Amenability and Unique Ergodicity of the Automorphism Groups of all Countable Homogeneous Directed Graphs

by

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Abstract

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We establish the amenability, unique ergodicity and nonamenability of various automorphism groups from Cherlin’s list of countable homogeneous directed graphs [7]. This marks a complete understanding of the amenability of the automorphism groups from this list, and except for the Semigeneric graph case, marks a complete understanding of the unique ergodicity of these groups.

Along the way we establish that a certain product of Fraïssé classes preserves amenability, unique ergodicity and the Hrushovski property. We also establish the unique ergodicity of various other automorphism groups of Fraïssé structures that do not appear on Cherlin’s list.
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A great thank you also goes to the author’s mom Lori Meilleur, dad Ivan Pawliuk and wife Janet Mowat.

The novel material presented here is based on joint work with the author and Miodrag Sokić following that conference, and for the sake of formality, the author will attempt to separate authorship of results where possible.
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Chapter 1

Introduction

1.1 History

Examining the amenability of topological groups is an old and ongoing endeavour. In the spirit of the Kechris-Pestov-Todorcevic correspondence [18], the amenability status has been determined for many automorphism groups of Fraïssé structures. In 1998, Cherlin published his complete list of countable homogeneous directed graphs [7], of which all infinite such graphs arise as Fraïssé limits.

Establishing the amenability of these groups through the KPT correspondence involves first finding an ordering (or more generally a precompact expansion) on each of the underlying finite substructures, which has some sort of coherence and Ramsey behaviour (a so-called “excellent expansion”). For the graphs on Cherlin’s list, some excellent expansions were found as early as 1977 (Nešetřil-Rödl, [23]), 1978 (Abramzon-Harrington, [1]) and 1984 (Paoli-Trotter-Walker, [28]). The next progress was made in 2005 with the KPT correspondence, where expansions as orderings were investigated. Recently, the rest of the list was essentially completed by Jasiński-Laflamme-Nguyen Van Thé-Sauer in [20, 26, 17]. One case - $\mathcal{T}$ - was still open after these works, and we complete it in Section 4.5. For more historical information see [17, Section 2].

The second ingredient to checking amenability in this way involves checking whether or not the natural mapping of a finite structure to the reciprocal of the number of expansions it has, is indeed an invariant probability measure. Besides the examples in [18], there are relatively few examples of such calculations in the literature [19, 38]. In this paper we complete the verification of amenability for the automorphism groups of graphs from Cherlin’s list.

Furthermore, once amenability has been established for a group, it is natural to ask about unique ergodicity (i.e. whether such a measure is unique). For Cherlin’s list, the first progress was made in 2002 (Glasner-Weiss, [11]). In 2012 (Angel-Kechris-Lyons, [3]), primarily using a quantitative version of the expansion property (for orderings, the $\text{QOP}$), unique ergodicity was established for numerous automorphism groups of graphs from Cherlin’s list (as well as some other structures). Adapting these methods to the more general setting of expansions (which may not be arbitrary linear orderings) has allowed us to verify unique ergodicity for more automorphism groups. We also introduce new methods for checking the unique ergodicity of the automorphism group of a structure which appears as a type of product of Fraïssé structures (Section 5.1). This completes the verification of unique ergodicity for the automorphism groups of graphs from Cherlin’s list, except for the one for the Semigeneric digraph.
In addition to the directed graphs from Cherlin’s list, we were able to verify the unique ergodicity and nonamenability of various other automorphism groups of Fraïssé structures, for example binary rooted trees with various order expansions (Section 4.2), the Semigeneric graph with various additional structure (Section 6.6), and $S(2)$ and $S(3)$ with linear order expansions instead of their usual expansions (Sections 3.2 and 3.3). Some of these are shown using a new sufficient condition on a dense subclass of a Fraïssé class (Theorem 4.1.1).

While investigating unique ergodicity of an automorphism group in this framework, it is natural to ask whether or not the underlying structure has a strong form of homogeneity called the Hrushovski property. In general, checking that a Fraïssé class has the Hrushovski property is very difficult (for example, the status of the class of finite tournaments is unknown). We verify that this property is preserved under a specific type of product of Fraïssé structures, thus ensuring that two specific graphs from Cherlin’s list have the Hrushovski property (Proposition 5.2.4).

1.2 Layout and Order - What to Expect

Some brief comments on the organization and presentation of this document, and what to expect.

Chapter 1 (Introduction) will continue with an introduction to the objects of interest, and a brief introduction of the language and notation for the non-expert. This section is intended to be a reference section.

Chapter 2 (Forms of Amenability) will present the reader with the relevant facts and background about various forms of amenability. It will also provide translations between various prominent forms of amenability.

Chapter 3 (Four Nonamenable Groups) will present short, combinatorial arguments for why four of the structures on Cherlin’s list have nonamenable automorphism groups. One of these results will be novel. This chapter is independent of the following chapters.

Chapter 4 (Amenability) contains verifications that most of the other structures on Cherlin’s list have amenable automorphism groups. This is done through establishing a lemma which guarantees amenability based on independence of expansions from embeddings. The results in this section are predominately novel. The results in Chapter 6 will depend on the results in this chapter.

Chapter 5 (The Product class $K[L]$) contains results about amenability and unique ergodicity related to the product class $K[L]$. We also introduce the Hrushovski property and verify that $K[L]$ interacts well with it. These results are novel. This chapter is independent of the others, besides the definition of the Hrushovski property.

Chapter 6 (Unique Ergodicity) establishes the unique ergodicity of the automorphism groups of many structures on Cherlin’s list. We first introduce the machinery needed (the Quantitative Ordering Property, the McDiarmid Inequality) and isolate the general strategy for proving unique ergodicity so that our arguments will not contain a lot of overlap. These results are largely novel. This chapter relies on Chapter 4.

Chapter 7 (Going Forward) presents three directions for future investigation. This chapter references various proofs and theorems from earlier sections.
1.3 Cherlin’s List

Let us describe some examples that we will find useful.

Denote by $I_n$ the edgeless directed graph on $n$ vertices, where $n \leq \omega$.

Denote by $C_3$ the directed 3-cycle. Specifically, $C_3 = (C_3, \rightarrow C_3)$ is the directed graph such that $C_3 = \{a, b, c\}$ with $a \rightarrow C_3 b, b \rightarrow C_3 c$ and $c \rightarrow C_3 a$.

The following is the complete (see [7]) list of countable homogeneous directed graphs. The infinite structures here are Fraïssé structures.

1. The finite digraphs $C_3$ and $I_n$ for $n < \omega$.
2. $I_\omega$ is the edgeless directed graph on $\omega$ vertices.
3. $Q, S(2)$ and $T_\omega$ are tournaments.
   (a) $Q$ is the set of rational numbers where we take $x \rightarrow Q y$ iff $x < y$.
   (b) $S(2)$ is the dense local ordering which may be seen as the set of points on the unit circle with rational arguments such that $e^{i\theta} \rightarrow S(2) e^{i\phi}$ iff $0 < \phi - \theta < \pi$. See Section 3.2.
   (c) $T_\omega$ is the generic tournament, i.e. the Fraïssé limit of the class of all finite tournaments.
4. $T[I_n], I_n[T]$, where $n \leq \omega$, and $T$ is one of the tournaments in (3) or $C_3$. This is a type of $n$-point “blowup” of the nodes of the tournament $T$. See Section 5.1.
5. $\hat{T}$, for $T = I_1, C_3, Q$ or $T_\omega$, is a type of “two point blowup” of the vertices of a tournament. See Section 4.5.
6. $D_n$, for $1 < n \leq \omega$, the complete $n$-partite directed graph with countably many nodes. See Sections 4.4 and 6.4.
7. $S$ is the Semigeneric graph. See Section 4.3.
8. $S(3)$ is a directed graph which may be seen as the set of points on the unit circle with rational arguments such that $e^{i\theta} \rightarrow S(3) e^{i\phi}$ iff $0 < \phi - \theta < \frac{2\pi}{3}$. This is not a tournament. See Section 3.3.
9. $P$ is the generic poset, i.e. the Fraïssé limit of the class of all finite posets such that $x \rightarrow P y$ iff $x < P y$.
10. $P(3)$ is the “twisted” generic poset in three parts. See Section 3.4.
11. $G_n$, for $n > 1$, is the generic directed graph with the property that $I_{n+1}$ can’t be embedded in $G_n$. See Section 6.7.
12. $L(T)$ is the Fraïssé limit of the class of finite directed graphs which do not embed any member of $T$, where $T$ is a fixed set of finite tournaments each of which has at least three vertices. See Section 6.7.

In this paper we go over Cherlin’s list and examine the automorphism group of each of these structures with respect to amenability and unique ergodicity. Along the way we will also check these properties for automorphism groups of some structures not appearing on this list. We consider each automorphism group as a topological group with the pointwise convergence topology, see [5] for more details.

First we give a list of known facts:
1. Aut($I_\omega$) ∼ $S_\infty$ is amenable and uniquely ergodic, see [11], [3, Proposition 10.1].

2. Aut($\mathbb{Q}$) and Aut($\mathbb{T}_\omega$) are amenable and uniquely ergodic, see [29], [3, Theorem 6.1, Theorem 2.2].

3. Aut($S(2)$) and Aut($S(3)$) are not amenable, see [38, Theorem 3.1] and private communication with Kechris.

4. Aut($\mathbb{P}$) is not amenable, see [19, Section 3].

In addition, the essential arguments for the amenability and unique ergodicity of Aut($G_n$) and Aut($L(T)$) are contained in [3, Theorem 5.1] which shows a slightly weaker theorem. Amenability and unique ergodicity of Aut($G_n$) and Aut($L(T)$) are not direct corollaries of their theorem or their proof, modifications had to be made.

### 1.4 Summary and Open Questions

The following summarizes what is now known, and what remains open, about the amenability of the automorphism groups of structures from Cherlin’s list. For readability, in the table we suppress Aut($X$) and simply write the structure $X$. We also suppress the finite structures.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Not Amenable</th>
<th>Amenable</th>
<th>Uniquely Ergodic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_\omega$</td>
<td>[11]</td>
<td>[11], [3]</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Q}$</td>
<td>[29]</td>
<td>[29]</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{T}_\omega$</td>
<td>[3]</td>
<td>[3]</td>
<td></td>
</tr>
<tr>
<td>$S(2)$</td>
<td>[38], Sect 3.2</td>
<td></td>
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</tr>
<tr>
<td>$\mathbb{T}(T)$</td>
<td>Sect 5.1</td>
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<tr>
<td>$\mathbb{P}$</td>
<td>Sect 4.5</td>
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<td>$\mathbb{P}(3)$</td>
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<td>$\mathbb{P}(3)$</td>
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<td>$L(T)$</td>
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</tbody>
</table>

### 1.5 Technical Preliminaries

Now we will describe the mathematical objects and notions that we will be using.

#### 1.5.1 Amenability

Let $G$ be a topological group. A continuous action of $G$ on a compact Hausdorff space is called a $G$-flow. A $G$-flow is **minimal** if the orbit of every point is dense. If every $G$-flow has a $G$-invariant Borel probability measure, then we say that $G$ is **amenable**. We say that $G$ is **uniquely ergodic** if every minimal $G$-flow has a unique $G$-invariant Borel probability measure. We will go into more depth about various equivalent versions of amenability in Chapter 2.
Throughout, we consider amenability and unique ergodicity for a collection of automorphism groups of countable structures related to directed graphs. These groups are not locally compact and they are not discrete, but they are non-Archimedian Polish groups, see [5] for more details.

1.5.2 Fraïssé Classes and Structures

Let $\mathcal{A}$ and $\mathcal{B}$ be given structures. If there is an embedding from $\mathcal{A}$ into $\mathcal{B}$ then we write $\mathcal{A} \hookrightarrow \mathcal{B}$, if $\mathcal{A}$ is a substructure of $\mathcal{B}$ then we write $\mathcal{A} \subseteq \mathcal{B}$, and if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then we write $\mathcal{A} \cong \mathcal{B}$. We write $\langle \mathcal{B} \rangle^\mathcal{A} = \{ \mathcal{C} \leq \mathcal{B} : \mathcal{C} \cong \mathcal{A} \}$. We say that a given structure is **locally finite** if each of its finitely generated substructures are finite. We denote by $\text{Age}(\mathcal{A})$ the collection of all finite substructures of $\mathcal{A}$. A structure $\mathcal{A}$ is **ultrahomogeneous** if every isomorphism between two finite substructures can be extended to an automorphism of $\mathcal{A}$. We say that $\mathcal{A}$ is a **Fraïssé structure** if it is countably infinite, locally finite and ultrahomogeneous.

Let $L$ be a signature and let $\mathcal{K}$ be a class of finite structures in $L$. Then $\mathcal{K}$ satisfies the:

- **Hereditary Property** (HP), if whenever $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{B} \in \mathcal{K}$, then $\mathcal{A} \in \mathcal{K}$.
- **Joint Embedding Property** (JEP), if for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ there is a $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{C}$ and $\mathcal{B} \hookrightarrow \mathcal{C}$.
- **Amalgamation Property** (AP), if for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{K}$ and all embeddings $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \to \mathcal{C}$ there is a $\mathcal{D} \in \mathcal{K}$ and embeddings $\overline{f} : \mathcal{B} \to \mathcal{D}$ and $\overline{g} : \mathcal{C} \to \mathcal{D}$ with $\overline{f} \circ f = \overline{g} \circ g$.
- **Strong Amalgamation Property** (SAP), if in addition to AP we have $\overline{f}(\mathcal{B}) \cap \overline{g}(\mathcal{C}) = \overline{f} \circ f(\mathcal{A})$.

We say that $\mathcal{K}$ is a **Fraïssé class** if it satisfies HP, JEP, AP, contains finite structures of arbitrarily large finite cardinality, and only countably many different isomorphism types. If $\mathcal{K}$ is a Fraïssé structure then its $\text{Age}(\mathcal{K})$ is a Fraïssé class. Given a Fraïssé class $\mathcal{K}$ we have its Fraïssé limit $\text{Flim}(\mathcal{K})$, which is a Fraïssé structure and is unique up to isomorphism. In this way there is a 1-1 correspondence between Fraïssé classes and Fraïssé structures. For more details, see [15].

We consider a structure as a tuple $\mathfrak{A} = (A, \{ R^A_i \}_{i \in I}, \{ f^A_j \}_{j \in J})$ where $A$ is the underlying set of the structure, $R^A_i$ is the interpretation of the relational symbol in $\mathfrak{A}$ and $f^A_j$ is the interpretation of the functional symbol in $\mathfrak{A}$ for all $i \in I$ and all $j \in J$. If $J = \emptyset$ then we say that the structure is **relational** (or that the signature is relational). All of the structures studied within are relational, so for ease of notation we will appropriate $J$ to also serve as an index set for a collection of relations.

In particular, we consider a directed graph (digraph) as a structure in the binary relational signature $\{ \rightarrow \}$. The symbol $\rightarrow$ is always interpreted as an irreflexive and asymmetric relation. For a directed graph $(A, \rightarrow^A)$ we sometimes use the symbol $\perp^A$ to denote $\neg(x \rightarrow^A y \lor y \rightarrow^A x)$. A **tournament** is a digraph $(A, \rightarrow^A)$ with the property that for every $x \neq y \in A$ we have either $x \rightarrow^A y$ or $y \rightarrow^A x$ (but not both).

In general, we will use the following typefaces: $\mathcal{L}, \mathcal{K}$ for classes, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{K}$ for structures, $A, B, C$ for universes (or underlying sets) of structures (with $L$ being reserved for the signature of a class) and $a, b, c$ for elements of underlying sets. Occasionally we will use $a, b$ for natural numbers that index the number of equivalence classes in $\mathcal{A}$ and $\mathcal{B}$. 

1.5.3 Reducts and the Expansion Property

Let $L \subseteq L^*$ be given signatures. Let $\mathcal{K}$ be a class of structures in $L$ and let $\mathcal{K}^*$ be a class of structures in $L^*$. If $A^* \in \mathcal{K}^*$ then we denote by $A^*/|L$ the structure in $\mathcal{K}$ obtained by dropping the interpretations of the symbols in $L^* \setminus L$ in $A^*$, and define $K^*|L := \{A^*|L : A^* \in \mathcal{K}^*\}$. We say that $\mathcal{K}^*$ is a precompact expansion of $\mathcal{K}$ provided that $\forall A \in \mathcal{K}$ there are only finitely many $A^* \in \mathcal{K}^*$ such that $A^*/|L = A$.

We say that $\mathcal{K}^*$ satisfies the expansion property (EP) (with respect to $\mathcal{K}$) if $\mathcal{K}^*/|L = \mathcal{K}$ and for every $A \in \mathcal{K}$ there is a $B \in \mathcal{K}$ such that for every $A^*,B^* \in \mathcal{K}^*$ with $A^*/|L = A$ and $B^*/|L = B$ we have $A^* \hookrightarrow B^*$.

We say that $\mathcal{K}^*$ is a reasonable expansion of $\mathcal{K}$ provided that it is a precompact expansion and $\forall A,B \in \mathcal{K}$, for every embedding $\pi : A \rightarrow B$, $\forall A^* \in \mathcal{K}^*$ with $A^*/|L = A$, there is a $B^* \in \mathcal{K}^*$ such that $B^*/|L = B$ and $\pi$ is also an embedding of $A^*$ into $B^*$.

1.5.4 Ramsey Property

We say that the class $\mathcal{K}$ satisfies the Ramsey Property (RP) (or is a Ramsey class) if for every (small) $A \in \mathcal{K}$ and every (medium) $B \in \mathcal{K}$ and every $r \in \mathbb{N}$ there is a (large) $C \in \mathcal{K}$ such that for every colouring $c : (\binom{C}{r}) \rightarrow \{1,\ldots,r\}$ there is a $\overline{B} \in (\binom{C}{r})$ such that $c|\overline{B}$ is a constant. We denote this using the arrow notation:

\[ C \rightarrow \overline{B}^A \]

Another notion that will always appear with RP is rigidity. We say that a given structure is rigid if it has no nontrivial automorphisms.

1.5.5 Consistent Random Expansions

Let $L \subseteq L^*$ be given signatures. Let $\mathcal{K}$ and $\mathcal{K}^*$ be classes of structures in $L$ and $L^*$ respectively such that $\mathcal{K}^*/|L = \mathcal{K}$. If $B \in \mathcal{K}^*$ then we write $B^* = (A \oplus A^*)$ where $A = B^*/|L$ and $A^* = B^*/(L^* \setminus L)$. Colloquially, “$A$ is the old stuff, and $A^*$ is the new stuff” when we use the representation $(A \oplus A^*)$. For $A \in \mathcal{K}$ we denote by $\mu_A$ a measure on the set

\[ \{A^* : (A \oplus A^*) \in \mathcal{K}^*\} \]

We will also have need for the related quantity

\[ \#(A)_\mathcal{K} := \{|A^* : (A \oplus A^*) \in \mathcal{K}^*\}| \]

which is the number of expansions of $A$ in $\mathcal{K}^*$. Let $A \subseteq B$ be structures in $\mathcal{K}$ and let $(A \oplus A^*) \in \mathcal{K}^*$. Then we write:

\[ \#_{\mathcal{K}^*}(A^*,B) := \{|B^* : (A \oplus A^*) \leq (B \oplus B^*) \in \mathcal{K}^*\}| \]

which is the number of expansions of $B$ in $\mathcal{K}^*$ that extend $(A \oplus A^*)$. If there is no confusion then we write $\#(A^*,B)$, or occasionally we will write $\#_{A,B}(A^*)$.

We say that the collection $\{\mu_A : A \in \mathcal{K}\}$ is a consistent random $\mathcal{K}^*$-expansion on $\mathcal{K}$ (when it is clear from context we suppress the reference to $\mathcal{K}^*$) if

where we have Probability measures, isomorphism Invariance and Extension properties. When it is clear from context we shall refer to a consistent random expansion as $(\mu_A)$, with no reference to $\mathcal{K}$.

We assume that for $A \cong B$ in $\mathcal{K}$ and $(A \oplus A^*) \in \mathcal{K}^*$ we have $B^*$ such that $(A \oplus A^*) \cong (B \oplus B^*)$. 

Each $\mu_A$ is a probability measure on $\{A^* : (A \oplus A^*) \in K^*\}$.

Whenever $\varphi : A \rightarrow B$ is an isomorphism then $\varphi_* \mu_A = \mu_B$, where $\varphi_* \mu_A$ is the push forward measure.

For every $A \leq B$ in $K$ and each $(A \oplus A^*) \in K^*$ we have,

$$\mu_A(\{A^*\}) = \sum \{\mu_B(\{B^*\}) : (A \oplus A^*) \leq (B \oplus B^*) \in K^*\}.$$ 

Let $L^* \setminus L$ be a relational signature, and let $K$ and $K^*$ be Fraïssé classes such that $K^*$ is a reasonable expansion of $K$. Then we say that $(K, K^*)$ is an excellent pair if:

1. $K^*$ is a Ramsey class of rigid structures, and
2. $K^*$ satisfies the expansion property relative to $K$.

### 1.5.6 Amenability via Expansions

The following is the key equivalence used to show amenability and nonamenability of the automorphism group of a Fraïssé structure. The version that appears as Proposition 9.2 in [3] is a special case of what we state, and the proof of this version is analogous. The proof will appear following Theorem 2.6.2.

**Proposition 1.5.1.** Let $(K, K^*)$ be an excellent pair. Then:

1. $\text{Aut}(\text{Flim}(K))$ is amenable iff $K$ has a consistent random $K^*$-expansion.
2. $\text{Aut}(\text{Flim}(K))$ is uniquely ergodic iff $K$ has a unique consistent random $K^*$-expansion.

We remark that a consistent random expansion $(\mu_A)$ cannot be degenerate, which means that when $(A \oplus A^*) \in K^*$, we have $\mu_A(\{A^*\}) \neq 0$. Otherwise, since Flim($K$) is separable, a degenerate measure for $A^*$ would give us a countable cover of the universal minimal flow by open sets each with measure 0. For more details, see the proof of [19, Proposition 2.1]. The connection between consistent random expansions, amenability and universal minimal flows will be made more explicit in Chapter 2.

### 1.5.7 Quotient Structures - $\mathcal{E}K$ and $K[\mathcal{L}]$

Now we introduce the product class $K[\mathcal{L}]$. This section is only relevant to Chapter 5, and may be safely skipped until then.

We mainly focus on the quotient structure $K[\mathcal{L}]$, but to introduce it we first mention the class $\mathcal{E}K$, which is used to define $K[\mathcal{L}]$. Intuitively, a structure in $K[\mathcal{L}]$ is a (horizontal) structure $\mathbb{K} \in K$ and associated to each point $k \in \mathbb{K}$ is a (vertical, possibly different) $\mathbb{L}_k \in \mathcal{L}$. The $K$-relations of elements in different $\mathbb{L}_k$ “columns” are given by looking at the $K$-relations of the corresponding $k \in \mathbb{K}$. In this way, if you “quotient out” by the equivalence relation of being in the same $\mathbb{L}_k$ column, then you get $\mathbb{K}$.

Alternatively, one can think of a structure in $K[\mathcal{L}]$ as taking a structure $\mathbb{K} \in K$ then “blowing-up” each of its points $k \in \mathbb{K}$ to a structure $\mathbb{L}_k \in \mathcal{L}$.

**Definition 1.5.2 ($\mathcal{E}K$).** Let $K$ be a class of structures in $L_I := \{R_i : i \in I\}$, a relational signature where each $R_i$ has arity $n_i$, and let $\sim$ be a binary relational symbol such that $\sim \not\in L_I$.

We denote by $\mathcal{E}K$ the class of relational structures of the form

$$A = (A, \{R_i^A\}_{i \in I}, \sim^A)$$

with the properties:
1. \(\sim^A\) is an equivalence relation on \(A\) with equivalence classes denoted by \([a]_{\sim^A}\).

2. For \(i \in I\) and \(x_1, \ldots, x_n, y_1, \ldots, y_n \in A\) with \(x_j \sim^A y_j\) (for all \(j \leq n_i\)) we have
\[
R_i^A(x_1, \ldots, x_n) \Leftrightarrow R_i^A(y_1, \ldots, y_n).
\]

3. Let \(A/\sim^A := \{[a]_{\sim^A} : a \in A\}\) be the set of equivalence classes. Let \(R_i^A/\sim^A\), for \(i \in I\), be the relation defined on the set \(A/\sim^A\) according to (2) with
\[
R_i^A/\sim^A([a_1]_{\sim^A}, \ldots, [a_n]_{\sim^A}) \Leftrightarrow R_i^A(x_1, \ldots, x_n)
\]
where \(a_j \sim^A x_j\) for all \(j \leq n_i\). Then we have
\[
\mathbb{A}/\sim^A := (A/\sim^A, \{R_i^A/\sim^A\}_{i \in I}) \in \mathbb{K}.
\]

**Definition 1.5.3 (\(\mathbb{K}[\mathcal{L}]\)).** Let \(L_I := \{R_i : i \in I\}\) and \(L_J := \{R_j : j \in J\}\) be disjoint relational signatures and let \(\sim\) be a binary relational symbol such that \(\sim \notin L_I \cup L_J\). Let \(\mathbb{K}\) and \(\mathcal{L}\) be classes of relational structures in \(L_I\) and \(L_J\) respectively.

We denote by \(\mathbb{K}[\mathcal{L}]\) the class of relational structures of the form
\[
\mathbb{A} = (A, \{R_i^A\}_{i \in I}, \{R_j^A\}_{j \in J}, \sim^A)
\]
with the properties:

1. \(\mathbb{A}|(L_I \cup \{\sim\}) \in \mathcal{E}\mathbb{K}\).

2. For \(j \in J\) and \(x_1, \ldots, x_n \in A\) we have
\[
R_j^A(x_1, \ldots, x_n) \Rightarrow [x_1]_{\sim^A} = \ldots = [x_n]_{\sim^A}.
\]

3. For \(a \in A\) we have
\[
([a]_{\sim^A}, \{R_j^A \cap ([a]_{\sim^A})^{n_j}\}_{j \in J}) \in \mathcal{L}.
\]

### 1.5.8 Expansions of \(\mathbb{K}[\mathcal{L}]\)

Let \(L_I^* \supset L_I\) and \(L_J^* \supset L_J\) be relational signatures such that \(L_i^* \cap L_j^* = \emptyset\) and \(\sim \notin L_i^* \cup L_j^*\). If \(\mathbb{K}^*\) and \(\mathcal{L}^*\) are expansions of the classes \(\mathbb{K}\) and \(\mathcal{L}\) such that \(\mathbb{K}^*|L_I = \mathbb{K}\) and \(\mathcal{L}^*|L_J = \mathcal{L}\) then we have that
\[
(\mathbb{K}^*|\mathcal{L}^*)|(L_I \cup L_J \cup \{\sim\}) = \mathbb{K}[\mathcal{L}].
\]

Let \(\mathbb{A} \in \mathbb{K}[\mathcal{L}]\) be the finite structure which has \(A_1, A_2, \ldots, A_a\) as its \(\sim^A\)-equivalence classes. Let \(\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_a\) be structures in \(\mathcal{L}\) which are placed on \(A_1, \ldots, A_a\) respectively and let \(\mathbb{B} \in \mathbb{K}\) be the structure given by representatives of the equivalence classes. Then we write \(\mathbb{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_a : \mathbb{B})\).

Similarly, an expansion \((\mathbb{A} \oplus \mathbb{A}^*) \in \mathbb{K}^*|\mathcal{L}^*\) of \(\mathbb{A} \in \mathbb{K}[\mathcal{L}]\) is given by the structures \((\mathbb{A}_1 \oplus \mathbb{A}_1^*), \ldots, (\mathbb{A}_a \oplus \mathbb{A}_a^*)\).
A_a^*) \in \mathcal{L}^* \text{ and } (B \oplus B^*) \in \mathcal{K}^*. \text{ So we write }

(A \oplus A^*) = ((A_1 \oplus A_1^*),\ldots,(A_a \oplus A_a^*):(B \oplus B^*))

or, if there is no confusion

$$A^* = (A_1^*,\ldots,A_a^*:B^*).$$

### 1.5.9 Excellent Pair Proposition

The following technical proposition ensures that $(\mathcal{K}[\mathcal{L}],\mathcal{K}^*[\mathcal{L}^*])$ is an excellent pair, thus we may apply Proposition 1.5.1 to verify amenability of $\text{Aut}(\text{Flim}(\mathcal{K}[\mathcal{L}]))$.

**Proposition 1.5.4.** Let $L_1^* \supset L_1$ and $L_J^* \supset L_J$ be relational signatures such that $L_1^* \cap L_J^* = \emptyset$ and let $\sim$ be a binary relational symbol such that $\sim \notin L_1^* \cup L_J^*$. Let $\mathcal{K},\mathcal{K}^*,\mathcal{L}$ and $\mathcal{L}^*$ be classes of finite relational structures in $L_1,L_1^*,L_J$ and $L_J^*$ respectively. Let $\mathcal{K}^|[L_1] = \mathcal{K}$ and $\mathcal{L}^|[L_J] = \mathcal{L}$. Then we have:

1. If $\mathcal{L}$ and $\mathcal{K}$ are Ramsey classes of rigid structures then $\mathcal{K}[\mathcal{L}]$ is a Ramsey class of rigid structures.

2. If $\mathcal{L}^*$ satisfies EP with respect to $\mathcal{L}$ and $\mathcal{K}^*$ satisfies EP with respect to $\mathcal{K}$, then $\mathcal{K}^*[\mathcal{L}^*]$ satisfies EP with respect to $\mathcal{K}[\mathcal{L}]$.

**Proof.** This follows by simple modifications of the proofs of Theorem 4.4 and Proposition 5.2 in [32]. \qed
Chapter 2

Forms of Amenability

Our goal in this section is to give five different formulations of amenability, extreme amenability and unique ergodicity of a group. They will be (1) in terms of the group acting on compact spaces, (2) the cardinality of the universal minimal flow $M(G)$, (3) the existence of a left-invariant measure on $M(G)$, (4) the existence of a consistent random expansion, and (5) the existence of a left-invariant mean. These notions will be explained and made precise in what follows.

For the most part unique ergodicity of a group will correspond to there being a unique witness to amenability in any of the previous formulations. Extreme amenability will be discussed here for the historical context, but will not be a key notion in the rest of the document. We will focus mostly on consistent random expansions as that is a setting that is well suited to the finite combinatorics of Fraïssé classes.

The material in this chapter is primarily a synthesis of material from [3], [30] and [27].

2.1 What is a Universal Minimal Flow $M(G)$?

A key object in the study of topological dynamics is the universal minimal flow $M(G)$. It is a topological space associated to a topological group $G$ that in some sense captures the essential features of the group’s action on compact Hausdorff spaces.

**Definition 2.1.1.** Let $G$ be a Hausdorff topological group, and let $X$ be a compact Hausdorff space. We say that $X$ is a $G$-flow provided that $G$ acts continuously on $X$. We denote this by $G \curvearrowright X$. We denote the action in the usual way with $g \cdot x$, for $g \in G$ and $x \in X$.

**Definition 2.1.2.** A $G$-flow $X$ is minimal provided that the orbit of each point in $X$ is dense. i.e. $\forall x \in X, G \cdot x = X$.

**Definition 2.1.3.** A $G$-flow $X$ is universal provided that every minimal $G$-flow $Y$ is the continuous image of $X$. i.e. for every $G \curvearrowright Y$ minimal, there is a continuous surjection $\pi : X \to Y$ such that $\forall g \in G, \forall x \in X$ we have $\pi(g \cdot x) = g \cdot \pi(x)$.

2.1.1 Existence of Universal Minimal Flows

We first remark on the existence of such objects.
Chapter 2. Forms of Amenability

Theorem 2.1.4 (Ellis 1969, [8]). Every $G$ has a unique universal, minimal $G$-flow denoted $M(G)$.

An easy application of the Zorn’s Lemma shows the following.

Theorem 2.1.5 (Ellis’ Lemma). Every compact (left semitopological) semigroup contains an idempotent.

In general a $G$-flow may have many minimal subflows. Ellis’ Lemma is used to show that any two minimal subflows of the greatest ambit $S(G)$ are isomorphic, which in turn gives the existence of a unique universal minimal $G$-flow. We will not have need for the greatest ambit here, so we will not provide a definition. For a more in-depth discussion, including all relevant definitions, see Section 6.1 in [30], Chapter 8 of [4], or [35].

2.1.2 Properties of $M(G)$

In the case of $G$ being a compact topological group itself, the situation is clear.

Proposition 2.1.6 (Folklore). If $G$ is a compact topological group, then $M(G) = G$.

In general, given a group $G$ it is difficult to give an explicit description of $M(G)$.

Theorem 2.1.7. Let $G$ be a topological group.

1. [8] If $G$ is an infinite discrete group, then $M(G)$ is not metrizable.
2. [18, A2.2] If $G$ is locally compact, non-compact, then $M(G)$ is not metrizable.
3. [35, 1.1] The action of $G$ on $M(G)$ is not 3-transitive, provided $|M(G)| \geq 3$.

In general (but not exclusively) we are concerned with groups that are subgroups of Homeo($X$) for $X$ a compact topological space, or Aut($X$) where $X$ is a first order structure in some (usually finite) language. Note that Homeo($X$) acts in the natural way on the compact space $2^X$ (given the product topology), and Aut($X$) acts in the natural way on the compact space $2^X$ (also given the product topology).

2.1.3 Examples of Universal Minimal Flows

Three early, pre-[18]-examples that were studied are the orientation preserving homeomorphisms of the circle Homeo$_+$($S^1$) [29], $S_\infty$ [11], and the homeomorphism group of the Cantor set Homeo($C$) [12].

Theorem 2.1.8 (Pestov 1998). If $G = \text{Homeo}_+ (S^1)$, then $M(G) = S^1$.

Theorem 2.1.9 (Glasner-Weiss 2002). If $G = S_\infty$, then $M(G) = \text{LO}(\mathbb{N})$, the collection of all linear orders on $\mathbb{N}$.

Theorem 2.1.10 (Glasner-Weiss 2003). If $G = \text{Homeo}(C)$, then $M(G)$ is the space of maximal chains in $C$.

A more general version of the 2003 Glasner-Weiss theorem can be found in [10].

2.2 Extreme Amenability

Pestov’s example was particularly interesting as it was the first known example of a non-compact group whose universal minimal flow is metrizable and is more than a single point.

Definition 2.2.1. A Hausdorff topological group $G$ is said to be extremely amenable provided that for all $G$-flows $X$ there is a fixed point $x_0 \in X$, i.e. $G \cdot x_0 = x_0$. 

2.2.1 Relation to Amenability

Extreme amenability should be contrasted with one particular definition of amenability, of which extreme amenability is an obvious strengthening:

**Definition 2.2.2.** A Hausdorff topological group $G$ is said to be **amenable** provided that for every affine continuous action of $G$ on any convex compact space $X$, there is a fixed point $x_0 \in X$, i.e. $G \cdot x_0 = x_0$.

2.2.2 Relation to $M(G)$

In the case of extremely amenable groups, the universal minimal flow is easy to compute; it is a single point.

**Theorem 2.2.3** *(Folklore).* $G$ is an extremely amenable group iff $|M(G)| = 1$.

**Proof.** Suppose first that $G$ is extremely amenable. Take any compact $G$-flow $X$, and let $x_0$ be a fixed point of this action. Clearly any minimal subflow must only be $\{x_0\}$. Thus the universal minimal flow is a single point.

Alternatively, suppose that $M(G) = \{e\}$. Let $X$ be a $G$-flow. Let $Y$ be a minimal subflow of $X$. By universality of $M(G)$, $Y$ must also be a single point that is fixed by $G$. \qed

2.2.3 Examples

We have already seen some examples of groups that are not extremely amenable, but now let us present some that are.

**Theorem 2.2.4** *(Pestov 1998, [29]).* The following groups are extremely amenable:

1. The orientation preserving order-isomorphisms of $\mathbb{Q}$, $\text{Aut}_+(\mathbb{Q})$.
2. The orientation preserving homeomorphisms of $[0,1]$ (or of $\mathbb{R}$).

Extreme amenability of $\text{Aut}_+(\mathbb{Q}, <)$ is actually equivalent to the classical Ramsey theorem. See section 1.5 of [29] for more details.

2.2.4 General Results

There are three general theorems which are of the flavour “topologically small groups cannot be extremely amenable”. To show that a group is not extremely amenable, it is enough to produce a $G$-flow $X$ without fixed points (i.e. $\forall x \in X, \exists g \in G$ such that $g \cdot x \neq x$). There is the stronger notion of $G$ acting **freely** on $X$, which means $\forall x \in X, \forall g \in G \setminus \{e_G\}, g \cdot x \neq x$. Clearly, if there is a compact space $X$ on which $G$ acts continuously and freely, then $G$ is not extremely amenable.

**Theorem 2.2.5.**

1. *(Folklore)* If $G$ is compact and nontrivial, then $G$ is not extremely amenable. Moreover, $G$ acts freely on itself by the group action.
2. [8] If $G$ is discrete, then $G$ is not extremely amenable. Moreover, $G$ admits a free action on $\beta(G)$.
3. [36] If $G$ is locally compact, then $G$ is not extremely amenable. Moreover, $G$ admits a free action on a compact space.
2.3 Relationship with Expansions

2.3.1 Computing $M(G)$ via Expansions

If $G$ is the group of automorphisms of a Fraïssé class, the construction of $M(G)$ comes via the Kechris-Pestov-Todorcevic correspondence. Computation of the minimal flow of $\text{Aut}(\text{Flim}(K))$ hinges on the existence of an expansion $K^*$, where $(K, K^*)$ is an excellent pair, even though $\text{Aut}(\text{Flim}(K))$ makes no reference to the expansion.

To do this we consider $X_{K^*}$ which is the space of all expansions $(K \oplus K^*)$ of $K := \text{Flim}(K)$, such that for any $A \in K$, $K^* \upharpoonright A \in K^*$. For example, if $K$ the collection of all finite graphs and $K^*$ is the collection of such graphs each equipped with a linear order on its vertices, then $X_{K^*}$ is the collection of all linear orders on the vertices of the random graph, that is, $X_{K^*} = \text{LO}(\mathbb{N})$.

In the case of a binary expansion, we consider $X_{K^*}$ as a subspace of the compact space $2^K$ (given the product topology). Note that $X_{K^*}$ itself is actually a closed and therefore compact subspace of $2^K$, since “not being an expansion” is detectable by using only finitely many pairs in $K^2$. In particular this happens when $K^*$ is an order expansion. When the expansion involves a relation of arity $n$, then we instead look at $2^K$.

We see that $\text{Aut}(\text{Flim}(K))$ acts continuously on this compact space $X_{K^*}$, so we have a $\text{Aut}(\text{Flim}(K))$-flow. The dynamics of this flow are given by the EP and RP of $K^*$. The following appears essentially in [18, 7.4, 10.8], but this statement is from [3, 9.1].

**Theorem 2.3.1** (Kechris-Pestov-Todorcevic 2005). Let $(K, K^*)$ be a reasonable pair in signature $L$. Then the following are equivalent:

1. $X_{K^*}$ is a minimal $\text{Aut}(\text{Flim}(K))$-flow;
2. $K^*$ satisfies EP (relative to $K$).

Moreover, the following are equivalent:

1. $X_{K^*}$ is the universal minimal flow of $\text{Aut}(\text{Flim}(K))$-flow;
2. $K^*$ satisfies EP and RP (relative to $K$).

In other words, given a Fraïssé class $K$, one can compute $M(\text{Aut}(\text{Flim}(K)))$ by finding an expansion class $K^*$ such that $(K, K^*)$ is an excellent pair. In terms of computing universal minimal flows this has been a very successful correspondence. See section 5 of [27] for a complete, historical survey of how this correspondence has been used.

2.3.2 Why Rigid Structures?

It is natural to ask about the use of rigid structures. Nguyen Van Thé explains the importance of rigidity as follows in [27]:

The importance of linear orderings, and more generally of rigidity, in relation to the Ramsey property was realized pretty early. A structure is rigid when it admits no non-trivial automorphism. Essentially, all Ramsey classes must be made of rigid structures. Sets, Boolean algebras and vector spaces do not fall into that category, but those being Ramsey is equivalent to some closely related classes of rigid structures being Ramsey. To use the common jargon, rigidity prevents the appearance of Sierpiński type colorings, which do not stabilize on large sets.
“Sierpiński type colorings” are the ones that come from fixing two linear orders on the same structure and colouring the pairs of elements red if the two orders agree, and blue if the orders disagree.

The importance of rigidity is further explained in the following theorem from [23]:

**Theorem 2.3.2** (Nešetřil-Rödl 1977). Let \( \mathcal{K} \) be a class of finite, rigid structures in language \( L \). If \( \mathcal{K} \) satisfies the HP, JEP and RP, then \( \mathcal{K} \) satisfies the AP, i.e. it is a Fraïssé class.

### 2.4 Amenability via Measure

Amenability can also be expressed in the language of measures. Later chapters will be in the setting of Fraïssé classes where we will use the more finitary notion of consistent random expansions.

**Definition 2.4.1.** Let \( G \) be a Hausdorff topological group acting on a topological space \( X \) with \( \mathcal{B} \) the Borel measurable subsets of \( X \). We say that \( \mu : \mathcal{B} \to [0,1] \) is a \((G\text{-invariant Borel Probability})\) measure on \( X \) provided that

1. \( \mu(\emptyset) = 0 \) and \( \mu(X) = 1 \) (probability);
2. \( \forall B \in \mathcal{B}, \forall g \in G \) we have \( \mu(g \cdot B) = \mu(B) \) (\( G \)-invariant);
3. \( \mu(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n) \) provided that all the \( B_n \in \mathcal{B} \) are pairwise disjoint (\( \sigma \)-additive).

We will sometimes refer to a measure as an invariant measure if we wish to emphasize the group action. For a historical discussion of the relation of amenability to existence of invariant measures, see Chapter 12 of [4].

**Proposition 2.4.2** (See above remark). Let \( G \) be a topological group. \( G \) is amenable iff every \( G \)-flow has an invariant measure.

For amenability, one only has to check that the universal minimal flow admits an invariant measure. The situation for unique ergodicity is not so simple, as a non-minimal \( G \)-flow might have two disjoint subflows each with its own measure, in which case any convex combination of these measures would yield a new measure. This leads to the following definition.

**Definition 2.4.3.** A topological group \( G \) is **uniquely ergodic** if every minimal \( G \)-flow admits a unique invariant measure.

There are some obvious connections between amenability, unique ergodicity and extreme amenability.

**Proposition 2.4.4** (Folklore). Let \( G \) be a topological group.

1. If \( G \) is extremely amenable, then \( G \) is uniquely ergodic.
2. If \( G \) is compact, then \( G \) is uniquely ergodic.

**Proof.** The first part is obvious since \( M(G) \) is just a single point. For the second part, \( G = M(G) \) since \( G \) is a compact topological group, so the Haar measure is the only \( G \)-invariant probability measure. \( \square \)

The following appears as [3, 8.1], although the first part is obvious.

**Theorem 2.4.5** (Angel-Kechris-Lyons 2012). Let \( G \) be a topological group.

1. \( G \) is amenable iff \( M(G) \) admits an invariant measure.
2. \( G \) is uniquely ergodic iff \( M(G) \) admits a unique invariant measure.
2.5 Amenability via Means

For the sake of completeness, we include the historical definition of amenable: “Admitting a mean”.

For a locally compact group \( G \), let \( L^\infty(G) \) be the collection of bounded measurable functions from \( G \) to \( \mathbb{C} \), with respect to Haar measure.

**Definition 2.5.1.** Let \( G \) be a locally compact group. A (multiplicative left \( G \)-invariant) mean is a linear functional \( \Lambda : L^\infty(G) \to \mathbb{C} \) such that

1. \( \Lambda \) is positive (\( f \geq 0 \) implies \( \Lambda(f) \geq 0 \)), of bounded norm 1 and \( \Lambda(\xi_G) = 1 \) (mean);
2. \( \Lambda(g \cdot f) = \Lambda(f) \) for all \( g \in G, f \in L^\infty(G) \) where \( (g \cdot f)(x) = f(g^{-1}x) \) (left \( G \)-invariant);
3. \( \Lambda(fh) = \Lambda(f)\Lambda(h) \) for all \( f, h \in L^\infty(G) \) (multiplicative).

There is an analogous definition for right \( G \)-invariant, and bi-invariant (both left and right invariant).

**Proposition 2.5.2** (Følner 1955). Let \( G \) be a locally compact group. \( G \) is amenable iff it admits a multiplicative \( G \)-bi-invariant mean.

The use of local compactness is a historical, practical one, not a necessary one; see [9]. Traditionally amenability was studied in the setting of locally compact groups, where amenability is equivalent to the so-called Følner condition. By replacing the domain of \( \Lambda \) with \( RUCB(G) \), the right uniformly continuous bounded functions from \( G \) to \( \mathbb{R} \), and by relaxing invariance to left-invariance, one gets the general definition of amenability.

2.6 Amenability via Consistent Random Expansions

Returning to \( X_{K^*} \), there is an alternate formulation of amenability which is well-suited to the finite combinatorics of Fraïssé classes \( K \).

**Definition 2.6.1.** We say that the collection \( \{\mu_A : A \in K\} \) is a consistent random \( K^* \)-expansion on \( K \) (when it is clear from context we suppress the reference to \( K^* \)) if

(P) Each \( \mu_A \) is a probability measure on \( \{A^* : (A \oplus A^*) \in K^*\} \).

(I) Whenever \( \varphi : A \to B \) is an isomorphism then \( \varphi_*\mu_A = \mu_B \), where \( \varphi_*\mu_A \) is the push forward measure.

(E) For every \( A \subseteq B \) in \( K \) and each \( (A \oplus A^*) \in K^* \) we have,

\[
\mu_A(\{A^*\}) = \sum \{\mu_B(\{B^*\}) : (A \oplus A^*) \leq (B \oplus B^*) \in K^*\}.
\]

When it is clear from context we shall refer to a consistent random expansion as \( (\mu_A) \), with no reference to \( K \).

The result that appears as Proposition 9.2 in [3] is a special case of the following theorem. This is the key equivalence through which we demonstrate the amenability and unique ergodicity of automorphism groups of structures on Cherlin’s list.
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Theorem 2.6.2 (Key Equivalence). Let \((\mathcal{K}, \mathcal{K}^*)\) be an excellent pair. Then:

1. \(\text{Aut}(\text{Flim}(\mathcal{K}))\) is amenable iff \(\mathcal{K}\) has a consistent random \(\mathcal{K}^*\)-expansion.

2. \(\text{Aut}(\text{Flim}(\mathcal{K}))\) is uniquely ergodic iff \(\mathcal{K}\) has a unique consistent random \(\mathcal{K}^*\)-expansion.

Before we give the proof, let us recall an important extension theorem for constructing measures. To that end we also need to define some terms only relevant to us for that theorem.

Definition 2.6.3. • A finitely additive pre-measure \(\mu\) is a measure with “\(\sigma\)-additive” weakened to “finitely-additive”.

• A finitely additive pre-measure \(\mu : \mathcal{A} \to [0, +\infty)\) is countably monotone if
  1. \(\mu(\bigcup_{n \in \mathbb{N}} A_n) = \lim_n \mu(A_n)\) provided that all the \(A_n \in \mathcal{A}\) and \(A_n \subseteq A_{n+1}\) for all \(n\); and
  2. \(\mu(\bigcap_{n \in \mathbb{N}} A_n) = \lim_n \mu(A_n)\) provided that all the \(A_n \in \mathcal{A}\) and \(A_{n+1} \subseteq A_n\) for all \(n\).

• A measure \(\mu\) (on a \(\sigma\)-algebra \(\mathcal{B}\) and set \(X\)) is \(\sigma\)-finite if \(X\) is the countable union of sets in \(\mathcal{B}\), each of finite measure.

• The outer measure \(\mu^*\) (relative to \(\mu\)) is the monotonic, countably subadditive map

\[
\mu^*(E) := \inf \left\{ \sum_{i \in \mathbb{N}} \mu(A_i) : A_i \in \mathcal{A}, E \subseteq \bigcup_{i \in \mathbb{N}} A_i \right\}.
\]

Theorem 2.6.4 (Carathéodory’s Extension Theorem). Let \(\mathcal{A}\) be an algebra of sets, let \(\mathcal{B}\) be the smallest \(\sigma\)-algebra generated by \(\mathcal{A}\), and let \(\mu : \mathcal{A} \to [0, +\infty)\) be a finitely additive pre-measure that is countably monotone. Then \(\mu\) extends to the (\(\sigma\)-additive) measure \(\mu^*\) on \(\mathcal{B}\). Moreover, if \(\mu\) is \(\sigma\)-finite, then the extension is unique.

Proof of Theorem 2.6.2. We will use the formulation of amenability that \(G\) is amenable iff \(M(G)\) admits an invariant measure. The idea here is that there is a one-to-one, natural correspondence between the structures in \(\mathcal{K}^*\) and the basic open sets in \(X_{\mathcal{K}^*}\). Let \(K\) be the underlying universe of \(X_{\mathcal{K}^*}\).

For \(A \subseteq K\) finite, and \((A \oplus A^*) \in \mathcal{K}^*\), define the basic open set

\[
N(A \oplus A^*) := \{(K \oplus K') \in X_{\mathcal{K}^*} : K' | A = A^*\}.
\]

Note that since \(\mathcal{K}^*\) is a precompact expansion, each structure \(A \in \mathcal{K}\) has only finitely many expansions, so \(N(A \oplus A^*)\) is in fact clopen. We consider the algebra \(\mathcal{A}\) generated by these clopen sets \(N(A \oplus A^*)\).

\[\Rightarrow\] Suppose that \(\mu\) is a \(\text{Aut}(\text{Flim}(\mathcal{K}))\)-invariant measure on \(X_{\mathcal{K}^*}\). For \(A \in \mathcal{K}\) and \((A \oplus A^*) \in \mathcal{K}^*\) define

\[
\mu_A(\{A^*\}) := \mu(N(A \oplus A^*)).
\]

We now check that \((\mu_A)\) is a consistent random expansion. (P) This follows immediately from the fact that \(\mu\) is a probability measure. (I) This follows immediately from \(\mu\) being \(\text{Aut}(\text{Flim}(\mathcal{K}))\)-invariant.
and \( \mathcal{K} \) being ultrahomogeneous. (E) Fix \( A \subseteq \mathbb{B} \) in \( \mathcal{K} \) and \( (A \oplus A^*) \in \mathcal{K}^* \). Observe,
\[
\mu_{\mathcal{K}}(\{A^*\}) = \mu(N(A \oplus A^*))
= \mu \left( \bigcup \{N(B \oplus B^*) : (A \oplus A^*) \leq (B \oplus B^*) \in \mathcal{K}^* \} \right)
= \sum \{\mu(N(B \oplus B^*)) : (A \oplus A^*) \leq (B \oplus B^*) \in \mathcal{K}^* \}
= \sum \{\mu_{\mathcal{K}}(B^*) : (A \oplus A^*) \leq (B \oplus B^*) \in \mathcal{K}^* \}
\]

Note that the union on the second line is disjoint, and finite since \( \mathcal{K}^* \) is a precompact expansion.

\([\Leftarrow]\) Suppose that \( (\mu_{\mathcal{K}}) \) is a consistent random expansion. Define \( \mu \) on the basic open sets of \( X_{\mathcal{K}^*} \) by:
\[
\mu(N(A \oplus A^*)) := \mu_{\mathcal{K}_0}(\{A_0^*\}),
\]
where \( A_0 \subseteq \mathcal{K} \) and \( (A \oplus A^*) \equiv (A_0 \oplus A_0^*) \). This is well defined since \( \mathcal{K} \) is ultrahomogeneous and by (I).

Extend \( \mu \) to all of \( \mathcal{A} \) by defining:
\[
\mu(N(A \oplus A^*) \cap N(B \oplus B^*)) := \sum \{\mu_{\mathcal{K}_0}(\{C_0^*\}) : (A_0 \oplus A_0^*), (B_0 \oplus B_0^*) \leq (C_0 \oplus C_0^*) \in \mathcal{K}^* \}
\]

where \( C_0 \subseteq \mathcal{K} \) and \( A, B \rightarrow \mathbb{C} \) (which is well defined for similar reasons to the previous case). There is an obvious extension of this definition to the intersection of finitely many \( N((A_i \oplus A_i^*)) \). Similarly, for \( A, B \in \mathcal{A} \) define:
\[
\mu(A \cup B) := \mu(A) + \mu(B) - \mu(A \cap B).
\]

Note that, by distributing, every element in \( \mathcal{A} \) can be written as the finite union of finite intersections of elements of the form \( N((A_i \oplus A_i^*)) \), so \( \mu \) is indeed defined on all of \( \mathcal{A} \).

(Rightness Check) Suppose \( (A \oplus A^*) \leq (B \oplus B^*) \), or equivalently \( N(B \oplus B^*) \subseteq N(A \oplus A^*) \). Then
\[
\mu(N(A \oplus A^*) \cap N(B \oplus B^*)) = \sum \{\mu_{\mathcal{K}_0}(\{C_0^*\}) : (A_0 \oplus A_0^*), (B_0 \oplus B_0^*) \leq (C_0 \oplus C_0^*) \in \mathcal{K}^* \}
= \sum \{\mu_{\mathcal{K}_0}(\{C_0^*\}) : (B_0 \oplus B_0^*) \leq (C_0 \oplus C_0^*) \in \mathcal{K}^* \}
= \mu(N(B \oplus B^*)),
\]

and,
\[
\mu(N(A \oplus A^*) \cup N(B \oplus B^*)) = \mu(N(A \oplus A^*)) + \mu(N(B \oplus B^*)) - \mu(N(A \oplus A^*) \cap N(B \oplus B^*))
= \mu(N(A \oplus A^*)) + \mu(N(B \oplus B^*)) - \mu(N(B \oplus B^*))
= \mu(N(A \oplus A^*)).
\]

We now check that \( \mu \) is a finitely additive pre-measure.

(Probability) Clearly \( \mu(\emptyset) = 0 \) and \( \mu(X_{\mathcal{K}^*}) = 1 \).

(Countably Monotone) Since \( X_{\mathcal{K}^*} \) is compact and the basic open sets are clopen, every increasing union of basic clopen sets is finite, and similarly every decreasing intersection of basic clopen sets is finite. Since \( \mu \) is obviously “finitely monotone”, we have that \( \mu \) is countably monotone.

(Finitely Additive) This is obvious from the definition.
(Invariance on \(A\)) Let \(g \in \text{Aut}(\text{Flim}(\mathcal{K}))\) and \((A \oplus A^*) \in \mathcal{K}^*\). By (I) we have that
\[
\mu(g \cdot N((A_0 \oplus A_0^*))) = \mu(N(g \cdot A_0 - g_* \cdot A_0^*)) = \mu(N((A_0 \oplus A_0^*))),
\]
where \(g_*\) is the induced map on the expansions. Now since \(g\) commutes with finite unions and intersections we have invariance on \(A\).

(Extension to \(\mu^*\)) Now we use the Carathéodory Extension Theorem to conclude that \(\mu^*\) is a \((\sigma\text{-additive})\) measure on the entire Borel \(\sigma\)-algebra \(\mathcal{B}\) of \(X_{\mathcal{K}}^*\). It only remains to verify that \(\mu^*\) is \(\text{Aut}(\text{Flim}(\mathcal{K}))\)-invariant on sets from \(\mathcal{B}\).

(Invariance on \(B\)) Let \(E \in \mathcal{B}\), and let \(A_i \in \mathcal{A}\), for all \(i \in \mathbb{N}\), with \(E \subseteq \bigcup_{i \in \mathbb{N}} A_i\), and let \(g \in \text{Aut}(\text{Flim}(\mathcal{K}))\). Note that
\[
g \cdot E \subseteq g \cdot \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \bigcup_{n \in \mathbb{N}} g \cdot A_n
\]
A similar argument, applying instead \(g^{-1}\), shows that \(E \subseteq \bigcup_{n \in \mathbb{N}} A_n\) iff \(E \subseteq \bigcup_{n \in \mathbb{N}} g \cdot A_n\). Thus
\[
\mu^*(E) = \inf \left\{ \sum_{i \in \mathbb{N}} \mu(A_i) : A_i \in \mathcal{A}, E \subseteq \bigcup_{i \in \mathbb{N}} A_i \right\}
= \inf \left\{ \sum_{i \in \mathbb{N}} \mu(g \cdot A_i) : A_i \in \mathcal{A}, E \subseteq \bigcup_{i \in \mathbb{N}} A_i \right\}
= \inf \left\{ \sum_{i \in \mathbb{N}} \mu(g \cdot A_i) : A_i \in \mathcal{A}, g \cdot E \subseteq \bigcup_{i \in \mathbb{N}} g \cdot A_i \right\}
= \mu^*(g \cdot E),
\]
where the final equality follows since \(g \cdot \mathcal{A} = \mathcal{A}\).

(Uniqueness) Since Carathéodory’s theorem guarantees uniqueness of the full measure, it is enough to remark that two distinct consistent random expansions would give rise to distinct measures.
2.7 Summary

We now capture all the equivalent notions of amenability, extreme amenability and unique ergodicity discussed previously. Note that there are many other equivalent versions of amenability in the case that the group is locally compact or discrete, but we will not have use for them presently, so they are omitted.

The historical attributions may be found earlier in the chapter.

Theorem 2.7.1 (Various). Let $G$ be a topological group. The following are equivalent.

1. $G$ is amenable.
2. Every $G$-flow admits an invariant measure.
3. $M(G)$ admits an invariant measure.
4. Every affine continuous action of $G$ on any convex compact space $X$ has a fixed point.
5. $G$ admits a multiplicative $G$-bi-invariant mean on $\text{RUCB}(G)$.

In the case that $G = \text{Aut} (\text{Flim}(K))$, this is equivalent to:
6. $K$ has a consistent random $K^*$-expansion.

Theorem 2.7.2 (Various). Let $G$ be a topological group. The following are equivalent.

1. $G$ is extremely amenable.
2. $M(G)$ is a single point.
3. Every continuous action of $G$ on a compact space $X$ has a fixed point.

Theorem 2.7.3 (Various). Let $G$ be a topological group. The following are equivalent.

1. $G$ is uniquely ergodic.
2. Every minimal $G$-flow admits a unique $G$-invariant measure.
3. $M(G)$ admits a unique $G$-invariant measure.

In the case that $G = \text{Aut} (\text{Flim}(K))$, this is equivalent to:
4. $K$ has a unique consistent random $K^*$-expansion.
In this chapter we will verify that four automorphism groups from Cherlin’s list are nonamenable: \( \text{Aut}(\mathcal{P}) \), \( \text{Aut}(\mathcal{S}(2)) \), \( \text{Aut}(\mathcal{S}(3)) \) and \( \text{Aut}(\mathcal{P}(3)) \). The first three were previously known, although our presentation will be in the language of consistent random expansions. A fifth group, \( \text{Aut}(\hat{\mathcal{Q}}) \) is also nonamenable, but its presentation is left for Section 4.5 and Theorem 4.5.3 where a related amenable group is discussed. Finally, in Section 3.5 we will discuss \( \text{Aut}(\mathcal{S}(n)) \), for \( n \geq 4 \), although these \( \mathcal{S}(n) \) do not appear on Cherlin’s list.

One advantage of using the language of consistent random expansions is that nonamenability is detectable by purely finite objects. In our case the finite objects that witness nonamenability will be a pair of directed graphs with very few nodes (fewer than 6 nodes each), an embedding of one into the other, and an expansion on one of the directed graphs. We answer the question “How many ways are there to expand the second graph subject to the prescribed expansion?” by an analysis of the geometry involved.

In the following cases, we often get that there is only one possible expansion if we choose an appropriate embedding. However, a different embedding will give at least two different expansions, one of which is the expansion we got from the first embedding.

Together this tells us that any consistent random expansion would have to assign measure 0 to some nontrivial structure, which can’t happen.

3.0.1 Attributions

Corollaries 3.2.2 and 3.2.2, and Theorem 3.4.1 are due to Sokić.

3.1 \( \mathcal{P} \), the Generic Partial Order

The first structure we will look at is the generic partial order \( \mathcal{P} \). The advantage of looking at this digraph first is that the structures, expansions and arguments will all be simple, so it serves as a good introduction to our notation.

Let \( \mathcal{P} \) be the class of all finite partial orders in signature \( \{\leq\} \), which is a Fraïssé class by [7]. Equivalently, we may think of \( \mathcal{P} \) as being the class of finite partial orders in the signature \( \rightarrow^P \), where

\[
a \rightarrow^P b \iff a < b.\]
This is the natural way to associate a directed graph structure to a partially ordered set.

Let $\mathcal{P}^*$ be the collection of structures $(P, <^P, \prec^P)$ where $(P, <^P) \in \mathcal{P}$ and $\prec^P$ is a linear order on $P$ that extends $<^P$. This is also a Fraïssé class by [28], and $(\mathcal{P}, \mathcal{P}^*)$ is an excellent pair by [23], or see [19, Section 3].

The main idea of the following proof is from [19, Section 3].

**Theorem 3.1.1.** $\text{Aut}(\mathcal{P})$ is not amenable.

**Proof.** Since $(\mathcal{P}, \mathcal{P}^*)$ is an excellent pair, it is enough to show that there is no consistent random $\mathcal{P}^*$ expansion of $\mathcal{P}$, by Proposition 1.5.1.

Suppose for the sake of contradiction that such an expansion $(\mu_k)$ exists. Consider the structures $A, B, C \in \mathcal{P}$ where $C = (C, <^C), C = \{a, b, c\}, a <^C b$, with $A = C \upharpoonright \{a, c\}$ and $B = C \upharpoonright \{b, c\}$.

Consider also expansions (in $\mathcal{P}^*$) of these structures. Let

- $(A \oplus A^*) = (A, \prec^A)$, where $a \prec^A c$;
- $(B \oplus B^*) = (B, \prec^B)$, where $b \prec^B c$.

![Figure 3.1](image1.png)

Figure 3.1: $A, (A \oplus A^*), B, (B \oplus B^*)$ and $C$.

Now we consider the possible expansions $(C \oplus C^*) = (C, <^C, \prec^C)$ with $(B \oplus B^*) \leq (C \oplus C^*)$. We claim that there is only one such expansion possible.

![Figure 3.2](image2.png)

Figure 3.2: $(C \oplus C^0)$ and $(C \oplus C^1)$.

We ask “Is $a \prec^C c$?”. A positive answer yields the expansion $C^0$, where $a \prec^C b \prec^C c$ and $a \prec^C c$, a linear order. A negative answer would yield the cycle $a \prec^C b \prec^C c \prec^C a$, which cannot happen in a linear order. (This is the key geometric observation about $\mathcal{P}^*$.)

Therefore:

$$
\mu_{B^*}(\{B^*\}) \equiv \sum \{\mu_C(\{C'\}) : (B \oplus B^*) \leq (C \oplus C') \in \mathcal{P}^*\} \\
= \mu_C(\{C^0\}),
$$

Now we consider the possible expansions $(C \oplus C^*) = (C, <^C, \prec^C)$ with $(A \oplus A^*) \leq (C \oplus C^*)$. We claim that there are two such expansions possible.
We ask “Is \( b \prec^C c \)?”. A positive answer yields the expansion \( C^0 \). A negative answer yields the expansion \( C^1 \), where \( a \prec^C c \prec^C b \) and \( a \prec^C b \), a linear order.

Therefore:

\[
\mu_k(\{A^*\}) \overset{\text{(3)}}{=} \sum \{ \mu_C(\{C'\}) : (A \oplus A^*) \leq (C \oplus C') \in \mathcal{P}^* \} \\
= \mu_C(\{C^0\}) + \mu_C(\{C^1\})
\]

Since \( A \cong B \) and \((A \oplus A^*) = (B \oplus B^*)\) we have

\[
\mu_C(\{C^0\}) = \mu_A(\{A^*\}) = \mu_B(\{B^*\}) = \mu_C(\{C^0\}) + \mu_C(\{C^1\})
\]

So \( \mu_C(\{C^1\}) = 0 \), which is impossible for a consistent random expansion. \( \square \)

A similar argument will be used to show that \( \mathbb{P}(3) \), the so-called “twisted generic poset” has nonamenable automorphism group. In that case there is an extra layer of notation which somewhat obscures the argument.

### 3.2 \( S(2) \), the Dense Local Ordering

Let \( L \) and \( R \) be unary relational symbols (\( \text{Left} \) and \( \text{Right} \)). We consider the structure \( S(2)^* \) in the signature \( \{ \to, L, R \} \) where

- \( S(2)^*|\{\to\} = S(2) \),
- \( L^{S(2)}(x) \iff x \in e^{i\theta} \) and \( \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \), and
- \( R^{S(2)}(x) \iff x \in e^{i\theta} \) and \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \).

Set \( S(2) := \text{Age}(S(2)) \) and \( S(2)^* := \text{Age}(S(2)^*) \).

The main idea of the following proof is from [38, Theorem 3.1].

**Theorem 3.2.1.** \( \text{Aut}(S(2)) \) is not amenable.

**Proof.** Since \((S(2), S(2)^*)\) is an excellent pair, see [20, 26], it is enough to show that there is no consistent random \( S(2)^* \) expansion of \( S(2) \), by Proposition 1.5.1.

Suppose for the sake of contradiction that such an expansion \( (\mu_k) \) exists. Consider the structures \( A, B, C \in S(2) \) where \( A \leq B, A \leq C \) and \( B \cap C = A \), which can be done by the \textbf{SAP}. Let:

- \( A = (A, \to^A), A = \{x, y\} \) and \( x \to^A y \);
- \( B = (B, \to^B), B = A \cup \{b\}, x \to^B b \) and \( b \to^B y \);
- \( C = (C, \to^C), C = A \cup \{c\}, x \to^C c \) and \( y \to^C c \).

Consider also expansions (in \( S(2)^* \)) of these structures. Let:

- \( (A \oplus A^*) = (A, L^A, R^A) \), where \( L^A = \{x, y\} \);
- \( (B \oplus B^*) = (B, L^B, R^B) \), where \( L^B = \{x, y, b\} \);
- \( (C \oplus C^*) = (C, L^C, R^C) \), where \( L^C = \{x, y, c\} \).
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Figure 3.3: \((A \oplus A^*), (B \oplus B^*), (C \oplus C^*)\) and \((C \oplus C^{**})\).

- \((C \oplus C^{**}) = (C, L^{**}, R^{**})\), where \(L^{**} = \{x, y\}, R^{**} = \{c\}\).

This gives us

\[
\mu_A(\{A^*\}) = \sum \mu_B(\{B^*\}) : (A \oplus A^*) \leq (B \oplus B^*) \in S(2)^*
\]

\[
= \mu_A(\{B^*\})
\]

since there is only one expansion of \(B\) in \(S(2)^*\) that extends \((A \oplus A^*)\), namely \((B \oplus B^*)\).

Also, we have

\[
\mu_A(\{A^*\}) = \sum \mu_C(\{C^*\}) : (A \oplus A^*) \leq (C \oplus C^*) \in S(2)^*
\]

\[
= \mu_C(\{C^*\}) + \mu_C(\{C^{**}\})
\]

since there are exactly two expansions of \(C\) in \(S(2)^*\) which extend \((A \oplus A^*)\), namely \((C \oplus C^*)\) and \((C \oplus C^{**})\).

Since \(B \cong C\) and \((B \oplus B^*) \cong (C \oplus C^*)\) we have

\[
\mu_C(\{C^*\}) = \mu_B(\{B^*\}) = \mu_C(\{C^*\}) + \mu_C(\{C^{**}\})
\]

So \(\mu_C(\{C^{**}\}) = 0\), which is impossible for a consistent random expansion.

We can also state a related result. Let \(\mathcal{OS}(2)\) be the class of structures of the form \((A, \rightarrow^A, \leq^A)\) where \((A, \rightarrow^A) \in S(2)\) and \(\leq^A\) is a linear order on \(A\). Let \(\mathcal{OS}(2)^*\) be the class of the structures of the form \((A, \rightarrow^A, \leq^A, L^A, R^A)\) where the structures \((A, \rightarrow^A, \leq^A) \in \mathcal{OS}(2)\) and \((A, \rightarrow^A, L^A, R^A) \in S(2)^*\). Then we have that \(\mathcal{OS}(2)\) and \(\mathcal{OS}(2)^*\) form an excellent pair of Fraïssé classes which both happen to satisfy the \(\text{SAP}\), see [32]. Using the argument in the proof of Theorem 3.2.1, we get the following corollary:

**Corollary 3.2.2.** \(\text{Aut}(\mathcal{OS}(2))\) is not amenable.

### 3.3 \(\mathcal{S}(3)\), the Circle with Three Parts

Let \(L, R\) and \(D\) be unary relational symbols (\(\text{Left}, \text{Right}\) and \(\text{Down}\)). We consider the structure \(\mathcal{S}(3)^*\) in the signature \(\{\rightarrow, L, R, D\}\) where:

- \(\mathcal{S}(3)^*|\{\rightarrow\} = \mathcal{S}(3)\).
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there are three vertices in the same third of the circle then they must form a tournament.

We have

\[ (S(3), S(3)^*) \] is an excellent pair, see [26], it is enough to show that there is no consistent random \( S(3)^* \) expansion of \( S(3) \), by Proposition 1.5.1.

Suppose for the sake of contradiction that such an expansion \( (\mu_k) \) exists. Consider the structures \( A, B, C \in S(3) \) where \( A \leq B \), \( A \leq C \) and \( B \cap C = A \), which can be done by the SAP. Let:

- \( A = (A, \rightarrow^A) \), \( A = \{x, y\} \), with no edges;
- \( B = (B, \rightarrow^B) \), \( B = A \cup \{s, t\} \), \( x \rightarrow^B s, s \rightarrow^B t \) and \( t \rightarrow^B y \);
- \( C = (C, \rightarrow^C) \), \( C = A \cup \{u, v\} \), \( x \rightarrow^C u, u \rightarrow^C y \) and \( y \rightarrow^C v \).

Consider also expansions (in \( S(3)^* \)) of these structures. Let:

- \( (A \oplus A^*) = (A, L^A, R^A, D^A) \), where \( L^A = \{y\}, R^A = \{x\} \);
- \( (B \oplus B^*) = (B, L^B, R^B, D^B) \), where \( L^B = \{y, t\}, R^B = \{x, s\} \);
- \( (C \oplus C^i) = (C, L^i, R^i, D^i) \), where \( i = 1, 2, 3 \) and \( L^i(u), R^i(x) \) and \( R^1(u), D^1(v), L^2(u), D^2(v), R^3(u), L^3(v) \).

This gives us

\[
\mu_A(\{A^*\}) \overset{(3)}{=} \sum \{ \mu_B(\{B^i\}) : (A \oplus A^*) \leq (B \oplus B^i) \in S(3)^* \} \\
= \mu_A(\{B^*\})
\]

since there is only one expansion of \( B \) in \( S(3)^* \) which extends \( (A \oplus A^*) \), namely \( (B \oplus B^*) \). Note that if there are three vertices in the same third of the circle then they must form a tournament.

Also, we have

\[
\mu_A(\{A^*\}) \overset{(3)}{=} \sum \{ \mu_C(\{C^i\}) : (A \oplus A^*) \leq (C \oplus C^i) \in S(3)^* \} \\
= \mu_C(\{C^1\}) + \mu_C(\{C^2\}) + \mu_C(\{C^3\})
\]
since there are exactly three expansions of $C$ in $S(3)^*$ which extend $(A \oplus A^*)$, namely $(C \oplus C^i)$, for $i = 1, 2, 3$.

Since $B \cong C$ and $(B \oplus B^*) \cong (C \oplus C^3)$ we have

$$\mu_C(C^3) = \mu_B(B^*) = \mu_C(C^1) + \mu_C(C^2) + \mu_C(C^3)$$

$$\Rightarrow \mu_C(C^1) + \mu_C(C^2) = 0$$

$$\Rightarrow \mu_C(C^1) = \mu_C(C^2) = 0.$$

So $\mu_C(C^1) = 0$, which is impossible for a consistent random expansion.

We can also state a related result. Let $OS(3)$ be the class of structures of the form $(A, \to_A, \leq_A)$ where $(A, \to_A, \leq_A) \in S(3)$ and $\leq_A$ is a linear order on $A$. Let $OS(3)^*$ be the class of the structures of the form $(A, \to_A, \leq_A, L_A, R_A, D_A)$ where $(A, \to_A, \leq_A) \in OS(3)$ and $(A, \to_A, L_A, R_A, D_A) \in S(3)^*$. Then we have that $OS(3)$ and $OS(3)^*$ form an excellent pair of Fraïssé classes which satisfy both the SAP, see [32]. Using the argument in the proof of Theorem 3.3.1, we get the following corollary:

**Corollary 3.3.2.** $\text{Aut}(OS(3))$ is not amenable.

### 3.4 $\mathbb{P}(3)$, the Twisted Generic Poset

Let $\mathcal{P}$ be the class of finite posets in signature $\{\leq\}$. Let $P_0, P_1, P_2$ be unary relational symbols and let $\mathcal{P}_3$ be the class of finite structures of the form $(A, \leq_A, P_0^A, P_1^A, P_2^A)$ where $(A, \leq_A) \in \mathcal{P}$ and $A = \bigsqcup_{i=0}^{2} \{x : P_i^A(x)\}$.

It is easy to see that $\mathcal{P}_3$ is a Fraïssé class with limit $\mathbb{P}_3 = (R, \leq_R, P_0^R, P_1^R, P_2^R)$. Using $\mathbb{P}_3$ we define the structure $\mathbb{P}(3) = (R, \to_R)$ such that for $x, y \in R$ with $P_i^R(x)$ and $P_j^R(y)$ we have $x \to_R y$ iff one of the following conditions is satisfied:

1. $j = i$ and $x <_R y$; or
2. $j = i + 1 \mod 3$ and $y <_R x$; or
3. $j = i + 2 \mod 3$ and $x$ is $<_R$-incompatible with $y$.

![Figure 3.5: A structure in $\mathcal{P}_3$ and its corresponding structure in $\mathbb{P}(3)$.](image)

The structure $\mathbb{P}(3)$ is a Fraïssé structure, see [7], which is called the generic twisted poset. Note that each $P_i$ induces a copy of $\mathbb{P}$ that is dense in $\mathbb{P}(3)$. We also consider $\mathbb{P}(3)^* = (R, \to_R, P_0^R, P_1^R, P_2^R, \leq_R)$, which is also a Fraïssé structure, where $\leq_R$ is a linear order on $R$ that extends the partial order $(R, \leq_R)$ given by untwisting $(R, \to_R, P_0^R, P_1^R, P_2^R)$. 

In what follows we will use $\leq$ for untwisted partial orders, $\rightarrow$ for the corresponding twisted directed graph, and $\preceq$ for the linear order that extends $\leq$. We will not refer to an untwisted partial order’s natural directed graph.

Let $\mathcal{P}(3) := \text{Age}(\mathcal{P}(3))$ and $\mathcal{P}(3)^* := \text{Age}(\mathcal{P}(3)^*)$.

**Theorem 3.4.1.** $\text{Aut}(\mathcal{P}(3))$ is not amenable.

**Proof.** Since $(\mathcal{P}(3), \mathcal{P}(3)^*)$ is an excellent pair, see [17, Theorem 9.3], it is enough to show that there is no consistent random $\mathcal{P}(3)^*$ expansion of $\mathcal{P}(3)$, by Proposition 1.5.1.

Suppose for the sake of contradiction that such an expansion $(\mu_K)$ exists. Consider the structures $A, B, C \in \mathcal{P}(3)$ where $C = (C, \leq^C)$, $C = \{a, b, c\}$, $a \rightarrow^C b$, with $A = C \upharpoonright \{a, c\}$ and $B = C \upharpoonright \{b, c\}$.

Consider also expansions (in $\mathcal{P}(3)^*$) of these structures. Let

- $(A \oplus A^*) = (A, P_A^0, P_A^1, P_A^2, \preceq^A)$, where $P_A^0 = \{a\}$, $P_A^1 = \{c\}$ and $a \preceq^A c$;
- $(B \oplus B^*) = (B, P_B^0, P_B^1, P_B^2, \preceq^B)$, where $P_B^0 = \{b\}$, $P_B^1 = \{c\}$ and $b \preceq^B c$.

Now we consider the possible expansions $(C \oplus C^*) = (C, P_C^0, P_C^1, P_C^2, \preceq^C)$ with $(A \oplus A^*) \leq (C \oplus C^*)$. There are three options for the label of $b$, namely: $P_C^0(b), P_C^1(b)$ and $P_C^2(b)$.

If $P_C^0(b)$, then we have $b \preceq^C a$ and $a \preceq^C c$ so $b \preceq^C c$. This contradicts the fact that $P_C^0(a), P_C^2(c)$ guarantee that $b$ and $c$ are $\preceq^C$ incomparable.

If $P_C^1(b)$, then we have $a \preceq^C c$ and $c \preceq^C b$ so $a \preceq^C b$. This contradicts the fact that $P_C^0(a), P_C^2(b)$ and $a \rightarrow^C b$ guarantee that $a$ and $b$ are $\preceq^C$ incomparable.

Therefore only $P_C^2(b)$ is possible, and so:

$$
\mu_A(\{A^*\}) \overset{(\ref{eq:muA})}{=} \sum \{\mu_C(\{C'\}) : (A \oplus A^*) \leq (C \oplus C') \in \mathcal{P}(3)^*\} \\
= \mu_C(\{C^0\}),
$$

where $C^0$ is given by $P_C^0(b), P_C^0(a), P_C^1(c)$, with $a \rightarrow^C c$ and $a \preceq^C b \preceq^C c$.\[\Box\]
On the other hand, there are many expansions \((C \oplus C^*)\) that respect \((B \oplus B^*)\), so
\[
\mu_3([B^*]) = \sum \mu_C([C']) : (B \oplus B^*) \leq (C \oplus C') \in \mathcal{P}(3)^*
\geq \mu_C([C^0]) + \mu_C([C^1]),
\]
where \(C^1\) is given by \(P_C^1(a), P_C^0(b), P_C^1(c)\), with \(a \rightarrow^C c\) and \(b \preceq^C c \preceq^C a\).

Since \(A \cong B\) and \((A \oplus A^*) \cong (B \oplus B^*)\) we have
\[
\mu_C([C^0]) = \mu_A([A^*]) = \mu_3([B^*]) \geq \mu_C([C^0]) + \mu_C([C^1]).
\]
So \(\mu_C([C^1]) = 0\), which is impossible for a consistent random expansion.

3.5 \(S(n)\), for \(n \geq 4\)

Fix \(n \geq 4\). \(S(n)\) is a directed graph which may be seen as the set of points on the unit circle with rational arguments such that \(e^{i\theta} \rightarrow^{S(n)} e^{i\phi}\) iff \(0 < \phi - \theta < \frac{2\pi}{n}\). This is not a tournament.

Let \(I_k\), for \(0 \leq k \leq n - 1\) be unary relational symbols. We consider the structure \(S(n)^*\) in the signature \(\{\rightarrow, I_0, \ldots, I_{n-1}\}\) where:

- \(S(n)^* | \{\rightarrow\} = S(n)\),
- \(I_k^{S(n)}(x) \Leftrightarrow x \in e^{i\theta} \text{ and } \theta \in (\frac{k\pi}{n}, \frac{(k+1)\pi}{n})\).

Set \(S(n) := \text{Age}(S(n))\) and \(S(n)^* := \text{Age}(S(n)^*)\).

For \(n \geq 4\), \(S(n)\) is not on Cherlin’s list. The following argument shows that 3-homogeneity fails. Following that we will show that \(\text{Aut}(S(n))\) is nonamenable.

**Proposition 3.5.1.** Fix \(n \geq 4\). \(\text{Aut}(S(n))\) is not 3-homogeneous, hence it is not ultrahomogeneous and \(S(n)\) is not a Fraïssé class.

**Proof.** Assume \(n = 4\), as the other cases are similar. Fix points \(a, b, c \in S(4)\) such that there are no edges between \(a, b\) and \(c\), and \(a \leq b \leq c\) in the circle order. For example, choose points very close to

- \(a\) with angle 0,
- \(b\) with angle \(\frac{7\pi}{16}\),
- \(c\) with angle \(\frac{23\pi}{16}\).
Notice that there is no point $x \in S(4)$ such that $x \to b$ and $c \to x$, but there is a point $y \in S(4)$ such that $b \to y$ and $y \to c$.

Let $f : \{a, b, c\} \to \{a, b, c\}$ be defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$. By the previous remark, $f$ does not extend to an automorphism of $S(n)$.

The question of amenability remains, and the previous proposition tells us that we will need to adapt our tools. In the case of showing nonamenability, our tool of consistent random expansions essentially emerges unscathed except for a slight modification to condition (I).

**Definition 3.5.2.** We say that the collection $\{\mu_A : A \in \mathcal{K}\}$ is a nearly-consistent random $\mathcal{K}^*$-expansion on $\mathcal{K}$ (when it is clear from context we suppress the reference to $\mathcal{K}^*$) if

- Each $\mu_A$ is a probability measure on $\{A^* : (A \oplus A^*) \in \mathcal{K}^*\}$.
- Whenever $\varphi : A \to B$ is an isomorphism that extends to Flim($\mathcal{K}$), then $\varphi_* \mu_A = \mu_B$, where $\varphi_* \mu_A$ is the push forward measure.
- For every $A \leq B$ in $\mathcal{K}$ and each $(A \oplus A^*) \in \mathcal{K}^*$ we have,
  $$\mu_A([A^*]) = \sum \{\mu_B([B^*]) : (A \oplus A^*) \leq (B \oplus B^*) \in \mathcal{K}^*\}.$$

**Lemma 3.5.3.** Let $\mathcal{K}$ be a relational class of finite structures (which need not be a Fraïssé class), and let $\mathcal{K}^*$ be a precompact expansion. If $\text{Aut}(\text{Flim}(\mathcal{K}))$ is amenable, then there is a nearly-consistent random $\mathcal{K}^*$-expansion on $\mathcal{K}$.

**Proof.** The following proof closely follows the $\Rightarrow$ direction of the proof of Theorem 2.6.2. We will use the equivalence that a group $G$ is amenable iff every $G$-flow admits an invariant measure.

Let $G := \text{Aut}(\text{Flim}(\mathcal{K}))$. Consider $X_{\mathcal{K}^*}$, which is a $G$-flow, although not necessarily the universal minimal flow since we are not assuming that $(\mathcal{K}, \mathcal{K}^*)$ is an excellent pair. Let $\mu$ be an invariant measure on $X_{\mathcal{K}^*}$.

Let $\mathcal{K}$ be the underlying universe of $X_{\mathcal{K}^*}$. For $A \subseteq \mathcal{K}$ finite, and $(A \oplus A^*) \in \mathcal{K}^*$, define the basic open set

$$N(A \oplus A^*) := \{(K \oplus K') \in X_{\mathcal{K}^*} : K' \upharpoonright A = A^*\}.$$

For $A \in \mathcal{K}$ and $(A \oplus A^*) \in \mathcal{K}^*$ define

$$\mu_A([A^*]) := \mu(N(A \oplus A^*)).$$

The collection $(\mu_A)$ obviously satisfies (P), and satisfies (E) for the same reasons as in the proof of theorem 2.6.2. For (I*) we cannot appeal to ultrahomogeneity as in the previously mentioned proof. However, the weakened definition of (I*) still allows us to proceed.
Let \( \varphi : \mathbb{A} \to \mathbb{B} \) be an isomorphism that extends to an isomorphism \( f : \text{Flim}(\mathbb{K}) \to \text{Flim}(\mathbb{K}) \). For isomorphic expansions \((\mathbb{A} \oplus \mathbb{A}^*), (\mathbb{B} \oplus \mathbb{B}^*) \in \mathbb{K}^*\), \( f \) naturally maps \( N(\mathbb{A} \oplus \mathbb{A}^*) \) to \( N(\mathbb{B} \oplus \mathbb{B}^*) \). Thus

\[
\mu_{\mathbb{A}}(\{\mathbb{A}^*\}) := \mu(N(\mathbb{A} \oplus \mathbb{A}^*)) = \mu(N(\mathbb{B} \oplus \mathbb{B}^*)) =: \mu_{\mathbb{B}}(\{\mathbb{B}^*\}),
\]

and moreover, these consistent random expansions are equal through the push forward of \( \varphi \).

We are now in a position to show that \( \text{Aut}(\mathbb{S}(n)) \) is nonamenable. The geometrical crux of the argument will be identical to the one in the proof of Theorem 3.2.1.

**Theorem 3.5.4.** Fix \( n \geq 4 \). \( \text{Aut}(\mathbb{S}(n)) \) is not amenable.

**Proof.** By the previous lemma it is enough to show that there is no consistent random \( \mathbb{S}(n)^* \)-expansion of \( \mathbb{S}(n) \).

Suppose for the sake of contradiction that such an expansion \((\mu_{\mathbb{K}})\) exists. Consider the structures \( \mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathbb{S}(n) \) where \( \mathbb{A} \leq \mathbb{B}, \mathbb{A} \leq \mathbb{C} \) and \( \mathbb{B} \cap \mathbb{C} = \mathbb{A} \), which can clearly be done. Let:

- \( \mathbb{A} = (A, \to^A), A = \{x, y\} \) and \( x \to^A y; \)
- \( \mathbb{B} = (B, \to^B), B = A \cup \{b\}, x \to^B b \) and \( b \to^B y; \)
- \( \mathbb{C} = (C, \to^C), C = A \cup \{c\}, x \to^C c \) and \( y \to^C c. \)

Note that \( \mathbb{B} \cong \mathbb{C} \), and the natural isomorphism (that leaves \( a \) fixed) extends to an isomorphism of \( \mathbb{S}(n) \) (which is essentially a partial rotation with stretching).

Consider also expansions \((\mathbb{S}(n)^*)\) of these structures. Let:

- \((\mathbb{A} \oplus \mathbb{A}^*) = (A, P_0^A, \ldots, P_{n-1}^A), \) where \( P_0^A = \{x, y\}; \)
- \((\mathbb{B} \oplus \mathbb{B}^*) = (B, P_0^B, \ldots, P_{n-1}^B), \) where \( P_0^B = \{x, y, b\}; \)
- \((\mathbb{C} \oplus \mathbb{C}^*) = (C, P_0^C, \ldots, P_{n-1}^C), \) where \( P_0^C = \{x, y, c\}; \)
- \((\mathbb{C} \oplus \mathbb{C}^{**}) = (C, P_0^*, \ldots, P_{n-1}^*), \) where \( P_0^* = \{x, y\}, P_1^* = \{c\}. \)

This gives us

\[
\mu_{\mathbb{A}}(\{\mathbb{A}^*\}) \equiv \sum \{\mu_{\mathbb{B}}(\{\mathbb{B}'\}) : (\mathbb{A} \oplus \mathbb{A}^*) \leq (\mathbb{B} \oplus \mathbb{B}') \in \mathbb{S}(n)^*\}
= \mu_{\mathbb{B}}(\{\mathbb{B}^*\})
\]

since there is only one expansion of \( \mathbb{B} \) in \( \mathbb{S}(n)^* \) that extends \((\mathbb{A} \oplus \mathbb{A}^*)\), namely \((\mathbb{B} \oplus \mathbb{B}^*)\).

Also, we have

\[
\mu_{\mathbb{A}}(\{\mathbb{A}^*\}) \equiv \sum \{\mu_{\mathbb{C}}(\{\mathbb{C}'\}) : (\mathbb{A} \oplus \mathbb{A}^*) \leq (\mathbb{C} \oplus \mathbb{C}') \in \mathbb{S}(n)^\star\}
= \mu_{\mathbb{C}}(\{\mathbb{C}^*\}) + \mu_{\mathbb{C}}(\{\mathbb{C}^{**}\})
\]

since there are exactly two expansions of \( \mathbb{C} \) in \( \mathbb{S}(n)^* \) which extend \((\mathbb{A} \oplus \mathbb{A}^*)\), namely \((\mathbb{C} \oplus \mathbb{C}^*)\) and \((\mathbb{C} \oplus \mathbb{C}^{**})\).
Since $\mathbb{B} \cong \mathbb{C}$ and $(\mathbb{B} \oplus \mathbb{B}^*) \cong (\mathbb{C} \oplus \mathbb{C}^*)$ we have

$$\mu_\mathbb{C}(\{\mathbb{C}^*\}) = \mu_\mathbb{B}(\{\mathbb{B}^*\}) = \mu_\mathbb{C}(\{\mathbb{C}^*\}) + \mu_\mathbb{C}(\{\mathbb{C}^{**}\})$$

So $\mu_\mathbb{C}(\{\mathbb{C}^{**}\}) = 0$, which is impossible for a nearly-consistent random expansion.
Amenability

We now move to establishing amenability of various automorphism groups \( \text{Aut}(\text{Flim}(K)) \), which will set the stage for later showing that these groups are uniquely ergodic. We do this by exhibiting a consistent random expansion of \( K \). The natural candidate for this is the uniform expansion, which is the reciprocal of the number of expansions on a structure. Theorem 4.1.1 establishes an obvious sufficient condition for the uniform expansion to actually be an expansion, namely that the number of expansions of \( B \) that respect an expansion of \( A \) does not depend on the particular embedding of \( A \) into \( B \).

In [3, Question 15.4] it is asked if this condition is necessary for an amenable group \( \text{Aut}(\text{Flim}(K)) \). For all of the examples we look at, the sufficient condition will be true, so they will not provide a negative answer to that question.

Theorem 4.1.1 also establishes a sufficient density condition on \( K \) for \( \text{Aut}(\text{Flim}(K)) \) to be uniquely ergodic. We will immediately use this in Section 4.2 to show that the automorphism group of a class of trees is uniquely ergodic.

Then we will use the sufficient condition for amenability to show that the automorphism groups of the following structures are amenable: (Section 4.3) \( S \), the Semigeneric digraph; (Section 4.4) \( D_n \), the complete \( n \)-partite digraph; (Section 4.5) \( \hat{T}_\omega \), a blowup of the generic tournament. We give explicit descriptions of these structures and their respective expansion classes. In particular we give a careful presentation of the Semigeneric digraph and its expansion class, which are subtle objects.

Along the way, in Section 4.5 we will also show that \( \text{Aut}(\hat{Q}) \) is nonamenable.

4.0.1 Attributions

Corollary 4.2.1, Proposition 4.6.3, Lemma 4.6.1 and Theorem 4.6.2 are due to Sokić. Theorems 4.1.1 and 4.5.3, Lemmas 4.5.1 and 4.5.2 are due to the current author. Theorems 4.3.3, 4.4.1 and 4.5.4 are jointly due to Sokić and the current author.

4.1 Density of Unique Ergodicity

Let \( \mathcal{F} \) be a class of finite structures and let \( \mathcal{D} \subseteq \mathcal{F} \). We say that \( \mathcal{D} \) is dense (or cofinal) in \( \mathcal{F} \) if for every \( A \in \mathcal{F} \) there is a \( D \in \mathcal{D} \) such that \( A \leq D \).
Theorem 4.1.1. Let \((\mathcal{K}, \mathcal{K}^*)\) be an excellent pair of Fraïssé classes. If for every \(A \leq B\) in \(\mathcal{K}\) and every \((A - A'), (A - A'') \in \mathcal{K}^*\) we have

\[
#(A', B) = #(A'', B)
\] (4.1)

then \(\text{Aut}(\text{Flim}(\mathcal{K}))\) is amenable.

Moreover, if there is a dense subclass \(D \subseteq \mathcal{K}\) with the property that for every \(D \in D\) and every \((D - D'), (D - D'') \in \mathcal{K}^*\) we have \((D - D') \cong (D - D'')\), then \(\text{Aut}(\text{Flim}(\mathcal{K}))\) is uniquely ergodic.

Proof. For \(A \in \mathcal{K}\) and \((A \oplus A^0) \in \mathcal{K}^*\) define

\[
\mu_A(\{A^0\}) := \frac{1}{\#(A)}.
\]

We check that \((\mu_A)_{A \in \mathcal{K}}\) defines a consistent random expansion. Conditions (P) and (I) are clear so it remains to check (E).

For \(A \leq B\) in \(\mathcal{K}\) and \((A \oplus A^0) \in \mathcal{K}^*\) we have

\[
#(A) \cdot \mu_A(\{A^0\}) = 1 = \sum\{\mu_B(\{B^*\}) : (A \oplus A^0) \leq (B \oplus B^*) \in \mathcal{K}^*\}
\]

\[
= \sum\{\mu_B(\{B^*\}) : (A \oplus A^0) \leq (B \oplus B^*) \in \mathcal{K}^*\}
\]

\[
= \#(A) \cdot \sum\{\mu_B(\{B^*\}) : (A \oplus A^0) \leq (B \oplus B^*) \in \mathcal{K}^*\}
\]

\[
\Rightarrow \mu_A(\{A^0\}) = \sum\{\mu_B(\{B^*\}) : (A \oplus A^0) \leq (B \oplus B^*) \in \mathcal{K}^*\}.
\]

Here, (4.1) is used for the fourth equality. By Proposition 1.5.1 we obtain that \(\text{Aut}(\text{Flim}(\mathcal{K}))\) is an amenable group.

In order to show unique ergodicity, it is enough to show uniqueness of a consistent random \(\mathcal{K}^*\) expansion by Proposition 1.5.1.

Let \((\mu_A)_{A \in \mathcal{K}}\) and \((\gamma_A)_{A \in \mathcal{K}}\) be two consistent random \(\mathcal{K}^*\) expansions. From our assumptions and (I) we obtain that \(\mu_D \equiv \gamma_D\) for all \(D \in D\). Fix \(A \in \mathcal{K}\), and find \(D \in D\) such that \(A \leq D\). For any fixed \((A \oplus A^0) \in \mathcal{K}^*\) we have:

\[
\mu_A(\{A^0\}) \overset{(\text{E})}{=} \sum\{\mu_D(\{D^*\}) : (A \oplus A^0) \leq (D \oplus D^*) \in \mathcal{K}^*\}
\]

\[
= \sum\{\gamma_D(\{D^*\}) : (A \oplus A^0) \leq (D \oplus D^*) \in \mathcal{K}^*\}
\]

\[
\overset{(\text{E})}{=} \gamma_A(\{A^0\})
\]

Therefore \(\mu_A \equiv \gamma_A\) for all \(A \in \mathcal{K}\) and uniqueness is verified. 

\(\Box\)
4.2 Binary Trees

Let $B$ be the class of finite rooted binary trees. For $B \in B$ we define $T(B)$ to be the set of terminal nodes of the tree $B$, and we define $\Delta(B)$ to be the structure $(T(B), C^B)$ where $C^B$ is a ternary relation on $T(B)$. We define $C^B$ such that for $x, y, z \in T(B)$ we have:

$$C^B(x, y, z) \iff x, y, z \text{ are distinct and the shortest path from } x \text{ to the root}$$
$$\text{is disjoint from the shortest path from } y \text{ to } z.$$

In this way we assign to each $B \in B$ a unique $\Delta(B)$, but also each structure $\Delta(B)$ gives the unique binary tree $T$ with the fewest nodes such that $\Delta(T) = \Delta(B)$.

Let $H$ be the class of the structures of the form $\Delta(B)$ for $B \in B$. Let $O\!H$ be the class of structures of the form $(A, C^A, \leq^A)$ where $(A, C^A) \in H$ and $\leq^A$ is a linear ordering of $A$. We say that $\leq^A$ is convex on $(A, \leq^A)$ if

$$C^A(x, y, z) \Rightarrow (x <^A y \land x <^A z) \lor (y <^A x \land z <^A x).$$

For $B \in B$ and $b \in B$ we write $\text{lev}(b) = n$ if the shortest path from $b$ to the root has $n$ edges. We also write $B(n) := \{b \in B : \text{lev}(b) = n\}$ and $B \mid n$ denotes the subtree given by $\{b \in B : \text{lev}(b) \leq n\}$. Define $B[b]$ to be the subtree of $B$ with root $b$ which contains vertices in $B$ whose shortest path to the root of $B$ contains $b$. We say that $B \in B$ is an $n$-nice tree if $T(B) = B(n)$ and $|B(n)| = 2^n$.

We consider the subclass $D \subseteq O\!H$ which is the collection of structures of the form $(A, C^A, \leq^A)$ for which there exists an $n$-nice tree $B \in B$ such that $\Delta(B) = (A, C^A)$. It is easy to see that $D$ is a dense subclass of $O\!H$.

Let $CH \subseteq O\!H$ that contains the structures with convex linear orderings. Let $CO\!H$ be the class of structures of the form $(A, C^A, \leq^A, \preceq^A)$ where $(A, C^A, \leq^A) \in O\!H$ and $(A, C^A, \leq^A) \in CH$. We have that $H, O\!H, CH$ and $CO\!H$ are all Fraïssé classes, see [2, 6]. Moreover, we have the following.

**Corollary 4.2.1.**

1. $\text{Aut}(\text{Flim}(H))$ is uniquely ergodic.
2. $\text{Aut}(\text{Flim}(O\!H))$ is amenable.

**Proof.** (1) We have that $(H, CH)$ is an excellent pair, see [22]. We will check the conditions in Theorem 4.1.1 for the dense subclass $D$.

Fix $A \leq B$ structures in $H$ and let $(A - A') \in CH$. Let $U \in B$ be the smallest tree such that $\Delta(U) = B$. Since this is the smallest tree, each non-terminal node has degree 3 or 2. Therefore $\#(B) = 2^b$, where $b$ is the number of non-terminal nodes in $U$. Similarly, if $V$ is the smallest tree such that $\Delta(V) = A$ then we have $\#(A) = 2^a$, where $a$ is the number of non-terminal nodes in $V$. Therefore we have that $\#(A', B) = 2^{b-a}$ only depends on $A$ and $B$, not $A'$.

(2) We have that $(O\!H, CO\!H)$ is an excellent pair, see [33]. The conditions in Theorem 4.1.1, part 1 follows as in the previous case, but because of rigidity, there is no dense class with the isomorphism condition as in part 2. We delay verifying unique ergodicity until Proposition 6.1.4.

\[\square\]
4.3 S, the Semigeneric Digraph. Aut(S) is Amenable.

4.3.1 Description of S

Let S be the class of finite directed graphs of the form (S, →S) with the following properties:

1. The binary relation ⊥S defined on S by x ⊥S y ⇔ ¬(x →S y ∨ y →S x) is an equivalence relation on S. (We call the equivalence classes columns.)

2. For x, y, tx, ty ∈ S, where x ⊥S tx and y ⊥S ty, we have that the number of edges directed from \{x, tx\} to \{y, ty\} is even.

Condition (1) ensures that the digraphs are n-partite, for some n. The parity condition (2) might seem artificial, but it has the following nice property which says “If you know three edges, then you know the fourth edge”.

**Lemma 4.3.1** (Three of four). Let \( A = (\{x, y, tx, ty\}, \rightarrow^S) \), with \( x \perp^S tx \) and \( y \perp^S ty \), we have that the number of edges directed from \{x, tx\} to \{y, ty\} is even.

Figure 4.1: The 4 possible digraphs, with 2 nodes on two columns, with the parity condition, up to reflection. The first is in “general position”.

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**Figure 4.1**: The 4 possible digraphs, with 2 nodes on two columns, with the parity condition, up to reflection. The first is in “general position”.

**Figure 4.2**: An example of the three of four lemma.

Let \( S^* \) be the class of finite structures of the form \((A, \rightarrow^A, R^A, \leq^A)\) where \((A, \rightarrow^A) \in S\), \( R^A \) is a binary relation on \( A \) and \( \leq^A \) is a linear ordering on \( A \) with the property that:

1. If \( a \) is the number of \( \perp^A \)-equivalence classes, then there is a linear ordering \( T = \{t_1 \leq t_2 \leq \ldots \leq t_a\} \), called a transversal, which we consider as a directed graph \( T = (T, \rightarrow^T) \in S \) given by \( t_i \to t_j \iff t_i < t_j \). Then there is a \( B = (B, \rightarrow^B) \in S \) such that \( A \leq^B T \leq^B B \) and \( B \) also has a many \( \perp^B \)-equivalence classes. Also, \( T \) must be defined on each of the columns of \( A \) (that is, \( \forall x \in A, \exists t_i \) such that \( x \perp t_i \)). See Lemma 4.3.2 for further discussion.
2. If $R^A(x, y)$, then $\neg(x \perp^A y)$. If $t_i \perp^B x$, then we have

$$R^A(x, y) \iff t_i \rightarrow^B y.$$ 

3. If $x \perp^A z$ and $x <^A y <^A z$ then $x \perp^A y \perp^A z$. If $t_i \perp^B x$ and $t_j \perp^B y$, then $x <^A y \iff t_i < t_j$.

(This is a type of convexity.)

The condition (2), and the three of four lemma, ensures that the digraph structure of $T$ amalgamated with $A$ can be reconstructed from the transversal and the relations $R^A(x, y)$ (and vice versa). The condition (3) says that the linear order is convex with respect to the $\perp^A$-equivalence classes.

The classes $S$ and $S^*$ are Fraïssé classes with limits $S$ and $S^*$ respectively, see [7, 17].

**Lemma 4.3.2.** Every structure $A \in S$ can be amalgamated with a transversal with the same columns. Moreover, the transversal can be chosen to respect an arbitrary linear order of the columns of $A$.

**Proof.** Let $C_i$ for $i \leq a$ be an enumeration of the columns of $A$. From each column choose a $c_i \in C_i$. We will amalgamate a transversal $T = (T, \rightarrow^T) \in S$, where $T = \{t_1 \leq t_2 \leq \ldots \leq t_a\}$, and $t_i \perp c_i$ for each $i \leq a$.

First we describe a digraph structure on the nodes $X = \{c_i : i \leq a\} \cup \{t_i : i \leq a\}$. Between $c_i$ and $c_j$ maintain the same edge direction as in $A$, and add an edge from $t_i$ to $t_j$ if $i < j$. For each $i \neq j$ there are many possible choices for the edges between $\{c_i, t_i\}$ and $\{c_j, t_j\}$ so that it respects the parity condition. Note that, in terms of the parity condition, the edges between $\{c_i, t_i\}$ and $\{c_j, t_j\}$ don’t interact with the edges to any other columns. Denote by $X$ this digraph structure on $X$.

By the Strong Amalgamation Property for $S$, $A$ and $X$ can be amalgamated along $A \upharpoonright \{c_i : i \leq a\}$, which yields the desired result. 

### 4.3.2 The Relation $R$

Fixing a point $x \in S$, the relation $R_x$ induces an equivalence relation with two classes on each other column $A$ which does not contain $x$, where

$$R_x(y) \iff R(x, y).$$ 

The three of four lemma ensures that if $x \perp x'$, then $R_x$ and $R_{x'}$ induce the same partition on $A$. Thus we refer to the **partition of a column** $A_i$ given by another column $A_j$. In fact, any $N$ columns $A_1, \ldots, A_N$ induce an equivalence relation on each other column $A$, which contains $2^N$ equivalence classes (some of which may be empty). These column partitions are equivalence relations that are finer than
the \(\perp\)-equivalence relation, but to avoid confusion we shall refer to *column partitions* and \(\perp\)-equivalence relations.

![Diagram](image)

Figure 4.4: The column partitions two columns induce on each other. Notice the relative positions of \(R_x\) and \(R_y\).

For more discussion about the relation \(R\) and column partitions see [17, Section 10].

### 4.3.3 Amenability

**Theorem 4.3.3.** \(\text{Aut}(\mathcal{S})\) is amenable.

**Proof.** Since \((\mathcal{S}, \mathcal{S}^*)\) is an excellent pair, see [17, Lemma 10.7, Lemma 10.8], it is enough to show that there is a consistent random \(\mathcal{S}^*\) expansion of \(\mathcal{S}\), by Proposition 1.5.1.

For each \(A \in \mathcal{S}\) we define a measure \(\mu_A\) by taking:

\[
\mu_A(\{A^*\}) := \frac{1}{\#(A^*)}.
\]

We check, using Theorem 4.1.1, that this is indeed a random consistent expansion of \(\mathcal{S}\). It is enough to show that for \(A \leq B\) in \(\mathcal{S}\) and \(A^*\) with \((A \oplus A^*) \in \mathcal{S}^*\) the number

\[
\#_{\mathcal{S}^*}(A^*, B) := \left| \{B^* : (A \oplus A^*) \leq (B \oplus B^*) \in \mathcal{S}^* \} \right|
\]

depends only on the isomorphism classes of \(A, B\) and \((A \oplus A^*)\), and notably not on the particular expansion of \(A\).

Let \(A_1, \ldots, A_a\) be the list of \(\perp^A\)-equivalence classes in \(A\), and let \(\leq^A\) be a linear ordering on \(A^*\) such that \(A_1 \leq^A \ldots \leq^A A_a\), and let \(R^A\) be the binary relation on \(A^*\). Similarly, let \(B_1, \ldots, B_b\) be the list of \(\perp^B\)-equivalence classes in \(B\), and let \(i_1 < i_2 < \ldots < i_a\) be such that \(A_j \subseteq B_{i_j}\) for \(1 \leq j \leq a\).

Let \(B^* = (B, R^B, \leq^B)\) be such that \((B \oplus B^*) \in \mathcal{S}^*\). If \((A \oplus A^*) \leq (B \oplus B^*)\) then \(\leq^B\) extends \(\leq^A\) and \(R^B\) extends \(R^A\). So we have:

\[
\#_{\mathcal{S}^*}(A^*, B) = \frac{b! \cdot \prod_{j=1}^a \frac{b_{i_j}!}{a_j!}}{a! \cdot \prod_{j=1}^b b_{i_j}! \cdot \prod_{1 \leq j \leq b, \ 1 \leq i \leq a} \frac{b_{i_j}!}{a_j!}} \cdot 2^\binom{b}{2} - \binom{a}{2}
\]

\[
= \left( b! \cdot 2^\binom{b}{2} \cdot \prod_{j=1}^b b_{i_j}! \right) \left( a! \cdot 2^\binom{a}{2} \cdot \prod_{j=1}^a a_j! \right)^{-1}.
\]

Clearly this quotient depends only on the isomorphism classes of \(A\) and \(B\).
Suppose that $B_k$ and $B_l$ are reaching $^B$-equivalence classes such that the linear ordering $\leq^B$ implies $B_k < B_l$. Then there are two ways to put $R^B$ between these two classes. Since $R^B$ extends $R^A$ we have to choose $R^B$ among $\binom{b}{2} - \binom{a}{2}$ pairs of column partitions, see Figure 4.4. This is why $2\binom{b}{2} - \binom{a}{2}$ shows up.

The unique ergodicity of $\text{Aut}(\mathbb{S})$ remains open, and in Section 6.6 we discuss some partial results in the direction of proving unique ergodicity of this group.

### 4.4 Complete $n$-partite Directed Graph

For $n \in \mathbb{N}$ let $\mathcal{D}_n$ be the class of finite digraphs $(A, \to^A)$ in which $\perp^A$ is an equivalence relation with at most $n$ many equivalence classes. We will also consider $\mathcal{D}_\omega := \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$. In this way we obtain Fraïssé classes $\mathcal{D}_n$, for $n \leq \omega$, with corresponding Fraïssé limits $\mathbb{D}_n$, for $n \leq \omega$.

We denote by $\mathcal{D}_n^*$ the class of finite structures of the form $(A, \to^A, \leq^A)$ where $(A, \to^A) \in \mathcal{D}_\omega$ and $\leq^A$ is a linear order on $A$ such that $\forall x, y, z \in A$ we have

$$x <^A y <^A z ; x \perp^A z \Rightarrow x \perp^A y \perp^A z.$$ 

This is a type of **convexity**.

For a finite $n$, we let $\mathcal{D}_n^*$ be the class of finite structures of the form $(A, \to^A, \leq^A, \{I_i^A\}_{i=1}^n)$ where $(A, \to^A, \leq^A) \in \mathcal{D}_\omega$ and each $I_i^A$ is a unary relation on $A$ such that $\forall x, y \in A$ we have:

- $(\exists i \leq n)(I_i^A(x))$,
- $I_i^A(x), x \perp^A y \Rightarrow I_i^A(y)$,
- $I_i^A(x), I_j^A(y), i < j \Rightarrow x <^A y$.

The $I_i^A$ indicate the $n$-parts, and the ordering is convex with respect to the parts.

**Theorem 4.4.1.** For $n \leq \omega$, $\text{Aut}(\mathbb{D}_n)$ is amenable.

**Proof.** Since $(\mathcal{D}_n, \mathcal{D}_n^*)$ is an excellent pair, see [17, Theorem 8.6] for $n = \omega$ and [17, Theorem 8.7] for $n < \omega$, it is enough to show that there is a consistent random $\mathcal{D}_n^*$ expansion of $\mathcal{D}_n$, by Proposition 1.5.1. This will be done using Theorem 4.1.1.

Let $A = (A, \to^A)$ and $B = (B, \to^B)$ be structures in $\mathcal{D}_\omega$ such that $A \leq B$, and let $\leq^A$ be a linear order on $A$ such that $(A, \leq^A) \in \mathcal{D}_\omega^*$.

Let $A_1, \ldots, A_a$ be the $\perp^A$-equivalence classes of $A$ and let $B_1, \ldots, B_b$ be $\perp^B$-equivalence classes of $B$. Without loss of generality we may assume that $\leq^A$ induces a linear ordering on the $\perp^A$-equivalence classes such that $A_1 \leq^A \ldots \leq^A A_a$. There are $1 \leq i_1 < i_2 < \ldots < i_a \leq b$ such that $A_j \subseteq B_{i_j}$ for each $1 \leq j \leq a$.

If $(B, \leq^B) \in \mathcal{D}_\omega^*$ is such that $(A, \leq^A) \leq (B, \leq^B)$, then $\leq^B$ and $\leq^A$ induce the same linear ordering on $\{B_{i_1}, \ldots, B_{i_a}\}$ and they agree on each $A_i$. 
Therefore we have:

\[
\#_{\mathcal{A}, \mathcal{B}}(\leq A) := \left| \left\{ \leq B : (\mathcal{A}, \leq A) \leq (\mathcal{B}, \leq B) \right\} \right|
\]

\[
= \frac{b!}{a!} \cdot \prod_{k=1}^{a} \frac{|B_k|!}{|A_k|!} \cdot \prod_{k \not\in \{i_1, \ldots, i_a\}} |B_k|!
\]

Moreover, for \(A_1, A_2 \in (\mathcal{B}_\mathcal{A})\) we have \(\#_{A_1, \mathcal{B}}(\leq A_1) = \#_{A_2, \mathcal{B}}(\leq A_2)\), so we have that \(\text{Aut}(\mathcal{D}_\omega)\) is amenable. Now suppose that \(A\) and \(B\) are structures in \(\mathcal{D}_n\) for a fixed \(n < \omega\). We have:

\[
\#_{\mathcal{A}, \mathcal{B}}(\leq A, \{I^A_i\}_{i=1}^a) := \left| \left\{ \leq B : (\mathcal{A}, \leq A) \leq (\mathcal{B}, \leq B) \right\} \right|
\]

\[
= \frac{(n-a)!}{(n-a-b)!} \cdot \prod_{k=1}^{a} \frac{|B_k|!}{|A_k|!} \cdot \prod_{k \not\in \{i_1, \ldots, i_a\}} |B_k|!
\]

Again, for \(A_1, A_2 \in (\mathcal{B}_\mathcal{A})\) we have \(\#_{A_1, \mathcal{B}}(\leq A_1, \{I^A_i\}_{i=1}^a) = \#_{A_2, \mathcal{B}}(\leq A_2, \{I^A_i\}_{i=1}^a)\), so we have that \(\text{Aut}(\mathcal{D}_n)\) is amenable.

In Section 6.4 we will verify that \(\text{Aut}(\mathcal{D}_n)\), for \(n \leq \omega\) is uniquely ergodic.

### 4.5 The Blowups \(\hat{T}\)

Here we discuss a way of blowing up points of a tournament so that it has much of the same structure, but it is no longer a tournament.

#### 4.5.1 The Blowup \(\hat{T}\)

Let \(T\) be one of the following tournaments: \(I_1, C_3, Q\) or \(T^\omega\), and let \(T\) be its underlying set, and let \(\mathcal{T} := \text{Age}(T)\). We consider the structure \(\hat{T}\) with underlying set \(\hat{T} = T \times \{C, P\}\) and edge relation \(\rightarrow\hat{T}\) given by:

\[
(x, i) \rightarrow \hat{T} (y, j) \leftrightarrow ((x \rightarrow T y, i \neq j) \lor (y \rightarrow T x, i = j))
\]

![Figure 4.5: The bottom graph is an example of \(T \in \mathcal{T}\), the upper graph is \(\hat{T} \in \hat{T}\).](image)

\(^1\text{Castor and Pollux are the Gemini twins.}\)
4.5.2 Description of $\hat{T}$ and $T^*$

For $T \in \{I_1, C_3\}$ the structure $\hat{T}$ is finite so $\text{Aut}(\hat{T})$ is finite, hence uniquely ergodic. For $T \in \{Q, T^\omega\}$, $\hat{T}$ is a Fraissé structure with corresponding Fraissé class $\hat{T} := \text{Age}(\hat{T})$.

Let $T^*$ be the collection of structures of the form $(A, \rightarrow^A, \leq^A)$ where $(A, \rightarrow^A) \in T$ and $\leq^A$ is a linear order on $A$. So $(T, T^*)$ is an excellent pair, by [1] and [23, 24, 25].

4.5.3 Description of $\hat{T}^*$

For each structure $(A, \rightarrow) \in \hat{T}$, the relation $\perp^A$ is an equivalence relation which gives the partition $A = A_1 \sqcup \ldots \sqcup A_k$ where each class has at most two elements. In the following we describe a Fraissé class $\hat{T}^*$ such that $(\hat{T}, \hat{T}^*)$ is an excellent pair.

Let $\hat{T}^*$ contain structures of the form $(A, \rightarrow, \leq^A, I_0^A, I_1^A)$ where:

- $(A, \rightarrow^A) \in \hat{T}$,
- $\leq^A$ is a linear order on $A$,
- $I_0^A$ and $I_1^A$ are unary relations on $A$ which partition $A$, and
- each $A_i$ is an interval with respect to $\leq^A$.

Though a slight abuse of notation, we denote by $\leq^A$ the linear ordering induced by $\leq^A$ on the set $\{A_1, \ldots, A_k\}$. The correct expansion of $\hat{T}$ for $T = \text{Age}(T^\omega)$ allows arbitrary partitions by $I_0$ and $I_1$ in each $\perp$-equivalence class, so long as the smallest element in each equivalence class, with respect to the linear ordering, belongs to $I_0$. For $T = \text{Age}(Q)$ there is an additional subtlety which we will explain in the following section; essentially the expansion must be given by a transversal that coheres with the inherent linear order of $Q$, but for $T^\omega$ there is no such linear order to cohere with. This is similar, although not identical, to how the correct expansions for the generic partial orders are the linear orders that extend the partial orders.

The proofs of the following two facts are relegated to the next section.

**Theorem** (Theorem 4.6.2). $\hat{T}^*$ is a Ramsey class.

**Proposition** (Proposition 4.6.3). $\hat{T}^*$ satisfies the EP with respect to $\hat{T}$.

4.5.4 Description of $\hat{T}^*$ for $T = \text{Age}(Q)$

Let $A = (A, \rightarrow^A) \in \hat{T}$ be a structure with $k$ many $\perp^A$-equivalence classes $A_1, \ldots, A_k$ such that $|A_i| = 2$ for all $i \leq k$. Suppose that $A_i = \{(i, C), (i, P)\}$ and that for $i \neq j$ we have:

- $(i, C) \rightarrow^A (j, C) \iff i > j$;
- $(i, P) \rightarrow^A (j, P) \iff i > j$;
- $(i, P) \rightarrow^A (j, C) \iff i < j$;
- $(i, C) \rightarrow^A (j, P) \iff i < j$.

Now we examine when a transversal $T \subseteq A$ forms a linear ordering, that is, a linear order on $T$ that also gives rise to the induced subgraph on $T$. The following lemma says that $T$ forms a linear order so long as there is at most one change of levels, and no zigzags.
Lemma 4.5.1. A sequence of vertices \((a_i)_{i=1}^k\) with \(a_i \in A_i\) forms a linear ordering iff

1. All \(a_i\) have the same second coordinate; or
2. There is \(l < k\) such that for all \(i < l\), the \(a_i\) have the same second coordinate \(m\), and for all \(i \geq l\) the \(a_i\) have the same second coordinate \(n \neq m\). (See Figure 4.8.)

Proof. It is enough to consider the following. Let \(i < j < k\) and \(m \neq n\). Then we have:

\[(i, m) \to^A (j, n) \to^A (k, m) \to^A (i, m)\]

so the directed graph induced by \((i, m), (j, n), (k, m)\) is not a linear ordering.

We define an expansion class \(\hat{\mathcal{T}}^*\) using such a sequence \(\vec{a} = (a_i)_{i=1}^k\) in \(A\), which forms a linear ordering \(\leq^{\vec{a}}\). Then we introduce indicators \(I_0^A\) and \(I_1^A\) such that for \(x \in A\) we have:

\[I_1^A(x) \Leftrightarrow x \in \{a_1, \ldots, a_k\}\]
\[I_0^A(x) \Leftrightarrow x \notin \{a_1, \ldots, a_k\}\]

We define a linear ordering \(\leq^A\) on \(A\) such that for \(x \in A_i\) and \(y \in A_j\) we have:

\[x <^A y \Leftrightarrow ((i = j, I_1^A(x)) \lor (a_i <^{\vec{a}} a_j))\]
In the case where some columns of \( A \) do not have two elements, we are a little more careful. For every structure \( A \in \hat{T} \) there is a unique structure \( B \in \hat{T} \), up to isomorphism, which contain the same number of \( \perp^B \)-equivalence classes of \( A \). Thus we may define an expansion of \( B \), then by taking the restriction to \( A \), we get an expansion of \( A \).

**Lemma 4.5.2.** For all \( A \in \hat{T} \) and all \( (A \oplus A^a), (A \oplus A^b) \in \hat{T}^* \), the structures \( (A \oplus A^a) \) and \( (A \oplus A^b) \) are isomorphic. Therefore \((\hat{T}, \hat{T}^*)\) satisfies the \( EP \).

**Proof.** Let \( \vec{b} = (b_i)_{i=1}^k \) be a sequence in \( A \) given by \( b_i = (i, C) \) for each \( i \leq k \). Then \( \vec{b} \) forms a linear ordering and it induces an expansion of \( A \), call it \( A^b = (A, I^b_0, I^b_1, \leq^b) \) (where the vector notation is dropped for readability). Let \( \vec{a} \) be another sequence in \( A \) which induces the expansion \( A^a \).

Consider the map \( \pi_{\vec{a}} : A \to A \) given by:

\[
\pi_{\vec{a}}(i, m) = \begin{cases} 
(i + l, m) & : 1 \leq i \leq l \\
(i - l, 1 - m) & : l + 1 \leq i \leq k 
\end{cases}
\]

where \( l \) is given by Lemma 4.5.1.

It is easy to see that \( \pi_{\vec{a}} \) is an automorphism of \( A \) and moreover that \( \pi_{\vec{a}} \) is an isomorphism between \((A \oplus A^a)\) and \((A \oplus A^b)\). Therefore any two expansions of \( A \) are isomorphic. \( \square \)

**4.5.5 Amenability Results for \( \text{Aut}(\hat{T}) \)**

**Theorem 4.5.3.** \( \text{Aut}(\hat{\mathbb{Q}}) \) is not amenable.

**Proof.** Let \( T := \text{Age}(\mathbb{Q}) \).

Suppose for the sake of contradiction that \( \text{Aut}(\text{Flim}(\hat{T})) \) is amenable. By Proposition 1.5.1, there is a consistent random expansion \((\mu_A)\). It is enough to work on the cofinal class of structures \( A \) that have exactly two elements in each \( \perp^A \)-equivalence class. Moreover, since for all \((A \oplus A^a), (A \oplus B^a) \in \hat{T}^* \) we have \((A \oplus A^a) \cong (A \oplus B^a) \) we must have:

\[\mu_A(\{A^a\}) = \frac{1}{2 \cdot k},\]

where \( k \) is the number of \( \perp^A \)-equivalence classes, and \( A = (A, \rightarrow^A) \).

Let \( B \in \hat{T} \) be such that each \( \perp^B \) equivalence class has two elements with the partition \( B = \bigcup_{i=1}^3 B_i \) such that \( B_i = \{(i, C), (i, P)\} \). Let the edges on \{\( (1, j), (2, j), (3, j) \)\} be given by the natural linear order for \( j = C, P \).

Let \( A \) be the substructure given by the initial segment \( B_1 \cup B_2 \).

Let \( A^* \) be obtained by the ordering \( B_1 <^A B_2 \) with \( I_1 = \{(1, C), (2, P)\} \) and \( I_0 = \{(1, P), (2, C)\} \).

![Figure 4.9: B, B*, A, A*.](image-url)
Then there is only one $\mathbb{B}^*$ such that $(A \oplus \mathbb{A}^*) \leq (\mathbb{B} \oplus \mathbb{B}^*) \in \hat{T}^*$, namely $I_1(3, P)$ and $I_0(3, C)$ thus:

$$
\frac{1}{4} = \mu_A(\{\mathbb{A}^*\}) = \sum_{B} \mu_B(\{B^*\}) : (A \oplus \mathbb{A}^*) \leq (B \oplus B^*) \in \hat{T}^*
$$

$$
= \mu_B(\{B^*\}) = \frac{1}{6}.
$$
a contradiction. \qedhere

**Theorem 4.5.4.** The group $\text{Aut}(\hat{T}^*)$ is amenable.

**Proof.** Since $(\mathcal{T}, \mathcal{T}^*)$ is an excellent pair, it is enough to show that there is a consistent random $\mathcal{T}^*$ expansion of $\mathcal{T}$, by Proposition 1.5.1.

For each $A \in \mathcal{T}$ we define a measure $\mu_A$ by taking:

$$
\mu_A(\{\mathbb{A}^*\}) := \frac{1}{\#(\mathbb{A}^*)}.
$$

We check, using Theorem 4.1.1, that this is indeed a random consistent expansion of $\mathcal{T}$. It is enough to show that for $A \leq B \in \mathcal{S}$ and $\mathbb{A}^*$ with $(A \oplus \mathbb{A}^*) \in \mathcal{S}$ the number

$$
\#_{\mathcal{T}^*}(\mathbb{A}^*, B) := |\{B^* : (A \oplus \mathbb{A}^*) \leq (B \oplus B^*) \in \mathcal{T}^*\}|
$$

depends only on the isomorphism classes of $A, B$ and $(A \oplus \mathbb{A}^*)$, and notably not on the particular embedding of $A$ into $B$.

Let $A = (A, \rightarrow^A)$ and $B = (B, \rightarrow^B)$ be structures in $\mathcal{T}$ such that $A \leq B$, and let $\leq^A$ be a linear order on $A$ such that $(A, \leq^A) \in \mathcal{T}^*$.

Let $A_1, \ldots, A_a$ be the $\perp^A$-equivalence classes of $A$ and let $B_1, \ldots, B_b$ be $\perp^B$-equivalence classes of $B$. Without loss of generality we may assume that $\leq^A$ induces a linear ordering on the $\perp^A$-equivalence classes such that $A_1 \perp^A \ldots \perp^A A_a$. Moreover we may assume that this linear order is induced by $l_i \in A_i$ where $1 \leq i \leq a$. We may also assume that $|A_i| = 2$ for all $1 \leq i \leq a$.

There are $1 \leq i_1 < i_2 < \ldots < i_a \leq b$ such that $A_j = B_{i_j}$ for each $1 \leq j \leq a$. Define $I := \{i_1, i_2, \ldots, i_a\}$ and $J := \{1, 2, \ldots, b\} \setminus I$.

If $(\mathbb{B} \oplus \mathbb{B}^*) = (B, \leq^B, I_0^B, I_1^B) \in \mathcal{T}^*$ is such that $(A \oplus \mathbb{A}^*) = (A, \leq^A, I_0^A, I_1^A) \leq (B, \leq^B, I_0^B, I_1^B)$, then $\leq^B$ and $\leq^A$ induce the same linear ordering on $\{B_{i_1}, \ldots, B_{i_a}\}$ and they agree on each $A_i$. Furthermore $I_0^A = I_0^B$ and $I_1^A = I_1^B$ on $A_i$ (for $1 \leq i \leq a$).

Therefore we have:

$$
\#_{\mathcal{T}^*}(\mathbb{A}^*, B) := |\{\mathbb{B}^* : (A \oplus \mathbb{A}^*) \leq (B \oplus \mathbb{B}^*) \in \mathcal{T}^*\}| = \frac{b!}{a!} \cdot 2^{|J|}.
$$

Clearly this does not depend on the particular expansion $\mathbb{A}^*$, or the particular embedding of $(A \oplus \mathbb{A}^*)$ into $(\mathbb{B} \oplus \mathbb{B}^*)$. Thus by Theorem 4.1.1, we have the desired amenability. \qedhere

### 4.6 Expansion Property for $\hat{T}$

As promised in Section 4.5 we will check that $\hat{T}^*$ satisfies **RP** and **EP**.
Chapter 4. Amenability

Lemma 4.6.1. There is a map $\Delta : \hat{T}^* \to T^*$ which is an injective assignment up to isomorphism, between structures in $\hat{T}^*$ whose $\perp^A$-equivalence classes have exactly two elements, and the class $T^*$ of finite ordered tournaments.

Proof. Let $(A, \rightarrow^A) \in \hat{T}$, with $\perp^A$-equivalence classes $A = A_1 \sqcup \ldots \sqcup A_k$ where each class has two elements. Consider a related structure

$$\Delta(A) := (\{1, 2, \ldots, k\}, \rightarrow^A, \leq^A)$$

such that $\leq^A$ is the natural ordering on the set $\{1, 2, \ldots, k\}$ and for $1 \leq i < j \leq k$ we have:

$$i \rightarrow^A j \Leftrightarrow (x \in A_i \land y \in A_j \land I^A_1(y) \land x \rightarrow^A y)$$

Clearly $\Delta(A) \in T^*$. For $B = (B, \rightarrow^B, \leq^B) \in T^*$ we may consider

$$\Delta^{-1}(B) := (B \times \{C, P\}, \rightarrow^B, I^B_0, I^B_1, \leq^B) \in \hat{T}^*$$

such that for $(x, i), (y, j) \in B \times \{C, P\}$ we have:

- $I^B((x, i)) \Leftrightarrow i = C$;
- $I^B_0((x, i)) \Leftrightarrow i = P$;
- $(x, i) <^B (y, j) \Leftrightarrow ((x = y, i < j) \lor (x <^B y))$;
- $(x, i) \rightarrow^B (y, j) \Leftrightarrow ((y \rightarrow^B x, i = j) \lor (x \rightarrow^B y, i \neq j))$.

Let $\hat{A} \leq A$ be such that $\hat{A}'$ and $\hat{A}$ have the same number of $\perp$-equivalence classes. Then $\left| \left(\frac{\hat{A}}{\hat{A}'}\right) \right| = 1$. Therefore, in order to verify the RP for $\hat{T}^*$ it is enough to consider only structures in $\hat{T}^*$ whose $\perp$-equivalence classes each have exactly two elements.

Theorem 4.6.2. $\hat{T}^*$ is a Ramsey Class.

Proof. Let $n$ be a natural number and let $A, B \in \hat{T}^*$ be such that $\left(\frac{B}{A}\right) \neq \emptyset$. Without loss of generality we may assume that all $\perp$-equivalence classes in $A$ and $B$ both have two elements each. Since $T^*$ is a Ramsey class, see [23, 24, 25] and [1], there is a (large) $C \in T^*$ such that:

$$C \longrightarrow (\Delta(B))^{\Delta(A)}_2.$$

Then we have:

$$\Delta^{-1}(C) \longrightarrow (B)^\hat{A}_2,$$

and so the verification of the Ramsey Property is complete.

Proposition 4.6.3. $\hat{T}^*$ satisfies EP with respect to $\hat{T}$.

Proof. We will verify that for each $\hat{A} = (A, \rightarrow^A, \leq^A, I^A_0, I^A_1) \in \hat{T}^*$ there is an $H \in \hat{T}$ such that for every $(H \oplus H^*) \in \hat{T}$ we have $\hat{A} \hookrightarrow (H \oplus H^*)$. 

\[ \square \]
Since $\hat{T}^*$ satisfies the JEP, it is enough to obtain EP. Without loss of generality we may assume that each $\perp$-equivalence class in $A$ contains exactly two elements.

We make use of the structures $X = (X, \rightarrow^X, \leq^X, I_0^X, I_1^X) \in \hat{T}^*$ and $Y = (Y, \rightarrow^Y, \leq^Y, I_0^Y, I_1^Y) \in \hat{T}^*$ such that:

- $X = \{1, 2\} \times \{P, C\}, Y = \{1\}$,
- $I_1^X((1, P)), I_1^X((2, P)), I_1^Y(1),$
- $(1, P) <^X (1, C) <^X (2, P) <^X (2, C),$
- $(1, P) \rightarrow^X (2, P), (1, C) \rightarrow^X (2, C), (2, C) \rightarrow^X (1, P), (2, P) \rightarrow^X (1, C)$.

Let $A_1, \ldots, A_k$ be $\perp^A$-equivalence classes which are linearly ordered such that $A_1 <^A A_2 <^A \ldots <^A A_k$. Then there is a $B \in \hat{T}^*$ such that $A \leq B$ and for every $1 \leq i < k$ there is a $\perp$-equivalence class $B_i'$ in $B$ such that:

1. $A_i <^B B_i' <^B A_{i+1}$; and
2. $B \upharpoonright (A_i \cup B_i') \cong B \upharpoonright (B_i' \cup A_{i+1}) \cong X$.

Then there is a $B' \in \hat{T}^*$ such that $B'$ and $B$ have the same underlying set and the same relations $I_0, I_1, \rightarrow$ but the linear ordering induced on the $\perp$-equivalence classes in $B'$ are opposite to the linear ordering induced on the $\perp$-equivalence classes in $B$. Since $\hat{T}^*$ satisfies the JEP there is a $C \in \hat{T}^*$ such that $B \hookrightarrow C$ and $B' \hookrightarrow C$.

Without loss of generality we may assume that each $\perp$-equivalence class in $C$ contains exactly two elements. There is a $C' \in \hat{T}^*$ which has the same underlying set as $C$, the same linear ordering of equivalence classes, the same $\rightarrow$ relation, but $I_0$ and $I_1$ are inverted. Since $\hat{T}^*$ satisfies the JEP, there is an $E \in \hat{T}^*$ such that $C \hookrightarrow E$ and $C' \hookrightarrow E$.

Since $\hat{T}^*$ is a Ramsey class there are $F, G \in \hat{T}^*$ such that:

$F \longrightarrow (E)_2^X$ and $G \longrightarrow (F)_2^Y$. 
Let $G = (G, \rightarrow^G, \leq^G, I^G_0, I^G_1)$. We claim that $H = (H, \rightarrow^H, \leq^H, I^H_0, I^H_1)$ verifies the EP for $A$. Let $H^* := (G, \rightarrow^G, \leq^G, I^G_0, I^G_1)$ be such that $(H \oplus H^*) \in T^*$. Then consider the colouring:

$$\xi_Y : (G_Y) \rightarrow \{0, 1\}$$

such that

$$\xi_Y(Y') = 1 \iff I^H_1 \downharpoonright Y' = I^G_1 \downharpoonright Y'$$

Consider also the colouring:

$$\xi_X : (G_X) \rightarrow \{0, 1\}$$

such that

$$\xi_X(X') = 1 \iff \leq^H \text{ and } \leq^G \text{ induce the same linear ordering on } \perp \text{-equivalence classes in } X'.$$

From the construction there are $F' \in (G_{F'})$ and $E' \in (G_{E'})$ such that $\xi_Y$ is constantly $c_Y$ on $(G_{F'})$ and $\xi_X$ is constantly $c_X$ on $(G_{E'})$.

In particular we have $\xi_Y$ is constant on $(G_{F'})$. Consider the following options for $(c_X, c_Y)$:

1. $(1, 1)$ Here $I^H_1, I^H_0, \leq^H$ agree with $I^G_0, I^G_1, \leq^G$ on $E'$ and we have that $A \hookrightarrow E'$, so $A \hookrightarrow (H \oplus H^*)$.

2. $(1, 0)$ Here $I^H_1$ and $I^H_0$ agree with $I^G_0$ and $I^G_1$ on $E'$ respectively, but $\leq^H$ and $\leq^G$ induce opposite linear orderings on $\perp$-equivalence classes. Since $E' \hookrightarrow E'$, this embedding produces $A \hookrightarrow (H \oplus H^*)$.

3. $(0, 1)$ Here $I^H_1$ and $I^H_0$ are opposite of $I^G_0$ and $I^G_1$ on $E'$ respectively, while $\leq^H$ and $\leq^G$ agree on $E'$.

Since $E' \hookrightarrow E'$, this embedding shows that $A \hookrightarrow (H \oplus H^*)$.

4. $(0, 0)$ Here $I^H_1, I^H_0, \leq^H$ are opposite of $I^G_1, I^G_0, \leq^G$ on $E'$ respectively. Since we have that $D' \hookrightarrow E'$ and $E' \hookrightarrow E'$, there is an embedding of $A \hookrightarrow (H \oplus H^*)$. 

$\Box$
Chapter 5

The Product Class $\mathcal{K}[\mathcal{L}]$

Recall the notation from Sections 1.5.7, 1.5.8 and 1.5.9.

The following theorem is the main result of this chapter. Establishing this theorem was the main goal of this entire project, and after it was established we expanded our aims to the other digraphs on Cherlin’s list.

**Theorem 5.0.4.** Let $(\mathcal{L}, \mathcal{L}^*)$ and $(\mathcal{K}, \mathcal{K}^*)$ be excellent pairs of classes of finite structures in distinct signatures. Then we have:

1. $\text{Aut}(\text{Flim}(\mathcal{K}[\mathcal{L}]))$ is amenable iff $\text{Aut}(\text{Flim}(\mathcal{L}))$ and $\text{Aut}(\text{Flim}(\mathcal{K}))$ are amenable.

2. $\text{Aut}(\text{Flim}(\mathcal{K}[\mathcal{L}]))$ is uniquely ergodic iff $\text{Aut}(\text{Flim}(\mathcal{L}))$ and $\text{Aut}(\text{Flim}(\mathcal{K}))$ are uniquely ergodic.

As an (almost) immediate corollary we get the unique ergodicity of $\text{Aut}((\mathbb{T}[\mathbb{I}_n]))$ and $\text{Aut}(\mathbb{I}_n[\mathbb{T}])$, which are both on Cherlin’s list.

The proof of this is broken up into five not entirely independent parts in Section 5.1. The consistent random expansions presented in the amenability proofs will be used in the unique ergodicity proofs. Moreover, in order to not overly repeat ourselves, detailed proofs that some maps are actually consistent random expansions will only appear in the amenability proofs. These proofs are “direct” in the sense that they do not rely on heavy machinery.

In Section 5.2 we discuss the Hrushovski property and show in Proposition 5.2.4 that this property behaves “exactly the way you’d want it to” with respect to the product class $\mathcal{K}[\mathcal{L}]$. The Hrushovski property plays an important role in the story of amenability and unique ergodicity which we will partly see in Proposition 5.2.2.

### 5.0.1 Attributions

The idea to prove Theorem 5.0.4 instead of Corollary 5.1.1 is due to Sokić, as is the technical fix presented in section 5.1.2. Theorem 5.0.4 is due to the current author. Proposition 5.2.4 is jointly due to Sokić and the current author.
5.1 The Proof for $\mathcal{K}[\mathcal{L}]$

**Proof of (1), $\Rightarrow$.** Assume that $\text{Aut}(\text{Flim}(\mathcal{L}))$ and $\text{Aut}(\text{Flim}(\mathcal{K}))$ are amenable. Then by Proposition 1.5.1 there are consistent random expansions $\nu$ and $\mu$ on $\mathcal{L}$ and $\mathcal{K}$ respectively. We will define a consistent random expansion $\nu \otimes \mu$ on $\mathcal{K}[\mathcal{L}]$.

Let $S = (S_1, \ldots, S_a : A) \in \mathcal{K}[\mathcal{L}]$ and $(S \oplus S^*) \in \mathcal{K}^*[\mathcal{L}^*]$ such that $S^* = (S_1^*, \ldots, S_a^* : A^*)$.

Define

$$(\nu \otimes \mu)_S([S^*]) := \mu([A^*]) \cdot \prod_{i=1}^a \nu_{S_i}([S_i^*]),$$

and we check the conditions for being a consistent random expansion.

**(P)** Observe that

$$\sum \{(\nu \otimes \mu)_S([S^*]) : (S \oplus S^*) \in \mathcal{K}^*[\mathcal{L}^*]\}
= \sum \left\{ \mu([A^*]) \cdot \prod_{i=1}^a \nu_{S_i}([S_i^*]) : (A \oplus A^*) \in \mathcal{K}^*, (S_i \oplus S_i^*) \in \mathcal{L}^*, \forall i \leq a \right\}
= \left( \sum_{(A \oplus A^*) \in \mathcal{K}^*} \mu([A^*]) \right) \cdot \sum \left\{ \prod_{i=1}^a \nu_{S_i}([S_i^*]) : (S_i \oplus S_i^*) \in \mathcal{L}^*, \forall i \leq a \right\}
= 1 \cdot \prod_{i=1}^a \left\{ \sum \nu_{S_i}([S_i^*]) : (S_i \oplus S_i^*) \in \mathcal{L}^* \right\}
= 1 \cdot \prod_{i=1}^a 1 = 1,$$

where the third equality follows from (P) on $\mu$, and the fourth equality follows from (P) on $\nu$. The second equality follows from a basic fact about sums and products.

**(I)** Let $S$ and $T$ be isomorphic structures in $\mathcal{K}[\mathcal{L}]$. Then we have $S = (S_1, \ldots, S_a : A), T = (T_1, \ldots, T_a : B)$ with $S_i \cong T_i$ and $A \cong B$. Let $S^* = (S_1^*, \ldots, S_a^* : A^*)$ and $T^* = (T_1^*, \ldots, T_a^* : B^*)$ be such that $(S \oplus S^*) \cong (T \oplus T^*)$. Then we have:

$$(\nu \otimes \mu)_S([S^*]) = \mu([A^*]) \cdot \prod_{i=1}^a \nu_{S_i}([S_i^*]) = \mu([B^*]) \cdot \prod_{i=1}^a \nu_{T_i}([T_i^*]) = (\nu \otimes \mu)_T([T^*]),$$

where the second equality follows from the fact that (I) is satisfied for $\mu$ and $\nu$. 

(E) Let \( S = (S_1, \ldots, S_a : A) \) and \( T = (T_1, \ldots, T_b : B) \) be structures in \( \mathcal{K}[L] \) such that \( S \leq T \). Let \( S^* = (S_1^*, \ldots, S_a^* : A^*) \) be such that \( (S \oplus S^*) \in \mathcal{K}^*[L^*] \). Since \( S \leq T \) there is an \( I \subseteq \{1, \ldots, b\} \) such that \( S_i \leq T_i \) if and only if \( i \in I \). Then

\[
\sum \left\{ \left( (\nu \otimes \mu)_T(\{T^*\}) : (S \oplus S^*) \leq (T \oplus T^*) \in \mathcal{K}^*[L^*] \right) \right\}
= \sum \left\{ \mu_B(\{B^*\}) \cdot \prod_{i=1}^{b} \nu_{T_i}(\{T_i^*\} : (S_i \oplus S_i^*) \leq (T_i \oplus T_i^*) \in \mathcal{L}^*, \text{ for } i \in I \right\}
= \left( \sum \{ \mu_B(\{B^*\}) : (A \oplus A^*) \leq (B \oplus B^*) \in \mathcal{K}^* \right) \cdot \prod_{i=1}^{b} \nu_{T_i}(\{T_i^*\} : (T_i \oplus T_i^*) \in \mathcal{L}^*, \text{ for } i \in I \right\}
= \mu_A(\{A^*\}) \cdot \prod_{i \in I} \nu_{S_i}(\{S_i^*\} : (S_i \oplus S_i^*) \leq (T_i \oplus T_i^*) \in \mathcal{L}^*)
= \mu_A(\{A^*\}) \cdot \prod_{i \in I} \nu_{S_i}(\{S_i^*\}) \cdot \prod_{j \in J} \nu_{T_j}(\{T_j^*\}) : (T_j \oplus T_j^*) \in \mathcal{L}^*)
= \mu_A(\{A^*\}) \cdot \prod_{i \in I} \nu_{S_i}(\{S_i^*\}) \cdot \prod_{j \in J} \nu_{T_j}(\{T_j^*\})
= \mu_A(\{A^*\}) \cdot \prod_{i \in I} \nu_{S_i}(\{S_i^*\}) \cdot \prod_{j \in J} \nu_{T_j}(\{T_j^*\}) = (\nu \otimes \mu)_S(\{S^*\}),
\]

where the third equality comes from (E) of \( \mu \), the fourth equality comes from (E) of \( \nu \) and the second equality uses the same basic fact about sums and products.

This completes the verification that \( \nu \otimes \mu \) is a consistent random expansion, and by Proposition 1.5.1 we have that \( \text{Aut}(\text{Flim}(\mathcal{K}[L])) \) is amenable. \( \square \)

**Proof of (1), \( \Rightarrow \text{Aut}(\text{Flim}(\mathcal{K})) \text{ is amenable.}** Assume that \( \text{Aut}(\text{Flim}(\mathcal{K}[L])) \) is amenable. By Proposition (1.5.1) there is a consistent random expansion \( \rho \) on \( \mathcal{K}[L] \). We will show that \( \text{Aut}(\text{Flim}(\mathcal{K})) \) is amenable.

Let \( K \in \mathcal{K} \) and \( (K \oplus K^*) \in \mathcal{K}^* \). Let \( S = (S_1, \ldots, S_a : K) \in \mathcal{K}[L] \). Consider

\[
\mu_{K,S}(\{K^*\}) := \sum \{ \rho_S(\{S^*\}) : S^* = (S_1^*, \ldots, S_a^* : K^*), (S \oplus S^*) \in \mathcal{K}^*[L^*] \}.
\]

First, we show that \( \mu_{K,S} \) is independent of our choice of \( S \). Let \( T = (T_1, \ldots, T_a : K) \) also be a structure in \( \mathcal{K}[L] \). If \( S \leq T \), then

\[
\mu_{K,T}(\{K^*\}) = \sum \{ \rho_T(\{T^*\}) : T^* = (T_1^*, \ldots, T_a^* : K), (T \oplus T^*) \in \mathcal{K}^*[L^*] \}
= \sum_{S^*=(S_1^*,\ldots,S_a^*:K^*)} \sum_{(S \oplus S^*) \in \mathcal{K}^*[L^*]} \{ \rho_T(\{T^*\}) : (S \oplus S^*) \leq (T \oplus T^*) \in \mathcal{K}^*[L^*] \}
= \sum_{S^*=(S_1^*,\ldots,S_a^*:K^*)} \rho_S(\{S^*\})
= \mu_{K,S}(\{K^*\})
\]

Where the third equality follows from (E) of \( \rho \). If \( S \) is not a substructure of \( T \), then by JEP for \( \mathcal{K}[L] \)
there is an $\mathcal{R} \in \mathcal{K}[\mathcal{L}]$ such that $S \leq \mathcal{R}$ and $T \leq \mathcal{R}$. So by the above we have:

$$\mu_{\mathcal{K},S}([\mathbb{K}^*]) = \mu_{\mathcal{K},T}([\mathbb{K}^*]) = \mu_{\mathcal{K},\mathcal{R}([\mathbb{K}^*])}$$

Therefore $\mu_{\mathcal{K},S}([\mathbb{K}^*])$ is independent of the choice of structure $S$, so without ambiguity, we write

$$\mu_{\mathcal{K}}([\mathbb{K}^*]) := \mu_{\mathcal{K},S}([\mathbb{K}^*])$$

where $S = (S_1, \ldots, S_n : \mathcal{K}) \in \mathcal{K}[\mathcal{L}]$.

Now we check that $\mu_{\mathcal{K}}$ is a consistent random expansion for $\mathcal{K}$.

(P) Fix any $S = (S_1, \ldots, S_n : \mathcal{K}) \in \mathcal{K}[\mathcal{L}]$. Observe that

$$\sum \{ \mu_{\mathcal{K}}([\mathbb{K}^*]) : ([\mathbb{K} \oplus \mathbb{K}^*]) \in \mathcal{K}^* \}$$

$$= \sum \{ \mu_{\mathcal{K},S}([\mathbb{K}^*]) : ([\mathbb{K} \oplus \mathbb{K}^*]) \in \mathcal{K}^* \}$$

$$= \sum \sum \{ \rho_S([S^*]) : (S \oplus S^*) \in \mathcal{K}^* \}$$

$$= \sum \{ \rho_S([S^*]) : (S \oplus S^*) \in \mathcal{K}^* \} = 1,$$

where the third equality follows from (P) for $\rho$.

(I) Fix $\mathcal{K}, \mathcal{L} \in \mathcal{K}$ such that $\mathbb{K} \cong \mathbb{L}$. Then there are (many) $S, T \in \mathcal{K}[\mathcal{L}]$ such that $S = (S_1, \ldots, S_n : \mathcal{K})$ and $T = (T_1, \ldots, T_n : \mathbb{L})$ with $S \cong T$. Then (I) for $\mu$ follows from (I) for $\rho$.

(E) Let $\mathbb{K} \leq \mathbb{L}$ be structures in $\mathcal{K}$ and let $([\mathbb{K} \oplus \mathbb{K}^*]) \in \mathcal{K}^*$. Let $S \leq T$ be structures in $\mathcal{K}[\mathcal{L}]$, with $S = (S_1, \ldots, S_n : \mathcal{K})$ and $T = (T_1, \ldots, T_n : \mathbb{L})$. Let $I := \{1 \leq i \leq n : S_i \leq T_i\}$ and $J := \{1, \ldots, n\} \setminus I$. Then we have

$$\sum \{ \mu_{\mathcal{L}}([L^*]) : ([\mathbb{K} \oplus \mathbb{K}^*]) \leq ([L \oplus L^*]) \in \mathcal{K}^* \}$$

$$= \sum \{ \mu_{\mathcal{L},T}([T^*]) : ([\mathbb{K} \oplus \mathbb{K}^*]) \leq ([L \oplus L^*]) \in \mathcal{K}^* \}$$

$$= \sum \sum \{ \rho_{T}([T^*]) : (T \oplus T^*) \in \mathcal{K}^* \}$$

$$= \sum \sum \{ \rho_{S}([S^*]) : (S \oplus S^*) \in \mathcal{K}^* \} = \mu_{\mathcal{K},S}([\mathbb{K}^*]) = \mu_{\mathcal{K}}([\mathbb{K}^*]),$$

where the fourth equality is by (E) of $\rho$. So we have shown that $\text{Aut}(\text{Flim}(\mathcal{K}))$ is amenable by Proposition 1.5.1. \hfill $\Box$

**Proof of (1), $\Rightarrow$ Aut(\text{Flim}(\mathcal{L}))$ is amenable.** Assume that $\text{Aut}(\text{Flim}(\mathcal{K}[\mathcal{L}])))$ is amenable. By Proposition (1.5.1) there is a consistent random expansion $\rho$ on $\mathcal{K}[\mathcal{L}]$. We will show that $\text{Aut}(\text{Flim}(\mathcal{L}))$ is amenable.

Let $\mathcal{P}$ be a one point structure in $\mathcal{K}$ and let $([\mathcal{P} \oplus \mathcal{P}^*]) \in \mathcal{K}^*$. For $L \in \mathcal{L}$ there is an $S \in \mathcal{K}[\mathcal{L}]$ such that $S = (L : \mathcal{P})$, and every $(L \oplus L^*) \in \mathcal{L}^*$ gives us an $S^* = (L^* : \mathcal{P}^*)$ such that $(S \oplus S^*) \in \mathcal{K}^* \mathcal{L}^*$. Using the
consistent random expansion $\mu$ we previously defined, we introduce

$$\gamma_L(\{L^*\}) := \rho_L(\{S^*\}) : \frac{1}{\mu_P(\{P^*\})}.$$ 

Note that we must have $\mu_P(\{P^*\}) \neq 0$ since $\rho_S(\{S^*\}) \neq 0$ for all $S \in \mathcal{K}[\mathcal{L}]$ and $(S \circ S^*) \in \mathcal{K}^*[\mathcal{L}^*]$. We prove that $(\gamma_L)$ is a consistent random expansion of $\mathcal{L}$ by checking $(P)$, $(I)$ and $(E)$.

$(P)$ Observe that

$$\sum \{\gamma_L(\{L^*\}) : (L \oplus L^*) \in \mathcal{L}^*\}$$

$$= \sum_{(L \oplus L^*) \in \mathcal{L}^*} \{\rho_S(\{S^*\}) : (S \oplus S^*) \in \mathcal{K}^*[\mathcal{L}^*], S^* = (L^* : P^*)\}$$

$$= \frac{1}{\mu_P(\{P^*\})} \cdot \sum_{(L \oplus L^*) \in \mathcal{L}^*} \{\rho_S(\{S^*\}) : (S \oplus S^*) \in \mathcal{K}^*[\mathcal{L}^*], S^* = (L^* : P^*)\}$$

$$= \frac{1}{\mu_P(\{P^*\})} \cdot \mu_P(\{P^*\}) = 1,$$

where the third equality follows from the definition of $\mu$.

$(I)$ Let $L \cong K$ be structures in $\mathcal{L}$. There are (many) $S \cong T \in \mathcal{K}[\mathcal{L}]$ such that $S = (L : P)$ and $T = (K : P)$. The property $(I)$ for $\gamma$ follows from $(I)$ for $\rho$.

$(E)$ Let $L \leq K$ be structures in $\mathcal{L}$. Then there are $S \leq T \in \mathcal{K}[\mathcal{L}]$ such that $S = (L : P)$ and $T = (K : P)$. Let $(L \oplus L^*) \in \mathcal{L}^*$. Then we have the following

$$\sum \{\gamma_K(\{K^*\}) : (L \oplus L^*) \leq (K \oplus K^*) \in \mathcal{L}^*\}$$

$$= \sum \left\{ \frac{1}{\mu_P(\{P^*\})} \cdot \rho_T(\{T^*\}) : (L \oplus L^*) \leq (K \oplus K^*) \in \mathcal{L}^*, T^* = (K^* : P^*) \right\}$$

$$= \frac{1}{\mu_P(\{P^*\})} \cdot \sum \{\rho_T(\{T^*\}) : (S \oplus S^*) \leq (T \oplus T^*), S^* = (L^* : P^*)\}$$

$$= \frac{1}{\mu_P(\{P^*\})} \cdot \rho_S(\{S^*\}) = \frac{1}{\mu_P(\{P^*\})} \cdot \mu_P(\{P^*\}) \cdot \gamma_L(\{L^*\}) = \gamma_L(\{L^*\}),$$

where the third equality follows from $(E)$ for $\rho$. This finishes the verification that $\text{Aut}(\text{Flim}(\mathcal{K}[\mathcal{L}]))$ is amenable.

\[ \square \]

Proof of (2), $\Rightarrow$. Assume that $\text{Aut}(\text{Flim}(\mathcal{K}[\mathcal{L}]))$ is uniquely ergodic. So we have that $\text{Aut}(\text{Flim}(\mathcal{L}))$ and $\text{Aut}(\text{Flim}(\mathcal{K}))$ are amenable, and by Proposition 1.5.1 there are consistent random expansions $\mu$ and $\gamma$ on $\mathcal{K}$ and $\mathcal{L}$ respectively. Suppose that one of $\text{Aut}(\text{Flim}(\mathcal{L}))$ or $\text{Aut}(\text{Flim}(\mathcal{K}))$ is not uniquely ergodic. Then there is a consistent random expansion $\mu'$ on $\mathcal{K}$ such that $\mu \neq \mu'$ or there is a consistent random expansion $\gamma'$ on $\mathcal{L}$ such that $\gamma \neq \gamma'$. Then there is a structure $K \in \mathcal{K}$ and an expansion $(K \oplus K^*) \in \mathcal{K}^*$ such that

$$\mu_K(\{K^*\}) \neq \mu_K(\{K^*\})$$

or there is a structure $L \in \mathcal{L}$ and an expansion $(L \oplus L^*) \in \mathcal{L}^*$ such that

$$\gamma_L(\{L^*\}) \neq \gamma_L(\{L^*\}).$$

Now consider the structure $S = (L, \ldots, L : K) \in \mathcal{K}[\mathcal{L}]$ with expansion $(S \oplus S^*) \in \mathcal{K}^*[\mathcal{L}^*]$ where $S^* = \ldots$
\((L^*, \ldots, L^* : K^*)\), with \(a := |K|\) many \(L\). Using similar arguments to the proof of [(1), \(\Leftarrow\)] of this theorem, we have that \(\gamma \otimes \mu, \gamma \otimes \mu'\) and \(\gamma' \otimes \mu\) are consistent random expansions on \(K[\mathcal{L}]\). In particular if \(\mu \neq \mu'\), we have

\[
(\gamma \otimes \mu)_S(\{S^*\}) = \mu_K(\{K^*\}) \cdot \prod_{i=1}^{a} \gamma_L(\{L^*_i\}) \\
\neq \mu'_K(\{K^*\}) \cdot \prod_{i=1}^{a} \gamma_L(\{L^*_i\}) = (\gamma \otimes \mu')_S(\{S^*\}),
\]

and if \(\gamma \neq \gamma'\), then we have

\[
(\gamma \otimes \mu)_S(\{S^*\}) = \mu_K(\{K^*\}) \cdot \prod_{i=1}^{a} \gamma_L(\{L^*_i\}) \\
\neq \mu'_K(\{K^*\}) \cdot \prod_{i=1}^{a} \gamma'_L(\{L^*_i\}) = (\gamma' \otimes \mu)_S(\{S^*\}).
\]

Therefore we have two distinct consistent random expansions on \(K[\mathcal{L}]\). This is in contradiction to the unique ergodicity of \(K[\mathcal{L}]\), according to Theorem 1.5.1, so \(\text{Aut}(\text{Flim}(\mathcal{L}))\) and \(\text{Aut}(\text{Flim}(K))\) must be uniquely ergodic.

\(\Box\)

**Proof of (2), \(\Leftarrow\).** Now assume that \(\text{Aut}(\text{Flim}(\mathcal{L}))\) and \(\text{Aut}(\text{Flim}(K))\) are uniquely ergodic, and let \(\mu\) and \(\gamma\) be the unique consistent random expansions on \(K\) and \(\mathcal{L}\) respectively. According to the first part of this theorem there is a consistent random expansion on \(K[\mathcal{L}]\). We will show that \(\rho := \gamma \otimes \mu\) is the unique consistent random expansion on \(K[\mathcal{L}]\), as defined in the previous part of the proof.

Let \(S = (S_1, \ldots, S_a : K)\) be a structure in \(K[\mathcal{L}]\) with expansion \((S \oplus S^*) \in K^*[\mathcal{L}^*]\) given by \(S^* = (S_1^*, \ldots, S_a^* : K^*)\). In the previous proof of [(1), \(\Rightarrow\)] \(\text{Aut}(\text{Flim}(K))\) is amenable [we described a consistent random expansion on \(K\) given by \(\rho\), so we may notice that by unique ergodicity, this is exactly \(\mu\).]

Fix any \(T_2, \ldots, T_a \in \mathcal{L}\) which will be used to define measures on \(\mathcal{L}\). They can be \(S_2, \ldots, S_a\) if you like, but for purposes of clarity we use \(T_2, \ldots, T_a\).

Define

\[
p_0 := \mu_K(\{K^*\}).
\]

Let \(L \in \mathcal{L}\) with \((L \oplus L^*) \in \mathcal{L}^*\) be given. Consider the map

\[
\gamma^L_1(\{L^*\}) := \frac{1}{p_0} \cdot \sum \left\{ \rho_{S_1}(X_1) : \begin{array}{c}
X_1 = (L, T_2, \ldots, T_a : K) \in K[\mathcal{L}] \\
X_1^* = (L^*, T_2^*, \ldots, T_a^* : K^*) \in K^*[\mathcal{L}^*]
\end{array} \right\}
\]

Notice that the sum does not run over \(L^*\) and \(K^*\), which are fixed.

In a similar way as the proof of [(1), \(\Rightarrow\)] \(\text{Aut}(\text{Flim}(\mathcal{L}))\) is amenable [we may conclude that \(\gamma^1\) is a consistent random expansion on \(\mathcal{L}\) that does not depend on the choice of \(T_2, \ldots, T_a\). Since \(\text{Aut}(\text{Flim}(\mathcal{L}))\) is uniquely ergodic we must have that \(\gamma^1 = \gamma\). In particular, for \(L = S_1\) we can define

\[
p_1 := p_0 \cdot \gamma_{S_1}(\{S_1^*\}).
\]

**Chapter 5. The Product Class \(K[\mathcal{L}]\)**
Now consider the map $\gamma^2$, for $L \in \mathcal{L}$ with $(L \oplus L^*) \in \mathcal{L}^*$ given by:

$$\gamma^2_L((L^*)) := \frac{1}{p_1} \cdot \sum \left\{ \rho_{L^*}(\{X^*_2\}) : X_2=(S_1, L, T_3, \ldots, T_a,K) \in \mathcal{K}[\mathcal{L}] \right\}$$

Notice that the sum does not run over $S^*_1, L^*$ and $K^*$, which are fixed.

Again similar arguments as in the previous proof show that $\gamma^2$ is a consistent random expansion on $\mathcal{L}$, and by unique ergodicity we have that $\gamma^2 = \gamma$. In particular we define:

$$p_2 := p_1 \cdot \rho_{S^*_2}(\{S^*_2\}).$$

Continuing on in this way we obtain

$$p_a := \mu_{\mathcal{K}}(\{K^*_a\}) \cdot \rho_{S^*_1}(\{S^*_1\}) \cdot \ldots \cdot \rho_{S^*_a}(\{S^*_a\})$$

with $p_a = \rho_{S^*}(\{S^*\})$. Therefore we have proved that $\rho = \gamma \otimes \mu$, so it must be unique. \qed

5.1.1 The Corollaries

For $n \leq \omega$, define $[n] := \{i \in \omega : i < n\}$.

Let $T$ be one of the tournaments $\mathcal{Q}, \mathcal{S}(2), \mathcal{T}^\omega$ or $\mathcal{C}_3$, and let $T$ be the underlying set of $T$. For $n \leq \omega$ we denote by $T[I_n]$ the directed graph with the underlying set $T \times [n]$ and the edge relation given by

$$(x, i) \to (y, j) \text{ iff } x \to y$$

and $I_n[T]$ the tournament with the underlying set $[n] \times T$ and edge relation given by

$$(i, x) \to (j, y) \text{ iff } (i = j, x \to y)$$

Consider $I_n$ as a structure on the empty signature. Therefore we have the following:

**Corollary 5.1.1.** Let $T$ be one of the tournaments $\mathcal{Q}, \mathcal{S}(2), \mathcal{T}^\omega$ or $\mathcal{C}_3$, and let $n \leq \omega$. Then,

1. $\text{Aut}(\mathcal{S}(2)[I_n])$ is not amenable. For $T \neq \mathcal{S}(2)$, $\text{Aut}(T[I_n])$ is uniquely ergodic.

2. $\text{Aut}(I_n[\mathcal{S}(2)])$ is not amenable. For $T \neq \mathcal{S}(2)$, $\text{Aut}(I_n[T])$ is uniquely ergodic.

**Proof.** In the case that $T[I_n]$ or $I_n[T]$ is a finite structure then its automorphism group is finite and therefore uniquely ergodic. So let us assume that they are infinite. Thus we may view the structures as Fraïssé limits of the form $\text{Flim}(\mathcal{K}[\mathcal{L}])$ where both $\mathcal{L}$ and $\mathcal{K}$ are Fraïssé structures, or one of them (e.g. $I_n$) is not a Fraïssé structure simply because it is not an infinite structure. This can be rectified by going through the proof of Theorem 5.0.4 and noticing that the assumption that $\mathcal{L}$ and $\mathcal{K}$ are infinite is not used. \qed

5.1.2 Finite Blowups

The question now arises if $I_n$ is special, or if we can blow up by any fixed finite structure and still have unique ergodicity of $\text{Aut}(\mathcal{K}[\mathcal{L}])$. The main obstacle is that a finite structure might not be ultrahomogeneous. We introduce a related structure to rectify this.
Let $M$ be a finite structure with underlying set $M$. We consider the action of the group $\text{Aut}(M)$ on the set $M^n$, $n \geq 1$. For all $n \geq 1$, and each orbit in $M^n$ which contains only tuples of distinct elements we introduce a new relational symbols and obtain a finite signature $L = \{ R_i \}_{i \in I}$, because $M$ is finite. We assume that $L$ is disjoint from the signature of the structure $M$. In this way we obtain a structure $\overline{M} = (M, \{ R^i_M \}_{i \in I})$ with the property $\text{Aut}(\overline{M}) = \text{Aut}(M)$.

Let $\mathcal{M}_M = \text{Age}(\overline{M})$. We also consider the unary relational symbols $\{ I^i_m \}_{m \in M}$ which don’t belong to $L$. For $e \in \text{Aut}(M)$, the identity, we consider structures $\overline{M}^e = (M, \{ I^i_m \}_{m \in M})$ such that for $x \in M$ we have

$$I^i_m(x) \Leftrightarrow x = m.$$  

For any $g \in \text{Aut}(M)$ we consider the structure $\overline{M}^g = (M, \{ I^i_m \}_{m \in M})$ such that for $x \in M$ we have

$$I^i_m(x) \Leftrightarrow I^i_m(g^{-1}(x)).$$

Let $\mathcal{M}^*_M = \text{Age}(\overline{M}^0) = \text{Age}(\overline{M})$.

**Lemma 5.1.2.** For a finite structure $M$ we have that $\mathcal{M}^*_M$ is a Ramsey class which satisfies EP with respect to $\mathcal{M}_M$.

**Proof.** Since for every $A^*, B^* \in \mathcal{M}^*_M$ we have $\left| \left( B^* \right)^{A^*} \right| \leq 1$ we have that $\mathcal{M}^*_M$ satisfies RP. For every $A \in \mathcal{M}_M$ and every $(A \oplus A^*) \in \mathcal{M}^*_M$ we have $(A \oplus A^*) \hookrightarrow \overline{M}^0$ which verifies EP. 

**Lemma 5.1.3.** Let $M$ be a finite structure with $\mathcal{M} := \text{Age}(M)$. Let $\mathcal{M}_M$ and $\mathcal{M}^*_M$ be as described. Let $(\mathcal{K}, \mathcal{K}^*)$ be an excellent pair.

1. $\text{Aut}(\text{Flim}(\mathcal{K}|\mathcal{M}))$ is amenable iff $\text{Aut}(\text{Flim}(\mathcal{K}))$ is amenable.

2. $\text{Aut}(\text{Flim}(\mathcal{K}|\mathcal{M}))$ is uniquely ergodic iff $\text{Aut}(\text{Flim}(\mathcal{K}))$ is uniquely ergodic.

3. $\text{Aut}(\text{Flim}(\mathcal{M}|\mathcal{K}))$ is amenable iff $\text{Aut}(\text{Flim}(\mathcal{K}))$ is amenable.

4. $\text{Aut}(\text{Flim}(\mathcal{M}|\mathcal{K}))$ is uniquely ergodic iff $\text{Aut}(\text{Flim}(\mathcal{K}))$ is uniquely ergodic.

**Proof.** This follows from the previous Lemma 5.1.2 and the proof of Theorem 5.0.4. We also use the fact that $\text{Aut}(M) = \text{Aut}(\overline{M})$.

We may also use the fact that $\text{Aut}(\text{Flim}(\mathcal{K}|\mathcal{M})) = \text{Aut}(M) \times \text{Aut}(\text{Flim}(\mathcal{K}))^{\lvert M \rvert}$. 

Let us also remark that the finiteness of $M$ is essential. Consider $M$ as an infinite structure, and suppose there is a set $A$ where the bijections pointwise fix everything outside of $A$. Even allowing points to be blown up to a two point set $I_2$ makes it so that $\text{Aut}(M)$ contains a closed copy of $\text{Aut}(I_2)^{\aleph_0}$ which is not amenable.

## 5.2 The Hrushovski Property

**Definition 5.2.1.** A class $\mathcal{K}$ of finite structures is a **Hrushovski class** if for any $K \in \mathcal{K}$ and any finite sequence of partial isomorphisms $\phi_i : A_i \rightarrow B_i$ (for $1 \leq i \leq k$) where $A_i, B_i \leq K$, there is a $C \in \mathcal{K}$, such that each $\phi_i$ (for $1 \leq i \leq k$) can be extended to an automorphism $\psi_i : C \rightarrow C$. 

Recall the following proposition which appears as Proposition 13.1 in [3] for the special case of order expansions.

**Proposition 5.2.2.** Let \((K, K^*)\) be an excellent pair. If \(K\) is a Hrushovski class, then \(\text{Aut}(\text{Flim}(K))\) is amenable.

We immediately get the following corollary.

**Corollary 5.2.3.** \(S(2), S(3), P, P(3), \text{Age}(S(2)[I_n]), \text{Age}(I_n[S(2)]), \text{and Age}(\hat{Q})\) are not Hrushovski classes.

Now we present a proposition which says that the Hrushovski property behaves “exactly the way you’d want it to” with respect to the product class \(K[L]\).

**Proposition 5.2.4.** Let \(K\) and \(L\) be classes of finite relational structures, such that \(L\) satisfies the JEP. Then \(K[L]\) is a Hrushovski class if and only if \(L\) and \(K\) are Hrushovski classes.

**Proof.** \(\Rightarrow\) Suppose that \(K[L]\) is a Hrushovski class.

First we show that \(L\) is a Hrushovski class. Let \(L \in L\), and let \(P \in K\) be a one point structure. Fix a finite sequence of partial isomorphisms \(\phi_i : A_i \rightarrow B_i\) (for \(1 \leq i \leq k\)) where \(A_i, B_i \leq L\). This gives a related partial isomorphism \(\phi'_i : (A_i : P) \rightarrow (B_i : P)\). By the Hrushovski property of \(K[L]\), there is an \(D \in K[L]\), and automorphisms \(\psi'_i : D \rightarrow D\), where \(\psi'_i\) extends \(\phi'_i\) for \(1 \leq i \leq k\). Without loss of generality we may assume that \(D = (C : P)\) and consequently, we must have automorphisms \(\psi_i : C \rightarrow C\) which extends \(\psi'_i\). So we have verified the Hrushovski property for \(L\).

Now we show that \(K\) is a Hrushovski class. Let \(K \in K\), with \(|K| = N\) and let \(\phi_i : A_i \rightarrow B_i\) (for \(1 \leq i \leq k\)) be a finite sequence of partial isomorphisms, where \(A_i, B_i \leq K\). Let \(Q \in L\) be a one-point structure. We consider the structure \(D := (Q, \ldots, Q : K) \in K[L]\). Clearly every \(\phi_i\) determines a unique \(\phi'_i : (Q, \ldots, Q : A_i) \rightarrow (Q, \ldots, Q : B_i)\), so by the Hrushovski property for \(K[L]\) there is a \(C \in K[L]\) and automorphisms \(\psi'_i : C \rightarrow C\) where \(\psi'_i\) extends \(\phi'_i\) for \(1 \leq i \leq k\). If \(C = (P_1, \ldots, P_N : R)\), then every \(\phi'_i\) determines an automorphism \(\psi_i : R \rightarrow R\) which extends \(\phi_i\).

\(\Leftarrow\) Suppose that \(L\) and \(K\) are Hrushovski classes. Let \(A = (S_1, \ldots, S_N : K) \in K[L]\) and let \(\phi_i : A_i \rightarrow B_i\) (for \(1 \leq i \leq k\)) be a finite sequence of partial isomorphisms in \(A\). Since \(L\) satisfies the JEP we may assume that \(S_1 = \ldots = S_k =: S\).

Each \(\phi_i\) is given by a partial isomorphism \(\phi'_i : A'_i \rightarrow B'_i\) where \(A'_i, B'_i \leq K\), and by a sequence of partial isomorphisms \((\phi_{i,s})_{s=1}^n\) inside \(S\). Now, by the Hrushovski property for \(K\) there is a \(S \leq D \in K\) and automorphisms \(\psi'_i : D \rightarrow D\) which extend the corresponding \(\phi'_i\). Moreover, there is an \(S \leq T \in L\) together with automorphisms \(\phi'_{i,s} : T \rightarrow T\) which extend the corresponding \(\phi'_{i,s}\).

The structure \(E := (T, \ldots, T : D)\) contains \(A\) and there are automorphisms \(\psi_i : E \rightarrow E\) given by \(\psi'_i\) and \(\psi'_{i,s}\) which extends \(\phi_i\). This completes the verification that \(K[L]\) is a Hrushovski class.

We are now in a position to give a strengthening of Theorem 5.0.4.1, with an alternate proof, in the special case that \(L\) and \(K\) are both Hrushovski Classes.

**Corollary 5.2.5.** Let \((L, L^*)\) and \((K, K^*)\) be excellent pairs of relational structures, where \(L\) and \(K\) are Hrushovski Classes. Then \(\text{Aut}(\text{Flim}(L)), \text{Aut}(\text{Flim}(K))\) and \(\text{Aut}(\text{Flim}(K[L]))\) are amenable.

**Proof.** This follows immediately from Proposition 5.2.2 and Proposition 5.2.4.
Chapter 6

Unique Ergodicity

Having dealt with the amenability of many automorphism groups, we now verify that these amenable groups are in fact uniquely ergodic. In the case of Aut(Flim(\(K[L]\))), checking unique ergodicity was direct, but for the remaining groups we will use the asymptotic condition \(QOP\), introduced in Section 6.1. This property allows us to verify unique ergodicity of Aut(Flim(\(K\))) by finding a combinatorially suitable structure in \(K\) with a large cardinality. The notion of “combinatorially suitable” will be made explicit in Section 6.3, but for now it is related to the number of expansions on the structure, together with the number of embeddings into that structure.

Before tackling the strategy of checking \(QOP\), we make a brief detour in Section 6.2 to describe some density lemmas for unique ergodicity.

After that we verify unique ergodicity of Aut(\(D_n\)) (Section 6.4), Aut(\(\overline{T}^\omega\)) (Section 6.5), and automorphism groups of slight variations of the semigeneric graph (Section 6.6).

This leaves us with Aut(\(G_n\)) and Aut(\(L(T)\)), whose unique ergodicity we check by using the so-called hypergraph method (Section 6.7). This amounts to checking \(QOP\), but is more delicate. We will need to be more clever about which embeddings we allow.

In addition, in Section 6.8 we provide an explicit, routine calculation that is used throughout.

6.0.1 Attributions

Propositions 6.1.4 and 6.1.5 are due to Sokić.

Lemma 6.3.2 and the discussion in that section is due to the current author.

Lemma 6.2.2, and the remaining theorems asserting unique ergodicity are jointly due to Sokić and the current author.

6.1 \(QOP\) and \(QOP^*\)

Here we look at two properties that allow us to push amenable automorphism groups up to uniquely ergodic. The following are Quantitative Ordering Properties, where the name comes from expansions that are linear orderings, but with suitable adaptations they apply to more general expansions, not just linear orderings.

For fixed structures A and B with expansions \((A \oplus A^*)\) and \((B \oplus B^*)\), and \(E\) a set of embeddings of
A into $B$, define

$$N_{\exp}(\mathcal{E}, A^*, B^*) := |\{ \phi \in \mathcal{E} : \phi \text{ embeds } (A \oplus A^*) \text{ into } (B \oplus B^*) \}|.$$ 

If $\mathcal{E}$ is clear from context, we shall denote this set by $N_{\exp}(A^*, B^*)$. Also define

$$N_{\emb}(A, B) := |\{ \phi : \phi \text{ embeds } A \text{ into } B \}|.$$ 

Note that this is $|\text{Aut}(A)| \cdot \left\lvert \binom{B}{A} \right\rvert$, and if $A$ is rigid, then this is just $\left\lvert \binom{B}{A} \right\rvert$.

### 6.1.1 Definitions

**Definition 6.1.1 (QOP).** Let $(\mathcal{K}, \mathcal{K}^*)$ be an excellent pair. We say that $\mathcal{K}^*$ satisfies the QOP if there is an isomorphism invariant map $\rho : \mathcal{K}^* \rightarrow [0, 1]$ such that for every $(A \oplus A^*) \in \mathcal{K}^*$ and every $\epsilon > 0$, there is a $B \in \mathcal{K}$ and a non empty set of embeddings $\mathcal{E}$, from $A$ into $B$ with the property that for every $(B \oplus B^*) \in \mathcal{K}^*$ we have:

$$\left\lvert \frac{N_{\exp}(\mathcal{E}, A^*, B^*)}{|\mathcal{E}|} - \rho(A \oplus A^*) \right\rvert < \epsilon.$$

Occasionally we will use the notation $a \approx b$ if $|a - b| < \epsilon$.

**Definition 6.1.2 (QOP*).** Let $(\mathcal{K}, \mathcal{K}^*)$ be an excellent pair. We say that $\mathcal{K}^*$ satisfies the QOP* if there is a $B \in \mathcal{K}$ and a non empty set of embeddings $\mathcal{E}$, from $A$ into $B$ with the property that for every $(A \oplus A^*), (B \oplus B^*) \in \mathcal{K}^*$ we have:

$$\left\lvert \frac{N_{\exp}(\mathcal{E}, A^*, B^*)}{|\mathcal{E}|} - \rho(A \oplus A^*) \right\rvert < \epsilon.$$

Note that in general QOP implies QOP* (because QOP works for an arbitrary expansion, but in QOP* you are working with a single expansion). Also, in a Hrushovski class, these are the same (see [3, Theorem 13.3]), and this is non-trivial.

### 6.1.2 General Results and the Main Tool

The following theorem is one of the main reasons that we examine QOP. It gives a method for ensuring that an amenable automorphism group is actually uniquely ergodic.

**Theorem 6.1.3.** Let $(\mathcal{K}, \mathcal{K}^*)$ be an excellent pair, and suppose that $\text{Aut}($Flim$(\mathcal{K}))$ is amenable and $\mathcal{K}^*$ satisfies the QOP*. Then $\text{Aut}($Flim$(\mathcal{K}))$ is uniquely ergodic.

**Proof.** With minor modifications, this follows from the Fubini-type argument presented in the proof of [3, Proposition 11.1]. It is also similar to the proof of Lemma 6.2.2. $\square$

### 6.1.3 Unique Ergodicity of $\text{Aut}(\mathcal{O}H)$

Let us illustrate a direct verification of the QOP* for a dense subclass of $\mathcal{O}H$. Recall the notation from Section 4.2.

**Proposition 6.1.4.** $\text{Aut}($Flim$(\mathcal{O}H))$ is uniquely ergodic.
Proof. Amenability was proved in Corollary 4.2.1.(ii). In order to prove unique ergodicity we will verify $QOP^*$ for the class $\mathcal{D}$, which is enough by Theorem 4.1.1.

Let $(A, C^A, \leq^A, \preceq^A) \in COH$ be given where $(A, C^A, \leq^A) \in \mathcal{D}$, and let $\epsilon > 0$. Let $\mathbb{B}$ be an $n$-nice tree such that $\Delta(\mathbb{B}) = (A, C^A)$. In particular, we may assume that $A = T(\mathbb{B})$. Let $\leq_1, \ldots, \leq_l$ be the list of all linear orderings on $A$ such that $(A, C^A, \leq_i) \in CH$ for all $1 \leq i \leq l$. Let $\mathbb{B}'$ be an $n + 1$-nice tree such that $\mathbb{B} \cup \mathbb{B}'$ is $n$-nice and $\mathbb{B} \subseteq \mathbb{B}'$. Now it is easy to see that in this way we can satisfy the condition of the

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Proof. Let $\mathbb{B}_i \in K$ be an $n$-nice tree such that $\Delta(\mathbb{B}_i) = (A, C^A)$. Let $\leq_i$ be the collection of subtrees of $\mathbb{B}'$ such that for $1 \leq i \leq l$ we have:

- $\mathbb{B}_i \cap n = \mathbb{B}$,
- $(\forall x \in \mathbb{B}(n))(\exists! x' \in T(\mathbb{B}'))[x' \in \mathbb{B}'[x]]$.

Now for $i \neq j$ we have $T(\mathbb{B}_i) \cap T(\mathbb{B}_j) = \emptyset$ and $\Delta(\mathbb{B}_i) \cong \Delta(\mathbb{B}_j)$.

Moreover, every linear ordering $\leq'$ such that $(\Delta(\mathbb{B}_i), \leq') \in CH$ is given by the unique linear ordering with the property that $(\Delta(T), \leq') \in CH$. More precisely, for $x, y \in T(\mathbb{B})$ and $x' \in T(\mathbb{B}'[x]), y' \in T(\mathbb{B}'[y])$ we have $x \leq y \iff x' \leq y'$. Therefore, $\leq'$ is given by one of the $\leq_i$, and without loss of generality we denote such $\leq'$ by $\leq_i$. On each $T(\mathbb{B}_i)$ we put a linear ordering $\leq_i$ such that

$$(T(\mathbb{B}_i), \leq_i) \cong (A, C^A, \leq^A, \preceq^A).$$

Let $\leq$ be a linear ordering on $T(\mathbb{B}')$ which extends each $\leq_i$. Note that this is possible since $\mathbb{B}_i \cap \mathbb{B}_j = \emptyset$ for $i \neq j$. Let $\phi_i$ be the unique embedding from $(A, C^A, \leq^A)$ into $(\Delta(\mathbb{B}_i), \leq_i)$ with image $(\Delta(\mathbb{B}_i), \leq_i)$. Let $\mathcal{E} := \{\phi_i : 1 \leq i \leq l\}$. Now we may take

$$\rho(A, C^A, \leq^A, \preceq^A) := \frac{1}{\#(A, C^A, \leq^A)}.$$ 

Now it is easy to see that in this way we can satisfy the condition of the $QOP^*$, since for a given $\leq$ with $(\Delta(\mathbb{B}'), \leq, \leq)$ there is only one $\phi_i$ such that $\phi_i$ embeds $(A, C^A, \leq^A, \preceq^A)$ into $(\Delta(\mathbb{B}'), \leq, \leq)$.

6.1.4 $QOP^*, QOP$ and $K[\mathcal{L}]$

We now show how the $QOP$ and the $QOP^*$ interact with $K[\mathcal{L}]$. This will give us an alternate way to check unique ergodicity of $\text{Aut}(\text{Flim}(K[\mathcal{L}]))$.

Proposition 6.1.5. Let $(\mathcal{K}, K^*)$ and $(\mathcal{L}, L^*)$ be excellent pairs. If $K^*$ and $L^*$ satisfy $QOP^*$, then $K[\mathcal{L}]$ satisfies the $QOP^*$.

Proof. Let $\mathbb{A} = (S_1, \ldots, S_k : T) \in K[\mathcal{L}]$. There is an $R \in \mathcal{K}$ and an $\mathcal{E}_0$, a collection of embeddings from $T$ into $R$ which witnesses the $QOP^*$ for $K^*$. Also, there are $L_i \in \mathcal{L}$ for each $i$ and $\mathcal{E}_i$, a collection of embeddings from $S_i$ into $L_i$ which witnesses the $QOP^*$ for $L^*$.

Consider the structure $\mathbb{B} = (L_1, \ldots, L_k : R) \in K[\mathcal{L}]$ with the collection $\mathcal{E}$ of all embeddings from $\mathbb{A}$ in $\mathcal{L}$. Each embedding from $\mathcal{E}$ is given by a member of $\mathcal{E}_0$ and a sequence of embeddings from $\mathcal{E}_1, \ldots, \mathcal{E}_k$.

Let $\mu$ and $\nu$ be maps on $\mathcal{K}$ and $\mathcal{L}$ respectively that verifies the $QOP^*$. We check that for $\mathbb{A}^* = (S_1^*, \ldots, S_k^* : T^*)$ with $(\mathbb{A} \oplus \mathbb{A}^*) \in K^*[\mathcal{L}^*]$ the following map verifies the $QOP^*$ for $K^*[\mathcal{L}^*]$

$$\rho(\mathbb{A} \oplus \mathbb{A}^*) := \mu(T \oplus T^*) \cdot \prod_{i=1}^{k} \nu((S_i \oplus S_i^*))$$
Notice

\[
\frac{N_{\exp}(E, A^*, B^*)}{|E|} = N_{\exp}(E_0, T^*, R^*) \cdot \prod_{i=1}^k N_{\exp}(E_i, S_i^*, L_i^*)
\approx (\mu(T \oplus T^*) \pm \epsilon) \cdot \prod_{i=1}^k (\nu((S_i \oplus S_i^*)) \pm \epsilon)
\approx \rho(A \oplus A^*)^{\epsilon^2 \cdot k+1}
\]

We trust that the reader can appropriately interpret the use of “≈” in the second line. Since \(\epsilon\) can be arbitrarily small, this completes the verification of the QOP\(^*\) for \(K^*[L^*]\).

**Proposition 6.1.6.** Let \((K, K^*)\) and \((L, L^*)\) be excellent pairs. If \(K^*\) and \(L^*\) satisfy the QOP, then \(K[L]\) satisfies the QOP.

**Corollary 6.1.7.** Let \((K, K^*)\) and \((L, L^*)\) be excellent pairs that satisfy the QOP. If Aut(Flim(L)) and Aut(Flim(K)) are amenable then Aut(Flim(K[L])) is uniquely ergodic.

**Proof.** This follows from Proposition 1.5.4 (that \((K[L], K^*[L^*])\)) is an excellent pair), Proposition 6.1.6, Theorem 5.0.4.1 and Theorem 6.1.3.

### 6.2 Lemmas for Proving Unique Ergodicity

The following obvious lemma is a way to prove unique ergodicity. It says that two consistent random expansions that agree on a cofinal subclass, agree on the whole class.

**Lemma 6.2.1.** Let \((K, K^*)\) be an excellent pair and let \(D \subseteq K\) be a cofinal subclass. Suppose that \((\mu_A)\) and \((\nu_A)\) are two consistent random expansions of \(K\) such that for every \(\epsilon > 0\), for every \(D \in D\) and \((D \oplus D^*) \in K^*\) we have

\[
|\mu_D(\{D^*\}) - \nu_D(\{D^*\})| \leq \epsilon.
\]

Then \(\mu_D \equiv \nu_D\) for \(D \in D\) and due to cofinality we have \(\mu_A \equiv \nu_A\) for all \(A \in K\).

Let \((K, K^*)\) be an excellent pair, and let \(G, H \in K\). Recall the notation of \(N_{\exp}(E, H^*, G^*)\) and \(N_{\text{emb}}(H, G)\) from Section 6.1.

For \(\mu, \nu\) two consistent random expansions, define the **Total Variation** between them (on \(H\)) as

\[
d_{TV}(\mu_H, \nu_H) := \max_A |\mu_H(\{A\}) - \nu_H(\{A\})|, \text{ where } A \in \{H^* : (H \oplus H^*) \in K^*\}.
\]

Note that this definition is slightly different from the one presented in [3].

The following lemma is based on Lemma 2.1 and Theorem 2.2 from [3]. Note that we are not assuming that \((\nu_H)\) is a consistent random expansion, although in the case that Aut(Flim(K)) is amenable and satisfies the QOP\(^*\), \((\nu_H)\) will a posteriori be the uniform consistent random expansion.

**Lemma 6.2.2.** Let \((K, K^*)\) be an excellent pair and suppose that \(\nu_H\) is the uniform probability measure on \(\{H^* : (H \oplus H^*) \in K^*\}\). Let \(G, H \in K\) be such that \(H \hookrightarrow G\). Let \(\delta > 0\) be such that for all
\[(H \oplus H^*), (G \oplus G^*) \in K^* \text{ we have:}\]
\[
\left| \frac{N_{\text{exp}}(H^*, G^*)}{N_{\text{emb}}(H, G)} - \nu_H(\{H^*\}) \right| \leq \delta. \tag{6.1}
\]

Let \(\mu\) be any consistent random expansion. Then for all \(H \in K\) we have
\[
d_{TV}(\mu_H, \nu_H) \leq \delta \cdot |\{H^* : (H \oplus H^*) \in K^*\}|.
\]

**Proof.** This is a Fubini-style argument. Consider the set \(G := \{G^* : (G \oplus G^*) \in K^*\}\) with measure \(\mu_G\) and let \(E\) be the set of embeddings \(\phi : H \rightarrow G\), with the uniform measure \(\rho(\phi) := \frac{1}{|E|}\). We consider \(G \times E\) with the product measure \(\mu_G \times \rho\). Fix \((H - H^0) \in K^*\). We will measure the following set in two ways:
\[
E := \{(G^*, \phi) : \phi \text{ embeds } (H \oplus H^0) \text{ into } (G \oplus G^*)\}.
\]

By property \((P)\) and \((E)\) for \(\rho\) we have:
\[
(\mu_G \times \rho)(E) = \sum_{\phi \in E} \rho(\phi) \cdot \mu_H(\{H^0\}) = \mu_H(\{H^0\}) \cdot \sum_{\phi \in E} \rho(\phi)
\]
\[
= \mu_H(\{H^0\}). \tag{6.2}
\]

Alternatively, we have
\[
(\mu_G \times \rho)(E) = \sum_{G^* \in G} \mu_G(\{G^*\}) \cdot \rho(\phi) \cdot N_{\text{exp}}(H^0, G^*). \tag{6.3}
\]

From 6.1 we have:
\[
\forall G^* \in G, \quad |N_{\text{exp}}(H^0, G^*) - \nu_H(\{H^0\}) \cdot N_{\text{emb}}(H, G)| \leq \delta \cdot N_{\text{emb}}(H, G)
\]

so by multiplying through by \(\mu_G(\{G^*\}) \cdot \rho(\phi)\) for each \(G^* \in G\), summing all of them, and 6.3 we get
\[
\left| (\mu_G \times \rho)(E) - \sum_{G^* \in G} \mu_G(\{G^*\}) \cdot \rho(\phi) \cdot \nu_H(\{H^0\}) \cdot N_{\text{emb}}(H, G) \right|
\]
\[
\leq \sum_{G^* \in G} \mu_G(\{G^*\}) \cdot \rho(\phi) \cdot \delta \cdot N_{\text{emb}}(H, G).
\]

After rearrangement, we have:
\[
\left| (\mu_G \times \rho)(E) - \nu_H(\{H^0\}) \cdot \rho(\phi) \cdot N_{\text{emb}}(H, G) \cdot \sum_{G^* \in G} \mu_G(\{G^*\}) \right|
\]
\[
\leq \delta \cdot \rho(\phi) \cdot N_{\text{emb}}(H, G) \cdot \sum_{G^* \in G} \mu_G(\{G^*\})
\]

Since \(\rho(\phi) \cdot N_{\text{emb}}(H, G) = 1\), and by \((P)\) for \(\mu_G\), we obtain:
\[
|(\mu_G \times \rho)(E) - \nu_H(\{H^0\})| \leq \delta. \tag{6.4}
\]
Therefore, from 6.2 and 6.4 we have:

\[ |\mu_H(\{H^0\}) - \nu_H(\{H^0\})| \leq \delta. \]

The final desired conclusion follows from the definition of \(d_{TV}\).

Therefore, in order to obtain unique ergodicity, in Lemma 6.2.2, it is enough to establish condition 6.1 for arbitrarily small \(\delta > 0\).

6.3 Strategy for \(D_n\) and \(\hat{T}^{\omega}\)

Let \(K\) be one of the directed graphs \(D_n\) or \(\hat{T}^{\omega}\). We will show that \(\text{Aut}(K)\) is uniquely ergodic using a method developed in [3, Section 3]. First we present a useful probabilistic inequality, and then we will discuss the general strategy.

6.3.1 McDiarmid’s Inequality

The following theorem appears as Lemma 1.2 in [21] and is a consequence of Azuma’s inequality.

**Theorem 6.3.1** (McDiarmid’s Inequality). Let \(\vec{X} = (X_1, \ldots, X_N)\) be a sequence of independent random variables and let \(f(X_1, \ldots, X_N)\) be a real-valued function such that there are positive constants \(a_i\), with

\[ |f(\vec{X}) - f(\vec{Y})| \leq a_i, \]

whenever the vectors \(\vec{X}\) and \(\vec{Y}\) differ only in the \(i^{th}\) coordinate. Then for \(\zeta = \mathbb{E}[f(\vec{X})]\) and all \(\epsilon > 0\) we have:

\[ P[|f(\vec{X}) - \zeta| \geq \epsilon] \leq 2\exp\left(\frac{-2\epsilon^2}{\sum_{i=1}^{N} a_i^2}\right). \]

Typically we will use \(\vec{X} = (X_1, X_2, \ldots, X_{\binom{n}{2}})\), to talk about the random directed tournament on \(n\) vertices, and \(\vec{X}\) will correspond to the direction of the \(\binom{n}{2}\) edges.

6.3.2 General Strategy

By Theorem 6.1.3, to show that \(\text{Aut}(K)\) is uniquely ergodic, it suffices to show that it is amenable and \(K := \text{Age}(\mathbb{K})\) satisfies the \(QOP^*\). Showing amenability will usually be direct, and in the case of \(K = D_n\) we have already shown amenability in Theorem 4.4.1. Showing that \(K\) satisfies the \(QOP^*\) is a more subtle affair.

For the \(QOP^*\), for a (small) fixed \(H \in \mathbb{K}\), (with around \(k \cdot m\) vertices), we will find a (large, somewhat “random”) \(G \in \mathbb{K}\), with \(n\) vertices. In general, \(\mathcal{E} := \{\phi : \phi \text{ embeds } A \text{ into } B\}\) will be the set of all embeddings from \(H\) into \(G\), so \(|\mathcal{E}| = N_{\text{emb}}(H, G)\), and \(\rho(H \oplus H^*) = \frac{1}{\#(H)}\), where \(\#(H)\) is the number of expansions of \(H\).

We use the notation of \(I(n, k, m) := \mathbb{E}[N_{\text{emb}}(H, G)]\), the expected value of the number of embeddings of \(H\) into \(G\). Note that \(I(n, k, m)\) may also depend on other aspects of \(H\) and \(G\), but in practice they won’t. In general only \(n\) will be allowed to vary, and we will be concerned with large \(n\).
Chapter 6. Unique Ergodicity

We will always establish two separate inequalities using the McDiarmid inequality. The first will be with the function

$$f(G) := \frac{N_{\text{emb}}(H, G)}{I(n, k, m)}$$

and we will establish that changing $G$ by a single edge changes $f(G)$ by at most $O(\frac{1}{n^2})$. It is clear that $E[f(G)] = 1$. McDiarmid’s inequality then yields

$$P \left[ \left| \frac{N_{\text{emb}}(H, G)}{I(n, m, k)} - 1 \right| \geq D \right] \leq 2\exp\left( \frac{-2D^2}{(\frac{n}{2})^2 n^4} \right) \leq 2\exp(-\delta_1 D^2 n^2), \quad (6.5)$$

where $D = \xi$, fixed at the beginning, and $\delta_1$ does not depend on $n$, and the $\epsilon_1$ comes from $O(\frac{1}{n^2})$. The second inequality will be similar, applying McDiarmid’s inequality to the function

$$f^*(G) := \frac{N_{\text{exp}}(H^*, G^*)}{I(n, k, m)},$$

and we will establish that changing $G$ by a single edge changes $f^*(G)$ by at most $O(\frac{1}{n^2})$. It is clear that $E[f^*(G)] = \rho(H \oplus H^*)$. Thus McDiarmid’s Inequality yields:

$$P \left[ \left| \frac{N_{\text{exp}}(H^*, G^*)}{I(n, m, k)} - \rho(H \oplus H^*) \right| \geq D \right] \leq 2\exp\left( \frac{-2D^2}{(\frac{n}{2})^2 n^4} \right) \leq 2\exp(-\delta_2 D^2 n^2) \quad (6.6)$$

where $\rho(H \oplus H^*)$ and $\delta_2$ do not depend on $n$, and the $\epsilon_2$ comes from $O(\frac{1}{n^2})$.

We then define the probability that $G$ is not a suitable candidate:

$$p = \frac{2 \cdot \#(H) \cdot \#(G)}{e^{n^2 D^2 n^2}},$$

which will go to 0 as $n$ gets large by Stirling’s approximation, since for us $\#(H) \cdot \#(G) = O(n!)$, or $O((n!)^k)$, which corresponds to the number of pairs of expansions on $H$ and $G$. See Lemma 6.8.1 for an explicit calculation.

So, except with probability $p$, by 6.5 and 6.6 we have, simultaneously:

$$\left| \frac{N_{\text{emb}}(H, G)}{I(n, m, k)} - 1 \right| < D \quad \text{and} \quad \left| \frac{N_{\text{exp}}(H^*, G^*)}{I(n, m, k)} - \rho(H \oplus H^*) \right| < D$$

for all $(H \oplus H^*), (G \oplus G^*) \in K^*$. For large enough $n$, we have $p < 1$, so a suitable $G$ will exist. The previous inequalities yield the following:

$$\left| \frac{N_{\text{exp}}(H^*, G^*)}{I(n, m, k)} - \frac{N_{\text{exp}}(H^*, G^*)}{N_{\text{emb}}(H, G)} \right| = \frac{N_{\text{exp}}(H^*, G^*)}{N_{\text{emb}}(H, G)} \cdot \left| \frac{N_{\text{emb}}(H, G)}{I(n, m, k)} - 1 \right| \leq 1 \cdot D = D.$$
Finally, from the triangle inequality, we have that $G$ witnesses the QOP:

$$
\left| \frac{N_{\text{exp}}(\mathbb{H}^*, G^*)}{N_{\text{emb}}(\mathbb{H}, G)} - \rho(\mathbb{H} \oplus \mathbb{H}^*) \right| \leq \left| \frac{N_{\text{exp}}(\mathbb{H}^*, G^*)}{N_{\text{emb}}(\mathbb{H}, G)} - \frac{N_{\text{exp}}(\mathbb{H}^*, G^*)}{I(n, m, k)} + \frac{N_{\text{exp}}(\mathbb{H}^*, G^*)}{I(n, m, k)} - \rho(\mathbb{H} \oplus \mathbb{H}^*) \right| 
\leq 2D \leq \epsilon.
$$

We summarize this in a lemma.

**Lemma 6.3.2.** Using the notation defined above, suppose that $\text{Aut}(\mathbb{K})$ is amenable, that changing $G$ by a single edge changes $f$ and $f^*$ by no more than $O(\frac{1}{n^2})$, and that $\#(G) \leq O(n!)$ or $O((n!)^k)$. Then $\text{Aut}(\mathbb{K})$ is uniquely ergodic.

### 6.4 Unique Ergodicity of $\text{Aut}(\mathbb{D}_n)$

We will show unique ergodicity of $\text{Aut}(\mathbb{D}_n)$ in two steps. First we consider the special case of $n = \omega$, then we adapt the proof for $n < \omega$.

Let $(n)_k$ be the number of injective maps from $\{1, \ldots, k\}$ into $\{1, \ldots, n\}$. Note that in general this is different from $\binom{n}{k}$.

**Theorem 6.4.1.** $\text{Aut}(\mathbb{D}_\omega)$ is uniquely ergodic.

**Proof.** Let $\mathbb{H} = (H, \rightarrow_H) \in \mathcal{D}_\omega$ have $k$ many $\perp H$-equivalence classes with respective cardinalities $a_1, \ldots, a_k$. Let $G$ be the set with partition $G = \bigcup_{i=1}^k G_i$, with $|G_i| = m$ for $i \leq k$.

We consider a sequence of independent random variables induced by a pair of elements $G(x, y)$ where $x \in G_i, y \in G_j$ and $i < j$. Each random variable indicates with probability $\frac{1}{2}$ that $x \rightarrow^G y$ and with probability $\frac{1}{2}$ that $y \rightarrow^G x$. In this way, the collection of random variables $G(x, y)$ gives a random directed graph $G = (G, \rightarrow^G) \in \mathcal{D}_\omega$.

Notice that $\#(G) = (n!)(m!)^n = O(n!)$ and $\#(\mathbb{H}) = k! \cdot a_1! \cdots a_k!$.

We have:

$$
I(n, m, k, \bar{a}) := \mathbb{E}[N_{\text{emb}}(\mathbb{H}, G)] = (n)_k \prod_{i=1}^k (m)_{a_i} \cdot 2^{-\sum_{i<j} a_i a_j},
$$

where $\bar{a} = (a_1, \ldots, a_k)$. For

$$
f(G) := \frac{N_{\text{emb}}(\mathbb{H}, G)}{I(n, m, k, \bar{a})}
$$

we have $\mathbb{E}[f(G)] = 1$. If we change the direction of only one edge then we change $N_{\text{emb}}(\mathbb{H}, G)$ by not more than

$$(k)_2 \cdot (n - 2)_{k-2} \prod_{i=1}^k (m)_{a_i},$$

and $f$ by not more than:

$$\frac{2}{n(n-1)} \frac{(k)_2 \cdot (n - 2)_{k-2} \prod_{i=1}^k (m)_{a_i}}{(n)_k \prod_{i=1}^k (m)_{a_i} \cdot 2^{-\sum_{i<j} a_i a_j}} = \frac{1}{n(n-1)} \frac{(k)_2 \cdot 2^{\sum_{i<j} a_i a_j}}{n^2} \leq \epsilon_1
$$

for large enough $n$ and some positive constant $\epsilon_1$. 

Let \((\mathbb{H}, \leq^H) \in \mathcal{D}_{\omega}^c\) and \((\mathbb{G}, \leq^G) \in \mathcal{D}_{\omega}^c\) be such that the \(\bot\)-equivalence classes are intervals with respect to \(\leq^G\). Then we have:

\[
\mathbb{E}[N_{\exp}(\leq^H, \leq^G)] = \frac{I(n, m, k, \bar{a})}{k!a_1! \cdots a_k!}.
\]

A change in the direction of one edge will change the function:

\[
f^*({\mathbb{G}}) := \frac{N_{\exp}(\leq^H, \leq^G)}{I(n, m, k, \bar{a})}
\]

by not more than

\[
\frac{(k) \cdot (n-2)^{k-2} \cdot \prod_{i=1}^k (m)_{a_i}}{(n)_k \cdot \prod_{i=1}^k (m)_{a_i}} \cdot 2^{-\sum_{i<j} a_i a_j} = \frac{1}{n(n-1)} \left(\frac{k}{2}\right) \cdot 2^{\sum_{i<j} a_i a_j} \leq \frac{\epsilon_2}{n^2}
\]

for large enough \(n\) and some positive constant \(\epsilon_2\). For the McDiarmid inequality we use \(\rho(\mathbb{H} \oplus \mathbb{H}^*) = \frac{1}{k!a_1! \cdots a_k!}\).

Thus we are finished by Lemma 6.3.2.

\[\Box\]

**Theorem 6.4.2.** For \(n < \omega\), \(\text{Aut}(\mathbb{D}_n)\) is uniquely ergodic.

**Proof.** Since it is more natural to let \(n\) vary, we will show that \(\text{Aut}(\mathbb{D}_N)\) is uniquely ergodic. For \(N\) finite we will consider a similar random directed graph \(\mathbb{G}\) given by parameters \(k\), the numbers of parts, and \(m\), the cardinality of the parts. In this case, \(N\) will be fixed and we will adjust \(m\).

Notice that \(\#(\mathbb{G}) = (N!)^k (m!)^N = O((m!)^N)\) and \(\#(\mathbb{H}) = k! \cdot a_1! \cdots a_k!\), where \(N\) is fixed, and \(m\) can vary. So \(\rho(\mathbb{H} \oplus \mathbb{H}^*) = \frac{1}{k!a_1! \cdots a_k!}\).

Let

\[
f(\mathbb{G}) := \frac{N_{\text{emb}}(\mathbb{H}, \mathbb{G})}{I(N, m, k, \bar{a})}.
\]

A single change in the direction of one edge of \(\mathbb{G}\) will change \(N_{\text{emb}}(\mathbb{H}, \mathbb{G})\) by not more than

\[
(k)_2 \cdot (N-2)^{k-2} \cdot a_i \cdot (m-1)_{a_i-1} \cdot a_j \cdot (m-1)_{a_j-1} \cdot \prod_{l \neq i,j}^m \left(\frac{m}{a_l}\right),
\]

and \(f\) by not more than:

\[
\frac{(k)_2 \cdot (N-2)^{k-2} \cdot a_i \cdot (m-1)_{a_i-1} \cdot a_j \cdot (m-1)_{a_j-1} \cdot \prod_{l \neq i,j} (m)_{a_l}}{(N)_k \cdot (m)_{a_1} \cdot \cdots \cdot (m)_{a_k} \cdot 2^{-\sum_{i<j} a_i a_j}}
\]

\[
= \frac{(k)_2}{N(N-1)} \cdot a_i a_j \cdot \frac{1}{m^2} \cdot 2^{\sum_{i<j} a_i a_j}
\]

\[
\leq \frac{\epsilon_1}{m^2}
\]

for a large enough \(m\) and some positive constant \(\epsilon_1\).

For \((\mathbb{H} \oplus \mathbb{H}^*), (\mathbb{G} \oplus \mathbb{G}^*) \in \mathcal{D}_N\), for

\[
f^*({\mathbb{G}}) := \frac{N_{\exp}(\mathbb{H}^*, \mathbb{G}^*)}{I(N, m, k, \bar{a})}
\]

\[\Box\]
we have:
\[
\mathbb{E}[N_{\exp}(\mathbb{H}^*, \mathbb{G}^*)] = \frac{I(N, m, k, \bar{d})}{(N)_k \cdot k! \cdot a_1! \cdots a_k!}.
\]

A single change in direction of one edge will change \(f^*(\mathbb{G})\) by not more than
\[
\frac{a_i \cdot (m-1)_{a_i-1} \cdot a_j \cdot (m-1)_{a_j-1} \cdot \prod_{i \neq j} (\binom{m}{a_i})}{(N)_k \cdot (m)_{a_1} \cdots (m)_{a_k} \cdot 2 \Pi_{i<j} a_i a_j} = \frac{1}{k^2} \cdot a_i a_j \frac{1}{m^2} \cdot 2 \Pi_{i<j} a_i a_j
\]

for large enough \(m\) and some positive constant \(\epsilon_2\).

Thus we are finished by Proposition 6.3.2.

\[\square\]

### 6.5 \(\hat{T}^{\omega}\) has a Uniquely Ergodic Automorphism Group

Recall the notation from Section 4.5. There we established that \(\text{Aut}(\hat{T}^{\omega})\) is amenable.

**Theorem 6.5.1.** The group \(\text{Aut}(\hat{T}^{\omega})\) is uniquely ergodic.

**Proof.** Let \(\mathcal{T} := \text{Age}(\hat{T}^{\omega})\). Since \((\hat{T}, \hat{T}^*)\) is an excellent pair, we may use Proposition 1.5.1 to establish unique ergodicity.

We established that \(\text{Aut}(\hat{T}^{\omega})\) is amenable in Theorem 4.5.4.

Let \(\mathbb{H} = (H, \rightarrow^H) \in \hat{T}\) and let \(H = H_1 \sqcup \ldots \sqcup H_k\) be the partition into \(\perp^H\)-equivalence classes, with \(|H_i| = 2\) for all \(i \leq k\).

We consider a sequence of independent random variables \(G(i,j)\) with \(1 \leq i < j \leq n\). Each random variable indicates with probability \(\frac{1}{2}\) that there is an edge between the equivalence classes \(G_i\) and \(G_j\), so that we obtain a graph in \(\mathcal{T}\). Observe that there are only two options since, for a given vertex and equivalence class, there is exactly one in and one out vertex in this class. In this way, the collection \(G(i,j)\) of random variables gives a directed graph \(\mathbb{G} = (G, \rightarrow^G) \in \mathcal{T}\).

Notice that \(\#(\mathbb{G}) = n! \cdot 2^n = O(n!)\) and \(\#(\mathbb{H}) = k! \cdot 2^k\) so \(\rho(\mathbb{H} \oplus \mathbb{H}^*) = \frac{1}{k^2 2^k}\).

In particular we have:

\[
I(n, k) := \mathbb{E}[N_{\text{emb}}(\mathbb{H}, \mathbb{G})] = (n)_k \cdot 2^k \cdot 2^{-\frac{k}{2}} = (n)_k \cdot 2^{-\frac{k}{2} + k}.
\]

Define

\[
f(\mathbb{G}) := \frac{N_{\text{emb}}(\mathbb{H}, \mathbb{G})}{I(n, k)}
\]

so we have \(\mathbb{E}[f(\mathbb{G})] = 1\). Changing a single value of a single \(G(i,j)\) changes \(N_{\text{emb}}(\mathbb{H}, \mathbb{G})\) by not more than:

\[
(k)_{2} \cdot (n-2)_{k-2} \cdot 2^k.
\]

and \(f(\mathbb{G})\) by not more than:

\[
\frac{(k)_{2} \cdot (n-2)_{k-2} \cdot 2^k}{(n)_k \cdot 2^{-\frac{k}{2} + k}} = \frac{1}{n(n-1)} \cdot \frac{(k)_{2} \cdot 2}{2^{-\frac{k}{2} - k}} \leq \frac{\epsilon_1}{n^2}
\]
for a large enough $n$ and some positive constant $\epsilon_1$.

Now let $((\mathbb{H}, \leq^H, I^H_1, I^H_2), (\mathbb{G}, \leq^G, I^G_1, I^G_2)) \in \mathcal{T}^*$ and let $\leq^G, I^G_1, I^G_2$ be given such that $(\mathbb{G}, \leq^G, I^G_1, I^G_2) \in \mathcal{T}^*$. That is, each set $G_i$ comes with a partition given by $I^G_1$ and $I^G_2$, where $\leq^G$ is a linear ordering such that $G_1 \leq^G \ldots \leq^G G_{m}$. Then we have:

$$E[N_{\exp}(\mathbb{H}^*, \mathbb{G}^*)] = nk \cdot 2^{-(\frac{k}{2})}.$$  

For

$$f^*(\mathbb{G}) = \frac{N_{\exp}(\mathbb{H}^*, \mathbb{G}^*)}{(n)_k \cdot 2^{-(\frac{k}{2})} + k},$$

we have

$$E[f^*(\mathbb{G})] = \frac{1}{k! \cdot 2^k}.$$

A change in a single $G(i, j)$ will change $f^*(\mathbb{G})$ by not more than

$$\frac{(k)_2 \cdot (n - 2)_{k-2}}{(n)_k \cdot 2^{-(\frac{k}{2})} + k} = \frac{1}{n(n - 1)} \cdot \frac{(k)_2}{2^{-(\frac{k}{2})} + k} \leq \frac{\epsilon_2}{n^2}$$

for a large enough $n$ and a fixed $k$.

Thus we are finished by Proposition 6.3.2.

\section{6.6 Expansions of the Semigeneric Digraph}

In Section 4.3 we established that the automorphism group of $\mathcal{S}$, the semigeneric digraph, is amenable. In this section we provide some related expansions of $\mathcal{S} := \text{Age}(\mathcal{S})$ and check that they satisfy the QOP. This is not enough to get unique ergodicity of $\text{Aut}(\mathcal{S})$, but provides a stepping stone to that result, and homes in on the difficulties it presents.

Consider the classes $\mathcal{S}_R := \mathcal{S}^* | \{\to, R\}$ and $\mathcal{S}_\leq := \mathcal{S}^* | \{\to, \leq\}$. It is not hard to see that $\mathcal{S}_R$ and $\mathcal{S}_\leq$ are Fraïssé classes, and that $(\mathcal{S}_R, \mathcal{S}^*)$ and $(\mathcal{S}_\leq, \mathcal{S}^*)$ are excellent pairs. Let $\mathcal{S}_R := \text{Flim}(\mathcal{S}_R)$ and $\mathcal{S}_\leq := \text{Flim}(\mathcal{S}_\leq)$.

\begin{thm}
$\text{Aut}(\mathcal{S}), \text{Aut}(\mathcal{S}_R)$ and $\text{Aut}(\mathcal{S}_\leq)$ are amenable.
\end{thm}

\begin{proof}
The cases of $\text{Aut}(\mathcal{S}_R)$ and $\text{Aut}(\mathcal{S}_\leq)$ are similar to the case of $\text{Aut}(\mathcal{S})$, which was shown in Theorem 4.3.3.
\end{proof}

\begin{thm}
$\text{Aut}(\mathcal{S}_R)$ and $\text{Aut}(\mathcal{S}_\leq)$ are uniquely ergodic.
\end{thm}

\begin{proof}
Since we already have that $\text{Aut}(\mathcal{S}_R)$ is amenable, it is enough to prove the uniqueness of a consistent random expansion on a dense subclass of $\mathcal{S}_R$.

Let $\mathbb{H} := (H, \to^H, R^H) \in \mathcal{S}_R$ and let $H = H_1 \sqcup \ldots \sqcup H_k$ be the partition into $\perp^H$-equivalence classes, with $M = |H_i| = 2^{k-1} \cdot m$ for all $i \leq k$, for some natural number $m \geq 1$. Moreover, assume that $(H, \to^H)$ is in the dense subclass of $\mathcal{S}$ where all parts in each column have the same size. By part in each column, we mean each element of the partition of a column given by the other columns.

Let $G = \bigcup_{i=1}^n G_i$ be a partition with $|G_i| = M$ for $i \leq n$. We consider a sequence of independent random variables $G(i, j)$ with $1 \leq i < j \leq n$. Each random variable $G(i, j)$ gives a pair of sets $(A, B)$, which is also given by $R$, such that:
• $A \subseteq G_i$,
• $B \subseteq G_j$, and
• $|A| = |B| = \frac{M}{2}$.

The partition given by $(A, B)$ is the same as the partition given by $(G_i \setminus A, G_j \setminus B)$. There are $\frac{1}{2} \cdot \left(\frac{M}{2}\right)^2$ such pairs and we assume that $G(i, j)$ has a uniform distribution, i.e. each pair occurs with probability

$$p = 2 \cdot \left(\frac{M}{2}\right)^{-2}.$$

Each pair $(A, B)$ describes a distribution of edges between $G_i$ and $G_j$ such that for $x \in G_i, y \in G_j$ we have

$$x \to^G y \iff (x \in A, y \in G_j \setminus B) \lor (x \in G_i \setminus A, y \in B)$$
$$y \to^G x \iff (x \in A, y \in B) \lor (x \in G_i \setminus A, y \in G_j \setminus B)$$
$$x \in G_i, y \in G_j \Rightarrow (R^G(x, y) \iff y \in B)$$
$$x \in G_i, y \in G_j \Rightarrow (R^G(y, x) \iff x \in A)$$

In this way we obtain a random graph $\mathbb{G} = (G, \to^G) \in \mathcal{S}_R$.

For the McDiarmid Inequality we take $\rho(H \oplus H^*) = \frac{1}{\frac{n}{M}}$.

Notice $\#(\mathbb{G}) = (M!)^n$ and $\#(\mathbb{H}) = (M!)^k$.

In particular we have:

$$I(n, M, k) := \mathbb{E}[\text{emb}(\mathbb{H}, \mathbb{G})] = (n)_k \cdot (M!)^k \cdot p^{(k)}$$

Then for

$$f(\mathbb{G}) := \frac{\text{emb}(\mathbb{H}, \mathbb{G})}{I(n, M, k)}$$

we have $\mathbb{E}[f(\mathbb{G})] = 1$.

Changing a single value of a single $G(i, j)$ changes $\text{emb}(\mathbb{H}, \mathbb{G})$ by not more than:

$$(k)_2 \cdot (n - 2)_{k-2} \cdot (M!)^k,$$

and changes $f(\mathbb{G})$ by at most

$$\left(\frac{k}{n} \cdot (M!)^k \cdot p^{(k)}\right)^2 = \frac{1}{n(n-1)} \cdot \left(\frac{k}{p^{(k)}}\right) \leq \frac{\epsilon_1}{n^2}$$

for a large enough $n$, a positive constant $\epsilon_1$ and a fixed $k$, since $p$ depends only on $M$.

Now let $(\mathbb{H} \oplus \mathbb{H}^*), (\mathbb{G} \oplus \mathbb{G}^*) \in \mathcal{S}_R$ where $H^* = (H, \leq_H)$ and $G^* = (G, \leq G)$.

Without loss of generality we may assume that $H_1 <^H \ldots <^H H_k$ and $G_1 <^G \ldots <^G G_n$.

Then we have:

$$\mathbb{E}[\text{exp}(\mathbb{H}^*, \mathbb{G}^*)] = \binom{n}{k} \cdot p^{(k)}.$$

For

$$f^*(\mathbb{G}) = \frac{\text{exp}(\mathbb{H}^*, \mathbb{G}^*)}{I(n, M, k)}$$
we have
\[ \mathbb{E}[f^*(G)] = \frac{1}{k! \cdot (M!)^k}. \]

A change in a single \( G(i, j) \) will change \( N_{\exp}(\mathbb{H}, \mathcal{G}^*) \) by not more than
\[ (k)_2 \cdot (n - 2)_{k-2} \]
and change \( f^*(G) \) by not more than
\[
\frac{(k)_2 \cdot (n - 2)_{k-2}}{(n)_k \cdot (M!)^k \cdot p^k(\mathcal{I})} = \frac{(k)_2}{n(n - 1)} \cdot \frac{\epsilon_2}{n^2} \leq \epsilon_2 \]
for a large enough \( n \), a positive constant \( \epsilon_2 \), and a fixed \( k \).

Thus we are finished by Proposition 6.3.2.

---

**Proof for \( S_{\leq} \).** The construction of \( \mathcal{G} \) is very similar to the previous case, so we only highlight the main differences.

Let \( \mathbb{H} = (H, \rightarrow^H, \preceq^H) \in S_{\leq} \) and \( (\mathbb{H} \oplus \mathbb{H}^*) \in S^* \) where \( \mathbb{H}^* = (H, R^H) \). Let \( H = H_1 \sqcup \ldots \sqcup H_k \) be the partition into \( \bot^H \)-equivalence classes, where \( H_1 <^H \ldots <^H H_k \). Let \( l = 2^L \) and let \( (\mathbb{H}^*_i)_{i=1}^k \) be the list of all expansions \( (H \oplus H^*) \in S^* \).

There is a structure \( \mathcal{G} = (G, \rightarrow^G, \preceq^G) \in S_{\leq} \) such that \( G = G_1 \sqcup \ldots \sqcup G_k \) is the partition into \( \bot^G \)-equivalence classes, and for each \( G_i \) we have
\[
G_i = \bigsqcup_{j=1}^l G_{i,j} \quad \text{ where } |G_{i,j}| = |H_i|.
\]

Now if \( \mathcal{A}_j \) is a substructure of \( \mathcal{G} \) given by \( (G_{i,j})_{i=1}^k \) then \( \mathcal{A}_j \cong \mathbb{H} \). Also, if \( (G \oplus G^*) \in S^* \) when \( G^* = (G, R^G) \) then for every \( 1 \leq j \leq l \) there is an \( \mathcal{A}_j \) such that
\[
(\mathbb{H} \oplus \mathbb{H}^*_j) \cong (\mathcal{A}_j, R^G \upharpoonright A_j^2)
\]
where \( A_j \) is the underlying set of \( \mathcal{A}_j \).

Now verifying \( \text{QOP}^* \) is similar to the case of \( S_R \), except we take
\[
\rho(\mathbb{H} \oplus \mathbb{H}^*_j) := \frac{1}{2^L}
\]
and then for any \( \epsilon > 0 \) we take a \( \mathcal{G} \) as in the case of \( S_R \). Since \( \text{Aut}(S_{\leq}) \) is amenable by Theorem 6.6.1, the \( \text{QOP}^* \) ensures that \( \text{Aut}(S_{\leq}) \) is uniquely ergodic.

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### 6.7 Hypergraph Method for \( \mathcal{G}_n \) and \( \mathcal{L}(T) \)

In this section we discuss a method for proving the \( \text{QOP} \) by using hypergraphs. This method was introduced in [3], and is different from the one presented in Section 6.3.2. The idea is to construct a large random object \( \mathcal{G} \) subject to some constraints. We first construct a hypergraph of large girth with many hyperedges. Then each hyperedge is replaced by a random copy of \( \mathbb{H} \). When checking the \( \text{QOP} \)
for this structure we only examine embeddings that map $\mathbb{H}$ entirely within a single hyperedge. We shall directly compute the QOP estimate and will only need a single application of McDiarmid’s Inequality.

There is some subtlety in constructing $\mathcal{G}$ from the hypergraph which is why we include proofs of unique ergodicity of $\text{Aut}(\mathcal{G}_n)$ and $\text{Aut}(\mathbb{L}(T))$, even though similar statements appear in Section 5 of [3]. Our proof of Theorem 6.7.1 should be compared to the proof of Theorem 5.2 in [3].

6.7.1 $\mathcal{G}_n$

Let $\mathcal{G}_n$ ($n \geq 2$) be the class of finite directed graphs $\mathcal{G}$ with the property that $\mathbb{I}_{n+1} \not\rightarrow \mathcal{G}$. This is a Fraïssé class with limit $\mathbb{G} := \text{Flim}(\mathcal{G}_n)$. Let $\mathcal{G}^*_n$ be the class of structures of the form $(A, \rightarrow, \leq)$ with the property that $(A, \rightarrow, \leq) \in \mathcal{G}_n$ and $\leq$ is a linear order on $A$. Then $\mathcal{G}^*_n$ is a Fraïssé class and moreover $(\mathcal{G}_n, \mathcal{G}^*_n)$ is an excellent pair. This can be seen by using a partite method construction as in [23, 24, 25], or by introducing new relation for “not edge”.

6.7.2 $\mathbb{L}(T)$

Let $T$ be a collection of finite tournaments with $|T| \geq 3$ for all $T \in \mathcal{T}$, and let $\mathbb{L}(T)$ denote the class of finite directed graphs $\mathcal{A}$ with the property that $\mathbb{B} \not\rightarrow \mathcal{A}$ for all $\mathbb{B} \in \mathcal{T}$. Then $\mathbb{L}(T)$ is a Fraïssé class with limit $\mathbb{L}(T) := \text{Flim}(\mathbb{L}(T))$. Let $\mathbb{L}^*(T)$ denote the class of finite structures of the form $(A, \rightarrow, \leq)$ with the property that $(A, \rightarrow, \leq) \in \mathbb{L}(T)$ and $\leq$ is a linear order on $A$. Then $\mathbb{L}^*(T)$ is a Fraïssé class and moreover $(\mathbb{L}(T), \mathbb{L}^*(T))$ is an excellent pair, again see [23, 24, 25] and [1].

6.7.3 Unique Ergodicity

**Theorem 6.7.1.** Let $n \geq 2$ be a natural number and let $T$ be a collection of finite tournaments, with $|T| \geq 3$, $\forall T \in \mathcal{T}$. Then $\text{Aut}(\mathcal{G}_n)$ and $\text{Aut}(\mathbb{L}(T))$ are uniquely ergodic.

**Proof.** Since it is more natural to have $n := |\mathcal{G}|$ vary, we shall fix $m \geq 2$ and let $n$ vary.

Since $(\mathbb{L}(T), \mathbb{L}^*(T))$ and $(\mathcal{G}_m, \mathcal{G}^*_m)$ are both excellent pairs, we may use Proposition 1.5.1 to establish amenability and unique ergodicity. Let $\mathbb{K} = \mathcal{G}_m$ or $\mathcal{L}(T)$, and let $\mathbb{K}^* = \mathcal{G}^*_m$ or $\mathcal{L}(T)^*$, as appropriate.

Let $\mathcal{A}$ be a structure in $\mathbb{K}$, and let $(\mathcal{A} \oplus \mathcal{A}^*) \in \mathbb{K}^*$. Consider

$$\mu_{\mathcal{A}}(\{\mathcal{A}^*\}) := \frac{1}{|\mathcal{A}|!}.$$ 

By [3, Proposition 9.3], $(\mu_{\mathcal{A}})$ is a consistent random expansion since the expansions $\mathbb{K}^*$ of $\mathbb{K}$ are just the linear orders. By Proposition 1.5.1 this ensures amenability of $\text{Aut}(\text{Flim}(\mathbb{K}))$.

We check unique ergodicity by verifying the QOP.

Let $\mathbb{H} \in \mathbb{K}$, $(\mathbb{H} \oplus \mathbb{H}^*) \in \mathbb{K}^*$, $|\mathbb{H}| = k$, $\rho(\mathbb{H} \oplus \mathbb{H}^*) = \frac{1}{k!}$ and $\epsilon > 0$. We will find a $\mathcal{G} \in \mathcal{K}$ and a collection $\mathcal{E}$ of embeddings from $\mathbb{H}$ into $\mathcal{G}$ such that for all $(\mathbb{H} \oplus \mathbb{H}^*) \in \mathbb{K}^*$ and $(\mathcal{G} \oplus \mathcal{G}^*) \in \mathbb{K}^*$ we have:

$$\left| \frac{N_{\text{exp}}(\mathcal{E}, \mathbb{H}^*, \mathcal{G}^*)}{|\mathcal{E}|} - \frac{1}{k!} \right| < \epsilon.$$ 

There is a constant $C$, which depends only on $k$, such that for all $n \geq k$ there is a $k$-uniform hypergraph on $n$ vertices with at least $Cn^{\frac{4}{3}}$ edges and with girth at least 5, see Lemma 4.1 in [3].

Let $n$ be large enough and let $G$ be the underlying set of one such graph, and let $E_1, \ldots, E_s$ be its edges. Since the girth of the graph is at least 5, for all $1 \leq i < j \leq s$ and $1 \leq l, l' \leq s$ we have:
1. $|E_i \cap E_j| \leq 1$;
2. If $E_i \cap E_j \neq \emptyset$ and $E_i \cap (E_i \setminus E_j) \neq \emptyset$, then $E_i \cap (E_j \setminus E_i) = \emptyset$;
3. If $E_i \cap E_j = \emptyset$, $E_i \cap E_i \neq \emptyset$, $E_i \cap E_j \neq \emptyset$, $E_i \cap E_i \neq \emptyset$, $E_i \cap E_j \neq \emptyset$, then $l = l'$.

Here (2) asserts that there are no 3-cycles in $G$, and (3) says there are no 4-cycles.

We will construct a random directed graph $G$ on the set $G$ by fixing some edges and choosing some others randomly. We distinguish two cases:

$[K = G_m]$ Let $x \neq y \in G$. Then:

- If there are $E_i$ and $E_j$ such that $x \in E_i \setminus E_j$ and $y \in E_j \setminus E_i$ then we fix an edge between $x$ and $y$, for example $x \rightarrow^G y$. This is well defined according to (1) and (2).
- If there is no $E_i$ such that $\{x, y\} \subseteq E_i$, and there are no $E_j$ and $E_k$ such that $x \in E_j \setminus E_k$ and $y \in E_k \setminus E_j$ then we also fix an edge between $x$ and $y$, e.g. $x \rightarrow^G y$. This is well-defined by (3) and (1).
- If there is some $E_i$ such that $\{x, y\} \subseteq E_i$, then we choose randomly an injective map $e_i : H \rightarrow E_i$ and take:
  
  $x \rightarrow^G y \Leftrightarrow (e^{-1}_i(x) \rightarrow^H e^{-1}_i(y))$.

This is well-defined according to (1). We choose each injective map from $H$ to $E_i$ with the same probability.

In this way we obtain a random directed graph $G = (G, \rightarrow^G)$. The construction of fixed edges, and properties (2) and (3), ensure that $I_3$ can be embedded only in a subgraph induced by $E_i$. However, this is also impossible, since $H \in G_m$, thus $G \in G_m$.

$[K = L(T)]$ Let $x \neq y \in G$. Then:

- If there are $E_i$ and $E_j$ such that $x \in E_i \setminus E_j$ and $y \in E_j \setminus E_i$ then $x \perp^G y$. This is well defined according to (1) and (2).
- If there is no $E_i$ such that $\{x, y\} \subseteq E_i$, and there are no $E_j$ and $E_k$ such that $x \in E_j \setminus E_k$ and $y \in E_k \setminus E_j$ then $x \perp^G y$. This is well-defined by (1) and (3).
- If there is some $E_i$ such that $\{x, y\} \subseteq E_i$, then we choose randomly an injective map $e_i : H \rightarrow E_i$ and take:
  
  $x \perp^G y \Leftrightarrow (e^{-1}_i(x) \rightarrow^H e^{-1}_i(y))$.

Figure 6.1: $G$ for $K = G_m$, and $G$ for $K = L(T)$. 
This is well-defined according to (1). We choose each injective map from $H$ to $E_i$ with the same probability.

In this way we obtain a random directed graph $G = (G, \rightarrow G)$. The construction, and properties (1), (2) and (3), ensure that an induced tournament can be embedded only in a subgraph induced by an $E_i$. However, this is also impossible, since $H \in G_n$, thus $G \in L(T)$.

Now let us check the QOP estimate. Let $\mathcal{E}$ denote the collection of embeddings of $H$ into $G$ whose image is completely in one of the $E_i$'s. Note that $N_{\text{emb}}(\mathcal{E}, H, G) = s \cdot L$, and its expected value has a binomial distribution with parameters $(s, \frac{L}{k!})$ where $L = |\text{Aut}(H)|$ and $s \geq C n^\frac{2}{3}$ is the number of hyperedges of $G$. Fix structures $(H \oplus H^*), (G \oplus G^*) \in K^*$.

For $f(G) := \frac{N_{\exp}(\mathcal{E}, H, G)}{N_{\text{emb}}(\mathcal{E}, H, G)}$

we have $\mathbb{E}[f(G)] = \frac{1}{k!}$.

Changing a single value of a single $e_i$ changes $f(G)$ by not more than:

$$\frac{1}{s \cdot L}$$

Thus by the McDiarmid inequality we have:

$$P \left[ \left| f(G) - \frac{1}{k!} \right| \geq D \right] \leq 2 \exp \left( \frac{-2D^2}{s \cdot (\frac{k!}{2}) \cdot \left( \frac{1}{s \cdot L} \right)^2} \right) \leq 2 \exp \left( \frac{-2 \cdot D^2 \cdot L^2 \cdot C \cdot n^\frac{2}{3}}{\frac{k!}{2}} \right) = 2 \exp(-\delta \cdot n^\frac{4}{3})$$

where $D, L, k$ and $C$ (hence $\delta$) do not depend on $n$. The same estimate holds for all expansions $(H \oplus H^*)$ and $(G \oplus G^*)$ in $K^*$. Therefore, since $\#(H) = k!$ and $\#(G) = n!$, except on a set of measure

$$k! \cdot n! \cdot 2 \exp(-\delta \cdot n^\frac{4}{3}),$$

which is less than 1 for large $n$, we have

$$\left| f(G) - \frac{1}{k!} \right| \leq D.$$

In particular, choosing $D = \epsilon$ and $n$ large enough, we have our desired graph $G$, which witnesses QOP.

6.8 A Stirling Calculation

Recall that Stirling’s Approximation asserts that $n! \approx \sqrt{n \left( \frac{n}{e} \right)^n}$.

The following is a routine calculation involving Stirling’s Approximation that is used throughout.
Lemma 6.8.1. Let $k \in \mathbb{N}, \epsilon > 0$ and $c > 0$. Then

$$\lim_{n \to \infty} \frac{(n!)^k \cdot c^n}{e^{c \cdot n^2}} = 0.$$ 

Proof. By Stirling’s Approximation, it is enough to investigate

$$\frac{n^k \left( \frac{n}{2} \right)^{kn} \cdot c^n}{e^{c \cdot n^2}}.$$ 

Taking the logarithm yields:

$$\frac{k}{2} \ln(n) + kn \ln(n) - kn + n \ln(c) - cn^2$$

which tends to $-\infty$ as $n \to \infty$. \qed
Chapter 7

Going Forward

We leave off with three possible directions for further research.

7.1 Semigeneric Digraph

The most obvious open question this work leaves is the following:

**Question 7.1.1.** Is the automorphism group of the Semigeneric digraph $\mathbb{S}$ uniquely ergodic?

Theorem 4.3.3 establishes that Aut($\mathbb{S}$) is amenable, and Theorem 6.6.2 establishes that the related structures $\mathbb{S}_R$ and $\mathbb{S}_\leq$ have uniquely ergodic automorphism groups. The latter result emphasizes the difficulties that the relation $R$ poses in checking unique ergodicity of Aut($\mathbb{S}$); there are $O(2^{n^2})$ many $R$-expansions of a semigeneric finite digraph with $n$ nodes, and that is too much for the McDiarmid inequality to be of use (See Lemma 6.3.2).

A positive answer to this question seems likely as Age($\mathbb{S}$) has a cofinal class of highly symmetric structures, which seems like a natural place to find witnesses for the $QOP$. A slightly more sensitive version of McDiarmid’s inequality might offer some insight, although in some sense McDiarmid’s inequality is the best possible inequality. In another direction, unique ergodicity might need to be verified more directly, either without passing through the $QOP$ (and using a different, equivalent notion of uniquely ergodic), or by showing the $QOP$ directly.

A negative answer would be surprising, and would be the only example on Cherlin’s list of a structure whose automorphism group is amenable but not uniquely ergodic.
7.2 Other Product Classes

In Chapter 5 we investigated the relationship of the product class $\mathcal{K}[\mathcal{L}]$ with amenability, unique ergodicity and the Hrushovski property. There we provided a full description of those relationships. There are other product classes for which these relationships remain open, in particular the product classes studied in [32]. Of those, one particular construction seems to isolate the main difficulty. In our notation, let $\mathcal{K}$ be a Fraïssé class in signature $L$, and for $n \in \mathbb{N}$, let $\mathcal{K}[n]$ be the collection of all structures $(\mathcal{K}, I_0, \ldots, I_{n-1})$ in the expanded signature $L \cup \{I_0, \ldots, I_{n-1}\}$ where

1. $\mathcal{K} \in \mathcal{K}$,
2. The $\{I_i : i < n\}$ partition the elements of $\mathcal{K}$.

This may be thought of as adding $n$ colours (of nodes) to the signature of $\mathcal{K}$. This is a Fraïssé class, and has expansion $\mathcal{K}[n]^*$ of convex (with respect to the colours) linear orderings, where $(\mathcal{K}[n], \mathcal{K}[n]^*)$ is an excellent pair by [32].

**Question 7.2.1.** If $n \geq 2$, is it true that “$\text{Aut}(\text{Flim} (\mathcal{K}[n]))$ is amenable iff $\text{Aut}(\text{Flim}(\mathcal{K}))$ is amenable”? What if ‘amenable’ is replaced by ‘uniquely ergodic’?

We notice that by fixing one colour in $\text{Flim}(\mathcal{K}[n])$ we get a copy of $\text{Flim}(\mathcal{K})$, so nonamenability of $\text{Aut}(\text{Flim}(\mathcal{K}))$ easily pushes up to nonamenability of $\text{Aut}(\text{Flim}(\mathcal{K}[n]))$. However, the “cross-terms” of how elements of different colours relate may behave very badly, thus making it difficult to create a consistent random expansion of $\mathcal{K}[n]$ from a consistent random expansion on $\mathcal{K}$. In the case of $\mathcal{K}[\mathcal{L}]$, these cross-terms were very well-behaved (they were uniform), which allowed the proof of Theorem Part (1), $\Leftarrow$ to go through.

In particular this question is still interesting in the case that $\mathcal{K}$ is the class of finite graphs.

Another neutral question, which mimics Theorem 5.2.4 is about the Hrushovski property.

**Question 7.2.2.** For $n \geq 2$, is it true that “$\mathcal{K}[n]$ is a Hrushovski class iff $\mathcal{K}$ is a Hrushovski class”?

Since any automorphism of $\text{Flim}(\mathcal{K}[n])$ (set-wise) fixes each colour, for reasons similar to the proof of the $\Rightarrow$ direction of Theorem 5.2.4, we have that $\mathcal{K}[n]$ is a Hrushovski class implies that $\mathcal{K}$ is a Hrushovski class.

7.3 Hrushovski Property

A rich set of combinatorial problems comes from the Hrushovski property. There are many Fraïssé classes for which the Hrushovski property is unknown. The following is probably the most pressing open question in this regard:

**Question 7.3.1.** Is the class of all finite tournaments $\mathcal{T}$ a Hrushovski class?

The proof that the class of all finite graphs $\mathcal{G}$ is a Hrushovski class uses the fact that the $\mathcal{G}$ has the Free Amalgamation Property, which $\mathcal{T}$ does not have. Moreover, the known concrete examples of Hrushovski classes are all symmetric relation classes. As a result, the following question should serve as a stepping stone:

**Question 7.3.2.** Does $\mathbb{D}_\omega$ have the Hrushovski property?
A more general question is:

**Question 7.3.3.** Which of the infinite structures on Cherlin’s list have the Hrushovski property?

Note that by Proposition 5.2.2, only the amenable groups are eligible to have the Hrushovski property. The most general open question is the one posed by Hrushovski in [16]:

**Question 7.3.4.** Which finite relational structures have the Hrushovski property?

So far the following interesting examples are known to have the Hrushovski property:

**Theorem 7.3.5.** The following classes of finite structures are Hrushovski classes.

2. [13], [14] $r$-uniform hypergraphs, $K_n$-free graphs.
3. [34] Metric spaces. (See also [37] and [31].)

These results are closely related to results about group structures. For example, the fact that graphs have the Hrushovski property is related to the fact that $\mathbb{Z}$ has the following property: for any finite $F \subseteq \mathbb{Z}$, there is a normal subgroup $N \trianglelefteq \mathbb{Z}$ such that no two elements of $F$ are identified in $\mathbb{Z}/N$. For metric spaces the related group is the free group on two generators, for which this property is true, but is not obvious.
Bibliography


