s and Buchweitz’s equivalences by the same \( \Phi \).

\[ \Phi : D^b(X) \rightarrow \text{MCM}_{\text{gr}}(R). \]

This equivalence of categories is our main tool for computing Betti numbers of MCM modules.
2.3 Indecomposable objects in the derived category $D^b(E)$

Let $E$ be a smooth elliptic curve, then the category of coherent sheaves $\text{Coh}(E)$ is a hereditary abelian category, namely $\text{Ext}^i(\mathcal{F}, \mathcal{G}) \cong 0$ for any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and any $i \geq 2$. The following theorem attributed to Dold (cf. [14]) shows that if we want to classify indecomposable objects in $D^b(E)$ then it is enough to classify indecomposable objects of $\text{Coh}(E)$.

Theorem 2.3.1. Let $A$ be an abelian hereditary category. Then any object $\mathcal{F}$ in the derived category $D^b(A)$ is formal, i.e., there is an isomorphism

$$\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} H^i(\mathcal{F})[-i].$$

Indecomposable objects in the abelian category $\text{Coh}(E)$ were classified in [3] by M. Atiyah. Every coherent sheaf $\mathcal{F} \in \text{Coh}(E)$ is isomorphic to direct sum of a torsion sheaf and a vector bundle $\mathcal{F}'$:

$$\mathcal{F} \cong \text{tors}(\mathcal{F}) \oplus \mathcal{F}'.$$

Therefore, it is enough to classify indecomposable torsion sheaves and indecomposable vector bundles. An indecomposable torsion sheaf is a skyscraper sheaf. To describe such sheaf we need two parameters: a point $\lambda \in E$ on the elliptic curve where the skyscraper sheaf is supported, and the degree of that sheaf $d \geq 1$. A torsion sheaf with these parameters is isomorphic to $\mathcal{O}_{E,\lambda}/m(\lambda)^{d+1}$, where $\lambda \in E$. Isomorphism classes of indecomposable vector bundles of given rank $r \geq 1$ and degree $d \in \mathbb{Z}$ are in bijection with $\text{Pic}^0(E)$.

On an elliptic curve $E$ there is a special discrete family of vector bundles, usually denoted by $F_r$ and parameterized by positive integer number $r \in \mathbb{Z}_{>0}$. By definition $F_1 = \mathcal{O}_E$, and the vector bundle $F_r$ is defined inductively through the unique non-trivial extension:

$$0 \to \mathcal{O}_E \to F_r \to F_{r-1} \to 0.$$ 

It is easy to see that $\text{deg}(F_r) = 0$ and $\text{rk}(F_r) = r$ for $r \in \mathbb{Z}_{>0}$. We call $F_r$, $r \in \mathbb{Z}_{>0}$, Atiyah bundles.

At the end of this subsection let us mention that computing cohomology of a vector bundle on an elliptic curve is very simple. In fact we are interested only in the dimensions of the cohomology groups. The following formula for the dimension of the zero cohomology group of an indecomposable vector bundle $\mathcal{F}$ of degree $d = \text{deg}(\mathcal{F})$ and rank $r = \text{rk}(\mathcal{F})$ can be found in [3].

$$\dim H^0(E, \mathcal{F}) = \begin{cases} 0, & \text{if } \text{deg}(\mathcal{F}) < 0, \\ 1, & \text{if } \text{deg}(\mathcal{F}) = 0 \text{ and } \mathcal{F} \cong F_r, \\ d, & \text{otherwise}. \end{cases}$$

For the dimensions of the first cohomology groups we use Serre duality in the form $\dim H^1(E, \mathcal{E}) = \dim H^0(E, \mathcal{E}^\vee)$. We see that the Atiyah bundles are exceptional cases in the previous formula. Moreover, Atiyah bundles can be characterized as vector bundles on $E$ with degree $d = 0$, rank $r \geq 1$, and $\dim H^0(E, F_r) = \dim H^1(E, F_r) = 1$. They are unique up to isomorphism.
2.4 Spherical twist functors for $D^b(E)$

We need a description of the automorphism group of the derived category $D^b(E)$. Spherical twist functors, introduced in [31] by Seidel and Thomas and thus sometimes called Seidel-Thomas twists, play an important role in the description. The reader can find the details and proofs of the statements of this subsection in [10] and [21].

On a smooth projective Calabi-Yau variety $X$ over a field $k$ an object $E \in D^b(X)$ is called spherical if

$$\text{Hom}(E, E[\ast]) \cong H^\ast(S_{\dim(X)}, k),$$

where $S_{\dim(X)}$ denotes a sphere of dimension $\dim(X)$. The definition of the spherical object can be extended to the non-Calabi-Yau case, but we don’t need it in this paper.

With any spherical object $E$ one can associate an endofunctor $T_E : D^b(X) \to D^b(X)$, called a spherical twist functor. It is defined as a cone of the evaluation map

$$T_E(F) = \text{cone}(\text{Hom}(E, F[\ast]) \otimes E \to F).$$

One of the main results of [31] is the following theorem.

**Theorem 2.4.1.** If $E$ is a spherical object in $D^b(X)$ of a smooth projective variety $X$, then the spherical twist

$$T_E : D^b(X) \to D^b(X)$$

is an autoequivalence of $D^b(X)$.

After this short deviation to spherical twists in general on Calabi-Yau varieties we return to the case of an elliptic curve $E$. Let us define the Euler form on $E, F \in D^b(E)$ as

$$\langle E, F \rangle = \dim H^0(E, F) - \dim H^1(E, F).$$

It is easy to see that the Euler form depends only on the classes of $[E]$ and $[F]$ in the Grothendieck group $K_0(D^b(E))$. Therefore, we treat the Euler form as a form on the Grothendieck group $K_0(D^b(E))$. The Riemann-Roch formula can be formulated as the following identity for the Euler form

$$\langle E, F \rangle = \chi(F) \text{rk}(E) - \chi(E) \text{rk}(F),$$

where $\chi(E)$ is the Euler characteristic of $E$:

$$\chi(E) = \langle O_E, E \rangle.$$

Note that $\chi(E) = \text{deg}(E)$ for any object $E$, because the genus $g(E) = 1$.

Since $E$ is a Calabi-Yau variety, the Euler form is skew-symmetric:

$$\langle E, F \rangle = -\langle F, E \rangle,$$

therefore, the left radical of the Euler form

$$1.\text{rad} = \{ F \in K_0(D^b(E)) | \langle F, - \rangle = 0 \}$$
coincides with the right radical

$$r. \text{rad} = \{ \mathcal{F} \in K_0(D^b(E)) | \langle - , \mathcal{F} \rangle = 0 \},$$

and we call it the radical of the Euler form and denote it $\text{rad}(\langle - , - \rangle)$.

Let us define the charge of $\mathcal{E} \in \text{Coh}(E)$ as

$$Z(\mathcal{E}) = \begin{pmatrix} \text{rk}(\mathcal{E}) \\ \text{deg}(\mathcal{E}) \end{pmatrix}.$$

The next proposition is crucial for the description of the automorphism group of the derived category $D^b(E)$ of an elliptic curve.

**Proposition 2.4.2.** The charge $Z$ induces an isomorphism of abelian groups

$$Z : K_0(E)/\text{rad}(\langle - , - \rangle) \to \mathbb{Z}^2.$$

Moreover, the charges of the structure sheaf $\mathcal{O}_E$ and of the residue field $k(x)$ at a point $x \in E$ form the standard basis for the lattice $\mathbb{Z}^2$

$$Z(\mathcal{O}_E) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Z(k(x)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

When we need a matrix of an endomorphism of $K_0(E)/\text{rad}(\langle - , - \rangle)$ we compute it in the standard basis for the lattice $K_0(E)/\text{rad}(\langle - , - \rangle) \cong \mathbb{Z}^2$.

Any automorphism of the derived category $D^b(E)$ induces an automorphism of the Grothendieck group and thus an automorphism of $K_0(E)/\text{rad}(\langle - , - \rangle)$. Such automorphisms preserve the Euler form, therefore, we get a group homomorphism

$$\pi : \text{Aut}(D^b(E)) \to \text{SP}_2(\mathbb{Z}) \cong \text{SL}_2(\mathbb{Z}).$$

Moreover, we observe that on an elliptic curve $E$ the objects $\mathcal{O}_E$ and $k(x)$ are spherical, thus we can define two endofunctors:

$$\mathbb{A} = T_{\mathcal{O}_E}, \quad \mathbb{B} = T_{k(x)}.$$

Another description for the functor $\mathbb{B}$ was given in [31]:

**Lemma 2.4.3.** There is an isomorphism of functors $\mathbb{B} \cong \mathcal{O}_E(x) \otimes -$.

The images of these automorphisms in $\text{SL}_2(\mathbb{Z})$ can be easily computed:

$$A = \pi(\mathbb{A}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B = \pi(\mathbb{B}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The category $\text{MCM}_{\text{gr}}(R)$ has a natural autoequivalence – the shift of internal degree (1). By Orlov’s equivalence it induces some autoequivalence $\sigma$ of $D^b(E)$

$$\sigma = \Phi^{-1} \circ (1) \circ \Phi : D^b(E) \to D^b(E).$$
In [24] (lemma 4.2.1) the functor $\sigma$ was completely described. Here we use that description to express $\sigma$ in terms of the spherical twists $\mathcal{A}$ and $\mathcal{B}$. The result depends on the degree $n$ of the elliptic normal curve.

**Lemma 2.4.4.** There is an isomorphism of functors $\sigma \cong \mathcal{B}^n \circ \mathcal{A}$.

Generally speaking, Orlov’s equivalence depends on choosing a cutoff $i \in \mathbb{Z}$, thus $\sigma$ also depends on the cutoff parameter $i$. We choose the cutoff parameter $i = 0$ in this paper. Note that in [24] a formula for $\sigma$ is given for any $i \in \mathbb{Z}$. 
Chapter 3

The Main Formula for Graded Betti numbers

In this section $R$ is a commutative connected Gorenstein algebra $R$ over a field $k$. We assume that the Gorenstein parameter $a$ of $R$ is is zero, $a = 0$. Orlov’s equivalence

$$\Phi : D^b(\text{Proj}(R)) \rightarrow \text{MCM}_{gr}(R)$$

of triangulated categories can be used to answer questions about MCM modules in terms of the geometry of $\text{Proj}(R)$. In particular, we use it to derive a general formula for the graded Betti numbers of MCM modules.

Let $M$ be a finitely generated graded $R$-module, and let $P^* \rightarrow M$ be a minimal free resolution of $M$ over $R$

$$\cdots \rightarrow P^1 \rightarrow P^0 \rightarrow M \rightarrow 0,$$

where each term $P^i$ of the resolution is a direct sum of free modules with generators in various degrees

$$P^i \cong \bigoplus_j R(-j)^{\beta_{i,j}}.$$

The exponents $\beta_{i,j}$ are positive integers, that are called the (graded) Betti numbers. The Betti numbers contain information about the shape of a minimal free resolution.

We start with the simple observation that the graded Betti numbers are given by the formula

$$\beta_{i,j}(M) = \dim \text{Ext}^i(M, k(-j)).$$

The extension groups $\text{Ext}^i(M, k(-j))$ are isomorphic to the cohomology groups of the complex $\text{Hom}(P^*, k(-j))$, but the resolution $P^*$ is minimal, therefore, differentials in the complex $\text{Hom}(P^*, k(-j))$ vanish.

In Orlov’s equivalence we deal with a stable category of MCM modules, so our next step is to express the Betti numbers in terms of dimensions of stable extension groups.

Lemma 3.0.5. Let $M$ be a finitely generated MCM module. Then the graded Betti numbers are equal
to the dimensions of the following stable extension groups

\[ \beta_{i,j}(M) = \dim \Ext^i(M,k^{st}(-j)) , \]

for \( i \geq 0 \).

**Proof.** By Lemma 2.1.3

\[ \Ext^i(M,k(-j)) \cong \Ext^i(M,k(-j)) , \]

for \( i \geq 0 \) and any MCM module \( M \) without free summands. Next by Lemma 2.1.4 we have

\[ \Ext^i(M,k(-j)) = \Ext^i(M,k^{st}(-j)) . \]

Combining these two lemmas we get the result.

**Remarks.** The formula above also can be used to compute graded Betti numbers of an MCM module \( M \) for \( i < 0 \). This means that we replace the projective resolution of \( M \) by a complete minimal resolution

\[ \text{CR}(M)^* = \ldots \leftarrow P_{-2} \leftarrow P_{-1} \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots , \]

and we use such a complete resolution to extend the definition of \( \Ext^i(M,k(-j)) \) for \( i < 0 \).

This is a useful reformulation, because stable extension groups can be computed on the geometric side of Orlov’s equivalence. We have the following result:

**Theorem 3.0.6.** The graded Betti numbers of an MCM module \( M \) are given by

\[ \beta_{i,j}(M) = \dim \Hom_{D^b(\Proj(R))}(\Phi^{-1}(M), \sigma^{-j}(\Phi^{-1}(k^{st}))[i]) . \]

**Proof.** This formula follows immediately from the previous lemma and the definition of the functor \( \sigma : D^b(\Proj(R)) \to D^b(\Proj(R)) \).
Chapter 4

The Cone over a plane cubic

4.1 Betti numbers of MCM modules

Let us fix a smooth elliptic curve $E$ over an algebraically closed field $k$ with distinguished point $x \in E$, and choose the invertible sheaf $\mathcal{L} = \mathcal{O}_E(1) = \mathcal{O}(3x)$. The complete linear system of $\mathcal{L}$ provides an embedding of $E$ into the projective plane $\mathbb{P}(W^\vee)$:

$$E \subset \mathbb{P}(W^\vee),$$

where $W = H^0(E, \mathcal{L})$. The homogeneous coordinate ring of this embedding

$$R_E = \bigoplus_{i \geq 0} H^0(E, \mathcal{L}^\otimes i)$$

is a hypersurface ring given by a cubic polynomial $f$, namely $R_E = k[x_0, x_1, x_2]/(f)$.

Recall that the endofunctor $\sigma$ on the geometric side represents the endofunctor of internal degree shift (1) on the stable category of graded MCM modules

$$\sigma = \Phi^{-1} \circ (1) \circ \Phi.$$  

In the case of a plane cubic the functor $\sigma$ is the following composition of spherical twist functors:

$$\sigma = \mathbb{B}^3 \circ \mathbb{A},$$

where $\mathbb{B} = T_{k(x)} \cong \mathcal{O}_E(x) \otimes -, \mathbb{A} \cong T_{\mathcal{O}_E}$.

We start with the following elementary, but useful lemma.

**Lemma 4.1.1.** There is a natural isomorphism of functors

$$\sigma^3 \cong [2]$$

on the derived category $D^b(E)$ of coherent sheaves on the elliptic curve $E$. 

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Proof. It follows immediately from the corresponding natural isomorphism

\[(3) \cong [2]\]

of functors on the stable category $\text{MCM}_\sigma(R_E)$ of MCM modules over $R_E$.

The isomorphism $(3) \cong \text{cosyz}^2$ is a graded version of the statement that an MCM module over a hypersurface ring has a 2-periodic resolution.

Let us denote $\mathcal{F} = \Phi^{-1}(M)$, then the main formula for graded Betti numbers applied to the elliptic curve $E$ takes the following form.

**Theorem 4.1.2.** There is an isomorphism

$$\Phi^{-1}(k^{st}) \cong \mathcal{O}_E[1]$$

in the category $D^b(E)$. The graded Betti numbers $\beta_{i,j}$ of the MCM module $M = \Phi(\mathcal{F})$ are given by the formula

$$\beta_{i,j}(M) = \dim \text{Hom}_{D^b(E)}(\mathcal{F}, \sigma^{-j}(\mathcal{O}_E[1])[i]).$$

Moreover, the graded Betti numbers have the following periodicity property $\beta_{i+2,j} = \beta_{i,j+3}$.

**Proof.** We need to show that $\Phi(\mathcal{O}_E) = \text{st}(\bigoplus_{i \geq 0} R \text{Hom}(\mathcal{O}_E, \mathcal{O}_E(i))) \cong k^{st}[−1]$.

Set $C = \bigoplus_{i \geq 0} R \text{Hom}(\mathcal{O}_E, \mathcal{O}_E(i))$. By the Grothendieck vanishing theorem the complex $C$ has cohomology only in degrees 0 and 1:

$$H^0(C) \cong \bigoplus_{i \geq 0} \text{Hom}(\mathcal{O}_E, \mathcal{O}_E(i)) \cong \bigoplus_{i \geq 0} H^0(E, \mathcal{O}_E(i)) \cong R_E,$$

and

$$H^1(C) \cong \bigoplus_{i \geq 0} \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E(i)) \cong \bigoplus_{i \geq 0} H^1(E, \mathcal{O}_E(i)) \cong k,$$

where we use $H^1(E, \mathcal{O}_E(i)) \cong 0$ if $i > 0$ and $H^1(E, \mathcal{O}_E) \cong k$.

We have an embedding of the zeroth cohomology

$$0 \to H^0(C) \to C^0 \to C^1 \to \ldots$$

We treat this as a morphism of complexes

$$H^0(C) \to C \to C',$$

where $C'$ is defined as a cokernel of the map $H^0(C) \to C$ above. The complex $C'$ has only non-trivial cohomology in degree 1, namely $H^*(C') \cong k[-1]$, therefore, $C' \cong k[-1]$ in the category $D^b(\text{gr}R_{i \geq 0})$.

Thus we get a distinguished triangle in $D^b(\text{gr}R_{i \geq 0})$:

$$R_E \to C \to k[-1] \to R_E[1] \to \ldots$$
Next we apply the stabilization functor $\text{st} : D^b(\text{gr} R_{\geq i}) \to D^b_{\text{gr}}(R)$ to this distinguished triangle:

$$\text{st}(R_E) \to \text{st}(C) \to \text{st}(k[-1]) \to \text{st}(R_E)[1] \to \cdots$$

Noting that $\text{st}(R_E) \cong 0$ we get

$$\Phi(O_E) = \text{st}(C) \cong k^{\text{st}}[-1].$$

Therefore, the object $\Phi^{-1}(k^{\text{st}})$, that we have in the main formula can be computed explicitly for an elliptic curve:

$$\Phi^{-1}(k^{\text{st}}) \cong O_E[1].$$

The periodicity of Betti numbers $\beta_{i+2,j} = \beta_{i,j-3}$ follows from the isomorphism of functors $\cong \cosy$ on the stable category of graded MCM modules $\text{MCM}_{\text{gr}}(R_E)$.

The property of periodicity in the theorem above is well known and can be proved by more elementary methods. It implies that for computing Betti tables it is enough to compute the Betti numbers $\beta_{0,*}$ and $\beta_{1,*}$ and then to apply periodicity. Our next step is to show that not only the homological index $i$ can be restricted to $i = 0, 1$, but the window of non-zero values for the inner index $j$ is also very small.

For this purpose we compute the iterations of the functor $\sigma$ on $D^b(E)$. Set

$$V_j = \sigma^j(O_E[1]),$$

for $j \in \mathbb{Z}$.

We start with the simple observation that $\mathbb{A}(O_E) = O_E$, which implies $V_1 = O_E(1)[1]$. In the following lemma we explicitly compute $V_2$.

**Lemma 4.1.3.** For the object $V_2$ the following is true:

$$V_2 = \sigma(V_1) \cong K'[2],$$

where $K' \cong \Omega_{\mathbb{P}^2}(2)|_E$, in particular $K'$ is a locally free sheaf and its charge is

$$Z(K') = \left( \begin{array}{c} 2 \\ 3 \end{array} \right).$$

**Proof.** For $\mathbb{A}(V_1)$ we have an exact triangle

$$\mathbb{R}\text{Hom}(O_E, V_1) \otimes O_E \xrightarrow{\text{ev}} V_1 \to \mathbb{A}(V_1) \to \cdots$$

Note that $\mathbb{R}\text{Hom}(O_E, V_1) \otimes O_E \cong W \otimes O_E[1]$, the evaluation map ev is surjective, and we have a short exact sequence of locally free sheaves on the elliptic curve:

$$0 \to K \to W \otimes O_E \to O_E(3x) \to 0.$$ 

We see that

$$\mathbb{A}(V_1) = K[2],$$
where $K$ is a locally free sheaf, and it is easy to see that its charge is

$$Z(K) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$ 

Moreover, comparing short exact sequence above with the Euler sequence on $\mathbb{P}^2$

$$0 \to \Omega^1_{\mathbb{P}^2}(1) \to W \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}(1) \to 0,$$

we can conclude that $K \cong \Omega^1_{\mathbb{P}^2}(1)_{|_E}$. Therefore, for $V_2$ we have

$$V_2 = B^3(A(V_1)) \cong K \otimes \mathcal{O}_E(1)[2].$$

If we define $K'$ as

$$K' = K \otimes \mathcal{O}_E(1) \cong \Omega^1_{\mathbb{P}^2}(2)_{|_E},$$

the object $V_2$ is given by

$$V_2 = K'[2],$$

as desired. □

For the rest of this section we use shorthand notation $K'$ for $\Omega^1_{\mathbb{P}^2}(2)_{|_E}$.

Knowing $V_0$, $V_1$ and $V_2$ and using $\sigma^3 \cong [2]$, we can compute all other objects $\sigma^{-j}(\mathcal{O}_E[1]) = V_j$. For instance,

$$V_{-1} = \sigma^{-1}(\mathcal{O}_E[1]) = \sigma^{-1} \circ \sigma^3(\mathcal{O}_E[-1]) = \sigma^2(\mathcal{O}_E[-1]) = \sigma^2(\mathcal{O}_E[1])[-2] = V_2[-2] = K',$$

and

$$V_{-2} = \sigma^{-2}(\mathcal{O}_E[1]) = \sigma^{-2} \circ \sigma^3(\mathcal{O}_E[-1]) = \sigma(\mathcal{O}_E)[-1] = \mathcal{O}_E(3x)[-1].$$

We summarize our results in the following proposition.

**Proposition 4.1.4.** The objects $V_j = \sigma^j(\mathcal{O}_E[1])$ are completely determined by

$$V_0 = \mathcal{O}_E[1], \quad V_1 = \mathcal{O}_E(3x)[1], \quad V_2 = K'[2],$$

and

$$V_{3i+j} = V_j[2i],$$

for $i \in \mathbb{Z}$ and $j \in \{0, 1, 2\}$.

Explicit formulae for $V_j$ can be used to prove the following vanishing property of Betti numbers.

**Proposition 4.1.5.** If $F$ is an indecomposable object of $D^b(\mathcal{I})$ concentrated in cohomological degree $l$, then the only potentially non-vanishing Betti numbers $\beta_{0,*}$ are $\beta_{0,l-1}$, $\beta_{0,l}$, $\beta_{0,l+1}$, and the only potentially non-vanishing Betti numbers $\beta_{1,*}$ are $\beta_{1,l+1}$, $\beta_{1,l+2}$, $\beta_{1,l+3}$.

**Proof.** The proposition follows from the formulae for $V_j$ and the observation that the groups $\text{Hom}_{D^b(\mathcal{I})}(X, Y[i])$ for vector bundles $X$ and $Y$ on $E$ vanish unless $i = 0, 1$. □
Without loss of generality we can restrict our attention to the objects concentrated in cohomological degrees 0 or 1. Moreover, the Betti table of the module $M(1)$ is a shift of the Betti table of $M$. On the geometric side it means that it is enough to study Betti numbers up to the action of the functor $\sigma$. The action of $\sigma$ induces an action on $K_0(E)$ and $\mathbb{Z}^2 \cong K_0(E)/\text{rad}(\cdot,\cdot)$. The matrix of $[\sigma]$ in the standard basis $\{Z(O_E), Z(k(x))\}$ for $\mathbb{Z}^2 \cong K_0(E)/\text{rad}(\cdot,\cdot)$ is given by

$$[\sigma] = B^3 \circ A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}. $$

The action of the cyclic group of order 3 generated by $[\sigma]$ can be extended to $\mathbb{R}^2$, that contains $\mathbb{Z}^2 \cong K_0(E)/\text{rad}(\cdot,\cdot)$ as the standard lattice. For the real plane $\mathbb{R}^2$ we continue to use the coordinates $(r,d)$. A fundamental domain for this action can be chosen in the following form:

$$r > 0, \quad 0 \leq d < 3r.$$

These conditions are illustrated in the Figure 1 below. On it, dots represent points of the integer lattice $\mathbb{Z}^2 \cong K_0(E)/\text{rad}(\cdot,\cdot)$ in the $(r,d)$ plane. The line $d = 3r$ is dotted because it is not included in the fundamental domain. The meaning of the second line $3r = 2d$ in the fundamental domain and the double points on the lines $d = 0$ and $3r = 2d$ will be made clear in the next proposition.

![Figure 4.1](image)

Figure 4.1. Fundamental domain in $(r,d)$-plane.

In other words, if we are interested in all possibilities for Betti tables up to shift (1) it is enough to consider sheaves with charges in this fundamental domain. In particular, it is enough to consider only vector bundles on $E$. The graded Betti numbers of MCM modules can now be expressed as dimensions of cohomology groups of vector bundles on $E$. Computing dimensions of cohomology groups in the case at hand is a simple exercise.

**Proposition 4.1.6.** Let $F$ be an indecomposable vector bundle with charge $Z(F) = \begin{pmatrix} r \\ d \end{pmatrix}$ in the fundamental domain. For points on the two rays $d = 0$ and $3r = 2d$ inside the fundamental domain there are two possibilities for the Betti numbers, and for any other point in the fundamental domain the Betti numbers are completely determined by the discrete parameters $r$ and $d$. Moreover, Betti numbers of an indecomposable MCM module $\Phi(F)$ can be expressed as dimensions of cohomology groups on the elliptic
curve in the following way:

\[
\begin{align*}
\beta_{0,-1} &= 0, \\
\beta_{0,0} &= \begin{cases} 
1, & \text{if } d = 0 \text{ and } \mathcal{F} \cong F_r, \\
\frac{d}{2}, & \text{otherwise.}
\end{cases} \\
\beta_{0,1} &= \begin{cases} 
0, & \text{if } 3r < 2d, \\
1, & \text{if } 3r = 2d \text{ and } \mathcal{F} \cong \sigma^{-1}(F_{r/2}[1]), \\
3r - 2d, & \text{otherwise.}
\end{cases} \\
\beta_{1,1} &= \begin{cases} 
0, & \text{if } 3r > 2d, \\
1, & \text{if } 3r = 2d \text{ and } \mathcal{F} \cong \sigma^{-1}(F_{r/2}[1]), \\
2d - 3r, & \text{otherwise.}
\end{cases} \\
\beta_{1,2} &= 3r - d. \\
\beta_{1,3} &= \begin{cases} 
1, & \text{if } d = 0 \text{ and } \mathcal{F} \cong F_r, \\
0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

**Proof.** First we recall a well-known formula for the degree of a tensor product of two vector bundles \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) on a smooth projective curve:

\[
\deg(\mathcal{F}_1 \otimes \mathcal{F}_2) = \deg(\mathcal{F}_1) \cdot \text{rk}(\mathcal{F}_2) + \text{rk}(\mathcal{F}_1) \cdot \deg(\mathcal{F}_2).
\]

The computation of the Betti number \( \beta_{0,-1} \) is straightforward:

\[
\beta_{0,-1} = \dim \text{Hom}_{D^b(E)}(\mathcal{F}, \mathcal{O}_E(3x)[1]) = \dim H^1(E, \mathcal{F}^\vee \otimes \mathcal{O}_E(3x)) = 0,
\]

because \( \deg(\mathcal{F}^\vee \otimes \mathcal{O}_E(3x)) = 3r - d > 0 \), for a pair \((r, d)\) in the fundamental domain.

To compute \( \beta_{0,0} \) we use Serre duality and get two cases:

\[
\beta_{0,0} = \dim \text{Hom}_{D^b(E)}(\mathcal{F}, \mathcal{O}_E[1]) = \dim H^0(E, \mathcal{F}) = \begin{cases} 
1, & \text{if } d = 0 \text{ and } \mathcal{F} \cong F_r, \\
\frac{d}{2}, & \text{otherwise.}
\end{cases}
\]

By analogous arguments we get the Betti numbers \( \beta_{1,2} \) and \( \beta_{1,3} \):

\[
\beta_{1,2} = \dim \text{Hom}_{D^b(E)}(\mathcal{F}, \mathcal{O}_E(3x)) = \dim H^0(E, \mathcal{F}^\vee \otimes \mathcal{O}_E(3x)) = 3r - d,
\]

while for \( \beta_{1,3} \) we again get two cases:

\[
\beta_{1,3} = \dim \text{Hom}_{D^b(E)}(\mathcal{F}, \mathcal{O}_E) = \dim H^1(E, \mathcal{F}) = \begin{cases} 
1, & \text{if } d = 0 \text{ and } \mathcal{F} \cong F_r, \\
0, & \text{otherwise.}
\end{cases}
\]

In order to compute \( \beta_{0,1} \) we use the fact that the left adjoint functor for the autoequivalence \( \sigma^{-1} \) is \( \sigma \).

Thus,

\[
\beta_{0,1} = \dim \text{Hom}_{D^b(E)}(\mathcal{F}, \sigma^{-1}(\mathcal{O}_E[1])) = \dim \text{Hom}_{D^b(E)}(\sigma(\mathcal{F}), \mathcal{O}_E[1]).
\]
Chapter 4. The Cone over a plane cubic

In the next step we use the derived adjunction between the functor \( \text{Hom} \) and the tensor product

\[
\text{Hom}_{D^b(E)}(X \otimes Y, Z) \cong \text{Hom}_{D^b(E)}(X, Z^\vee \otimes Y),
\]

where \( X, Y \) and \( Z \) are objects in \( D^b(E) \) and \( Z^\vee = R\text{Hom}(Z, \mathcal{O}_E) \). Therefore,

\[
\beta_{0,1} = \dim \text{Hom}_{D^b(E)}(\mathcal{O}_E, \sigma(\mathcal{F})^\vee[1]).
\]

Rank \( r' \) and degree \( d' \) of \( \sigma(\mathcal{F})^\vee[1] \) can be computed as

\[
\left(\begin{array}{c}
r' \\
d'
\end{array}\right) = -\left(\begin{array}{c}
[\sigma] \\
d
\end{array}\right) \left(\begin{array}{c}
d - r \\
3r - 2d
\end{array}\right),
\]

where the operation \((-)^\vee\) is induced by the sheaf-dual \((-)^\vee\), it changes the sign of the second component of the charge, while it leaves invariant the first component.

We need to analyze several cases for \( \sigma(\mathcal{F})^\vee[1] \). If \( d' = 3r - 2d < 0 \), then \( r' = d - r > 0 \), and \( \sigma(\mathcal{F})^\vee[1] \) is a vector bundle of negative degree. Therefore,

\[
\beta_{0,1} = \dim \text{Hom}_{D^b(E)}(\mathcal{O}_E, \sigma(\mathcal{F})^\vee[1]) = \dim H^0(E, \sigma(\mathcal{F})^\vee[1]) = 0.
\]

If \( d' = 3r - 2d > 0 \), then \( r' = d - r \) is not necessarily positive. Thus \( \sigma(\mathcal{F})^\vee[1] \) is a vector bundle or translation of a vector bundle by \([1]\) depending on the sign of \( r' \). If \( \sigma(\mathcal{F})^\vee[1] \) is a vector bundle,

\[
\beta_{0,1} = \dim \text{Hom}_{D^b(E)}(\mathcal{O}_E, \sigma(\mathcal{F})^\vee[1]) = \dim H^0(E, \sigma(\mathcal{F})^\vee[1]) = d' = 3r - 2d.
\]

If \( \sigma(\mathcal{F})^\vee[1] \) is a translation of a vector bundle, in other words \( \sigma(\mathcal{F})^\vee \) is a vector bundle with \( \text{rk} \sigma(\mathcal{F})^\vee = -r' > 0 \), \( \deg \sigma(\mathcal{F})^\vee = -d' < 0 \). We obtain the same answer for \( \beta_{0,1} \) as in the previous case:

\[
\beta_{0,1} = \dim \text{Hom}_{D^b(E)}(\mathcal{O}_E, \sigma(\mathcal{F})^\vee[1]) = \dim H^1(E, \sigma(\mathcal{F})^\vee) = d' = 3r - 2d.
\]

Finally, we need to analyze the case \( d' = 3r - 2d = 0 \). The solutions of the diophantine equation \( 3r - 2d = 0 \) are of the form \( r = 2l, d = 3l \), where \( l \in \mathbb{Z} \), and we are only interested in positive solutions. Therefore, we consider only \( l \geq 1 \). In particular, \( \text{rk} \sigma(\mathcal{F})^\vee[1] = r' = 3l - 2l = l \).

If \( \sigma(\mathcal{F})^\vee[1] \) is isomorphic to the Atiyah bundle,

\[
\sigma(\mathcal{F})^\vee[1] \cong F_l,
\]

we have

\[
\beta_{0,1} = \dim \text{Hom}_{D^b(E)}(\mathcal{O}_E, F_l) = 1.
\]

The condition \( \sigma(\mathcal{F})^\vee[1] \cong F_l \) determines \( \mathcal{F} \) uniquely up to isomorphism. Indeed, this condition is equivalent to

\[
\sigma(\mathcal{F}) \cong (F_l[-1])^\vee \cong F_l[1],
\]

where the second isomorphism is based on the fact that Atiyah bundles are self-dual \( F_l^\vee \cong F_l \). Therefore, for \( \mathcal{F} \) we have

\[
\mathcal{F} \cong \sigma^{-1}(F_l[-1]).
\]
If $\sigma(F)^\vee[1]$ is not isomorphic to an Atiyah bundle, the Betti number $\beta_{0,1}$ vanishes $\beta_{0,1} = 3r - 2d = 0$. Combining all cases we get

$$\beta_{0,1} = \begin{cases} 
0, & \text{if } 3r < 2d, \\
1, & \text{if } 3r = 2d \text{ and } F \cong \sigma^{-1}(F_{r/2}[-1]), \\
3r - 2d, & \text{otherwise}.
\end{cases}$$

Analyzing in the same way all cases for $\beta_{1,1}$ we get the formula

$$\beta_{1,1} = \dim \text{Hom}_{D^b(E)}(F, V_{-1}[1]) = \begin{cases} 
0, & \text{if } 3r > 2d, \\
1, & \text{if } 3r = 2d \text{ and } F \cong \sigma^{-1}(F_{r/2}[-1]), \\
2d - 3r, & \text{otherwise}.
\end{cases}$$

Remarks. 1. The line $3r = 2d$ in Figure 1 above represents a line where the special cases $F \cong \sigma^{-1}(F_{r/2}[-1])$ can occur.

2. Encircled points in Figure 1 represent charges in the fundamental domain for which the Betti tables are not completely determined by the discrete parameters $r$ and $d$.

3. We denote $S_l = \sigma^{-1}(F_l[-1])$ the vector bundles corresponding to the special cases $3r = 2d$. Note that $\text{rk } S_l = 2l$, $\deg S_l = 3l$.

4. The condition $3r = 2d$ listed for $\beta_{0,1}$ and $\beta_{1,1}$ in the proposition is superfluous, as $F \cong \sigma^{-1}(F_{r/2}[-1])$ implies it. But we keep it for clarity of exposition.

The results of Proposition 4.1.6 can be expressed in a more readable form as Betti tables.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$i = 0$</th>
<th>$i = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\beta_{0,0}$</td>
<td>$\beta_{1,1}$</td>
</tr>
<tr>
<td>1</td>
<td>$\beta_{0,1}$</td>
<td>$\beta_{1,2}$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\beta_{1,3}$</td>
</tr>
</tbody>
</table>

Table 4.1.

We keep only the part of the Betti table corresponding to the cohomological indices $i = 0, 1$, as for other values of $i$ the table can be continued by 2-periodicity.

In the corollary below we continue to assume that the point $(r, d)$ is in the fundamental domain.

Corollary 4.1.7. There are four cases of Betti tables of indecomposable MCM modules over the ring $R_E$. We summarize these cases in the following table
In particular, we see that the discrete parameters of the vector bundle $F$ can be read off the Betti table of the MCM module $M = \Phi(F)$ in all four cases.

Proof. This follows immediately from the proposition 4.1.6.

The most convenient way to present the results is in the form of complete resolutions of indecomposable MCM modules. Numbers in the vertices of the diagrams below represent ranks of free modules occurring in the complete resolutions.

**Corollary 4.1.8.** The Betti diagram of a complete resolution of an indecomposable MCM module over the homogeneous coordinate ring $R_E$ has one of the following four forms, that occur in two discrete and two continuous families. On the diagrams below $[0]$ indicates cohomological degree $0$ of a complex.

1. The first discrete family $F_r$, where $r$ is a positive integer.

   ![Diagram 1](image1)

2. The second discrete family $S_l$, where $l$ is a positive integer.

   ![Diagram 2](image2)

3. The first continuous family $G_\lambda(r, d)$. Elements in the family $G_\lambda(r, d)$ are parameterized by a pair
of integers \((r,d)\), satisfying conditions \(r > 0, d \geq 0, 3r - 2d > 0\), and a point \(\lambda \in E\).

\[
\begin{align*}
\cdots & \leftarrow d \\
3r - 2d & \leftarrow 3r - d \leftarrow d \\
3r - 2d & \leftarrow 3r - d \leftarrow \cdots \\
\end{align*}
\]

4. The second continuous family \(H_\lambda(r,d)\). Elements in the family \(H_\lambda(r,d)\) are parameterized by a pair of integers \((r,d)\), satisfying conditions \(r > 0, 3r - d > 0, 3r - 2d \leq 0\), and a point \(\lambda \in E\).

\[
\begin{align*}
\cdots & \leftarrow d \leftarrow 2d - 3r \\
3r - d & \leftarrow d \leftarrow 2d - 3r \\
3r - d & \leftarrow \cdots \\
\end{align*}
\]

**Proof.** This is just a 2-periodic extension of the Betti tables we obtained above.

**Corollary 4.1.9.** The two discrete families and the two continuous families are syzygies of each other, respectively. More precisely,

\[
\text{syz} \Phi(F_r) \cong \Phi(S_r)(1),
\]

for the discrete families and

\[
\Phi(H_\lambda(r,d))(1) \cong \text{syz} \Phi(G_\lambda(d - r, 2d - 3r)).
\]

for the continuous families.

**Proof.** For the discrete families the given formula is the same as was used to define the vector bundles \(S_r = \sigma^{-1}(F_r[-1])\). For the continuous families we have the isomorphism of vector bundles

\[
\sigma(H_\lambda(r,d)[1]) = G_\lambda(d - r, 2d - 3r),
\]

which is equivalent to the second formula of the corollary.

**Remarks.** If we want to study Betti tables not only up to shifts in internal degree but also up to taking syzygies we can restrict to a smaller fundamental domain. More precisely, we need to change the fundamental domain of the cyclic group of order 3 generated by \([\sigma]\) to the fundamental domain of the
cyclic group of order 6 generated by \(-[\sigma]\). The latter fundamental domain can be chosen in the form
\[ r > 0, \]
\[ 3r > 2d \geq 0. \]

### 4.2 The Hilbert Series of MCM modules

Our next goal is to compute the Hilbert series \( H_M(t) = \sum \dim(M_i)t^i \) in all four cases. Let
\[ 0 \leftarrow M \leftarrow F \leftarrow F(-3) \leftarrow \ldots \]
be a minimal 2–periodic free resolution of the MCM module \( M \) over the ring \( R_E \). Then the Hilbert series of \( M \) can be computed as
\[
H_M(t) = \frac{1}{1-t^3} (H_F(t) - H_G(t)).
\]
We use the isomorphisms \( F \cong \oplus_j R(-j)^{\beta_{0,j}} \) and \( G \cong \oplus_j R(-j)^{\beta_{1,j}} \) to conclude that
\[
H_F(t) = \left( \sum_j \beta_{0,j}t^j \right) H_{R_E}(t),
\]
and
\[
H_G(t) = \left( \sum_j \beta_{1,j}t^j \right) H_{R_E}(t).
\]
It is easy to see that
\[
H_{R_E}(t) = \frac{1+t+t^2}{(1-t)^2}.
\]
Then the Hilbert series of \( M \) can be written as
\[
H_M(t) = H_{R_E}(t) \left( \sum_j \beta_{0,j}t^j - \sum_j \beta_{1,j}t^j \right) = \frac{1}{(1-t)^2} \left( \frac{B(t)}{1-t} \right) = \frac{1}{(1-t)^2} P(t),
\]
where
\[
B(t) = \sum_j \beta_{0,j}t^j - \sum_j \beta_{1,j}t^j.
\]
Note that \( B(1) = 0 \) and, therefore, \( P(t) = \frac{B(t)}{1-t} \) is a polynomial for a module \( M \) generated in positive degrees (\( P(t) \) is a Laurent polynomial in general). Moreover, the multiplicity of an MCM module \( M \) is given by
\[
e(M) = P(1).
\]
Consequently, we can read off the Betti tables the Hilbert series and, in particular, the multiplicity in all cases. The multiplicity of an MCM module \( M \) is related to the multiplicity of the ring \( e(M) = \text{rk}(M)e(R) \). Note that in this formula \( \text{rk} \) means rank of the module \( M \) (not to be confused with the rank of the corresponding vector bundle). In the case of the homogeneous coordinate ring of a smooth
cubic \( e(R) = 3 \), thus, ranks of MCM modules also can be extracted from the Hilbert series. The number of generators \( \mu(M) \) of a module \( M \) can be easily computed as a sum of Betti numbers

\[
\mu(M) = \sum_j \beta_{i,j},
\]

for any \( i \in \mathbb{Z} \). In particular, we can choose \( i = 0 \).

Therefore, the numerical invariants of an MCM module \( M = \Phi(\mathcal{F}) \) can be expressed in terms of the discrete invariants \( r = \text{rk}(\mathcal{F}) \) and \( d = \deg(\mathcal{F}) \) of the vector bundle \( \mathcal{F} \). We present the result in the form of a table:

<table>
<thead>
<tr>
<th>case</th>
<th>( P(t) )</th>
<th>( e(M) )</th>
<th>( \mu(M) )</th>
<th>( \text{rk}(M) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_r )</td>
<td>( 1 + t + 3rt + t^2 )</td>
<td>( 3r + 3 )</td>
<td>( 3r + 1 )</td>
<td>( r + 1 )</td>
</tr>
<tr>
<td>( S_l )</td>
<td>( 3l + 3lt )</td>
<td>( 6l )</td>
<td>( 3l + 1 )</td>
<td>( 2l )</td>
</tr>
<tr>
<td>( 3r - 2d \geq 0 )</td>
<td>( d + (3r - d)t )</td>
<td>( 3r )</td>
<td>( 3r - d )</td>
<td>( r )</td>
</tr>
<tr>
<td>( 3r - 2d &lt; 0 )</td>
<td>( d + (3r - d)t )</td>
<td>( 3r )</td>
<td>( d )</td>
<td>( r )</td>
</tr>
</tbody>
</table>

Table 4.3.

We finish this section with a discussion of maximally generated MCM modules or Ulrich modules over the ring \( R_E \). Recall that a module over a Cohen-Macaulay ring is Cohen-Macaulay if and only if \( e(M) = \text{length}(M/xM) \), for a maximal regular sequence \( x = (x_1, x_2, \ldots, x_n) \). Then we have

\[
\mu(M) = \text{length}(M/mM) \leq \text{length}(M/xM) = e(M),
\]

where \( m \) is the maximal ideal for the local case or the irrelevant ideal for the graded case. For details see [30] and [33].

MCM modules for which the upper bound is achieved are called maximally generated or Ulrich MCM modules. Existence of such modules over a ring \( R \) is usually a hard problem. Here we want to show that Orlov’s equivalence allows us to prove the existence almost immediately for the ring \( R_E \).

From the table above we see that if the charge of \( \mathcal{F} \) is in the fundamental domain, then \( M = \Phi(\mathcal{F}) \) corresponding to the third case and \( d = 0 \) is maximally generated, unless we get a module corresponding to the Atiyah bundle. But it is convenient to work with modules generated in degree 0 that projective cover of \( M \) looks like

\[
\mathcal{O}_E^d \to M \to 0.
\]

We can archive for this Ulrich modules if we consider modules on the other boundary of the fundamental domain \( d = 3r \) (boundary that we exclude from the domain).

**Proposition 4.2.1.** Let \( \mathcal{F} \) be an indecomposable vector bundle on an elliptic curve \( E \). If \( \deg(\mathcal{F}) = 3 \text{rk}(\mathcal{F}) \) and \( \mathcal{F} \) is not isomorphic to \( \sigma(F_r) \), then the MCM module \( M = \Phi(\mathcal{F}) \) is Ulrich and generated in degree 0.

Conversely, every indecomposable Ulrich module over \( R_E \) generated in degree 0 is isomorphic to \( \Phi(\mathcal{F}) \) for some indecomposable \( \mathcal{F} \) with \( \deg(\mathcal{F}) = 3 \text{rk}(\mathcal{F}) \).

**Proof.** The computation can be done in the same spirit as in the proof of the proposition 4.1.6.
Note that for an Ulrich MCM module the matrix factorization is given by a matrix with linear entries and by a matrix with quadratic entries, this observation also follows from a more general result of A. Beauville for spectral curves in the case of line bundles.
Chapter 5

Examples of Matrix factorizations of a smooth cubic

A complete description of matrix factorizations of rank one MCM modules over the Fermat cubic can be found in [25], and that for rank one MCM modules over a general elliptic curve in Weierstrass form in [18].

In this section we discuss several examples in order to illustrate our general formulae for graded Betti numbers. So far in the presentation of the ring $R_E = k[x_0, x_1, x_2]/(f)$ we did not specify the cubic polynomial $f$. In this section we choose $f$ to be in Hesse form:

$$f(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2.$$  

The following theorem is classical.

**Theorem 5.0.2.** Relative to suitable homogeneous coordinates, each non-singular cubic has an equation of the form

$$f(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2,$$

and the parameter $\psi$ satisfies $\psi^3 \neq 1$.

**Proof.** See, for example, p.293 in [9].

5.1 A matrix factorization of $\text{cosy}_z(k^{st})$

In this section we show that a matrix factorization for $\text{cosy}_z(k^{st})$ can be obtained as a Koszul matrix factorization. Let us recall that a Koszul matrix factorization $\{A, B\}$ of a polynomial $f$ of the form

$$f = \sum_{i=1}^l a_ib_i$$

is given as a tensor product of matrix factorizations [33]:

$$\{A, B\} = \{a_1, b_1\} \otimes \{a_2, b_2\} \otimes \ldots \otimes \{a_l, b_l\}.$$
For a Hesse cubic, choosing the simplest possible presentation as a sum of the products of linear and quadratic terms,
\[ x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = x_0(x_0^2) + x_1(x_1^2) + (x_2^2 - 3\psi x_0 x_1)x_2, \]
we get the following matrix factorization:
\[ \{A, B\} = \{x_0, x_0^2\} \otimes \{x_1, x_1\} \otimes \{x_2^2 - 3\psi x_0 x_1, x_2\}, \]
where the matrices \( A \) and \( B \) are given by the formulae
\[
A = \begin{pmatrix}
    x_0 & x_1 & -3\psi x_0 x_1 + x_2^2 & 0 \\
    -x_1 & x_0^2 & 0 & -3\psi x_0 x_1 + x_2^2 \\
    -x_2 & 0 & x_0^3 & -x_1^2 \\
    0 & -x_2 & x_1 & x_0
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
    x_0^2 & -x_1^2 & 3\psi x_0 x_1 - x_2^2 & 0 \\
    x_1 & x_0 & 0 & 3\psi x_0 x_1 - x_2^2 \\
    x_2 & 0 & x_0 & x_1^2 \\
    0 & x_2 & -x_1 & x_0^2
\end{pmatrix}.
\]
We see that this matrix factorization corresponds to the first case \( r = 1 \) in the discrete series \( F_r \). In other words, it is a matrix factorization of \( \Phi(\mathcal{O}_E) = \cosy(k^{st}) \).

### 5.2 Matrix factorizations of skyscraper sheaves of degree one

Let \( \lambda \in E \) be a closed point. By a skyscraper sheaf of degree one we mean a skyscraper sheaf of the form \( k(\lambda) = \mathcal{O}_{E,\lambda}/m(\lambda) \). We assume that the point \( \lambda \) can be obtained as intersection of \( E \) with two lines, given as zeroes of linear forms \( l_1 \) and \( l_2 \). In other words, we have an inclusion of ideals:
\[ (f) \subset (l_1, l_2). \]
This implies that there are quadratic forms \( f_1 \) and \( f_2 \) such that
\[ f = l_1 f_1 + l_2 f_2. \]
Let
\[ A = \begin{pmatrix} l_2 & f_1 \\ -l_1 & f_2 \end{pmatrix}. \]
It is easy to see that
\[ \text{coker } A \cong (k(\lambda)^{st}). \]
The second matrix \( B \) is determined by the matrix \( A \) if \( \det(A) = f \) because in such case \( B = fA^{-1} = \)
\[ (A) \]. Therefore, the matrix \( B \) is given as
\[
B = \begin{pmatrix}
  f_2 & -f_1 \\
  l_1 & l_2
\end{pmatrix}.
\]

Such matrix factorizations \( \{A, B\} \) are parameterized by the \( G_{\lambda}(1,1) \) family.

### 5.3 Moore matrices as Matrix Factorizations

We fix a line bundle \( \mathcal{L} \) on \( E \) of degree \( \deg \mathcal{L} = 3 \). The canonical evaluation morphism \( \text{Hom}(\mathcal{O}_E, \mathcal{L}) \otimes \mathcal{O}_E \to \mathcal{L} \) is epimorphic, so we get a short exact sequence of vector bundles on \( E \):
\[
0 \to \mathcal{K} \to \text{Hom}(\mathcal{O}_E, \mathcal{L}) \otimes \mathcal{O}_E \to \mathcal{L} \to 0,
\]
where \( \mathcal{K} \) is defined as the kernel of the evaluation map. Tensoring this short exact sequence with \( \mathcal{O}_E(1) \), we get
\[
0 \to \mathcal{K}(1) \to \text{Hom}(\mathcal{O}_E(1), \mathcal{L}(1)) \otimes \mathcal{O}_E(1) \to \mathcal{L}(1) \to 0.
\]

It is easy to see that the charge of the vector bundle \( \mathcal{K}(1) \) is
\[
Z(\mathcal{K}(1)) = \begin{pmatrix} 2 \\ 3 \end{pmatrix},
\]
thus \( H^1(E, \mathcal{K}(1)) \cong 0 \), and we get a short exact sequence of cohomology groups
\[
0 \to H^0(E, \mathcal{K}(1)) \to H^0(E) \otimes H^0(\mathcal{O}_E(1)) \to H^0(E, \mathcal{L}(1)) \to 0.
\]

In the previous section we introduced the notation \( W = H^0(\mathcal{O}_E(1)) \). Let us also denote \( U = H^0(E, \mathcal{L}) \) and \( V = H^0(E, \mathcal{K}(1)) \). The vector spaces \( U, V \) and \( W \) are three dimensional, as easily follows from the formula for dimensions of cohomology groups in terms of charges. The short exact sequence above thus yields the following map of vector spaces
\[
\phi : V \to U \otimes W;
\]
coming from an elliptic curve and two line bundles of degree three. Such maps were studied extensively under the name of elliptic tensors (see [5] and references therein). We want to describe this map in more concrete terms. For this we choose a basis in each of \( U, V \), and \( W \). To do this we need to use the action of the finite Heisenberg group \( H_3 \) on these spaces.

Recall that if \( \mathcal{F} \) is a vector bundle of degree \( n > 0 \), then \( (t_x)^* \mathcal{F} \cong \mathcal{F} \) if and only if \( x \in E[n] \), where \( t_x \) is a translation on \( x \) in the group structure of \( E \), and \( E[n] \) denotes the subgroup of points of order \( n \). The group \( E[n] \) can be described explicitly as
\[
E[n] \cong \mathbb{Z}_n \times \mathbb{Z}_n.
\]

Another way of stating this observation is that \( \mathcal{F} \) is an invariant vector bundle on \( E \) with respect to the natural \( E[n] \) action. But \( \mathcal{F} \) is not an equivariant vector bundle. It means that only some central
extension of $E[n]$ will act on the cohomology groups of $F$. The universal central extension of $\mathbb{Z}_n \times \mathbb{Z}_n$ by $\mathbb{Z}_n$ is called the Heisenberg group $H_n$.

$$0 \to \mathbb{Z}_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 0.$$ 

It is a finite nonabelian group of order $n^3$. We denote elements mapping to generators of $\mathbb{Z}_n \times \mathbb{Z}_n$ by $\sigma$ and $\tau$, and the generating central element by $\epsilon$. The center of the Heisenberg group is the cyclic subgroup generated by $\epsilon$. The Heisenberg group $H_n$ acts on the cohomology $H^0(E,F)$. If, for instance, $F$ is a line bundle, then $H^0(E,F)$ is isomorphic to a standard irreducible Schröedinger representation of $H_n$. For a detailed description of $H_n$, its representations and actions on the cohomology groups see [22].

The map $\phi$ constructed above is a map of $H_3$ representations. Moreover, we can think about this map as a map given by some $3 \times 3$ matrix representing a linear map from $V$ to $W$ with entries given by some linear forms in the variables $x_0, x_1, x_2$. Note that these coordinates (and thus some basis) for $W$ were fixed before, but we still have the freedom to choose bases for $V$ and $U$.

For the Schröedinger representation $V$ we choose a standard basis $\{e_i\}_{i \in \mathbb{Z}_3}$. In this basis the generators act as

$$\tau e_i = \rho^i e_i,$$
$$\sigma e_i = e_{i-1},$$

where $\rho$ is a primitive third root of unity in the field $k$.

The images of the vectors $e_i$ are vectors $v_i$ that satisfy $\tau v_i = \rho^i v_i$ and $\sigma v_i = v_{i-1}$. Then it is clear that such vectors are of the form

$$v_0 = \begin{pmatrix} a_0 x_0 \\ a_2 x_2 \\ a_1 x_2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} a_2 x_2 \\ a_1 x_0 \\ a_0 x_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_1 x_1 \\ a_0 x_2 \\ a_2 x_0 \end{pmatrix}.$$

Our notation requires some explanation. The three vectors $v_0, v_1$ and $v_2$ represent elements of $U \otimes W$, if we fix a basis in $U$ and interpret tensors in $U \otimes W$ as column vectors with entries from $W$. We treat the map $V \to U \otimes W$ using the same logic as a $3 \times 3$ matrix with entries from $W$. Explicitly, in the chosen bases, the map $V \to U \otimes W$ has the matrix

$$A = \begin{pmatrix} a_0 x_0 & a_2 x_2 & a_1 x_1 \\ a_2 x_1 & a_1 x_0 & a_0 x_2 \\ a_1 x_2 & a_0 x_1 & a_2 x_0 \end{pmatrix}.$$ 

We get thus an explicit formula for the mapping on the level of zeroth cohomology, extending it to the morphism of sheaves we get

$$0 \to \mathcal{K}(1) \xrightarrow{A(1)} \text{Hom}(\mathcal{O}_E, \mathcal{L}) \otimes \mathcal{O}_E(1) \to \mathcal{L}(1) \to 0,$$

or, equivalently,

$$0 \to \mathcal{K} \xrightarrow{A} \text{Hom}(\mathcal{O}_E, \mathcal{L}) \otimes \mathcal{O}_E \to \mathcal{L} \to 0,$$
where, of course, $A$ and $A(1)$ coincide as matrices.

There is an alternative, more geometric, interpretation of the matrix $A$. Sections of the two line bundles $\mathcal{O}_E(1)$ and $\mathcal{L}$ provide an embedding of the elliptic curve $E$ into the multi-projective space $\mathbb{P}(W) \times \mathbb{P}(U)$

![Diagram of the embedding](image)

We want to find equations of the elliptic curve $E$ in the multi-projective space $\mathbb{P}(U) \times \mathbb{P}(W)$. Equations of bi-degree $(1, 1)$ span the kernel of the map

$$\ldots \to H^0(\mathbb{P}(U) \times \mathbb{P}(W), \mathcal{O}(1, 1)) \to H^0(E, \phi^*\mathcal{O}(1, 1)) \to 0.$$ 

Noting that $\phi^*\mathcal{O}(1, 1) \cong \mathcal{O}_E(1) \otimes \mathcal{L} \cong \mathcal{L}(1)$ and $H^0(\mathbb{P}(U) \times \mathbb{P}(W), \mathcal{O}(1, 1)) \cong U \otimes W$, we can conclude that we have the same short exact sequence as before

$$0 \to V \to U \otimes W \to H^0(E, \mathcal{L}(1)) \to 0.$$ 

In other words the same map $V \to U \otimes W$ describes $(1, 1)$ equations of the elliptic curve $E$ in the multi-projective space $\mathbb{P}(U) \times \mathbb{P}(W)$, but this time we interpret this map differently: a basis vector of $V$ is mapped to a tensor in $U \otimes W$ which is one of the equations of $E$. It is easy to check that equations of bi-degree $(1, 1)$ generate the bi-graded ideal of $E$. In this context the matrix $A$ appears in [1], where it was shown that the parameters $a_0$, $a_1$ and $a_2$ appearing in $A$ should be interpreted as projective coordinates of a point $a$ on $E$ embedded into $\mathbb{P}(W)$. Moreover, it was shown that equations with parameter $a$ correspond to the situation where the embeddings $p_2\phi$ and $p_1\phi$ are related as follows

$$p_1\phi = p_2\phi t_a.$$ 

Equivalently, we can say that $\mathcal{L} \cong (t_a)^*\mathcal{O}_E(1)$. If we choose $x \in E$ as the origin of that abelian variety and write $\mathcal{O}_E(1) \cong \mathcal{O}_E(3x)$ then $\mathcal{L} \cong \mathcal{O}_E(3a)$. This is a serious disadvantage of the method: instead of parametrizing matrices $A$ by points on the elliptic curve $E$ we parametrize them by points of the degree nine isogeny:

$$E \to E$$

$$a \mapsto 3a$$

In particular, it means that if we add any of the 9 points of order 3 on $E$ to $a$ we get a different matrix $A$ that corresponds to the same line bundle $\mathcal{L} \cong \mathcal{O}_E(3a)$.

Finally, we are going to find a complement matrix $B$ such that the pair $(A, B)$ forms a matrix factorization of $f$:

$$AB = BA = f \text{Id}.$$
We return to the short exact sequence

\[ 0 \rightarrow \mathcal{K} \xrightarrow{A} \text{Hom}(\mathcal{O}_E, \mathcal{L}) \otimes \mathcal{O}_E \rightarrow \mathcal{L} \rightarrow 0, \]

and use again the evaluation map, this time for \( K \). We get

\[ \ldots \rightarrow \text{Hom}(\mathcal{O}_E, \mathcal{K}(1)) \otimes \mathcal{O}_E \rightarrow \mathcal{K}(1) \rightarrow 0. \]

Thus \( A \) is the first map in the resolution of \( \mathcal{L} \)

\[ \ldots \rightarrow \text{Hom}(\mathcal{O}_E, \mathcal{K}(1)) \otimes \mathcal{O}_E(-1) \xrightarrow{A} \text{Hom}(\mathcal{O}_E, \mathcal{L}(1)) \otimes \mathcal{O}_E \rightarrow \mathcal{L} \rightarrow 0. \]

This resolution is 2-periodic, next map is given by a matrix \( B \), but to get an explicit formula for \( B \) we use a different approach.

The matrix \( B \) can be computed using the identity

\[ AB = \det(A) = a_0a_1a_2(x_0^3 + x_1^3 + x_2^3) - (a_0^3 + a_1^3 + a_2^3)x_0x_1x_2, \]

if we assume, as before, that \( a = [a_0 : a_1 : a_2] \) is a point of the elliptic curve \( E \subset \mathbb{P}(W) \) and \( a_0a_1a_2 \neq 0 \). The last condition means that \( a \) is not a point of order 3 that is \( a \notin E[3] \), indeed that would mean \( \mathcal{L} \cong \mathcal{O}_E(3a) \cong \mathcal{O}_E(1). \) Therefore, for \( B = \text{Adj}(A) \) we can use the following formula

\[
B = \frac{1}{a_0a_1a_2}
\begin{pmatrix}
    a_1a_2x_0^2 - a_0^2x_1x_2 & a_0a_1x_1^2 - a_0^2x_0x_2 & a_0a_2x_2^2 - a_1^2x_0x_1 \\
    a_0a_1x_2^2 - a_0^2x_0x_1 & a_0a_2x_0^2 - a_1^2x_1x_2 & a_1a_2x_1^2 - a_2^2x_0x_2 \\
    a_0a_2x_1^2 - a_1^2x_0x_2 & a_1a_2x_2^2 - a_0^2x_1x_1 & a_0a_1x_0^2 - a_2^2x_1x_2
\end{pmatrix}.
\]

These two matrices provide a matrix factorization of the MCM module \( \Phi(\mathcal{L}) \). Note that the charge of \( \mathcal{L} \) is not included in our fundamental domain, but the matrix factorization of a line bundle of degree 0 will only differ by a shift in internal degree (3). Therefore, we can consider this matrix factorization as an example that corresponds to the point (1,0) in the fundamental domain. We summarize our discussion in the following

**Theorem 5.3.1.** A matrix factorization of the MCM module \( \Phi(\mathcal{O}_E(3a)) \) is given by the matrix

\[
A = \begin{pmatrix}
    a_0x_0 & a_2x_2 & a_1x_1 \\
    a_2x_1 & a_1x_0 & a_0x_2 \\
    a_1x_2 & a_0x_1 & a_2x_0
\end{pmatrix},
\]

and the matrix

\[
B = \frac{1}{a_0a_1a_2}
\begin{pmatrix}
    a_1a_2x_0^2 - a_0^2x_1x_2 & a_0a_1x_1^2 - a_0^2x_0x_2 & a_0a_2x_2^2 - a_1^2x_0x_1 \\
    a_0a_1x_2^2 - a_0^2x_0x_1 & a_0a_2x_0^2 - a_1^2x_1x_2 & a_1a_2x_1^2 - a_2^2x_0x_2 \\
    a_0a_2x_1^2 - a_1^2x_0x_2 & a_1a_2x_2^2 - a_0^2x_1x_1 & a_0a_1x_0^2 - a_2^2x_1x_2
\end{pmatrix},
\]

where \( a = [a_0 : a_1 : a_2] \in E \subset \mathbb{P}(W) \), and \( a \) is not a point of order 3 on \( E \) as abelian variety with \([0 : -1 : 1]\) as origin.

We want to emphasize that we get such a nice description for these matrix factorizations when the
line bundle $\mathcal{L}$ is described as the pullback along translation by $a$ that is $\mathcal{L} \cong (t_a)^*\mathcal{O}_E(1)$. On one hand, the translation parameter $a$ is determined only up to addition of a point of order 3 and a more natural parametrization of a line bundle of degree 3 would be $\mathcal{O}_E(2x + p)$, where $x$ is the origin and $p \in E$ is some parameter. On the other hand, this very description in terms of translation is used in the majority of the literature on abelian varieties, where in general the parameter $a$ is called a characteristic.

Though it is certainly possible to explicitly compute matrix factorizations of the Hesse cubic corresponding to the line bundle $\mathcal{O}_E(2x + p)$, using $p$ as a parameter instead of $a$, the formulas we were able to get are much more complicated. Besides, the question formulated for $\mathcal{O}_E(2x + p)$ does not look like a natural one for the Hesse canonical form.

Matrices of the same type as $A$ above are known as Moore matrices (the matrix $A$ above is an example of a $3 \times 3$ Moore matrix). They have already appeared in the literature in a variety of contexts: to describe equations of projective embeddings of elliptic curves see [19]; to give an explicit formula for the group operation on a cubic in Hesse form see [17] and [28]; as differential in a projective resolution of the field over elliptic algebras see [1]. Although the matrices $A$ and $B$ above are known to be a matrix factorization of a Hesse cubic cf. [10] example 3.6.5], their relation to line bundles of degree 3 via representations of the Hesse canonical form seems never to have appeared in the literature before.

### 5.4 Explicit Matrix Factorizations of skyscraper sheaves

We want to indicate that explicit formulae of the same type as for Moore matrices can be also obtained for skyscraper sheaves of degree 1. For the next example we choose a point on the Hesse cubic $a = [a_0 : a_1 : a_2] \in V(f)$, so that we have an inclusion of ideals $(f) \subset (a_1x_2 - a_2x_1, a_0x_1 - a_1x_0).$ The first matrix in a matrix factorization $\{A, B\}$ of a skyscraper sheaf $k(a) = \mathcal{O}_a/m_a$ can be explicitly written as

$$A = \begin{pmatrix} a_1x_2 - a_2x_1 & a_0x_1 - a_1x_0 \\ a_0a_2x_0^2 + a_2^2x_0x_2 - a_1^2x_1x_2 - a_2^2x_2^2 & a_0a_2x_2 + a_0a_2x_0^2 - a_0a_1x_1^2 \end{pmatrix},$$

where the forms $a_1x_2 - a_2x_1$ and $a_0x_1 - a_1x_0$ cut out the point $a$ on $E$. The matrix $B$ can be computed by the formula

$$(a_0a_1a_2)AB = \det(A) = a_0a_1a_2(x_0^3 + x_1^3 + x_2^3) - (a_0^3 + a_1^3 + a_2^3)x_0x_1x_2,$$

if we assume that $a_0a_1a_2 \neq 0$. Last condition means that $a$ is not one of the inflection points of the Hesse pencil. The matrix $B$ is given by the following formula:

$$B = \frac{1}{a_0a_1a_2} \begin{pmatrix} a_0^2x_0^2 + a_0a_1x_1^2 - a_2^2x_0x_2 - a_0a_2x_2^2 & a_0x_0 + a_0x_1 \\ a_0a_2x_0^2 + a_0^2x_0x_2 - a_1^2x_1x_2 - a_2^2x_2^2 & a_2x_1 - a_1x_2 \end{pmatrix}.$$ 

### 5.5 Determinantal presentations of a Weierstrass cubic

In this sections we choose a smooth cubic in $\mathbb{P}^2$ in Weierstrass form

$$R = k[x, y, z]/(f),$$
where \( f(x, y, z) = y^2z - x^3 - axz^2 - bz^3 \), and \( a, b \in k \) are parameters. We assume that the curve \( E = \text{Proj}(R) \) is smooth or equivalently that the parameters satisfy the condition \(-4a^3 - 27b^2 \neq 0\).

Determinantal presentations of \( f \) are special cases of matrix factorizations: if \( \det(A) = f \) then for \( B = \text{adj}(A) \) we have \( AB = BA = f \). By definition of a determinantal presentation the entries of the matrix \( A \) are linear, therefore, determinantal presentations are in bijection with Ulrich line bundles. Thus, we can say that the goal of this section is to find matrix factorizations of Ulrich line bundles over the Weiestrass cubic.

First we briefly recall a general procedure of finding determinantal presentations of plane curves, with all details to be found in [13]. Let \( L \) and \( M \) be vector bundles of degree 3 on \( E \) such that \( L \otimes M \cong O_E(2) \) and \( L \not\cong O_E(1) \not\cong M \).

Such a pair of line bundles gives us a pair of embeddings of \( E \) into a projective plane:

\[
l : E \to \mathbb{P}(U), \quad \text{and} \quad r : E \to \mathbb{P}(V),
\]

where \( U = H^0(E, L)^\vee \) and \( V = H^0(E, M)^\vee \).

Composing these maps with the Segre embedding \( s_2 : \mathbb{P}(U) \times \mathbb{P}(V) \to \mathbb{P}(U \otimes V) \) we get

\[
\psi : E \xrightarrow{(l,r)} \mathbb{P}(U) \times \mathbb{P}(V) \xrightarrow{s_2} \mathbb{P}(U \otimes V).
\]

**Theorem 5.5.1.** The map \( \psi \) can be extended uniquely to \( \mathbb{P}^2 \) and the extended map is represented uniquely up to multiplication by a nonzero scalar by the matrix \( B \) of a matrix factorization,

\[
B : \mathbb{P}^2 \to \mathbb{P}(U \otimes V), \quad \text{and} \quad B|_E = \psi.
\]

For smooth cubics this general recipe immediately gives explicit answers. Any line bundle \( L \) of degree 3 on \( E \) can be uniquely presented as

\[
L \cong O_E(2x + p),
\]

where \( x \in E \) is a point playing the role of the origin in \( E \) as an abelian variety, and \( p \in E \) is a point playing the role of a moduli parameter. The condition \( L \not\cong O_E(1) \) is equivalent to \( p \neq x \). Note that, for a Weiestrass cubics there are no reasons to use a characteristic \( a \) and to present \( L \) as \( O_E(3a) \) that we used extensively for a Hesse cubic. The second line bundle \( M \) is determined from the equation \( L \otimes M \cong O_E(2) \) as follows

\[
M \cong O_E(2) \otimes L^{-1} \cong O_E(6x - 2x - p) \cong O_E(2x + (\oplus p)),
\]

where we used Abel’s theorem and \( \oplus p \) stands for the inversion of \( p \) in the abelian variety \( E \).

We choose the homogeneous coordinates of the origin \( x \in E \subset \mathbb{P}^2 \) to be \([0 : 1 : 0]\). If we choose homogeneous coordinates of \( p \in E \subset \mathbb{P}^2 \) as \([\lambda : \mu : 1]\) then the homogeneous coordinates of \( \oplus p \) are \([\lambda : -\mu : 1]\).

The line bundle \( O_E(1) \) has \( x, y \) and \( z \) as a basis of global sections. Then it is easy to see that the
Changing sections of Chapter 5. Examples of Matrix factorizations of a smooth cubic

Rational functions $\frac{y}{z}$, $\frac{y+\mu z}{x-\lambda z}$ and 1 restricted from $\mathbb{P}^2$ to $E$ form a basis of global sections of $L$ on $E$. Changing $\mu$ to $-\mu$ in these formulas we obtain a basis of global sections of $M$.

The map $l$ is given in such a basis by

$$l([x : y : z]) = \left[\frac{y+\mu z}{x-\lambda z} : \frac{x}{z} : 1\right] = [z(y+\mu z) : x(x-\lambda z) : z(x-\lambda z)],$$

where $[x : y : z] \in E$. Changing $\mu$ to $-\mu$ we get a similar formula for $r$.

**Remarks.** In the analytical theory of elliptic curves the Weierstrass form is intimately connected with the Weierstrass $\wp$-function. In fact the functions 1, $\wp(z)$ and $\wp'(z)$ form a basis of global sections of $\mathcal{O}_E(1)$ used to embed $E$ into the projective plane. Now, sections of $L$ are meromorphic functions that have a pole of order at most 2 at the origin and possibly a pole at $p$, so we can choose a basis of global sections of $L$ as 1, $\wp(z)$ and $\frac{\wp'(z)+\wp'(p)}{\wp(z)-\wp(p)}$. One can check that this approach gives the same basis of global sections of $L$.

The matrix $B$ has rank 1 and by the previous theorem it can be computed as the Segre matrix

$$B = \begin{pmatrix}
  z(y-\mu z) \\
  x(x-\lambda z) \\
  z(x-\lambda z)
\end{pmatrix} \begin{pmatrix}
  z(y+\mu z) & x(x-\lambda z) & z(x-\lambda z) \\
  \frac{z^2(y^2-\mu^2 z^2)}{xz(x-\lambda z)(y-\mu z)} & \frac{z^2(y-\mu z)(x-\lambda z)}{xz(x-\lambda z)} & \frac{z^2(y-\mu z)(x-\lambda z)}{xz(x-\lambda z)} \\
  \frac{z^2(x-\lambda z)(y+\mu z)}{xz(x-\lambda z)} & \frac{x^2(x-\lambda z)}{xz(x-\lambda z)} & \frac{z^2(x-\lambda z)^2}{xz(x-\lambda z)}
\end{pmatrix} =
\begin{pmatrix}
  z(x-\lambda z) & \left(\frac{x^2+\lambda xz+(a+\lambda^2)z^2}{x(y+\mu z)}\right) & \left(\frac{x(y-\mu z)}{z(y+\mu z)}\right) & \left(\frac{z(y-\mu z)}{z(x+\lambda z)}\right) \\
  \left(\frac{x^2+\lambda xz+(a+\lambda^2)z^2}{x(y+\mu z)}\right) & \left(\frac{y^2-axz-bz^2}{x(x-\lambda z)}\right) & \left(\frac{y^2-axz-bz^2}{x(x-\lambda z)}\right) & \left(\frac{z(x-\lambda z)}{z(x-\lambda z)}\right)
\end{pmatrix}.$$

In the last equality we use the equation of the cubic. The common factor $z(x-\lambda z)$ does not play any essential role, as we consider projective embeddings. Thus, we set

$$B = -\begin{pmatrix}
  \frac{x^2+\lambda xz+(a+\lambda^2)z^2}{x(y+\mu z)} & \frac{x(y-\mu z)}{z(y+\mu z)} & \frac{z(y-\mu z)}{z(x+\lambda z)} \\
  \frac{x(y+\mu z)}{y^2-axz-bz^2} & \frac{y^2-axz-bz^2}{x(x-\lambda z)} & \frac{z(x-\lambda z)}{z(x-\lambda z)}
\end{pmatrix}.$$

One can easily check that $\det(B) = f^2$.

For the matrix $A$ we have

$$A = \frac{1}{f} \text{adj}(B) = \begin{pmatrix}
  \frac{\lambda^2 z + \lambda x + az}{-x} & -x & \mu z - y \\
  -x & -z & 0 \\
  -\mu z - y & 0 & -\lambda z + x
\end{pmatrix}.$$

Similar formulas (up to elementary row and column operations on the matrices $A$ and $B$) were obtained by L. Galinat [18] by a different method.
Chapter 6

The cone over an elliptic normal curve of degree $n > 3$

6.1 Betti numbers of MCM modules

In this section we generalize the formulae for Betti numbers for arbitrary embeddings of elliptic curves. We fix a line bundle $\mathcal{L} = \mathcal{O}_E(1) = \mathcal{O}(nx)$, where $n > 3$. The goal of this section is to describe Betti numbers of MCM modules over the homogeneous coordinate ring

$$R_E = \bigoplus_{i \geq 0} H^0(E, \mathcal{L}^i),$$

of $E$ embedded in $\mathbb{P}^{n-1}$ as an elliptic normal curve.

As for the case of a plane cubic we start with the general formula for Betti numbers

$$\beta_{i,j}(\Phi(\mathcal{F})) = \dim \text{Hom}_{D^b(E)}(\mathcal{F}, V_{-j}[i]),$$

where as before we set $V_j = \sigma^j(\mathcal{O}_E[1])$. Let us express object $V_j$ as the translation of an indecomposable sheaf. We consider separately two cases, $j > 0$ and $j \leq 0$. For $j > 0$ it is easy to see that

$$V_j = K_j[j],$$

where $K_j$ is a sheaf. But for $j \leq 0$ the translation is different:

$$V_j = K_j[j + 1],$$

where again $K_j$ is a sheaf. For instance, $V_0 \cong \mathcal{O}_E[1]$, hence $K_0 = \mathcal{O}_E$ and $V_1 \cong \mathcal{O}_E(1)[1]$, hence $K_1 = \mathcal{O}_E(1)$.

The properties of the sheaves $K_j$ for $j \in \mathbb{Z}$ are summarized in the following proposition.

**Proposition 6.1.1.** The object $K_j$ is a vector bundle over elliptic curve $E$. If we denote the charge
$Z(K_j) = \begin{pmatrix} r_j \\ d_j \end{pmatrix}$, then for $j > 0$ rank and degree satisfy the recursion formulae

\[ r_{j+1} = (n-2)r_j - r_{j-1}, \]
\[ d_{j+1} = (n-2)d_j - d_{j-1}, \]

and the sequences start with $r_1 = 1$, $r_2 = n - 1$, $d_1 = n$ and $d_2 = n^2 - 2n$. For $j \leq 0$ we have the same recursion formulae:

\[ r_{j-1} = (n-2)r_j - r_{j+1}, \]
\[ d_{j-1} = (n-2)d_j - d_{j+1}, \]

and these sequences start with $r_0 = 1$, $r_{-1} = n - 1$, $d_0 = 0$ and $d_{-1} = n$. In particular if $j > 0$, $\deg(K_j) = \deg(K_{-j})$, and $\rk(K_{j+1}) = \rk(K_{-j})$.

Proof. Note that $\sigma = B^n \circ \mathbb{A}$, and the induced transformation of $K_0(E)/\rad(-,-)$ is

\[ c_n = B^n \circ A = \begin{pmatrix} 1 & -1 \\ n & 1 - n \end{pmatrix}. \]

First consider the case $j > 0$ and use induction, where the base $j = 1$ of the induction is clear. We see that $r_1 = 1$ and $d_1 = n$, and we can also directly compute $r_2$ and $d_2$.

\[ \begin{pmatrix} r_2 \\ d_2 \end{pmatrix} = Z(K_2) = Z(V_2) = c_n Z(V_1) = c_n \begin{pmatrix} -1 \\ -n \end{pmatrix} = \begin{pmatrix} n - 1 \\ n^2 - 2n \end{pmatrix}. \]

Consequently, $d_1 > 0$ and it is easy to check that

\[ \frac{n - 2 + \sqrt{n^2 - 4n}}{2} d_2 > d_1. \]

Lemma 6.1.2 below guarantees that $d_j$ is an increasing sequence of positive numbers.

Now we are ready to prove the induction step. Suppose that the statement is true for $j$, then we have an exact triangle

\[ \text{RHom}(\mathcal{O}_E, K_j) \otimes \mathcal{O}_E \to K_j \to C_{j+1} \to \ldots, \]

where $C_{j+1}$ is a cone of the evaluation map. By the induction step we know that the degree of $K_j$ is high enough so that

\[ \text{RHom}(\mathcal{O}_E, K_j) \cong H^0(E, K_j)[0]. \]

Moreover, by [3, Lemma 8] the evaluation map is surjective, therefore, there is an exact sequence

\[ 0 \to K'_{j+1} \to H^0(E, K_j) \otimes \mathcal{O}_E \to K_j \to 0 \]

of vector bundles on $E$. This implies that $C_{j+1} \cong K'_{j+1}[1]$ is the translation of a vector bundle, but

\[ V_{j+1} = C_{j+1} \otimes \mathcal{O}_E(nx)[j] \cong K'_{j+1} \otimes \mathcal{O}_E(1)[j+1] = K_{j+1}[j+1], \]
where we define $K_{j+1}$ as $K_{j+1} = K'_{j+1} \otimes O_E(1)$.

Observe that
\[ Z(V_j) = (-1)^j Z(K_j). \]

Therefore,
\[
\begin{pmatrix} r_j \\ d_j \end{pmatrix} = (-c_n)^j \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -n & n-1 \end{pmatrix}^j \begin{pmatrix} -1 \\ 0 \end{pmatrix},
\]
that is equivalent to the given recursion formula.

The case of $j \leq 0$ is completely analogous, but we use a dual sequence of vector bundles on the induction step and the recursion formula follows from
\[
\begin{pmatrix} r_{-j} \\ d_{-j} \end{pmatrix} = (-c_n^{-1})^j \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} n-1 & -1 \\ n & -1 \end{pmatrix}^j \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

\[ \square \]

Note that although for $j \leq 0$ and $j > 0$ the recursive sequences are given by the same formulae we can not unify them into one sequence because of the difference in the initial values.

Lemma 6.1.2. Let $\{s_j\}_{j=1}^\infty$ be a recursive sequence defined by
\[ s_{j+1} = (n-2)s_j - s_{j-1}, \]
and denote $\mu = \frac{n^2 - 4n}{2}$. Then
\begin{enumerate}
\item $s_1 > 0$ and $\mu s_2 - s_1 \geq 0$ if and only if the sequence $\{s_j\}$ is an increasing sequence of positive numbers.
\item $s_1 < 0$ and $\mu s_2 - s_1 \leq 0$ if and only if the sequence $\{s_j\}$ is a decreasing sequence of negative numbers.
\end{enumerate}

Proof. The characteristic equation of the recursive sequence is
\[ \lambda^2 - (n-2)\lambda + 1 = 0. \]

The roots of the equation are $\mu = \frac{n^2 - 4n}{2} > 1$ and $\mu^{-1} < 1$ if $n > 4$, while for $n = 4$ the equation has a root of multiplicity two at the point $\lambda = 1$. If $n > 4$ the general solution of the recursive sequence is given by the formula
\[ s_j = A\mu^j + B\mu^{-j}, \]
where $A$ and $B$ are some constants uniquely determined by $s_1$ and $s_2$ by the following formulae:
\[
A = \frac{\mu s_2 - s_1}{\mu^2 - 1},
B = \frac{\mu^2 s_1 - s_2}{\mu^2 - 1}.
\]

If $n = 4$, then the general solution is
\[ s_j = Aj + B, \]
where \( A = s_2 - s_1, B = 2s_1 - s_2 \). Now the lemma easily follows from these explicit formulae.

When we study Betti tables of graded modules it is enough to describe all possibilities up to a shift of internal degree \((1)\). Therefore, the following proposition is very useful, because it shows that it is enough to consider vector bundles and their translations on \( E \).

**Proposition 6.1.3.** Every indecomposable MCM module over \( R_E \) after some shift \((1)\) corresponds under Orlov’s equivalence \( \Phi : D^b(E) \to MCM_{gr}(R_E) \) to an indecomposable vector bundle, possibly translated into some cohomological degree.

**Proof.** The classification of the indecomposable objects of \( D^b(E) \) implies that it is enough to show that in the orbit of the functor \( \sigma \) there is at least one object with non-zero rank, but the last statement follows from the explicit recursion relation for ranks. \( \square \)

If \( \mathcal{F} \) is a vector bundle over \( E \), it is clear that \( \text{Hom}_{D^b(E)}(O_E, \mathcal{F}[i]) \) can be non zero for at most two values of the parameter \( i \). Indeed,

\[
\text{Hom}_{D^b(E)}(O_E, \mathcal{F}[i]) = \begin{cases}
\dim H^0(E, \mathcal{F}), & \text{if } i = 0, \\
\dim H^1(E, \mathcal{F}), & \text{if } i = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

We apply this observation to the general formula for the Betti numbers and get the following corollary.

**Corollary 6.1.4.** Let \( \mathcal{F}[l] \in D^b(E) \) be an indecomposable object, where \( \mathcal{F} \) is a vector bundle, then the Betti numbers of \( \Phi(\mathcal{F}) \) are given by the following formulae.

1. If \( j \geq 0 \), then \( \beta_{i,j} = \begin{cases}
\dim H^0(E, \mathcal{F}^\vee \otimes K_{-j}), & \text{if } i - j = l - 1, \\
\dim H^1(E, \mathcal{F}^\vee \otimes K_{-j}), & \text{if } i - j = l.
\end{cases} \)

2. If \( j < 0 \), then \( \beta_{i,j} = \begin{cases}
\dim H^0(E, \mathcal{F}^\vee \otimes K_{-j}), & \text{if } i - j = l, \\
\dim H^1(E, \mathcal{F}^\vee \otimes K_{-j}), & \text{if } i - j = l + 1.
\end{cases} \)

In particular, if the two discrete parameters \( r \) and \( d \) are fixed, there are only finitely many possibilities for the Betti tables of graded MCM modules over the ring \( R_E \).

**Proof.** Immediately follows from previous results. \( \square \)

Note that now we have much more concrete formulae for the Betti numbers, because if the charge of \( \mathcal{F} \) is fixed, the cohomology groups can be easily calculated, and we have at most two possibilities for each answer depending on whether \( \mathcal{F}^\vee \otimes K_{-j} \cong F_{\text{rk}(\mathcal{F})d_{-j}} \), here \( F_{\text{rk}(\mathcal{F})d_{-j}} \) denotes Atiyah’s bundle of rank \( \text{rk}(\mathcal{F})d_{-j} \).

**Corollary 6.1.5.** Let \( i \gg 0 \). Then the Betti numbers \( \beta_{i,j} \) of a finitely generated graded module over \( R_E \) grow linearly if \( n = 4 \) and exponentially if \( n > 4 \).

**Proof.** First, by the depth lemma we know that after finitely many steps in a projective resolution the syzygy modules become MCM, thus it is enough to know the asymptotical behavior of the Betti numbers of MCM modules. The condition \( i \gg 0 \) implies \( j \gg 0 \) because \( i \) and \( j \) differs by some constant. The
previous Corollary and the formula for the dimensions of cohomology groups for a vector bundle over an elliptic curve give us
\[ \beta_{i,j} \sim \deg(\mathcal{F}^\vee \otimes K_{-j}). \]

If we fix the charge \( Z(\mathcal{F}) = \left( \begin{array}{c} p \\ q \end{array} \right) \), then we get
\[ \beta_{i,j} \sim pd_{-j} - qr_{-j}. \]

Therefore, the result follows from explicit the formulae in the proof of Lemma 6.1.2.

Our next goal is to describe the possible shapes of the Betti tables. We will see that all potentially nonzero Betti numbers lie on the line \( D_l = \{(i, j) \in \mathbb{Z}^2 | j = i - l\}, \) the half-line \( U_l = \{(i, j) \in \mathbb{Z}^2 | j = i - l + 1, i \geq -1\}, \) and the half-line \( B_l = \{(i, j) \in \mathbb{Z}^2 | j = i - l - 1, i \leq 0\}. \)

From the figure we see that the differentials have only linear and quadratic components, with the possible exception at \( i = 0. \)

Because the calculation of the Betti numbers was reduced to the calculation of dimensions of cohomology groups of \( \mathcal{F}^\vee \otimes K_{-j}, \) and on an elliptic curve dimensions of cohomology group are almost (except in the special case of Atiyah bundles) defined by degrees, we are going to study the associated sequence of degrees. For a vector bundle \( \mathcal{F} \) with charge \( Z(\mathcal{F}) = \left( \begin{array}{c} p \\ q \end{array} \right) \) we define sequence of integer numbers by the following formula:
\[ s_j(\mathcal{F}) = \deg(\mathcal{F}^\vee \otimes K_j) = pd_j - qr_j, \]
where \( j \in \mathbb{Z}. \) We will write just \( s_j \) if \( \mathcal{F} \) is understood. The sequence \( \{s_j\} \) satisfies the same recursive relation that can be checked directly:
\[ (n - 2)s_j - s_{j-1} = (n - 2)(pd_{-j} - qr_{-j}) - pd_{-j+1} + qr_{-j+1} = \\
= p((n - 2)d_{-j} - d_{-j+1}) - q((n - 2)r_{-j} - r_{-j+1}) = pd_{-j-1} - qr_{-j-1} = s_{j+1}. \]
Moreover, the formulae for degrees and ranks give us the initial values of the recursive sequences

\[
\begin{align*}
s_2 &= p(n^2 - 2n) - q(n - 1), \\
s_1 &= pn - q, \\
s_0 &= -q, \\
s_{-1} &= pm - q(n - 1).
\end{align*}
\]

As in the case of the degrees of the \( K_j \) we have the same formula for \( s_j \) for \( j > 0 \) and \( j \leq 0 \), but two initial values need to be modified by direct computation, for instance \( s_{-1} \neq (n - 2)s_0 - s_1 \). Note that all numbers \( s_j \) are linear functions of the variables \( p \) and \( q \). When we write some condition for \( s_j(\mathcal{F}) \), we mean that condition on the charge of the vector bundle \( \mathcal{F} \). The properties of the Betti numbers can be extracted from the properties of the sequence \( s_j \) by some elementary analytical calculations.

**Proposition 6.1.6.** For fixed \( i \in \mathbb{Z} \) the Betti numbers \( \beta_{i,j} \) are non-zero for only one value of \( j \), except for finitely many values of \( i \). In particular, if \( |i| \) is big enough, non-vanishing Betti numbers concentrate along only one of the lines \( U_l \) or \( D_l \) for \( i > 0 \) and \( D_l \) or \( B_l \) for \( i < 0 \). Moreover, at most one jump between the lines \( U_l \) and \( D_l \) or \( D_l \) and \( B_l \) can occur.

**Proof.** We start with the generic case \( n > 4 \). Because we modify the initial values it makes sense to consider the sequences \( \{s_j\}_{j>0} \) and \( \{s_j\}_{j\leq0} \) separately. For \( j > 0 \) we have

\[
s_j = A_+ \mu^j + B_+ \mu^{-j},
\]

where

\[
A_+ = \frac{\mu s_2 - s_1}{\mu^2 - 1} \quad B_+ = \frac{\mu^2 s_1 - s_2}{\mu^2 - 1}.
\]

For \( j \leq 0 \)

\[
s_j = A_- \mu^j + B_- \mu^{-j},
\]

where

\[
A_- = \frac{\mu s_0 - s_1}{\mu^2 - 1} \mu, \quad B_- = \frac{\mu s_{-1} - s_0}{\mu^2 - 1}.
\]

Thus for \( j \gg 0 \) approximately \( s_j \sim A_+ \mu^j \), and for \( j \ll 0 \) approximately \( s_j \sim B_- \mu^{-j} \). For example, if \( A_+ \) is positive and \( j > 0 \), then after finitely many steps \( s_j \) becomes an increasing sequence of positive numbers. It means that all non-vanishing Betti numbers lie on the line \( U_l \). The other cases can be treated similarly.

A jump of the Betti numbers corresponds to a change of sign in the sequence \( s_j \), therefore, for some \( j \in \mathbb{R} \) we should have \( s_j = 0 \), and such an equation can be solved explicitly. For \( j > 0 \) the solution is given by the formula

\[
j = \frac{1}{2} \log_\mu \left( -\frac{B_+}{A_+} \right) = \frac{1}{2} \log_\mu \left( \frac{\mu^2 s_1 - s_2}{s_1 - s_2} \right) + \frac{3}{2},
\]
and a solution exists if and only if
\[
\begin{cases}
  s_1 > \mu s_2 \\
  \mu^2 s_1 - s_2 > \mu^{-3}(s_1 - \mu s_2),
\end{cases}
\quad \text{or} \quad
\begin{cases}
  s_1 < \mu s_2 \\
  \mu^2 s_1 - s_2 < \mu^{-3}(s_1 - \mu s_2).
\end{cases}
\]

These conditions are linear in the variables \(p\) and \(q\), thus the plane \(K_0(E)/\text{rad}(\cdot, \cdot) \otimes \mathbb{R}\) is divided into four chambers by two lines \(s_1 - \mu s_2 = 0\) and \(\mu^2 s_1 - s_2 - \mu^{-3}(s_1 - \mu s_2) = 0\). Moreover, we work under the assumption that \(F\) is a vector bundle, that is \(p > 0\). If \((p, q)\) is an integer point in one of the two chambers described by the system of inequalities above and \(p > 0\), then the Betti numbers jump between the lines \(U_l\) or \(D_l\).

For \(j \leq 0\) a solution of the equation is
\[
j = \frac{1}{2} \log_\mu \left( \frac{B_-}{A_-} \right) = \frac{1}{2} \log_\mu \left( \frac{s_0 - \mu s_{-1}}{\mu s_0 - s_{-1}} \right) - \frac{1}{2}.
\]
First note that \(s_0 = 0\) is an obvious solution, and the condition \(s_0 = 0\) is equivalent to \(j = 0\). In terms of the parameters \(p\) and \(q\) it means that for a vector bundle \(F\) such that the degree is zero, \(s_0 = q = \text{deg}(F) = 0\), a jump always happens at the \(j = 0\) level. Other solutions exist if and only if
\[
\begin{cases}
  s_0 - \mu s_{-1} > 0 \\
  s_0 < 0,
\end{cases}
\quad \text{or} \quad
\begin{cases}
  s_0 - \mu s_{-1} < 0 \\
  s_0 > 0.
\end{cases}
\]

The description of such solutions in terms of the geometry of \(K_0(E) \otimes \mathbb{R}\) is similar to the case \(j > 0\).

For \(n = 4\) we repeat the same argument, but instead of logarithmic conditions we get linear conditions. For completeness of exposition we derive inequalities that describe the jumps of the Betti numbers. For \(j > 0\)
\[
s_j = A_+ j + B_+,
\]
where
\[
A_+ = s_2 - s_1, \quad B_+ = 2s_1 - s_2.
\]
The same argument as before shows that the conditions for a jump are
\[
\begin{cases}
  s_2 - s_1 < 0 \\
  s_2 - 2s_1 < 0,
\end{cases}
\quad \text{or} \quad
\begin{cases}
  s_2 - s_1 > 0 \\
  s_2 - 2s_1 > 0.
\end{cases}
\]

Finally, for \(j \leq 0\)
\[
s_j = A_- j + B_-,
\]
where
\[
A_+ = s_0 - s_{-1}, \quad B_+ = s_0.
\]
Again we consider the case \(s_0 = 0\) (or equivalently \(j = 0\)) separately, and the conclusion in this case is
the same as for \( n > 4 \). For the other cases the following condition is equivalent to a jump:

\[
\begin{cases}
  s_0 < 0, & \text{or} \\
  s_0 - s_1 < 0,
\end{cases}
\]

\[
\begin{cases}
  s_0 > 0, \\
  s_0 - s_1 > 0.
\end{cases}
\]

Note that if \( j \), given by the previous formulae, is an integer number and \( F^\vee \otimes K_{-j} \cong F_{pr-j} \) at a jump point, then there are two non-vanishing Betti numbers, both equal to one, otherwise there is only one non-vanishing number located on one of the lines.

### 6.2 Koszulity of MCM modules over the ring \( R_E \)

In this section we apply the methods developed earlier to derive Koszulity criteria for MCM modules over the ring \( R_E \). Let us recall the definition of the Koszul and CoKoszul properties.

**Definition.** Let \( R = \bigoplus_{i \geq 0} R_i \) be a connected graded ring, and let \( M \) be a finitely generated graded module over \( R \). Then

- \( M \) is called Koszul if \( \beta_{i,j}(M) = 0 \) for \( i \neq j \) for \( i \geq 0 \)
- \( M \) is called Cokoszul if \( \beta_{i,j}(M) = 0 \) for \( i \neq j \) for \( i \leq 0 \)

We start with a Koszulity criterion for MCM modules over \( R_E \).

**Theorem 6.2.1.** An indecomposable MCM module \( M = \Phi(F[l]) \) is Koszul if and only if \( l = 0 \) and \( F \) is an indecomposable vector bundle with charge \( Z(F) = \left( \begin{array}{c} p \\ q \end{array} \right) \), satisfying the conditions

- \( s_1 \geq 0 \), and if \( s_1 = 0 \) then \( F \not\cong F_p(1) \),
- \( s_0 < 0 \), \( \mu s_{-1} - s_0 \leq 0 \).

**Proof.** Let \( M = \Phi(F[l]) \) be an indecomposable Koszul module, then \( F \) is an indecomposable object of \( D^b(E) \). There are two possibilities for \( F \): \( F \) is a vector bundle, or \( F \) is a skyscraper sheaf. We start with the case of a vector bundle. By the description of the shape of the Betti table given earlier, we conclude that \( M = \Phi(F[l]) \) can be Koszul only for \( l = 0 \), or \( l = 1 \). First let us assume that \( l = 1 \). Corollary 6.1.4 in this case shows that \( \beta_{0,-2} = 0 \) implies \( s_2 \geq 0 \), and \( \beta_{0,-1} = 0, \beta_{1,-1} = 0 \) implies that \( s_1 = 0 \). Thus we get the conditions

\[
\begin{cases}
  s_2 = p(n^2 - 2n) - q(n - 1) \geq 0 \\
  s_1 = pm - q = 0,
\end{cases}
\]

which imply that \( np \leq 0 \), but we assume that \( F \) is a vector bundle, that is \( p > 0 \). This contradiction shows that there are no Koszul MCM modules of the form \( \Phi(F[1]) \).
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For modules of the form $\Phi(\mathcal{F})$ Koszulity is equivalent to two conditions: $\beta_{0,-1} = 0$ and $\beta_{i,i+1} = 0$ for $i \geq 0$. The first condition is equivalent to the condition $s_1 \geq 0$, and if $s_1 = 0$, we need to assume that $F^\vee \otimes K_1 \not\cong F_{pr_1}$. Note that $r_1 = 1$ and the last condition is equivalent to

$$F \not\cong F_p(1).$$

The second condition is equivalent to $s_{-j} \leq 0$ for $j \geq 1$, where, in principle, for the cases $s_{-j} = 0$ we again need the additional assumption that we stay away from Atiyah bundles, but we are going to show that $s_{-j} < 0$. Note also that we need to assume $s_0 \leq 0$, because otherwise $\beta_{0,0} = 0$, and we get a trivial module. In fact, if $s_0 = 0$, then $s_1 = pn > 0$, and we arrive at a contradiction, and, by induction, if $s_{-j} = 0$ for some $j \geq 0$, then $s_{-(j+1)} > 0$ which contradicts our assumptions. Applying lemma 6.1.2 we conclude that the second condition is equivalent to $s_0 < 0$ and $s_{-1} - s_0 \leq 0$.

Let $\mathcal{F}$ be a skyscraper sheaf. Since $\mathcal{F}$ is indecomposable, it has the form

$$\mathcal{F} = \mathcal{O}_z/m_z^d,$$

for some point $z \in E$ and $d \geq 1$. Let us apply the functor $\sigma$ to $\mathcal{F}$

$$\sigma(\mathcal{F}) = \text{cone}(\text{RHom}(\mathcal{O}_E, \mathcal{F}) \otimes \mathcal{O}_E \to \mathcal{F}) \otimes \mathcal{O}_E(nx).$$

Higher cohomology groups of a skyscraper sheaf vanish, $\text{RHom}(\mathcal{O}_E, \mathcal{F}) \cong k^d[0]$. The evaluation map $k^d \otimes \mathcal{O}_E \to \mathcal{O}_z/m_z^d$ is surjective, and there is a short exact sequence

$$0 \to P' \to k^d \otimes \mathcal{O}_E \to \mathcal{O}_z/m_z^d \to 0,$$

where $P'$ is the kernel of the evaluation map. Therefore, $\sigma(\mathcal{F}) = P' \otimes \mathcal{O}_E(nx)[1]$, because $\sigma$ is an autoequivalence and $\mathcal{F}$ is indecomposable $\sigma(\mathcal{F})$ is indecomposable and $P'$ is also indecomposable. The short exact sequence above implies that the rank of $P'$ is positive, thus $P'$ is an indecomposable vector bundle. It is more convenient to work with the vector bundle

$$P = P' \otimes \mathcal{O}_E(nx),$$

it is easy to see that $\text{rk}(P) = d$ and $\text{deg}(P) = d(n-1)$. In terms of the vector bundle $P$ the module $M$ is given by the following isomorphism

$$\Phi(P[l + 1]) = M(1).$$

We reduced now the skyscraper sheaf case to the vector bundle case where all previous results can be applied. The Betti numbers $\beta_{i,j}(M(1)) = \beta_{i,j+1}(M)$ of $M(1)$ are only non-zero along the line $j = i - 1$, which is only possible for $l = 0, 1$. If $l = 0$, one of the conditions $\beta_{0,-2}(M(1)) = 0$ is equivalent to $s_2(P) > 0$, but explicit calculation shows that $s_2(P) = d(1-2n) < 0$ if $n > 0$, so we arrive at a contradiction. If $l = 1$ conditions $\beta_{2,-1}(M(1)) = \beta_{1,-1}(M(1)) = 0$ imply $s_1(P) = 0$, but again by explicit calculation $s_1(P) = d$, thus $d = 0$, and we arrive at another contradiction. Therefore, the equivalence $\Phi$ never maps skyscraper sheaves to Koszul MCM modules.

For convenience of the reader we reformulate the conditions of the Theorem in terms of variables $p$
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and \( q \): \( M = \Phi(\mathcal{F}) \) is Koszul if and only if the charge of the vector bundle \( \mathcal{F} \) satisfies

\[
\begin{cases}
  p > 0 \\
  q > 0 \\
  \mu(pn - q(n - 1)) - q \leq 0 \\
  np - q \geq 0
\end{cases}
\]

On the plane \((p, q)\) charges of vector bundles corresponding to Koszul modules coincide with the integer points in the closed domain between the two lines \( \mu(pn - q(n - 1)) - q = 0 \) and \( np - q = 0 \) in the first quadrant.

We illustrate this description with the example \( n = 5 \) on the Figure 2. On this figure the light grey area represents two half-spaces given by inequalities \( \mu(pn - q(n - 1)) - q \leq 0 \) and \( np - q \geq 0 \), while the dark grey area represents their intersection.

The prove of the following Cokoszulity criterion is based on the same case-by-case analysis and does not involve any new ideas.

**Theorem 6.2.2.** An indecomposable MCM module \( M = \Phi(\mathcal{F}[l]) \) is Cokoszul if and only if \( l = 0 \) and \( \mathcal{F} \) is an indecomposable vector bundle with charge \( Z(\mathcal{F}) = \begin{pmatrix} p \\ q \end{pmatrix} \), satisfying conditions

- \( s_{-1} \leq 0 \), and if \( s_{-1} = 0 \) then \( \mathcal{F} \otimes K_{-1}^\vee \not\cong F_{p(n-1)} \),
- \( s_0 > 0 \), \( \mu s_1 - s_0 \geq 0 \).

Our next goal in this section is to compute the Hilbert series for a Koszul module \( M = \Phi(\mathcal{F}) \) in terms of the rank \( p \) and degree \( q \) of a vector bundle \( \mathcal{F} \). For a Koszul module \( M \) we have the following simple relation between Hilbert series \( H_M(t) = \sum_i \dim(M_i)t^i \) and Poincaré series \( S(t) = \sum_i \beta_{i,i}t^i \):

\[
H_M(t) = \sum_{i \geq 0} (-1)^i H_{R(-i)^{a_{-1}}} = \sum_{i \geq 0} (-1)^i \beta_{i,i}t^i H_{R_E}(t) = S(-t)H_{R_E}(t).
\]
On the other hand, the relation between the degrees \( s_i \) and the Betti numbers in the Koszul case takes the simplest possible form:

\[ \beta_{i,i}(M) = -s_{-i}(F). \]

Now we can see that the conditions \( s_0 < 0, \mu s_1 - s_0 \leq 0 \) of the Koszulity criterion mean exactly that \( \beta_{i,i} > 0 \) for \( i \geq 0 \). We use the generating function for the sequence \( s_j \) to express the Poincaré series in the form

\[ S(t) = -\frac{s_0 + (s_1 - (n-2)s_0)t}{t^2 - (n-2)t + 1}. \]

This formula for the Poincaré series together with the formula for the Hilbert series of the ring \( R_E \)

\[ H_{R_E} = \frac{1 + (n-2)t + t^2}{1-t^2}, \]

gives us the Hilbert series of \( M \)

\[ H_M(t) = \frac{-s_0 + (s_1 - (n-2)s_0)t}{(1-t)^2} = \frac{q + (pn - q)t}{(1-t)^2}. \]

In particular, we can extract the formula for multiplicity: \( e(M) = pn \), and interpret the condition \( s_1 \geq 0 \) of the Koszulity criterion as Ulrich’s bound \( \mu(M) \leq e(M) \) for the Koszul module \( M = \Phi(F) \).

We can apply our results to identify the maximally generated MCM modules.

**Proposition 6.2.3.** A Koszul module \( M = \Phi(F) \) is a maximally generated MCM module if and only if the following relation between rank \( p \) and degree \( q \) of a vector bundle \( F \) holds:

\[ pn = q. \]

The Hilbert series of such a module is given by the formula

\[ H_M(t) = \frac{q}{(1-t)^2}, \]

and the Poincaré series is

\[ S(t) = \frac{q}{t^2 - (n-2)t + 1}. \]

By [27, p.44] the algebra \( R_E \) is Koszul, and by [23, Theorem 3.4] any maximally generated MCM module over a Koszul algebra is a Koszul module. Consequently, the previous Proposition gives the complete description of maximally generated MCM modules over the homogeneous coordinate ring \( R_E \) of an elliptic curve \( E \).

For a maximally generated MCM module \( M = \Phi(F) \) we can give a more explicit description.

**Theorem 6.2.4.** Any maximally generated MCM module over \( R_E \) is isomorphic to \( \Gamma_*(F) = \bigoplus_{i \in \mathbb{Z}} H^0(E, F(i)) \), where \( F \) is a vector bundle, whose rank \( p \) and degree \( q \) of \( F \) satisfy the condition \( pn = q \), and \( \det(F) \not\cong \mathcal{O}_E(p) \).

**Proof.** First, we compute the degrees of the Serre twists \( \deg(F(i)) = pi \deg(\mathcal{O}_E(1)) + q = q(i+1) \). Therefore, \( H^0(E, F(i)) \cong 0 \) for \( i \leq -1 \), where we use \( \det(F) \not\cong \mathcal{O}_E(p) \) for \( i = -1 \) and \( H^1(E, F(i)) \cong 0 \) for \( i \geq 0 \), thus

\[ \Gamma_*(F) = \bigoplus_{i \in \mathbb{Z}} H^0(E, F(i)) \cong \bigoplus_{i \geq 0} \text{RHom}(\mathcal{O}_E, F(i)). \]
Moreover, as \( \Gamma_*(\mathcal{F}) \) is an MCM module if \( \mathcal{F} \) is vector bundle, it means that

\[
\Gamma_*(\mathcal{F}) \cong \Phi(\mathcal{F}).
\]
Chapter 7

Embeddings into weighted projective spaces

7.1 The Elliptic Singularity \( \widetilde{E}_8 \)

So far we only considered graded rings

\[ R_{E,n} = \oplus_{i \geq 0} H^i(E, \mathcal{O}_E(i \mathfrak{n})) , \]

for \( n \geq 3 \). Now we are going to study two more cases, \( n = 1, 2 \). For these cases \( \mathcal{O}_E(n \mathfrak{n}) \) is no longer a very ample line bundle and, consequently, \( R_{E,1} \) and \( R_{E,2} \) are homogeneous coordinate rings of an elliptic curve embedded into a weighted projective space. These two cases were extensively studied by K. Saito \([29]\), under the name \( \widetilde{E}_8 \) (\( n = 1 \)) and \( \widetilde{E}_7 \) (\( n = 2 \)) singularities. In this section we consider the \( \widetilde{E}_8 \) singularity, and the singularity \( \widetilde{E}_7 \) is considered in the next section.

We start with the equation of the \( \widetilde{E}_8 \) singularity in Hesse form:

\[ R_{E,1} \cong k[x_0, x_1, x_2]/(x_0^6 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2) , \]

where \( \deg(x_0) = 1, \deg(x_1) = 2, \deg(x_2) = 3 \), and the parameter \( \psi \) satisfies \( \psi^3 \neq 1 \).

Our first goal is to show that we have an equivalence of abelian categories \( \text{Proj} R_{E,1} \cong \text{Proj} R_{E,3} \). Here the notation \( \text{Proj} \) denote the same category as \( \text{qgr} \). Note that the homogeneous coordinate ring of the Hesse cubic \( R_{E,3} \) is the third Veronese subring of \( R_{E,1} \):

\[ R_{E,3} \cong R_{E,1}^{(3)} , \]

where in general we use \( A^{(n)} \) to denote the \( n \)-th Veronese subalgebra of \( A \). S.P. Smith in \([32]\) studied the relationship between \( \text{Proj} A \) and \( \text{Proj} A^{(n)} \), for an algebra \( A \) that is not necessarily commutative, nor necessarily generated in degree 1. We use the following proposition from that paper.

**Proposition 7.1.1.** Let \( A \) be an \( \mathbb{N} \)-graded algebra, and fix \( n \geq 2 \). Denote \( I_r = A^{(n)+r} \) for \( r \in \mathbb{Z} \) the indicated ideals in \( A \). Let

\[ I = \bigcap_{r \in \mathbb{Z}} I_r = I_1 \cap I_2 \cap \ldots \cap I_n . \]
Then there is an isomorphism
\[ \text{Proj } A \setminus Z \cong \text{Proj } A^{(n)} \setminus Z', \]
where \( Z' \) and \( Z \) are the loci of \( I \) and \( I^{(n)} \) respectively.

**Proof.** This is proposition 4.8 of [32]. \( \square \)

We apply this proposition to the case at hand.

**Corollary 7.1.2.** There is an equivalence of abelian categories
\[ \text{Proj } R_{E,1} \cong \text{Proj } R_{E,3} . \]

**Proof.** For the case of \( R_E \) it is easy to find generators of the ideals \( I_1, I_2, I_3 \) and \( I \) explicitly. One has
\[ I_1 = A^{(3)+1}A = (x_0^3, x_0^2x_1, x_0x_2, x_1^3), \]
\[ I_2 = A^{(3)+2}A = (x_0^4, x_0^3x_1, x_0^2x_2, x_1x_2), \]
\[ I_3 = A^{(3)}A = (x_0^3, x_0x_1, x_2). \]
The intersection is \( I = I_1 \cap I_2 \cap I_3 = (x_0^3, x_0^2x_1, x_0x_2, x_1x_2) \). It is clear that the ideal \( I \) is cofinite, and thus so is the ideal \( I^{(n)} \). Therefore, \( Z = Z' = \emptyset \).

Next we are going to show that the calculations of the Betti numbers and the Hilbert series of MCM modules over the \( \tilde{E}_8 \) singularity are almost the same as the calculations for a cone over a plane cubic curve.

The functor \( \sigma \) in this case takes the form
\[ \sigma = B \circ A, \]
where \( B = T_{k(x)} = \mathcal{O}_E(x) \otimes - \), \( A = T_{\mathcal{O}_E} \). Because \( R_{E,1} \) is a hypersurface of degree 6 we have a natural isomorphism [2] \( \cong \sigma^6 \) of functors in \( D^b(E) \). In [31] p. 68, another isomorphism was established:
\[ \sigma^3 \cong \text{FM}_{\mathcal{P}}^{-2}, \]
where \( \text{FM}_{\mathcal{P}} \) denotes the autoequivalence of \( D^b(E) \) given by the Fourier-Mukai transform with kernel represented by the Poincaré bundle \( \mathcal{P} \). If we denote \( \iota : E \to E \) the map taking inverses in \( E \) as an algebraic group, then we can continue the previous isomorphism:
\[ \sigma^3 \cong \text{FM}_{\mathcal{P}}^{-2} \cong \iota^* \circ [1] . \]

To calculate Betti numbers we need to calculate \( \sigma^j(\mathcal{O}_E[1]) \) for \( j \in \mathbb{Z} \), but \( \sigma^3(\mathcal{O}_E[1]) \cong \iota^*\mathcal{O}_E[2] \cong \mathcal{O}_E[2] \), therefore, the sequence of objects \( \sigma^j(\mathcal{O}_E[1]) \) is not 6-periodic, but already 3-periodic up to translation [1].

**Lemma 7.1.3.** We have isomorphisms \( \sigma^{-1}(\mathcal{O}_E[1]) \cong k(x), \sigma^{-2}(\mathcal{O}_E[1]) \cong \mathcal{O}_E(x), \sigma^{-3}(\mathcal{O}_E[1]) \cong \mathcal{O}_E \), and any other object in the sequence \( \sigma^j(\mathcal{O}_E[1]) \) can be determined from the formula
\[ \sigma^{3j+j}(\mathcal{O}_E[1]) \cong \sigma^j(\mathcal{O}_E[1])[i], \]
where \( j \in \{-3, -2, -1\} \);
Proof. It is clear that \( \sigma(O_E[1]) \cong O_E(x) \), and then \( \sigma^2(O_E[1]) \) is a cone of the following evaluation map:

\[
\sigma^2(O_E[1]) = \text{cone}(R\text{Hom}(O_E, O_E(x)) \otimes O_E \to O_E(x)).
\]

It is easy to see that \( R\text{Hom}(O_E, O_E(x)) \otimes O_E \cong O_E \), and the evaluation map has cokernel \( k(x) = O_x/m_x \).

We translate properties of \( \sigma^{-j}(O_E[1]) \) into properties of Betti numbers of \( M = \Phi(F) \). We see that

\[
\beta_{i+1,j} = \beta_{i,j}(M) = \dim \text{Ext}^{i+1}_{D^b(E)}(F, \sigma^{-j}(O_E[1])) =
\dim \text{Ext}^i_{D^b(E)}(F, \sigma^{-j+3}(O_E[1])) = \beta_{i,j-3}.
\]

In other words, the minimal free resolution of \( M \) over \( R_{E,1} \) has the form

\[0 \leftarrow M \leftarrow F \leftarrow F(-3) \leftarrow F(-6) \leftarrow \ldots\]

For a suitable finitely generated module free module \( F \). In particular, if we want to compute the Betti table, it is enough to compute only \( \beta_{0,*} \).

**Proposition 7.1.4.** If \( M = \Phi(F[l]) \), where \( F \in \text{Coh}(D^b) \), then \( \beta_{0,j}(M) = 0 \), except for \(-2 - 3l \leq j \leq 3 - 3l \). In particular, if \( l \leq -1 \) then \( M \) is generated in positive degrees.

The functor \( \sigma \) induces the linear map

\[
[\sigma] = \begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}
\]

on \( K(E)/\text{rad}(-,-) \). It is of order six, \( [\sigma]^6 = 1 \), and we can choose a fundamental domain for this action to be

\[ r > 0, \]
\[ 0 \leq d < r. \]

In this fundamental domain the rank of \( F \) is positive, therefore, \( F \) is a vector bundle. In fact, our previous result can be improved: the module \( M = \Phi(F) \), corresponding to the case \( l = 0 \) in the proposition above, is still generated in positive degrees. This improvement is based on the explicit computation of the Betti numbers in term of the charge of a vector bundle as before. As we compute Betti numbers, we again get several cases that should be treated separately. A summary of these computations is given in the next proposition.

**Proposition 7.1.5.** Let \( F \) be a vector bundle with the charge \( Z(F) = \begin{pmatrix} r \\ d \end{pmatrix} \) in the fundamental domain. The Betti numbers of the MCM module \( M = \Phi(F) \) can be expressed as dimensions of the cohomology
groups on an elliptic curve in the following way

\[ \beta_{0,0} = \dim H^1(E, F^\vee) = \begin{cases} 1, & \text{if } F \cong \mathcal{O}_E; \\ 0, & \text{otherwise.} \end{cases} \]

\[ \beta_{0,1} = \dim H^0(E, F^\vee \otimes k(x)) = r. \]

\[ \beta_{0,2} = \dim H^0(E, F^\vee \otimes \mathcal{O}_E(x)) = r - d. \]

\[ \beta_{0,3} = \dim H^0(E, F^\vee) = \begin{cases} 1, & \text{if } F \cong \mathcal{O}_E; \\ d, & \text{otherwise.} \end{cases} \]

**Proof.** The preceding results guarantee that we get zero Betti numbers, \( \beta_{0,j} = 0 \), for \( j < -2 \) and \( j > 3 \).

But the explicit calculation of \( \beta_{0,-2} \) as

\[ \beta_{0,-2} = \dim \text{Hom}(F, \sigma^2(\mathcal{O}_E[1])) = \dim \text{Hom}(F, \sigma^{-1}(\mathcal{O}_E[1])[1]) = \\
= \dim \text{Hom}(F, k(x)[1]) = \dim H^1(E, F^\vee \otimes k(x)) = 0, \]

and the explicit calculation of \( \beta_{0,-1} \) as

\[ \beta_{0,-1} = \dim \text{Hom}(F, \sigma^1(\mathcal{O}_E[1])) = \dim \text{Hom}(F, \sigma^{-2}(\mathcal{O}_E[1])[1]) = \\
= \dim \text{Hom}(F, \mathcal{O}_E(x)[1]) = \dim H^1(F^\vee, \mathcal{O}_E(x)) = 0, \]

allow to restrict the window of non-trivial Betti numbers to \( 0 \leq j \leq 3 \). The computations of the other Betti numbers use our standard tools and are straightforward:

\[ \beta_{0,0} = \dim \text{Hom}(F, \mathcal{O}_E[1]) = \dim H^1(E, F^\vee) = \begin{cases} 1, & \text{if } F \cong \mathcal{F}_r; \\ 0, & \text{otherwise.} \end{cases} \]

\[ \beta_{0,1} = \dim \text{Hom}(F, \sigma^{-1}(\mathcal{O}_E[1])) = \dim \text{Hom}(F, k(x)) = \\
= \dim H^0(E, F^\vee \otimes k(x)) = r. \]

\[ \beta_{0,2} = \dim \text{Hom}(F, \sigma^{-2}(\mathcal{O}_E[1])) = \dim \text{Hom}(F, \mathcal{O}_E(x)) = \\
= \dim H^0(E, F^\vee \otimes \mathcal{O}_E(x)) = r - d. \]

\[ \beta_{0,3} = \dim \text{Hom}(F, \sigma^{-3}(\mathcal{O}_E[1])) = \dim \text{Hom}(F, \mathcal{O}_E) = \\
= \dim H^0(E, F^\vee) = \begin{cases} 1, & \text{if } F \cong \mathcal{F}_r; \\ d, & \text{otherwise.} \end{cases} \]

\[ \square \]

From this Proposition we see that there are two cases: in the first \( F \) is the Atiyah bundle, and in the second case \( F \) is generic.

As in the case of a plane cubic we can read off the rank and degree of \( F \) from the minimal free resolution of \( M \). As before, it is more convenient to present the results of our computations in form of the two possible shapes of the Betti table:
Assume that $M$ is as in the foregoing proposition. Then the Hilbert series is easy to compute:

$$H_M(t) = \frac{H_F(t)}{1 + t^3} = \frac{B(t) H_{R_{E,1}}(t)}{1 + t^3},$$

where $B(t) = \sum_j \beta_{0,j} t^j$, and

$$H_{R_{E,1}} = \frac{1 - t^6}{(1-t)(1-t^2)(1-t^3)} = \frac{1 - t^2}{(1-t)^2}.$$

The ring $R_{E,1}$ has multiplicity $e(R_{E,1}) = 2$, because the multiplicity of any hypersurface is equal to the minimum of the degrees of a non-zero monomials in the equation of the hypersurface. The rank of a module can be computed as

$$\text{rk} M = \lim_{t \to 1} \frac{H_M(t)}{H_R(t)} = \frac{B(1)}{2} = \frac{\mu(M)}{2}.$$

The multiplicities of a ring and a module are related as before by

$$e(M) = e(R_{E,1}) \text{rk}(M) = 2 \text{rk}(M).$$

We present the numerical information, namely rank, multiplicity and minimal number of generators, of an MCM module $M$ as a table:

<table>
<thead>
<tr>
<th>case</th>
<th>$e(M)$</th>
<th>$\mu(M)$</th>
<th>$\text{rk}(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F} \cong \mathcal{F}_r$</td>
<td>$2(r+1)$</td>
<td>$2(r+1)$</td>
<td>$r+1$</td>
</tr>
<tr>
<td>Generic $\mathcal{F}$</td>
<td>$2r$</td>
<td>$2r$</td>
<td>$r$</td>
</tr>
</tbody>
</table>

Table 7.2.

In particular, we see that every MCM module is maximally generated in this case. More generally, over a ring of multiplicity two any MCM module is maximally generated, this was shown in [26, Corollary 1.4].
7.2 The Elliptic Singularity $\widetilde{E}_7$

In this section we briefly summarize the results for an elliptic singularity of type $\widetilde{E}_7$, as they are only slightly different from the results of the $\widetilde{E}_8$ case. The Hesse form of a $\widetilde{E}_7$ singularity is

$$R_{E,2} \cong k[x_0, x_1, x_2]/(x_0^4 + x_1^4 + x_2^2 + 3\psi x_0 x_1 x_2),$$

where $\deg(x_0) = 1$, $\deg(x_1) = 1$, $\deg(x_2) = 2$, and the parameter $\psi$ satisfies $\psi^3 \neq 1$. Note that as for the $\widetilde{E}_8$ case the multiplicity of $R_{E,2}$ is two.

The result of S.P. Smith quoted in above [7.1.1] implies the analogous corollary for the abelian category $\text{Proj} R_{E,2}$.

**Corollary 7.2.1.** There is an equivalence of abelian categories

$$\text{Proj} R_{E,2} \cong \text{Proj} R_{E,3}.$$

In this case, $\sigma^2 \cong i[1]$, and we have analogously

**Lemma 7.2.2.** There are isomorphisms $\sigma^{-1}(O_E[1]) \cong O_E(2x)$, $\sigma^{-2}(O_E[1]) \cong O_E$ and any other object in the sequence $\sigma^j(O_E[1])$ can be determined from the formula

$$\sigma^{2i+j}(O_E[1]) \cong \sigma^j(O_E[1])[i],$$

where $j \in \{-2, -1\}$.

A MCM module $M$ has thus a resolution of the form

$$0 \leftarrow M \leftarrow F \leftarrow F(-2) \leftarrow F(-4) \leftarrow \ldots$$

For a suitable finitely generated module free module $F$. In particular, it is enough to calculate $\beta_{0,*}$.

**Proposition 7.2.3.** If $M = \Phi(\mathcal{F}[l])$, where $\mathcal{F} \in \text{Coh}(D^b)$, then $\beta_{0,j}(M) = 0$, except for $-1 - 2l \leq j \leq 2 - 2l$. In particular, if $l \leq -1$, then $M$ is generated in positive degrees.

The functor $\sigma$ induces the linear map

$$[\sigma] = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$$

on $K_0(E)/\text{rad}\langle -, - \rangle$. It is of order four, $[\sigma]^4 = 1$, and we can choose a fundamental domain for this action to be

$$r > 0,$$

$$0 \leq d < 2r.$$

In such a fundamental domain the rank of $\mathcal{F}$ is positive, therefore, $\mathcal{F}$ is a vector bundle. As before, the result of generation in positive degrees can be extended to the case $l = 0$ by explicit computation. A summary of these computations is given in the following proposition.
Proposition 7.2.4. Let \( \mathcal{F} \) be a vector bundle with the charge \( Z(\mathcal{F}) = \left( \begin{array}{c} r \\ d \end{array} \right) \) in the fundamental domain. The Betti numbers of an MCM module \( M = \Phi(\mathcal{F}) \) can be expressed as dimensions of cohomology groups on the elliptic curve in the following way:

\[
\beta_{0,0} = \dim H^1(E,F^\vee) = \begin{cases} 
1, & \text{if } \mathcal{F} \cong F_r, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
\beta_{0,1} = \dim H^0(E,F^\vee \otimes O_E(2r)) = 2r - d.
\]

\[
\beta_{0,2} = \dim H^0(E,F^\vee) = \begin{cases} 
1, & \text{if } \mathcal{F} \cong F_r, \\
d, & \text{otherwise}.
\end{cases}
\]

As before, we see that there are two cases: in the first case \( \mathcal{F} \) is the Atiyah bundle, and in the second case \( \mathcal{F} \) is generic. As in the case of a plane cubic we can read off the rank and degree of \( \mathcal{F} \) from the minimal free resolution of \( M \). We present the two possible shapes of Betti tables below.

<table>
<thead>
<tr>
<th>( \mathcal{F} \cong F_r )</th>
<th></th>
<th>Generic ( \mathcal{F} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
<td>( i = 0 )</td>
<td>( i = 0 )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2r</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2r</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7.3.

With \( M \) as in the proposition above, the Hilbert series is

\[
H_M(t) = \frac{H_F(t)}{1 + t^2} = \frac{B(t)H_{R_{E,2}}(t)}{1 + t^2},
\]

where \( B(t) = \sum \beta_{0,j} t^j \), and

\[
H_{R_{E,2}} = \frac{1 - t^4}{(1 - t)^2(1 - t^2)^2} = \frac{1 + t^2}{(1 - t)^2}.
\]

The computation of the numerical invariants for MCM modules over the ring \( R_{E,2} \) carries over verbatim and the table of numerical invariants is exactly the same as for the case \( \tilde{E}_8 \).
Bibliography


