Lusztig Slices in the Affine Grassmannian and Nilpotent Matrices

by

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Abstract

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The focus of this dissertation is to present some new results related to an isomorphism of Mirković-Vybornov between Lusztig slices in the affine Grassmannian and varieties of nilpotent matrices. We give an explicit alternate Mirković-Vybornov isomorphism in the $GL_n$-affine Grassmannian and an explicit isomorphism in the $GL_2$-affine Grassmannian. We also transport an additive action on the Lusztig slice through the Mirković-Vybornov isomorphism and show that it corresponds to a residual conjugation action. We go on to verify that this additive action matches a previously discovered action on the Slodowy slice.
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Chapter 1

Introduction

The motivating force behind this dissertation is the connection between lattices and nilpotent matrices. We use the word "lattice" to refer to an element of the $GL_n$-affine Grassmannian $GL_n(K)/GL_n(O)$, where $K := \mathbb{C}((z))$ and $O := \mathbb{C}[[z]]$. The use of the word lattice in this context reflects an analogy with the space $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$, whose elements are lattices in the original sense of the term.

To be precise in our case of the $GL_n$-affine Grassmannian, elements are free, rank $n$, $O$-modules inside $K^n$. If we take two lattices $L_1$ and $L_2$ in the $GL_n$-affine Grassmannian satisfying $L_1 \subseteq L_2$, then $L_2/L_1$ is a finite dimensional $\mathbb{C}$-vector space equipped with a nilpotent operator $z$. Exploring this relationship between lattices and nilpotents will be our primary goal.

New Results. There are three main results in this thesis. In chapter 4 we find an explicit description of the Mirković-Vybornov isomorphism between
Lusztig slices in the $GL_2$-affine Grassmannian and varieties of nilpotent matrices. Then we find an explicit “alternate” Mirković-Vybornov isomorphism in the case of the $GL_n$-affine Grassmannian. In chapter 5 we transport a certain $\mathbb{G}_a$-action on the Lusztig slice over to the nilpotent cone using the Mirković-Vybornov isomorphism and show it to be a residual conjugation action. Then we verify that this matches a $\mathbb{G}_a$-action on a Slodowy slice as studied in [Mor14].

Another goal of this thesis is to contribute pedagogically to the literature on the affine Grassmannian. This goal has been achieved if any researcher interested in the affine Grassmannian (whether as the richly-structured space that it is, or because of its connections to representation theory) reads this thesis in order to make a pleasant acquaintance with this very interesting mathematical object.

**Historical Context.** The affine Grassmannian has surfaced during the study of a variety of subjects by a variety of mathematicians. In the context of the representation theory of $GL_n(\mathbb{F}_q)$ over $\mathbb{C}$, Lusztig [Lus81] was led to consider a subset of the $\mathbb{F}_q$-affine Grassmannian and identified it with the variety of nilpotent matrices in $\mathfrak{gl}_n(\mathbb{F}_q)$. Lusztig later generalized this result in [Lus90a] (see also [Mag02]) and once again demonstrated the importance of using the affine Grassmannian for studying questions related to nilpotent orbits. See [KLMW06] for how Lusztig’s isomorphism is applied to study nilpotent orbit closures. However, it is the closely related work of Mirković-Vybornov in [MV07] and [MV08] which forms the bedrock of this thesis. One of the many
results of these papers is what we shall call the Mirković-Vybornov isomorphism. It is an isomorphism of varieties between Lusztig slices in the affine Grassmannian (see 3.2) and a certain affine space of nilpotent matrices (see 3.3.2).

The affine Grassmannian has also appeared in the context of geometric representation theory, most notably in the geometric Satake isomorphism of Lusztig [Lus], Ginzburg [Gin95], and Mirković-Vilonen [MV07]. Here the theories of intersection cohomology and perverse sheaves are used to relate the geometry of the affine Grassmannian $G(\mathbb{K})/G(\mathfrak{o})$ to the representation theory of the Langlands dual group $\tilde{G}(\mathbb{C})$. The geometric Satake isomorphism was itself an outgrowth of number theory, specifically the Langlands program, in which the Satake isomorphism [Sat63] was studied as an unramified local Langlands correspondence (see also [Gro98]).

The affine Grassmannian is also an object of rich combinatorics in its own right. The study of Mirković-Vilonen cycles and polytopes, which are objects that arise from the affine Grassmannian, lead to an understanding of the combinatorics of bases for the highest weight representations of reductive groups over $\mathbb{C}$, see [Kam10] (and also [Kam09] and [AK04]).

The affine Grassmannian also has close connections to the subject of loop groups, which have been studied by many mathematicians from a variety of perspectives (see [PS86] and [Fre07]).
Below you can find more detailed summaries of our goals in each chapter.

**Chapter 2** This chapter serves as an introduction to affine Grassmannians $\text{Gr}_G$ when the group $G$ is equal to $GL_n$, $SL_n$ or $PGL_n$, as well as the interpretation of $\text{Gr}_G$ as a space of *lattices*. This chapter also examines the $G(\mathcal{O})$-orbits on the affine Grassmannian and how this relates to the Jordan type of a natural nilpotent operator associated to a lattice. The important relationship of lattice containment is also examined in detail.

**Chapter 3** The aim of this chapter is to give the definition of a Lusztig slice and to state the Mirković-Vybornov isomorphism. Roughly speaking, the Mirković-Vybornov isomorphism is a one-to-one correspondence between the Lusztig slice and a special variety of nilpotent matrices.

**Chapter 4** Two of the three main results of this dissertation are in this chapter. First is an explicit realization of the Mirković-Vybornov isomorphism in the case of $GL_2$. The Lusztig slice is interpreted as matrices in $GL_2(\mathcal{K})$ with a condition on their determinant. Then we explicitly calculate the action of the nilpotent operator $z$ on a finite dimensional vector space. This yields an explicit Mirković-Vybornov isomorphism. Secondly we have an explicit realization of an “alternate” Mirković-Vybornov isomorphism for $GL_n$ in the case when lattices $L$ are contained in the standard lattice. This amounts to representing the Lusztig slice as certain matrices in $GL_n(\mathcal{K})$ with a condition on their determinant, and then working out the action of the nilpotent operator $z$ on a finite dimensional vector space. Many examples are given of both of
Chapter 1. Introduction

our explicit maps in this chapter.

Chapter 5 The third main result of this dissertation is in this chapter. It concerns a $\mathbb{G}_a$-action on the Lusztig slice that was discovered by Kamnitzer, that is conjectured to give a Hamiltonian reduction between “neighboring” Lusztig slices. We determine the form this action takes on the Mirković-Vybornov slice intersect the nilpotent cone by transporting the $\mathbb{G}_a$-action through the Mirković-Vybornov isomorphism. Then we show that this action coincides with the action on the Slodowy slice corresponding to a certain pyramid that is found in Morgan’s thesis [Mor14]. Our result of this chapter says that the actions of $\mathbb{G}_a$ on the Lusztig slice and on the Slodowy slice are identical.

Chapter 6 In this final chapter, we highlight some questions that come out of various parts of this thesis and the future research directions that they point.
Chapter 2

Type A Affine Grassmannians

In this chapter we focus on three reductive groups over $\mathbb{C}$: $GL_n$, $SL_n$, and $PGL_n$. We will write $G$ when we do not want to specify a particular one of the three. We denote the $\mathbb{C}$-algebras $\mathbb{C}((z))$ and $\mathbb{C}[[z]]$ by $\mathcal{K}$ and $\mathcal{O}$, respectively. We take $\mathbb{C}((z))$ to be the field formal Laurent series in one variable, and $\mathbb{C}[[z]]$ the ring of formal power series in one variable.

Throughout this chapter, the symbol $V$ will stand for an $n$-dimensional vector space over $\mathcal{K}$.

2.1 Coset Definition

Taking the points of $G$ in the $\mathbb{C}$-algebras $\mathcal{K}$ and $\mathcal{O}$, we obtain the group $G(\mathcal{K})$ with the subgroup $G(\mathcal{O})$. We can consider $G(\mathcal{O})$ right-acting on $G(\mathcal{K})$.

Definition 2.1.1 (Affine Grassmannian). The coset space $G(\mathcal{K})/G(\mathcal{O})$ is called the affine Grassmannian of $G$ and will be denoted by $\text{Gr}_G$. 

6
2.2 Lattices and $\text{Gr}_{GL_n}$

In this section we will discuss the relationship between lattices and the $GL_n$-affine Grassmannian.

**Definition 2.2.1** (Lattices in $V$). A lattice in $V$ is a free $\mathcal{O}$-submodule $L$ in $V$ satisfying $L \otimes_\mathcal{O} \mathcal{K} = V$.

A lattice $L$ may be thought of as a free $\mathcal{O}$-submodule of $V$ that has $\mathcal{O}$-module rank equal to the dimension of $V$ as a $\mathcal{K}$-vector space. Any lattice in $V$ can therefore be expressed as $\text{span}_\mathcal{O}(v_1, ..., v_n)$ where $\{v_1, ..., v_n\}$ is a $\mathcal{K}$-basis for $V$. The collection of all $\mathcal{K}$-bases of $V$ is a $GL_n(\mathcal{K})$-torsor and the subgroup of $GL_n(\mathcal{K})$ fixing a particular lattice is conjugate to $GL_n(\mathcal{O})$. Therefore the coset space $\text{Gr}_{GL_n} = GL_n(\mathcal{K})/GL_n(\mathcal{O})$ gives a non-canonical realization of the collection of all lattices in $V$. It is non-canonical because there is an arbitrary choice of a “base lattice” represented by the identity coset in $GL_n(\mathcal{K})/GL_n(\mathcal{O})$.

If we fix a $\mathcal{K}$-basis of $V$, say $\{e_1, ..., e_n\}$, which identifies $V$ with $\mathcal{K}^n$, then the collection of lattices in $V$ is unambiguously identified with $GL_n(\mathcal{K})/GL_n(\mathcal{O})$, where the identity coset corresponds to the “standard lattice” $\text{span}_\mathcal{O}(e_1, ..., e_n)$. However the choice of the basis $\{e_1, ..., e_n\}$ was arbitrary.

**Definition 2.2.2** (The standard lattice). Fix a $\mathcal{K}$-basis of $V$, say $\{e_1, ..., e_n\}$. We call the lattice $\text{span}_\mathcal{O}(e_1, ..., e_n)$ the standard lattice and denote it by $L_0$.

2.3 Unimodular Lattices and $\text{Gr}_{SL_n}$

In this section we will discuss the relationship between unimodular lattices
and the $SL_n$-affine Grassmannian.

If we make a choice of $\mathfrak{K}$-basis for $V$, say $\{e_1, ..., e_n\}$, then the one-dimensional $\mathfrak{K}$-vector space $\bigwedge^n V$ develops a distinguished basis, namely $\{e_1 \wedge ... \wedge e_n\}$. Then the collection of $\mathfrak{K}$-bases $\{v_1, ..., v_n\}$ of $V$ with the property that $v_1 \wedge ... \wedge v_n = e_1 \wedge ... \wedge e_n$ is a torsor for the group $SL_n(\mathfrak{K})$.

**Definition 2.3.1 (Unimodular lattices in $V$).** A *unimodular lattice* in $V$ is a lattice $L$ in $V$ having a basis $\{v_1, ..., v_n\}$ with the property $v_1 \wedge ... \wedge v_n = e_1 \wedge ... \wedge e_n$.

A unimodular lattice is a lattice $L$ in $V$ that can be represented by a matrix in $SL_n(\mathfrak{K})$. The collection of all unimodular lattices in $V$ can be identified with the coset space $Gr_{SL_n} = SL_n(\mathfrak{K})/SL_n(\mathfrak{O})$.

### 2.4 Lattice Homothety Classes and $Gr_{PGL_n}$

In this section we will discuss the relationship between lattices up to $\mathfrak{K}^*$-scaling and the $PGL_n$-affine Grassmannian.

The set of $\mathfrak{K}$-bases of $V$ up to $\mathfrak{K}^*$-scaling is a torsor for $PGL_n(\mathfrak{K})$. The group fixing a given $\mathfrak{K}^*$-scale-class of a lattice is conjugate to $GL_n(\mathfrak{O})/\mathfrak{O}^* = PGL_n(\mathfrak{O})$. Therefore the collection of lattices in $V$ up to $\mathfrak{K}^*$-scaling can be identified with the coset space $Gr_{PGL_n} = PGL_n(\mathfrak{K})/PGL_n(\mathfrak{O})$.

**Definition 2.4.1 (Homothety classes of lattices in $V$).** The collection of lattices equivalent to a lattice $L$ by $\mathfrak{K}^*$-scaling is called its *homothety class*. 

The collection of lattices equivalent to $L$ by $\mathcal{K}^*$-scaling can be identified with $\mathcal{K}^*/\mathcal{O}^*$ which can then be identified with $\{z^aL : a \in \mathbb{Z}\}$. Therefore an equivalent definition of the homothety class of $L$ is the collection of lattices of the form $z^aL$ for $a \in \mathbb{Z}$.

### 2.5 $GL_n(\mathcal{O})$-Orbits on $\text{Gr}_{GL_n}$

Consider the group $GL_n(\mathcal{O})$ left-acting on the affine Grassmannian $\text{Gr}_{GL_n}$. In this section, we will discuss the two equivalent statements 2.5.1 and 2.5.2.

**Proposition 2.5.1.** The collection of $GL_n(\mathcal{O})$-orbits on $\text{Gr}_{GL_n}$ is in bijection with $X_+(GL_n)$, the set of dominant coweights of $GL_n$. Specifically, our orbit decomposition is

$$\text{Gr}_{GL_n} = \bigsqcup_{\lambda \in X_+} GL_n(\mathcal{O})L_{\lambda}$$

where $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$ is a dominant coweight of $GL_n$, and $L_{\lambda} = [z^\lambda]$ is the lattice represented by the matrix

$$z^\lambda = \begin{bmatrix} z^{\lambda_1} & & \\ & \ddots & \\ & & z^{\lambda_n} \end{bmatrix} \in GL_n(\mathcal{K}).$$

This statement admits the following equivalent form:

**Proposition 2.5.2.** Every matrix in $GL_n(\mathcal{K})$ can be reduced by $GL_n(\mathcal{O})$-row and $GL_n(\mathcal{O})$-column operations to a matrix of the form
Chapter 2. Type A Affine Grassmannians

\[
z^\lambda = \begin{bmatrix}
z^{\lambda_1} \\
\vdots \\
z^{\lambda_n}
\end{bmatrix} \in GL_n(\mathcal{K}),
\]

where \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n\) and the entries of \(\lambda\) are ordered.

\[\square\]

Clarification. Notice that in 2.5.2 we say that the entries of \(\lambda\) are ordered, and in 2.5.1 we say that \(\lambda\) is a dominant coweight of \(GL_n\). In 2.5.1 we are only asserting that dominant coweights are in bijection with the set of \(GL_n(\mathcal{O})\)-orbits on \(\text{Gr}_{GL_n}\). This is important because at various points of this thesis we will sometimes prefer to represent a \(GL_n(\mathcal{O})\)-orbit using \(L_\lambda\) with \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\) and sometimes using \(L_\lambda\) with \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\). We will be very explicit about which choice is being made in each instance.

One way to understand the \(GL_n(\mathcal{O})\)-orbits on \(\text{Gr}_{GL_n}\) is to ask: what are the possible “echelon forms” of matrices in \(GL_n(\mathcal{K})\) up to \(GL_n(\mathcal{O})\)-row and \(GL_n(\mathcal{O})\)-column operations?

Given a matrix in \(GL_n(\mathcal{K})\), we can determine the coset in

\[GL_n(\mathcal{O}) \backslash GL_n(\mathcal{K}) / GL_n(\mathcal{O})\]

to which it belongs. We begin by looking for the entry with a lowest power of \(z\), factoring out the lowest power of \(z\), and then zeroing everything in the corresponding row and column using matrices in \(GL_n(\mathcal{O})\). Repeating this process reduces the original matrix to one of the form \(\sigma z^{\lambda'} \tau\), where \(\lambda' \in \mathbb{Z}^n\), and \(\sigma\) and \(\tau\) are permutation matrices. Left and right-multiplication by appropriate
permutation matrices will reduce to a matrix $z^\lambda$ where the entries of $\lambda$ are ordered.

**Example 2.5.3.** We will work with the matrix

$$g = \begin{bmatrix} z^2 & 1 \\ z^{-3} & z \end{bmatrix} \in GL_2(K).$$

We perform $GL_2(\mathcal{O})$-row and $GL_2(\mathcal{O})$-column operations:

$$\begin{bmatrix} z^2 & 1 \\ z^{-3} & z \end{bmatrix} \xrightarrow{c_2 \to z^4 c_1 + c_2} \begin{bmatrix} z^2 & -z^6 + 1 \\ z^{-3} & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - z^5 R_2} \begin{bmatrix} 0 & -z^6 + 1 \\ z^{-3} & 0 \end{bmatrix} \xrightarrow{R_1 \to (1-z^6)^{-1} R_1} \begin{bmatrix} 0 & 1 \\ z^{-3} & 0 \end{bmatrix},$$

where the last operation was to multiply row one by the power series reciprocal of $1 - z^6$. Swapping rows results in the matrix:

$$z^{(-3,0)} = \begin{bmatrix} z^{-3} & 0 \\ 0 & z^0 \end{bmatrix}.$$  

We can either say that the matrix $z^{(-3,0)}$ represents the double coset of $g \in GL_2(K)$, or we can say that the lattice $L_{(-3,0)} = [z^{(-3,0)}]$ represents the $GL_2(\mathcal{O})$-orbit of the lattice $L = [g]$. \hfill \square

### 2.6 The Containment Relation

Throughout this thesis we will use $\mathbb{Z}_+$ to denote the positive integers including zero and $\mathbb{Z}_-$ to denote the negative integers including zero.
Throughout this thesis we will use $\Lambda$ to denote the set $\mathbb{Z}^n$, $\Lambda^+$ to denote the set of $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, and $\Lambda^-$ to denote the set of $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \leq \ldots \leq \lambda_n \leq 0$.

We know by 2.5.2 that $GL_n(\mathcal{O})$-orbits on $Gr_{GL_n}$ are of the form $GL_n(\mathcal{O})L_\lambda$, for some $\lambda \in \Lambda/S_n$.

If $\lambda \in \Lambda^-$ then $L_\lambda$ contains the standard lattice $L_0$, and in fact any lattice $L \in GL_n(\mathcal{O})L_\lambda$ also contains $L_0$. This and many other statements about lattice containment are corollaries of the following statement.

**Proposition 2.6.1.** Two lattices $L_a$ and $L_b$ satisfy the containment relation $L_a \subseteq L_b$ precisely when matrices $a$ and $b$ in $GL_n(\mathfrak{K})$ representing them satisfy $b^{-1}a \in \mathfrak{gl}_n(\mathcal{O})$.

**Proof.** Suppose $L_a = \text{span}_\mathcal{O}(v_1, \ldots, v_n)$ and $L_b = \text{span}_\mathcal{O}(w_1, \ldots, w_n)$ where $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are bases for $\mathfrak{K}^n$. Note that $a = [v_1 \ldots v_n]$ and $b = [w_1 \ldots w_n]$ are in $GL_n(\mathfrak{K})$. Notice that $L_a \subseteq L_b$ precisely when there is an $n \times n$ matrix $F = (f_{ij})$ defined over $\mathcal{O}$ satisfying

\[
\begin{align*}
v_1 &= f_{11}w_1 + f_{12}w_2 + \ldots + f_{1n}w_n \\
\vdots \\
v_n &= f_{n1}w_1 + f_{n2}w_2 + \ldots + f_{nn}w_n.
\end{align*}
\]

This can be expressed as $a = bF^t$. Therefore we see that $L_a \subseteq L_b$ precisely when $b^{-1}a \in \mathfrak{gl}_n(\mathcal{O})$. Of course this condition is well-defined; if $\tilde{a}, \tilde{b} \in GL_n(\mathfrak{K})$ also represent $L_a$ and $L_b$ respectively, then $\tilde{a} = ag_1$ and $\tilde{b} = bg_2$, where $g_1, g_2 \in GL_n(\mathcal{O})$, and we can see that the condition $(\tilde{b})^{-1}\tilde{a} \in \mathfrak{gl}_n(\mathcal{O})$ is preserved. \qed
It is important to note that a pair of arbitrary lattices do not necessarily have any containment relations. For example, if $G = GL_2$, neither of the lattices $L_{(-1,1)}$ and $L_0$ contains the other.

**Corollary 2.6.2.** A lattice $L$ satisfies $L_0 \subseteq L$ precisely when any matrix $g \in GL_n(\mathcal{K})$ representing $L$ satisfies $g^{-1} \in \mathfrak{gl}_n(\emptyset)$.

**Corollary 2.6.3.** A lattice $L$ satisfies $L \subseteq L_0$ precisely when any matrix $g \in GL_n(\mathcal{K})$ representing $L$ satisfies $g \in \mathfrak{gl}_n(\emptyset)$.

**Corollary 2.6.4.** Let $g$ be a matrix in $GL_n(\mathcal{K})$. Two lattices $L_a$ and $L_b$ satisfy $L_a \subseteq L_b$ precisely when $gL_a \subseteq gL_b$.

**Corollary 2.6.5.** A lattice $L$ satisfies $L_0 \subseteq L$ precisely when it belongs to an orbit $GL_n(\emptyset)L_\lambda$ with $L_0 \subseteq L_\lambda$. Equivalently, $L_0 \subseteq L$ precisely when $\lambda \in \Lambda^-$.  

**Corollary 2.6.6.** A lattice $L$ satisfies $L \subseteq L_0$ precisely when it belongs to an orbit $GL_n(\emptyset)L_\lambda$ with $L_\lambda \subseteq L_0$. Equivalently, $L \subseteq L_0$ precisely when $\lambda \in \Lambda^+$.

**Corollary 2.6.7.** Let $L$ be an arbitrary lattice. There is an integer $d$ such that $L \subseteq z^dL_0$.

**Proof.** Let $g \in GL_n(\mathcal{K})$ be a matrix representing $L$. Since the entries of $g$ are in $\mathcal{K}$, there is an integer $d$ so that $z^{-d}g \in \mathfrak{gl}_n(\emptyset)$, and by 2.6.3 this means that $z^{-d}L \subseteq L_0$. Then 2.6.4 implies that $L \subseteq z^dL_0$.  

**Corollary 2.6.8.** Let $L_a$ and $L_b$ be two lattices. There is an integer $d$ such that $L_a \subseteq z^dL_b$. 

Proof. Suppose that $L_a$ and $L_b$ are represented by $a$ and $b$ respectively, with $a, b \in GL_n(K)$. Using 2.6.7 on the lattice represented by $b^{-1}a$ we get some integer $d$ such that $b^{-1}aL_0 \subseteq z^dL_0$. By 2.6.4 we then multiply through by $b$ to find $aL_0 \subseteq z^dbL_0$, which means that $L_a \subseteq z^dL_b$.

Corollary 2.6.9. Let $L$ be an arbitrary lattice. There are integers $d$ and $e$ such that $z^eL_0 \subseteq L \subseteq z^dL_0$.

Proof. We may write $L = gL_\lambda$ for some $g \in GL_n(\mathcal{O})$ and some $\lambda \in \Lambda$. By taking the integer $e$ to be the maximum entry of $\lambda$ and $d$ to be the minimum entry of $\lambda$ we have $z^eL_0 \subseteq L_\lambda \subseteq z^dL_0$. By 2.6.4 and the fact that $gL_0 = L_0$, we may multiply through by $g$ to obtain $z^eL_0 \subseteq L \subseteq z^dL_0$.

Corollary 2.6.10. Let $L_a$ and $L_b$ be two lattices. There are integers $d$ and $e$ such that $z^eL_a \subseteq L_b \subseteq z^dL_a$.

2.7 $G(\mathcal{O})$-Orbits on $Gr_G$

Remember that we are using $G$ to denote the reductive groups $GL_n$, $SL_n$ and $PGL_n$ over $\mathbb{C}$. We have already determined that $Gr_{GL_n}$ represents the collection of $\mathcal{O}$-lattices in an $n$-dimensional $K$-vector space, and that every $GL_n(\mathcal{O})$-orbit is of the form $\{gL_\lambda : g \in GL_n(\mathcal{O})\}$ where $L_\lambda$ is the lattice represented by a matrix $z^\lambda \in GL_n(K)$ with $\lambda \in \Lambda/S_n$. In this section, we will work out the $G(\mathcal{O})$-orbits on $Gr_G$, showing that the dominant coweights $X_\lambda(G)$ are an appropriate index set for the orbits.
Proposition 2.7.1. The left $G(\emptyset)$-orbits on $Gr_G$ can be described as follows:

$$Gr_G = \bigsqcup_{\lambda \in X_+(G)} G(\emptyset)[z^\lambda]$$

where $[z^\lambda]$ represents the coset of $z^\lambda$ in $Gr_G$.

Proof. ($G = SL_n$) Take an arbitrary $L \in Gr_{SL_n}$. The lattice $L$ is represented by a matrix $g \in SL_n(K)$. We apply the algorithm of 2.5.2 to $g$, except we perform $GL_n(\emptyset)$-row and $GL_n(\emptyset)$-column operations having determinant one, i.e. $SL_n(\emptyset)$-row and $SL_n(\emptyset)$-column operations. Using these operations, the matrix $g$ will reduce to one of the form:

$$uz^\lambda = \begin{bmatrix} u_1 z_1^\lambda \\ \vdots \\ u_n z_n^\lambda \end{bmatrix} \in SL_n(K),$$

where the entries of $\lambda$ are in order, and $u_1, ..., u_n$ are in $\emptyset^\ast$. The lattice represented by $uz^\lambda$ is the same as the lattice represented by the matrix $z^\lambda$. Therefore, $[z^\lambda] \in Gr_{SL_n}$ is a unimodular lattice that represents the $SL_n(\emptyset)$-orbit of $L \in Gr_{SL_n}$. Hence $z^\lambda \in SL_n(K)$, which implies that $\sum \lambda = 0$. Therefore, it is equivalent to describe $\lambda$ as a dominant coweight of $SL_n$.

($G = PGL_n$) Take an arbitrary lattice homothety class, i.e. an arbitrary point $[L] \in Gr_{PGL_n}$. By 2.5.1 the lattice $L$ can be written as $L = gL_\lambda$ for some $\lambda \in X_+(GL_n)$ and $g \in GL_n(\emptyset)$. We therefore can write the following equations on homothety classes:

$$[L] = [gL_\lambda] = g[L_\lambda] = [g][L_\lambda]$$

where the last equality holds because the action of $g \in GL_n(\emptyset)$ on $[L_\lambda]$ will depend only on its class $[g] \in PGL_n(\emptyset)$. Therefore $[L_\lambda]$ represents the left
Chapter 2. Type A Affine Grassmannians

$PGL_n(\mathcal{O})$-orbit of $[L] \in \text{Gr}_{PGL_n}$. The homothety class of the lattice $L_\lambda$ satisfies the relation $[L_\lambda] = [z^{\pm 1}L_\lambda] = [z^{\pm 1}z^\lambda] = [z^{\lambda z(1,\ldots,1)}]$, and therefore $\lambda$ can be regarded as an element of $\mathbb{Z}^n/(1,\ldots,1)$. Therefore, it is equivalent to describe $\lambda$ as a dominant coweight of $PGL_n$.

2.8 Closures of $G(\mathcal{O})$-Orbits

We have that left $G(\mathcal{O})$-orbits on $\text{Gr}_G$ are of the form $G(\mathcal{O})[z^\lambda]$ with $\lambda \in X_+(G)$, $z^\lambda$ the matrix in $G(\mathcal{K})$ from 2.5.1, and $[z^\lambda]$ the point in $\text{Gr}_G$. We will be using the notation $\text{Gr}_{\mathcal{O}}^\lambda := G(\mathcal{O})[z^\lambda]$ to denote the orbit of $[z^\lambda]$. The closure of the orbit will be denoted by $\text{Gr}_{\mathcal{O}}^\lambda$. We have:

$$\text{Gr}_{\mathcal{O}}^\lambda := \overline{\text{Gr}_{\mathcal{O}}^\lambda} = \bigsqcup_{\nu \subseteq \lambda} \text{Gr}_{\mathcal{O}}^\nu.$$  

The closure of an orbit $\text{Gr}_{\mathcal{O}}^\lambda$ consists of all orbits $\text{Gr}_{\mathcal{O}}^\nu$ where $\nu$ is a dominant coweight with $\lambda - \nu$ equal to a sum of positive coroots.

2.9 Valuations

Definition 2.9.1 (Valuation). The valuation of a element of $\mathcal{K}^*$ is the integer representing its class in $\mathcal{K}^*/\mathcal{O}^* \cong \mathbb{Z}$. Equivalently, the valuation of an element $f \in \mathcal{K}^*$ is the integer $a$ that appears when we write $f$ in the form $f = z^au$, with $u \in \mathcal{O}^*$.

We will let $\text{val}(f)$ denote the valuation of $f \in \mathcal{K}$, where we adopt the con-
vention that \( \text{val}(0) = -\infty \).

Let \( L \in \text{Gr}_{GL_n} \) be a lattice, and suppose that \( L \) is represented by a matrix \( g \in GL_n(\mathcal{K}) \). The valuation of the determinant of \( g \), denoted \( \text{valdet}(g) \in \mathbb{Z} \), is a well-defined property of the lattice \( L \). In fact, it is a well-defined property of the entire \( GL_n(\mathcal{O}) \)-orbit of \( L \). This is expressed in the following proposition.

**Proposition 2.9.2.** The function \( \text{valdet} : GL_n(\mathcal{K}) \rightarrow \mathbb{Z} \) is constant on the orbits \( GL_n(\mathcal{O})z^\lambda GL_n(\mathcal{O}) \).

**Proof.** Let \( g, h \in GL_n(\mathcal{O}) \) and consider \( \text{valdet}(gz^\lambda h) \).

\[
\text{valdet}(gz^\lambda h) = \text{val}(\det(g) \det(z^\lambda) \det(h)) \\
= \text{valdet}(g) + \text{valdet}(z^\lambda) + \text{valdet}(h) \\
= 0 + \sum \lambda + 0 \\
= \sum \lambda \\
= \text{valdet}(z^\lambda).
\]

\[\square\]

## 2.10 Dimension

The following proposition about lattices is an important one to notice.

**Proposition 2.10.1.** Let \( L_a \) and \( L_b \) be two lattices in \( V \). If \( L_a \subseteq L_b \), then \( L_b/L_a \) is a finite dimensional \( \mathbb{C} \)-vector space that is equipped with a nilpotent operator \( z \circ L_b/L_a \).
Proof. By 2.6.10, there are integers $e$ and $d$ such that $z^e L_a \subseteq L_b \subseteq z^d L_a$. Since $L_a \subseteq L_b$ we can take $e = 1$ and $d$ must be a negative integer. Therefore we have that $L_a \subseteq L_b \subseteq z^d L_a$, and hence $L_b/L_a \subseteq z^d L_a/L_a$. This proves that $L_b/L_a$ is finite dimensional vector space over $\mathbb{C}$, and in addition it shows that multiplication by $z$ is a nilpotent operator $z \subseteq L_b/L_a$.

If $L_a$ and $L_b$ are arbitrary lattices (not necessarily having a containment relation) then the lattice $L_a \cap L_b$ is contained in each of them, and we can define the relative dimension of $L_a$ and $L_b$ to be:

$$\dim(L_a, L_b) := \dim(L_b/L_a \cap L_a) - \dim(L_a/L_a \cap L_b) \in \mathbb{Z}.$$ 

Represent a lattice $L \in \text{Gr}_{GL_n}$ with a matrix $g \in GL_n(\mathbb{K})$. Suppose that $L$ belongs to the $GL_n(\mathbb{O})$-orbit $\text{Gr}_\lambda^\lambda$ for $\lambda \in X_+(GL_n)$. Notice that:

$$\text{valdet}(g) = \text{valdet}(z^\lambda) = \sum \lambda = \dim(L, L_0) \in \mathbb{Z}.$$
Chapter 3

Lusztig Slices and Nilpotent Matrices

In this chapter we will study Lusztig slices in the case of $G = GL_n$. Lusztig slices are intriguing finite-dimensional subvarieties of the affine Grassmannian. Via the Mirković-Vybornov isomorphism, they correspond to certain varieties of nilpotent matrices.

3.1 The Subset $G_1[z^{-1}] \subseteq GL_n(\mathcal{K})$

The definition of a Lusztig slice first requires an understanding of the subset $G_1[z^{-1}] \subseteq GL_n(\mathcal{K})$. Before we define this subset, let us develop another interpretation of the group $GL_n(\mathcal{O})$.

Proposition 3.1.1. The group $GL_n(\mathcal{O})$ can be interpreted as the set of formal
power series with matrix coefficients having invertible constant term:

\[ GL_n(\emptyset) = \left\{ A_0 + A_1 z + A_2 z^2 + \ldots \mid A_i \in \mathfrak{gl}_n(\mathbb{C}), A_0 \in GL_n(\mathbb{C}) \right\} \]

**Proof.** The group \( GL_n(\emptyset) \) is the set of matrices with \( \emptyset \)-coefficients having invertible determinant, i.e. the determinant is a power series with nonzero constant term. Therefore if \( g \in GL_n(\emptyset) \), we have that \( \det(g) \in \emptyset^* \) and so \( \det(g) \) evaluated at \( z = 0 \) is not equal to zero. Since the determinant is polynomial function in the entries of \( g \) we have that \( 0 \neq \det(g)(0) = \det(g(0)) = \det(A_0) \).

In summary, a power series with matrix coefficients \( A_0 + A_1 z + A_2 z^2 + \ldots \) belongs to \( GL_n(\emptyset) \) exactly when \( A_0 \) belongs to \( GL_n(\mathbb{C}) \).

Now consider the subset \( GL_n(\mathbb{C}[z^{-1}]) \subseteq GL_n(K) \) consisting of all \( \mathbb{C}[z^{-1}] \)-points of \( GL_n \). In other words \( GL_n(\mathbb{C}[z^{-1}]) \) is the set of matrices over the \( \mathbb{C} \)-algebra \( \mathbb{C}[z^{-1}] \) having invertible determinant, i.e. the determinant is a nonzero scalar.

We can now define the subset \( G_1[z^{-1}] \subseteq GL_n(K) \).

**Definition 3.1.2.** We define \( G_1[z^{-1}] \) to be the subset of \( GL_n(\mathbb{C}[z^{-1}]) \) consisting of the elements that map to the identity under the evaluation map \( \mathfrak{gl}_n(\mathbb{C}[z^{-1}]) \to \mathfrak{gl}_n(\mathbb{C}) \) induced by \( z^{-1} = 0 \).

There is another perspective from which to look at the subset \( G_1[z^{-1}] \).

**Proposition 3.1.3.** The subset \( G_1[z^{-1}] \subseteq GL_n(K) \) can be interpreted as the set of polynomials in the variable \( z^{-1} \) with matrix coefficients, having identity
constant term and nonzero scalar determinant:

\[ G_1[z^{-1}] = \{ g = I + A_{-1}z^{-1} + \ldots + A_{-k}z^{-k} \mid k \in \mathbb{N}, A_i \in \mathfrak{gl}_n(\mathbb{C}), \det(g) \in \mathbb{C}^* \} \]

### 3.2 The Lusztig Slices $\text{Gr}_{\mu}^\lambda$

We will focus on Lusztig slices arising when $\lambda - \mu$ is a sum of positive coroots, which implies that $\sum \lambda = \sum \mu$, and we will use $d$ to denote the integer $d := \sum \lambda = \sum \mu$. We will also assume that either $\lambda, \mu \in \Lambda^-$ or $\lambda, \mu \in \Lambda^+$.

**Definition 3.2.1** (Lusztig slice). For arbitrary coweights $\lambda$ and $\mu$, we define the *Lusztig slice* $\text{Gr}_{\mu}^\lambda$ as follows:

\[
\text{Gr}_{\mu}^\lambda := G_1[z^{-1}]L_\mu \cap \text{Gr}^\lambda.
\]

Lusztig slices $\text{Gr}_{\mu}^\lambda$ are created by intersecting the space of lattices of the form $G_1[z^{-1}]L_\mu$ with an orbit closure $\text{Gr}^\lambda$. Lusztig slices correspond with certain subvarieties of the nilpotent cone $\mathcal{N} \subseteq \mathfrak{gl}_{|d|}(\mathbb{C})$. When $d \in -\mathbb{N}$ the correspondence goes through the Mirković-Vybornov isomorphism, and when $d \in \mathbb{N}$ the correspondence goes through our alternate Mirkovic-Vybornov isomorphism (see section 4.4).

### 3.3 The Mirković-Vybornov Isomorphism

In this section we will work in the case of $\lambda, \mu \in \Lambda^-$. Let $d := \sum \lambda = \sum \mu$. 

As we have already mentioned, the Mirković-Vybornov isomorphism is about the relationship between the Lusztig slices $Gr^\lambda$ and certain subvarieties of nilpotent matrices in $\mathfrak{gl}_n(\mathbb{C})$. Consider the (somewhat) natural basis of $L_\mu/L_0$:

$$A = \{ [z^{-1}e_1], ..., [z^{\mu_1}e_1], [z^{-1}e_2], ..., [z^{\mu_2}e_2], ..., [z^{-1}e_n], ..., [z^{\mu_n}e_n] \}.$$ 

The idea begins with noticing that lattices $L$ inside $Gr^\lambda_\mu$ satisfy $L_0 \subseteq L$ and also admit a distinguished vector space isomorphism:

$$\begin{array}{ccc}
L/L_0 & \overset{\pi_L}{\sim} & L_\mu/L_0 \ \\
\pi_L^{-1} & \overset{\sim}{\longleftarrow} & 
\end{array}$$

One way to define $\pi_L$ is to first consider the map of vector spaces:

$$\mathcal{K}^n/L_0 \xrightarrow{\pi} L_\mu/L_0$$

$$\pi[z^i e_j] = \begin{cases} 
[z^i e_j] & \mu_j \leq i < 0 \\
0 & \text{otherwise} 
\end{cases},$$

then define $\pi_L$ to be the restriction of $\pi$ to the subspace $L/L_0 \subset \mathcal{K}^n/L_0$.

Alternatively, we can view $\pi_L^{-1}$ as defined by the condition (for $[z^i e_s] \in A$):
\[
\pi^{-1}_L[z^r e_s] = \left[ z^r e_s + \sum_{1 \leq j \leq n} \left( z^i e_j \right) \right]
\]

where the asterisks * are a system of uniquely determined coefficients.

Here is an example.

**Example 3.3.1.** Let \( G = GL_2 \), \( \lambda = (-3, 0) \), and \( \mu = (-2, -1) \). Consider the lattice \( L \in Gr^\lambda_\mu \) represented by the matrix

\[
\begin{bmatrix}
    z^{-2} + z^{-3} & z^{-3} \\
    z^{-3} & z^{-1} - z^{-2} + z^{-3}
\end{bmatrix}.
\]

In this case the basis \( A \) of \( L/L_0 \) is given by \( A = \{ [z^{-1}e_1], [z^{-2}e_1], [z^{-1}e_2] \} \). We will now describe the isomorphism \( L/L_0 \cong L_/L_0 \). The vector \( \pi^{-1}_L[z^{-1}e_2] \in L/L_0 \) can be observed directly:

\[
\pi^{-1}_L[z^{-1}e_2] = [z^{-1}e_2 - z^{-2}e_2 + z^{-3}e_2 + z^{-3}e_1].
\]

The vector \( \pi_L[z^{-2}e_1] \) can also be observed directly:

\[
\pi^{-1}_L[z^{-2}e_1] = [z^{-2}e_1 + z^{-3}e_1 + z^{-3}e_2].
\]

To find the vector \( \pi^{-1}_L[z^{-1}e_1] \) we need to do the calculation \( \pi^{-1}_L[z^{-1}e_1] = z (\pi^{-1}_L[z^{-2}e_1]) - \pi^{-1}_L[z^{-2}e_1] \), yielding:

\[
\pi^{-1}_L[z^{-1}e_1] = [z^{-1}e_1 - z^{-3}e_1 + z^{-2}e_2 - z^{-3}e_2].
\]
We may represent the operator $\pi_L \pi_L^{-1} \subseteq \mathcal{L}_\mu / \mathcal{L}_0$ as a nilpotent matrix by using the basis $A$ for the vector space $\mathcal{L}_\mu / \mathcal{L}_0$. It turns out that this process maps lattices $L \in \text{Gr}_\mu^\lambda$ to nilpotent matrices in an affine subspace $M_\mu \subseteq \mathfrak{gl}_{|d|}(\mathbb{C})$ called the Mirković-Vybornov slice [MV08, 3.3]. In [MV08, 1.2] Mirković and Vybornov show that the nilpotent matrices inside $M_\mu \cap \overline{\mathcal{O}}_\lambda$ (where $\overline{\mathcal{O}}_\lambda$ denotes the closure of the nilpotent orbit $\mathcal{O}_\lambda$ of type $\lambda$) are in a one-to-one correspondence with the lattices in the Lusztig slice $\text{Gr}_\mu^\lambda$.

**Definition 3.3.2 (The Mirković-Vybornov slice).** Let $\mu \in \Lambda^-$ and $d := \sum \mu$. Define the Mirković-Vybornov slice to be the affine subspace $M_\mu \subseteq \mathfrak{gl}_{|d|}(\mathbb{C})$ consisting of $[|\mu_i| \times |\mu_j|]$ block matrices $(M_{ij})$ where: if $i = j$ then $M_{ij}$ has ones on the super-diagonal and entries along the entire bottom row; if $i > j$ then $M_{ij}$ has $|\mu_i|$ left-justified entries along the bottom row; and if $i < j$ then $M_{ij}$ has entries along the bottom row.

**Example 3.3.3.**

$$M_{(-2,-1)} = \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \\ * & 0 & * \end{bmatrix} \quad M_{(-2,-2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 0 & 1 \\ * & * & * & * \end{bmatrix}$$

The Mirković-Vybornov slice $M_\mu$ is a particular example of an affine space that is transverse to the conjugacy class of nilpotent matrices $\mathcal{O}_\mu \subseteq \mathfrak{gl}_{|d|}(\mathbb{C})$ of type $\mu$. The significance of $M_\mu$ is that it is a natural receptacle for the nilpotent matrices that come from lattices in the Lusztig slice.
The following is a recollection of [MV08, 1.2 \sim \psi] which we have been referring to as the Mirković-Vybornov isomorphism.

**Theorem 3.3.4** (Mirković-Vybornov isomorphism). Let $G = GL_n$, $\lambda, \mu \in \Lambda^-$, and $d := \Sigma \lambda = \Sigma \mu$. There is an isomorphism of algebraic varieties:

$$Gr^\lambda\mu \cong M_\mu \cap \overline{0}_\lambda$$

$$L \mapsto [\pi_L \overline{z} \pi_L^{-1}]_A$$

between the Lusztig slice $Gr^\lambda\mu$ and the subvariety $M_\mu \cap \overline{0}_\lambda \subseteq N_{\emptyset_{d!}}(C)$.

In the next chapter we will focus on the the case of $\lambda = d\omega_1 = (d, 0, \ldots, 0)$, where $\omega_1$ is the first fundamental coweight. Notice that in this case the Mirković-Vybornov isomorphism specializes to:

$$Gr^{d\omega_1}_\mu \cong M_\mu \cap \overline{0}_{d\omega_1} \cong M_\mu \cap N_{\emptyset_{d!}}(C).$$
Chapter 4

Explicit Isomorphisms

In this chapter we develop explicit isomorphisms between Lusztig slices and varieties of nilpotent matrices. In 4.3 we give an explicit realization of the Mirković-Vybornov isomorphism $G_{\mu}^{\text{dac}} \cong M_\mu \cap \mathbb{N}_{\mathfrak{gl}_d}(\mathbb{C})$ in the case of $G = \text{GL}_2$ and $\mu \in \Lambda^-$. In 4.4 we work with lattices contained in the standard lattice (rather than with lattices containing the standard lattice) and give an explicit realization of an alternate Mirković-Vybornov isomorphism $G_{\mu}^{\text{dac}} \cong \tilde{M}_\mu \cap \mathbb{N}_{\mathfrak{gl}_d}(\mathbb{C})$ for $G = \text{GL}_n$ and $\mu \in \Lambda^+$. Here, $\tilde{M}_\mu$ denotes a transposed version of the Mirković-Vybornov slice.

4.1 The Subset $G_{\mu}^{s} \subseteq G_{\text{GL}_n}$

In this section $\mu$ will either be in $\Lambda^-$ or in $\Lambda^+$.

We will use $G_{\mu}$ to denote the collection of lattices of the form $G_1[z^{-1}]L_\mu$. Note that lattices in $G_{\mu}$ are represented by matrices over the polynomial ring.
\( \mathbb{C}[z^{-1}] \) of the form \( gz^\mu \) where \( g = I + g_{-1}z^{-1} + g_{-2}z^{-2} + \ldots + g_{-k}z^{-k} \), with \( g_i \in \mathfrak{gl}_n(\mathbb{C}) \), and \( \det(g) \in \mathbb{C}^\ast \).

For \( s \in \mathbb{Z} \), we will define subsets \( \text{Gr}^s_\mu \subseteq \text{Gr}_\mu \) that behave like “truncations at valuation \( s \)” of \( \text{Gr}_\mu \).

**Definition 4.1.1.** For \( s \in \mathbb{Z} \) define the subset \( \text{Gr}^s_\mu \subseteq \text{Gr}_\mu \) to be those lattices in \( \text{Gr}_\mu \) whose representative matrices in \( GL_n(K) \) have entries of valuation greater than or equal to \( s \).

The condition that the entries of \( g \in GL_n(K) \) have valuation greater than or equal to \( s \) is unaffected if we left or right-multiply \( g \) by matrices in \( GL_n(O) \). Therefore this condition is well-defined on the lattice \( L = [g] \), and in addition, is well-defined on the entire \( GL_n(O) \)-orbit of \( L \).

It is an important observation that lattices in \( \text{Gr}_\mu \) do not necessarily satisfy any containment relations with the standard lattice, as the following example illustrates.

**Example 4.1.2.** Work in \( \text{Gr}_{GL_2} \). Consider the lattice \( L \) in \( G_1[z^{-1}]L_{(-1,-1)} \) represented by the matrix:

\[
\begin{bmatrix}
z^{-1} + z^{-3} & z^{-2} \\
z^{-2} & z^{-1}
\end{bmatrix}.
\]

We want to illustrate that \( L \) has no containment relation with the standard lattice. We proceed by performing \( GL_n(O) \)-row and \( GL_n(O) \)-column operations to reduce this matrix to
\[
\begin{bmatrix}
z^{-3} & 0 \\
0 & z
\end{bmatrix}.
\]

By corollaries 2.6.5 and 2.6.6, we may conclude that \( L \) neither contains nor is contained in the standard lattice.

\[\square\]

### 4.2 The Affine Space \( G^s_\mu \subseteq GL_n(K) \)

In this section \( \mu \) will either be in \( \Lambda^- \) or in \( \Lambda^+ \).

We are going to exhibit an affine space \( G^s_\mu \subseteq GL_n(K) \) of matrices that represent the lattices in \( \text{Gr}^s_\mu \).

**Definition 4.2.1.** Assume \( s \in \mathbb{Z} \) and \( \mu \in \Lambda^- \). We define \( G^s_\mu \subseteq GL_n(K) \) to be the affine space of matrices:

\[ z^\mu + Q, \]

where \( Q = (q_{ij}) \) is a matrix of polynomials \( q_{ij}(z^{-1}) \in \mathbb{C}[z^{-1}] \) of the form:

\[
q_{ij}(z^{-1}) = \begin{cases} 
q_{ij}^1 z^{\mu j-1} + q_{ij}^2 z^{\mu j-2} + \ldots + q_{ij}^{\mu j-s} z^s & i \geq j \\
n_{ij}^1 z^{\mu i-1} + q_{ij}^2 z^{\mu i-2} + \ldots + q_{ij}^{\mu i-s} z^s & i < j 
\end{cases}
\]

Notice that \( G^s_\mu \) only makes sense when \( s \leq \mu_1 \).

**Proposition 4.2.2.** Assume \( \mu \in \Lambda^- \) and \( s \leq \mu_1 \). Every lattice \( L \) in \( \text{Gr}^s_\mu \) admits a matrix representative in the affine space \( G^s_\mu \).
Proof. We can represent $L$ with a the matrix of the form $z^\mu + (g_{ij})$ for polynomials $g_{ij}$ in the variable $z^{-1}$ that are of the form:

$$g_{ij}(z^{-1}) = * z^\mu i - 1 + ... + * z^s.$$ 

Then an evident algorithm of $GL_n(\mathcal{O})$-column operations can be used to bring the matrix $z^\mu + (g_{ij})$ to a matrix of the form $z^\mu + Q \in G^s_{\mu}$. \hfill \Box

Now we are going to give a definition of $G^s_{\mu}$ in the case of $\mu \in \Lambda^+$. 

**Definition 4.2.3.** Assume $s \in \mathbb{Z}_+$ and $\mu \in \Lambda^+$. We define $G^s_{\mu} \subseteq GL_n(\mathcal{K})$ to be the affine space of matrices:

$$z^\mu + P,$$

where $P = (p_{ij})$ is a matrix of polynomials $p_{ij}(z) \in \mathbb{C}[z]$ of the form:

$$p_{ij}(z) = \begin{cases} p_{ij}^1 z^{\mu_i - 1} + p_{ij}^2 z^{\mu_i - 2} + ... + p_{ij}^{\mu_i - s} z^s & i \geq j \\ p_{ij}^1 z^{\mu_j - 1} + p_{ij}^2 z^{\mu_j - 2} + ... + p_{ij}^{\mu_j - s} z^s & i < j \end{cases}.$$

Notice that $G^s_{\mu}$ only makes sense when $s \leq \mu_n$.

**Proposition 4.2.4.** Assume $\mu \in \Lambda^+$ and $s \leq \mu_n$. Every lattice $L$ in $\text{Gr}^s_{\mu}$ admits a matrix representative in the affine space $G^s_{\mu}$.

**Proof.** We can represent $L$ with a the matrix of the form $z^\mu + (g_{ij})$ for polynomials $g_{ij}$ in the variable $z$ that are of the form:

$$g_{ij}(z) = * z^{\mu_i - 1} + ... + * z^s.$$
Then an evident algorithm of $GL_n(\mathcal{O})$-column operations can be used to bring the matrix $z^\mu + (g_{ij})$ to a matrix of the form $z^\mu + P \in G_{\mu}^s$.

### 4.3 Explicit Mirković-Vybornov

Let $\mu \in \Lambda^+$ and $d = \sum \mu$. In this section we give an explicit realization of the Mirković-Vybornov isomorphism $Gr_{\mu}^\lambda \cong M_{\mu} \cap \mathcal{O}_\lambda$ in the case of $GL_2$ and $\lambda = d\omega_1$. We explain a general approach that could potentially be adapted to yield an explicit realization of the isomorphism $Gr_{\mu}^{d\omega_1} \cong M_{\mu} \cap N_{gl_d}(\mathbb{C})$ in the case of $GL_n$.

**The Idea.** Let $\mu \in \Lambda^-$ and $d = \sum \mu$. Recall that for $s \leq \mu_1$ the affine space $G_{\mu}^s \subseteq GL_n(\mathcal{K})$ consists of representatives for lattices in $Gr_{\mu}^s$. Consider the determinant map $\det : G_{\mu}^s \to \mathcal{K}^*$, which takes values in monic polynomials in $z^{-1}$ of degree $d$. Lattices $L \in Gr_{\mu}^s$ are of the form $[z^\mu + Q]$ (see 4.2.2), and the matrix $z^\mu + Q$ can be seen to have determinant $\det(z^\mu + Q) = z^d$. Therefore if we descend the map $\det : G_{\mu}^s \to \mathcal{K}^*$ to lattices in $Gr_{\mu}^s$ we obtain $\valdet : Gr_{\mu}^s \to \mathcal{K}^*/\mathcal{O}^* \cong \mathbb{Z}$ that takes only the value $\{d\}$. We will prove that $Gr_{\mu}^s = Gr_{\mu}^{\lambda(s)}$ for a particular $\lambda(s)$ that depends upon $s$. In the case of $G = GL_2$ and $s = d$, this result specializes to $Gr_{\mu}^{d\omega_1} = Gr_{\mu}^{d\omega_1}$. What we gain from this is a realization of the Lusztig slice $Gr_{\mu}^{d\omega_1}$ as represented by matrices in $Gr_{\mu}^d$ with determinant equal to $z^d$. We can then map $Gr_{\mu}^{d\omega_1}$ to $M_{\mu} \cap N_{gl_d}(\mathbb{C})$ by using the idea of Mirković-Vybornov (see section 3.3), which is to represent the nilpotent operator $\pi_L z \pi_L^{-1} \subseteq L_{\mu}/L_0$ using the basis $A$. This yields an explicit description of the Mirković-Vybornov isomorphism.
Proposition 4.3.1. Let $G = GL_n$, $\mu \in \Lambda$ with $\mu_1 \leq \ldots \leq \mu_n$, $d := \sum \mu$, and $s \in \mathbb{Z}$ with $s \leq \mu_1$. We have:

$$\text{Gr}^s_\mu = G_1[z^{-1}]L_\mu \cap \left( \bigcap_{\nu \in W} \text{Gr}^\nu_\nu \right),$$

where the set $W$ consists of all coweights $\nu \in \Lambda$ that satisfy $\nu_1 \leq \ldots \leq \nu_n$, $s \leq \nu_1 \leq \mu_1$, and $\sum \nu = d$.

This in particular implies that:

$$\text{Gr}^s_\mu = \text{Gr}^\lambda(s)_\mu,$$

where $\lambda(s) \in \Lambda$ is the unique coweight in $W$ such that for every $\nu \in W$, $\lambda(s) - \nu$ is a sum of coroots of the form $\omega_i - \omega_{i-1}$ for $i = 2, \ldots, n$.

Proof. Suppose $L \in \text{Gr}^s_\mu$, and let us write $L = [g]$ where $g \in G^s_\mu$. We may assume that $\det(g) = z^d$. Suppose the minimal valuation of the entries of $g$ is $\nu_1$. It follows that $L \in \text{Gr}^\nu_\nu$ for some $\nu$ with $\nu_1 \leq \ldots \leq \nu_n$ and $s \leq \nu_1 \leq \mu_1$. By 2.10, valdet is well defined on the $GL_n(\mathbb{O})$-orbit of $L$, and therefore $\sum \nu = d$.

Conversely, if $\nu$ satisfies $s \leq \nu_1 \leq \ldots \leq \nu_n$, then every lattice in $Gr^\nu_\nu \cap G_1[z^{-1}]L_\mu$ is contained in $\text{Gr}^s_\mu$. This completes the proof. \hfill \Box

Example 4.3.2. Let $G = GL_3$, $\mu = (-1, -1, -1)$, and $s = -2$. We have:

$$\text{Gr}^{-2}_\mu = G_1[z^{-1}]L_\mu \cap \left( \text{Gr}^{(-2,-2,1)}_\mu \cup \text{Gr}^{(-2,-1,0)}_\mu \cup \text{Gr}^{(-1,-1,-1)}_\mu \right),$$

For example, the lattice represented by the matrix
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\[
\begin{bmatrix}
  z^{-1} & z^{-2} & 0 \\
  0 & z^{-1} & z^{-2} \\
  0 & 0 & z^{-1}
\end{bmatrix}
\]

belongs to \( \text{Gr}_s^{(-2,-2,1)} \), and the lattice represented by the matrix

\[
\begin{bmatrix}
  z^{-1} & z^{-2} & 0 \\
  0 & z^{-1} & 0 \\
  0 & 0 & z^{-1}
\end{bmatrix}
\]

belongs to \( \text{Gr}_s^{(-2,-1,0)} \).

Consider the maps \( \det : G_s^\mu \to \mathbb{C}[z^{-1}] \) and \( \valdet : \text{Gr}_s^\mu \to \mathbb{Z} \).

**Corollary 4.3.3** (Of 4.3.1). If \( G = GL_2, \mu \in \mathbb{Z}^2 \) with \( \mu_1 \leq \mu_2 \), and \( s \leq \mu_1 \), then
\( \lambda(s) = (s, d-s) \). Therefore we may write:

\[ \valdet^{-1}(d) = \text{Gr}_s^\mu = \text{Gr}_s^{(s,d-s)}. \]

**Corollary 4.3.4** (Of 4.3.1). If \( G = GL_2, \mu \in \Lambda^- \), and \( s = d \), then \( \lambda(d) = (d, 0) = d\omega_1 \). We have therefore:

\[ \valdet^{-1}(d) = \text{Gr}_d^\mu = \text{Gr}_d^{d\omega_1}. \]

This fact leads to our explicit Mirković-Vybornov isomorphism 4.3.5.

**Theorem 4.3.5.** Let \( G = GL_2 \) and \( \mu \in \Lambda^- \). The isomorphism of affine spaces
\( G_s^d \leftrightarrow M_\mu \) given by
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\[ z^\mu + Q \leftrightarrow E_\mu + \begin{bmatrix}
- q_{22}^{-\mu_1} & \ldots & - q_{22}^2 & q_{11}^1 \\
\frac{q_{22}^{-\mu_2}}{q_{21}} & \ldots & \frac{q_{22}^1}{q_{21}} & q_{12}^{-\mu_2} & \ldots & q_{12}^2 & q_{12}^1
\end{bmatrix} \]

restricts to the Mirković-Vybornov isomorphism \( \text{Gr}_{d+1}^\mu = (\text{valdet})^{-1}(d) \cong M_{\mu} \cap N_{\mu}(\mathbb{C}) \), where \( Q \) is the \( 2 \times 2 \) matrix over \( \mathbb{C}[z^{-1}] \) given by:

\[
\begin{bmatrix}
q_{11}^1 z_{\mu_1}^{-1} + \ldots + q_{11}^{-\mu_2} z_d & q_{12}^1 z_{\mu_1}^{-1} + \ldots + q_{12}^{-\mu_2} z_d \\
q_{21}^1 z_{\mu_1}^{-1} + \ldots + q_{21}^{-\mu_2} z_d & q_{22}^1 z_{\mu_2}^{-1} + \ldots + q_{22}^{-\mu_1} z_d
\end{bmatrix}
\]

and \( E_\mu \) is the nilpotent of type \( \mu \) in Jordan form.

**Proof.** Let \( L \) be a lattice in \( \text{Gr}_{d+1}^\mu \). We may be represent \( L \) by the matrix \( z^\mu + Q \in \text{Gr}_d^\mu \). Notice that \( \text{det}(z^\mu + Q) \) has the form \( z^d + * z^{d-1} + \ldots + * z^2 d \). The lattice \( L = [z^\mu + Q] \) belongs to \( \text{Gr}_{d+1}^\mu \) if and only if \( \text{det}(z^\mu + Q) = z^d \). So all of the lower coefficients of \( \text{det}(z^\mu + Q) \) must be zero. Consider the distinguished isomorphism \( L/L_0 \xrightarrow{\pi_L^{-1}} L_\mu/L_0 \) and the distinguished basis \( A = \{[z^{-1} e_1], \ldots, [z^{\mu_1} e_1], [z^{-1} e_2], \ldots, [z^{\mu_2} e_2]\} \) for \( L_\mu/L_0 \) explained in 3.3. Let \( \pi_L^{-1}(A) = \{v_{-1}, \ldots, v_{\mu_1}, w_{-1}, \ldots, w_{\mu_2}\} \) be the basis for \( L/L_0 \) that corresponds with \( A \) via the isomorphism \( \pi_L \). If we represent the operator \( \pi_L \zeta \pi_L^{-1} \subset L_\mu/L_0 \) in the basis \( A \) we get a nilpotent matrix \( [\pi_L \zeta \pi_L^{-1}]_A \) in the Mirković-Vybornov slice.
We need to calculate the matrix $[\pi_L z \pi_L^{-1}]_A$ for the lattice $L = [z^\mu + Q] \in \text{Gr}_{L\mu}^{d\omega_1}$.

One can see that $v_{\mu_1}$ and $v_{\mu_2}$ are the two columns of $z^\mu + Q$, and so the $\mu_1$-st and the last column of $[\pi_L z \pi_L^{-1}]_A$ are evident. Now we need to find the remaining columns of the matrix $[\pi_L z \pi_L^{-1}]_A$. Let $(z^{\mu_1-1} e_1)$ and $(z^{\mu_2-1} e_2)$ be the functions that extract those coefficients of vectors in $L/L_0$. To complete the matrix $[\pi_L z \pi_L^{-1}]_A$ we need the coefficients $(z^{\mu_1-1} e_1)$ and $(z^{\mu_2-1} e_2)$ of the vectors $v_{\mu_1+1}, \ldots, v_{-1} \text{ and } w_{\mu_2+1}, \ldots, w_{-1}$. We have that

$$\det(z^\mu + Q) = z^d + (q_{11}^1 + q_{22}^1)z^{d-1} + \sum_{k=2}^{d} C_k z^{d-k}$$

where $q_{11}^1 + q_{22}^1 = 0$ and all the coefficients $C_k$, for $k = 2, \ldots, d$, have the form (and must all be zero):

$$0 = C_k = \begin{cases} 
\sum_{\alpha + \beta = k} q_{11}^\alpha q_{22}^\beta & 2\mu_1 - 2 - (d - k) < 0 \\
\sum_{\alpha + \beta = k} q_{11}^\alpha q_{22}^\beta - \sum_{\gamma + \delta = 2\mu_1 - 2 -(d-k)} q_{12}^\gamma q_{21}^\delta & 2\mu_1 - 2 - (d - k) \geq 0
\end{cases}.$$

For the vectors $v_{\mu_1+1}, \ldots, v_{\mu_2-1}$ we have that:

$$(z^{\mu_1-1} e_1)v_{\mu_1+k} = C_{k+1} - q_{22}^{k+1} = -q_{22}^{k+1}$$

$$(z^{\mu_2-1} e_2)v_{\mu_1+k} = 0.$$

For the vectors $v_{\mu_2}, \ldots, v_{-1}$ we have that:
\[(z^{\mu_1-1}e_1)v_{\mu_2+k} = C_{\mu_2-\mu_1+k+1} - q_{22}^{\mu_2-\mu_1+k+1} = -q_{22}^{\mu_2-\mu_1+k+1}\]

\[(z^{\mu_2-1}e_2)v_{\mu_2+k} = q_{21}^{k+1} - q_{21}^{k}(q_{11}^1 + q_{22}^1) = q_{21}^{k+1}\]

For the vectors \(w_{\mu_2+1}, \ldots, w_{-1}\) we have that:

\[(z^{\mu_1-1}e_1)w_{\mu_2+k} = q_{12}^{k+1} - q_{12}^{k}(q_{11}^1 + q_{22}^1) = q_{12}^{k+1}\]

\[(z^{\mu_2-1}e_2)w_{\mu_2+k} = (z^{\mu_2-1}e_2)(zw_{\mu_2+k-1} - w_{\mu_2+k-1}) = C_{k+1} - q_{11}^{k+1} = -q_{11}^{k+1}.\]

Therefore the nilpotent matrix \([\pi_L z \pi_L^{-1}]_A\) is as stated in the theorem. The form the map takes clearly has an inverse, so we have found a variety isomorphism \(Gr_{\mu}^{d\omega_1} \cong M_{\mu} \cap N_{gl_d}(C)\).

**Example 4.3.6.** Let \(\mu = (-2, -1)\). The isomorphism \(Gr_{\mu}^{-3\omega_1} \cong M_{\mu} \cap N_{gl_3}(C)\) takes the form:
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\[
\begin{bmatrix}
z^{-2} + az^{-3} & cz^{-3} \\
bz^{-3} & z^{-1} + dz^{-2} + ez^{-3}
\end{bmatrix}
\] : \( \det = z^{-3} \)

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-e & a & c & 0 \\
b & 0 & d & 0
\end{bmatrix}
\cap \mathcal{N}_{\text{gl}_3(\mathbb{C})}.
\]

**Example 4.3.7.** Let \( \mu = (-2, -2) \). The isomorphism \( \text{Gr}_{\mu}^{-4\omega_1} \cong M_{\mu} \cap \mathcal{N}_{\text{gl}_4(\mathbb{C})} \) takes the form:

\[
\begin{bmatrix}
z^{-2} + az^{-3} + bz^{-4} & ez^{-3} + fz^{-4} \\
cz^{-3} + dz^{-4} & z^{-2} + gz^{-3} + hz^{-4}
\end{bmatrix}
\] : \( \det = z^{-4} \)

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-h & a & f & e & 0 \\
0 & 0 & 0 & 1 & 0 \\
d & c & -b & g & 0
\end{bmatrix}
\cap \mathcal{N}_{\text{gl}_4(\mathbb{C})}.
\]

**Example 4.3.8.** Let \( \mu = (-3, -2) \). The isomorphism \( \text{Gr}_{\mu}^{-5\omega_1} \cong M_{\mu} \cap \mathcal{N}_{\text{gl}_5(\mathbb{C})} \)
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takes the form:

\[
\begin{cases}
    \begin{bmatrix}
        z^{-3} + az^{-4} + bz^{-5} & ez^{-4} + fz^{-5} \\
        cz^{-4} + dz^{-5} & z^{-2} + gz^{-3} + hz^{-4} + iz^{-5}
    \end{bmatrix} : \det = z^{-5}
\end{cases}
\]

\[
\begin{bmatrix}
    0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    -i & -h & a & f & e \\
    0 & 0 & 0 & 0 & 1 \\
    d & c & 0 & -b & g
\end{bmatrix}
\cap \mathcal{N}_{\text{gl}_5(C)}.
\]

Example 4.3.9. Let \( \mu = (d, 0) = d\omega_1 \). Notice that \( \text{Gr}^{d\omega_1}_{d\omega_1} \) consists of a single point, \( L_{d\omega_1} \). The isomorphism \( \text{Gr}^{d\omega_1}_{d\omega_1} \cong M_4 \cap \mathcal{N}_{\text{gl}_4(C)} \) takes the form:

\[
\begin{cases}
    \{ L_{d\omega_1} \} \\
    \uparrow \\
    \{ E_{d\omega_1} \}
\end{cases}
\]

where \( E_{d\omega_1} \) is the \(|d| \times |d|\) regular nilpotent matrix with ones on the super-diagonal.

Corollary 4.3.10 (Of 4.3.5). In the case of \( G = GL_2 \), the set of matrices in \( G^d_\mu \) having determinant equal to \( z^d \) is in one-to-one correspondence with \( \text{Gr}^{d\omega_1}_{\mu} \).
There is an interesting potential relationship between the determinant of matrices in \( G^d_\mu \) and the characteristic polynomial of matrices in the Mirković-Vybornov slice \( M_\mu \subseteq \mathfrak{gl}_d(\mathbb{C}) \). Let \( \chi \) be the map that associates to a matrix \( A \) its characteristic polynomial \( \chi_A(z) = \det(zI - A) \), and let \( \rho \) denote the map from 4.3.5. Consider the following diagram.

\[
\begin{array}{ccc}
G^d_\mu & \xrightarrow{\rho} & M_\mu \\
\downarrow{\det} & \quad & \downarrow{\chi} \\
\mathbb{C}[z, z^{-1}] & & \\
\end{array}
\]

After some experimentation, it appears plausible to have a formula like

\[
\det(g) = z^{2d} \chi_{\rho(g)}(-z).
\]

However, the following example illustrates a case where \( \det(g) \) is comes close to, but does not equal \( z^{-6} \chi_{\rho(g)}(-z) \). It would be interesting to find an equation that relates determinants of matrices over \( \mathbb{C}[z^{-1}] \) to characteristic polynomials either by using an adapted slice or finding another map similar to \( \rho \). What makes this question interesting is that such a formula does exist for the alternate Mirković-Vybornov isomorphism that is discussed in 4.4.

**Example 4.3.11.** Let \( \mu = (-2, -1) \). Then we have:
det \[
\begin{bmatrix}
 z^{-2} + az^{-3} & cz^{-3} \\
 bz^{-3} & z^{-1} + dz^{-2} + ez^{-3}
\end{bmatrix}
\] = z^{-3} + (a + d)z^{-4} + (e + ad)z^{-5} + (ae - bc)z^{-6}

\[
\chi \begin{bmatrix}
 0 & 1 & 0 \\
 -e & a & c \\
 b & 0 & d
\end{bmatrix} = z^3 - (a + d)z^2 + (e + ad)z - (ed + bc)
\]

4.4 Explicit Alternate Mirković-Vybornov

Let \( \mu \) be in \( \Lambda^+ \) and \( d = \sum \mu \). In this section, we will give an explicit realization of an alternate Mirković-Vybornov isomorphism \( \text{Gr}_\mu^\lambda \cong M_\mu \cap \mathcal{O}_\lambda \) in the case of \( G = GL_n \) and \( \lambda = d\omega_1 = (d, 0, ..., 0) \). Note that while Mirković and Vybornov considered lattices containing the standard lattice, this section deals with lattices that are contained in the standard lattice.

**Definition 4.4.1 (Alternate Mirković-Vybornov slice).** Let \( \mu \in \Lambda^+ \) and \( d = \sum \mu \). Define the alternate Mirković-Vybornov slice to be the affine subspace \( \tilde{M}_\mu \subseteq \mathfrak{gl}_d(\mathbb{C}) \) consisting of \( \mu_i \times \mu_j \) block matrices \( \tilde{M}_{ij} \) where: if \( i = j \) then \( \tilde{M}_{ij} \) has ones on the sub-diagonal and arbitrary entries along the last column; and if \( i \neq j \) then \( \tilde{M}_{ij} \) has entries in the last column but not below row \( \mu_j \).

Notice that \( \tilde{M}_\mu \) is the transpose of the Mirković-Vybornov slice \( M_{-\mu} \).
The Idea. Let $\mu \in \Lambda^+$ and $d = \sum \mu$. The same principles from 4.3 will continue to apply. Every lattice in $\text{Gr}_\mu^0$ is represented by a matrix in $G_\mu^0$ and we notice that $G_\mu^0 \subseteq \mathfrak{gl}_n(0)$. By 2.6.3 this implies that every lattice in $\text{Gr}_\mu^0$ is contained in the standard lattice $L_0$. Using the ideas of 4.3.1, but instead with the condition $\mu_1 \geq \ldots \geq \mu_n \geq 0$, we can see that $\text{Gr}_\mu^0 = \text{Gr}_{\mu^\omega_1}$. We then proceed by taking $L \in \text{Gr}_{\mu^\omega_1}$ and examining the action of the nilpotent operator $z \in L_0/L$ in the basis $B = \{ [e_1], \ldots, [z^{\mu_1-1}e_1], \ldots, [e_n], \ldots, [z^{\mu_n-1}e_n] \}$ for $L_0/L$. This associates to $L$ a nilpotent matrix. The significance of the alternate slice $\tilde{M}_\mu$ introduced in definition 4.4.1 is that it is a natural receptacle for the nilpotent matrices that come from the Lusztig slice $\text{Gr}_{\mu^\omega_1}$. This idea turns out to give an explicit isomorphism $\text{Gr}_{\mu^\omega_1} \cong \tilde{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_d}(\mathbb{C})$.

**Theorem 4.4.2.** Let $G = GL_n$ and $\mu \in \Lambda^+$. The isomorphism of affine spaces $G_\mu^0 \leftrightarrow \tilde{M}_\mu$ given by:

$$
\begin{align*}
\begin{bmatrix}
\vdots & \vdots & \vdots \\
0 & 0 & -p_{ij}^2 \\
0 & \ldots & 0 & -p_{ij}^1 \\
0 & 0 & 0 & \vdots & \vdots \\
0 & 0 & 0 & \mu_i & \mu_j
\end{bmatrix}
\end{align*}
$$

restricts to an isomorphism $\text{Gr}_{\mu^\omega_1} \cong \tilde{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_d}(\mathbb{C})$, where $P$ is the $n \times n$ matrix over $\mathbb{C}[z]$ introduced in 4.2.3 and given by:
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Let $z$ matrix respect to the basis $B$ and $d$ other matrix is a and $z$ determine that $z$ \begin{equation}
\begin{bmatrix}
p_{11}z^{\mu_1-1} + \cdots + p_{11}^\mu & p_{12}z^{\mu_2-1} + \cdots + p_{12}^\mu \\
p_{21}z^{\mu_2-1} + \cdots + p_{21}^\mu & p_{22}z^{\mu_2-1} + \cdots + p_{22}^\mu \\
& \vdots \\
p_{n1}z^{\mu_n-1} + \cdots + p_{n1}^\mu & p_{n2}z^{\mu_n-1} + \cdots + p_{n2}^\mu \\
& \cdots \end{bmatrix},
\end{equation}

and $\tilde{E}_\mu$ is the $d \times d$ nilpotent of type $\mu$ with ones on the sub-diagonal and the other matrix is a $d \times d$ matrix that is given by a $\mu_i \times \mu_j$ block description.

Proof. Let $L$ be a lattice in $G_\mu^{d\omega_1}$. We may be represent $L = [z^\mu + P]$ by the matrix $z^\mu + P \in G_\mu^0$. Let us find the matrix $[z]_B$ of the operator $z \in L_0/L$ with respect to the basis $B = \{[e_1], \ldots, [z^{\mu_1-1}e_1], \ldots, [e_n], \ldots, [z^{\mu_n-1}e_n]\}$ for $L_0/L$. We have that $z[z^i e_j] = [z^{i+1} e_j]$ for $0 \leq i < \mu_j - 1$ and $1 \leq j \leq n$. It remains to determine $z[z^{\mu_j-1} e_j]$ in terms of the basis $B$. We calculate:

$z[z^{\mu_1-1} e_1] = [z^{\mu_1} e_1]$

$= -\sum_{s=1}^{\mu_1} p_{11}^s [z^{\mu_1-s} e_1] - \sum_{s=1}^{\mu_2} p_{21}^s [z^{\mu_2-s} e_2] - \cdots - \sum_{s=1}^{\mu_n} p_{n1}^s [z^{\mu_n-s} e_n]$

$z[z^{\mu_2-1} e_2] = [z^{\mu_2} e_2]$

$= -\sum_{s=1}^{\mu_2} p_{12}^s [z^{\mu_2-s} e_1] - \sum_{s=1}^{\mu_2} p_{22}^s [z^{\mu_2-s} e_2] - \cdots - \sum_{s=1}^{\mu_n} p_{n2}^s [z^{\mu_n-s} e_n]$

$\vdots$

$z[z^{\mu_n-1} e_n] = [z^{\mu_n} e_n]$

$= -\sum_{s=1}^{\mu_n} p_{1n}^s [z^{\mu_n-s} e_1] - \sum_{s=1}^{\mu_n} p_{2n}^s [z^{\mu_n-s} e_2] - \cdots - \sum_{s=1}^{\mu_n} p_{nn}^s [z^{\mu_n-s} e_n].$
Therefore \([z]_B\) is a nilpotent matrix of the form described in the theorem. The form that the map takes clearly has an inverse, and so we have found a variety isomorphism \(\text{Gr}^{d\omega_1}_\mu \cong \tilde{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}\).

**Example 4.4.3.** Let \(\mu = \omega_1 + \ldots + \omega_n = (1, \ldots, 1)\), then \(\tilde{M}_\mu = \mathfrak{gl}_n(\mathbb{C})\). The isomorphism \(\text{Gr}^{n\omega_1}_\mu \cong \mathcal{N}_{\mathfrak{gl}_n(\mathbb{C})}\) takes the form:

\[
\{ z^{\omega_1 + \ldots + \omega_n} + P : \det = z^n \} \leftrightarrow -P.
\]

**Example 4.4.4.** Let \(\mu = (2, 1)\). The isomorphism \(\text{Gr}^{3\omega_1}_\mu \cong \tilde{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}\) takes the form:

\[
\begin{bmatrix}
  z^2 + az + b & d \\
  c & z + e
\end{bmatrix} : \det = z^3
\]

\[
\begin{bmatrix}
  0 & -b & -d \\
  1 & -a & 0 \\
  0 & -c & -e
\end{bmatrix} \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}.
\]

**Example 4.4.5.** Let \(\mu = (2, 2, 1)\). The isomorphism \(\text{Gr}^{5\omega_1}_\mu \cong \tilde{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_5(\mathbb{C})}\) takes the form:
Example 4.4.6. Let $\mu = (d,0,\ldots,0) = d\omega_1$. Notice that $Gr_{d\omega_1}^{d\omega_1}$ consists of a single lattice, namely $L_{d\omega_1}$. The isomorphism takes the form:
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\{ L_{d
omega_1} \} = \left\{ \begin{bmatrix} z^d + a_1 z^{d-1} + \ldots + a_d \\ 1 \\ \vdots \\ 1 \end{bmatrix} : \det = z^d \right\}

\left\{ \tilde{E}_{d
omega_1} \right\} = \left\{ \begin{bmatrix} -a_d \\ 1 \\ \vdots \\ -a_2 \\ \vdots \\ 1 - a_1 \end{bmatrix} : a_1 = a_2 = \ldots = a_d = 0 \right\}

where \( \tilde{E}_{d
omega_1} \) is the \( d \times d \) regular nilpotent matrix with ones on the sub-diagonal.

\[ \square \]

Corollary 4.4.7 (Of 4.4.2). In the case of \( G = GL_n \), the set of matrices in \( G^0 \) having determinant equal to \( z^d \) is in one-to-one correspondence with \( G^{1
omega_1}_\mu \).

In this section, as opposed to 4.3, there is a direct relationship between the determinant of matrices in \( G^0 \) and the characteristic polynomial of matrices in the alternate Mirković-Vybornov slice \( \tilde{M}_\mu \subseteq \mathfrak{gl}_d(\mathbb{C}) \). Let \( \chi \) be the map that associates to a matrix \( A \in \mathfrak{gl}_d(\mathbb{C}) \) its characteristic polynomial \( \chi_A(z) = \det(zI - A) \), and let \( \kappa \) denote the isomorphism from 4.4.2. Consider the diagram:
One can verify that the diagram commutes:
\[ \det(g) = \chi_{\kappa(g)}(z). \]

Below is an example of this phenomenon.

**Example 4.4.8.** Let \( \mu = (2, 2) \). Then we have:

\[
\begin{vmatrix}
z^2 + az + b & ez + f \\
z^2 + gz + h & cz + d
\end{vmatrix}
\]

\[ = z^4 + (a + g)z^3 + (b + h + ag - ec)z^2 + (ah + gb - de - cf)z + (bh - df) \]

\[
\begin{pmatrix}
0 & -b & 0 & -f \\
1 & -a & 0 & -e \\
0 & -d & 0 & -h \\
0 & -c & 1 & -g
\end{pmatrix}
\]

\[ = z^4 - (-a - g)z^3 + (b + h + ag - ec)z^2 - (-bg + cf - ah + ed)z + (bh - df). \]
Chapter 5

$\mathbb{G}_a$-Actions

In this section we will introduce a certain $\mathbb{G}_a$-action on the Lusztig slices $\text{Gr}^\lambda_{\mu}$ that is due to Kamnitzer. We will focus on the case of $G = GL_n$ and $\lambda, \mu \in \Lambda^-.$

5.1 The Action $\mathbb{G}_a \circlearrowleft \text{Gr}^\lambda_{\mu}$

Let $\alpha$ be a simple coroot of $GL_n$ with its corresponding additive subgroup $x_\alpha : \mathbb{G}_a \to N.$ For each simple coroot $\alpha,$ define an action $\mathbb{G}_a \circlearrowleft \text{Gr}_G$ as follows:

$$a \cdot L = x_\alpha(a)L.$$ 

Example 5.1.1. The action $\mathbb{G}_a \circlearrowleft \text{Gr}_{GL_2}$ is given by:

$$L \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} L.$$
Example 5.1.2. The two actions $\mathbb{G}_a \circlearrowleft \text{Gr}_{GL_3}$ corresponding to the two simple coroots of $GL_3$ are given by:

$$L \rightarrow \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} L \quad \quad L \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} L.$$ 

\[\square\]

Proposition 5.1.3. This $\mathbb{G}_a$-action restricts to an action $\mathbb{G}_a \circlearrowleft \text{Gr}_\mu^\lambda$.

Proof. Clearly $\mathbb{G}_a$ preserves $\text{Gr}_\mu^\lambda$. We must only show that $\mathbb{G}_a$ preserves $G_1[z^{-1}]L_\mu$. To this end, notice that $x_\alpha(a)L_\mu = L_\mu$, and so if $g \in G_1[z^{-1}]$ we have:

$$x_\alpha(a)gL_\mu = x_\alpha(a)gx_\alpha(-a)L_\mu.$$ 

The proof is completed by noticing that $g \in G_1[z^{-1}]$ implies $x_\alpha(a)gx_\alpha(-a) \in G_1[z^{-1}]$. \[\square\]

The following conjecture is a motivating force behind this chapter.

Conjecture 5.1.4. Assume that $\mu + \alpha$ is in $\Lambda^-$. There is an isomorphism of Poisson varieties:

$$\text{Gr}_\mu^\lambda //_1 \mathbb{G}_a \cong \text{Gr}_{\mu+\alpha}^\lambda.$$
5.2 The Action $\mathbb{G}_a \subset M_\mu \cap N_{\mathfrak{g}l_d}(\mathbb{C})$

Let $G = GL_n$ and $\mu \in \Lambda^-$. Consider the Mirković-Vybornov isomorphism from 3.3 where $d := \sum \mu$ and $\lambda = d\omega_1$:

$$Gr^{d\omega_1}_\mu \longrightarrow M_\mu \cap N_{\mathfrak{g}l_d}(\mathbb{C})$$

where $A$ is the basis $A = \{[z^{-1}e_1], ..., [z^\mu e_1], ..., [z^{-1}e_n], ..., [z^\mu e_n]\}$ for $L_\mu/L_0$ and $\pi_L$ is the distinguished isomorphism $\pi_L : L/L_0 \cong L_\mu/L_0$.

We are going to transport the action $\mathbb{G}_a \subset Gr^{d\omega_1}_\mu$ through the Mirković-Vybornov isomorphism. First we will look at an example, and then we will explain the general case. The action $\mathbb{G}_a \subset M_\mu \cap N_{\mathfrak{g}l_d}(\mathbb{C})$ turns out to have the form $X \rightarrow a \cdot X$, where

$$a \cdot X = h(a, X) a^{-1} X a h(a, X)^{-1}.$$ 

We first conjugate $X \in M_\mu \cap N_{\mathfrak{g}l_d}(\mathbb{C})$ by a unipotent matrix $a \in GL_{d}(\mathbb{C})$ induced from the action of $a \in \mathbb{G}_a$ on the vector space $L_\mu/L_0$, and then we conjugate by another unipotent matrix $h \in GL_d(\mathbb{C})$, depending on both $a$ and on $X$, in order to bring the matrix $a^{-1} X a$ back into the Mirković-Vybornov slice $M_\mu$.

Here is a complete example.

**Example 5.2.1.** Let $G = GL_2$, $\mu = (-2, -1)$, and $A = \{[z^{-1}e_1], [z^{-2}e_1], [z^{-1}e_2]\}$ be our basis for $L_\mu/L_0$. By 4.3.6 the isomorphism $Gr^{2\omega_1}_\mu \cong M_\mu \cap N_{\mathfrak{g}l_2}(\mathbb{C})$ is given by:
which yields the following equations:

\[
0 = a + d \\
0 = e + ad \\
0 = ae - bc.
\]

Putting everything in terms of \( b, c \) and \( d \) we have:
\begin{align*}
\left\{ \begin{bmatrix} z^{-2} - dz^{-3} & cz^{-3} \\ bz^{-3} & z^{-1} + dz^{-2} + d^2z^{-3} \end{bmatrix} : d^3 + bc = 0 \right\}
\end{align*}

For $L \in \text{Gr}_{\mu}^{-3\omega_1}$ we will employ “box-diagrams” to represent the distinguished basis for $L/L_0$ (here the boxes in row $j$ correspond to $[z^{-3}e_j],[z^{-2}e_j],[z^{-1}e_j]$ and the entries in the boxes correspond to the coefficients):

\begin{align*}
\{v_1, v_2, v_3\} := \begin{bmatrix} -d^2 & 0 & 1 \\ bd & b & 0 \end{bmatrix}, \quad \begin{bmatrix} -d & 1 & 0 \\ b & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} c & 0 & 0 \\ d^2 & d & 1 \end{bmatrix}.
\end{align*}

Recall that the action $G_a \trianglelefteq \text{Gr}_{\mu}^{-3\omega_1}$ is given by:

\begin{align*}
L \to \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} L.
\end{align*}

The action of $a \in G_a$ sends $\{v_1, v_2, v_3\}$ to a basis for $aL/L_0$, namely:

\begin{align*}
\{w_1, w_2, w_3\} := \begin{bmatrix} -d^2 + abd & ba & 1 \\ bd & b & 0 \end{bmatrix}, \quad \begin{bmatrix} -d + ab & 1 & 0 \\ b & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} c + ad^2 & da & a \\ d^2 & d & 1 \end{bmatrix}.
\end{align*}
However the basis \( \{w_1, w_2, w_3\} \) for \( aL/L_0 \) is not ready to be identified with the basis \( A \) for \( L_\mu/L_0 \). Instead we must “correct” the basis \( \{w_1, w_2, w_3\} \) if we wish to see the action \( G_a \circlearrowright M \cap N_{gl_3(\mathbb{C})} \). The first correction is \( \{w_1, w_2, w_3\} \rightarrow \{w'_1, w'_2, w'_3\} = \{w_1, w_2, w_3 - aw_1\} \). We have:

\[
\{w'_1, w'_2, w'_3\} = \begin{pmatrix}
-d^2 + abd & ba & 1 \\
bd & b & 0 \\
-\frac{d}{a} + ab & 1 & 0 \\
c + 2ad - a^2bd & ad - a^2b & 0 \\
\end{pmatrix}\.
\]

This corresponds to the conjugation:

\[
M_\mu \cap N_{gl_3(\mathbb{C})} \ni X \mapsto a^{-1}Xa
\]

where the matrix \( a \) is induced from \( a \) acting on \( L_\mu/L_0 \) in the basis \( A \):

\[
a = \begin{bmatrix}
1 & 0 & a \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \in GL_3(\mathbb{C}).
\]

Notice that the matrix \( a^{-1}Xa \) does not necessarily remain in the Mirković-Vybornov slice. We must conjugate it by another unipotent matrix \( h(a, X) \in GL_3(\mathbb{C}) \) to bring it back into the slice. This corresponds to performing the remaining corrections:
\[
\begin{align*}
\{w'_1, w'_2, w'_3\} & \rightarrow \{w'_1, w'_2, w'_3 - (ad - a^2b)w'_2\} = \{w''_1, w''_2, w''_3\} \\
\{w''_1, w''_2, w''_3\} & \rightarrow \{w''_1 - (ba)w''_2, w''_2, w''_3\}.
\end{align*}
\]

This results in the following basis for \( aL/L_0 \):

\[
\left\{ \begin{array}{ccc}
-(d - ab)^2 & 0 & 1 \\
bd - ab & b & 0 \end{array} \right\},
\left\{ \begin{array}{ccc}
-(d - ab) & 1 & 0 \\
bd & b & 0 \end{array} \right\},
\left\{ \begin{array}{ccc}
-b^{-1}(d - ab)^3 & 0 & 0 \\
(d - ab)^2 & d - ab & 1 \end{array} \right\}.
\]

We have therefore calculated the action \( \mathbb{G}_a \triangleleft M_{\mu} \cap N_{\text{gl}_3(\mathbb{C})} \)

\[
X \mapsto h(a, X)a^{-1}Xah(a, X)^{-1}
\]

\[
\begin{pmatrix}
0 & 1 & 0 \\
-d^2 & -d & c \\
b & 0 & d
\end{pmatrix}
\mapsto
\begin{pmatrix}
0 & 1 & 0 \\
-(d - ab)^2 & -(d - ab) & -b^{-1}(d - ab)^3 \\
b & 0 & d - ab
\end{pmatrix}.
\]

\[\square\]

**Definition 5.2.2** (An additive matrix group \( U_\alpha \)). Let \( U_\alpha \subseteq GL_{|d|}(\mathbb{C}) \) be the matrix representation of \( \mathbb{G}_a \) induced on the vector space \( L_\mu/L_0 \) (as in 5.2.1) from the action \( \mathbb{G}_a \triangleleft \text{Gr}_{GL_n} \) coming from the coroot \( \alpha \).

**Definition 5.2.3** (The unipotent correction \( h \)). Consider our action \( \mathbb{G}_a \triangleleft \text{Gr}_{GL_n} \). Let \( L \in \text{Gr}_\mu^{d\omega_1} \) and \( X \in M_{\mu} \cap N_{\text{gl}_d(\mathbb{C})} \) be a lattice and nilpotent matrix corresponding to one another under the Mirković-Vybornov isomorphism. Let \( \{v_1, \ldots, v_{-d}\} = \pi^{-1}_L(A) \) be the distinguished basis for \( L/L_0 \). For \( a \in \mathbb{G}_a \),
let \( \{w_1, \ldots, w_{-d}\} \) be the resultant basis for \( aL/L_0 \). We define \( h \in GL_{|d|}(\mathbb{C}) \), depending on \( a \) and \( X \), to be the matrix that implements the corrections that we must do to the basis \( \{w_1, \ldots, w_{-d}\} \) proceeding the initial correction \( a \in GL_{|d|}(\mathbb{C}) \) when finding a basis for \( aL/L_0 \) to identify with \( A \) (see example 5.2.1).

Let \( L \in \text{Gr}_{\mu}^{d, \omega} \) and \( X \in M_\mu \cap N_{\theta}|_{d}(\mathbb{C}) \) be a lattice and nilpotent matrix corresponding to one another under the Mirković-Vybornov isomorphism. The reason that the action \( G_a \circlearrowright M_\mu \cap N_{\theta}|_{d}(\mathbb{C}) \), denoted by \( X \rightarrow a \cdot X \), has the form that it does, is because the isomorphism \( \pi_{aL} : aL/L_0 \cong L_\mu/L_0 \) is related to \( \pi_L : L/L_0 \cong L_\mu/L_0 \) through the formula \( \pi_{aL} = h(a, X)\tilde{a}^{-1}\pi_L\tilde{a}^{-1} \), where \( \tilde{a} \in U_\alpha \) and \( h(a, X) \in GL_{|d|}(\mathbb{C}) \) are the unipotent matrices consisting of the operations needed before we can identify the basis of \( aL/L_0 \) with our basis \( A \) of \( L_\mu/L_0 \).

Therefore

\[
a \cdot X = [\pi_{aL}z\pi_{aL}^{-1}]_A = h(a, X)\tilde{a}^{-1}[\pi_Lz\pi_L^{-1}]_A\tilde{a}h(a, X)^{-1} = h(a, X)\tilde{a}^{-1}X\tilde{a}h(a, X)^{-1}.
\]

### 5.3 Equality of \( G_a \circlearrowright M_\mu \cap N \) and \( G_a \circlearrowright S(P_\mu) \)

We will work in the case of \( G = GL_n \) and with Lusztig slices \( \text{Gr}_{\mu}^{d, \omega} \) where \( \mu \in \Lambda^- \) and \( d := \Sigma \mu \).
The following conjecture is related to 5.1.4 through the Mirković-Vybornov isomorphism.

**Conjecture 5.3.1.** Assume that \( \mu, \mu + \alpha \in \Lambda^- \). There is an isomorphism of Poisson varieties:

\[
(M_\mu \cap N_{\mathfrak{gl}(\mathbb{C})}) / / \mathbb{G}_a \cong M_{\mu + \alpha} \cap N_{\mathfrak{gl}(\mathbb{C})}.
\]

A similar question was studied by Morgan [Mor14]. Instead of the Mirković-Vybornov slice intersected with the nilpotent cone, he considered Slodowy slices \( S(P) \) with respect to certain pyramids \( P \) (described in [Mor14]). Slodowy slices, like Mirković-Vybornov slices, are examples of affine subspaces of \( \mathfrak{gl}_{|d|}(\mathbb{C}) \) which are transverse to the nilpotent orbits \( \mathcal{O}_\mu \). In 5.3.5 we show that Slodowy slices are Poisson isomorphic with Mirković-Vybornov slices. While the reader should see [Mor14, 3.2] for the general definition of a pyramid, it will suffice for us to let a pyramid be a numbered right-justified Young diagram.

In the context of the Mirković-Vybornov isomorphism we were led to consider the basis \( A = \{ [z^{-1}e_1], \ldots, [z^\mu e_1], \ldots, [z^{-1}e_n], \ldots, [z^\mu e_n] \} \) for \( L_\mu / L_0 \). Let us depict the basis \( A \) as a numbered right-justified Young diagram. In a similar fashion as [Mor14, 3.4.1] we are going to regard this Young diagram as a pyramid \( P_\mu \).

**Example 5.3.2.** Let \( G = GL_3 \) and \( \mu = (-3,-2,-1) \). Our basis for \( L_\mu / L_0 \) is \( A = \{ [z^{-1}e_1], [z^{-2}e_1], [z^{-3}e_1], [z^{-1}e_2], [z^2e_2], [z^{-1}e_3] \} \). Depict \( A \) as a numbered right-justified Young diagram and regard it as a pyramid:
Let $\alpha$ be a simple coroot of $GL_n$ and suppose that $\mu$ and $\mu + \alpha$ are in $\Lambda^-$. Recall that $a \cdot L = x_{\alpha}(a)L$ gives an action $G_a \subseteq Gr_{GL_n}$ which restricts to an action on the Lusztig slices.

In proposition 5.3.5, following [GG02] and a yet-to-be-published paper of Weekes-Yacobi, we show that the Mirković-Vybornov slice $M_\mu$ is Poisson isomorphic to a Hamiltonian reduction of $\mathfrak{gl}_{|d|}(C)$ by a unipotent group $M(P_\mu)$ called the Premet group with respect to the pyramid $P_\mu$. In 5.3.6 we show that our action $G_a \subseteq M_\mu \cap N_{\mathfrak{gl}_{|d|}(C)}$ coincides with the action $G_a \subseteq S(P_\mu)$ on the Slodowy slice that is described in [Mor14, 3.4.3].

**Clarification.** We are going to be numbering the rows and columns of the pyramid $P_\mu$ in such a way that the row numbers increase as you move up, and the column numbers increase as you move right to left.

**Definition 5.3.3** (Premet group for a pyramid). The Premet group $M(P_\mu)$ with respect to the pyramid $P_\mu$ is the following unipotent subgroup of $GL_{|d|}(C)$:

$$M(P_\mu) := I_{|d|} + \mathbb{C}\{E_{lk} : \text{col}(l) > \text{col}(k)\}.$$ 

**Definition 5.3.4** (Premet algebra for a pyramid). The Premet Lie algebra $m(P_\mu)$ with respect to the pyramid $P_\mu$ is the following nilpotent Lie subalgebra
of $\mathfrak{gl}_{|d|}(\mathbb{C})$:

$$m(P_\mu) := \mathbb{C}\{E_{lk} : \text{col}(l) > \text{col}(k)\}.$$  

**Proposition 5.3.5.** We have isomorphisms of Poisson varieties:

$$M_\mu \cong (E_\mu + m(P_\mu)^\perp)/M(P_\mu) \cong S(P_\mu)$$

**Proof.** Let $E := E_\mu$ be a nilpotent matrix of type $\mu$ in Jordan form, and $\{E, F, H\}$ be an $\mathfrak{sl}_2(\mathbb{C})$-triple containing $E$. Also let $M := M(P_\mu)$ be the Premet group for the pyramid $P_\mu$ with lie algebra $m := m(P_\mu)$ (see [Mor14, 3.4.1]). We will argue that any affine subspace of the form $E + C$, where $C \subseteq \mathfrak{gl}_{|d|}(\mathbb{C})$ is a subspace that satisfies $\mathfrak{gl}_{|d|}(\mathbb{C}) = [\mathfrak{gl}_{|d|}(\mathbb{C}), E] \oplus C$ with all the eigenvalues of $\text{ad}_H$ on $C$ being $\leq -2$, is necessarily Poisson isomorphic to $(E + m^\perp)/M$. This will suffice, as one can check that both $M_\mu$ and $S(P_\mu)$ are of the form $E + C$ where $C$ satisfies the required conditions (see [MV08, 1.4]).

For example, in the case of $S(P_\mu)$ we have that $C = \mathfrak{z}(F)$. Let us consider a general affine subspace $E + C \subseteq \mathfrak{gl}_{|d|}(\mathbb{C})$ with $C$ satisfying the conditions. We are assuming that $C$ must be contained within the $\leq -2$ weight spaces for $\text{ad}_H$, i.e. $C \subset \bigoplus_{i \leq -2} \mathfrak{g}(i) \subset m^\perp$, and therefore the image of the adjoint-action map $M \times (E + C) \to \mathfrak{gl}_{|d|}(\mathbb{C})$ is contained in $E + m^\perp$. Since $\mathfrak{gl}_{|d|}(\mathbb{C}) = [\mathfrak{gl}_{|d|}(\mathbb{C}), E] \oplus C$ it follows that $[m, E] \cap C = 0$. Since both spaces $\mathfrak{z}(F)$ and $C$ are complimentary to $[\mathfrak{gl}_{|d|}(\mathbb{C}), E]$ in $\mathfrak{gl}_{|d|}(\mathbb{C})$ they must have the same dimension. Therefore we calculate:

$$\dim m^\perp = \dim [m, E] + \dim \mathfrak{z}(F) = \dim [m, E] + C.$$  

This proves that $m^\perp = [m, E] \oplus C$. The rest of the proof goes through exactly
as in [GG02, Lemma 2.1].

**Theorem 5.3.6.** Let $\alpha$ be a simple coroot of $GL_n$ and suppose that $\mu$ and $\mu + \alpha$ are in $\Lambda^-$. Let $P_\mu$ be the pyramid associated to $\mu$. Then the action of $\mathbb{G}_a$ on $M_\mu \cap N_{\mathfrak{sl}_d}(\mathbb{C})$ is equal to the action of $\mathbb{G}_a$ on the Slodowy slice $S(P_\mu)$ as described in [Mor14, 3.4.3].

**Proof.** We are supposing that $\alpha$ is a simple coroot of $GL_n$ and that $\mu$ and $\mu + \alpha$ satisfy $\mu_1 \leq \ldots \leq \mu_n$ and $\mu_1 + \alpha_1 \leq \ldots \leq \mu_n + \alpha_n$. Therefore, we are in the case where a box moves upwards a single row. Therefore $i$ and $j$ from [Mor14, theorem 3.4.3] are as follows: $j$ is the row of $P_\mu$ that has the box which is moving, and $i = j + 1$ is the row that it moves to. It is very important to clarify that we are considering a numbering of rows and columns of $P_\mu$ where the row numbers increase as you move up and the column numbers increase as you move right to left. The additive group that acts on $S(P_\mu)$ predicted in theorem [Mor14, 3.4.3] has Lie algebra $\mathfrak{k} = \mathbb{C}\{E_1\}$ where

$$E_1 = \sum_{\text{row}(l)=j+1, \text{row}(k)=j, \text{col}(k)=\text{col}(l)} E_{lk}.$$  

One can see that the additive group $K$ with Lie algebra $\mathfrak{k}$ is precisely our matrix group $U_\alpha \subseteq GL_{|d|}(\mathbb{C})$ defined in 5.2.2. We consider the Premet group $M(P_\mu)$ relative to $P_\mu$ which has the Lie algebra $\mathfrak{m}(P_\mu)$ generated as follows:

$$\mathfrak{m}(P_\mu) = \mathbb{C}\{E_{lk} : \text{col}(l) > \text{col}(k)\}.$$  

Therefore the Premet group is:

$$M(P_\mu) = I_{|d|} + \mathbb{C}\{E_{lk} : \text{col}(l) > \text{col}(k)\}.$$
Let \( a \in \mathbb{G}_a \) and consider the lattices \( L \) and \( aL \) in \( \Gamma_M^{d\omega_1} \) along with the corresponding matrices \( X \) and \( a \cdot X \) in \( M_\mu \cap N_{\mathfrak{g}_{[\mu]}(\mathbb{C})} \) via the Mirković-Vybornov isomorphism. Consider the distinguished basis \( \{v_1, ..., v_{-d}\} \) for \( L/L_0 \) as in 5.2.1. Let \( \{w_1, ..., w_{-d}\} \) be the basis for \( aL/L_0 \) achieved by acting with \( a \in \mathbb{G}_a \) on \( \{v_1, ..., v_{-d}\} \). The basis \( \{w_1, ..., w_{-d}\} \) now must be “corrected” if we are to see the matrix \( a \cdot X \). The initial corrections are of the form:

\[
w_l \mapsto w_l - aw_k
\]

for \( \text{row}(l) = j + 1 \), \( \text{row}(k) = j \), and \( \text{col}(l) = \text{col}(k) \). Once this correction has been made, the vectors in a basis for \( aL/L_0 \) have “leading ones”, which means everything in the column with a leading one is zero (see 5.2.1). Therefore the remaining corrections on the basis for \( aL/L_0 \) to be performed must be of the form:

\[
w_l \mapsto w_l + *w_k
\]

for \( \text{col}(l) > \text{col}(k) \). By the considerations of 5.2, we have that \( h(a, X) \in M(P_\mu) \) and \( a \cdot X = h(a, X)a^{-1}Xaha(a, X)^{-1} \). We may therefore conclude that the two actions \( \mathbb{G}_a \subseteq M_\mu \cap N_{\mathfrak{g}_{[\mu]}(\mathbb{C})} \) and \( \mathbb{G}_a \subseteq S(P_\mu) \) are identical. \( \square \)
Chapter 6

Future Directions

6.1 Characteristic Polynomials

There is an interesting connection between determinants of matrices in $G^*_\mu$ and the characteristic polynomials of matrices in these “affine slices”. In 4.4, when $\mu \in \Lambda^+$ and $s = 0$, we saw that the diagram below was commutative.

In 4.3 we were in the case of $G = GL_2$, $\mu \in \Lambda^-$, and $s = d$, and we had the diagram below.
However, this diagram was not commutative. There was only a vague connection between the determinant of matrices in $G^d_\mu$ and the characteristic polynomials of their images under $\rho$ in the Mirković-Vybornov slice.

This makes one curious if there might be some $s \in \mathbb{Z}$, with affine slices $X^s_\mu \subseteq G^s_\mu$ and $S_\mu \subseteq \mathfrak{g}_{[d]}(\mathbb{C})$ transverse to the nilpotent orbit of type $\mu$, as well as an isomorphism $\nu$, so that the diagram

admits some nice functional relationship between $\det$, $\nu$, and $\chi$.

Here are two questions potentially connected to these issues.

**Question 1.** Can 4.3.5 be deduced from 4.4.2?

**Question 2.** For $G = GL_n$ and $\mu \in \Lambda^-$, can we find an explicit Mirković-Vybornov isomorphism $G^d_\mu \cong M_\mu \cap \mathcal{N}_{\mathfrak{g}_{[d]}(\mathbb{C})}$?
6.2 Explicit Description of $\text{Gr}^\lambda_\mu$

In 4.3 and 4.4 we were able to realize $\text{Gr}^{d\omega_1}_\mu$ as represented by matrices in $G^d_\mu$ having determinant equal to $z^d$. If we want an explicit description of $\text{Gr}^\lambda_\mu$ in the general case (i.e. for $\lambda \neq d\omega_1$), we would need to extract the subset $\text{Gr}^\lambda_\mu \subseteq \text{Gr}^s_\mu$ by imposing some conditions on matrices in $G^s_\mu$ beyond merely $\det = z^d$.

One approach might be to consider extracting the subset $\text{Gr}^\lambda_\mu \subseteq \text{Gr}^s_\mu$ by looking at those matrices in $G^s_\mu$ with various conditions on their minors that are adapted to the coweight $\lambda$.

6.3 The Reduction $\text{Gr}^\lambda_\mu / \!/ _1G_a \cong \text{Gr}^\lambda_{\mu+\alpha}$

Another goal is to prove conjecture 5.1.4. One can read [KWWY14, 2.6] for a description of the Poisson structure on $\text{Gr}^\lambda_\mu$. Then, given the action $G_a \subset \text{Gr}^\lambda_\mu$ introduced in 5.1, one must show that it is Hamiltonian and find a description of the momentum map.

After this is completed, one should try to find a $G_a$-invariant map of Poisson varieties $\text{Gr}^s_\mu \to \text{Gr}^s_{\mu+\alpha}$ that descends to an isomorphism $\text{Gr}^\lambda_\mu / \!/ _1G_a \cong \text{Gr}^\lambda_{\mu+\alpha}$. 
References


[Lus] George Lusztig. Singularities, character formulas, and a $q$-analog of weight multiplicities. In *Analysis and topology on
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