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Dynamical analysis of autonomous system in entropic cosmology

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The entropic cosmology model is an alternative method to explain the accelerated expansion of the universe. In this paper, we discuss the dynamical system in two types of entropic cosmology model, i.e. $\Lambda(t)$ type and BV type. We found that, the stability properties of fixed points are affected by H^2 term, while the H term and constant term have no influence on stability properties of fixed points. We also found that, the dynamical properties of C-version model are same as H-version model.

Keyword: entropic cosmology; dark energy; dynamical system.

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I. INTRODUCTION

Observations indicate that the expansion rate of universe is accelerating [1–5]. There are several methods to explain the accelerated expansion of the late universe[6]. The most important method is the introduction of dark energy [7, 8]. The simplest dark energy is cosmology constant Λ . However, the energy density of cosmology constant which is needed by cosmologists is much smaller than the energy of the vacuum in quantum theory. The commonly used method is to introduce scalar fields to replace the cosmological constant.

Recently, an alternative method, named entropic cosmology model, is proposed to explain the accelerated expansion of the universe [9, 10]. The basic idea of entropic cosmology is to consider the entropic-force terms which are derived from the usually neglected surface terms on the horizon of the universe [9]. (The idea of the entropic-force term on the horizon can explain the accelerating universe in [9] should be distinguished from the idea that gravity itself is an entropic force [11]. Although they use the same terms, they are two very different ideas.) Different entropic-force terms are derived from different classes of entropy. In original entropic cosmology model [12, 13], the entropic terms include H^2 terms and $\dot{H}$ terms. H^2 terms are derived from Bekenstein entropy (area entropy), where H is Hubble parameter. Further research shows that, the background equations with H^2 term and $\dot{H}$ term can be made equivalent to a model with a dark energy component[14]. Other works [13, 15, 16] suggest that, the behavior of H^2 terms and $\dot{H}$ terms are found to be same, therefore the $\dot{H}$ term can be neglected [15]. Besides the H^2 terms, there are H terms in entropic-force terms and they are derived from volume entropy [15]. Moreover, in [16], the author shows that the H^2 terms do not describe cosmological fluctuations properly without the inclusion of constant terms which are derived from entropy proportional to the square of area entropy[15], therefore it is necessary to introduce constant entropic-force terms.

Moreover, in some discussions [17], H^2 terms have been neglected, while H terms and constant terms are considered separately. In this case, the entropic cosmology also can be divided into two categories, i.e. H-version and C-version.

According to different forms of evolution equations, there are two types of entropic cosmology model [17]. The first type is so-called $\Lambda(t)$ type, which is similar to $\Lambda(t)$ CDM[18–25]. The entropic-force terms are included both in Friedmann equation and acceleration equation in this type. The second type is BV type, which is similar to bulk viscous models[26–41]. In this type, the entropic-force terms appear only in acceleration equation.

The entropic cosmology model, which is different from dark energy, is another method to explain the accelerated expansion of the universe. Therefore, it is necessary to study the dynamical properties of entropic cosmology model.

However, this does not mean that dark energy is not necessary in entropic cosmology model. In addition to explaining the accelerated expansion of the universe, dark energy is an indispensable component of the universe. Measurements of anisotropy of cosmology microwave background indicate that, the space of universe is flat, which means that $\Omega_0 \approx 1$. Meanwhile, a large number of observational facts show that, the radiation and nonrelativistic matter (mainly baryons and nonbaryonic dark matter) only contribute about 30% of Ω_0 , which means that dark energy should contribute about 70% of Ω_0 . Therefore, in this paper, we consider the entropic cosmology model with dark energy. We introduce a scalar field in two types of entropic cosmology model and discuss the influence of different entropic-force terms on the dynamical properties of autonomous system in entropic cosmology.

The paper is organized as follows: In Sec.II we sum up on the dynamical equations of entropic cosmology, in which we introduce a scalar field ϕ . We discuss the autonomous systems in $\Lambda(t)$ type model in Sec.III and BV type in Sec.IV. The summary and discussion is in Sec.V.

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II. DYNAMICAL SYSTEM IN ENTROPIC COSMOLOGY

The Friedmann equation of a homogeneous, isotropic, spatially flat entropic cosmology is [17]

$$\frac{\dot{a}}{a} = H^2 = \frac{\kappa^2}{3}\rho(t) + f(t) \quad (1)$$

and the acceleration equation is [17]

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{\kappa^2}{6}[\rho(t) + p(t)] + g(t) \quad (2)$$

where $\kappa^2 = 8\pi G$, and G is gravitational constant. H is Hubble parameter and defined by

$$H(t) \equiv \frac{da(t)/dt}{a(t)} = \frac{\dot{a}(t)}{a(t)} \quad (3)$$

$f(t)$ and $g(t)$ are entropic-force terms. In general model of entropic cosmology, the form of entropic-force terms are [17]

$$f(t) = \alpha_1 H^2 + \alpha_3 H_0 H + \alpha_4 H_0^2 \quad (4)$$

$$g(t) = \beta_1 H^2 + \beta_3 H_0 H + \beta_4 H_0^2 \quad (5)$$

where H_0 is the Hubble parameter at the present time.

The coefficients $\alpha_1, \alpha_3, \alpha_4$ and $\beta_1, \beta_3, \beta_4$ are dimensionless constants. In [10, 12], it was argued that the value of coefficients is likely

$$\alpha_1 = \beta_1 = \frac{3}{2\pi} \quad (6)$$

In this paper, we do not restrict the value of this coefficients, i.e. we discuss the general case of entropic cosmology.

One can obtain the continuity equation from Eq.(1,2) [17, 42]

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = Q \quad (7)$$

where

$$Q = \frac{6}{\kappa^2} H \left[-f(t) - \frac{\dot{f}(t)}{2H} + g(t) \right] \quad (8)$$

Q can be seen as the energy exchange between different component of universe [17, 43]. We consider the cosmology model which is filled by some positive pressure matter with density ρ_m and a scalar field ϕ with exponential form potential $V = V_0 \exp(-\lambda\kappa\phi)$.

It should be noted that the exponential form potential itself can be used to explain the accelerating expansion of the universe. However, we are still using the exponential form potential in entropic cosmology model based on the following two reasons: First, such potentials arise naturally in supergravity models [44], and it play an important role in driving a period of early inflation [45], ekpyrotic or pre big bang collapse [46]. Second: The

purpose of this paper is to study the influence of various entropy-force terms on the dynamical properties of the universe. Meanwhile, the exponential form potential is one of the most widely used in the cosmological model. Therefore the using of such potentials in entropic cosmology model makes it convenient for us to compare the conclusions in this paper with the conclusions of other cosmology models.

To simplify the discussion, we restrict the value of λ to positive. So we have

$$\rho = \rho_m + \rho_\phi = \rho_m + \frac{\dot{\phi}^2}{2} + V \quad (9)$$

$$p = p_m + p_\phi = p_m + \frac{\dot{\phi}^2}{2} - V \quad (10)$$

and the continuity equation of ϕ is

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = \mathcal{Z}Q \quad (11)$$

we introduce a parameter $\mathcal{Z}$ to denote the different cases of energy exchange between scalar field and the matter. When $\mathcal{Z} = 1$, there is energy exchange. When $\mathcal{Z} = 0$, there is no energy exchange.

Eq.(11) lead to the K-G equation

$$\ddot{\phi} + 3H\dot{\phi} - \lambda\kappa V = \mathcal{Z}\frac{Q}{\dot{\phi}} \quad (12)$$

Insert Eq.(9,10) to Eq.(1,2), we have

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \frac{\dot{\phi}^2}{2} + V) + f(t) \quad (13)$$

$$\dot{H} = -\frac{\kappa^2}{2}(\gamma\rho_m + \dot{\phi}^2) + g(t) - f(t) \quad (14)$$

where we use the state equation of matter $p_m = w\rho_m$ and set $\gamma = 1 + w$. γ is adiabatic index, it satisfies $0 < \gamma < 2$. Its value is 4/3 when there is radiation, and is 1 for standard matter or dark matter.

A more general situation of γ is $0 \leq \gamma \leq 2$ [47]. If $\gamma = 0$ or $\gamma = 2$, the eigenvalues corresponding to some fixed points will have zero elements. To analyze the stability of these points, we need more complex methods just as we do in the Appendix. In this paper, we just consider the barotropic fluid which includes radiation and dust, and $\gamma \neq 0, 2$.

Eq.(12,13,14) characterize an autonomous system. We will analyze the dynamical behavior of this system in two type models of entropic cosmology.

III. $\Lambda(t)$ TYPE

A. Basic equations

In $\Lambda(t)$ type model of entropic cosmology, $f(t) = g(t)$ [17]. Therefore the Eq.(12,13,14) change to

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \frac{\dot{\phi}^2}{2} + V) + f(t) \quad (15)$$

$$\ddot{\phi} + 3H\dot{\phi} - \lambda\kappa V = -\mathcal{Z} \frac{3}{\kappa^2\dot{\phi}} \dot{f}(t) \quad (16)$$

$$\dot{H} = -\frac{\kappa^2}{2}(\gamma\rho_m + \dot{\phi}^2) \quad (17)$$

To rewrite the Eq.(15,16,17), we introduce following dimensionless variables: [48–50]

$$x^2 = \frac{\kappa^2\dot{\phi}^2}{6H^2}, \quad y^2 = \frac{\kappa^2 V}{3H^2}, \quad z = \frac{H_0}{H_0 + H} \quad (18)$$

The variable z ensures that the physical phasespace is compact. The Hubble parameter $H \rightarrow 0$ in the early time universe and $H \rightarrow \infty$ for the late time universe. Therefore [50]

$$z = \begin{cases} 1 & (H \rightarrow 0) \\ 1/2 & (H = H_0) \\ 0 & (H \rightarrow \infty) \end{cases} \quad (19)$$

Using these new variables, one can obtain

$$\frac{\kappa^2\rho_m}{3H^2} = 1 - \alpha_1 - x^2 - y^2 - \alpha_3 \frac{z}{1-z} - \alpha_4 \left(\frac{z}{1-z} \right)^2 \quad (20)$$

Using the new variables, and considering Eq.(20), one can rewrite Eq.(15,16,17) in the following forms:

$$\begin{aligned} \frac{dx}{dN} = & -3x + \sqrt{\frac{3}{2}}\lambda y^2 + \frac{3}{2} \left\{ \gamma \left[1 - \alpha_1 - x^2 - y^2 - \alpha_3 \frac{z}{1-z} - \alpha_4 \left(\frac{z}{1-z} \right)^2 \right] + 2x^2 \right\} \\ & \times \left\{ x + \frac{\mathcal{Z}}{x} \left[\alpha_1 + \alpha_3 \frac{z}{2(1-z)} \right] \right\} \end{aligned} \quad (21)$$

$$\frac{dy}{dN} = -\sqrt{\frac{3}{2}}\lambda xy + \frac{3}{2}y \left\{ \gamma \left[1 - \alpha_1 - x^2 - y^2 - \alpha_3 \frac{z}{1-z} - \alpha_4 \left(\frac{z}{1-z} \right)^2 \right] + 2x^2 \right\} \quad (22)$$

$$\frac{dz}{dN} = \frac{3}{2}z(1-z) \left\{ \gamma \left[1 - \alpha_1 - x^2 - y^2 - \alpha_3 \frac{z}{1-z} - \alpha_4 \left(\frac{z}{1-z} \right)^2 \right] + 2x^2 \right\} \quad (23)$$

where $N \equiv \ln a$.

If we selected the suitable value for $\lambda, \gamma, \alpha_1, \alpha_3, \alpha_4$, the right-hand of Eq.(21,22,23) only depend on x, y, z , but not on N . Such a dynamical system is usually called an autonomous system. Inspection of the dynamical equations shows that system Eq.(21,22,23) is invariant under the transformation $y \mapsto -y$. This implies that we can restrict our analysis on the half phase space with $y > 0$.

We consider two different cases of energy exchange, i.e. $\mathcal{Z} = 1$ and $\mathcal{Z} = 0$. The fixed point of this autonomous system are shown in Table I for $\mathcal{Z} = 1$ and Table II for $\mathcal{Z} = 0$.

where

$$x_{\Lambda_2^\pm} = \frac{\lambda^2 + 3\gamma \pm \sqrt{(\lambda^2 - 3\gamma)^2 + 12\alpha_1\lambda^2\gamma}}{2\sqrt{6}\lambda} \quad (24)$$

$$y_{\Lambda_2^\pm} = \left[\left(\frac{2}{\gamma} - 1 \right) \left(x_{\Lambda_2^\pm} \right)^2 - \sqrt{\frac{2}{3}} \frac{\lambda}{\gamma} x_{\Lambda_2^\pm} + 1 - \alpha_1 \right]^{1/2} \quad (25)$$

$$z_{\Lambda_3} = \frac{2 - 2\alpha_1 + \alpha_3 - \sqrt{\alpha_3^2 + 4\alpha_4 - 4\alpha_1\alpha_4}}{2(1 - \alpha_1 + \alpha_3 - \alpha_4)} \quad (26)$$

There are some remarks on Table I and II:

TABLE I. Fixed point for $\mathcal{Z} = 1$ in $\Lambda(t)$ -type

Fixed point	x_f	y_f	z_f	Existence
Γ_1^+	$\sqrt{\frac{\alpha_1\gamma}{2-\gamma}}$	0	0	$\forall \lambda$ and γ
Γ_1^-	$-\sqrt{\frac{\alpha_1\gamma}{2-\gamma}}$	0	0	$\forall \lambda$ and γ
Γ_2^+	$\sqrt{1-\alpha_1}$	0	0	$0 \leq \alpha_1 \leq 1$
Γ_2^-	$-\sqrt{1-\alpha_1}$	0	0	$0 \leq \alpha_1 \leq 1$
Γ_3	$\lambda/\sqrt{6}$	$\sqrt{1-\alpha_1-\frac{\lambda^2}{6}}$	0	$\lambda^2 < 6(1-\alpha_1)$
Γ_4	$\sqrt{\frac{3}{2}} \frac{\gamma}{\lambda}$	$\sqrt{\frac{3(2-\gamma)\gamma-2\alpha_1\lambda^2}{2\lambda^2}}$	0	$\lambda^2 > 3\gamma$

- When $\mathcal{Z} = 1$, the fixed points are all in $x-y$ plane. If the value of λ and γ have been selected, the position of fixed points depend on α_1 only. In other words, the value of α_3 and α_4 dose not affect the position of fixed points.
- When $\mathcal{Z} = 0$, the fixed points are in $x-y$ plane except Λ_3 . The positon of all the fixed points in $x-y$ plane depend only on α_1 like in $\mathcal{Z} = 1$ case.

TABLE II. Fixed point for $\mathcal{Z} = 0$ in $\Lambda(t)$ -type

Fixed point	x_f	y_f	z_f	Existence
Λ_0	0	0	0	$\forall \lambda$ and γ
Λ_1^+	$\sqrt{1 + \frac{\alpha_1 \gamma}{2 - \gamma}}$	0	0	$\forall \lambda$ and γ
Λ_1^-	$-\sqrt{1 + \frac{\alpha_1 \gamma}{2 - \gamma}}$	0	0	$\forall \lambda$ and γ
Λ_2^+	$x_{\Lambda_2^+}$	$y_{\Lambda_2^+}$	0	$(\lambda, \gamma) \in \{y_{\Lambda_2^+} \geq 0\}$
Λ_2^-	$x_{\Lambda_2^-}$	$y_{\Lambda_2^-}$	0	$(\lambda, \gamma) \in \{y_{\Lambda_2^-} \geq 0\}$
Λ_3	0	0	z_{Λ_3}	$\forall \lambda$ and γ

The influences of α_3 and α_4 on fixed points is only reflected in the position of Λ_3 .

- For C-version ($\alpha_1 = \alpha_3 = 0$) and H-version ($\alpha_1 = \alpha_4 = 0$) of $\Lambda(t)$ -type, $\Gamma_1^\pm$ are origin of phase space, and $\Gamma_2^\pm$, Γ_3 , Γ_4 are same as in normal cosmology. It means that, when $\mathcal{Z} = 1$ the appearance of H and constant terms does not affect the position of fixed points.
- Λ_3 is $(0, 0, \frac{1}{1+\sqrt{\alpha_4}})$ for C-version and is $(0, 0, \frac{1}{1+\alpha_3})$ for H-version.

B. Stability of fixed points

For simplicity, we define $\frac{dx}{dN} = F_1(x, y, z)$, $\frac{dy}{dN} = F_2(x, y, z)$ and $\frac{dz}{dN} = F_3(x, y, z)$. The properties of each fixed points are determined by the eigenvalues of the Jacobian matrix[51, 52]

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix} \bigg|_{(x_f, y_f, z_f)} \quad (27)$$

We will discuss the eigenvalues of J in two different cases of energy exchange, i.e. $\mathcal{Z} = 1$ and $\mathcal{Z} = 0$.

1. $\mathcal{Z} = 1$

Eigenvalues of J for $\mathcal{Z} = 1$ in $\Lambda(t)$ type shown in Table III.

where

$$\alpha_1^* = 1 - \alpha_1 \quad (28)$$

$$\mathcal{M} = -3\lambda\gamma(2 - \gamma) - 2\alpha_1\lambda(\lambda^2 - 6\gamma) \quad (29)$$

$$\begin{aligned} \mathcal{N} &= \{9(2 - \gamma)\gamma^2[\lambda^2(2 - 9\lambda) + 24\gamma^2] + 4\alpha_1^2\lambda^2(\lambda^2 - 6\gamma)^2 \\ &\quad + 12\alpha_1\lambda^2\gamma[\lambda^2(2 + 3\gamma) - 6\lambda(2 + \gamma)]\}^{1/2} \\ &= \sqrt{\mathcal{M}^2 + \mathcal{R}} \end{aligned} \quad (30)$$

and

$$\begin{aligned} \mathcal{R} &= 9(2 - \gamma)\gamma^2(\lambda^2\gamma + 24\gamma^2 - 9\lambda^3) + 24\alpha_1\lambda^2\gamma[2\lambda^2\gamma \\ &\quad + 3\lambda(2 + \gamma) + 3\gamma(2 - \gamma)] \end{aligned} \quad (31)$$

TABLE III. Eigenvalues of J for $\mathcal{Z} = 1$ in $\Lambda(t)$ -type

Fixed point	Eigenvalues
Γ_1^+	$\left\{ \frac{3\gamma}{2}, \frac{1}{2} \left(3\gamma - \sqrt{6\lambda} \sqrt{\frac{\alpha_1}{2-\gamma}} \right), 3\gamma - 6\alpha_1^* \right\}$
Γ_1^-	$\left\{ \frac{3\gamma}{2}, \frac{1}{2} \left(3\gamma + \sqrt{6\lambda} \sqrt{\frac{\alpha_1}{2-\gamma}} \right), 3\gamma - 6\alpha_1^* \right\}$
Γ_2^+	$\left\{ 3\alpha_1^*, 3\alpha_1^* - \sqrt{\frac{3}{2}} \sqrt{\alpha_1^*} \lambda, 6\alpha_1^* - 3\gamma \right\}$
Γ_2^-	$\left\{ 3\alpha_1^*, 3\alpha_1^* + \sqrt{\frac{3}{2}} \sqrt{\alpha_1^*} \lambda, 6\alpha_1^* - 3\gamma \right\}$
Γ_3	$\left\{ \frac{1}{2}(\lambda^2 - 6\alpha_1^*), \lambda^2 - 3\gamma, \frac{\lambda^2}{2} \right\}$
Γ_4	$\left\{ \frac{3\gamma}{2}, \frac{\mathcal{M} - \mathcal{N}}{4\lambda\gamma}, \frac{\mathcal{M} + \mathcal{N}}{4\lambda\gamma} \right\}$

We can conclude that:

- (1) If $\lambda^2\alpha_1 > 3/2$, then $3\gamma - \sqrt{6\lambda} \sqrt{\frac{\alpha_1}{2-\gamma}} < 0$ for all $0 < \gamma < 2$. So Γ_1^+ is a saddle fixed point, its stability property does not depend on the γ .
- (2) If $\lambda^2\alpha_1 < 3/2$, then the stability property of Γ_1^+ is depend on the value of γ . It may be noted that, when $1 - \sqrt{1 - \frac{2}{3}\lambda^2\alpha_1} < \gamma < 1 + \sqrt{1 - \frac{2}{3}\lambda^2\alpha_1}$, $3\gamma - \sqrt{6\lambda} \sqrt{\frac{\alpha_1}{2-\gamma}} > 0$. On the other hand, when $\gamma > 2(1 - \alpha_1)$, $3\gamma - 6\alpha_1^* > 0$. So, when $\lambda^2\alpha_1 < 3/2$ and $\gamma \in \mathcal{P} := (1 - \sqrt{1 - \frac{2}{3}\lambda^2\alpha_1}, 1 + \sqrt{1 - \frac{2}{3}\lambda^2\alpha_1}) \cap (2\alpha_1^*, 2)$, Γ_1^+ is a unstable fixed point. If $0 < \alpha_1 < 1/2$, then $\mathcal{P} = (2\alpha_1^*, 1 + \sqrt{1 - \frac{2}{3}\lambda^2\alpha_1})$. If $\alpha_1 > 1$, then $\mathcal{P} = (1 - \sqrt{1 - \frac{2}{3}\lambda^2\alpha_1}, 1 + \sqrt{1 - \frac{2}{3}\lambda^2\alpha_1})$.
- (3) If $\alpha_1 > 1$, then $3\gamma - 6\alpha_1^*$ is always positive. So Γ_1^- is always an unstable fixed point. Its stability property does not depend on γ and λ . If, $0 < \alpha_1 < 1$, then the stability of Γ_1^- is depend on γ . When $\gamma \in (2\alpha_1^*, 2)$, Γ_1^- is unstable fixed point. When $\gamma \in (0, 2\alpha_1^*)$, Γ_1^- is saddle fixed point. One can notice that, the form of potential (the value of λ) will not affect the stability of Γ_1^- .
- (4) If $0 < \alpha_1 < 1$, and $\lambda^2 > \max[6\alpha_1^*, 3\gamma]$, Γ_3 will be a unstable fixed point. Otherwise, Γ_3 will be a saddle fixed point. However, the appearing condition for Γ_3 is $\lambda^2 < 6\alpha_1^*$. So if $6\alpha_1^* > 3\gamma$, i.e. $\alpha_1 < 1 - 3\gamma/2$, Γ_3 is always a saddle fixed point, and if $6\alpha_1^* > 3\gamma$, i.e. $1 - 3\gamma/2 < \alpha_1 < 1$, Γ is unstable when $\lambda^2 > 3\gamma$, and Γ is saddle when $\lambda^2 < 3\gamma$.
- (5) If $\alpha_1 > 1$, and $\lambda^2 > 3\gamma$, Γ_3 will be a unstable fixed point. Otherwise, Γ_3 will be a saddle fixed point.
- (6) If $\mathcal{M} > \mathcal{N}$, then Γ_4 is a unstable fixed point. From Eq.(30) we can see that, $\mathcal{N} \geq 0$. It means that $\mathcal{M}$ should be positive, and $\mathcal{R}$ should be negative. From Fig.1, we can see that, there is no intersection between positive $\mathcal{M}$ zone and negative $\mathcal{R}$ zone. Therefore, Γ_4 is a saddle fixed point always.

TABLE IV. Eigenvalues of J for $\mathcal{Z} = 0$ in $\Lambda(t)$ -type

Fixed point	Eigenvalues
Λ_0	$\left\{ \frac{3\gamma}{2}\alpha_1^*, \frac{3\gamma}{2}\alpha_1^*, \frac{3\gamma}{2}\alpha_1^* - 3 \right\}$
Λ_1^+	$\left\{ 3, 6 - 3\alpha_1^*\gamma, 3 - \lambda\sqrt{\frac{3}{2}\frac{2-\gamma+\alpha_1\gamma}{2-\gamma}} \right\}$
Λ_1^-	$\left\{ 3, 6 - 3\alpha_1^*\gamma, 3 + \lambda\sqrt{\frac{3}{2}\frac{2-\gamma+\alpha_1\gamma}{2-\gamma}} \right\}$
$\Lambda_2^+(\alpha_1 = 0)$	$\left\{ \frac{1}{2}(\lambda^2 - 6), \lambda^2 - 3\gamma, \lambda^2/2 \right\}$
$\Lambda_2^-(\alpha_1 = 0)$	$\left\{ \frac{3\gamma}{2}, \frac{\mathcal{M}_0 - \mathcal{N}_0}{4\lambda\gamma}, \frac{\mathcal{M}_0 + \mathcal{N}_0}{4\lambda\gamma} \right\}$
Λ_3^C	$\{0, -3, -3\gamma\}$
Λ_3^H	$\{0, -3, -\frac{3\gamma}{2}\}$

2. $\mathcal{Z} = 0$

Eigenvalues of J for $\mathcal{Z} = 0$ in $\Lambda(t)$ -type shown in Table IV. where we consider the special case of $\alpha_1 = 0$ for $\Lambda_2^\pm$ and

$$\mathcal{M}_0 = -3\gamma\lambda(2 - \gamma) \quad (32)$$

$$\mathcal{N}_0 = 3\gamma\sqrt{(2 - \gamma)[\lambda^2(2 - 9\lambda) + 24\gamma^2]} \quad (33)$$

The superscript "C" and "H" of Λ_3 denote the C-version and H-version. We can conclude that:

- (1) For $\gamma \in (0, 2)$, $\frac{3\gamma}{2}\alpha_1^* - 3$ is always negative, therefore the stability of Λ_0 depend only on α_1 . If $\alpha_1 \in (0, 1)$, Λ_0 is a saddle fixed point. If $\alpha_1 > 1$, Λ_0 is an stable fixed point, it is a late time attractor.
- (2) For $\gamma \in (0, 2)$, $6 - 3\alpha_1^*\gamma$ is always positive. If $\lambda^2 < \frac{6(2-\gamma)}{2-\gamma+\alpha_1\gamma}$, Λ_1^+ is an unstable fixed point. If $\lambda^2 > \frac{6(2-\gamma)}{2-\gamma+\alpha_1\gamma}$, Λ_1^+ is an saddle fixed point.
- (3) Λ_1^- is always unstable fixed point, it's stability does not depend on the value of $\lambda, \gamma, \alpha_1$.
- (4) If $\lambda^2 > 6$, then $\lambda^2 - 3\gamma$ is always positive and Λ_2^+ is an unstable fixed point. If $\lambda^2 < 6$, Λ_2^+ is a saddle fixed point.

FIG. 1. The parameter space for Γ_4 is unstable fixed point

(5) $\mathcal{M}_0 - \mathcal{N}_0$ is always negative. Therefore Λ_2^- is always a saddle fixed point.

(6) Whether in C-version or H-version, there is an zero eigenvalue for J of Λ_3 . It means that Λ_3 is a non-hyperbolic point. Linear stability theory fails for non-hyperbolic points, and the stability property for this point should be determined by centre manifold theory [51, 52]. We will get the stability property of Λ_3 in Appendix by using centre manifold theory. We can conclude from Appendix that: Λ_3 is a stable fixed point.

From the above discussion, we find that, the stability properties of fixed points depend on the values of the constants λ, γ and α_1 . It is worth noting that, the values of α_3 and α_4 only affect the position of Λ_3 . This means the H^2 terms (area entropy) decide the dynamical properties of the dynamical system Eq.(21,22,23) when λ and γ have been selected.

Therefore, it can be expected that the dynamical properties in C-version and H-version of entropic cosmology model (where $\alpha_1 = 0$) are similar.

IV. BV-TYPE

In BV type model of entropic cosmology, $f(t) = 0$. Therefore, Eq.(12,13,14) change to

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \frac{\dot{\phi}^2}{2} + V) \quad (34)$$

$$\ddot{\phi} + 3H\dot{\phi} - \lambda\kappa V = \mathcal{Z} \frac{6H}{\kappa^2\dot{\phi}} g(t) \quad (35)$$

$$\dot{H} = -\frac{\kappa^2}{2}(\gamma\rho_m + \dot{\phi}^2) + g(t) \quad (36)$$

As in Section III, we introduce dimensionless variables Eq.(18). From Eq.(34), one can obtain

$$\frac{\kappa^2\rho_m}{3H^2} = 1 - x^2 - y^2 \geq 0 \quad (37)$$

This lead to $0 \leq x^2 + y^2 \leq 1$, which means the phase-space corresponds to a half-cylinder of unit height and unit radius.

Using the new variables and considering Eq.(37), one can rewrite Eq.(34,35,36) into following forms:

$$\frac{dx}{dN} = -3x + \sqrt{\frac{3}{2}}\lambda y^2 + \frac{3}{2}x[\gamma(1-x^2-y^2) + 2x^2] + \left[\frac{\mathcal{Z}}{x} - x\right] \left[\beta_1 + \beta_3 \frac{z}{1-z} + \beta_4 \left(\frac{z}{1-z}\right)^2\right] \quad (38)$$

$$\frac{dy}{dN} = -\sqrt{\frac{3}{2}}\lambda xy + \frac{3}{2}y[\gamma(1-x^2-y^2) + 2x^2] - y \left[\beta_1 + \beta_3 \frac{z}{1-z} + \beta_4 \left(\frac{z}{1-z}\right)^2\right] \quad (39)$$

$$\frac{dz}{dN} = z(1-z) \left\{ \frac{3}{2}[\gamma(1-x^2-y^2) + 2x^2] - \left[\beta_1 + \beta_3 \frac{z}{1-z} + \beta_4 \left(\frac{z}{1-z}\right)^2\right] \right\} \quad (40)$$

TABLE V. Fixed point for $\mathcal{Z} = 1$ in BV-type

Fixed point	x_f	y_f	z_f	Existence
Ω_1^+	1	0	0	$\forall \lambda$ and γ
Ω_1^-	-1	0	0	$\forall \lambda$ and γ
Ω_2	$\lambda/\sqrt{6}$	$\sqrt{1-\frac{\lambda^2}{6}}$	0	$\lambda^2 < 6$
Ω_3	$\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}$	$\sqrt{\frac{3(2-\gamma)\gamma}{2\lambda^2}}$	0	$\lambda^2 > 3\gamma$
Ω_4^+	1	0	z_{Ω_4}	$\forall \lambda$ and γ
Ω_4^-	-1	0	z_{Ω_4}	$\forall \lambda$ and γ

TABLE VI. Fixed point for $\mathcal{Z} = 0$ in BV-type

Fixed point	x_f	y_f	z_f	Existence
Π_0	0	0	0	$\forall \lambda$ and γ
Π_1^+	1	0	0	$\forall \lambda$ and γ
Π_1^-	-1	0	0	$\forall \lambda$ and γ
Π_2	$\lambda/\sqrt{6}$	$\sqrt{1-\frac{\lambda^2}{6}}$	0	$\lambda^2 < 6$
Π_3	$\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}$	$\sqrt{\frac{3(2-\gamma)\gamma}{2\lambda^2}}$	0	$\lambda^2 > 3\gamma$
Π_4	0	0	z_{Π_4}	$\forall \lambda$ and γ

Eq.(38,39,40) form an autonomous system.

To discuss the H terms and the constant terms in dynamical system, we consider the H-version and C-version of the entropic cosmology, i.e. we set $\beta_1 = 0$.

The fixed points of this autonomous system shown in Table V for $\mathcal{Z} = 1$ and Table VI for $\mathcal{Z} = 0$. where

$$z_{\Omega_4} = \begin{cases} \frac{3}{3+\sqrt{3}\beta_4} & \text{C-version} \\ \frac{3}{3+\beta_3} & \text{H-version} \end{cases} \quad (41)$$

and

$$z_{\Pi_4} = \begin{cases} \frac{3\gamma}{3\gamma+\sqrt{6}\gamma\beta_4} & \text{C-version} \\ \frac{3\gamma}{3\gamma+2\beta_3} & \text{H-version} \end{cases} \quad (42)$$

The eigenvalues of Jacobian matrix J are shown in Table VII where

$$\sigma = \begin{cases} 6 & \text{C-version} \\ 3 & \text{H-version} \end{cases} \quad \text{and} \quad \theta = \begin{cases} 3\gamma & \text{C-version} \\ 3\gamma/2 & \text{H-version} \end{cases} \quad (43)$$

We can conclude that:

TABLE VII. Eigenvalues of J for BV-type

Fixed point(s)	Eigenvalues
Π_0	$\{\frac{3}{2}(\gamma-2), \frac{3}{2}\gamma, \frac{3}{2}\gamma\}$
$\Omega_1^\pm, \Pi_1^\pm$	$\{3(2-\gamma), 3 \mp \sqrt{\frac{3}{2}}\lambda, 3\}$
Ω_2, Π_2	$\{\lambda^2/2, (\lambda^2-6)/2, \lambda^2-3\gamma\}$
Ω_3, Π_3	$\{\frac{3\gamma}{2}, \frac{\mathcal{M}_0-\mathcal{N}_0}{4\lambda\gamma}, \frac{\mathcal{M}_0+\mathcal{N}_0}{4\lambda\gamma}\}$
$\Omega_4^\pm$	$\{-\sigma, -3\gamma, \mp\sqrt{\frac{3}{2}}\lambda\}$
Π_4	$\{-3, 0, -\theta\}$

- (1) For $\gamma \in (0, 2)$, Π_0 is always an saddle fixed point. The stability properties of the Π_0 is same as Λ_0 when $\alpha_1 = 0$.
- (2) For $\gamma \in (0, 2)$, Ω_1^- and Π_1^- are always unstable fixed point.
- (3) If $\lambda^2 < 6$, then $3 - \sqrt{\frac{3}{2}}\lambda > 0$. It means that Ω_1^+ and Π_1^+ are unstable fixed point. If $\lambda^2 > 6$, then Ω_1^+ and Π_1^+ are saddle fixed point.
- (4) The existence of Ω_2 and Π_2 is $\lambda^2 < 6$, which means that Ω_2 and Π_2 are saddle fixed point.
- (5) For $\gamma \in (0, 2)$, Ω_3 and Π_3 are saddle point.
- (6) Ω_4^+ is always a stable fixed point, and Ω_4^- is always a saddle fixed point.
- (7) Π_4 is a non-hyperbolic point, its stability property is determined by centre manifold theory. We discuss the stability of Π_4 in Appendix, and we can conclude that Π_4 is a stable fixed point.

From the discussion of this section, we can see that, the dynamical structure of H-version and C-version are same. It mean that, H terms and the constant terms have the same influence on dynamical properties.

V. SUMMARY AND DISCUSSION

In this paper, we discuss the stability properties of autonomous system in entropic cosmology. It can be found

that, different entropic terms have different effect on the stability properties of fixed points.

The H^2 terms do not describe cosmological fluctuation and is neglected in some discussions[17]. However, the influences of H^2 terms on dynamical behavior of ϕ are important.

From the discussion of $\Lambda(t)$ type in Sec.II, one can find that, α_3 and α_4 only affect the position of fixed point Λ_3 (see Eq.26). Furthermore, the stability property of Λ_3 is decided by α_1 . If we neglect the H^2 term, i.e. $\alpha_1 = 0$, the fixed point Λ_3 is always a stable point, its stability property has nothing to do with H term and constant term.

Therefore, it is not surprising that the eigenvalues of J in C-version and H-version of BV type model are independent with β_3 and β_4 (see VII). H terms and constant terms also have no influence on stability property of fixed points.

On the other hand, from the discussion of BV type model in Sec.III, we can see that there is no essential difference between dynamical properties of H-version and C-version. The only difference between two versions of model is the position of fixed points $\Omega_4^\pm$ and Π_4 .

In summary, from the perspective of dynamics, the entropic-force terms can be divided into two classes. The first class of entropic-force terms is proportional to H^2 (area entropy), they affect the dynamical properties of autonomous system in entropic cosmology. The second class is the linear combination of H terms (volume entropy) and constant terms, they only affect the position of fixed points, they have no influences on dynamical structure.

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Appendix A: The stability properties of the non-hyperbolic points Λ_3 and Π_4

As we have point out in Section III and IV, if the eigenvalues of the Jacobi matrix had one or more eigenvalues with zero real parts while the rest of the eigenvalues had negative real parts, then linearization failed to determine the stability properties of this critical point [51, 52]. Λ_3 and Π_4 are such points. In this Appendix, we determine the stability properties of these points by using centre manifold theory [52].

1. Λ_3

We transfer Λ_3 to $\Lambda'_3(x_f = 0, y_f = 0, z_f^* = z_f - z_{\Lambda_3} = 0)$ for convenience at first, where

$$z_{\Lambda_3} = \begin{cases} \frac{1}{\sqrt{\alpha_4+1}} & \text{C-version} \\ \frac{1}{\alpha_3+1} & \text{H-version} \end{cases} \quad (\text{A1})$$

Then Eq.(21,22,23) can be rewritten as

$$\frac{dx}{dN} = \frac{3}{2}x [\gamma(1 - x^2 - y^2 - \mathcal{T}(z^*)) + 2x^2] - 3x + \sqrt{\frac{3}{2}}\lambda y^2 \quad (\text{A2})$$

$$\frac{dy}{dN} = \frac{3}{2}y [\gamma(1 - x^2 - y^2 - \mathcal{T}(z^*)) + 2x^2] - \sqrt{\frac{3}{2}}\lambda xy \quad (\text{A3})$$

$$\frac{dz^*}{dN} = \frac{3}{2} [\gamma(1 - x^2 - y^2 - \mathcal{T}(z^*)) + 2x^2] \times (z^* + z_{\Lambda_3})(1 - z^* - z_{\Lambda_3}) \quad (\text{A4})$$

where

$$\mathcal{T}(z^*) = \begin{cases} \alpha_4 \left(\frac{z^* + z_{\Lambda_3}}{1 - z^* - z_{\Lambda_3}} \right)^2 & \text{C-version} \\ \alpha_3 \frac{z^* + z_{\Lambda_3}}{1 - z^* - z_{\Lambda_3}} & \text{H-version} \end{cases} \quad (\text{A5})$$

The Jacobi matrix J of dynamical system Eq.(A2,A3,A4) at point Λ'_3 is

$$J = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\theta \end{pmatrix} \quad (\text{A6})$$

where

$$\theta = \begin{cases} 3\gamma & \text{C-version} \\ 3\gamma/2 & \text{H-version} \end{cases} \quad (\text{A7})$$

The eigenvectors for eigenvalues $(-3, 0, -\theta)$ are

$$(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T \quad (\text{A8})$$

respectively.

The stable subspaces $\mathcal{E}^s$ is spanned by the eigenvectors of J which have negative real part. In the case of our discussion, $\mathcal{E}^s$ is $x - z^*$ plane. The centre subspaces $\mathcal{E}^c$ is spanned by the eigenvectors of J which have zero real part. In our case, $\mathcal{E}^c$ is y axis. Linear stability theory is sufficient to understand the dynamics of trajectories in $\mathcal{E}^s$, and the dynamics of trajectories in $\mathcal{E}^c$ is determined by centre manifold theory [52].

Therefore, we set centre manifold W^c is

$$W^c(\Lambda'_3) = \{(h_1(y), y, h_2(y)) : y \in [0, \infty)\} \quad (\text{A9})$$

where

$$h_1(0) = h_2(0) = \nabla h_1(0) = \nabla h_2(0) = 0 \quad (\text{A10})$$

Therefore, we can assume that

$$h_1(y) = a_1 y^2 + a_2 y^3 + \dots \quad (\text{A11})$$

$$h_2(y) = a_3 y^2 + a_4 y^3 + \dots \quad (\text{A12})$$

The vector $(\frac{dx}{dN}, \frac{dy}{dN}, \frac{dz}{dN})$ on a trajectory in W^c must be tangent to W^c . This means

$$x(N) = h_1(y(N)), \quad z^*(N) = h_2(y(N)) \quad (\text{A13})$$

i.e.

$$\frac{dx}{dN} = \nabla h_1(y) \frac{dy}{dN}, \quad \frac{dz^*}{dN} = \nabla h_2(y) \frac{dy}{dN} \quad (\text{A14})$$

Substitute Eq.(A2,A3,A4) in Eq.(A14) and using Eq.(A11,A12):

$$\begin{aligned} & -3x + \sqrt{\frac{3}{2}} \lambda y^2 + \frac{3}{2} x [\gamma (1 - x^2 - y^2 - \mathcal{R}(z^*)) + 2x^2] \\ & = (2a_1 y + 3a_2 y^2) \left\{ -\sqrt{\frac{3}{2}} \lambda x y + \frac{3}{2} y [\gamma (1 - x^2 - y^2 - \mathcal{R}(z^*)) + 2x^2] \right\} \end{aligned} \quad (\text{A15})$$

$$\begin{aligned} & \frac{3}{2} [\gamma (1 - x^2 - y^2 - \mathcal{R}(z^*)) + 2x^2] (z^* + z_{\Lambda_3})(1 - z^* - z_{\Lambda_3}) \\ & = (2a_3 y + 3a_4 y^2) \left\{ -\sqrt{\frac{3}{2}} \lambda x y + \frac{3}{2} y [\gamma (1 - x^2 - y^2 - \mathcal{R}(z^*)) + 2x^2] \right\} \end{aligned} \quad (\text{A16})$$

Substitute $x = h_1(y), z^* = h_2(y)$ into Eq.(A15,A16) and notice that the coefficients of each power of y must be zero, we can solve that

$$a_1 = \frac{\lambda}{\sqrt{6}}, a_2 = a_4 = 0, a_3 = \begin{cases} \frac{\sqrt{\alpha_4}}{2(1+\sqrt{\alpha_4})^2} & \text{C-version} \\ -\frac{\alpha_3}{(1+\alpha_3)^2} & \text{H-version} \end{cases}$$

Therefore the dynamics equation for y is

$$\frac{dy}{dN} = -\frac{\lambda^2}{2} y^3 + O(y^4) \quad (\text{A17})$$

We conclude that, for y sufficiently small, $y = 0$ is stable, therefore Λ_3 is an stable fixed point.

2. Π_4

The discussion of the Π_4 is same as Λ_3 . We transfer Π_4 to $\Pi'_4(x_f = 0, y_f = 0, z_f^* = z_f - z_{\Pi_4} = 0)$. Then the Eq.(38,39,40) change to

$$\begin{aligned} \frac{dx}{dN} &= \frac{3}{2} x [\gamma (1 - x^2 - y^2) + 2x^2] - x \mathcal{S}(z^*) \\ & - 3x + \sqrt{\frac{3}{2}} \lambda y^2 \end{aligned} \quad (\text{A18})$$

$$\begin{aligned} \frac{dy}{dN} &= \frac{3}{2} y [\gamma (1 - x^2 - y^2) + 2x^2] - y \mathcal{S}(z^*) \\ & - \sqrt{\frac{3}{2}} \lambda x y \end{aligned} \quad (\text{A19})$$

$$\begin{aligned} \frac{dz^*}{dN} &= \left\{ \frac{3}{2} [\gamma (1 - x^2 - y^2) + 2x^2] - \mathcal{S}(z^*) \right\} \\ & \times (z^* + z_{\Pi_4})(1 - z^* - z_{\Pi_4}) \end{aligned} \quad (\text{A20})$$

where

$$\mathcal{S}(z^*) = \begin{cases} \beta_4 \left(\frac{z^* + z_{\Pi_4}}{1 - z^* - z_{\Pi_4}} \right)^2 & \text{C-version} \\ \beta_3 \frac{z^* + z_{\Pi_4}}{1 - z^* - z_{\Pi_4}} & \text{H-version} \end{cases} \quad (\text{A21})$$

The Jacobi matrix J of dynamical system Eq.(A18,A19,A20) at point Π'_4 is

$$J = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\theta \end{pmatrix} \quad (\text{A22})$$

The eigenvectors for eigenvalues $(-3, 0, -\theta)$ are

$$(1, 0, 0)^T, \quad (0, 1, 0)^T, \quad (0, 0, 1)^T \quad (\text{A23})$$

respectively.

We set centre manifold W^c is

$$W^c(\Lambda'_3) = \{(h'_1(y), y, h'_2(y)) : y \in [0, \infty)\} \quad (\text{A24})$$

where

$$h'_1(0) = h'_2(0) = \nabla h'_1(0) = \nabla h'_2(0) = 0 \quad (\text{A25})$$

Therefore, we can assume that

$$h'_1(y) = b_1 y^2 + b_2 y^3 + \dots \quad (\text{A26})$$

$$h'_2(y) = b_3 y^2 + b_4 y^3 + \dots \quad (\text{A27})$$

The vector $(\frac{dx}{dN}, \frac{dy}{dN}, \frac{dz^*}{dN})$ on a trajectory in W^c must be tangent to W^c . This means

$$x(N) = h'_1(y(N)), \quad z^*(N) = h'_2(y(N)) \quad (\text{A28})$$

i.e.

$$\frac{dx}{dN} = \nabla h'_1(y) \frac{dy}{dN}, \quad \frac{dz^*}{dN} = \nabla h'_2(y) \frac{dy}{dN} \quad (\text{A29})$$

Substitute Eq.(A18,A19,A20) in Eq.(A29) and using Eq.(A26,A27):

$$\begin{aligned} & \frac{3}{2}x[\gamma(1-x^2-y^2)+2x^2]-x\mathcal{S}(z^*)-3x+\sqrt{\frac{3}{2}}\lambda y^2 \\ &= (2b_1y+3b_2y^2)\left\{\frac{3}{2}y[\gamma(1-x^2-y^2)+2x^2]-y\mathcal{S}(z^*)-\sqrt{\frac{3}{2}}\lambda xy\right\} \end{aligned} \quad (\text{A30})$$

$$\begin{aligned} & \left\{\frac{3}{2}[\gamma(1-x^2-y^2)+2x^2]-\mathcal{S}(z^*)\right\}(z^*+z_{\Pi_4})(1-z^*-z_{\Pi_4}) \\ &= (2b_3y+3b_4y^2)\left\{\frac{3}{2}y[\gamma(1-x^2-y^2)+2x^2]-y\mathcal{S}(z^*)-\sqrt{\frac{3}{2}}\lambda xy\right\} \end{aligned} \quad (\text{A31})$$

Substitute $x = h'_1(y)$, $z^* = h'_2(y)$ into Eq.(A30,A31) and notice that the coefficients of each power of y must be zero, we can solve that

$$b_1 = \frac{\lambda}{\sqrt{6}}, b_2 = b_4 = 0, b_3 = \begin{cases} \frac{3\gamma\sqrt{6\gamma\beta_4}}{2(3\gamma+\sqrt{6\gamma\beta_4})^2} & \text{C-version} \\ -\frac{6\beta_3\gamma}{(3\gamma+2\beta_3)^2} & \text{H-version} \end{cases}$$

Therefore the dynamics equation for y is

$$\frac{dy}{dN} = -\frac{\lambda^2}{2}y^3 + O(y^4) \quad (\text{A32})$$

We conclude that, for y sufficiently small, $y = 0$ is stable, therefore Π_4 is an stable fixed point.

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