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Bose-Einstein Condensation in One-Dimensional Optical Lattices: Bogoliubov's Approximation and Beyond

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Abstract

Bose-Einstein condensation in a finite one-dimensional atomic Bose gas trapped in an optical lattice is studied within Bogoliubov's approximation and then beyond this approximation, within the static fluctuation approximation. A Bose-Hubbard model is used to construct the Hamiltonian of the system. The effect of the potential strength on the condensate fraction is explored at different temperatures; so is the effect of temperature on this fraction at different potential strengths. The role of the number of lattice points (the size effect) at constant number density (the filling factor) is examined; so is the effect of the number density on the condensate fraction. The results obtained are compared to other published results wherever possible.

Keyword: Bose-Einstein Condensation, One-Dimensional Optical Lattices, Bogoliubov's approximation, Thermodynamic Properties, Static fluctuation approximation.

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1. Introduction

The neutral Bose ultra-cold atomic gas in an optical lattice represents a most interesting physical system. It has opened a new avenue for exploring quantum phase transitions and quantum fluctuations [1, 2]. Optical lattices also allow the control of dimensionality; in particular, they enable experimentalists to realize one-dimensional quantum systems [3].

Laser-cooling and evaporation-cooling techniques led in 1995 to the experimental observation of Bose-Einstein condensation (BEC) in dilute alkali gases [4-7]. This achievement was a breakthrough where atomic physics and quantum optics were fused in condensed-matter physics, opening new areas of research. It has been known for some time that, in infinite weakly-interacting dilute Bose gases in the ground state, BEC is well-described by the mean-field Bogoliubov-de Gennes theory [8]. However, the finiteness and dimensionality of a trapped system play a crucial role in determining its physical properties; and the mean-field approach is no longer adequate for treating such systems.

An optical lattice is prepared by creating a periodic potential of atoms, using intersecting laser beams. Optical potentials are created by the interaction between the oscillating electric field of a laser and the electric dipole moment it induces in an atom. The atoms can be confined to periodic lattice sites, and the interaction between these atoms can be controlled by the laser parameters. The properties and geometry of the optical lattice itself can be controlled by tuning the laser parameters and configuration [9]. The possibility of controlling the Hamiltonian

parameters gives researchers an opportunity to probe various regimes of the system. It also allows one to examine some fundamental phenomena of such systems – including BEC, superfluidity and quantum magnetism [10-12].

A basic model describing the Hamiltonian of a Bose atomic gas trapped in an optical lattice is the Bose-Hubbard model [13, 14]. However, solving the corresponding quantum dynamics exactly is possible only for a few-particle system, even if the gas is weakly-interacting; approximations are, in general, necessary. Most of the techniques used to investigate such a system are based on variational or numerical calculations [15-18]; hence the dearth of analytical results here. On the other hand, a perturbation method was used [19] to study this system with a hard-core Bose-Hubbard potential. Further, the Gross-Pitaevskii equation remains a main theoretical tool.

In addition to the extensive experimental work [9, 20-26], further theoretical work has been undertaken for this system. A Bose-Hubbard model was used to study dynamical correlation functions [27]. The current and kinetic-energy correlation functions for the Mott-insulator regime were determined. The condensate fraction for a few-particle system (less than 13 particles) was calculated numerically by an exact diagonalization method [15]. It was observed, not surprisingly, that the condensate fraction decreased as the potential strength increased. The existence of Bose condensation phase transition in a three-dimensional lattice was investigated by the infrared bound method [28]. It was found that BEC occurred for any non-negative chemical potential and for a small enough potential strength. The phase diagram of the quasi- one-dimensional optical lattice was also investigated, within

the density-matrix renormalization group technique [29] as well as the diffusion Monte Carlo technique [30]. The effect of the magnetic field on the superfluid-insulator transition for the system was studied within a mean-field theory; it was found that the critical hopping strength increased with the applied field [31]. The fact remains, however, that quantum fluctuations become large in one-dimensional systems, as compared to three- and two-dimensional systems; accordingly, the theoretical treatment becomes more difficult [32].

In this work, we shall study BEC in a finite one-dimensional atomic Bose gas in an optical lattice. A Bose-Hubbard model [32-37] is used to describe the Hamiltonian of the system. Bogoliubov's approximation (BA) has been used in many previous works to explore such systems [33, 37, 38]. In this approximation, the creation and annihilation operators in the ground state are replaced with a c-number, and the number of particles in excited states to quadratic and higher orders is neglected. The main purpose of the present work is to consider the neglected terms in BA by using the static fluctuation approximation (SFA). Specifically, the condensate fraction will be calculated within both approximations. The results will give a measure of the validity of BA in such systems.

In SFA, the *square* of the local-field operator is replaced with its mean value [39-45]; whereas in the mean-field approach, the operator itself is replaced with its mean value. The point is that in SFA, the fluctuations in both the number of particles and the energy spectrum are taken into account.

This paper is organized as follows. In Section two, the Bose-Hubbard model and SFA are outlined. In Section three, the results for the condensate fraction in our system are presented and discussed within both approximations: BA and SFA. Some closing remarks follow in Section four.

2. Theoretical Formalism

2.1 Bose-Hubbard Model

We consider Bose atoms in an optical lattice with a repulsive interatomic interaction. These can be described by the Bose-Hubbard model [46]:

$$H = -J \sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c.) - \mu \sum_i b_i^\dagger b_i + \frac{U}{2} \sum_i b_i^\dagger b_i (b_i^\dagger b_i - 1). \quad (1)$$

The first term on the right-hand side is the kinetic energy: it describes tunneling from one site to another; the symbol $\langle ij \rangle$ indicates that the summation is restricted to nearest-neighbor sites. The second term is the chemical potential μ which controls the occupation number; $b_i^\dagger$ and b_i are the respective Bose creation and annihilation operators acting at site i . The final term describes the interaction between two atoms in the same site. The parameters J and U can be conveniently adjusted by various methods – e.g., the Feshbach-resonance technique; or by changing the intensity of the laser beams. We shall consider a regular one-dimensional lattice, consisting of M sites. In the weak-interaction limit, $U/J \rightarrow 0$, most particles are in the ground state, and the number of ‘condensed’ atoms N_0 is almost equal to the total number of atoms N . As the interatomic interaction

increases, atoms are gradually ‘depleted’ from the condensate to excited states. To describe the condensation, it is convenient to rewrite the Hamiltonian in momentum space, using Fourier transforms; b_j can then be written as

$$b_j = \frac{1}{\sqrt{M}} \sum_k \exp\left(-i \frac{2\pi k}{M} j\right) a_k. \quad (2)$$

The operator a_k annihilates a particle in the Bloch state with quasi-momentum $\frac{2\pi k}{M}$.

In terms of a_k and a_k^+ , the Hamiltonian becomes

$$H = -\sum_{k=0}^{M-1} \left(2J \cos\left(\frac{2\pi k}{M}\right) + \mu \right) a_k^+ a_k + \frac{U}{2M} \sum_{k,q,p=0}^{M-1} a_k^+ a_q^+ a_p a_{p-k-q}. \quad (3)$$

This can be written as a superposition of three terms: $H = H_0 + H_1 + H_2$, where

$$H_0 = -(2J + \mu) a_0^+ a_0 + \frac{U}{2M} a_0^+ a_0^+ a_0 a_0; \quad (4)$$

$$H_1 = -\sum_{k \neq 0}^{M-1} \left(2J \cos\left(\frac{2\pi k}{M}\right) + \mu \right) a_k^+ a_k + \frac{U}{2M} \sum_{k \neq 0}^{M-1} \left(a_0^+ a_0^+ a_k a_{-k} + a_0 a_0 a_k^+ a_{-k}^+ + 4a_0^+ a_0 a_k^+ a_k \right); \quad (5)$$

and

$$H_2 = \frac{U}{M} \sum_{k \neq 0, q \neq 0}^{M-1} \left(a_k^+ a_q^+ a_0 a_{k+q} + a_k^+ a_0^+ a_q a_{k-q} \right) + \frac{U}{2M} \sum_{k \neq 0, q \neq 0, p \neq 0}^{M-1} \left(a_k^+ a_q^+ a_p a_{k+q-p} \right). \quad (6)$$

The terms H_0 and H_1 represent BA; whereas the third term H_2 denotes the interaction between particles in excited states. In the following sections, we shall investigate this approximation; but the effect of H_2 on the physical properties of the system will be studied within SFA.

2.2 Bogoliubov's Approximation (BA)

This is known to be valid for infinite dilute gases at very low temperatures T . Recently, however, there have also been several studies for finite systems within

this approximation and beyond it [33, 38, 47, 48]. The main assumption in the approximation is that the lowest-energy state (the condensate) is occupied by a macroscopic number of particles. Thus, the operators a_0 and a_0^+ can each be replaced with a c-number, $\sqrt{N_0}$. Our starting point will be the ground state, where most particles are in the condensate state. It is justified, then, to write: $a_0 = \sqrt{N_0} + \delta a_0$ and $a_0^+ = \sqrt{N_0} + \delta a_0^+$, respectively. At very low T , the deviation of the operators from the mean value $\sqrt{N_0}$ is small; so that the quadratic and higher orders of δa_0 and δa_0^+ can be neglected.

From Eqs.(4) and (5), the Hamiltonian becomes

$$H = N_0 \left(-2J - \mu + \frac{UN_0}{2M} \right) + \sqrt{N_0} \left(-2J - \mu + \frac{UN_0}{M} \right) (\delta a_0^+ + \delta a_0) - \sum_{k \neq 0}^{M-1} \left(2J \cos \left(\frac{2\pi k}{M} \right) + \mu \right) a_k^+ a_k + \frac{UN_0}{2M} \sum_{k \neq 0}^{M-1} (a_k a_{-k} + a_k^+ a_{-k}^+ + 4a_k^+ a_k) \quad (7)$$

The second term on the right can be eliminated by setting $\mu = -2J + \frac{UN_0}{M}$. The other terms can then be diagonalized by the Bogoliubov (canonical) transformation:

$$c_k^+ = u_k a_k^+ - v_k a_{-k}; \quad (8)$$

$$c_k = u_k a_k - v_k a_{-k}^+. \quad (9)$$

The creation c_k^+ and annihilation c_k operators satisfy the Bose commutation relations: $[c_k, c_p^+] = \delta_{k,p}$ and $[c_k, c_p] = 0$, from which one finds: $u_k^2 - v_k^2 = 1$. Equation (7) becomes

$$\begin{aligned}
H_b = & -\frac{Un_0N_0}{2} - \sum_{k \neq 0}^{M-1} \left((\varepsilon(k) - 2J - Un_0)v_k^2 - Un_0u_kv_k \right) \\
& + \sum_{k \neq 0}^{M-1} \left(-(\varepsilon(k) - 2J - Un_0)(u_k^2 + v_k^2) + 2Un_0u_kv_k \right) c_k^+ c_k \\
& + \sum_{k \neq 0}^{M-1} \left(-(\varepsilon(k) - 2J - Un_0)u_kv_k + \frac{Un_0}{2}(u_k^2 + v_k^2) \right) (c_k^+ c_{-k}^+ + c_k c_{-k})
\end{aligned} \quad (10)$$

where $n_0 = \frac{N_0}{M}$ is the number density in the condensate state.

The Hamiltonian can be diagonalized under the condition

$$-\left(\varepsilon(k) - 2J - \frac{UN_0}{M} \right) u_k v_k + \frac{UN_0}{2M} (u_k^2 + v_k^2) = 0. \quad (11)$$

This implies that

$$u_k^2 = v_k^2 + 1 = \frac{1}{2} \left(1 + \frac{2J - \varepsilon(k) + Un_0}{\hbar\omega_k} \right), \quad (12)$$

where

$$\hbar\omega_k = \sqrt{(2J - \varepsilon(k))^2 + 2Un_0(2J - \varepsilon(k))}. \quad (13)$$

and $\varepsilon(k) = 2J \cos\left(\frac{2\pi k}{M}\right)$.

After simple algebra, H_b reduces to

$$H_b = -\frac{UN_0n_0}{2} + \frac{1}{2} \sum_{k \neq 0}^{M-1} (\hbar\omega_k + \varepsilon(k) - 2J - Un_0) + \sum_{k \neq 0}^{M-1} \hbar\omega_k c_k^+ c_k. \quad (14)$$

The total-number-of-particles operator N can be written in terms of c_k^+ and c_k as

$$N = N_0 + \sum_{k \neq 0}^{M-1} v_k^2 + (u_k^2 + v_k^2) \langle c_k^+ c_k \rangle. \quad (15)$$

From Eq. (14), using Bose-Einstein statistics:

$$\langle c_k^+ c_k \rangle = \frac{1}{\exp(\hbar\omega_k) - 1}. \quad (16)$$

At absolute zero, $\langle c_k^+ c_k \rangle$ is zero; so that

$$N = N_0 + \sum_{k \neq 0}^{M-1} v_k^2 = N_0 + \frac{1}{2} \sum_{k \neq 0}^{M-1} \left(\frac{2J - \varepsilon(k) + Un_0}{\sqrt{(2J - \varepsilon(k))^2 + 2Un_0(2J - \varepsilon(k))}} - 1 \right). \quad (20)$$

The physical properties of the system (such as the energy, heat capacity, condensate fraction, ...) at both zero and finite low T can be calculated using Eqs. (12-16). It should be emphasized that reliable results are obtained so long as $N_0(T) \sim N$, as dictated by the basic assumption of BA.

2.3 Beyond Bogoliubov's Approximation: Static Fluctuation Approximation (SFA)

According to BA, the interaction between two particles in excited states, that remain in these states after the interaction is completely neglected. Also, the interaction between two particles, one of which is in the ground state and the other in an excited state such that the scattered particles end up in excited states, is again neglected. These interactions are represented by H_2 in Eq.(6).

With the aid of SFA, the neglected terms can be incorporated into the picture. In this approach, H_2 is rewritten as $H_2 = \sum_{k \neq 0} \hat{E}_k c_k^+ c_k$ [40]. SFA can, in principle, be applied to any many-body system. In fact, it has been applied to both strongly- and weakly-interacting systems [39-45]. However, it is essentially a modified mean-field approach or an independent-particle model. Thus, strictly speaking,

it can hardly be applicable to dense and strongly-interacting systems – except at ‘sufficiently low’ T [40]. The local-field (energy) operator in this formalism is written in terms of its mean value and the corresponding fluctuations: $\hat{E}_k = \langle \hat{E}_k \rangle + \Delta \hat{E}_k$. The basic assumption in SFA is that the square of the fluctuations in this operator can be replaced with its mean value: $\Delta \hat{E}_k^2 = \langle \Delta \hat{E}_k^2 \rangle = \Omega_k^2$. In the mean-field approximation, it is the local-field operator itself that is replaced with its mean value $\langle \hat{E}_{k\lambda} \rangle$. SFA, then, goes one step further than the mean-field approach. Its validity depends on the amount of the fluctuation relative to the mean value of the local-field operator $\Omega_k / \langle \hat{E}_k \rangle$. These fluctuations increase with both T and U . At low T or for weakly-interacting systems, SFA is expected to give credible results.

The hermitian local-field operator $\hat{E}_k$ is determined from Heisenberg’s equation of motion and the assumption that it commutes with the creation and annihilation operators [40-45].

The Heisenberg equation of motion is

$$\frac{dc_s^+}{d\tau} = [H_2, c_s^+] = \hat{E}_s c_s^+, \quad (21)$$

where $\tau = it$, t being the time. $\hat{E}_s$ is then [40]

$$\hat{E}_s = \frac{2U}{M} \left(\frac{2J - \varepsilon(s) + Un_0}{\hbar\omega_s} \hat{B}_1 + \frac{Un_0}{2\hbar\omega_s} \hat{B}_2 \right); \quad (22)$$

the operators $\hat{B}_1$ and $\hat{B}_2$ have the forms

$$\hat{B}_1 = \sum_{k \neq 0}^{M-1} A_1(k) \left(c_k^+ c_k + \frac{1}{2} \right) - \frac{M-1}{2}; \quad (23)$$

$$\hat{B}_2 = \sum_{k \neq 0}^{M-1} A_2(k) \left(c_k^+ c_k + \frac{1}{2} \right), \quad (24)$$

where $A_1(k) = \frac{2J - \varepsilon(k) + Un_0}{\hbar\omega_k}$ and $A_2(k) = \frac{Un_0}{\hbar\omega_k}$.

One can obtain all the necessary correlation functions from the following ‘generating’ equation [40-45], which relates $\hat{m}_k = c_k^+ c_k$ to $\hat{E}_k$:

$$\langle \hat{m}_k \hat{A} \rangle = \eta_0(k) \langle \hat{A} \rangle + \eta_1(k) \langle \Delta \hat{E}_k \hat{A} \rangle, \quad (25)$$

$\hat{A}$ being an arbitrary operator which commutes with the creation and annihilation operators; $\sqrt{\langle (\Delta \hat{E}_k)^2 \rangle}$ has only the characteristic values $\pm \Phi_k$;

$$\eta_0(k) = \frac{1}{2} \left(\frac{1}{\exp[\beta(\hbar\omega_k + \langle \hat{E}_k \rangle + \Phi_k)] - 1} + \frac{1}{\exp[\beta(\hbar\omega_k + \langle \hat{E}_k \rangle - \Phi_k)] - 1} \right); \quad (26)$$

$$\eta_1(k) = \frac{1}{2\Phi_k} \left(\frac{1}{\exp[\beta(\hbar\omega_k + \langle \hat{E}_k \rangle + \Phi_k)] - 1} - \frac{1}{\exp[\beta(\hbar\omega_k + \langle \hat{E}_k \rangle - \Phi_k)] - 1} \right),$$

(27)

where $\beta = 1/k_B T$, k_B being Boltzmann’s constant.

The function $\hat{m}(k)$, which is related to the distribution function, is obtained by putting $\hat{A} = 1$ in Eq. (25):

$$\langle \hat{m}(k) \rangle = \eta_0(k). \quad (28)$$

The fluctuations Φ_k in the operator $\hat{E}_k$ are obtained by putting $\hat{A} = \Delta \hat{E}_k$ again in Eq. (25):

$$\eta_1(k) \Phi_k^2 = \langle \hat{m}_k \Delta \hat{E}_k \rangle. \quad (29)$$

From Eqs.(22) and (29), we have

$$\eta_1(k) \Phi_k^2 = \frac{2U}{M} \sum_{p \neq 0}^{M-1} \left(A_1(k) A_1(p) + \frac{A_2(k) A_2(p)}{2} \right) \left(\langle \hat{m}_k \hat{m}_p \rangle - \langle \hat{m}_k \rangle \langle \hat{m}_p \rangle \right). \quad (30)$$

Putting in Eq. (25) $\hat{A} = \hat{m}_p$, we obtain the pair correlation function $\langle \hat{m}_k \hat{m}_p \rangle_c$, the index c denoting the true correlations ($k \neq p$) involved:

$$\langle \hat{m}_k \hat{m}_p \rangle_c = \langle \hat{m}_k \rangle \langle \hat{m}_p \rangle + \eta_1(k) \frac{2U}{M} \sum_{q \neq 0}^{M-1} \left(A_1(k) A_1(q) + \frac{A_2(k) A_2(q)}{2} \right) \left(\langle \hat{m}_p \hat{m}_q \rangle - \langle \hat{m}_q \rangle \langle \hat{m}_p \rangle \right). \quad (31)$$

The unknown value $\langle \hat{m}_p^2 \rangle$ was derived in [40]:

$$\langle \hat{m}_p^2 \rangle = \langle \hat{m}_p \rangle (1 + 2 \langle \hat{m}_p \rangle) + 2\eta_1(p) \frac{2U}{M} \sum_{q \neq 0}^{M-1} \left(A_1(p) A_1(q) + \frac{A_2(p) A_2(q)}{2} \right) \left(\langle \hat{m}_p \hat{m}_q \rangle - \langle \hat{m}_p \rangle \langle \hat{m}_q \rangle \right). \quad (32)$$

The closed system of coupled equations consisting of $\langle \hat{E}_k \rangle$, $\langle \hat{m}_k \rangle$, $\langle \hat{m}_k^2 \rangle$, Φ_k and $\langle \hat{m}_k \hat{m}_p \rangle_c$ can be solved numerically by an iteration method.

3. Results and Discussion

The condensate fraction as calculated within both SFA and BA, under different conditions, is presented in Figs. 1 and 2. The effect of U on this fraction at different

T is shown in Fig. 1a. The role of the number of lattice points (the size effect) as a function of U , at constant number density (the filling factor) and at different T , is examined in Fig. 1b. The condensate fraction at different number densities is displayed in Fig. 1c. Finally, this fraction is shown in Fig. 2 as a function of T at different U . Let us now elaborate.

Figure 1a shows the condensate fraction for $N = M = 100$ at different T . The repulsive interaction $U > 0$ between particles in a site increases the depletion rate; the condensate fraction decreases. In BA, this fraction decreases smoothly with $\frac{U}{J}$; in SFA, it decreases faster. Both approximations give close results in the very-weak-interaction limit, $\frac{U}{J} \ll 1$. In BA, the condensate fraction vanishes when $\frac{U}{J} \approx 50$; but in SFA, it almost vanishes around $\frac{U}{J} \approx 3.3$ at $T = 1 \times 10^{-6}$ K. For comparison, according to variational theory, within the framework of the path-integral formalism [48], it almost vanishes at $\frac{U}{J} = 4$ for $N = M$. This behavior of the condensate fraction could be interpreted as a quantum phase transition from the superfluid to Mott-insulator phase [48]. In passing, we note that this decrease of the condensate fraction with increasing U/J also occurs in two and three dimensions in the zero- T limit [49].

In Fig. 1a, three temperatures have been chosen: 1×10^{-6} K, 3×10^{-5} K and 5×10^{-5} K. Below 1×10^{-6} K, the condensate fraction is almost temperature-independent and behaves as for $T = 1 \times 10^{-6}$ K in both SFA and BA. The system can then be considered to be in the ground state below 1×10^{-6} K. The condensate fraction

decreases as $\frac{U}{J} \rightarrow 0$, as T increases. Our results indicate that both BA and SFA work well at $T \ll J$. Moreover, it is observed that the variations in the condensate fraction in both BA and SFA increase as U becomes stronger, especially as T increases. The fluctuations reduce the energy difference between the ground state and the excited states, especially as U increases. Therefore, the condensate fraction in SFA decreases faster than in BA as U or T increases.

Now, it is well known that BEC does *not* occur in *infinite* one-dimensional systems. Therefore it is in order to explore the size effect of our ‘chain’. This is displayed in Fig. 1b at $T = 1 \times 10^{-9}$ K and $J = 3 \times 10^{-4}$ K for $N = M = 100, 500$, and 1000 . For very weak interactions, $\frac{U}{J} \rightarrow 0$, $\frac{N_0}{N} \approx 0.95, 0.90$, and < 0.90 for $M = 100, 500$, and 1000 , respectively. This indicates that the condensate fraction decreases as the number of lattice points increases (i.e., the chain size increases). The condensate fraction decreases quite fast in both approximations, BA and SFA, as the chain size increases. In BA, at $\frac{U}{J} = 5$, $\frac{N_0}{N}$ is $0.50, 0.35$, and 0.30 for $M = 100, 500$, and 1000 , respectively. In SFA, $\frac{N_0}{N}$ vanishes at: $\frac{U}{J} = 3.3, 1.7$, and 1.4 for $M = 100, 500$, and 1000 , respectively. These results indicate that as $M \rightarrow \infty$, the condensate fraction vanishes. This occurs at lower U in SFA than in BA.

Figure 1c shows the effect of the number density in a site, N/M , on N_0/N . In BA, N_0/N is weakly-dependent of N/M – unlike the case for SFA, where N_0/N *does*

depend on N/M . For very weak interactions, $\frac{U}{J} \ll 1$, SFA results are quite close to BA results. In SFA, the condensate fraction vanishes faster at lower density. It vanishes around $\frac{U}{J} = 2.2, 3.3$, and 7.0 at $N/M = 0.5, 1$, and 2 , respectively. These results mean that the transition temperature increases with increasing number density. This is not surprising in view of the dependence of the effective interaction itself on the density. It is clear that the condensate fraction is more sensitive to the density in SFA than in BA.

In variational theory [48], however, it turns out that the condensate fraction is insensitive to the density.

Finally, Fig. 2 (a, b, and c) shows the condensate fraction at $N/M = 1$ and $J = 3 \times 10^{-4} \text{K}$, at different values of $\frac{U}{J} = 0.5, 1$, and 2 . It is observed that the discrepancy between BA and SFA results become appreciable when $\frac{U}{J} > 1$, as expected, since BA is valid only at low T and for weak interactions. The neglected terms in BA become quite significant as the interaction becomes stronger. Thus, the condensate fraction and transition temperature are almost the same in both approximations for weak interactions: $\frac{U}{J} = 0.5$; whereas in the strong-interaction limit, $\frac{U}{J} > 1$, the BA condensate fraction is greater than SFA's. In view of the neglect of important terms in BA, but not in SFA, the SFA results should be more reliable. At $\frac{U}{J} \leq 1$, the condensate fraction in SFA, at any T , is close to that in BA.

However, at $\frac{U}{J} = 2$, Fig. 2c shows that this fraction, even as $T \rightarrow 0$, differs in SFA from that in BA. This difference manifests itself as U increases, even at very low T . Also, the transition temperature in SFA is less than that in BA. These results confirm that the mean-field approximation is workable at very low T and weak U ; but, as T or U increases, the fluctuations in the local-field operator cannot be ignored.

4. Conclusions

BEC has been investigated for a finite one-dimensional atomic Bose gas trapped in an optical lattice under different physical conditions within two approximations, BA and SFA. The condensate fraction has been calculated as a function of potential strength, temperature, and number density. The BA and SFA results have been found to be quite close only for very weak interactions. However, in SFA, the condensate fraction generally vanishes at lower values of $\frac{U}{J}$, compared to BA. Our SFA results for the condensate fraction as a function of potential strength are consistent with those obtained by the path-integral formalism [48]. However, this fraction has been found to be dependent on the number density – unlike the case in [48]. Since SFA goes beyond BA, one must conclude that SFA results are more reliable than BA's.

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Figure 1: The condensate fraction as a function of $\frac{U}{J}$ at $J = 3 \times 10^{-4} \text{ K}$.

a) $N = M = 100$, and temperature $T = y \times 10^{-6} \text{ K}$, where $y = 1, 30$, and 50 .

b) $T = 5 \times 10^{-9} \text{ K}$, $\frac{N}{M} = 1$ at different N .

c) $T = 1 \times 10^{-6} \text{ K}$ at different filling factor $f = \frac{N}{M} = 2, 1$, and 0.5 .

Figure 2: The condensate fraction as a function of temperature T , at $J = 3 \times 10^{-4}$ K, $N = 100$ and $M = 50$; a) $\frac{U}{J} = 0.5$, b) $\frac{U}{J} = 1$, and c) $\frac{U}{J} = 2$.