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EM RESPONSE OF AN ARBITRARY
SOURCE ON A LAYERED EARTH; A
NEW COMPUTATIONAL APPROACH.

by

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ABSTRACT

The Fast Fourier transform is used to compute the EM induction response in a layered earth by an arbitrary grounded or ungrounded source (e.g. dipole, Turam loop, grounded wire). The induction problem is expressed in the horizontal wavenumber domain, the 2-D Fourier transform of the x-y plane, where a source term and an earth transfer function may be considered separately. This new procedure results in a significant reduction in computation time and flexible straightforward programming. Several examples are described to demonstrate the advantages and limitations of the method.

INTRODUCTION

The layered earth is a classic problem in geophysics and has received considerable attention in the literature (Sunde, 1949; Bhattacharyya, 1955; Wait, 1962; and others). In all these articles, the standard procedure of finding the EM response to a dipole source is to develop integral expressions of the form

$$\int_0^{\infty} C(\lambda) J(\lambda) d\lambda$$

where C is a complex function of the layered earth and source parameters, J is a Bessel function and λ is the variable of integration. Since the Bessel function is a rapidly oscillating function of λ , the numerical integration can be very expensive and subject to truncation errors.

Our method solves the induction problem in the wavenumber domain, i.e. the Fourier transform of the horizontal x-y plane, and allows a completely arbitrary source current distribution. In the wavenumber domain, the induction problem is multiplicative, i.e.

$$\begin{pmatrix} \text{EM} \\ \text{RESPONSE} \end{pmatrix} = \begin{pmatrix} \text{SOURCE} \\ \text{TERM} \end{pmatrix} \times \begin{pmatrix} \text{TRANSFER FUNCTION} \\ \text{OF} \\ \text{EARTH MODEL} \end{pmatrix}$$

where the source term and transfer function are independent.

The complete spatial distribution of the electromagnetic field response on a horizontal plane, anywhere inside or outside the earth, is then found by simply taking the inverse two-dimensional Fast Fourier Transform (FFT).

The procedure is very flexible and the computer programming is straightforward. Numerical accuracy is limited only by the sampling interval and length of record used, in common with any discrete Fourier transform technique.

The basic theory for a general time-varying, nongrounded source over a halfspace is described in a paper by Weaver (1970). In our paper, the theory is simplified to consider only harmonic time dependence, but is generalized to include grounded sources and a layered earth model. Our only restriction is that the source not be located below ground level, for example a wire at the bottom of a drill hole. A summary of the theoretical development is presented, followed by several examples demonstrating the scope and practical aspects of this technique.

MATHEMATICAL NOTATION (1)

x, y, z	Coordinate axes; z positive down
u, v	Wavenumbers in x and y directions respectively
$\underline{E}(x, y, z)$	Electric field intensity
$\underline{B}(x, y, z)$	Magnetic induction
$\underline{H}(x, y, z)$	Magnetic field intensity
$\underline{\Pi}(x, y, z)$	Electric Hertz potential
$\underline{\Gamma}(x, y, z)$	Magnetic Hertz potential
$P(u, v, z)$	Fourier transform of $\underline{\Pi}(x, y, z)$
$G(u, v, z)$	Fourier transform of $\underline{\Gamma}(x, y, z)$
$\hat{E}(u, v, z)$	Fourier transform of $\underline{E}(x, y, z)$
$\hat{B}(u, v, z)$	Fourier transform of $\underline{B}(x, y, z)$
$\hat{H}(u, v, z)$	Fourier transform of $\underline{H}(x, y, z)$
$\underline{K}(x, y)$	Surface current
σ	Conductivity
μ	Permeability
μ_0	Permeability of free space
ϵ_0	Permittivity of free space
K	Relative permeability
ω	Angular frequency
t	Layer thickness

(1) A wavy line drawn under a symbol indicates a vector quantity.

- Partial Derivative Notation:

To simplify the mathematical text, the partial derivatives of the Hertz potentials Γ and Π , or G and P , are indicated by the subscripts 1,2,3 corresponding to partial derivatives with respect to x, y, z respectively:

$$\begin{aligned} \text{e.g.} \quad \Pi &= \Pi(x, y, z) \\ \text{then} \quad \partial \Pi / \partial x &= \Pi_1 \\ \text{and} \quad \partial^2 \Pi / \partial z \partial y &= \Pi_{32} \end{aligned}$$

On the other hand, x, y, z subscripts refer to the vector components.

- Derivative and Integral Operators:

$$\begin{array}{ccc} \text{Space Domain } (x, y) & \longleftrightarrow & \text{Wavenumber domain } (u, v) \\ \partial / \partial x, \int dx & \longleftrightarrow & iu, (iu)^{-1} \\ \partial / \partial y, \int dy & \longleftrightarrow & iv, (iv)^{-1} \end{array} \quad (1)$$

- Compact Notation:

$$\text{if } \underline{r} = x \underline{i} + y \underline{j} \quad \text{and} \quad \underline{p} = u \underline{i} + v \underline{j},$$

$$\text{then } \phi(\underline{r}, z) \equiv \phi(x, y, z) \quad \text{and} \quad \phi(\underline{p}, z) \equiv \phi(u, v, z),$$

$$\text{where } r = |\underline{r}| = \sqrt{(x^2 + y^2)} \quad \text{and} \quad p = |\underline{p}| = \sqrt{(u^2 + v^2)}.$$

- The Fourier Transform Pair:

$$\begin{aligned}\tilde{F}(u, v, z) &= \iint_{-\infty}^{+\infty} \tilde{\phi}(x, y, z) \exp(-i(ux + vy)) \, dx \, dy \\ \tilde{\phi}(x, y, z) &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \tilde{F}(u, v, z) \exp(+i(ux + vy)) \, du \, dv\end{aligned}\quad (2)$$

where u and v are in radians per unit distance.

THEORY

The layered earth model is shown in Figure 1. The electrical properties in each layer are assumed to be homogeneous and isotropic. Since we are only interested in relatively low frequencies, we shall neglect displacement currents in the conductive layers. However, in the air above the conductor, where the conductivity is zero, we equate $\sigma_0 = i \omega \epsilon_0$. The time dependence is $e^{i\omega t}$ and is implicit.

Fig. 1. The layered earth model.

The electric and magnetic Hertz potentials are defined by the following relations:

$$\underline{\underline{E}}(x, y, z) = \underline{\underline{\nabla}} \times \underline{\underline{\nabla}} \times \underline{\underline{\Pi}}(x, y, z) - \underline{\underline{\nabla}} \times (i \omega \underline{\underline{\Gamma}}(x, y, z)) \quad (3)$$

and

$$\underline{\underline{B}}(x, y, z) = \underline{\underline{\nabla}} \times \underline{\underline{\nabla}} \times \underline{\underline{\Gamma}}(x, y, z) + \mu \sigma \underline{\underline{\nabla}} \times \underline{\underline{\Pi}}(x, y, z) \quad (4)$$

where each component of $\underline{\underline{E}}$, $\underline{\underline{B}}$, $\underline{\underline{\Pi}}$ and $\underline{\underline{\Gamma}}$ satisfies the steady state diffusion equation,

$$\nabla^2 \Theta(x, y, z) = i \omega \mu \sigma \Theta(x, y, z) \quad (5)$$

in each conductive layer, and, as discussed above,

$$\nabla^2 \Theta(x, y, z) = -\mu \epsilon_0 \omega^2 \Theta(x, y, z) \quad (6)$$

in the air.

For the layered earth problem, the Hertz vectors are taken to be in the z direction, i.e.,

$$\underline{\underline{\Pi}}(x, y, z) = \Pi(x, y, z) \underline{\underline{k}} \quad (7)$$

and

$$\underline{\underline{\Gamma}}(x, y, z) = \Gamma(x, y, z) \underline{\underline{k}} \quad (8)$$

Equations (7) and (8) are substituted into (3) and (4) to give:

$$\begin{aligned} \underline{\underline{E}} &= \underline{\underline{i}} \left[\pi_{31} - i \omega \Gamma_2 \right] \\ &+ \underline{\underline{j}} \left[\pi_{32} + i \omega \Gamma_1 \right] \\ &- \underline{\underline{k}} \left[\pi_{11} + \pi_{22} \right] \end{aligned} \quad (9)$$

and

$$\begin{aligned} \underline{\underline{B}} &= \underline{\underline{i}} \left[\Gamma_{31} + \mu \sigma \pi_2 \right] \\ &+ \underline{\underline{j}} \left[\Gamma_{32} - \mu \sigma \pi_1 \right] \\ &- \underline{\underline{k}} \left[\Gamma_{11} + \Gamma_{22} \right] \end{aligned} \quad (10)$$

The induction problem can now be described by the two Hertz potentials $\Gamma(x, y, z)$ and $\Pi(x, y, z)$. Alternately, equations (9) and (10) may be expressed in the wavenumber domain, transforming the derivative operators according to (1). In the wavenumber domain, $\underline{\underline{\hat{E}}}(u, v, z)$ and $\underline{\underline{\hat{B}}}(u, v, z)$ are obtained from $P(u, v, z)$ and $G(u, v, z)$ by multiplicative operations.

The problem is to find the response of the layered earth model to any type of grounded or ungrounded source. It will be solved in the horizontal wavenumber domain and expressed in terms of the magnetic and electric Hertz

potentials, $G(u,v,z)$ and $P_3(u,v,z)$. First of all, then, it will be shown how the source current distribution may be (equivalently) expressed by Hertz potentials in the wavenumber domain. These may be thought of as input Hertz potentials. Secondly, magnetic and electric transfer functions $TFM(u,v,z)$ and $TFE(u,v,z)$ will be derived. The transfer functions are only dependent on the parameters of the layered earth model under consideration, and will be multiplied with the input Hertz potentials to give the output Hertz potentials in the wavenumber domain.

General Boundary Conditions

In this section, the boundary conditions are derived for the Hertz potentials at a source free boundary, $z = d_i$.

Following Weaver's example (1970), we define:

$$\Lambda(r, d_i) = \Gamma(r, d_i^+) - \Gamma(r, d_i^-) \quad , \quad (11)$$

$$\Upsilon(r, d_i) = \frac{1}{K_{i+1}} \Gamma_3(r, d_i^+) - \frac{1}{K_i} \Gamma_3(r, d_i^-) \quad , \quad (12)$$

$$\Psi(r, d_i) = \Pi_3(r, d_i^+) - \Pi_3(r, d_i^-) \quad , \quad (13)$$

and

$$X(r, d_i) = \sigma_{i+1} \Pi(r, d_i^+) - \sigma_i \Pi(r, d_i^-) \quad . \quad (14)$$

Applying the conditions of continuity of tangential $\underline{E}$ and $\underline{H}$, and normal $\underline{B}$, to equations (9) and (10), it is readily found that Λ, Υ, Ψ and X all satisfy Laplace's equation:

$$\Theta_{11}(r, d_i) + \Theta_{22}(r, d_i) = 0 \quad . \quad (15)$$

Λ, Υ, Ψ and X are then harmonic functions.

Since there are no sources on the plane $z = d_i$, and since the fields must vanish as $r \rightarrow \infty$, we must have that

$$\Lambda(r, d_i) \equiv \Upsilon(r, d_i) \equiv \Psi(r, d_i) \equiv X(r, d_i) \equiv 0 \quad (16)$$

Taking the Fourier transforms of equations (11) to (14), the source free boundary conditions reduce to:

$$G(p, d_i^+) = \dot{G}(p, d_i^-) \quad , \quad (17)$$

$$\frac{1}{\kappa_{i+1}} G_3(p, d_i^+) = \frac{1}{\kappa_i} G_3(p, d_i^-) \quad , \quad (18)$$

$$P_3(p, d_i^+) = P_3(p, d_i^-) \quad , \quad (19)$$

and

$$\sigma_{i+1} P(p, d_i^+) = \sigma_i P(p, d_i^-) \quad . \quad (20)$$

The Generalized Source

In this section, we shall consider the behaviour of any type of source current distribution in the region $z \leq 0$, and how it may be expressed by Hertz potentials. The source currents may be grounded, partly grounded, or ungrounded. First of all, it may be demonstrated by the equivalence principle (Harrington, 1961), that, as far as induction in the region $z > 0$ is concerned, any source current distribution within the region $z \leq 0$ may be uniquely replaced by an equivalent surface current in the plane $z = 0$.

Of course, in many cases the source current is actually lying in the $z = 0$ plane. If there is a surface current in the boundary, the boundary conditions in equations (17) to (20) must be revised. The boundary condition at the $z = 0$ surface is then (Harrington, 1961)

$$\hat{n} \times [\underline{H}(x, y, 0^+) - \underline{H}(x, y, 0^-)] = \underline{K}(x, y, 0), \quad (21)$$

where $\hat{n}$ is a unit vector in the z direction. Written for scalar components, and using equations (10), (12), and (14) this becomes

$$K_x(r, 0) = \gamma_2(r, 0) - X_1(r, 0) \quad (22)$$

and

$$-K_y(r, 0) = \gamma_1(r, 0) + X_2(r, 0) \quad (23)$$

By differentiating equation (22) with respect to x and equation (23) with respect to y and subtracting, we obtain

$$X_{11}(r, 0) + X_{22}(r, 0) = -\operatorname{div} \underline{K}(r, 0). \quad (24)$$

Similarly, we may obtain

$$Y_{11}(r, 0) + Y_{22}(r, 0) = -\operatorname{curl} \underline{K}(r, 0). \quad (25)$$

Using the Green's function for the two dimensional Laplacian operator, the solutions of equations (24) and (25) are respectively,

$$\begin{aligned} X(r, 0) &= \sigma_1 \Pi(r, 0^+) - \sigma_0 \Pi(r, 0^-) \\ &= \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \ln \sqrt{(x-x')^2 + (y-y')^2} \cdot \operatorname{div} K(x', y', 0) \, dx' dy' \end{aligned} \quad (26)$$

and

$$\begin{aligned} Y(r, 0) &= \frac{1}{K_1} \Gamma_3(r, 0^+) - \frac{1}{K_0} \Gamma_3(r, 0^-) \\ &= \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \ln \sqrt{(x-x')^2 + (y-y')^2} \cdot \operatorname{curl} K(x', y', 0) \, dx' dy'. \end{aligned} \quad (27)$$

From equation (26), $\sigma \Pi$ is associated with the divergence of the surface current distribution corresponding to the grounded parts. On the other hand, equation (27) shows that Γ/K is associated with the curl of the surface current distribution corresponding to the nongrounded part.

The problem may therefore be broken into two parts:

- 1) the divergence free part of the surface current distribution which generates Γ , the magnetic Hertz potential.
- 2) the grounded electrical contacts which generate Π , the electric Hertz potential.

Of course, one must remember that from equations (9) and (10), the horizontal components of $\underline{E}$ and $\underline{H}$ involve derivatives of both Γ and Π .

In the following sections we might have used $\sigma_F(\text{div } K(x,y))$ and $\sigma_F(\text{curl } K(x,y))$ as the source and input to the problem. However, it is obviously more practical to describe a magnetic source by its free space magnetic field and, likewise, an electric source by its input current.

Expressing equations (24) and (25) in the wavenumber domain, we obtain

$$\hat{X}(p,0) = \sigma_1 P(p,0^+) - \sigma_0 P(p,0^-) = \hat{D}/p^2 \quad (28)$$

and
$$\hat{Y}(p,0) = \frac{1}{K_1} G_3(p,0^+) - \frac{1}{K_0} G_3(p,0^-) = \hat{C}/p^2, \quad (29)$$

where
$$\hat{D} = \sigma_F(\text{div } \hat{K})$$

and
$$\hat{C} = \sigma_F(\text{curl } \hat{K})$$

To express $\hat{C}$ and $\hat{D}$ of the source currents in terms of Hertz potentials, let us consider the particular case of free space on both sides of the $z=0$ boundary. Then from (12) and (19) we obtain, respectively,

$$\hat{\gamma}(p, 0) = -2 \frac{k_0}{k_0} G'(p, 0^+) \quad (30)$$

and

$$P'(p, 0^+) = -P'(p, 0^-) \quad (31)$$

where the superscripts denote a source term.

From equations (28), (29), (30) and (31) it follows that

$$\hat{D} = -2 \sigma_0 \frac{p^2}{k_0} P_3'(p, 0^+) \quad (32)$$

and

$$\hat{C} = -2 \frac{k_0}{k_0} p^2 G'(p, 0^+) \quad (33)$$

where

$$k_0 = \sqrt{p^2 + i\sigma_0 \omega \mu}$$

For the case of low frequencies this reduces to

$$\hat{D} \simeq -2 \sigma_0' p P_3'(p, 0^+) \quad (34)$$

and

$$\hat{C} \simeq -2 \frac{p^3}{k_0} G'(p, 0^+) \quad (35)$$

Obviously then, the source may be defined equivalently by $\hat{C}$ and $\hat{D}$, or $G'(p, 0^+)$ and $P_3'(p, 0)$ respectively.

The Magnetic Transfer Function

We wish to obtain a wavenumber domain expression for $G(\underline{p}, z)$ in the form

$$G(\underline{p}, z) = \left(\begin{array}{c} \text{TRANSFER} \\ \text{FUNCTION} \end{array} \right) \times \left(\begin{array}{c} \text{SOURCE} \\ \text{TERM} \end{array} \right), \quad (36)$$

where the source term consists of the divergence free part and the transfer function depends only on the earth model parameters shown in Figure 1.

Let us define $\Gamma'(\underline{r}, z)$ and its transform $G'(\underline{p}, z)$ as the free space magnetic Hertz potential of the divergence free source currents. This will generate the total Hertz potential denoted by $\Gamma(\underline{r}, z)$ and its transform $G(\underline{p}, z)$ and therefore leads to the following definition of the secondary fields:

$$\Gamma^2(\underline{r}, z) = \Gamma(\underline{r}, z) - \Gamma'(\underline{r}, z) \quad (37)$$

and
$$G^2(\underline{p}, z) = G(\underline{p}, z) - G'(\underline{p}, z). \quad (38)$$

The magnetic transfer function $TFM(\underline{p}, z)$, is now defined by

$$G(\underline{p}, z) = TFM(\underline{p}, z) G'(\underline{p}, z). \quad (39)$$

Usually, we are interested in the plane $z=0^-$ and we may define

$$G(\underline{p}, 0^-) = \text{TFM}(\underline{p}, 0^-) G'(\underline{p}, 0^-) \quad , \quad (40)$$

or

$$G^2(\underline{p}, 0^-) = [\text{TFM}(\underline{p}, 0^-) - 1] G'(\underline{p}, 0^-) \quad . \quad (41)$$

To derive $\text{TFM}(\underline{p}, 0^-)$, we begin by noting that, from equation (5),

$$G_{33}(\underline{p}, z) = k^2 G(\underline{p}, z) \quad , \quad (42)$$

where

$$k = \sqrt{p^2 + i\omega\mu\sigma}$$

Therefore, in the i th layer, G must take the form

$$G(\underline{p}, z) = A(\underline{p}) \exp(-k_i z) + B(\underline{p}) \exp(+k_i z) \quad . \quad (43)$$

The exponential terms in equation (43) change from one layer to the next and thus also the functions $A(\underline{p})$ and $B(\underline{p})$. It will, however, be useful to think of $G(\underline{p}, z)$ as having two parts, namely

$$G^D(\underline{p}, z) = A(\underline{p}) \exp[-k_i(z - d_{i-1})] \quad (44)$$

and

$$G^U(\underline{p}, z) = B(\underline{p}) \exp[+k_i(z - d_{i-1})] \quad , \quad (45)$$

the superscripts U and D symbolizing up and down.

G^u and G^D may be thought of as, respectively, the fields of all currents below and all currents above the plane of observation. Thus at $z=0$, $G^D = G^u$ and $G^u = G^z$ and our desired transfer function relationship is

$$G(\underline{p}, 0^-) = TFM(\underline{p}, 0^-) \cdot G^D(\underline{p}, 0^-) \quad (46)$$

Returning to an arbitrary layer i , the boundary conditions (17) and (18) require that $G(\underline{p}, z)$ and $k_i^{-1} G_3(\underline{p}, z)$ be continuous from one layer to the next. Therefore in the i th layer,

$$G_3(\underline{p}, z) = -k_i \left[G^D(\underline{p}, z) - G^u(\underline{p}, z) \right] \quad (47)$$

$$= k_i \left[G(\underline{p}, z) - 2 G^D(\underline{p}, z) \right] \quad (48)$$

Our objective is to obtain the relationship between G and G^D at $z=0^-$. To do this for a multilayer case, we start by obtaining a similar relationship at $z=d_{n-1}$,

$$G(\underline{p}, d_{n-1}^-) = TFM(\underline{p}, d_{n-1}^-) G^D(\underline{p}, d_{n-1}^-) \quad (49)$$

In the bottom half space, $G(\underline{p}, z)$ must vanish as $z \rightarrow \infty$. It follows that $G^u(\underline{p}, z) = 0$, and that

$$G(\underline{p}, z) = G^D(\underline{p}, z) \quad (50)$$

and

$$G_3(\underline{p}, z) = -k_n G^D(\underline{p}, z) \quad (51)$$

Thus, at $z = d_{n-1}$,

$$G(\underline{p}, d_{n-1}^-) = G^0(\underline{p}, d_{n-1}^+) \quad (52)$$

and

$$k_n k_{n-1} [G(\underline{p}, d_{n-1}^-) - 2G^0(\underline{p}, d_{n-1}^-)] = -k_{n-1} k_n G^0(\underline{p}, d_{n-1}^+) \quad (53)$$

Eliminating $G^0(\underline{p}, d_{n-1}^+)$, we have

$$G(\underline{p}, d_{n-1}^-) = \frac{2 k_n k_{n-1}}{k_{n-1} k_n + k_n k_{n-1}} \cdot G^0(\underline{p}, d_{n-1}^-), \quad (54)$$

i.e.

$$TFM(\underline{p}, d_{n-1}^-) = \frac{2 k_n k_{n-1}}{k_{n-1} k_n + k_n k_{n-1}} \quad (55)$$

Recursion Relation

A recursion relation is now required so that $TFM(\underline{p}, d_i^-)$ can be obtained from $TFM(\underline{p}, d_i^-)$. At the bottom of the i th layer, where $z = d_i^-$,

$$G^D(\underline{p}, d_i^-) = G^D(\underline{p}, d_{i-1}^+) \exp(-k_i t_i) \quad (56)$$

and

$$G^U(\underline{p}, d_i^-) = G^U(\underline{p}, d_{i-1}^+) \exp(+k_i t_i), \quad (57)$$

where $t_i = d_i - d_{i-1}$.

Applying equation (57), we obtain a relation at $z = d_{i-1}^+$,

$$G(\underline{p}, d_{i-1}^+) = \left[\frac{\text{TFM}(\underline{p}, d_{i-1}) \exp(-k_i t_i) + 2 \sinh k_i t_i}{\exp(k_i t_i)} \right] G^D(\underline{p}, d_{n-1}^+) \quad (58)$$

Applying now the boundary condition at $z = d_{i-1}$, we obtain

$$G(\underline{p}, d_{i-1}) = \text{TFM}(\underline{p}, d_{i-1}) \cdot G^D(\underline{p}, d_{i-1}), \quad (59)$$

where

$$\text{TFM}(\underline{p}, d_{i-1}) = \frac{2 A_M}{A_M (1 - B_M) + 2}, \quad (60)$$

$$A_M = \frac{\text{TFM}(\underline{p}, d_i) \exp(-k_i t_i) + 2 \sinh(k_i t_i)}{\exp(k_i t_i)},$$

$$\text{and } B_M = \frac{k_i k_{i-1}}{k_{i-1} k_i}.$$

We can therefore obtain $\text{TFM}(\underline{p}, 0)$ by starting with equation (55) and successively applying equation (60) for $i = n-2, \dots, 1, 0$.

Since the magnetic fields are derivable from the magnetic Hertz potential via the linear operators in the wavenumber domain, the transfer function TFM may be used to equate directly the input and output magnetic field components,

for example,

$$\hat{H}_x(\underline{P}, 0^-) = \text{TF}(\underline{P}, 0^-) \cdot \hat{H}_x^D(\underline{P}, 0^-) \quad (61)$$

However, in the latter case, a problem arises if the source currents are also in the plane $z=0^-$, resulting in a singularity in the fields. This may be dealt with in two ways. The first is an approximation and consists of raising the source currents slightly to remove the singularity in the $z=0^-$ plane. The second alternative applies in the case of simple sources such as dipoles where it is possible to express the source directly in the wavenumber domain. The singularity in space is depicted in the wavenumber domain by increasingly large amplitudes at high wavenumbers. This singularity is removed when the transfer function (TFM - 1) is used to compute the anomalous response as in equation (41). Although the total field response is plagued by the singularity, the anomalous response is not. If the total field response is desired, the source field may be calculated directly and added on.

The Electric Transfer Function

Since the electric fields may be obtained from the divergence of $\underline{P}_3$, the electric transfer function and recursion relation are derived in terms of $\underline{P}_3$ rather than $\underline{P}$.

At low frequencies we may neglect the term $\sigma_n P(p, 0^-)$ in equation (28) which contributes to the horizontal magnetic components in the air.

Following a development similar to that of the previous section, the following relationship may be derived at the interface of the homogeneous half space in the layered earth model:

$$P_3(p, d_{n-1}^-) = \frac{2 k_n \sigma_{n-1}}{k_n \sigma_{n-1} + k_{n-1} \sigma_n} P_3^0(p, d_{n-1}^-), \quad (62)$$

and therefore,

$$TFE(p, d_{n-1}^-) = \frac{2 k_n \sigma_{n-1}}{k_n \sigma_{n-1} + k_{n-1} \sigma_n} \quad (63)$$

Now using the boundary conditions for P_3 and P , we obtain the following recursion relation for the electric transfer function:

$$TFE(p, d_{i-1}) = \frac{2 A_E B_E}{A_E (B_E - 1) + 2} \quad (64)$$

where

$$A_E = \frac{TFE(p, d_i) \exp(-k_i t_i) + 2 \sinh(k_i t_i)}{\exp(k_i t_i)}$$

$$\text{and } B_E = \frac{k_i \sigma_{i-1}}{\sigma_i k_{i-1}}$$

As in the magnetic case, we may obtain the electric transfer function at the surface, $TFE(p, 0)$ by starting with equation (63) and successively applying

equation (64) for $i = n-2, \dots, 1, 0$. Note however that if we neglect σ_0 for low frequencies the electric transfer function at the surface is given by

$$TFE(p, 0) = \frac{2 \frac{\sigma_0 k_i}{\sigma_i p}}{\sigma_i p \left[\frac{2 \exp(k_i t)}{TFE(p, d_i) \exp(-k_i t) + 2 \sinh(k_i \tau)} - 1 \right]} \quad (65)$$

COMPUTATIONAL RESULTS

The Coaxial Dipole System

The coaxial dipole system (Figure 2) is used to compare our computed results with those obtained by numerical integration techniques (Frischknecht, 1967). It also serves to illustrate the limitations and problems inherent in using the Fourier transform approach.

Fig. 2. The coaxial dipole system.

In this example, the source is ungrounded and hence only the magnetic Hertz potential and related transfer function are considered. An expression for the source field in the wavenumber domain is easily derived. Consequently,

the numerical Fourier transform is used only in executing the inversion to the space domain. Choosing the x axis as the axis of the coaxial system, a grid of size 256×64 is used to increase the accuracy in the x direction at the expense of that in the y direction.

The free space magnetic induction of a horizontal, x-oriented dipole at the origin is given by:

$$\underline{B}(\underline{r}) = - \underline{M} \underline{\nabla} \cdot \left(\frac{\underline{x}}{R^3} \right) \quad (66)$$

where

$$\underline{M} = \frac{m \mu_0}{4\pi}$$

m = magnetic dipole moment ,

and
$$R = \sqrt{x^2 + y^2 + z^2}$$

The corresponding wavenumber domain expression for the magnetic Hertz potential at the surface $z=0^-$, due to a horizontal x-oriented dipole at $z=-h$, is

$$G^1(\underline{p}, 0^-) = 2\pi i \underline{M} u p^{-2} \exp(-ph) \quad (67)$$

From (10), we have that

$$H_x(\underline{p}, z) = \Gamma_{13}(\underline{p}, z) / \mu_0 \quad (68)$$

and using the derivative operators from equation (1),

the corresponding expression in the wavenumber domain is

$$\hat{H}_x(\underline{p}, 0^-) = \frac{1}{\mu_0} \{-i u\} \{p\} G(\underline{p}, 0) \quad (69)$$

From (67) and (69), we may define

$$\hat{H}_x^0(\rho, 0^-) = 2\pi M u^2 \rho^{-1} \exp[-\rho h] \quad (70)$$

For the coaxial dipole system, the fields are required on the surface $z = -h$, which also contains the source. To eliminate this singularity, only the anomalous fields are considered. If the total field response is desired, the primary field may be calculated directly in the space domain and added to the anomalous response.

Hence,

$$\hat{H}_x^2(\rho, 0^-) = [\text{TFM}(\rho, 0) - 1] \cdot \hat{H}_x^0(\rho, 0^-) \quad (71)$$

From (70) and (71), continuing the fields to $z = -h$, we have that

$$\hat{H}_x^2(\rho, -h) = 2\pi M u^2 \rho^{-2} \exp(-2\rho h) \cdot \{\text{TFM}(\rho, 0) - 1\} \quad (72)$$

To obtain the anomalous $H^2(\rho, -h)$ field distribution, the expression (72) is computed in the wavenumber domain, and the two dimensional Fourier transform is executed.

The tabulated data by Frischknecht is in the form of mutual coupling ratios. The in-phase and quadrature components are obtained by simply specifying a source field with the appropriate magnetic moment.

In the following examples, skin depth (δ), source height (H), and layer thickness (D) are all given in units of the sampling interval $\Delta x = \Delta y = 1$. The conductivity contrast (C) is that of the half space with respect to the first layer.

Figure 3a is a plot of equation (72), the anomalous response of the coaxial dipole system in the wavenumber domain, shown along the u wavenumber axis only. The earth model is homogeneous ($C=1$), with skin depth of three sampling units ($\delta=3$) and coil height of 0.3 sampling units ($H=0.3$). For such a case, the wavenumber response is seen to be well behaved. Figures 3b and 3c compare the computed inverse transform of 3a with the available tabulated data (Frischkrecht). The linear interpolation between points plotted by Calcomp is evident. Figure 4 shows the response for a conducting layer over an insulating half space which has been cubic spline interpolated.

Figure 5a corresponds to the same model as that of Figure 3a but with both coils on the surface $Z=0$. The downward continuation is apparent when comparing the transforms of Figures 3a and 5a. The I.P. transform in Figure 5a is still well behaved resulting in a good fit between calculated and tabulated data in Figure 5b. The quadrature transform however has a cutoff at the Nyquist wavenumber. Now, if the response in Figure 3c is downward continued to the surface $Z=0$, there will result a very sharp negative spike at zero coil separation. However, the truncation of the quadrature transform in Figure 5a is equivalent to convolving the true spatial response with the sinc function. This results in a broadening of the negative central peak in Figure 5c.

COAXIAL DIPOLE RESPONSE IN THE WAVENUMBER DOMAIN

(U AXIS)

$\delta=3.0$

$C=1.0$

$H=0.3$

Fig. 3a. Plot of equation (72) on the u wavenumber axis.

Fig. 3b. The inverse Fourier transform of the in-phase wavenumber domain response shown in Figure 3a.

Fig. 3c. The inverse Fourier transform of the quadrature wavenumber domain response shown in Figure 3a.

COAXIAL DIPOLE RESPONSE IN THE WAVENUMBER DOMAIN

(U AXIS)

$\delta=3.0$
 $C=0.1$
 $H=0.6$
 $D=0.6$

+ IN PHASE

x QUADRATURE

Fig. 4a. Plot of equation (72) on the u wavenumber axis.

Fig. 4b. The inverse Fourier transform of the in-phase wavenumber domain response shown in Figure 4a, with cubic spline interpolation applied.

Fig. 4c. The inverse Fourier transform of the quadrature wavenumber domain response shown in Figure 4a, with cubic spline interpolation applied.

COAXIAL DIPOLE RESPONSE IN THE WAVENUMBER DOMAIN

(U AXIS)

$\delta=3.0$

$C=1.0$

$H=0.0$

Fig. 5a. Plot of equation (72) on the u wavenumber axis.

Fig. 5b. The inverse Fourier transform of the in-phase wavenumber domain response shown in Figure 5a.

Fig. 5c. The inverse Fourier transform of the quadrature wavenumber domain response shown in Figure 5a.

Figures 6 and 7 illustrate why care must be taken in choosing the skin depth to sampling interval ratio. In Figure 6a, the skin depth is 20 sampling units. For this case the response in the wavenumber domain (Figure 6a) is such that the lower wavenumbers are undersampled, especially the I.P. response. This error in definition for low wavenumbers components is self evident in Figure 6b.

Figure 7 illustrates the other extreme, the skin depth to sampling interval ratio being too small. In this case, the high wavenumber truncation is the source of error. The inverse transforms as shown in Figures 7b and 7c suffer the effect of convolution with the sinc function.

The problems of truncation are at their worst when the coil system is on the surface of the earth as in the previous examples. Raising the source and/or the receiver will introduce an exponential decay in the transform such that the high wavenumber components tend more rapidly to zero.

These examples demonstrate the care that must be taken in obtaining satisfactory results with this technique. The parameters and scale of the problem must be adjusted such that the product of source term and transfer function is well behaved in the wavenumber domain. The accuracy will depend directly on adequate sampling at low wavenumbers

COAXIAL DIPOLE RESPONSE IN THE WAVELENGTH DOMAIN

(U AXIS)

 $\delta=20.0$ $C=1.0$ $H=0.0$

Fig. 4a. Plot of equation (72) on the u wavenumber axis.

Fig. 4b. The inverse Fourier transform of the in-phase wavenumber domain response shown in Figure 4a.

Fig. 4c. The inverse Fourier transform of the quadrature wavenumber domain response shown in Figure 4a.

COAXIAL DIPOLE RESPONSE IN THE WAVELENGTH DOMAIN

Fig. 7a. Plot of equation (72) on the u wavelength axis.

Fig. 7b. The inverse Fourier transform of the in-phase wavenumber domain response shown in Figure 7a.

Fig. 7c. The inverse Fourier transform of the quadrature wavenumber domain response shown in Figure 7a.

and the degree of high wavenumber cutoff. This example has been computed without any form of smoothing. In dealing with complicated source current distributions, the whole transform in the u-v wavenumber domain can be inspected before inverting to the space domain.

The Barringer Input System

This example describes how the procedure was used to calibrate the flight height dependence of the electromagnetic response in the six channels of the Barringer Input System. In November 1971, a test flight was flown at different altitudes over Lake Ontario. A one-layer model (water over a deep conductive clay lake bottom) was used to try to fit the observed data.

First, the response at ten frequencies for each flight height was calculated using the Fourier transform process. For each value computed, a grid of size 128×128 was used. On first thought, this may seem rather inefficient, but not so when considering the speed, flexibility, and low cost of calculating the frequency response in this manner, rather than by numerical integration.

The frequency response was interpolated by cubic spline and multiplied by the Fourier transform of the Barringer source function (Palacky, 1972). This was then inverted to the time domain where the channel amplitudes were computed. The best fit was obtained with a water layer

CHANGE OF AMPLITUDE
WITH FLIGHT HEIGHT
TEST FLIGHT OVER LAKE ONTARIO

Fig. 8. The dependence of the Barringer Input System on flight height. (from Palacky, 1972)

of conductivity 0.035 mhos/m and thickness of $100'$, and a lower half-space of conductivity 0.1 mho/m representing the clay lake bottom. Figure 8 shows the fit obtained with the observed data.

The Current Loop

This example demonstrates how the development is not restricted to simple sources such as dipoles but can deal with any known finite source current distribution. It also serves as a useful pictorial example of the INPUT - OUTPUT nature of the process.

The current source is an odd shaped four sided loop raised 2 grid units above the $z=0$ surface. In units of the sampling interval, the skin depths and layer thicknesses for the earth model are as follows:

$$\begin{array}{ll} \delta_1 = 7. & t_1 = 3. \\ \delta_2 = 2. & t_2 = 4. \\ \delta_3 = 10. & t_3 = \infty \end{array}$$

First, the source magnetic field $H_z(x, y, 0^-)$, due to the current loop alone, is computed and shown in Figure 9a. $\hat{H}_z(u, v, 0^-)$ is then obtained via FFT.

Using the derivative operators in equation (1) and the appropriate magnetic transfer function for the layered

Fig. 9a. $H_z(x, y, 0^-)$ source field.

Fig. 9b. $H_x(x, y, 0^-)$ total field in-phase.

Fig. 9c. $H_y(x, y, 0^-)$ anomalous quadrature.

Fig. 9d. The quadrature current flow pattern in the plane $z=0^+$.

The Grounded Wire

In this example, the electromagnetic response of the finite grounded wire is derived and extended to the limits of a short dipole and infinite line.

Using the transfer relationship, we have that

$$P_3(\underline{r}, 0) = TFE(\underline{r}, 0) P_3^0(\underline{r}, 0) \quad (76)$$

The source electric Hertz potential is given by

$$P_3^0(\underline{r}, 0) = \frac{-q}{4\pi\epsilon_0} \frac{1}{r} \left\{ \sqrt{(x-x_1)^2 + (y-y_1)^2} - \sqrt{(x-x_2)^2 + (y-y_2)^2} \right\} \quad (77)$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the grounded points, and $q = q_0 \exp(i\omega t)$ is the electric charge input and output at the grounded points where

$$\text{div } \underline{K} \neq 0$$

For simplicity, we let

$$\begin{aligned} x_1 &= -x_0, & x_2 &= +x_0, \\ y_1 &= 0, & \text{and } y_2 &= 0. \end{aligned}$$

Then, equation (77) reduces to

$$P_3^0(\underline{r}, 0) = -\frac{q}{\epsilon_0} i \frac{\sin x_0 u}{\rho} \quad (78)$$

Considering the electric transfer function for the homogeneous earth and the relationship $\underline{I} = i\omega q$, the

total electric Hertz potential is given by

$$P_3(\rho, 0) = -i \frac{2 k_1 I \sin(x_0 u)}{\sigma \rho^2} \quad (79)$$

In a similar manner, the magnetic Hertz potential may be derived as

$$G(\rho, 0) = i \frac{2 v I \sin(x_0 u)}{\rho^2 u (\rho + k_1)} \quad (80)$$

From equations (79) and (80) and the derivative operators we may obtain the electromagnetic field components.

If we let $x_0 \rightarrow 0$, i.e. the dipole, equations (79) and (80) reduce to

$$P_3(\rho, 0) = -\frac{I ds k_1 i u}{\sigma_1 \rho^2} \quad (81)$$

and

$$G(\rho, 0) = \frac{I ds u_0 i v}{\rho^2 (\rho + k_1)} \quad (82)$$

If we let $x_0 \rightarrow \infty$, i.e. the infinite line, equations (79) and (80) reduce to

$$P_3(\rho, 0) = -\frac{i I k_1}{\sigma_1 \rho^2} \delta(u) \cdot u \equiv 0 \quad (83)$$

and

$$G(\rho, 0) = i \frac{I v u_0}{\rho^2 (\rho + k_1)} \delta(u) \quad (84)$$

Hence, $P_3(\rho, 0)$ tends to zero as $x_0 \rightarrow \infty$, an expected result.

The total electromagnetic field in the plane of the grounded line source at $z=0$ is singular. Therefore, for computational purposes, we must separate the problem into two parts, singular and non-singular. We choose the

singular part to be the DC ($\omega=0$) response of a homogeneous earth having the conductivity of the first layer. This is readily computed directly in the space domain. The remaining non-singular part is then computed via the 2-D Fourier transform.

To give an example of the results that can be obtained for the grounded line, we have investigated the earth model consisting of a thin layer over a homogeneous half space. This model is important in representing the relatively thin weathered layer which is usually the case in EM surveys.

The results are presented in Figures 10 and 11 where the variables are L , the length of the line, H the thickness of the first layer, and ρ_2/ρ_1 the resistivity contrast. The curves for the infinite line ($L=\infty$) are particularly easy to compute since only a one dimensional Fourier transform is required. In general, the region of maximum secondary field intensity moves closer to the source as L , H and ρ_2/ρ_1 become larger. The factors controlling the accuracy of the solutions are the same as for the magnetic source.

INFINITE LINE $L = \infty$

FINITE LINE $L = 8\delta_1$

SHORT LINE $L = 0.5\delta_1$

IN-PHASE

QUADRATURE

① $\rho_2/\rho_1 = 0.1$ ② $\rho_2/\rho_1 = 1$ ③ $\rho_2/\rho_1 = 10$ ④ $\rho_2/\rho_1 = 100$

$H = \delta_1/10$

Fig. 10. The response of the one-layer earth to the grounded line source.

HORIZONTAL COMPONENT PERPENDICULAR TO THE WIRE $H_Y / |H_{ZP}|$

INFINITE LINE $L = \infty$

FINITE LINE $L = 8\delta_1$

SHORT LINE $L = 0.5\delta_1$

IN-PHASE

QUADRATURE

① $\rho_2 / \rho_1 = 0.1$ ② $\rho_2 / \rho_1 = 1$ ③ $\rho_2 / \rho_1 = 10$ ④ $\rho_2 / \rho_1 = 100$

$H = \delta_1 / 10$

Fig. 11. The response of the one-layer earth to the grounded line source.

CONCLUSION

We hope the procedure described in this paper will greatly facilitate numerical modelling of electromagnetic prospecting problems involving a layered earth. With it, any complicated source geometry may be used as long as the scale of the problem may be adjusted to obtain a well behaved transform. The restrictions can be better understood by referring to corresponding problems which arise in analog modelling in an electrolytic tank. The length of record or array size in the Fourier transform process is analogous to the size of the tank. If the tank is too small, edge effects will be a problem. Alternately, the sampling interval is analogous to the size of the receiver coil. If the receiver coil is large relative to the spatial variations of the electromagnetic fields, the output of the coil will represent an average rather than the true value.

Our method requires a minimal amount of computation time. For example, in the coaxial dipole results, a grid of size 256×64 was used, from which $128 \times 32 = 4096$ independent complex values of the response were obtained. The execution time for generating the transform and inverting to the space domain is less than 2.5 seconds on the IBM 370/Model 145 computer at the U. of Toronto. To calculate this number of values by numerical integration methods, the time and cost may be orders of magnitude

higher. The saving is especially great when the field values are required at many positions around a single source.

The programming is simple and straightforward and readily lends itself to rapid computations for model studies and fitting field data. An extra saving in computation time can result, for example, when the same source function is used with many earth models, in which case the source function may be stored on disk. On the other hand, as was done with the Barringer example, the transfer function of a particular earth model can be stored on disk and used with a source at varying heights.

The theory presented in this paper may be readily extended to consider electric and magnetic sources anywhere in the stack of layers. In particular, the response of the buried electric dipole is just the Green's function for the layered earth. It can be readily obtained over a whole x-y plane in one operation using our procedure. This has obvious advantages in reducing the computation time for the three dimensional electromagnetic modelling of a finite inhomogeneity in the layered earth; we hope this to be the subject of a future paper.

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