The Calculation of Electromagnetic Fields from an Arbitrary Source in a Horizontally Layered Earth

by

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Abstract

In order to explore sedimentary environments with electromagnetics techniques, a greater understanding of the benefits and problems with the available methods must be obtained. In this thesis, an efficient method of calculating the response of a horizontally layered earth to an electric or magnetic dipole source has been developed. The computational effort involved in calculating response of a large grounded or ungrounded source was significantly reduced by using the concept of lagged convolution, and a novel integration technique. The integration technique permits the evaluation of the fields due to any arbitrary source on the surface of a layered earth. The combination of the two techniques will hopefully reverse the present thinking that it is much too expensive to do routine interpretation of large source responses using generalized inversion schemes.

Since many rocks exhibit anisotropy and induced polarization, both effects are described in terms of the fields of a grounded electric bipole on the surface of a layered earth. Furthermore, the concept of considering a grounded source as two individual sources is used to explain the field responses to halfspace and layered earth models. In general, the galvanic part of the source is sensitive to anisotropy and IP, while the inductive part of the source is not.

The major advantage of a bipole source is that the fields due to subtle changes in conductivity can be discriminated. This makes a grounded bipole source suitable for EM exploration of sedimentary environments where various conductivity contrasts may exist between different layers in a stratigraphic section.
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List of Symbols

\begin{align*}
i & \quad \sqrt{-1} \\
\delta(z) & \quad \text{Dirac Delta function} \\
J_n(\lambda \rho) & \quad \text{Bessel function of the first kind, of order } n \\
p, q & \quad \text{rectangular Fourier wavenumbers in the } x \text{ and } y \text{ direction} \\
\lambda, \varphi & \quad \text{cylindrical Fourier wavenumbers in the } \rho \text{ and } \theta \text{ direction} \\
\rho & \quad \text{radial distance in the space domain } \rho = \sqrt{x^2 + y^2} \\
\theta & \quad \text{angle the receiver makes with the source} \\
\omega & \quad \text{angular frequency} \\
\varepsilon & \quad \text{electric permeability} \\
\mu & \quad \text{magnetic permittivity} \\
\alpha_h, \alpha_v & \quad \text{admittivity in horizontal and vertical directions } \alpha_h = \sigma_h + i\omega\varepsilon_h \\
\beta & \quad \text{the impedivity } \beta = i\omega\mu \\
\gamma^2 & \quad \text{propagation constant } \gamma^2 = \beta \alpha_h \\
\kappa & \quad \text{generalized notation for a kernel function} \\
g(\rho, \theta) & \quad \text{generalized notation for the geometrical factors associated} \\
& \quad \text{with a kernel function} \\
P & \quad \text{generalized notation for the vertical component of a Hertz potential} \\
R_P(i) & \quad \text{the } i^{th} \text{ reflection coefficient for the Hertz potential } P \\
K & \quad \text{coefficient of anisotropy } K = \alpha_h / \alpha_v \\
\tau & \quad \text{time constant} \\
\sigma_h, \sigma_v & \quad \text{horizontal and vertical conductivities} \\
\sigma_L, \sigma_H & \quad \text{conductivities at } \omega = 0 \text{ and } \omega = \infty \text{ respectively} \\
\phi & \quad \text{generalized notation for a local basis function}
\end{align*}
Vectors

$E$  Electric field vector
$H$  Magnetic field vector
$J$  Electric source current vector
$M$  Magnetic source current vector
$\Pi$  Electric Hertz potential vector
$\Gamma$  Magnetic Hertz potential vector

Mathematical Notation

$F$  vectors indicated by boldface type
$\vec{F}$  matrices indicated by underscoring
$\vec{\vec{F}}$  tensors indicated by double underscoring
$F_x, F_y, F_z$  spatial coordinates indicated by a coordinate subscript on the RHS of the symbol
$F_x, F_y, F_z$  partial derivatives with respect to coordinates indicated by a number on the LHS of the symbol (ie $F = \partial F/\partial x$)
$\hat{F}$  Fourier transform of $F$
$\tilde{F}$  Hankel transform of $F$
Chapter 1

Introduction

1.1 Electromagnetic Prospecting

Electromagnetic prospecting basically consists of introducing a man-made, low frequency electromagnetic field into the earth and mapping some aspects of this field over the area of interest. From an understanding of the behaviour of the fields, information about the electrical properties of the subsurface can be deduced from the measured data resulting in a geoelectric section. A geophysicist then may interpret the geoelectric section within geological constraints in order to deduce the geological structure of the subsurface.

Electromagnetic techniques can provide valuable information about a large variety of geological environments, but are particularly useful in the search for massive base metal mineralization which is often vastly more conductive than common mineralized rock. Base metal ores are often sought in areas of complex geological structure, although some occur in horizontally stratified environments. It is rapidly becoming clear that the versatility and depth penetration of EM methods might make them applicable to some problems of hydrocarbon exploration, and so the subtleties of the behaviour of EM fields in a stratified earth must be understood in some detail. This also implies that a greater emphasis must be placed on interpreting the electrical structure of the host geology from the field data instead of simply detecting anomalously conductive features and deducing their geometry and electrical properties.

This thesis addresses the problems of modelling electromagnetic fields in a horizon-
tally stratified earth of arbitrary structure. The general anisotropic layered earth problem is solved for grounded or ungrounded (electric or magnetic) dipole sources which may exist anywhere within the section. By taking a unified approach to the problem, a single algorithm has been developed which is suitable for numerical calculations in the frequency or time domain involving any practical source and stratification. Solutions for the layered earth problem are certainly not novel. The particular virtue of the one given here is its generality and computational efficiency.

A general anisotropic layered earth EM modelling algorithm is required for many different problems in electromagnetic interpretation. An example is integral equation modelling of an electrical inhomogeneity in a stratified host medium where sources are used to represent electric current flowing in a region of anomalous electrical conductivity and the layering is often used to represent overburden. In this thesis, the EM fields of a grounded electric dipole on a layered earth are calculated. Whereas magnetic (ungrounded) sources are most widely applied in base metal massive sulfide exploration, the dipole source is most practical for EM exploration of sedimentary environments. A dipole source consists of two electrodes in ohmic contact with the ground which are separated by distances of up to several kilometers. The electrodes are connected electrically by a straight wire which carries an alternating current. The advantage of this type of source is it can discriminate the effects of several different electrical structures.

One problem with modelling a dipole source (or any large source for that matter), is the number of calculations required to construct the source from elemental dipoles. Although the integration over the source region is not a problem when only a few field calculations are required, it can make the use of iterative model fitting procedures impractical. In order to improve the computational efficiency, a special integration technique (Integration by Weighted Sums - IWS) was developed which dramatically
reduces the time required for successive re-integration over the source. The fields from any large source (ie. loops or bipoles) can be calculated using IWS.

Rocks often display anisotropy which occurs when the properties of a medium depend on the direction in which they are measured. The electrical conductivity and magnetic permeability of rocks are often anisotropic and will be considered in some detail. When these properties are anisotropic, the admittivity \((\sigma + i\omega e)\) and impedance \((i\omega \mu)\) of the medium should be expressed as tensors. However, in the present case, the impedance will be assumed to have a scalar form since the relative permeability of most rocks and soil is generally very close to one.

The admittivity tensor has a symmetric matrix form and is diagonal if the principle axes of anisotropy are identical to the coordinate axes. The diagonal case is an important one. A sedimentation process will deposit elongated particles parallel to the bedding and lithification may promote preferential crystal growth, thus the conductivities which are parallel to the bedding can differ from conductivities which are perpendicular to the bedding. This is especially true when the interstitial fluid is very conductive since the layering of the particles will tend to create elongated pore spaces which are also parallel to the bedding. When the bedding is horizontal, the principle axes of anisotropy and the coordinate system axes will coincide and the tensor will be diagonal.

There are essentially two types of anisotropy; micro-anisotropy, which is determined by the fabric of the rock, and macro-anisotropy, which arises from the inability of the measuring system to resolve thin layers. An example of macro-anisotropy is sandstone which is often interbedded with thin layers of shale. A macro-anisotropic sequence may then be modelled with a single anisotropic layer when the thickness of the individual layers in the sequence is small compared with the resolution of the measuring system.
It is an important advantage to be able to incorporate anisotropy within the layered earth model in terms of computational efficiency since the alternative is to construct a macro-anisotropic layer from many thin layers. Anisotropy is also important if inverse theory is used to interpret the model, since an isotropic earth model cannot properly describe an anisotropic earth.

1.2 Previous Modelling Work

The formal solution for an electromagnetic source over a layered halfspace has been in the literature for many years (Sommerfeld (1909)). However, the difficulty in obtaining numerical data from the solution has only recently been overcome with the use of digital computers. The complexity of the problem is indicated by the fact that the vast majority of geophysical electromagnetic literature is devoted almost entirely to theory. Even the solution for excitation on a halfspace may only be expressed in closed form when the frequencies are low enough that the quasistatic assumption is valid (i.e. displacement currents in the air may be neglected when all distances are small compared to the free-space wavelength). At low frequencies (< 10^6 Hz) electromagnetic fields exhibit a diffusive character, and the assumption is quite accurate.

The origins of electromagnetic exploration with an electric dipole source are to be found in classical DC resistivity methods which date back to the turn of the century. Resistivity sounding has been performed on a routine basis for many years. Rather complete descriptions of resistivity methods can be found in Maillet (1947), Kunetz (1966) and Kocfield (1979), as well as several general geophysical texts. DC resistivity surveying is somewhat time consuming because of the number of readings required to
construct a depth-sounding curve, however, the simplicity of the interpretation and instrumentation still make it an attractive method for certain problems.

With the great improvement of portable electronic instrumentation in the last decade, it is not surprising that the extension to electromagnetic sounding is being attempted. Electromagnetic techniques take advantage of the possibility of measuring the fields at various frequencies at the same receiver position. Since the penetration of the EM fields is dependent upon the frequency, this is somewhat equivalent to measuring the DC field at a number of positions. Even more information can be obtained by measuring EM frequency response data at a number of receiver positions.

Many initial efforts at EM sounding in the Western hemisphere were directed towards the use of simple inductive sources, which to a good approximation acted as magnetic dipoles (eg. Patra and Mallick (1980)). The limitations of the magnetic dipole sources are that insufficient strength is generally achieved for kilometer scale depth penetration, and that resistive features cannot be detected. The strength problem is overcome by using a geometrically large loop (diameter comparable to distance to receiver) (Lamontagne (1975), Patra and Mallick (1980) and Hoversten and Morrison (1983)), but this makes the interpretation problem more complicated. The earliest regular use of EM sounding techniques was in the Soviet Union (Vanyan (1966)), where large inductive sources and grounded bipole sources have been studied extensively. Electric bipole sources for EM sounding have seen widespread use only in the last decade (Keller and Frischknecht (1966), Keller (1971), Duncan et al. 1980), and Gomez-Trevino and Edwards (1983)). The reason for the delay seems to have been partly a lack if suitable instrumentation and partly the lack of adequate modelling routines to permit reliable interpretation.

The first EM solutions for finite grounded wires on the surface of a stratified earth were provided by Foster (1931) and Riordan and Sunde (1933), but numerical
investigations of the fields about a grounded wire were not possible until the advent of digital computers. Much of the early work was limited to halfspace or two layer cases, and can be found in Sunde (1968). Sunde's nomenclature has been retained in the more recent grounded bipole solutions (Nabulsi and Wait (1982), Wait (1982) and Dey and Morrison (1973)), indicating the mathematical completeness of the early solutions for simple models. Recent efforts by Wait (1968,1982) and Dey and Morrison (1973) have extended the early solutions to the the case of general layering.

In the numerical aspect of the problem, the major advances that have been made are mostly in the area of evaluating the Bessel function integrals which occur invariably in layered earth problems. These integrals represent the transformation from the Hankel domain (a two dimensional wavenumber domain), to the space domain. Brute force numerical integration of the oscillating integrands is expensive and error prone. Lajoie et. al. (1975) exploited the fact that the Hankel domain is just the cylindrical Fourier domain, and used a 2-D Fast Fourier Transform to evaluate the Bessel function integrals. The development of linear digital filtering theory for Hankel transforms (Ghosh (1971) and reviewed by Koe foed (1979)) was a major step towards permitting efficient calculation of quasistatic electromagnetic fields. This theory allowed routine evaluations of the Hankel transform integrals by simply calculating a weighted sum in the form of a numerical convolution. This can be an order of magnitude faster than direct integration for certain classes of models.

The problem of transverse isotropy in a horizontally layered structure has been treated a number of times. It was found in DC resistivity work that transverse isotropy could not be directly detected from surface resistivity methods but introduced ambiguity into the interpretation (Maillet (1947)). Wait (1966) extended the EM solutions of a dipole over a halfspace to include anisotropy in the halfspace, and then went on to generalize the solution to account for an arbitrarily layered earth (Wait (1982)).
Wait (1966) also solved the problem of having a dipole source within an anisotropic halfspace, but very little work has followed on having the source within an arbitrary anisotropic layer. Wynn (1979) attempted to model an anisotropic layered earth but failed to obtain the correct boundary conditions. A transient (time domain) solution for grounded wires on an anisotropic halfspace was recently given by Nabusli and Wait (1982). Unfortunately, although the mathematics of transverse isotropy is well understood, the actual physical processes involved have not been described in detail.

1.3 Thesis Outline

In developing the material in this thesis, I have attempted to provide a layered earth solution that is capable of efficiently modelling a wide variety of geological environments. Towards this end, I have extended an existing layered earth solution for dipole sources (Weaver (1971), Walker (1981)) to include transverse isotropy and induced polarization effects. In order that the solution be completely general, I also devised a method that efficiently integrates the EM fields of a dipole to obtain the fields of a large source. The remainder of the thesis involves using the algorithms developed to study the effects of anisotropy on the EM fields measured on the surface of the earth due to a grounded electric dipole. Also, a minor study of the effects of induced polarization on electric dipole surveys is conducted.

Chapter two presents the development of the solution for an arbitrary dipole source in an uniform, anisotropic space in terms of Hertz potentials. This simple problem contains all the different aspects of the more general problem. The full solution for dipole excitation of a layered earth is presented in Appendix B, and the analytical, closed form solution for an anisotropic halfspace is given in Appendix
C. The layered earth solution is developed in the Fourier domain (frequency and two horizontal wavenumbers). Thus a transformation to the space domain must be performed to calculate the field quantities. The spatial transform is obtained by the convolution method of Johansen and Sørenson (1979). Improvements in computational efficiency by using the concept of lagged convolution for the Fast Hankel transform are also discussed in Chapter two. Since many electromagnetic systems operate in the time domain, the final topic in Chapter two discusses a suitable method of transforming the layered earth response from the frequency domain to the time domain.

Chapter three demonstrates the specific solution to the problem of a horizontal electric dipole on the surface of the earth as a simplification of the general solution. Also, a new method of performing the integration of the elementary dipole solution to obtain the solution for a long grounded dipole is introduced (Integration by Weighted Sums - IWS). The final section in Chapter three demonstrates that the integration procedure used to find the long wire response is also suitable for calculating the response of a large loop systems.

In Chapter four, the response of an anisotropic layered earth is discussed in terms of the dipole source. The simple response of a halfspace is described for the isotropic and anisotropic cases. The concepts used to describe these responses are then elaborated upon to explain the effects of layers within the halfspace. Finally, the problem of resolving anisotropy within a layer is considered.

Chapter five is similar to Chapter four in design, but is concerned with the response of a polarizable earth. Again, the simple halfspace models are described and expanded on to cover stratified earths. A brief look at the problem of separating IP and EM responses is also given in this chapter.

The final chapter, Chapter six, contains the conclusions reached after performing
this work and suggestions for further studies.
Chapter 2

Solution of the Layered Earth Problem

Even though the general solutions for the layered earth problem have been in the literature for many years, the computational effort involved in calculating some of the responses is still quite significant. In fact, some parts of the solutions can have numerical instabilities which makes it exceedingly difficult to obtain accurate answers. Also, the mathematics of the derivations tend to obscure the physical process that are being modelled. While a large amount of the mathematics is unavoidable, it is possible to present the solution in a straightforward manner, and at the same time show the effects of different types of sources on different field components. The purpose of this chapter is to outline a procedure for finding the response to a layered earth which is computationally efficient, and relatively simple to calculate.

2.1 General Strategy

The development of the layered earth solution follows Weaver (1970), Lajoie et al. (1975), and Walker (1981) with several improvements and the introduction of anisotropy. Therefore, the majority of the derivation is described in Appendix B, only the introductory equations and the source terms are derived here. The solution is derived without making a quasistatic approximation. It is valid for sources and receivers which are anywhere within the layered earth, and is therefore applicable to a variety of EM problems from borehole work, to ground electromagnetics, to airborne electromagnetics.
The solution is developed in the two-dimensional wave-number domain \((p,q,z)\), using Hertz potential vectors as field variables. These particular potential vectors simplify the problem considerably by separating the sources into galvanically produced (electric current) and inductively produced (magnetic current) components. The Hertz potential vectors are redundant; in fact, Jones (1947) has shown that any electromagnetic field generated from a single source can be uniquely represented by two scalars. In principle, any two scalars can be used. The vector components \(\Pi_z\) and \(\Gamma_z\) are chosen since \(z\) is the untransformed working variable of the problem and to be consistent with the introduction of electric and magnetic sources.

The first step in the development is to find the Hertz potential vectors for an arbitrary dipole source in a uniform, anisotropic space. Since the development of the primary potentials is somewhat complicated and not presented in this form in the literature, it is outlined in some detail. The next step involves expressing the fields in terms of the vertical components of the Hertz potential vectors which essentially reduces the number of working variables from six to two.

From the primary vertical potentials, the secondary (or scattered) vertical potentials due to the interfaces are found from the boundary conditions imposed by the stratification. The total vertical potentials (primary + secondary) are initially found at the source level and then are found at the receiver level by a continuation operation across each layer and by a transmission operation across each interface between the source and receiver. Once the vertical potentials are known at the receiver level, the field components may be found from the relations defining the potentials.

Finally, the fields must be transformed to the space domain. To avoid a 2-D Fourier transformation to the space domain, the field components are rewritten in the Hankel domain (the cylindrical Fourier domain). This takes advantage of the simple axial symmetry of the fields when the sources are points. A Fast Hankel Transform
(Johnansen and Sørenson (1979)) is then used to find the components of the field in the space domain. Use of the Fast Hankel transform algorithm is restricted to the cases where a quasistatic assumption is appropriate. In other cases, a more general quadrature method must be applied.

2.2 The Helmholtz Equations for the Hertz Potential Vectors

In this section, the relationships between the Hertz potentials and the electromagnetic fields are developed. From these relationships, the equations which govern the behaviour of the Hertz potentials in an anisotropic space (the Helmholtz equations) are derived. Throughout the derivation, a time harmonic variation of the form $e^{i\omega t}$ is assumed. Maxwell's equations for a uniform, homogeneous space are

\[ \nabla \times \mathbf{H} = \alpha \mathbf{E} + \mathbf{J} \]

(2.2.1)

\[ \nabla \times \mathbf{E} = \beta \mathbf{H} + \mathbf{M} \]

(2.2.2)

\[ \nabla \cdot \mu \mathbf{H} = 0 \]

(2.2.3)

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \]

(2.2.4)

where $\alpha = \sigma + i\omega \epsilon$ (the admittivity) and $\beta = i\omega \mu$ (the impedivity). $\mathbf{J}$ is the \textit{applied} electric current source, while $\mathbf{M}$ is the \textit{applied} magnetic current source which is analogous to the electric current source. The concept of a magnetic source is extremely useful in geophysics and is introduced to make equations (2.2.1) and (2.2.2) symmetric.

Since electrical transverse isotropy is to be modelled, the admittivity must be
expressed as a diagonal tensor of the form,

$$
\mathbf{\sigma} = \begin{bmatrix}
\sigma_h + i\omega \varepsilon_h & 0 & 0 \\
0 & \sigma_h + i\omega \varepsilon_h & 0 \\
0 & 0 & \sigma_v + i\omega \varepsilon_v \\
\end{bmatrix}
$$

(2.2.5)

In other words, the principal axes of the anisotropy coincide with the coordinate system axes, and the properties in the horizontal planes differ from the properties in the vertical plane. Using this tensor, equation (2.2.1) becomes

$$
\nabla \times \mathbf{H} = \mathbf{\alpha E} + \mathbf{J}
$$

(2.2.6)

It is now convenient to consider two cases; electric sources, and magnetic sources. When only electric sources are present, \( \mathbf{M} = 0 \) and equation (2.2.2) is written

$$
\nabla \times \mathbf{E} = -\beta \mathbf{H}
$$

(2.2.7)

Introducing the vector potential \( \mathbf{A} \) in the usual manner

$$
\mathbf{H} = \nabla \times \mathbf{A}
$$

(2.2.8)

results in a degree of freedom in the conditions on \( \mathbf{A} \) which is accounted for by the scalar potential \( \psi \), where

$$
\mathbf{E} = -\beta \mathbf{A} - \nabla \psi
$$

(2.2.9)

\( \psi \) is chosen such that it is related to \( \mathbf{A} \) by a gauge condition. The primary consideration in choosing the gauge condition is that it is desirable to have each of the rectangular components of the sources directly related to their respective potential vector components. In other words, the potential vector should be oriented along the axis of the source. With a certain amount of foresight, the gauge condition can be
chosen as
\[ \nabla \cdot \mathbf{A} = -\alpha_h \psi. \] (2.2.10)

Now it is possible to derive the Helmholtz equations for a transversely isotropic medium. Eliminating \( \mathbf{H} \) from equation (2.2.6) using (2.2.8) gives;
\[ \nabla \times \nabla \times \mathbf{A} = \alpha \mathbf{E} + \mathbf{J}. \] (2.2.11)

Another equation for \( \mathbf{E} \) in terms of \( \mathbf{A} \) only may be obtained by eliminating \( \psi \) between equations (2.2.9) and (2.2.10).
\[ \mathbf{E} = -\beta \mathbf{A} + \alpha_h^{-1} \nabla \nabla \cdot \mathbf{A} \] (2.2.12)

By eliminating \( \mathbf{E} \) from (2.2.11) and (2.2.12), the required vector equation in \( \mathbf{A} \) is obtained. This vector equation may be separated into components by representing the source term as;
\[ \mathbf{J} = \mathbf{J} \delta(r - r_s) = (J_x \mathbf{i} + J_y \mathbf{j} + J_z \mathbf{k}) \delta(r - r_s) \] (2.2.13)

Also, the electric Hertz potential \( (\Pi = (\Pi_x, \Pi_y, \Pi_z)) \) may be introduced in the form
\[ \mathbf{A} = \alpha \Pi \] (2.2.14)

The Helmholtz equations in terms of electric sources are then;
\[ 11\Pi_x + 22\Pi_x + 33\Pi_x - \gamma^2 \Pi_x = -\frac{J_x}{\alpha_h} \delta(r - r_s) \] (2.2.15)
\[ 11\Pi_y + 22\Pi_y + 33\Pi_y - \gamma^2 \Pi_y = -\frac{J_y}{\alpha_h} \delta(r - r_s) \] (2.2.16)
\[ K(11\Pi_x + 22\Pi_x) + 33\Pi_x - \gamma^2 \Pi_x = -\frac{J_z K}{\alpha_h} \delta(r - r_s) + (K - 1)(13\Pi_x + 23\Pi_y) \] (2.2.17)
where $K = \alpha_k/\alpha_s$ and is called the coefficient of anisotropy. Here a numeric subscript represents the partial derivative of the variable with respect to a space coordinate, while the alphabetic subscript refers to the space coordinate of the variable.

A very similar development is used to find the Helmholtz equations in terms of magnetic sources. In this case, the magnetic vector potential $\mathbf{F}$ is introduced such that

$$\alpha \mathbf{E} = -\alpha_k \nabla \times \mathbf{F}$$  \hspace{1cm} (2.2.18)$$

which is analogous to equation (2.2.8). This definition arises from recognizing that in the absence of electric sources, the divergence of (2.2.1) gives

$$\nabla \cdot (\alpha \mathbf{E}) = 0$$  \hspace{1cm} (2.2.19)$$

Therefore, the $\nabla \times \mathbf{F}$ may be equated to $\alpha \mathbf{E}$ since $\nabla \cdot \mathbf{F} = 0$. The best gauge condition found is that of equation (2.2.20).

$$\nabla \cdot (\alpha \frac{1}{\alpha_s} \mathbf{F}) = \beta \phi$$  \hspace{1cm} (2.2.20)$$

Using this condition, and the definition of the magnetic Hertz potential ($\Gamma = (\Gamma_x, \Gamma_y, \Gamma_z)$),

$$\mathbf{F} = \beta \Gamma$$  \hspace{1cm} (2.2.21)$$

Therefore, Helmholtz equations for magnetic sources are:

$$K(11\Gamma_x + 22\Gamma_x) + 33\Gamma_x - \gamma^2 \Gamma_x = -\frac{M_x}{\beta} \delta(r - r_s)$$  \hspace{1cm} (2.2.22)$$

$$K(11\Gamma_y + 22\Gamma_y) + 33\Gamma_y - \gamma^2 \Gamma_y = -\frac{M_y}{\beta} \delta(r - r_s)$$  \hspace{1cm} (2.2.23)$$

$$11\Gamma_x + 22\Gamma_z + 33\Gamma_z - \gamma^2 \Gamma_z = -\frac{M_z}{\beta} \delta(r - r_s) + (K - 1)(13\Gamma_z + 23\Gamma_y)$$  \hspace{1cm} (2.2.24)$$
Obviously, the horizontal Helmholtz equations for electric sources are very similar to the vertical Helmholtz equations for magnetic sources, and vice versa. The similarity is as a result of the magnetic current introduced in (2.2.2), and is one aspect of the general principle of *reciprocity*.

It is possible to find the relationships between the electric and magnetic fields and the Hertz potential vectors from the equations used to find the Helmholtz equations. The general solution can be written as

\[
E = -\gamma^2 \Pi + \alpha_h^{-1} \nabla \nabla \cdot (\alpha \Pi) - \alpha_h \alpha^{-1} \beta \nabla \times \Gamma \tag{2.2.25}
\]

\[
H = -\gamma^2 \Gamma + \alpha_h^{-1} \nabla \nabla \cdot (\alpha \Gamma) + \nabla \times (\alpha \Pi) \tag{2.2.26}
\]

When the medium is isotropic, the above equations reduce to the usual relationships between the Hertz potentials and the electric and magnetic fields.

\[
E = \nabla \times \nabla \times \Pi - \beta \nabla \times \Gamma \tag{2.2.27}
\]

\[
H = \nabla \times \nabla \times \Gamma + \alpha \nabla \times \Pi \tag{2.2.28}
\]

### 2.3 Dipole Sources in a Uniform, Anisotropic Space

The principle advantage in using the Hertz potentials is the simple way in which the source terms are related to the components of the potentials. In an isotropic medium an electric source generates the electric Hertz potential directed along the axis of the source. Similarly, a magnetic source will generate the magnetic Hertz potential in the source direction. This *uncoupling* of the potentials is due to the chosen gauge condition and simplifies the mathematics tremendously.
In the transversely isotropic case, such a clear uncoupling of the potentials does not arise. The magnetic and electric sources are still related to the respective Hertz potentials, but the vertical components of the potentials are coupled to the horizontal source terms. This means that a horizontal source generates two potentials; one along the axis of the source, and the other in the vertical direction. A vertical source on the other hand, generates only a vertical component of the potential.

To facilitate the development, the problem is transformed into the two-dimensional Fourier domain \((p,q,z)\). The transformation of the Helmholtz equation results in a one-dimensional differential equation. The Helmholtz equation in the Fourier domain for \(\hat{\Pi}_z\) at the source point is given by

\[
\frac{\partial^2 \hat{\Pi}_z}{\partial z^2} - u^2 \hat{\Pi}_z = \frac{-J_z \delta(z)}{\alpha_h}
\] (2.3.1)

where the forcing function has the form of a constant times a delta function. In this equation \(u^2 = \lambda^2 + \gamma^2\), and \(\lambda^2 = p^2 + q^2\). The solution of this type of differential equation is by no means trivial, but is sufficiently straightforward that it will not be reproduced here. An example of the method used to solve this type of equation may be found in Butkov, page 512. Equation (2.3.1) has the solution

\[
\hat{\Pi}_z = \frac{J_z}{2u\alpha_h} e^{-\text{sgn}(z)uz}
\] (2.3.2)

where

\[
\text{sgn}(z) = \begin{cases} 
+1, & \text{if } z > 0 \\
0, & \text{if } z = 0 \\
-1, & \text{if } z < 0
\end{cases}
\]

Similarly, the solutions for the other horizontal potentials have the same form.

\[
\hat{\Pi}_\nu = \frac{J_\nu}{2u\alpha_h} e^{-\text{sgn}(z)uz}
\] (2.3.3)
\[ \hat{\Gamma}_z = \frac{M_z}{2v\beta} e^{-\text{sgn}(z) vz} \quad (2.3.4) \]

\[ \hat{\Gamma}_v = \frac{M_v}{2v\beta} e^{-\text{sgn}(z) vz} \quad (2.3.5) \]

The more complicated solution for the vertical potentials will be shown for \( \hat{\Pi}_z \) only, since the procedure is identical for \( \hat{\Gamma}_z \). The Helmholtz equation for \( \hat{\Pi}_z \) is

\[ \frac{\partial^2 \hat{\Pi}_z}{\partial z^2} - v^2 \hat{\Pi}_z = -\frac{K J_z \delta(z)}{\alpha_v} + (K - 1)(\hat{\Pi}_{13} + \hat{\Pi}_{23}) \quad (2.3.6) \]

where the \( \hat{\Pi}_{13} \) and \( \hat{\Pi}_{23} \) terms act as sources for the vertical potential. The solution to this equation is the sum of the solutions for the three forcing functions. The solution for equations with the delta forcing function is already known.

\[ \frac{\partial^2 \hat{\Pi}_z}{\partial z^2} - v^2 \hat{\Pi}_z = -\frac{K J_z \delta(z)}{\alpha_v} \quad (2.3.7) \]

\[ \hat{\Pi}_z = \frac{J_z K}{2v\alpha_v} e^{-\text{sgn}(z) vz} \quad (2.3.8) \]

The other type of forcing function to consider is of the form

\[ \frac{\partial^2 \hat{\Pi}_z}{\partial z^2} - v^2 \hat{\Pi}_z = \mathcal{F}(z) \quad (2.3.9) \]

where \( \mathcal{F}(z) \) is a known forcing function of \( z \). By considering equation (2.3.6), the forcing function can be expressed as a constant times an exponential.

\[ \frac{\partial^2 \hat{\Pi}_z}{\partial z^2} - v^2 \hat{\Pi}_z = \frac{ip(K - 1)}{2\alpha_h} \text{sgn}(z) J_z e^{-\text{sgn}(z) uz} \quad (2.3.10) \]

This term may be rewritten using the convolution theorem for a delta function.

\[ \frac{ip(K - 1)}{2\alpha_h} \text{sgn}(z) J_z e^{-\text{sgn}(z) uz} = \frac{ip(K - 1)}{2\alpha_h} J_z \int_{-\infty}^{+\infty} \delta(\zeta) \text{sgn}(\zeta) e^{-\text{sgn}(z - \zeta) u(z - \zeta)} d\zeta \quad (2.3.11) \]

Now the forcing function is a convolution of the delta forcing function with an exponential distribution, and since the solution for the delta forcing function is known,
the solution of equation (2.3.10) can immediately be written down.

\[ \hat{P}_x = -\frac{ip(K-1)}{2\alpha_h} J_x \int_{-\infty}^{+\infty} \text{sgn}[\xi] \left[ -\frac{e^{-\text{sgn}[\xi]u\xi}}{2v} \right] e^{-\text{sgn}[x-s]v(x-s)} d\xi \]  

(2.3.12)

In this equation, the first term after the integral sign in square brackets is the solution to the differential equation with a delta forcing function. The integral (2.3.12) is evaluated in a piecewise sense as follows;

\[ \hat{P}_x = \int_{-\infty}^{0} d\xi + \int_{0}^{e=x} d\xi + \int_{e=x}^{+\infty} d\xi = I_1 + I_2 + I_3 \]  

(2.3.13)

The solutions of these three integrals are given in equations (2.3.14), (2.3.15), and (2.3.16).

\[ I_1 = \frac{ip(K-1)J_x}{4v\alpha_h} \left[ -\frac{e^{-\text{sgn}[x]v\xi}}{v+u} \right] \]  

(2.3.14)

\[ I_2 = \frac{ip(K-1)J_x}{4v\alpha_h} \left[ \frac{e^{-\text{sgn}[x]v\xi} - e^{-\text{sgn}[x]u\xi}}{v-u} \right] \]  

(2.3.15)

\[ I_3 = \frac{ip(K-1)J_x}{4v\alpha_h} \left[ \frac{e^{-u\xi \text{sgn}[x]}}{v+u} \right] \]  

(2.3.16)

The solution to equation (2.3.12) is then the sum of the three equations.

\[ \hat{P}_x = \frac{-ipJ_y}{2\alpha_h\lambda^2} \left[ e^{-\text{sgn}[x]u\xi} - e^{-\text{sgn}[x]v\xi} \right] \]  

(2.3.17)

The result is easily verified by direct substitution into equation (2.3.10). A similar development results in an expression for a \( J_y \) source.

\[ \hat{P}_x = \frac{-iqJ_y}{2\alpha_h\lambda^2} \left[ e^{-\text{sgn}[x]u\xi} - e^{-\text{sgn}[x]v\xi} \right] \]  

(2.3.18)

The vertical potential due to any horizontal source can be constructed from equations (2.3.17), and (2.3.18).

These source terms have a very simple form, and at first glance it would seem possible to compute the fields in an anisotropic earth by simply placing a suitable
set of sources in an isotropic earth. After performing some algebra, one finds that the potentials generated by a horizontal source in an anisotropic earth are identical to the potentials of a single source in an isotropic earth. The form of this source is shown in (2.3.19) for the case of a dipole oriented in the x-direction.

\[
\frac{J_x}{2u\alpha_h} e^{-\text{sgn}(z)u z} \quad (2.3.19)
\]

In order for the source to be able to physically exist within the isotropic medium, it must show an exponential dependence of the form \(e^{-\text{sgn}(z)u z}\). Therefore, the source term can be rewritten as;

\[
\frac{J_x}{2u\alpha_h} e^{-\text{sgn}(z)u(ax)} \quad (2.3.20)
\]

where

\[
a = \frac{v}{u} = \sqrt{\frac{K\lambda^2 + \beta\alpha_h}{\lambda^2 + \beta\alpha_h}} \quad (2.3.21)
\]

In the DC limit, \(a = \sqrt{K}\), indicating that an anisotropic medium can be modelled by an isotropic medium in which the z direction is scaled by an amount equal to \(\sqrt{K}\). This agrees with results found for DC resistivity (Maillet (1947) and Kunetz (1966)). Obviously, it is impossible to correctly interpret DC resistivity measurements for a transversely isotropic earth since the response will be identical to an isotropic earth with a scaled z dimension. Therefore, if the anisotropy is neglected, the thickness of any anisotropic layer will appear different than its actual thickness by a factor of \(\sqrt{K}\).

It is not possible to scale the potentials in the EM case, since a set of sources with an \(e^{-\text{sgn}(z)u z}\) dependence will not act collectively as a single source with an \(e^{-\text{sgn}(z)u z}\) dependence. The matter is further complicated by the fact that \(a\) is complex, and changes with frequency, conductivity, and wavenumber. However, the direct current case is useful for explaining some of the features of the low frequency response of
the electromagnetic fields. Another interesting point is that at very high frequency, the value of $a$ approaches unity and the presence of anisotropy has little effect on the source terms. What is seen is the horizontal conductivity of the medium because of the $\alpha_{\alpha}$ term in the source expressions. This same effect is seen at large distances, and as usual the product $|\gamma\rho|$ is more appropriate for describing the behaviour of the fields, than either the distance, frequency or conductivity alone.

2.4 Representing EM Fields in Terms of Two Scalar Potentials

In the previous section, the manner in which various distributions of sources create components of the Hertz potential vectors was introduced. In this section, the redundancy of the Hertz potential vectors is eliminated by finding a representation of the fields in terms of only the vertical components of the Hertz potentials.

In order to represent the source potentials in terms of two scalar potentials, there must be some relationship between the components of the Hertz potential vectors and the chosen scalar potentials. This is complicated somewhat by the fact that the scalar potentials chosen to represent the fields are actually the vertical components of the Hertz potentials. To find the relationships between the components of the fields and the two scalars, an imaginary boundary is introduced. This boundary does not actually represent a property contrast, but at any position in space, the usual boundary conditions apply to the fields along this surface. This is equivalent to saying that all the components of the fields are equal on either side of the boundary. Therefore, the boundary conditions relate the two alternate representations of the fields (in terms of vectors or in terms of scalars). Algebraically this can be written as;

$$\hat{F}_m(\hat{\Pi}, \hat{\Gamma}) = \hat{F}_m(\hat{\Pi}_z, \hat{\Gamma}_z) \quad (2.4.1)$$
where \( \hat{F}_m \) represents the \( m^{th} \) component of either \( \mathbf{E} \) or \( \mathbf{H} \).

The following example is for the \( x \)-component of the electric Hertz potential. By including the \( y \)-component of the electric Hertz potential no generality is added to the solution, but the equations are lengthened considerably. A set of equations relating the \( x \)-component of the electric Hertz potential vector to the vertical components of the electric and magnetic Hertz potentials can be derived from equations (B.14) and (B.15). Because the scalar representation of the fields is in terms of the vertical components of the vector representation, there must be a way of distinguishing the two representations. Therefore, primed variables will indicate representation by the vector potentials (which were derived in the previous section), and unprimed variables will indicate representation by the scalar potentials. In this development, it is important to recall that a horizontal source generates both horizontal and vertical components of the vector potential. From the boundary condition (2.4.1), the six field components give six equations in terms of the Hertz potentials.

\[
-\gamma^2\hat{\Pi}'_x - p^2\hat{\Pi}'_z - \frac{ip}{K^3}\hat{\Pi}'_z = \frac{iv u \text{ sgn}[z]}{K} \hat{\Pi}_z + iq\beta \hat{\Gamma}_z \tag{2.4.2}
\]

\[
-pq\hat{\Pi}'_z - \frac{iq}{K^2}\hat{\Pi}'_z = \frac{iv u \text{ sgn}[z]}{K} \hat{\Pi}_z - iq\beta \hat{\Gamma}_z \tag{2.4.3}
\]

\[
iv u \text{ sgn}[z]\hat{\Pi}'_z - \frac{\gamma^2}{K}\hat{\Pi}'_z + \frac{1}{K^2}\hat{\Pi}'_z = \lambda^2\hat{\Pi}_z \tag{2.4.4}
\]

\[
-iq\alpha_v \hat{\Pi}'_x = -iq\alpha_v \hat{\Pi}_x + ivu \text{ sgn}[z] \hat{\Gamma}_x \tag{2.4.5}
\]

\[
-iv\alpha_h \hat{\Pi}'_x + iv\alpha_v \hat{\Pi}'_z = iv\alpha_v \hat{\Pi}_x + ivu \text{ sgn}[z] \hat{\Gamma}_x \tag{2.4.6}
\]

\[
iq\alpha_h \hat{\Pi}'_x = \lambda^2\hat{\Gamma}_x \tag{2.4.7}
\]

At first sight, the scalar potentials appear greatly overdetermined. However, the five equations (2.4.2 - 2.4.6) reduce to three different equations when \( \hat{\Gamma}_z \) is eliminated.
using (2.4.7).

\[
\hat{\Pi}_z = \frac{1}{\lambda^2} \left[ i p u \text{sgn}[z] \hat{\Pi}'_z - \frac{\gamma^2}{K} \hat{\Pi}'_z + \frac{3\gamma^2}{K} \hat{\Pi}_z \right] \tag{2.4.8}
\]

\[
\hat{\Pi}_x = \frac{i p u K \text{sgn}[z]}{\lambda^2 v} \hat{\Pi}'_z - \frac{\text{sgn}[z]}{v} 3\hat{\Pi}_z \tag{2.4.9}
\]

\[
\hat{\Pi}_z = \frac{i p u K \text{sgn}[z]}{\lambda^2} \hat{\Pi}'_z + \hat{\Pi}'_z \tag{2.4.10}
\]

While it is not immediately obvious, (2.4.8) and (2.4.9) are two different forms of (2.4.10). To demonstrate this requires that the partial derivative operations be carried out and this is always possible as long as the direction of propagation of the fields is known (i.e. all sources are on one side of the field point). Therefore, equation (2.4.10) relates the vector potential components to the scalar potential \( \hat{\Pi}_z \). At the source level \( z = 0 \), equation (2.4.10) can be simplified by using equation (2.3.17) to find the components of the potentials.

\[
\hat{\Pi}'_z = 0 \tag{2.4.11}
\]

\[
\hat{\Pi}'_x = \frac{\text{sgn}[z]}{2u \alpha_h} J_z \tag{2.4.12}
\]

The vertical electric scalar potential is

\[
\hat{\Pi}_z = \frac{i p \text{sgn}[z]}{2\lambda^2 \alpha_v} J_z \tag{2.4.13}
\]

A similar development leads to the vertical magnetic scalar potential

\[
\hat{\Gamma}_z = \frac{i q}{2\lambda^2 u} J_z \tag{2.4.14}
\]

Following the procedure outlined in this section, the primary potentials at the source level, above or below the source plane due to magnetic and electric sources can be found for the two scalar potentials \( \hat{\Pi}_z \) and \( \hat{\Pi}_x \). These source terms can be
combined in a matrix notation as in equations (2.4.15) and (2.4.16).

\[
\begin{align*}
\begin{bmatrix}
\hat{\Gamma}_{x}^{\text{out}} \\
\hat{\Gamma}_{x}^{\text{in}}
\end{bmatrix} &=
\begin{bmatrix}
\frac{ip}{2\lambda^2 \alpha_v} & \frac{iq}{2\lambda^2 \alpha_v} & K & \frac{ipK}{2\lambda^2 v} & -\frac{iqK}{2\lambda^2 v} \\
-\frac{ip}{2\lambda^2 \alpha_v} & -\frac{iq}{2\lambda^2 \alpha_v} & K & \frac{ipK}{2\lambda^2 v} & -\frac{iqK}{2\lambda^2 v}
\end{bmatrix}
\begin{bmatrix}
J_x \\
J_y \\
J_z \\
M_y \\
M_z
\end{bmatrix}
\end{align*}
\] (2.4.15)

\[
\begin{align*}
\begin{bmatrix}
\hat{\Gamma}_{x}^{\text{out}} \\
\hat{\Gamma}_{x}^{\text{in}}
\end{bmatrix} &=
\begin{bmatrix}
\frac{ip}{2\lambda^2 \beta} & \frac{iq}{2\lambda^2 \beta} & \frac{1}{2\beta u} & -\frac{ip}{2\lambda^2 u} & \frac{iq}{2\lambda^2 u} \\
-\frac{ip}{2\lambda^2 \beta} & -\frac{iq}{2\lambda^2 \beta} & \frac{1}{2\beta u} & -\frac{ip}{2\lambda^2 u} & \frac{iq}{2\lambda^2 u}
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z \\
J_y \\
J_z
\end{bmatrix}
\end{align*}
\] (2.4.16)

In the above equations, the superscripts in and out refer to potentials which are travelling towards or away from the source respectively.

Now that the source terms have been developed for the two scalar potentials, the solution of the layered earth problem follows closely work that has already been published (Walker (1981)). The next stages in the development involve including the effects of layers within the earth, and it is from this point in the development that Appendix B starts.

### 2.5 The Fast Hankel Transform

In order to transform the Fourier domain solution back to the space domain, the cylindrical symmetry exhibited by the dipole source and layered earth can be exploited. Instead of performing a two-dimensional Fourier transform, a one-dimensional Hankel transform is used. In order to do this, the solution in the Fourier domain is
transformed to the Hankel domain as described in Appendix B (cf. equation B.48). The result of transforming the solution from the Hankel domain to the space domain is that the solution of the layered earth problem is expressed in terms of infinite integrals over Bessel functions of order 0 and 1 (cf. equation B.50). The classical method of evaluating these oscillatory integrals is by direct integration between the zero crossings of the Bessel function. Clearly, this involves the evaluation of Bessel function at many points in every half cycle available when the function to be transformed is _diffusive_ in its behaviour. The procedure can be very tedious.

An alternate method of evaluating these integrals is called the Fast Hankel Transform (FHT) and it has become the standard in geophysical applications. The FHT is a digital filter method that operates on a logarithmically sampled representation of the function to be integrated. It is relatively accurate and may be an order of magnitude faster than direct integration in some cases. The development of the method is well documented in the literature (Ghosh (1971), Koepeoed (1972), Johansen and Sørenson (1979), Verma (1982), and Anderson (1982)) and will only be outlined here.

The standard form of the Hankel transform integral is;

$$ f(\rho) = \int_{0}^{\infty} k(\lambda) \lambda J_n(\lambda \rho) \, d\lambda $$  \hspace{1cm} (2.5.1)

where $k(\lambda)$ is referred to as the kernel function, $J_n(\lambda \rho)$ is a Bessel function of order $n$, and $f(\rho)$ is the output function. By introducing the following change of variables, equation (2.5.1) may be written as a convolution integral.

$$ u = \ln \left( \frac{1}{\lambda} \right) , \quad v = \ln(\rho) $$

$$ F(v) = \int_{-\infty}^{\infty} K(u) H_n(v - u) \, du $$  \hspace{1cm} (2.5.2)
where,

\[ K(u) = e^{-u} k(e^{-u}), \quad F(v) = e^v f(e^v), \]

and

\[ H_n(v - u) = e^{v-u} J_n(e^{v-u}) \]

The commutative law for convolution allows an alternate form for this integral in which the Bessel function term is independent of \( \rho \).

\[ F(v) = \int_{-\infty}^{\infty} K(v - u)H_n(u) \, du \quad (2.5.3) \]

Provided that an equi-spaced sampling of \( K \) at a finite interval adequately represents the function (i.e. that \( K \) is band-limited), equation (2.5.3) may then written as a finite convolution sum where the Bessel function term acts as a digital filter through which the kernel function is passed to achieve the desired output function. Algebraically, this operation is written as;

\[ F(v) \approx \sum_{i=1}^{N} K(v - u_i)H_n(u_i) \quad (2.5.4) \]

The method of finding the filter coefficients (the \( H_n(u_i) \) terms), was until recently, something of an art. An elaborate description of the trial and error procedure used to find the coefficients may be found in Verma (1982) or Anderson (1982). In 1979, Johansen and Sørenson published an analytical method of deriving the filter coefficients which is also very efficient. They also gave limits for the errors expected in the coefficients generated with their algorithm. Unfortunately, it would appear that the method of Johansen and Sørenson has been ignored by many.

Once the filter weights have been determined, the transformation is found by evaluating \( K(v - u_i) \) at the values of \( u_i \) required by the coefficients and calculating
the sum as expressed in equation (2.5.4). Obviously, the most expensive part of this procedure is the evaluation of the kernel function, although an adaptive convolution procedure may reduce the number of evaluations required. When the output function is to be found at a number of different $\rho$ values, tremendous savings in function evaluations may be realized by performing a lagged convolution.

The concept of lagged convolution as applied to FHT is perhaps greatly under used in geophysics, but has been in the literature for a few years (Tsubota and Wait (1980), Anderson (1982)). It results from the sampling procedures that are used to find the filter coefficients which in turn dictate the sampling in $\lambda$ space for the evaluation of the input function. The sampling in a FHT is exponential so that,

$$\lambda_{i+1} = c\lambda_i$$

where $c$ is the sample spacing in log space which is determined by the filter design. If a transformation of a function is performed for $\rho_k$, then the kernel function must be evaluated at $\lambda_i$ for $i = 1$ to $N$. Consider that the same function be transformed at $\rho_{k+1} = c\rho_k$. Then it should be obvious that the function has already been evaluated at $N - 1$ of the $N$ points required for the transformation, since

$$K(v_k - u_i) = K(v_{k+1} - u_{i-1})$$ (2.5.5)

Therefore, by saving the kernel function evaluations, the transformation consists of one extra function evaluation, and the calculation of the sum in equation (2.5.4). This is equivalent to forcing the input function to lag behind the filter weights in the summation. Following this procedure, a $\rho$ spectrum consisting of $M$ points may be found with $N + M$ function evaluations.

The FHT procedure does have some disadvantages. The kernel function that is to be transformed greatly affects the filter accuracy (and length if it is used adaptively).
Perhaps the greatest drawback is that the kernel functions must not be divergent at high and low $\lambda$, and it is for this reason that the analytical half-space solution is subtracted from the kernel before transformation. Since the halfspace solution and the layered earth solution approach the same asymptote, the halfspace solution is subtracted from the full solution to force convergence of the kernel function. After the transformation, the analytical halfspace solution is added back into the kernel to obtain the true solution. A recent paper by Chave (1983) indicates that it is possible to integrate divergent kernel functions accurately using continued fractions.

2.6 Time Domain Transformations of Electromagnetic Fields

The earth response to an electromagnetic field is spread over many decades of frequency which makes it difficult to transform to the time domain. A conventional Fast Fourier Transform (FFT) requires data points to be linearly spaced over the transform range, but the EM response is more naturally expressed in exponential space. This makes the FFT impractical for transforming EM data. One way of circumventing this problem was given by Lamontagne (1975), and adapted by Holladay (1981) (program YVSEFT). The frequency domain response is fitted to a weighted sum of basis functions which are chosen to be representative of EM fields, but which have analytical transforms. Since the Fourier transform is linear, the time domain response is simply the weighted sum of the transformed basis functions.

Lamontagne (1975) and Holladay (1981) chose the simple pole response for the frequency domain basis function as given in equation (2.6.1).

$$R(\omega) = \sum_{i=1}^{\infty} a_i \left[ \frac{\omega^2}{\omega^2 + \tau_i^{-2}} + i \frac{\omega/\tau_i}{\omega^2 + \tau_i^{-2}} \right]$$  \hspace{1cm} (2.6.1)
Figure 2-1. The simple pole basis function used to represent the frequency domain response of a layered earth.

Figure 2-2. The exponential basis function used to represent the time domain response of a layered earth.
The time domain step response equivalent to this function is a simple exponential.

\[ r(t) = \sum_{i=1}^{\infty} a_i e^{-t/\tau_i} \]  \hspace{1cm} (2.6.2)

Equations (2.6.1) and (2.6.2) are shown plotted in figures 2-1, and 2-2. A singular-value decomposition (SVD) is used to fit the frequency domain response to equation (2.6.1) for a suitable set of time constants \((\tau_i)\) at every frequency in the spectrum. Then the time domain response is generated by evaluating equation (2.6.2) at the desired times \((t)\) using the values of \(a_i\) determined by the SVD.

This method must be modified to account for the direct-current component of a grounded source. When the frequency approaches zero, the EM fields of a grounded source becomes purely in-phase (i.e. the imaginary component goes to zero) and approaches the DC limit. This is not at all like the simple pole response in which both the real and imaginary components go to zero as \(\omega \rightarrow 0\). Therefore, one should not expect the simple pole basis functions to provide an adequate representation of the frequency domain response. By subtracting the DC term from the frequency spectrum before the transformation, the frequency domain response begins to have the same characteristics as the simple pole response. In order to further stabilize the transformation, a constant term is introduced as an auxiliary basis function. This constant term may be handled as an exponential with an infinite time constant (or equivalently a simple pole response at zero frequency) and may be included in the computer program without any modifications.

Using this method of transformation a spectrum of \(a_i\) values is obtained which provides a great deal of information about the time and frequency responses. These values show the emphasis placed on certain time constants, and subtle changes in the amplitude of the time domain response. An example of a typical frequency response, time response, and a-value spectrum is shown in figure 2-3. This figure shows the
Figure 2-3. An example of a typical frequency to time domain transformations using YVESFT.
frequency domain data before transformation, the transformed response, and the amplitudes of each time constant that were required for the transformation.
Chapter 3

The Grounded Bipole Electromagnetic System

A variety of different source and receiver types and configurations are used in electromagnetic prospecting. Some configurations are extremely simple to model in a layered earth environment. The ease with which an EM system is modelled is a function of the complexity of the source. A dipole source may be modelled by directly applying the solutions derived in Appendix B (cf. Patra and Mallick (1980)), while a large source must be constructed from elemental dipole sources. This chapter deals with the problem of representing a long grounded wire in terms of electric dipole sources, and introduces a method whereby any type of large source response may be conveniently calculated. It is hoped that this new approach to calculating the response of a large source will permit regular inversion of EM sounding data by substantially reducing the amount of computational effort required.

3.1 A Horizontal Electric Dipole on the Earth's Surface

The layered earth solution presented in Chapter 2 and Appendix B may be simplified considerably when the source and receiver exist on the same plane. This is true in most practical applications of grounded EM sources where the source and receiver are actually on the surface of the earth. Since the fields are dependent upon the properties of the medium in which the sources exist, the calculations simplify even further if the source-receiver plane is considered to be just above the earth in the air halfspace. Clearly, this is permissible, since the boundary conditions insure that the
fields calculated will be the same when the source-receiver plane is an infinitesimal
distance on either side of the air-earth interface.

When the source is just above the layered earth, there is no interface in the air
halfspace above the source and therefore no downward travelling secondary potential
at the source level so that \( R^*(1) = 0 \) (cf. figure B-2). The asterisk indicates the
reflection coefficient on the air side of the earth-air interface. Therefore, equation
(B.28) is reduced to

\[
\begin{bmatrix}
\hat{P}^*_{\text{Total}} \\
\hat{P}^*_{\text{Total}}
\end{bmatrix} =
\begin{bmatrix}
1 & R_P(1) \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{P}^*_{\text{Primary}} \\
\hat{P}^*_{\text{Primary}}
\end{bmatrix}.
\] (3.1.1)

Now the propagation matrices developed in Appendix B have a very simple form.

\[
T =
\begin{bmatrix}
1 & R_P(1) \\
0 & 0
\end{bmatrix}
\] (3.1.2)

\[
S =
\begin{bmatrix}
1 & R_T(1) \\
0 & 0
\end{bmatrix}
\] (3.1.3)

The other simplification is that the air halfspace is isotropic (since \( \sigma_{\text{air}} = 0 \))

\[ K_{\text{air}} = 1 \quad \text{and} \quad u_{\text{air}} = v_{\text{air}}. \]

Using the above simplifications, the components of the electric and magnetic fields can
be written in terms of certain kernel functions. Eadie (1981) decided that the horizontal
electric fields, and the vertical magnetic field were the most desirable components
to measure under field conditions. Therefore, these components are considered in
detail and can be written in terms of three kernel functions, eg.

\[
E_z = \frac{\cos 2\theta}{4\pi} \left[ \int_0^\infty \frac{\lambda K_1(\lambda)}{2} J_0(\lambda \rho) \, d\lambda + \int_0^\infty \frac{-K_1(\lambda)}{\rho} J_1(\lambda \rho) \, d\lambda \right] \]

\[ - \frac{1}{8\pi} \int_0^\infty \lambda K_2(\lambda) J_0(\lambda \rho) \, d\lambda \] (3.1.4)
\[ E_y = \frac{\sin 2\theta}{4\pi} \left[ \int_0^\infty \lambda K_1(\lambda) J_0(\lambda \rho) \, d\lambda + \int_0^\infty \frac{-K_1(\lambda)}{\rho} J_1(\lambda \rho) \, d\lambda \right] \] (3.1.5)

\[ H_x = \frac{\sin \theta}{4\pi} \int_0^\infty \lambda K_3(\lambda) J_1(\lambda \rho) \, d\lambda \] (3.1.6)

where the three kernel functions are just

\[ K_1(\lambda) = \left[ \frac{\beta(1 + R_\Gamma(1))}{u} - \frac{u(1 - R_\Pi(1))}{\alpha_v} \right] J_x \] (3.1.7)

\[ K_2(\lambda) = \left[ \frac{\beta(1 + R_\Gamma(1))}{u} + \frac{u(1 - R_\Pi(1))}{\alpha_v} \right] J_x \] (3.1.8)

\[ K_3(\lambda) = (1 + R_\Gamma(1)) J_x \] (3.1.9)

In fact, four Hankel transformations must be performed because \( K_1 \) occurs with two different orders of Bessel functions. The recursion relation by which the reflection coefficients are found is generally the most expensive part of the calculation. Therefore, once the reflection coefficients have been found, there is very little extra work in calculating the kernels for any fields. Clearly, it is an advantage to find several field components simultaneously.

### 3.2 Integration of Current Elements into a Long Wire

In the development of the layered earth solution, the source considered was an infinitesimal electric or magnetic dipole. By connecting a set of electric dipoles together, the fields caused by the divergence of the source current will cancel at the point where two dipoles meet. This leaves the frequency dependent magnetic field about the wire which induces electric currents in the medium. At the ends of a string of dipoles, the diverging electric field represents galvanic current flow which is entering
or leaving the wire. Thus, one constructs a long-wire source on the surface of the earth by integrating the response of horizontal electric dipoles over the path taken by the wire.

Integration along a line source can be expensive computationally, especially if the receiver point is near the wire. Some authors (Dey and Morrison (1973), Nabulsi and Wait (1982), and Wait (1982)) prefer to keep part of the transformed kernel functions in a second derivative form, and then integrate to give the *grounding* terms analytically. This, however, still leaves the numerical integration of the remaining transformed kernel functions. The usual method of integration is by adaptive quadrature. However, because the integration is repeated for every frequency of each model and each receiver-source orientation, it is not difficult to see that this is a very time consuming part of the calculations. The purpose of this section is to develop a method of numerical integration that requires substantially less computational effort.

Every integral can be approximated by the sum

\[ I = \sum_{i=1}^{m} w_i F_i \]  

(3.2.1)

where \( w_i \) is some weighting function, and \( F_i \) is the \( i^{th} \) value of the function to be integrated. In general, the integration weights will depend on the integration procedure used, the discretization of the function, and the limits of integration. Ideally, the weights should depend only on the receiver-transmitter orientation, and not any part of the earth model. If the \( w_i \) terms can be evaluated and *saved*, the integration becomes a matter of calculating the summation in equation (3.2.1) for any given receiver-transmitter orientation. The following is a description of the procedure used to find the \( w_i \) terms.

As can be seen from equations (3.1.4) to (3.1.9), it is possible to find a radial distribution of the transformed kernel functions (field functions) *independently* of
the geometrical factors. For example, consider the vertical magnetic field.

\[ H_z = \sin \theta \frac{1}{4\pi} \int_0^\infty K_3(\lambda)J_1(\lambda \rho) \, d\lambda \]  
(3.2.2)

This can be written as

\[ H_z = g(\rho, \theta)f(\rho) \]  
(3.2.3)

where

\[ g(\rho, \theta) = \sin \theta \quad \text{and} \quad f(\rho) = \frac{1}{4\pi} \int_0^\infty K_3(\lambda)J_1(\lambda \rho) \, d\lambda. \]

When the lagged convolution method is used to perform the Hankel transform, a radial distribution equi-spaced in \( \log(\rho) \) space is generated for each of the field functions \( f(\rho) \). These field functions are very well behaved in \( \log(\rho) \) space and so it should be possible to interpolate between the sample locations to generate enough points for the integration. A set of local basis functions can be used to interpolate over a small range of field function values. This interpolation procedure can be written for a single point at radius \( \rho_j \) as

\[ f(\rho_j) \approx \sum_{i=1}^n \frac{F_{k+1}}{2} \phi_i(\rho_j), \quad \rho_k \leq \rho_j \leq \rho_{k+1} \]  
(3.2.4)

where \( \phi_i(\rho_j) \) is a part of the local basis function and \( n \) is the number of nodes in the basis function. This representation is only valid in the region \( \rho_k \leq \rho_j \leq \rho_{k+1} \).

The choice of the interpolating function is important to this procedure and depends on the type of function to be interpolated. At small \( \rho \), the field functions change rapidly, but in a smoothly varying fashion. At large \( \rho \), the fields functions tend to be linear in \( \log(\rho) \) space, so that the interpolating functions must be able to fit a straight line through collinear points. A quintic spline \( (n = 4) \) developed by Lamontange (1975) at the University of Toronto was chosen as the basis function for this work. This spline actually consists of two quintic polynomials \( S_1 \) and \( S_2 \) that
are pieced together in four sections to form the basis function (see figure 3-1). The components of the spline are:

\[ \phi_1(\rho_j) = S_2(R) \]  \hspace{1cm} (3.2.5)

\[ \phi_2(\rho_j) = S_1(R) \]  \hspace{1cm} (3.2.6)

\[ \phi_3(\rho_j) = S_1(1 - R) \]  \hspace{1cm} (3.2.7)

\[ \phi_4(\rho_j) = S_2(1 - R) \]  \hspace{1cm} (3.2.8)

where

\[ R = \frac{\rho_j - \rho_k}{\rho_{k+1} - \rho_k} \]  \hspace{1cm} (3.2.9)

(assuming that the function is known at equally spaced intervals). The equations for the polynomials are:

\[ S_1(z) = 1 - \frac{9}{4} z^2 \left( 1 - \frac{2}{9} z \left( 1 + \frac{5}{2} z \left( 1 - \frac{2}{5} z \right) \right) \right) \]  \hspace{1cm} (3.2.10)

\[ S_2(z) = -\frac{1}{2} z \left( 1 - \frac{11}{6} z \left( 1 - \frac{2}{11} z \left( 1 + \frac{5}{2} z \left( 1 - \frac{2}{5} z \right) \right) \right) \right) \]  \hspace{1cm} (3.2.11)

The integration of the elemental dipole fields can be written as

\[ I = \int_a^b g(\rho, \theta) f(\rho) \, dl \]  \hspace{1cm} (3.2.12)

where \( f(\rho) \) is the continuous representation of the field function, and \( g(\rho, \theta) \) is the geometric factor associated with the field function. This integration is performed along the path taken by the wire, which is represented by the elemental segment \( dl \).

Since the integration operator is linear, equation (3.2.12) can be written as a sum of integrations over smaller intervals.

\[ I = \sum_{k=1}^{N} \int_{\rho_k}^{\rho_{k+1}} g(\rho, \theta) f(\rho) \, dl \]  \hspace{1cm} (3.2.13)

This is done in order to insure the proper limits for the interpolating functions (ie \( \rho_k \leq \rho_j \leq \rho_{k+1} \)). The interpolated approximation to the continuous function is now
Figure 3-1. The basis function used in the integration by weighted sums (IWS) procedure. The quintic polynomials from which the basis function is constructed are indicated by $S_1$ and $S_2$. 
introduced into the integration.

\[ I \approx I' = \sum_{k=1}^{N} \int_{\rho_k}^{\rho_{k+1}} \sum_{i=1}^{4} g(\rho, \theta) F_{k+i-2} \phi_i(\rho) \, dl \]  \hspace{1cm} (3.2.14)

Equation (3.2.14) can be rearranged to give

\[ I' = \sum_{k=1}^{N} \sum_{i=1}^{4} F_{k+i-2} \int_{\rho_k}^{\rho_{k+1}} g(\rho, \theta) \phi_i(\rho) \, dl \]  \hspace{1cm} (3.2.15).

This equation can be written in the same form as equation (3.2.1) where the weighting functions can be found from the term

\[ \int_{\rho_k}^{\rho_{k+1}} g(\rho, \theta) \phi_i(\rho) \, dl. \]

This term can be integrated for any \( k \) without actually knowing anything about the field function except the sampling interval. This integral may be evaluated analytically if \( g(\rho, \theta) \) has a simple form, or numerically if \( g(\rho, \theta) \) is complicated. It is essential in using this method that the field function spectrum is well sampled, and that the basis functions are able to properly represent the spectrum.

The method of Integration by Weighted Sums (IWS) has had great success in performing simple integrations. The present program evaluates the integrals

\[ \int_{\rho_k}^{\rho_{k+1}} g(\rho, \theta) \phi_i(\rho) \, dl \]

numerically using Simpson's rule, however, it should be possible to use more sophisticated quadrature rules to improve the convergence and possibly the accuracy of the program. The IWS routine was tested against functions with analytical integrals, the fields of a wire in free space, the fields of a small loop of wire and compared to the results of horizontal loop programs, and by comparing the results with other long wire dipole programs (Eadie (1981) and Anderson (1975)). The agreement in all the
Figure 3-2. A comparison between an adaptive quadrature integration, and the integration routine developed in section 3.2. (The square symbols are the saved integration result).

Figure 3-3. This figure shows how the same dipole response that was used in figure 3-2 can be combined with different integration weights to give accurate results for any receiver position.
above cases was never worse than 3.0% and usually was much better. Two examples of comparisons of IWS with direct numerical integration are shown in figure 3-2, and figure 3-3. In these figures, the solid and dotted lines represent the real and imaginary parts, respectively, of the $E_x$ field over a halfspace as found by direct numerical integration. The square symbols are the response calculated with the IWS procedure. It is encouraging that a single set of integration weights can be used to find the fields at every frequency of interest. By storing sets of commonly used integration weights in a data base, routine evaluation of the fields about a long wire would be very simple and efficient. The same set of integration weights will give equally good results for various layered earth models (as shown in figure 3-3), and so are completely general.

The IWS method of evaluating the integrals takes approximately 20-25% of the CPU time required by a conventional adaptive quadrature routine. For example, to calculate the results shown in figure 3-2, The IWS procedure took 0.25 CPU seconds on a VAX 11/780, while the adaptive quadrature routine took 0.98 CPU seconds. Once the integration weights are calculated it takes of the order of 0.05 CPU seconds to find the integrated response. Clearly, the time saved in only having to evaluate the integral only once is enormous. Even greater savings can be realized when using the routine as the forward model for a joint geometrical-frequency inversion routine where several source-receiver orientations must be modelled a number of times (Vozoff and Jupp (1975)).

Consider an order of magnitude comparison for the case of joint geometrical-frequency inversion. Assume that the frequency spectrum is sampled ten times at each of ten different receiver positions. If the inversion procedure required ten iterations to converge, then a thousand integrations would have to be made, excluding the evaluation of the Jacobian matrix. In contrast, with the saved integration procedure, only ten integrations would be required, and quite possibly a saving in CPU time of
almost two orders of magnitude could be realized. Clearly, this integration procedure becomes more attractive as the size of the inversion data set increases.

All the integrations calculated by the IWS routine are for a point receiver which in practice is not a serious limitation. It is usually not desirable to have a receiver that is sufficiently large that the fields being measured change by any significant amount over the length of the receiver. In cases where a large receiver is required the problem becomes one of double integration. The procedure for a double integration is exactly analogous to the one outlined above, except that a second integral is evaluated along the path of the receiver wire.

3.3 Survey Configurations

In typical exploration problems, the survey configuration consists of a fixed position dipole and a small receiver which is free to move around the transmitter. For electric field measurements, the receiver would be a short, grounded dipole, while magnetic field measurements would require a magnetometer, or a coil sensor (which measures the time derivative of the magnetic field). All the results shown here are for the total vertical magnetic field, and for a dipole receive length of one meter.

When the earth is perfectly stratified, the electromagnetic fields about an electric dipole will show two-fold symmetry. With due consideration to signs, the entire field may be represented in a single quadrant as shown in figure 3-4. Field verification of symmetry would be useful for determining the presence of lateral inhomogeneities which would influence the interpretation process. For the purposes of illustration, a single receiver-transmitter orientation was chosen and shown in figure 3-4 by the symbol at point A. All the results presented within this chapter and the following
Figure 3-4. The DC electric field on the surface of the earth about a grounded dipole of length 1.0 km. The vector amplitudes are shown on a logarithmic scale, and the position denoted by "A" indicates the receiver position used in the model studies.
Figure 3-5. The integration weights used to calculate the response at point A in figure 3-4. The plots are arranged according to the geometric factors of the kernels; (a) is $\cos 2\theta$, (b) is 1, (c) is $\sin 2\theta$, and (d) is $\sin \theta$. 
chapters are for this particular orientation.

The integration weights for the configuration shown in figure 3-4 are plotted in figure 3-5. Weights for all four field functions are shown to illustrate the purely geometrical effects of the receiver position. The radial distance over which the integration weights are spread is controlled by the distances from the receiver to the nearest and furthest points of the transmitter. Since the points at which the field is known are exponentially spaced many more integration weights are required when the receiver is near the wire than when the receiver is far away (relative to the length of the transmitter). Also, the four sets of integration weights change dramatically for various positions about the transmitter, mostly due to the nature of the \( \cos 2\theta \) and \( \sin 2\theta \) terms.

### 3.4 Integration Procedures for Large Loop EM Systems

In principle, any electromagnetic source can be composed of electric dipoles, so that considerable cost savings may be realized in modelling other EM systems in a one-dimensional enviroment. If the earth was not laterally homogeneous, it would not be possible to separate the geometrical factors from the field functions, and this method of integration will not be applicable.

The method of integration described in section 3.2 is general and not limited to grounded wire sources. One of the most common EM sounding systems is that which utilizes an ungrounded loop. Unfortunately, considerable effort is required to calculate the large loop response by integrating elementary current dipole solution and this tends to prevent attempts to interpret field data by inverse methods. A large loop may be thought of as a series of straight line segments that are connected
together without any contact with the ground. For each segment that makes up the loop, the integration weights may be found by the IWS described in section 3.2. The sum of the integration weights for each segment results in the integration weights for the entire loop.

There is no computational advantage in calculating the response of an arbitrarily oriented dipole source since a simple rotation transforms any linear section of wire into a wire oriented along the x-axis. After the fields are found, the inverse rotation will give the correct fields for the orientation of the wire segment. The matrix describing this rotation is;

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

(3.3.1)

where \( \theta \) is the angle the wire makes with the x-axis. By performing his simple rotation, it is possible to represent bent wires of arbitrary orientation.

Typically, the response of an inductive source may be calculated using magnetic dipoles as the elementary source. Using magnetic dipoles avoids the problem of having to ensure that the diverging current at the ends of each wire segment are cancelled exactly by the adjacent wire segment. However, when using vertical magnetic dipoles to model a large loop, one must integrate over the surface area of the loop, which can be somewhat complicated. Also, it might be more accurate to model large loop systems with current dipoles instead of magnetic dipoles. This is because it is difficult to position the magnetic dipoles properly and still be able to integrate the elementary solution efficiently. On the other hand, it is quite simple to construct the required wire segments and find the required integration weights.

In order to test the validity of modelling a large loop with straight wire segments, some models were compared against published results (Hoversten and Morrison (1982)). One such comparison is shown in figure 3-6. The response indicated with
Figure 3-6. A comparison between the $H_x$ field response calculated for a large circular loop by Hoversten and Morrison (1982) (squares), and the IWS procedure (solid line).

Figure 3-7. The set of integration weights used to calculate the vertical magnetic field in figure 3-6.
square symbols is that reported by Hoversten and Morrison (1982) for a circular loop of radius 56.4 meters \( (10^4 m^2) \), with a unit dipole moment. The receiver position is 300 meters from the center of the loop, and the current in the loop has a square waveform. The solid line represents the response of an octagonal loop of approximately the same area, as calculated using the IWS procedure. The difference between the two curves is small, and is mostly due to difference in dipole moments between the two loop models. The set of integration weights used in the calculation of the large loop response are shown in figure 3-7. The vertical magnetic field is a special case in that the integration weights are additive. For any horizontal field, this is not the case and each set of integration weights must be saved separately. Even so, large loop systems can be efficiently modelled using the IWS procedure.
Chapter 4

Anisotropy in Electromagnetic Surveys

Electromagnetic data from an anisotropic environment would contain information about the anisotropy and consequently could not be interpreted properly in terms of isotropic models. Therefore, understanding the effects of anisotropy is essential for accurate interpretation of EM sounding data and the ability to recognize an anisotropic layer would add to the interpretive power of the geophysicist. Whether an anisotropic sequence is of economic importance remains to be seen; however if a layer exhibits transverse isotropy, it might be an effective trapping layer for hydrocarbons (ie oil shales).

All the responses presented in this chapter were calculated assuming a transmitter bipole extending for one kilometer along the x-axis and carrying a current of one ampere. A point receiver is situated at \((250 \text{ m}, 250 \text{ m})\) as shown in figure 3-4. Also, following Eadie (1981), only the horizontal electric, and vertical magnetic field components are considered.

4.1 The Electromagnetic Response of an Anisotropic Halfspace

The simplest case to consider is that of a homogeneous, isotropic halfspace. Graphs of the three field components are shown in figure 4-1 for the frequency domain, and in figure 4-2 for the time domain step (off) response. There is no essential difference between the two domains, but the frequency domain is more easily understood in terms of simple mathematics, whereas the time domain is more practical
in terms of instrumentation and so is favored in actual sounding work. Therefore, response curves in both domains will be shown in this section, although the frequency domain data will be used to explain the physics.

The most striking feature of the plots is that the electric field perpendicular to the wire is purely real, and constant with respect to frequency. In the time domain, this corresponds to the $E_y$ field decaying to zero immediately after the current in the wire is shut off. In order to describe this effect, consider the fields on an isotropic halfspace due to a horizontal electric dipole source (derived in Appendix C).

$$E_x = \frac{3 \cos 2\theta}{4\pi \rho^3 \alpha} + \frac{1 + (1 + \gamma\rho) e^{-\gamma\rho}}{4\pi \rho^3 \alpha}$$  \hspace{1cm} (4.1.1)

$$E_y = \frac{3 \sin 2\theta}{4\pi \rho^3 \alpha}$$  \hspace{1cm} (4.1.2)

$$H_z = \frac{\sin \theta}{2\pi \rho^2 \gamma^2} \left[ 3 - (3 + 3\gamma\rho + \gamma^2\rho^2) e^{-\gamma\rho} \right]$$  \hspace{1cm} (4.1.3)

Under the quasistatic assumption for the halfspace, $\alpha = \sigma$ and the $E_y$ field is purely real and has no dependence on frequency (i.e., no dependence on $\gamma$). Also, the $E_x$ fields is composed of a frequency dependent part and a constant term, while $H_z$ has only a frequency dependent term. Clearly, each field contains different information about the halfspace.

The behaviour of the fields can be explained by considering the dipole source to be two distinct sources. One of the sources is inductive. An alternating magnetic field is generated by the current travelling through the wire and this magnetic field induces a frequency dependent current flow within the halfspace. The other source is a galvanic source, by which current enters or leaves the medium through the ends of the wire. This current flow creates a charge distribution on the surface of the halfspace. The two sources combine to produce the observed electric and magnetic fields.
Figure 4.1: The frequency domain electromagnetic fields due to an isotropic half-space of conductivity $\sigma = 1$. 

\[ (g_{-01x}) \frac{\omega}{\Lambda} x E \]

\[ (g_{-01x}) \frac{\omega}{\Lambda} x H \]
Figure 4-2. The time domain electromagnetic fields due to an isotropic halfspace of conductivity $\sigma = 1$. 
Charge distributions exist where the normal component of an electric field crosses a conductivity interface, (or more generally, on conductivity gradients), so that when the halfspace is isotropic, the charge distribution exists only on the air-halfspace interface. In the case of a horizontal line current at the air-halfspace boundary the charge is located only at the grounding points (since the line current creates no vertical electromotance). The electric field of the charge is created instantaneously by the source current, so that the fields generated directly by the charge distribution will have no frequency dependence. This would indicate that the $E_y$ field is a measure of the charge distribution due to current being injected into the ground and may be considered a grounding term.

Maxwell's equation for magnetic fields (2.1.1) dictates that the magnetic field generated directly by the current in the wire circulates around the wire. Since the induced current flows so as to oppose the inducing field, the induced electric current flow will always be parallel to a straight wire. Therefore, the $E_x$ field shows the effect of the charge distribution and the frequency dependent induced currents.

The magnetic field shows a dependence on $\gamma^2 \rho^4$ instead of the $\alpha \rho^3$ dependence shown by the electric fields. By recalling the definition of $\gamma^2$, it is easy to see that the vertical magnetic field has only frequency dependent terms. Therefore, it must reflect only the induced current flow within the halfspace, and shows no effect of the charge distribution. An important advantage of the grounded source method is that the source acts in two different ways, and that the effects of the two effects may be distinguished by the $E_y$ and $H_x$ fields, while the $E_x$ field shows both the effects.

A transversely isotropic halfspace is the next more complicated model to consider. The field components due to a transversely isotropic halfspace are shown in figure 4-3 for the frequency domain, and in figure 4-1 for the time domain. When the halfspace is anisotropic, the $E_y$ field shows a frequency component, or in other words, the charge
distribution has taken a finite time to be created. The reason for this is that the anisotropy provides the equivalent of interfaces within the halfspace on which charge can build up, and that the frequency dependence is due to the time that it takes for changes in current to reach the interfaces. However, no actual interfaces were included in the model. Clearly the assumption that a transversely isotropic medium can be modelled by a set of thin layers of alternating high and low conductivity is valid.

The electric field parallel to the wire has not been changed so dramatically by the introduction of anisotropy. The amplitude of the curve has been decreased, but no other discernable effects can be seen. An interesting point is that at high frequencies, the anisotropic halfspace begins to appear as if it were an isotropic halfspace with conductivity \( \sigma_h \). In this limit, the fields do not penetrate very far into the earth and appear to be plane waves with a wavefront which is perpendicular to the earth's surface. The depth of penetration of the fields is determined by the skin depth of the halfspace;

\[
\delta = \sqrt{\frac{2}{\sigma \mu \omega}} \tag{4.1.4}
\]

where \( \delta \) is the distance (meters) which the fields can penetrate the medium before being attenuated by a factor of \( 1/e \). From these considerations it is obvious that only the horizontal conductivity of the halfspace will be seen at high frequencies.

The vertical magnetic field is sensitive only the horizontal conductivity of the halfspace and is unchanged from that shown in figures 4-1 and 4-2. This should not be too surprising since \( H_z \) is created by horizontally circulating current flow. Therefore, no change is observed in the \( H_z \) field when the vertical conductivity was changed between the model shown in figures 4-1 and 4-2, and the model shown in figures 4-3 and 4-4.
Figure 4-3. The frequency domain electromagnetic fields due to an anisotropic halfspace where $\sigma_h = 1$, and $\sigma_v = 0.25$. 
Figure 4-4. The time domain electromagnetic fields due to an anisotropic halfspace where $\sigma_h = 1$, and $\sigma_v = 0.25$. 
4.2 Electromagnetic Response of an Anisotropic Layer

The physics involved in understanding anisotropy can be applied to either a layer or a halfspace. However, the position within the earth of the anisotropic layer does influence the effect of the anisotropy on the fields measured on the surface of the earth. The reason for this influence is the basic nature of the EM fields which actually reach the anisotropic layer. Gomez-Trevino and Edwards (1983) show in a very graphic manner that direct current resistivity will not see below a highly resistive layer. The reason for this is that the huge conductivity contrast between the resistive layer and the layer directly above it causes a large amount of charge to build up on this interface. This charge effectively prevents current from flowing through the layer. In order to see below this layer, one must rely on induction, which is relatively insensitive to resistive layers. In contrast, very conductive layers can act to mask inductive fields, since the fields tend to expend their energy in creating currents within the conductive layer. Hoversten and Morrison (1983) show this effect very clearly with a set of diagrams which show the electric field strength as a function of depth for a large loop source over a layered earth. Conductive layers also tend to channel the galvanic current so that it is often difficult to penetrate a very conductive layer with any type of EM field.

Except in the case of a very conductive layer, the two sources which make up a grounded wire can provide complementary information; when one source effect fails to recognize a layer, the other source effect can identify it. The only difficulty with the method being that severe conductivity contrasts can limit the depth of penetration.

It was shown in the previous section that the anisotropy of a halfspace is most strongly reflected in the electric fields. The majority of the anisotropic contribution
to the electric fields is from the grounding terms because the galvanic current has strong horizontal and vertical components. In general inductive surface sources show no unambiguous response to anisotropy since no vertical current density is created within the anisotropic layer. Therefore, one should not expect to detect anisotropic behaviour due to a layer that is overlain by a very resistive layer simply because not enough of the galvanic current flow would reach the layer to provide information about it.

In order to demonstrate the effect of an overlying resistive layer, the fields shown in figures 4-5 and 4-6 were calculated. In both these figures, the response has been computed for an isotropic layer within a stratigraphic section, and compared to the response of the same model when this layer was anisotropic. The conductivity as a function of depth is also illustrated, with the horizontal conductivity drawn as a solid line, and the vertical conductivity drawn as a dotted line. The difference between the sets of figures is that in figure 4-6, the overlying layer is an order of magnitude more resistive than in figure 4-5 and the results very clearly show the masking effect of this resistive layer. Not surprisingly, the $E_y$ field shows stronger variations due the anisotropy than the $E_z$ field, since $E_y$ is purely as a result of the galvanic current flow. In figure 4-6, the effect of anisotropy is not seen in the $E_y$ field, but a minute variation in the $E_z$ field can be seen. This is a result of the inductive response of the $E_z$ field. Again, the $H_z$ field is not altered by the anisotropy because the horizontal conductivity was kept constant in the comparisons. Also, the difference in the conductivity of the overlying layer between the two models is seen in all three field components.

Aside from the extreme cases of huge conductivity contrasts where anisotropy will definitely not be detected, the ability of a grounded EM system to find an anisotropic layer has not been properly tested. There are several features of the fields that appear
Figure 4-5. The frequency domain response of a layered earth with a resistive layer overlying an anisotropic layer. Compare with figure 4-6.
Figure 4-6. The frequency domain response of a layered earth with a resistive layer overlying an anisotropic layer. Compare with figure 4-5.
to provide unambiguous information about the extent of anisotropy. First of all, the vertical magnetic field is dependent only upon the horizontal conductivities of the layers in the stratigraphic section, and so separate inversions of electric and magnetic field data using isotropic models would lead to apparent inconsistencies between the models. These inconsistencies would provide grounds for assuming anisotropy is present in the layering.

At very low frequencies, both components of the electric fields show the DC response of the stratification, and any interpretation of the thickness of an anisotropic layer using DC data will be in error by a factor of $\sqrt{K}$ (cf. section 2.4). Therefore, when a model is fitted to the low frequency data, and another model is fit to the data at higher frequencies, the two models will undoubtedly be different. This again would lead the interpreter to assume some form of anisotropy.

A third comparison could lead to the same conclusion as the above two by considering the response at large distances from the wire. This is equivalent to considering the fields at high frequencies since the response curves are essentially scaled by the parameter $|\gamma \rho|$. Figure 4-7 shows that irrespective of the coefficient of anisotropy, the fields at high frequencies tend to the limit shown by fields for an isotropic earth of conductivity $\sigma_h$ (cf. section 4.1). In this figure, the curves marked with an a are for an isotropic halfspace; b indicates a halfspace with $\sigma_h = 1.0$ and $\sigma_v = 0.25$; and c indicates a halfspace with $\sigma_h = 1.0$ and $\sigma_v = 0.1$. Again, model fitting at large distances should prove inconsistent with models for the fields closer to the wire.

Although the above arguments seem to indicate that separate inversions of different sections of the data from an exploration problem should indicate anisotropy, this may not be the case. The data will errors due to noise, errors in positioning the receiver and transmitter, lateral inhomogeneities, subsurface topography, as well
Figure 4.7. A frequency domain comparison between anisotropic halfspaces with a common horizontal conductivity $\sigma_h = 1$, and vertical conductivities of (a) $\sigma_v = 1$, (b) $\sigma_v = 0.25$, and (c) $\sigma_v = 0.1$. 

$\rho = \{1, \ldots, 5\}$.
as surface topography. All these effects serve to make any interpretation process difficult. Whether the inconsistencies suggested above can actually be determined to be outside the standard errors for an inverted model has yet to be determined.

Another approach to the determining the anisotropy would be to invert a model with respect to the coefficient of anisotropy using a joint inversion with the specific intention of including anisotropy. The concepts of joint inversions have been around for some time (Jupp and Vozoff, 1975; Vozoff and Jupp 1975), and were part of the motivation for developing the integration scheme described in section 3.2. Joint inversion does require a significant effort to calculate the forward model for all the components that go into the inversion. In the case of determining anisotropy, a joint inversion of the EM data would require;

1. the electric and magnetic field components described in this thesis
2. knowing the components of the fields over a broad frequency range,

and

3. knowing this information at several receiver positions.

While this type of inversion requires a substantial effort to carry out, the gains in terms of numerical stability and extra information make it well worth while. After the inversion, an estimate of the anisotropy of each layer would be available to the interpreter, and from this a decision about the validity of the anisotropic assumption could be made. Even in the case where the layers are all isotropic, having some idea of the apparent anisotropy of a layer would indicate when layers should be added to, or removed from the model.
Chapter 5

Induced Polarization

Frequently, electrical surveys are carried out to measure the low frequency electrical phenomenon often exhibited by rocks known as Induced Polarization (IP). In fact, some recent work has been applied to relating IP to the formation factor of a rock unit (cf. Worthington and Collar (1982)). The fact that most rocks exhibit an IP effect to some degree indicates that IP should be considered when performing any type of electrical survey. An important problem with electrical surveys is that the electromagnetic induction effects and IP effects can occupy similar regions in the frequency domain. In general, induction effects arise at higher frequencies or shorter decay times than IP effects and have different frequency dependence, but often the effects overlap and are not easily interpretable.

The usual treatment of IP and EM induction considers the processes as separate, contributing additively to the total response. The IP response is calculated using DC theory, and the electromagnetic response by using Maxwell's equations and assuming non-dispersive conductivities and no interface impedances. However, such an approximation is unnecessary for the stratified earth algorithm developed in this thesis, where complex conductivities are permitted.

Most of the work on IP responses of a layered earth has specifically dealt with determining the EM response, and methods of removing it from field data. The purpose of this chapter is to model some simple IP responses in a halfspace, and from these models, deduce some of the effects of induced polarization on EM sounding problems and check the validity of the approximate additive calculation method.
5.1 Induced Polarization Models

Induced polarization is an electrochemical phenomena which causes frequency dispersive behaviour of electromagnetic fields in rocks. Induced polarization within a macroscopically uniform medium may be represented by very simple models, even though the subtleties of the theory are not well understood. The conductivity of most rocks is largely dependent upon the quantity and salinity of the interstitial fluid. Usually, the major component of current flowing through the ground is in the form of ions. The diffusion of the ions may be impeded by electrochemical processes which are dependent upon the frequency of the applied field. Quite simply, at high frequencies there is usually less impedance from the electrochemical processes and the conductivity of the medium will be larger than that at low frequencies.

Materials which show IP effects are often classified into two groups;

(1) those which exhibit electrode polarization, and

(2) those which exhibit membrane polarization.

Electrode polarization results from charge transfer reactions at metal-electrolyte surfaces. Such boundaries arise when grains of semi-conducting minerals such as pyrite are situated in the electrolytically conducting pore network of a rock, or where a large semi-conducting formation is surrounded by electrolytically conducting host rock. The interfacial phenomena are well represented by complicated electrochemical models (Wong (1979)), or in a simpler manner by a surface impedance (Wait (1983)). The two representations are basically equivalent, with the former somewhat more suited to representing disseminated and the latter massive mineralization.

Membrane polarization does not show such large changes of conductivity with respect to frequency that can characterize electrode polarization, but it is more
widespread in sedimentary environments (Roy and Elliot (1980)). Clays are the most important mineral group causing membrane polarization, although certainly not the only one. For example, Roy and Elliot (1980) suggest small amounts of shales can significantly increase the IP response of a rock, and Nelson et. al.,(1979) find carbonaceous siltstones and zeolites give IP responses. Since the major groups of clay minerals occur in sedimentary rocks of all ages and in almost all proportions, membrane polarization is almost ubiquitous in sedimentary environments. The proposed mechanism for this effect is that a cationic cloud tends to gather near charge deficiencies on the clay-electrolyte interface. The deficiencies arise from isomorphic substitution of Al$^{3+}$ for Si$^{4+}$, or Mg$^{2+}$ for Al$^{3+}$ within the clay lattice. If the pore space containing the electrolyte is small enough, the cationic cloud may act to block anion diffusion through the pore which results in a resistance to current flow. The barrier is frequency dependent and thus may be considered an impedance rather than a resistance. In igneous and metamorphic rocks membrane polarization is due to surface effects in microcracks. Since membrane polarization is generally not a local feature, it is appropriate to describe a stratigraphic unit by a single average frequency dependent conductivity.

Several empirical models for frequency dependent conductivity have been proposed, but the Cole-Cole model (Cole and Cole (1949)) has gained the most acceptance on the geophysical community. This model was originally proposed to describe dispersive behaviour in dielectrics, but is easily adapted to conductivity models. Several other models have been proposed, however, most may be considered a subset of the Cole-Colé model (Pelton (1977)).

The Cole-Cole model for frequency dependent conductivity may be written as

$$\sigma(\omega) = \frac{\sigma_L [1 + (i\omega\tau)^\delta]}{1 + (1 - m)(i\omega\tau)^\varepsilon},$$

(5.1.1)
where $\omega$ is the angular frequency. The model contains four parameters $(m, \sigma_L, \tau, c)$ which must be chosen to give a physically realistic model of a conductivity spectrum. The time constant, $\tau$, determines the frequency $(2\pi\tau)^{-1}$ about which the conductivity spectrum is centered. $c$ is called the frequency dependence and controls the range over which the conductivity spectrum is spread. The dimensionless parameter $m$ controls the strength of the IP effect and is the chargeability proposed by Seigel (1959), defined as

$$m = \sigma_H - \sigma_L \over \sigma_H$$

(5.1.2)

where $\sigma_L$ is the conductivity at very low frequency and $\sigma_H$ is the conductivity at very high frequency.

The physical ranges of the model parameters have been studied in great detail by Pelton (1977), and from these results, some general limits may be set.

$$0.10 \leq c \leq 0.5$$
$$10^{-5} \leq \tau \leq 10^5$$
$$0 \leq m \leq 1.0$$

Since only membrane polarization is to be considered here, the conductivity variation with frequency should not be greater than 20 to 30% and is frequently much less, consequently, $m$ was chosen to have a value of 0.3. Admittedly, this is an extreme value for membrane polarization, but it is useful for the purposes of illustration of IP effects. Choosing the time constant is more difficult, but different time constants only have the effect of shifting the conductivity spectrum in time or frequency. Therefore, a single value representative of $\tau$ will not be chosen a priori. The frequency dependence is also difficult to choose, although Pelton suggests an average of 0.25 for electrode polarization of mineralized samples. Figures 5-1 and 5-2 show the amplitude and phase spectra of the Cole-Cole model for $m = 0.3$, $\sigma_L = 1.0$.
and two different time constants. The frequency dependence was chosen to be 0.5 in order that the conductivity spectrum would be relatively compact in the frequency domain. A frequency dependence of 0.25 (which was used in the model studies) results in a much broader conductivity spectrum.

5.2 Induced Polarization in a Layered Earth

In order to understand induced polarization effects in an EM sounding problem, it is highly desirable to examine the EM response and the IP response separately. The EM response is easily calculated from a non-polarizable layered earth model. As mentioned, IP responses are usually calculated using DC resistivity theory (i.e., no magnetic induction) since the algorithms are much simpler than the EM induction case. Such a calculation can be effected without any changes to the EM algorithm by introducing a frequency dependent conductivity with an extremely large time constant ($\tau = 10^{+8}$) for the layers that is to be polarizable. A large time constant insures that the IP response will be positioned at very low frequencies (of the order $10^{-7}$ Hz) where the EM response will be essentially constant (at the DC value). Since the IP time constant is independent of scale or DC conductivity, the calculated IP response (independent of the EM response) is known for any time constant. The is clearly shown in figures 5-1 and 5-2. The example also demonstrates that when the EM time constant and IP time constant are sufficiently different, the IP response would be easily interpreted.

Figure 5-3 shows a halfspace IP response (with an IP time constant of 0.1 seconds) without any EM induction. The model was calculated with $\sigma_L = 0.7$ S/m, $m = 0.3$, $c = 0.25$, and $\tau = 10^8$ seconds and the spectrum was shifted seven
**Figure 5-1** The amplitude of the conductivity spectrum for a Cole-Cole frequency dependent conductivity model.

**Figure 5-2** The phase of the conductivity spectrum for a Cole-Cole frequency dependent conductivity model.
Figure 5-3. The induced polarization response of a halfspace with no EM induction.
decades in frequency. The response incorporates the effects of the receiver-transmitter orientation, the IP response of the halfspace, and the direct current component of the fields.

In this model the behaviour of the two electric field components is very similar, while the vertical magnetic field shows no response at all in the time domain (i.e. constant in the frequency domain). Since EM induction is not included, and the discharge current system of the IP creates no external magnetic field, it is not surprising that there is no magnetic field (Edwards, 1983).

In order to show how the total (EM+IP) response of a polarizable halfspace differs from the response of a non-polarizable halfspace, a comparison between the two models was made. A polarizable halfspace with Cole-Cole parameters $\sigma_L = 0.7$, $m = 0.3$, $c = 0.25$, and $\tau = 0.1$ was compared with a non-polarizable halfspace of conductivity $\sigma = 1.0$, and figure 5-4 shows the comparison between the $E_z$, $E_y$ and $H_z$ fields. In this figure, the solid line represents the response of a non-polarizable halfspace, while the dashed lines represent the response of the polarizable halfspace. The conductivity range of the polarizable earth was chosen such that the conductivity of the models was the same at high frequency (early time).

The $E_y$ component demonstrates the most diagnostic change due to the frequency dependent conductivity. The $E_y$ component has an instantaneous decay for a non-polarizable halfspace, but shows a decay when the halfspace is polarizable. Obviously, when the conductivity of the halfspace changes with frequency, the conductivity contrast between the air and halfspace also changes. Once the source current is turned off, no charge exists on the interface and there is a slowly changing dipole distribution of charge inside the medium (not unlike the effect seen in an anisotropic medium). This implies that the charge distribution is decaying with conductivity. The form of the decay is not like an EM decay due to layering within the halfspace, so that the
Figure 5-4. A comparison between the response of a polarizable halfspace (dotted line) and a non-polarizable halfspace (solid line).
physical distribution of charge is not altered, but rather the amplitude is decreased with time. Also, the amplitude of the $E_y$ field is an order of magnitude smaller than the $E_z$ field, indicating the relative magnitudes of the IP response and the EM response.

Now that the behaviour of the galvanic current has been qualitatively described, one must turn to the $H_z$ field to deduce the IP effect due to the induced currents in the halfspace. The only change in the response between the two models for the $H_z$ component is a shift in time of the response curve. This shift is again a direct result of the conductivity change with respect to time, and can be described in terms of the EM response parameter $|\gamma_0|$. Since none of the other factors in the response parameter are changed, the net result is to shift the curve in frequency and hence in time. Unfortunately, this effect would be difficult to detect in actual field data.

The $E_z$ field is a combination of the above effects. However, if the IP effect was unrecognized, the basic shape of the response curve might appear to be due to layering. The time decay due to IP in the galvanic component of $E_z$ is for the most part lost in the time decay of the halfspace response, although some distortion of the early time response can be seen. The fact that the IP response can be present, yet so easily masked by the EM response indicates something of the importance of recognizing that any strong IP effects are present when interpreting the EM transient.

There are several additional complications when the halfspace is stratified. The $E_y$ component will be determined by the charge distribution within the halfspace, and so will exhibit a slow decay. The decay of the charge may tend to obscure the IP response but not as severely as in the $E_z$ component. Also, the majority of the information about the IP response is contained in the galvanic component of the field. Since the galvanic component of the source was also the most important in resolving anisotropy, one should expect that the same arguments about the basic
nature of the EM fields within a layered halfspace given in Chapter 4 may be applied here. Specifically, a resistive layer which overlies a polarizable layer will prevent the galvanic current from reaching the polarizable layer, thereby reducing the ability to detect the IP response.

5.3 Interpretation of IP Responses in a Layered Earth

There is a great deal of information contained in both the EM response, and in the IP response of a stratified earth. Unfortunately, not all of this information is used in presently existing electrical methods. For example in the full spectrum IP method (complex resistivity method), measurements are made over a wide range of frequencies which are generally below the EM response range, and only the IP data is interpreted. However, since the IP response is generally spread over many decades of frequency, it is rare that the entire IP spectrum can be recorded.

The problem of separating the IP and EM responses has received a great deal of consideration in the literature (eg. Dey and Morrison (1973), Hallof (1974), Pelton (1977), and Major and Silic (1981)). The usual assumption made about IP and inductive EM responses is that they are additive at least in the time or frequency range where the EM response is of similar intensity to the IP response, and this forms the basis for much of the interpretation of field data. Since it is possible to calculate the responses independently, it is an interesting exercise to test this hypothesis.

Consider first the response of a polarizable halfspace when the IP time constant has different values as shown in Figure 5-5. The parameters of the model are the same as those for figure 5-1 with the exception of the changing IP time constants. The figures show a number of interesting features. The \( E_p \) component simply displays the
Figure 5.5. The response of a polarizable halfspace for various IP time constants.
changing conductivity of the halfspace, while the $E_z$ field is more complicated. When the IP time constant is 1 second, the IP effect is only seen at late times as a change of sign in the decay curve (Lee (1981)). This decay does not interfere significantly with the EM response which has a time constant of approximately 0.05 seconds. The IP time constants for the other two curves are very close to the EM time constant. Since the time constant of the EM response depends on the conductivity of the medium which is changing at a comparable rate, the IP effects and EM effects interact, resulting in a more complicated decay waveform. Interpretation of a decay curve like that for $\tau = 0.1$ or 0.01 would be complicated because it would be difficult to determine if the shape of the decay curve was due to layering or due to IP effects. In a case where the change in conductivity with frequency is very large, the response curves become so distorted that IP effects would be recognized, but even if the change in response could be detected, it would be very difficult to separate the effects.

Consider next whether the addition of the separately calculated IP and EM responses correctly models the total response of a polarizable halfspace. The curve shown in figure 5-6 indicated by a dotted line represent the IP response in the absence of EM induction. The dashed line indicates the EM response of the halfspace, and the solid line is the sum of the responses due to a polarizable halfspace independent of IP effects (cf. figure 5-3), and the EM response of a non-polarizable halfspace. In both cases, the time constant for the IP response was $\tau = 0.1$, and the other model parameters are unchanged from the previous models.

In figure 5-7, the summed EM and IP responses are compared with the true response of a polarizable halfspace. The $E_y$ components for the two models are identical, however, the $H_z$ components appear shifted slightly, and the $E_z$ fields are not at all similar. That the $E_y$ field appears unchanged should not be too surprising.
Figure 5-6. This figure shows a comparison between the IP response independent of EM induction (dotted line), the EM response (dashed line) and the sum of the IP and EM responses (solid line) for a halfspace.
Figure 5-7. A comparison between the additive model for the IP and EM response (dotted line) and the response of a polarizable halfspace (solid line).
The summation of the IP response with the isotropic halfspace response results in exactly the IP response, since there is no \(E_y\) component for an isotropic halfspace. Therefore, in this special case of an isotropic halfspace, the \(E_y\) response curves are additive.

For the \(H_z\) component of the fields, the IP response is zero. Therefore, the observed response over a polarizable halfspace is simply the EM response with an adjusted time constant that is due to the changing conductivity. The time constant for the \(H_z\) field is dependent upon the conductivity of the medium, and so the changing conductivity does shift the response in the time spectrum. Therefore, the IP and EM responses for the vertical magnetic field are never additive since, the pure IP response of this field will always be very small, or zero.

The \(E_z\) component is more complicated than either of the other two components. In this component, neither of the separate responses are zero, and clearly the summation of separately calculated responses does not give a good approximation to the field. This will also be true of the \(E_y\) field when the earth is layered. In fact, since the inductive component of the current flow in the earth has such a small effect on the \(E_z\) field, the \(E_z\) and \(E_y\) fields should be affected in an almost identical manner.

Obviously, the separation of EM responses and IP responses is difficult when their time constant spectra overlap significantly, and it is not simply a matter of finding the individual responses and correcting for one or the other by subtraction. It should be emphasized that the models presented here are extreme, and that in most cases, the IP and EM time constants are well separated. Figures 5-8 and 5-9 show a comparison between the EM halfspace response (the solid line) and the response of a polarizable halfspace with \(\tau = 1\) (the dotted line). With the IP time constant larger than the EM time constant, it is quite easy to distinguish the presence of IP due to the change in sign of the curve (dashed line).
Figure 5-8. A comparison between the $E_z$ fields for a non-polarizable halfspace (solid line) and a polarizable halfspace (dotted line; the dashed line indicates a negative response).
Figure 5-9. A comparison between the $E_x$ fields for a non-polarizable halfspace (solid line) and a polarizable halfspace (dotted line; the dashed line indicates a negative response).
When the EM and IP time constants are different, the responses are not completely additive. Therefore, any interpretation using data from which either the EM or IP response was empirically subtracted may be unable to determine the correct conductivity of the polarizable layer. The extent of the error in interpretation will depend upon the strength of the IP response, as well as the separation of time constants of the two responses. A more reasonable approach might be to estimate the parameters of a simple conductivity model and fit the data to a model which combines IP and EM effects. The resulting layered earth parameters or IP parameters could be modelled individually to interpret the separate effects.

Since the time constant of the EM response increases when the distance from the transmitter to the receiver increases, it may be possible to detect the high frequency part of the IP effects in electric fields measurements made close to the wire, whereas only the lower frequency part can be determined at larger distances.

One method of detecting whether there is an IP response in the field data is using the method of transformation between time and frequency domain outlined in section 2.6. This routine finds a set of time constants of simple loop responses that may be added together to obtained the EM response. When IP effects are present, it should be possible to detect two sets of time constants which are required to fit the EM and IP responses. Again, the ability to detect the IP response will depend on the separation of the time constants.

An example of this procedure is shown in figure 5-10, where an IP response with time constant \( \tau = 0.01 \) is seen in the \( E_x \) data. Plotted with this figure are the time constants required to transform the frequency domain data into time domain. There are two sets of time constants which are recognizable with the dominant set being the EM time constants. The other set has very low amplitude and is spread over a large frequency range (\( 10^{-3} \) to \( 10^9 \) sec\(^{-1} \)), and corresponds to the change of sign in
Figure 5-10  An example of using YVESFT to determine the time constant of the IP response.
the decay curve at 0.1 seconds. Although the amplitude of the IP time constant set is small, it is clearly recognizable as not being part of the EM decay.
Chapter 6

Conclusions

6.1 Summary

A major problem with electromagnetic exploration in a sedimentary environment is that the response of a layered earth is difficult to interpret. The field responses are subtle and consequently, direct inversion of EM sounding data is an extremely powerful tool for interpretation of layer thicknesses and conductivities. To implement an inversion scheme, one requires an efficient algorithm which can calculate the response of an arbitrarily layered earth to a large source EM system. Also, the basic physics behind the EM layered earth problem can be easily investigated by the use of such an algorithm. Therefore, in this thesis, a completely general solution to the layered earth problem was developed. The ability to model anisotropic or polarizable layers was included as a first step towards modelling realistic geological environments.

In order to obtain computational efficiency, the concept of lagged convolution using a Fast Hankel Transform algorithm was employed. This permits the evaluation of several Hankel transforms at various radii from the source for slightly more effort than goes into calculating the transform at a single point.

The ability to evaluate the fields at several radii is augmented by an integration procedure developed specifically to find the response of any large source. The Integration by Weighted Sums (IWS) procedure is sufficiently general that it may be used to model a number of sources, but was mainly used to model the grounded electric dipole source. The IWS method essentially evaluates an integral once by
finding a general set of integration weights which can be used to find the response of any layered earth model for a given receiver-transmitter configuration. The combination of these two techniques will hopefully reverse the present thinking that it is much too expensive to do routine interpretation of large source responses using generalized inversion schemes.

The concept of considering a grounded source as having both galvanic and inductive components is useful when explaining the field responses to halfspace and layered earth models. In fact, the entire layered earth solution (developed in Chapter two and Appendix B) was developed in terms of galvanic and inductive source components because of their conceptual simplicity.

Three components of the electric and magnetic fields were considered; $E_z$, $E_y$, and $H_z$ (Eadie (1981)). It was deduced that the vertical magnetic field is generated by the inductive component of the source. The electric field perpendicular to the wire is generated by the galvanic component, while the electric field parallel to the wire is a combination of the two components.

The effective penetration into the earth is different for the fields from the two types of sources, and so a grounded bipole can provide information about a wide variety of geological environments. Very conductive layers act to mask the inductive (and galvanic to some extent) source components, while very resistive layers mask the galvanic source components. Clearly, the advantage of the grounded source is that the two source components act in a complementary manner.

In general, the galvanic component of the source is sensitive to both anisotropy and IP, while the inductive component is not. Therefore, by interpreting fields with consideration to which type of source they arise from, it should be possible to resolve both anisotropy and IP in a layered earth. The condition on the ability to resolve
anisotropy is that a significant galvanic component of the current must penetrate the anisotropic layer. From the examples presented in Chapters 4 and 5, it is obvious that some account of anisotropy and IP must be taken in order to interpret field data properly.

It would appear that the IP response and EM response are not simply additive, and more work should be done on determining a procedure for separating the responses and interpreting the effects individually. When the IP time constant is sufficiently different from the EM time constant, the additive model may be useful for interpretation purposes. The most likely approach to the separation problem when the IP and EM time constants are similar would be to find the parameters of a simple frequency dependent conductivity model by inversion, and use this to calculate the individual responses.

### 6.2 Future Work

A vast amount of work on the problem of a grounded bipole remains to be done. More work should be done on understanding the EM fields within the earth about a grounded bipole source. It may be that a borehole receiver (or bipole source) would provide invaluable data about deeply buried layers. There are innumerable applications of the algorithms developed in this work, and a great deal can be learned about the methods which optimize the data collected in a particular geological setting.

The IWS algorithm has worked so well for integrating EM fields with respect to distance, that other applications of the procedure should be considered. For example, Walker (personal communication) is attempting to find the volume integral of EM sources in a conductive plate imbedded in a conductive medium using an adaptation
of the IWS procedure.

Inversion algorithms should be written to test some of the ideas outlined in Chapters 4 and 5. The routines should be able to invert the data with respect to the horizontal and vertical conductivities of a layer, and the layer thickness. Ideally, a joint inversion process should be constructed that makes use of data from various receiver orientations. Also, the parameter information density, and parameter correlation matrices which result from the inversion process would provide invaluable information on exactly what the resolution capabilities of a grounded bipole really are. To facilitate the inversion schemes, the optimum receiver-transmitter configurations for a particular problem should be considered, and a data-base of integration weight sets should be constructed. Since most inversion schemes are iterative, it might be possible to store some simple models for a first approximation. This would substantially reduce the effort of converging to a set of parameters by effectively reducing the number of iterations.

Another major consideration is obtaining good quality field data in controlled environments so that a comparison with the mathematical modelling can be done. The importance of this can not be over-emphasized, since the final test of any exploration method is its success under actual field conditions. Field data and laboratory data are needed in order to estimate the importance of anisotropy and induced polarization within a sedimentary environment. Also, the logistics involved in performing actual surveys must be thought out in order to maximize the information obtained, and minimize the costs involved.
References


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Appendix A

Bessel Function Relationships

Recursive Relationship

\[
J_{n+1}(\lambda \rho) = \frac{2}{\lambda \rho} J_n(\lambda \rho) - J_{n-1}(\lambda \rho)
\]  
(A.1)

Differential Relationships

\[
\frac{\partial J_0(\lambda \rho)}{\partial \rho} = -\lambda J_1(\lambda \rho)
\]  
(A.2)

\[
\frac{\partial J_1(\lambda \rho)}{\partial \rho} = \lambda J_0(\lambda \rho) - \frac{J_1(\lambda \rho)}{\rho}
\]  
(A.3)

\[
\frac{\partial J_0(\lambda \rho)}{\partial x} = -\lambda \cos \theta J_1(\lambda \rho)
\]  
(A.4)

\[
\frac{\partial J_0(\lambda \rho)}{\partial y} = -\lambda \sin \theta J_1(\lambda \rho)
\]  
(A.5)

\[
\frac{\partial J_0(\lambda \rho)}{\partial x \partial y} = \sin 2\theta \left[ \frac{\lambda}{\rho} J_1(\lambda \rho) - \frac{\lambda^2}{2} J_0(\lambda \rho) \right]
\]  
(A.6)

\[
\frac{\partial^2 J_0(\lambda \rho)}{\partial x^2} = \cos 2\theta \left[ \frac{\lambda}{\rho} J_1(\lambda \rho) - \frac{\lambda^2}{2} J_0(\lambda \rho) \right] - \frac{\lambda^2}{2} J_0(\lambda \rho)
\]  
(A.7)
Integral Relationships

\[ \int_0^\infty J_0(\lambda \rho) \, d\lambda = \frac{1}{\rho} \]  \hspace{1cm} (A.8)

\[ \int_0^\infty \frac{\lambda}{\lambda + u} J_0(\lambda \rho) \, d\lambda = \frac{1}{\gamma^2 \rho^3} \left[ 1 - (1 + \gamma \rho) e^{-\gamma \rho} \right] \]  \hspace{1cm} (A.9)

\[ \int_0^\infty u J_1(\lambda \rho) \, d\lambda = \frac{\gamma}{\rho} \left[ 1 + \frac{e^{-\gamma \rho}}{\gamma \rho} \right] \]  \hspace{1cm} (A.10)

\[ \int_0^\infty (v - u) J_1(\lambda \rho) \, d\lambda = \frac{1}{\rho^2} \left[ K^{\frac{1}{2}} e^{-\gamma \rho K^{-\frac{1}{2}}} - e^{-\gamma \rho} \right] \]  \hspace{1cm} (A.11)
Appendix B

The General Layered Earth Solution

The solution to the layered earth problem involves finding the equations describing the electric and magnetic fields in a whole space in terms of the vector potentials \( \Pi \) and \( \Gamma \). From Chapter 2, the potentials that are generated by an arbitrary electric or magnetic source in a whole space are given by equations (2.4.15) and (2.4.16). Now the potentials at any point due to a given source distribution must be found. Then the total transformed potentials can be found at the source coordinates and used to find the fields at the receiver coordinates. This is done in two stages; first the transformed potentials are propagated in the \( z \) direction to the receiver level, and then the fields are found from the potentials and converted to the space domain by a 2-D Fourier transformation (or equivalently, a Hankel Transform).

The introduction of layering to this problem results in two complexities. The first is that the total potential at the source level is the sum of the primary potential due to the source, and a scattered potential that is the result of the layering. The second problem is that the total potential must be propagated across layers of different properties, and hence interfaces between these layers.

The coordinate system used here is chosen to make the mathematics comprehensive. In order that a potential which propagates away from the source be represented in a consistent manner, a new variable \( d \) is introduced such that

\[
d = |z - \varsigma|
\]  

(B.1)

where \( \varsigma \) is the \( z \)-coordinate of the source. The numbering scheme that will describe the layering is shown in figure B-1. This numbering scheme, together with the newly defined coordinate system allows the layering on either side of the source to be treated in the same manner, regardless of the relative positions of the source and receiver.
Figure B-1. The general numbering scheme for the layered earth problem. Note that the numbering scheme is identical for layers on either side of the source level.
The solution in $\lambda$ space of the homogeneous Helmholtz equations for $\Pi_z$ and $\Gamma_z$ can be obtained by the method of separation of variables as

$$\hat{\Pi}_z(p, q, z - \zeta) = \hat{\Pi}_z(0, 0, -\zeta) e^{-ipx} e^{-iqy} e^{-ud} \quad (B.2)$$

$$\hat{\Gamma}_z(p, q, z - \zeta) = \hat{\Gamma}_z(0, 0, -\zeta) e^{-ipx} e^{-iqy} e^{-ud} \quad (B.3)$$

Knowing these functional forms, it is easy to deduce the spatial derivatives of the potentials. Also, the potentials may be continued vertically from any level at which they are known, to any other level in the same medium (as long as the source level is not crossed).

The first step in the solution of the layered earth problem is to find the total potential at the source point. The source itself generates a primary potential that propagates away from the source. When this primary potential encounters an interface (property contrast), part of it is transmitted through the interface, while the rest is reflected back towards the source. Therefore, the total potential at the source consists of the primary potential plus the reflected part due to the layering on both sides of the source. A convenient way of representing the effects of an interface is through a reflection coefficient.

$$R_{\Pi}(i) \equiv \frac{\hat{\Pi}^\text{in}_z(d_{i-1} + 0)}{\hat{\Pi}^\text{out}_z(d_{i-1} + 0)}, \quad R_{\Gamma}(i) \equiv \frac{\hat{\Gamma}^\text{in}_z(d_{i-1} + 0)}{\hat{\Gamma}^\text{out}_z(d_{i-1} + 0)} \quad (B.4a, 4b)$$

where the subscripts $\text{in}$ and $\text{out}$ refer to potentials which are traveling towards or away from the source (cf. figure B-2). These reflection coefficients are defined on the far side (i.e., the side furthest away from the source) of the $i^{th} - 1$ interface as indicated by the notation $(d_{i-1} + 0)$. By using ratios of the potentials, one need not consider the multiple reflections of potentials since these reflections are implicitly included in the potential ratios. Also, the reflection coefficient $R$ is a property of the potentials and not the interface, however, it does depend on the environment in which the interface is situated.

An extremely useful property of reflection coefficients is that they may be calculated in a recursive manner. That is, the reflection coefficient for one interface is
directly related to the reflection coefficient for an adjacent interface. Therefore, the potentials at the $d_{i-1}$ and $d_i$ interfaces bounding the $i^{th}$ layer can be related to derive the reflection coefficients for each of the two interfaces.

\[
\tilde{\Pi}_x^\text{out}(d_{i-1} + 0) = \tilde{\Pi}_x^\text{out}(d_i - 0) e^{-v_i t_i}; \quad (B.5)
\]

\[
\tilde{\Pi}_x^\text{in}(d_{i-1} + 0) = \tilde{\Pi}_x^\text{in}(d_i - 0) e^{v_i t_i}; \quad (B.6)
\]

where $t_i$ is the thickness of the $i^{th}$ layer. From these two equations, it follows that

\[
R_{\Pi}(i) = \frac{\tilde{\Pi}_x^\text{in}(d_i - 0)}{\tilde{\Pi}_x^\text{out}(d_i - 0)} e^{2v_i t_i}. \quad (B.7)
\]

The potentials at any point are composed of an outward component, and an inward component so that the potential and the vertical derivative of the potentials in terms of these components are;

\[
\tilde{\Pi}_x(d_i - 0) = \tilde{\Pi}_x^\text{out}(d_i - 0) + \tilde{\Pi}_x^\text{in}(d_i - 0) \quad (B.8)
\]

\[
\tilde{s}\tilde{\Pi}_x(d_i - 0) = -v_i \tilde{\Pi}_x^\text{out}(d_i - 0) + v_i \tilde{\Pi}_x^\text{in}(d_i - 0) \quad (B.9)
\]

Combining equations (B.7) with (B.8) and (B.9) results in two new equations,

\[
\tilde{\Pi}_x(d_i - 0) = \left[ 1 + \frac{e^{2v_i t_i}}{R_{\Pi}(i)} \right] \tilde{\Pi}_x^\text{in}(d_i - 0) \quad (B.10)
\]

\[
\tilde{s}\tilde{\Pi}_x(d_i - 0) = v_i \left[ 1 - \frac{e^{2v_i t_i}}{R_{\Pi}(i)} \right] \tilde{\Pi}_x^\text{in}(d_i - 0) \quad (B.11)
\]

Now, in order to account for the property contrast that the $i^{th}$ interface actually represents, the boundary conditions must be applied. The boundary conditions insist that the tangential components of $\mathbf{E}$ and $\mathbf{H}$, and the normal components of $\mathbf{J}'$ (the total current; displacement + conduction) and $\mathbf{B}$ be continuous across the interface. These conditions can be represented in terms of the potentials by recalling
the definitions of the fields.

\[
E = -\gamma^2 \Pi + \alpha_h^{-1} \nabla \cdot (\alpha \Pi) - \alpha_h \alpha^{-1} \beta \nabla \times \Gamma \quad (B.12)
\]

\[
H = -\gamma^2 \Gamma + \alpha_h^{-1} \nabla \cdot (\alpha \Gamma) + \nabla \times (\alpha \Pi) \quad (B.13)
\]

Writing the components of equations (B.12) and (B.13),

\[
E = \begin{bmatrix}
-\alpha_h \beta \Pi_x + 11 \Pi_x + 12 \Pi_y + \frac{1}{K} \left( \frac{1}{2} \right) 13 \Pi_z - \beta \left( \frac{2}{5} \Pi_x + \frac{3}{5} \Pi_y \right) \\
-\alpha_h \beta \Pi_y + 12 \Pi_x + 22 \Pi_y + \frac{1}{K} \left( \frac{1}{2} \right) 23 \Pi_z - \beta \left( \frac{3}{5} \Pi_x + \frac{1}{5} \Pi_z \right) \\
-\alpha_v \beta \Pi_z + 13 \Pi_x + 23 \Pi_y + \frac{1}{K} \left( \frac{1}{2} \right) 33 \Pi_z - \beta K \left( \frac{1}{5} \Pi_y + \frac{2}{5} \Pi_z \right)
\end{bmatrix} \quad (B.14)
\]

\[
H = \begin{bmatrix}
-\alpha_h \beta \Gamma_x + K \left( \frac{1}{2} \Pi_x + \frac{1}{2} \Pi_y \right) + 13 \Pi_z + \alpha_v 2 \Pi_x - \alpha_h 3 \Pi_y \\
-\alpha_h \beta \Gamma_y + K \left( \frac{1}{2} \Pi_y + \frac{1}{2} \Pi_z \right) + 23 \Pi_z + \alpha_h 3 \Pi_x - \alpha_v 1 \Pi_z \\
-\alpha_h \beta \Gamma_z + K \left( \frac{1}{2} \Pi_z + \frac{1}{2} \Pi_y \right) + 33 \Pi_z + \alpha_h 1 \Pi_y - \alpha_v 2 \Pi_x
\end{bmatrix} \quad (B.15)
\]

Since only the \(\Pi_z\) and \(\Gamma_z\) components of the potentials are required to describe the fields, the above equations can be simplified as shown in (B.16) and (B.17).

\[
E = \begin{bmatrix}
\frac{1}{K} 13 \Pi_z - \beta 2 \Pi_x \\
\frac{1}{K} 23 \Pi_x + \beta 1 \Pi_z \\
\frac{1}{K} \left[ 33 \Pi_z - \gamma^2 \Pi_z \right]
\end{bmatrix} \quad (B.16)
\]

\[
H = \begin{bmatrix}
13 \Gamma_z + \alpha_v 2 \Pi_x \\
23 \Gamma_x - \alpha_v 1 \Pi_z \\
33 \Gamma_z - \gamma^2 \Gamma_z
\end{bmatrix} \quad (B.17)
\]

Alternate forms for the \(z\) components of the fields can be found from the homogeneous Helmholtz equation.

\[
\frac{1}{K} \left[ 33 \Pi_z - \gamma^2 \Pi_z \right] = 11 \Pi_z + 22 \Pi_z \quad (B.18)
\]
\[ s_3 \Gamma_z - \gamma^2 \Gamma_z = 11 \Gamma_z + 22 \Gamma_z \quad (B.19) \]

At any given position \((x, y, d_i + 0)\) on the \(i^{th}\) interface, the derivatives with respect to \(x\) and \(y\) will be unchanged from the derivatives at \((x, y, d_i - 0)\). Therefore, the continuity of the fields is assured only if

\[ \alpha_v \Pi_z, \quad \frac{v}{K} \Pi_z, \quad \beta \Gamma_z, \quad u \Gamma_z \quad (B.20) \]

are continuous everywhere. Using the above boundary conditions, equations (B.10) and (B.11) can be written as,

\[ \hat{\Pi}^\text{out}_z (d_i + 0) + \hat{\Pi}^\text{in}_z (d_i + 0) = A_i \left[ 1 + \frac{e^{2v_i t_i}}{R_{\Pi}(i)} \right] \hat{\Pi}^\text{in}_z (d_i - 0) \quad (B.21) \]

\[ \hat{\Pi}^\text{out}_z (d_i + 0) - \hat{\Pi}^\text{in}_z (d_i + 0) = V_i \left[ 1 - \frac{e^{2v_i t_i}}{R_{\Pi}(i)} \right] \hat{\Pi}^\text{in}_z (d_i - 0) \quad (B.22) \]

where

\[ A_i \equiv \frac{\alpha_{v,i}}{\alpha_{v,i+1}}, \quad V_i \equiv \frac{v_i}{v_{i+1}} \frac{K_{i+1}}{K_i} \]

By eliminating the \(\hat{\Pi}^\text{in}_z (d_i - 0)\) term and using the definition for \(R_{\Pi}(i+1)\), the two equations can be solved to give

\[ R_{\Pi}(i) = \left[ \frac{(V_i - A_i) + R_{\Pi}(i+1)(V_i + A_i)}{(V_i + A_i) + R_{\Pi}(i+1)(V_i - A_i)} \right] e^{-2v_i t_i} \quad (B.23a) \]

This form of the reflection coefficient is numerically unstable for large conductivity contrasts, and can be rearranged to a more robust form.

\[ R_{\Pi}(i) = \left[ \frac{V_i (R_{\Pi}(i+1) + 1) + A_i (R_{\Pi}(i+1) - 1)}{V_i (R_{\Pi}(i+1) + 1) - A_i (R_{\Pi}(i+1) - 1)} \right] e^{-2v_i t_i} \quad (B.23b) \]

The development for \(R_{\Gamma}\) follows the above argument, and the reflection coefficient is

\[ R_{\Gamma}(i) = \left[ \frac{U_i (R_{\Gamma}(i+1) + 1) + B_i (R_{\Gamma}(i+1) - 1)}{U_i (R_{\Gamma}(i+1) + 1) - B_i (R_{\Gamma}(i+1) - 1)} \right] e^{-2v_i t_i} \quad (B.24) \]
where

\[ U_i \equiv \frac{u_i}{u_{i+1}}, \quad B_i \equiv \frac{\beta_i}{\beta_{i+1}} \]

The reflection coefficients for any interface can be evaluated in a recursive fashion if a reflection coefficient is known at any interface within the stratified earth. Since the reflection coefficients are defined on the outward side of an interface (ie. at \( d_i + 0 \)),

\[ R(n + 1) = 0 \]

That is, there are no boundaries past \( r_n + 0 \) which can act as reflectors, so that the secondary (inward) potential at \( r_n + 0 \) is zero.

Figure B-2 shows a schematic diagram of the potentials at the source level, as well as a generalized notation for reflection coefficients found on either side of the source level. Here the asterisk serves to distinguish between the outward travelling potentials on either side of the source, and \( P \) represents either of the two scalar Hertz potential. The asterisk denotes the potential that is on the same side of the source as the receiver. From the definitions of the reflection coefficients,

\[ R_P^*(1) = \frac{\hat{P}_{\text{Secondary}}}{\hat{P}_{\text{Primary}} + \hat{P}_{\text{Secondary}}} \quad (B.25) \]

\[ R_P(1) = \frac{\hat{P}_{\text{Secondary}}}{\hat{P}_{\text{Primary}} + \hat{P}_{\text{Secondary}}} \quad (B.26) \]

Solving these two equations gives

\[
\begin{bmatrix}
\hat{P}_{\text{Secondary}}^* \\
\hat{P}_{\text{Secondary}}
\end{bmatrix} = \frac{1}{1 - R_P(1)R_P^*(1)} \begin{bmatrix}
R_P(1)R_P^*(1) & R_P(1) \\
R_P^*(1) & R_P(1)R_P^*(1)
\end{bmatrix} \begin{bmatrix}
\hat{P}_{\text{Primary}}^* \\
\hat{P}_{\text{Primary}}
\end{bmatrix} \quad (B.27)
\]

This matrix may be rewritten in terms of the total potential at the source point by noting that

\[ P_{\text{Total}} = P_{\text{Primary}} + P_{\text{Secondary}} \]
Figure B-2. A schematic diagram of the potentials at the source level. The asterisk indicates which side of the source level the receiver is on.
\[
\begin{bmatrix}
\hat{P}_{\text{Total}}^* \\
\hat{P}_{\text{Total}}^*
\end{bmatrix} = \frac{1}{1 - R_P(1)R_P^*(1)} \begin{bmatrix}
1 & R_P(1) \\
R_P^*(1) & R_P(1)R_P^*(1)
\end{bmatrix} \begin{bmatrix}
\hat{P}_{\text{Primary}}^* \\
\hat{P}_{\text{Primary}}^*
\end{bmatrix}
\] (B.28)

Using (B.28) the total potentials at the source level may be found by substituting the source terms (equations (2.4.15) and (2.4.16)) in for \( \hat{P}_{\text{Primary}}^* \) and \( \hat{P}_{\text{Secondary}}^* \).

Now that the total potential has been found at the source, it must be propagated to the receiver level through a series of operations. Firstly, the potentials must be continued across a homogeneous layer of thickness \( t_i \). This is done by using equations (B.2) and (B.3), viz.

\[
\begin{bmatrix}
\hat{\Pi}_z^{\text{out}}(d_i - 0) \\
\hat{\Pi}_z^{\text{in}}(d_i - 0)
\end{bmatrix} = \begin{bmatrix}
e^{-\nu_i t_i} & 0 \\
0 & e^{+\nu_i t_i}
\end{bmatrix} \begin{bmatrix}
\hat{\Pi}_z^{\text{out}}(d_{i-1} + 0) \\
\hat{\Pi}_z^{\text{in}}(d_{i-1} + 0)
\end{bmatrix}
\] (B.29)

\[
\begin{bmatrix}
\hat{\Gamma}_z^{\text{out}}(d_i - 0) \\
\hat{\Gamma}_z^{\text{in}}(d_i - 0)
\end{bmatrix} = \begin{bmatrix}
e^{-u_i t_i} & 0 \\
0 & e^{+u_i t_i}
\end{bmatrix} \begin{bmatrix}
\hat{\Gamma}_z^{\text{out}}(d_{i-1} + 0) \\
\hat{\Gamma}_z^{\text{in}}(d_{i-1} + 0)
\end{bmatrix}
\] (B.30)

The second operation to perform is the transmission of the potentials through an interface. For this operation, the boundary conditions for the \( i^{th} \) interface must be written explicitly. From (B.20),

\[
\alpha_{i+1} \left[ \hat{\Pi}_z^{\text{out}}(d_i + 0) + \hat{\Pi}_z^{\text{in}}(d_i + 0) \right] = \alpha_i \left[ \hat{\Pi}_z^{\text{out}}(d_i - 0) + \hat{\Pi}_z^{\text{in}}(d_i - 0) \right]
\] (B.31)

\[
\frac{\nu_{i+1}}{K_{i+1}} \left[ \hat{\Pi}_z^{\text{out}}(d_i + 0) + \hat{\Pi}_z^{\text{in}}(d_i + 0) \right] = \frac{\nu_i}{K_i} \left[ \hat{\Pi}_z^{\text{out}}(d_i - 0) + \hat{\Pi}_z^{\text{in}}(d_i - 0) \right]
\] (B.32)

\[
\beta_{i+1} \left[ \hat{\Gamma}_z^{\text{out}}(d_i + 0) + \hat{\Gamma}_z^{\text{in}}(d_i + 0) \right] = \beta_i \left[ \hat{\Gamma}_z^{\text{out}}(d_i - 0) + \hat{\Gamma}_z^{\text{in}}(d_i - 0) \right]
\] (B.33)

\[
u_{i+1} \left[ \hat{\Gamma}_z^{\text{out}}(d_i + 0) + \hat{\Gamma}_z^{\text{in}}(d_i + 0) \right] = \nu_i \left[ \hat{\Gamma}_z^{\text{out}}(d_i - 0) + \hat{\Gamma}_z^{\text{in}}(d_i - 0) \right]
\] (B.34)

The solutions of these equations is

\[
\begin{bmatrix}
\hat{\Pi}_z^{\text{out}}(d_i + 0) \\
\hat{\Pi}_z^{\text{in}}(d_i + 0)
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(A_i + V_i) & (A_i - V_i) \\
R_i(i + 1)(A_i + V_i) & R_i(i + 1)(A_i - V_i)
\end{bmatrix} \begin{bmatrix}
\hat{\Pi}_z^{\text{out}}(d_i - 0) \\
\hat{\Pi}_z^{\text{in}}(d_i - 0)
\end{bmatrix}
\] (B.35)
\[ \begin{bmatrix} \hat{\Gamma}_z^{out}(d_i + 0) \\ \hat{\Gamma}_z^{in}(d_i + 0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (B_i + U_i) & (B_i - U_i) \\ R_T(i + 1)(B_i + U_i) & R_T(i + 1)(B_i - U_i) \end{bmatrix} \begin{bmatrix} \hat{\Gamma}_z^{out}(d_i - 0) \\ \hat{\Gamma}_z^{in}(d_i - 0) \end{bmatrix} \]  \quad (B.36)

The total potentials at the source level can now be propagated to the receiver level. The matrix equation (B.28) converts the primary potential into the total potential at the source level. To obtain the potential away from the source level, additional matrices multiply (B.28) to form a single propagation matrix for any level within the medium. The results of these multiplications are called propagation matrices. Each multiplication represents the continuation or transmission across a layer or interface, and so successive multiplication of (B.29) or (B.30) and (B.35) or (B.36) are used to construct the propagation matrices. The propagation matrices for the electric and magnetic hertz potentials are denoted by \( T \) and \( S \) respectively.

The propagation matrices relate the primary fields at the source level to the total field receiver level and therefore must be multiplied by the primary potentials as described by equations (2.1.15) and (2.1.16). The result of this multiplication is shown in Table B-1 for both potentials. Here the elements \( T_{i,j}, S_{i,j} \) of the two propagation matrices are indicated by the row and column subscripts on the right hand side of the symbols. The subscript \( s \) on the other parameters indicates that the quantities subscripted are for the layer containing the source.

Now that the scalar potentials can be found anywhere within the layered earth, it is necessary to convert the potentials into the electric and magnetic fields. This is done by substituting equations (B.37) and (B.38) into the field equations for the scalar potentials in the Fourier domain.

\[
\begin{align*}
E &= \begin{bmatrix} i pu \left( \hat{\Gamma}_z^{out} - \hat{\Gamma}_z^{in} \right) + iq \beta \left( \hat{\Gamma}_z^{out} + \hat{\Gamma}_z^{in} \right) \\ i qu \left( \hat{\Gamma}_z^{out} - \hat{\Gamma}_z^{in} \right) - ip \alpha \left( \hat{\Gamma}_z^{out} + \hat{\Gamma}_z^{in} \right) \\ \lambda^2 \left( \hat{\Gamma}_z^{out} + \hat{\Gamma}_z^{in} \right) \end{bmatrix} \\
H &= \begin{bmatrix} i pu \left( \hat{\Gamma}_z^{out} - \hat{\Gamma}_z^{in} \right) - iq \alpha \left( \hat{\Gamma}_z^{out} + \hat{\Gamma}_z^{in} \right) \\ i qu \left( \hat{\Gamma}_z^{out} - \hat{\Gamma}_z^{in} \right) + ip \alpha \left( \hat{\Gamma}_z^{out} + \hat{\Gamma}_z^{in} \right) \\ \lambda^2 \left( \hat{\Gamma}_z^{out} + \hat{\Gamma}_z^{in} \right) \end{bmatrix}
\end{align*}
\]  \quad (B.39)
\[
\begin{bmatrix}
\hat{\Pi}_z^{\text{out}} \\
\hat{\Pi}_z^{\text{in}}
\end{bmatrix}
= \begin{bmatrix}
\frac{ip(T_{1,1} - T_{1,2})}{2\lambda^2 \alpha_{\nu,s}} & \frac{iq(T_{1,1} - T_{1,2})}{2\lambda^2 \alpha_{\nu,s}} & K_s(T_{1,1} + T_{1,2}) & \frac{ipK_s(T_{1,1} + T_{1,2})}{2\lambda^2 v_s} & -\frac{iqK_s(T_{1,1} + T_{1,2})}{2\lambda^2 v_s} \\
\frac{ip(T_{2,1} - T_{2,2})}{2\lambda^2 \alpha_{\nu,s}} & \frac{iq(T_{2,1} - T_{2,2})}{2\lambda^2 \alpha_{\nu,s}} & K_s(T_{2,1} + T_{2,2}) & \frac{ipK_s(T_{2,1} + T_{2,2})}{2\lambda^2 v_s} & -\frac{iqK_s(T_{2,1} + T_{2,2})}{2\lambda^2 v_s}
\end{bmatrix}
\begin{bmatrix}
J_z \\
J_y \\
J_z \\
M_y \\
M_z
\end{bmatrix}
\] (B.37)

\[
\begin{bmatrix}
\hat{\Gamma}_z^{\text{out}} \\
\hat{\Gamma}_z^{\text{in}}
\end{bmatrix}
= \begin{bmatrix}
\frac{ip(S_{1,1} - S_{1,2})}{2\lambda^2 \beta_s} & \frac{iq(S_{1,1} - S_{1,2})}{2\lambda^2 \beta_s} & S_{1,1} + S_{1,2} & \frac{-ip(S_{1,1} + S_{1,2})}{2\lambda^2 u_s} & \frac{iq(S_{1,1} + S_{1,2})}{2\lambda^2 u_s} \\
\frac{ip(S_{2,1} - S_{2,2})}{2\lambda^2 \beta_s} & \frac{iq(S_{2,1} - S_{2,2})}{2\lambda^2 \beta_s} & S_{2,1} + S_{2,2} & \frac{-ip(S_{2,1} + S_{2,2})}{2\lambda^2 u_s} & \frac{iq(S_{2,1} + S_{2,2})}{2\lambda^2 u_s}
\end{bmatrix}
\begin{bmatrix}
M_z \\
M_y \\
M_z \\
J_y \\
J_z
\end{bmatrix}
\] (B.38)

Table B-1. The Hertz potential source terms after multiplication by the propagation matrices.
where the subscript \( f \) indicates the variable exists in the layer containing the receiver. After some algebra, the consolidated result for the fields in the Fourier domains is obtained in terms of the two types of sources

\[
\mathbf{E} = \left[ \hat{\mathbf{L}}_J + \hat{\mathbf{L}}_M \right] \hat{\mathbf{w}} \quad (B.41)
\]

\[
\mathbf{H} = \left[ \hat{\mathbf{K}}_J + \hat{\mathbf{K}}_M \right] \hat{\mathbf{w}} \quad (B.42)
\]

where

\[
\hat{\mathbf{w}} = \begin{bmatrix}
p^2 \\
q^2 \\
pq \\
p \\
q \\
1
\end{bmatrix} \quad (B.43)
\]

and where matrices \( \hat{\mathbf{L}}_J \) and \( \hat{\mathbf{L}}_M \) appear in table B-2, and \( \hat{\mathbf{K}}_J \) and \( \hat{\mathbf{K}}_M \) appear in table B-3. The following terms also appear in these tables, and here the subscripts refer to the row and column of the propagation matrices.

\[
T(1) = T_{1,1} + T_{1,2} + T_{2,1} + T_{2,2}
\]

\[
T(2) = T_{1,1} + T_{1,2} - T_{2,1} - T_{2,2}
\]

\[
T(3) = T_{1,1} - T_{1,2} - T_{2,1} + T_{2,2}
\]

\[
T(4) = T_{1,1} - T_{1,2} + T_{2,1} - T_{2,2}
\]

\[
S(1) = S_{1,1} + S_{1,2} + S_{2,1} + S_{2,2}
\]

\[
S(2) = S_{1,1} + S_{1,2} - S_{2,1} - S_{2,2}
\]

\[
S(3) = S_{1,1} - S_{1,2} - S_{2,1} + S_{2,2}
\]

\[
S(4) = S_{1,1} - S_{1,2} + S_{2,1} - S_{2,2}
\]

The solution in the Fourier domain is expressed in terms of the products of wavenumbers in order to simplify the transformation into the Hankel domain. Since the Hankel domain is just the cylindrical Fourier domain, a two dimensional Fourier transform over \( p \) and \( q \) is avoided by finding a one dimensional transform over \( \lambda \). All of the equations (B.44) to (B.47) are in terms of \( \lambda \), so that only the vector \( \hat{\mathbf{w}} \).
The electric field component matrices in the radial Fourier Domain (the Hankel Domain) for electric and magnetic sources.

Table B.2.
Table B-3. The magnetic field component matrices in the radial Fourier Domain (the Hankel Domain) for electric and magnetic sources.
has to be transformed to \( \lambda \) space. The mathematical relationship between the radial and rectangular Fourier domains is given in equation (B.48), where \( p = \lambda \cos \varphi \) and \( q = \lambda \sin \varphi \), and \( n \) is the order of a Bessel function of the first kind.

\[
\tilde{F}_n(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i n (\varphi - \pi/2)} \hat{f}(\lambda \cos \varphi, \lambda \sin \varphi) d\varphi \quad (B.48)
\]

In order to find the Hankel domain equivalent of \( \mathbf{\tilde{w}} \), the term \( \hat{f}(\lambda \cos \varphi, \lambda \sin \varphi) \) is replaced by the required component of \( \mathbf{\tilde{w}} \), and the integral is evaluated. The transformation must in principle be performed for \( n = -\infty \) to \( \infty \). However, for simple dipole source elements the only non-zero results are for

\[-2 \leq n \leq 2\]

The result of the transformation leads to the matrix \( \mathbf{\tilde{W}} \), where

\[
\mathbf{\tilde{W}} = \begin{bmatrix}
-\frac{\lambda^2}{4} & 0 & 0 & 0 \\
0 & -\frac{\lambda^2}{2} & 0 & 0 \\
0 & 0 & 0 & -i \frac{\lambda^2}{4} \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad (B.49)
\]

Now it is desirable to express the solution in terms of Bessel functions of order 0 and 1 only, and to convert the final expressions to the space domain. Equation (B.50) demonstrates the relationship between the Hankel domain and the space domain, while the recursion relation (A.1) permits the compression of the Bessel functions into orders 0 and 1.

\[
\mathbf{F}(r, \theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \tilde{F}_n(\lambda) e^{-in\delta} J_n(\lambda \rho) \lambda \, d\lambda \quad (B.50)
\]
The result of these two operations is expressed in equation (B.51) in terms of the matrix $\tilde{C}$.

$$\tilde{C} = \frac{\lambda}{2\pi} \begin{bmatrix} -e^{2i\theta} & \frac{2e^{2i\theta}}{\lambda\rho} \\ 0 & -e^{i\theta} \\ 1 & 0 \\ 0 & e^{i\theta} \\ -e^{-2i\theta} & \frac{2e^{-2i\theta}}{\lambda\rho} \end{bmatrix} \tag{B.51}$$

The full solution in the space domain may be written succinctly as

$$\mathbf{F} (r, \theta) = \int_0^{+\infty} \hat{\mathbf{F}} \tilde{W}\tilde{C} \begin{bmatrix} J_0(\lambda\rho) \\ J_1(\lambda\rho) \end{bmatrix} \, d\lambda \tag{B.52}$$

where $\hat{\mathbf{F}}$ is a generalized notation for one of the matrices $\hat{\mathbf{L}}_J$, $\hat{\mathbf{L}}_M$, $\hat{\mathbf{K}}_J$, or $\hat{\mathbf{K}}_M$. One further step of algebra finds the product $\tilde{W}\tilde{C}$ which relates the geometrical effects of the kernels to the fields components.

$$\tilde{W}\tilde{C} = \frac{\lambda}{2\pi} \begin{bmatrix} \frac{\lambda^2}{2}(\cos 2\theta + 1) & -\lambda \cos 2\theta \\ -\frac{\lambda^2}{2}(\cos 2\theta - 1) & \lambda \cos 2\theta \\ \frac{\lambda^2}{2} \sin 2\theta & -\lambda \sin 2\theta \\ 0 & -i\lambda \cos \theta \\ 0 & -i\lambda \sin \theta \\ 1 & 0 \end{bmatrix} \tag{B.53}$$

From this representation, it is quite simple to find any particular field component from a single arbitrary source by finding the required propagation matrices. This is accomplished by multiplying the Hankel domain field matrix which is appropriate to the source used ([B.44] to [B.47]) by this matrix and then by the Bessel function matrix in (B.52).
Appendix C

Analytical Solution for a Uniform Anisotropic Halfspace

Wait (1966,1982) solved the problem of horizontal electric dipole excitation of a transversely isotropic halfspace analytically. This solution is useful directly if no layering is required in the earth model. It is also required in order to force convergence of the Hankel transform as described in section 2.5. To be able to do this, the halfspace solution must be directly compatible with the kernel functions to be transformed. Wait presented his solution in terms of the anisotropic and isotropic components of the fields, whereas the solution derived in this thesis is in a form that is more computationally efficient. This means that slightly different, but equivalent expressions must be obtained for the halfspace fields.

Wait chose to solve the problem in terms of two components of the vector potential \( \mathbf{A} \) instead of the Hertz potentials. The derivation in the papers by Wait is complete and will not be reproduced here, but it should be pointed out that the quasi-static assumption has been made. From the Helmholtz equation for the vector potential \( \mathbf{A} \), the electric field can be expressed as:

\[
E_z = \frac{1}{\sigma_k} \left[ -\gamma^2 A_z + \frac{\partial}{\partial z} \left( \frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial z} \right) \right] \tag{C.1}
\]

\[
E_y = \frac{1}{\sigma_k} \frac{\partial}{\partial y} \left( \frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial z} \right) \tag{C.2}
\]

\[
E_x = \frac{1}{\sigma_k} \left[ -\gamma^2 A_z + \frac{\partial}{\partial z} \left( \frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial z} \right) \right] \tag{C.3}
\]

Also, from the definition of the magnetic vector potential, the magnetic fields can be seen to be
\[ H_z = \frac{\partial A_z}{\partial y} \]  
\[ H_y = \frac{\partial A_z}{\partial z} - \frac{\partial A_z}{\partial x} \]  
\[ H_x = -\frac{\partial A_z}{\partial y}. \]  
(C.4)  
(C.5)  
(C.6)

In the above expressions, the x and z components of the vector potential are

\[ A_x = \frac{1}{2\pi} \int_0^\infty \frac{\lambda}{\lambda + u} e^{u \lambda} J_0(\lambda \rho) \, d\lambda, \]  
(C.7)

and

\[ A_x = \frac{1}{2\pi} \frac{\partial}{\partial x} \int_0^\infty e^{u \lambda} - \frac{u}{\lambda + u} e^{u \lambda} J_0(\lambda \rho) \, d\lambda. \]  
(C.8)

The above equations can be solved to obtain closed forms for the electric and magnetic fields on an anisotropic halfspace. The electric field parallel to the source dipole can be written as;

\[
E_z = \frac{1}{4\pi \sigma_h \rho^3} \left[ (K^{1/2} + \gamma \rho) e^{-\gamma \rho K^{-1/2}} - (1 - (1 + \gamma \rho) e^{-\gamma K}) \right] \\
+ \frac{\cos 2\theta}{4\pi \sigma_h \rho^3} \left[ 3 - (3 + \gamma \rho) e^{-\gamma \rho} + (3K^{1/2} + \gamma \rho) e^{-\gamma K^{-1/2}} \right]
\]  
(C.9)

The electric field perpendicular to the source dipole can be seen to be identical to the second term in equation (C.9) with the exception of the \( \sin 2\theta \) and \( \cos 2\theta \) terms describing the geometrical dependence of the fields.

\[
E_y = \frac{\sin 2\theta}{4\pi \sigma_h \rho^3} \left[ 3 - (3 + \gamma \rho) e^{-\gamma \rho} + (3K^{1/2} + \gamma \rho) e^{-\gamma K^{-1/2}} \right] 
\]  
(C.10)

The vertical electric field can only be expressed in terms of modified Bessel functions of the first and second kinds (Bannister, 1966). The complex arguments of the Bessel functions are \( \gamma \rho / 2 \), and are written without the arguments for the sake of brevity.

\[
E_z = \frac{\beta \cos \theta}{2\pi \rho} I_1 K_1 
\]  
(C.11)
The horizontal magnetic fields are also written in terms of the modified Bessel functions.

\[
H_z = \frac{\sin 2\theta}{4\pi \rho^2} \left[ 4I_1K_1 - \frac{7\rho}{2} \{I_0K_1 - I_1K_0\} \right] \tag{C.12}
\]

As with the electric fields, the two horizontal magnetic fields have a common term with the exception of a geometrical factor.

\[
H_y = \frac{1}{4\pi \rho^2} \left[ 2I_1K_1 - \frac{7\rho}{2} \{I_0K_1 - I_1K_0\} \right]
- \frac{\cos 2\theta}{4\pi \rho^2} \left[ 4I_1K_1 - \frac{7\rho}{2} \{I_0K_1 - I_1K_0\} \right] \tag{C.13}
\]

The vertical magnetic field can be simply written in terms of exponentials.

\[
H_z = \frac{2\sin \theta}{4\pi \gamma^2 \rho^4} \left[ 3 - (3 + 3\gamma \rho + \gamma^2 \rho^2) e^{-\gamma \rho} \right] \tag{C.14}
\]

In many cases, the high frequency limit of the fields can be found using much simpler expressions. The asymptotic solutions of the exponentials and Bessel functions can be evaluated to give the fields at large values of the product $|\gamma \rho|$. The electric fields can be seen to be

\[
E_x = \frac{3 \cos 2\theta}{4\pi \sigma_h \rho^3} - \frac{1}{4\pi \sigma_h \rho^3} \tag{C.15}
\]

\[
E_y = -\frac{\sin 2\theta}{4\pi \sigma_h \rho^3} \tag{C.16}
\]

\[
E_z = \frac{\beta \cos \theta}{2\pi \gamma \rho^3} \tag{C.17}
\]

while the magnetic fields may be expressed as

\[
H_x = \frac{3 \sin 2\theta}{4\pi \gamma \rho^3} \tag{C.18}
\]

\[
H_y = \frac{1}{4\pi \gamma \rho^3} - \frac{3 \cos 2\theta}{4\pi \gamma \rho^3} \tag{C.19}
\]

\[
H_z = \frac{3 \sin \theta}{2\pi \gamma^2 \rho^4} \tag{C.20}
\]
It is also useful to obtain the low frequency fields. In the DC approximation, it is possible to derive the electric fields as

\[ E_z = \frac{K^{1/2}}{4\pi \sigma \rho^3} (3 \cos 2\theta + 1) \quad (C.21) \]

\[ E_y = \frac{K^{1/2}}{4\pi \sigma \rho^3} 3 \sin 2\theta \quad (C.22) \]

\[ E_x = 0 \quad (C.23) \]

The magnetic fields can be similarly derived as

\[ H_z = \frac{1}{8\pi \rho^3} 3 \sin 2\theta \quad (C.24) \]

\[ H_y = -\frac{1}{8\pi \rho^3} (3 \cos 2\theta + 1) \quad (C.25) \]

\[ H_x = \frac{\sin \theta}{4\pi \rho^2} \quad (C.26) \]

The equations presented in this appendix are useful in several respects. First and foremost, they provide a means by which the Hankel convolution of divergent kernels may be evaluated using the lagged convolution concept. It is for this reason that the direct numerical approach of Chave (1983) was not used even though it is possibly more accurate, and does not need the quasistatic assumption. The only reason the quasistatic assumption is introduced in this thesis is to allow the use of the Fast Hankel Transform, but this forces the full solution to become quasistatic so that the same behaviour is observed at the extreme limits of \( \lambda \).

Another reason why the halfspace equations proved useful is that they provide a check on the mathematics in the full layered earth solution. It is essential that the derived results reduce to the analytical halfspace results. The equations are also instructive about the basic physics of the layered earth, and are computationally efficient. One final point that should be made clear is that when \( K = 1 \), the solutions reduce to those of an isotropic halfspace.
Appendix D

Examples of Integration Weight Sets for a Grounded Bipole

In this appendix some examples of the integration weights used to calculate the field response at different receiver positions are given. In all cases, the transmitter bipole is one kilometer in length and is oriented along the x-axis. The integration weights are not changed significantly if the distances involved in the receiver-transmitter configuration are scaled by some factor. Also, there is a certain amount of symmetry involved in any configuration of a linear bipole and a point receiver. The following list describes the receiver position in terms of its x and y coordinates. All distances are in meters.

Figure D-1 \((-100, 0)\)
Figure D-2 \((-100, 100)\)
Figure D-3 \((-100, 200)\)
Figure D-4 \((0, 100)\)
Figure D-5 \((0, 200)\)
Figure D-6 \((500, 100)\)
Figure D-7 \((500, 200)\)

The format of presentation is the same as in Chapter three, namely that geometrical factors for the kernel functions are indicated on the plots using the following convention;

(a) \(\cos 2\theta\)
(b) 1
(c) \(\sin 2\theta\)
(d) \(\sin \theta\)
Figure D.1. Integration weights for a point receiver at (-100,0) meters and a transmitter dipole that extends from (0,0) to (1000,0).
Figure D-2. Integration weights for a point receiver at \((-100, 100)\) meters and a transmitter dipole that extends from \((0,0)\) to \((1000,0)\).
Figure D-3. Integration weights for a point receiver at (-100,200) meters and a transmitter bipole that extends from (0,0) to (1000,0).
Figure D-4. Integration weights for a point receiver at (0,100) meters and a transmitter dipole that extends from (0,0) to (1000,0).
Figure D-5. Integration weights for a point receiver at (0, 200) meters and a transmitter bipole that extends from (0, 0) to (1000, 0).
Figure D-6. Integration weights for a point receiver at (500,100) meters and a transmitter bipole that extends from (0,0) to (1000,0).
Figure D-7. Integration weights for a point receiver at (500, 200) meters and a transmitter dipole that extends from (0, 0) to (1000, 0).