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DRAINED AND UNDRAINED ANALYSES OF CYLINDRICAL CAVITY CONTRACTIONS BY BOUNDING SURFACE PLASTICITY

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Draft

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Abstract: This paper develops a rigorous semi-analytical approach for drained/undrained cylindrical cavity contraction problems in bounding surface elastoplastic geomaterials. For undrained situation, the effective radial, tangential, and vertical component stresses can be directly solved from the constitutive governing differential equations as an initial value problem, the excess pore pressure subsequently being determined from the radial equilibrium equation. Whereas for the drained case, the Eulerian radial equilibrium equation must be first transformed into an equivalent one in Lagrangian description via the introduction of an auxiliary variable, and then be solved together with the elastoplastic constitutive relation for the three stress components as well as the specific volume. It is observed that during drained/undrained contraction processes, plastic deformations occur immediately as a direct result of employing the bounding surface model, so outside the cavity there exists no purely elastic zone. The computed stress distributions and in particular the stress path prediction through an example analysis capture well the anticipated elastoplastic and failure behaviour of the geomaterials surrounding the cavity. The validity and accuracy of the proposed semi-analytical elastoplastic solutions are justified through comparison with ABAQUS numerical results, and its applicability to the tunnel excavation and wellbore drilling problems is also demonstrated.

Keywords: cylindrical cavity contraction; bounding surface model; semi-analytical solutions; stress path; tunnelling; wellbore drilling

Introduction

The cavity contraction theory has wide applications to the design and construction of tunnels in civil engineering and to the stress and displacement modelling around the drilled wellbore in petroleum engineering as well. However, in contrast to the cavity expansion problem (Gibson and Anderson 1961; Vesic 1972; Randolph et al. 1979; Carter et al. 1986; Yu 1990; Collins and Stimpson 1994; Salgado and Randolph 2001; Russell and Khalili 2006), much less investigations of cavity contraction in soil and rock have been conducted, which is particularly the case when the critical state plasticity models (Wood 1990; Dafalias and Herrmann 1980) are involved.

Stress and strain analyses for the cavity contraction problem incorporating elasoplastic behaviour of geomaterials usually are obtained through numerical methods (Carter 1978; Potts and Zdravkovic 1999). So far scarce analytical/semi-analytical plasticity solutions exist in the literature for such fundamental problem even when major approximations are assumed. The earlier analytical contributions to the tunnel/wellbore opening problems usually modelled the soil/rock as associated or non-associated elastic-perfectly plastic materials like linear Mohr-Coulomb and nonlinear Hoek-Brown yield criteria (Brown et al. 1983), and under isotropic in situ stress condition. Later Graziani and Ribacchi (1993) proposed an analytical approach to determine the state of stress and strain for a circular opening excavated in a rock mass, where the rock is assumed to obey the non-associated strain softening Mohr-Coulomb model with the softening behaviour linked to a simple plastic shear strain through the so-called softening modulus parameter. Papanastasiou and Durban (1997) made a further substantial extension to the cylindrical cavity solutions for Drucker-Prager and Mohr-Coulomb geomaterials exhibiting arbitrary strain hardening behaviour. However, for the case of Mohr-Coulomb model in

Papanastasiou and Durban (1997), the assumption of the axial stress being the intermediate principal stress is not necessarily true during the cavity expansion/contraction and may easily violate the plane strain condition in the axial direction (Reed 1988; Yu and Rowe 1999).

In addition to the Drucker-Prager, Mohr-Coulomb, and Hoek-Brown models (the latter two are quite suitable for modelling the granular geomaterials as they capture reasonably well the particle rearrangement and the frictional sliding between material particles), the analytical cavity contraction solutions based on the critical state plasticity models have also been developed, though quite limited. For example, Charlez and Roatesi (1999) derived an approximate analytical solution for the wellbore stability problem under undrained condition using a very simplistic, idealized Cam Clay model where the elliptical yield surface was replaced by two straight lines, and the solution is restricted to the volumetric strain hardening rocks with overconsolidation ratio less than 2. Concurrently, Yu and Rowe (1999) provided a set of comprehensive yet still approximate analytical/semi-analytical solutions for the undrained circular excavation problem using the well known Cam Clay critical state theories (Wood 1990). Nevertheless, as noted in Chen and Abousleiman (2012; 2013), the main drawback in their approach is that the deviatoric and mean effective stresses were treated in some approximate fashion by enforcing both the two stress invariants independent of the axial stress, simply to obtain the possible closed form solutions.

This paper considers the cavity contraction boundary value problem involving the bounding surface plastic model (Dafalias and Herrmann 1980; 1982), under both undrained and drained conditions. The prominent feature of this critical state based model, with the seminal concept of the bounding surface in stress space, is that inelastic deformation can occur for stress points within the bounding surface at a pace depending on the proximity of the current stress state to the

bounding surface (Kaliakin et al. 1987), so that the realistic non-recoverable behaviour of soils/rocks observed on unloading and reloading can be well recovered. As with Chen and Abousleiman (2012; 2013), the key step in the formulation of cavity contraction in geomaterials using bounding surface plasticity model is to establish an appropriate incremental relationship between the effective radial, tangential, and vertical stresses and the corresponding strain components, i.e., the elastoplastic constitutive equations, and then reduce them to a set of differential equations valid for any material point in the plastic zone. For undrained condition, the three stresses can be directly solved from these governing differential equations as an initial value problem, the excess pore pressure then being determined from the radial equilibrium equation. Whereas for the drained condition, the Eulerian radial equilibrium equation must be first transformed into an equivalent one in Lagrangian description, which can be accomplished with the introduction of an auxiliary variable. This transformed equation, together with the aforementioned elastoplastic constitutive relation, again constitute a set of differential equations. The three stress components as well as the specific volume thus can be readily solved. The computed stress distributions and in particular the stress path prediction through an example analysis, for different values of overconsolidation ratio (OCR), capture well the anticipated elastoplastic to failure behaviour of the soils/rocks surrounding the cavity. As an application to the practical tunnel and wellbore opening problems, the semi-analytical elastoplastic solutions developed is utilized to predict the critical support pressure that is required to maintain the tunnel/wellbore stability.

Bounding surface model

The well established bounding surface concept was originally introduced by Dafalias and Popov (1975; 1976) and later on reformulated within the framework of critical state soil

plasticity by Dafalias and Herrmann (1980; 1982; 1986). The basic idea of their formulation is that an isotropically expanding or contracting surface, rather than a yield surface, is used in the model. This surface is known as the bounding surface which depends only on the plastic volumetric strain (or plastic void ratio). In addition, the bounding surface model uses a simple radial mapping to determine a unique image point on the bounding surface that corresponds to the current stress point inside the bounding surface. The value of the plastic modulus is assumed to be a function of the distance between the stress point and its image, while the gradient of the bounding surface at the image point defines the loading-unloading direction. The salient feature of this model is that plastic deformation may occur for stress point even inside the surface.

Fig. 1 shows a schematic illustration of the bounding surface model. A stress state σ'_{ij} lies always within or on the bounding surface which, mathematically, can be expressed as

$$\bar{F}(\bar{\sigma}'_{ij}, e^p) = 0 \quad (1)$$

where a bar over effective stress quantities σ'_{ij} indicates points on the bounding surface $\bar{F} = 0$, and e^p , the plastic void ratio, is the only plastic internal variable defining the hardening/softening behaviour of the bounding surface. Note that the plastic void ratio e^p is related to the plastic volumetric strain ε_p^p and the elastic void ratio e^e as follows

$$de^p = -(1 + e)d\varepsilon_p^p \quad (2)$$

$$e^e + e^p = v - v_0 \quad (3)$$

where e is the total void ratio; v is the specific volume ($v = 1 + e$); v_0 is the initial specific volume; and de^p and $d\varepsilon_p^p$ are the increments of plastic void ratio and plastic volumetric strain, respectively. It should be pointed out that in Eq. (2) the current (varying) void ratio e , instead of the initial void ratio e_i (Dafalias and Herrmann 1980; 1986), has been adopted for a rigorous definition of de^p . This will be able to more suitably accommodate the large plastic deformations

of the geomaterials that most likely occur during the cavity contraction process.

In addition to the bounding surface, at any stress point σ'_{ij} a surface homeothetic to the bounding surface with respect to the origin o can be indirectly defined (shown by a dashed line in Fig. 1)

$$F(\sigma'_{ij}) = 0 \quad (4)$$

Such a surface is referred to as the loading surface in Dafalias and Herrmann (1980; 1982), which specifies a quasi-elastic domain but is actually not a yield surface since an inward motion of σ'_{ij} will eventually induce loading after an initial path of unloading before σ'_{ij} reaches the surface again.

The plastic constitutive relations for the bounding surface model, in the general stress space, can be expressed as (Dafalias and Herrmann 1980)

$$d\varepsilon_{ij}^p = \langle L \rangle n_{ij} \quad (5)$$

$$L = \frac{1}{K} d\sigma'_{kl} n_{kl} = \frac{1}{K_b} d\bar{\sigma}'_{kl} n_{kl} \quad (6)$$

where n_{ij} (or n_{kl}) denotes the unit normal at stress point σ'_{ij} or its image point $\bar{\sigma}'_{ij}$ on the bounding surface, see Fig. 1; K is the actual plastic modulus associated with $d\sigma'_{kl}$; K_b is a plastic modulus on the bounding surface associated with $d\bar{\sigma}'_{kl}$; the Macauley bracket $\langle \cdot \rangle$ define the operation $\langle L \rangle = h(L) \cdot L$, h being the heaviside step function; L is usually called the loading function. The bounding surface plastic modulus K_b is obtained from the consistency condition

$$d\bar{F} = \frac{\partial \bar{F}}{\partial \bar{\sigma}'_{ij}} d\bar{\sigma}'_{ij} + \frac{\partial \bar{F}}{\partial e^p} de^p = 0 \quad (7)$$

with the aid of Eq. (2), as follows

$$K_b = \frac{1+e}{\left[\frac{\partial \bar{F}}{\partial \bar{\sigma}'_{ij}} \frac{\partial \bar{F}}{\partial \bar{\sigma}'_{ij}} \right]} \frac{\partial \bar{F}}{\partial e^p} \frac{\partial \bar{F}}{\partial \bar{\sigma}'_{kk}} \quad (8)$$

Note that in the above two equations the summation convention over repeated indices applies.

The plastic modulus K needed to determine the plastic strain increment at the current stress point σ'_{ij} is related to K_b by

$$K = K_b + H(\sigma'_{ij}, e^p) \frac{\delta}{\delta_0(\sigma'_{ij}, e^p) - \delta} \quad (9)$$

where H is the hardening function; δ is the distance between σ'_{ij} and $\bar{\sigma}'_{ij}$; and δ_0 is a properly chosen reference stress.

Elastoplastic constitutive equations

It must be emphasized that the major objective of this paper is to provide a rigorous semi-analytical solution for the cavity contraction boundary value problem in generic boundary surface soils/rocks, instead of focusing on the constitutive model itself. Therefore, to simplify the formulation yet still retaining the essential ingredients of the bounding surface plasticity (Dafalias and Herrmann 1980; 1982; 1986; Kaliakin and Dafalias 1989; Manzari and Nour 1997), the initial and specific version of the bounding surface model by Dafalias and Herrmann (1980) will be chosen for the cavity analysis. The bounding surface is further assumed to consist of one single ellipse (Kaliakin and Dafalias 1989; Manzari and Nour 1997), as shown in Fig. 1. However, the formulation and derivation presented in this work, with little or minor modification, shall be applicable to the “composite” surface consisting of two ellipses and a hyperbola (Dafalias and Herrmann 1980; 1982; 1986) and later improved versions of the bounding surface models as well involving three stress invariants (Dafalias and Herrmann 1986) and more sophisticated hardening function H (Kaliakin and Dafalias 1989).

In accordance with Dafalias and Herrmann (1980; 1986), the bounding and loading surfaces, $\bar{F}$ and F (see Fig. 1), can be expressed as

$$\bar{F}(\bar{\sigma}'_{ij}, e^p) \equiv \bar{F}(\bar{p}', \bar{q}, e^p) = \left(\frac{\bar{p}'}{\bar{p}'_C}\right)^2 + \frac{(R-1)^2}{M^2} \left(\frac{\bar{q}}{\bar{p}'_C}\right)^2 - \frac{2}{R} \frac{\bar{p}'}{\bar{p}'_C} + \frac{2-R}{R} = 0 \quad (10)$$

$$F(\sigma'_{ij}) \equiv F(p', q) = \left(\frac{p'}{p'_A}\right)^2 + \frac{(R-1)^2}{M^2} \left(\frac{q}{p'_A}\right)^2 - \frac{2}{R} \frac{p'}{p'_A} + \frac{2-R}{R} = 0 \quad (11)$$

where

$$p' = \frac{1}{3}(\sigma'_r + \sigma'_\theta + \sigma'_z) \quad (12a)$$

$$\bar{p}' = \frac{1}{3}(\bar{\sigma}'_r + \bar{\sigma}'_\theta + \bar{\sigma}'_z) \quad (12b)$$

$$q = \sqrt{\frac{1}{2}[(\sigma'_r - \sigma'_\theta)^2 + (\sigma'_r - \sigma'_z)^2 + (\sigma'_\theta - \sigma'_z)^2]} \quad (13a)$$

$$\bar{q} = \sqrt{\frac{1}{2}[(\bar{\sigma}'_r - \bar{\sigma}'_\theta)^2 + (\bar{\sigma}'_r - \bar{\sigma}'_z)^2 + (\bar{\sigma}'_\theta - \bar{\sigma}'_z)^2]} \quad (13b)$$

with σ'_r , σ'_θ , σ'_z and $\bar{\sigma}'_r$, $\bar{\sigma}'_\theta$, $\bar{\sigma}'_z$ denoting, respectively, the actual and image effective radial, tangential, and vertical stresses, which are the three principal stresses for axisymmetric cylindrical cavity problem and considered positive here for compression; M is the slope of critical state line; R is a constant model parameter and essentially defines the shape of the bounding surface; $\bar{p}'_C$ represents the intersection of the current bounding surface with the p' axis, which is, in fact, the value of p' for isotropic consolidation and is connected with the plastic volumetric strain by

$$\frac{d\bar{p}'_C}{\bar{p}'_C} = \frac{(1+e)}{\lambda-\kappa} d\varepsilon_p^p \quad (14)$$

where λ and κ are the slopes of normal compression and swelling lines in $v - \ln p'$ plane (Wood 1990); and p'_A is determined as the intersection between the loading surface passing through the current stress point (p', q) and the p' axis.

According to Eqs. (4)-(6) and (11), the three components of incremental plastic strain can be expressed as

$$d\varepsilon_r^p = \frac{1}{K} (d\sigma'_r n_r + d\sigma'_\theta n_\theta + d\sigma'_z n_z) n_r = \frac{1}{K} \frac{\sqrt{(\partial F/\partial p')^2 + (\partial F/\partial q)^2}}{\sqrt{\frac{1}{3}(\partial F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2}} (dp' n_p + dq n_q) n_r \quad (15)$$

$$d\varepsilon_\theta^p = \frac{1}{K} (d\sigma'_r n_r + d\sigma'_\theta n_\theta + d\sigma'_z n_z) n_\theta = \frac{1}{K} \frac{\sqrt{(\partial F/\partial p')^2 + (\partial F/\partial q)^2}}{\sqrt{\frac{1}{3}(\partial F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2}} (dp' n_p + dq n_q) n_\theta \quad (16)$$

$$d\varepsilon_z^p = \frac{1}{K} (d\sigma'_r n_r + d\sigma'_\theta n_\theta + d\sigma'_z n_z) n_z = \frac{1}{K} \frac{\sqrt{(\partial F/\partial p')^2 + (\partial F/\partial q)^2}}{\sqrt{\frac{1}{3}(\partial F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2}} (dp' n_p + dq n_q) n_z \quad (17)$$

where

$$n_r = \frac{\frac{1}{3} \frac{\partial F}{\partial p'} + \frac{3(\sigma'_r - p')}{2q} \frac{\partial F}{\partial q}}{\sqrt{\frac{1}{3}(\partial F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2}}, \quad n_\theta = \frac{\frac{1}{3} \frac{\partial F}{\partial p'} + \frac{3(\sigma'_\theta - p')}{2q} \frac{\partial F}{\partial q}}{\sqrt{\frac{1}{3}(\partial F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2}}, \quad n_z = \frac{\frac{1}{3} \frac{\partial F}{\partial p'} + \frac{3(\sigma'_z - p')}{2q} \frac{\partial F}{\partial q}}{\sqrt{\frac{1}{3}(\partial F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2}} \quad (18a)$$

are the components of the unit normal to the loading surface $F(p', q) = 0$ or to the bounding surface $\bar{F}(\bar{p}', \bar{q}, e^p) = 0$ in r , θ , and z directions, respectively;

$$n_p = \frac{\partial F/\partial p'}{\sqrt{(\partial F/\partial p')^2 + (\partial F/\partial q)^2}} = \frac{1}{g} \left[f(\eta) - \frac{1}{R} \right], \quad n_q = \frac{\partial F/\partial q}{\sqrt{(\partial F/\partial p')^2 + (\partial F/\partial q)^2}} = \frac{1}{g} \eta f(\eta) \left(\frac{R-1}{M} \right)^2 \quad (18b)$$

are the components of the unit normal along p' and q axes, where

$$f(\eta) = \frac{1 + (R-1)\sqrt{1 + R(R-2)x^2}}{R[1 + x^2 + R(R-2)x^2]} \quad (18c)$$

$$\eta = \frac{q}{p'}, \text{ and } x = \frac{\eta}{M},$$

$$g = \sqrt{\left[f(\eta) - \frac{1}{R} \right]^2 + \eta^2 f^2(\eta) \left(\frac{R-1}{M} \right)^4} \quad (18d)$$

By adding Eqs. (15)-(17), the incremental plastic volumetric strain can be determined as

$$d\varepsilon_p^p = d\varepsilon_r^p + d\varepsilon_\theta^p + d\varepsilon_z^p = \frac{1}{K} \frac{(\partial F/\partial p')^2 + (\partial F/\partial q)^2}{\left[\frac{1}{3}(\partial F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2 \right]} (dp' n_p + dq n_q) n_p \quad (19)$$

On the other hand, the expression of plastic volumetric strain increment, $d\varepsilon_p^p$, is shown to be (Dafalias and Herrmann 1980)

$$d\varepsilon_p^p = \frac{1}{K^*} (dp' n_p + dq n_q) n_p \quad (20)$$

where

$$K^* = K_b^* + \frac{1+e}{\lambda-\kappa} h p_a \left(1 + \left|\frac{M}{\eta}\right|^m\right) \frac{\delta}{\delta_0 - \delta} \quad (21)$$

$$K_b^* = \frac{1+e}{\lambda-\kappa} \frac{\bar{p}'_c}{R} \frac{1}{g^2} [R + f(\eta) - 2] \left[f(\eta) - \frac{1}{R}\right] \quad (22)$$

where h and m are dimensionless material constants; p_a represents the atmospheric pressure with the proper stress unit; δ_0 is defined as the maximum isotropic stress, i.e., $\delta_0 = \bar{p}'_c$; and δ can be calculated as

$$\delta = \sqrt{1 + \eta^2} [f(\eta) \bar{p}'_c - p'] \quad (23)$$

Through comparison of Eqs. (19) and (20), one can find

$$K = \frac{\left(\frac{\partial F}{\partial p'}\right)^2 + \left(\frac{\partial F}{\partial q}\right)^2}{\frac{1}{3}(F/\partial p')^2 + \frac{3}{2}(\partial F/\partial q)^2} K^* \quad (24)$$

and its substitution into Eqs. (15)-(17) yields

$$d\varepsilon_r^p = \frac{1}{K^*} (dp' n_p + dq n_q) \left[\frac{1}{3} n_p + \frac{3(\sigma'_r - p')}{2q} n_q\right] \quad (25)$$

$$d\varepsilon_\theta^p = \frac{1}{K^*} (dp' n_p + dq n_q) \left[\frac{1}{3} n_p + \frac{3(\sigma'_\theta - p')}{2q} n_q\right] \quad (26)$$

$$d\varepsilon_z^p = \frac{1}{K^*} (dp' n_p + dq n_q) \left[\frac{1}{3} n_p + \frac{3(\sigma'_z - p')}{2q} n_q\right] \quad (27)$$

Note further that

$$dp' = \frac{1}{3} (d\sigma'_r + d\sigma'_\theta + d\sigma'_z) \quad (28a)$$

$$dq = \frac{\partial q}{\partial \sigma'_r} d\sigma'_r + \frac{\partial q}{\partial \sigma'_\theta} d\sigma'_\theta + \frac{\partial q}{\partial \sigma'_z} d\sigma'_z \quad (28b)$$

and introduce the notations

$$a_r = \frac{1}{3} n_p + \frac{3(\sigma'_r - p')}{2q} n_q \quad (29a)$$

$$a_\theta = \frac{1}{3}n_p + \frac{3(\sigma'_\theta - p')}{2q}n_q \quad (29b)$$

$$a_z = \frac{1}{3}n_p + \frac{3(\sigma'_z - p')}{2q}n_q \quad (29c)$$

the plastic stress strain response can therefore be written in a matrix form

$$\begin{Bmatrix} d\varepsilon_r^p \\ d\varepsilon_\theta^p \\ d\varepsilon_z^p \end{Bmatrix} = \frac{1}{K^*} \begin{bmatrix} a_r^2 & a_r a_\theta & a_r a_z \\ a_\theta a_r & a_\theta^2 & a_\theta a_z \\ a_z a_r & a_z a_\theta & a_z^2 \end{bmatrix} \cdot \begin{Bmatrix} d\sigma_r' \\ d\sigma_\theta' \\ d\sigma_z' \end{Bmatrix} \quad (30)$$

It should be remarked here that a_r , a_θ , a_z , and K^* on the right hand side of the above equation all could be explicitly expressed as functions of the three stress components σ_r' , σ_θ' , and σ_z' , which is of vital importance to ensure an analytical solution for the cavity contraction elastoplastic boundary value problem. Such a desirable feature for the plastic modulus K^* will be demonstrated in the next section.

On the other hand, the elastic stress strain relationship can be written in incremental form as

$$\begin{Bmatrix} d\varepsilon_r^e \\ d\varepsilon_\theta^e \\ d\varepsilon_z^e \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \cdot \begin{Bmatrix} d\sigma_r' \\ d\sigma_\theta' \\ d\sigma_z' \end{Bmatrix} \quad (31)$$

where $d\varepsilon_r^e$, $d\varepsilon_\theta^e$, and $d\varepsilon_z^e$ are the elastic strain increments in r , θ , and z directions, respectively; ν is the Poisson's ratio; and the Young's modulus E [or shear modulus $G = \frac{E}{2(1+\nu)}$], is a function of the mean stress p' and specific volume v :

$$E = \frac{3(1-2\nu)vp'}{\kappa} \quad (32)$$

Combining Eqs. (30) and (31), and by inverting, one finally has the following elastoplastic constitutive equations for the bounding surface model

$$\begin{Bmatrix} d\sigma_r' \\ d\sigma_\theta' \\ d\sigma_z' \end{Bmatrix} = \frac{1}{\Delta} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \cdot \begin{Bmatrix} d\varepsilon_r \\ d\varepsilon_\theta \\ d\varepsilon_z \end{Bmatrix} \quad (33)$$

where

$$b_{11} = \frac{1}{E^2} \left[1 - \nu^2 + E a_\theta^2 \frac{1}{K^*} + 2E \nu a_\theta a_z \frac{1}{K^*} + E a_z^2 \frac{1}{K^*} \right] \quad (34a)$$

$$b_{12} = \frac{1}{E^2} \left[-E a_r (a_\theta + \nu a_z) \frac{1}{K^*} + \nu (1 + \nu - E a_\theta a_z \frac{1}{K^*} + E a_z^2 \frac{1}{K^*}) \right] \quad (34b)$$

$$b_{13} = \frac{1}{E^2} \left[-E a_r (\nu a_\theta + a_z) \frac{1}{K^*} + \nu (1 + \nu + E a_\theta^2 \frac{1}{K^*} - E a_\theta a_z \frac{1}{K^*}) \right] \quad (34c)$$

$$b_{22} = \frac{1}{E^2} \left[1 - \nu^2 + E a_r^2 \frac{1}{K^*} + 2E \nu a_r a_z \frac{1}{K^*} + E a_z^2 \frac{1}{K^*} \right] \quad (34d)$$

$$b_{23} = \frac{1}{E^2} \left[\nu + \nu^2 + E \nu a_r^2 \frac{1}{K^*} - E a_\theta a_z \frac{1}{K^*} - E \nu a_r (a_\theta + a_z) \frac{1}{K^*} \right] \quad (34e)$$

$$b_{33} = \frac{1}{E^2} \left[1 - \nu^2 + E a_r^2 \frac{1}{K^*} + 2E \nu a_r a_\theta \frac{1}{K^*} + E a_\theta^2 \frac{1}{K^*} \right] \quad (34f)$$

$$b_{21} = b_{12} \quad (34g)$$

$$b_{31} = b_{13} \quad (34h)$$

$$b_{32} = b_{23} \quad (34i)$$

$$\Delta = -\frac{1+\nu}{E^3} \left[(-1 + \nu + 2\nu^2) + E(-1 + \nu) a_r^2 \frac{1}{K^*} + E(-1 + \nu) a_\theta^2 \frac{1}{K^*} - 2E \nu a_\theta a_z \frac{1}{K^*} \right. \\ \left. - E a_z^2 \frac{1}{K^*} + E \nu a_z^2 \frac{1}{K^*} - 2E \nu a_r (a_\theta + a_z) \frac{1}{K^*} \right] \quad (34j)$$

Cavity contraction boundary value problem

The scenario of cylindrical cavity contraction in an infinite elastoplastic porous medium is shown in Fig. 2. Initially the geomaterial is subjected to an in-plane (horizontal) stress σ_h , an out-of-plane (vertical) stress σ_v , and a pore pressure u_0 , respectively. The cavity has an initial radius a_0 and is contracted to the current radius a when the cavity pressure is decreased from σ_h to σ_a . At this instant a typical particle initially at radial distance r_{x0} will have moved inward to a new position denoted by r_x .

It is noteworthy that for a cavity contracted in a bounding surface geomaterial, plastic

deformations occur immediately for all the material particles, after the reduction of the internal cavity pressure, so there is no elastic zone existing around the cavity, see Fig. 2. This is the direct result of employing bounding surface model for which as described earlier yielding may occur as soon as loading commences in the stress space.

UNDRAINED CASE

For undrained condition,

$$d\varepsilon_v = d\varepsilon_p^e + d\varepsilon_p^p = 0 \quad (35)$$

where $d\varepsilon_p^e$ denotes the elastic volumetric strain increment and, from Eqs. (31) and (32), can be expressed as

$$d\varepsilon_p^e = d\varepsilon_r^e + d\varepsilon_\theta^e + d\varepsilon_z^e = \frac{\kappa dp'}{vp'} \quad (36)$$

Substituting Eq. (36) into Eq. (14) yields

$$\frac{d\bar{p}'_C}{\bar{p}'_C} = -\frac{(1+e)}{\lambda-\kappa} d\varepsilon_p^e = -\frac{\kappa}{\lambda-\kappa} \frac{dp'}{p'} \quad (37)$$

which, after integration, gives

$$\bar{p}'_C = \bar{p}'_{C,0} \left(\frac{p'}{p'_0} \right)^{-\frac{\kappa}{\lambda-\kappa}} \quad (38)$$

Here p'_0 corresponds to the initial mean effective stress and $\bar{p}'_{C,0}$ is the initial value of $\bar{p}'_C$ before the contraction of cavity. Eq. (38) shows clearly that $\bar{p}'_C$, the size of the bounding surface, is expressible with respect to the current stress state p' , and hence to the three stress components. This in conjunction with Eqs. (21)-(22) indicates that K^* can also be explicitly expressed in terms of σ'_r , σ'_θ , and σ'_z only, an ascertainment having been made indeed in the previous section.

Note that in Fig. 1, p'_A is determined as the intersection between an ellipse passing through the current stress point (p', q) and the p' axis. Its initial value of $p'_{A,0}$ relevant to the initial stress

state (p'_0, q_0) , together with $\bar{p}'_{C,0}$, thus define the useful concept of overconsolidation ratio for the soils and rocks

$$OCR = \frac{\bar{p}'_{C,0}}{p'_{A,0}} \quad (39)$$

For undrained cylindrical cavity contraction problem $d\varepsilon_v = 0$, and also $d\varepsilon_z = 0$ as a result of plane strain deformation, the incremental radial and tangential logarithmic strains thus can be expressed as (Chen and Abousleiman 2012)

$$d\varepsilon_r = -d\varepsilon_\theta = \frac{dr}{r} \quad (40)$$

where r is the radial coordinate associated with a given material particle and dr is the infinitesimal change in position of that particle (Lagrangian description). Turning back to the elastoplastic constitutive equation (33), one obtains three first order governing differential equations with the three stress unknowns being functions of single variable r as follows

$$\frac{D\sigma'_r}{Dr} - \frac{b_{11}-b_{12}}{\Delta r} = 0 \quad (41a)$$

$$\frac{D\sigma'_\theta}{Dr} - \frac{b_{21}-b_{22}}{\Delta r} = 0 \quad (41b)$$

$$\frac{D\sigma'_z}{Dr} - \frac{b_{31}-b_{32}}{\Delta r} = 0 \quad (41c)$$

where $\frac{D}{Dr}$ denotes the material derivative taken along the particle motion path (Lagrangian description).

The final point that needs to be addressed now is the specification of suitable initial conditions for the above differential equations. Recall that, for the bounding surface model, there is no purely elastic deformation zone existing around the cavity. Therefore, Eqs. (41a)-(41c) are in principle valid for any material point with current position r_x , whose original position, r_{x0} , can be identified from the kinematic constraint of undrained deformation as

$$\frac{r_{x0}}{a} = \sqrt{\left(\frac{r_x}{a}\right)^2 + \left(\frac{a_0}{a}\right)^2 - 1} \quad (42)$$

and the corresponding initial stress conditions are simply given by

$$\sigma'_r(r_{x0}) = \sigma'_{r0} = \sigma_h - u_0, \quad \sigma'_\theta(r_{x0}) = \sigma'_{\theta0} = \sigma_h - u_0, \quad \sigma'_z(r_{x0}) = \sigma'_{z0} = \sigma_v - u_0 \quad (43)$$

where σ'_{r0} , $\sigma'_{\theta0}$, and σ'_{z0} are the initial effective stresses.

Once the effective stresses are solved from Eqs. (41a)-(41c), subject to the initial condition (43), the distribution of pore pressure $u(r_x)$ can be easily calculated from the equilibrium equation

$$\frac{\partial \sigma'_r}{\partial r} + \frac{\partial u}{\partial r} + \frac{\sigma'_r - \sigma'_\theta}{r} = 0 \quad (44)$$

and the excess pore pressure then determined from

$$\Delta u(r_x) = u(r_x) - u_0 \quad (45)$$

DRAINED CASE

When the cavity is contracted under drained condition, the pore pressure u is eventually constant equal to the initial value of u_0 and can be subtracted out of the analysis. The equilibrium equation (44) therefore reduces to

$$\frac{\partial \sigma'_r}{\partial r} + \frac{\sigma'_r - \sigma'_\theta}{r} = 0 \quad (46)$$

For the drained situation, the elastoplastic stress strain response, Eq. (33), is still valid as the constitutive relation for geomaterials is formulated based on the concept of effective stresses. However, it should be pointed out that in this case the void ratio e (or specific volume v), is no longer a constant but instead changes with the deformation so itself needs to be determined during the contraction process. The influences of varying e (or v) are well reflected in Eqs. (21) and (22) for the plastic moduli K^* and K_b^* . Additionally, it will affect indirectly the hardening

parameter $\bar{p}'_C$ which essentially controls the size of the current bounding surface. As a matter of fact, Eq. (38) for the determination of $\bar{p}'_C$ is no longer applicable for the drained case since this equation is derived based on the constant specific volume condition, $d\varepsilon_v = 0$.

Consider now the expression of $\bar{p}'_C$ for the drained condition. Substituting Eq. (2) into (14), and integrating, gives

$$\ln \frac{\bar{p}'_C}{\bar{p}'_{C,0}} = -\frac{1}{\lambda-\kappa} e^p \quad (47)$$

Using Eq. (3), thus,

$$\ln \frac{\bar{p}'_C}{\bar{p}'_{C,0}} = -\frac{1}{\lambda-\kappa} (v - v_0 - \int_0^e de^e) \quad (48)$$

Here

$$\int_0^e de^e = \int_0^e -(1+e)d\varepsilon_p^e = \int_0^e -(1+e) \frac{dp' \cdot \kappa}{(1+e)p'} = -\kappa \ln \frac{p'}{p'_0} \quad (49)$$

so that

$$\frac{\bar{p}'_C}{\bar{p}'_{C,0}} = e^{-\frac{v-v_0}{\lambda-\kappa} \left(\frac{p'}{p'_0}\right)^{-\frac{\kappa}{\lambda-\kappa}}} \quad (50)$$

which again expresses $\bar{p}'_C$ as an explicit function of the current stress state as well as the current specific volume v , and so does the plastic moduli K^* from Eqs. (21)-(22). Note that when $v = v_0$, the above equation correctly reduces to Eq. (38) for the undrained case.

Let us now move on to the formulation of the governing differential equations for the drained condition. Given the elastoplastic incremental constitutive relation, Eq. (33), and noting that $d\varepsilon_v = -\frac{dv}{v}$ and $d\varepsilon_\theta = -\frac{dr}{r}$, it follows that

$$d\sigma'_r = \frac{1}{\Delta} [b_{11}d\varepsilon_v + (b_{12} - b_{11})d\varepsilon_\theta] = \frac{1}{\Delta} [b_{11}\left(-\frac{dv}{v}\right) + (b_{12} - b_{11})\left(-\frac{dr}{r}\right)] \quad (51a)$$

$$d\sigma'_\theta = \frac{1}{\Delta} [b_{21}d\varepsilon_v + (b_{22} - b_{21})d\varepsilon_\theta] = \frac{1}{\Delta} [b_{21}\left(-\frac{dv}{v}\right) + (b_{22} - b_{21})\left(-\frac{dr}{r}\right)] \quad (51b)$$

$$d\sigma'_z = \frac{1}{\Delta} [b_{31}d\varepsilon_v + (b_{32} - b_{31})d\varepsilon_\theta] = \frac{1}{\Delta} [b_{31}\left(-\frac{dv}{v}\right) + (b_{32} - b_{31})\left(-\frac{dr}{r}\right)] \quad (51c)$$

Introduce an auxiliary independent variable ξ , defined as

$$\xi = \frac{u_r}{r} = \frac{r-r_0}{r} \quad (52)$$

with u_r known as the radial displacement and r_0 the original position of the particle, and then follow the same procedure as outlined in Chen (2012) for the drained cavity analysis, one can finally derive the desirable four governing differential equations

$$\frac{D\sigma'_r}{D\xi} = -\frac{\sigma'_r - \sigma'_\theta}{1 - \xi - \frac{v_0}{v(1-\xi)}} \quad (53a)$$

$$\frac{D\sigma'_\theta}{D\xi} = -\frac{b_{21}}{b_{11}} \left[\frac{\sigma'_r - \sigma'_\theta}{1 - \xi - \frac{v_0}{v(1-\xi)}} + \frac{b_{11} - b_{12}}{\Delta(1-\xi)} \right] - \frac{b_{22} - b_{21}}{\Delta(1-\xi)} \quad (53b)$$

$$\frac{D\sigma'_z}{D\xi} = -\frac{b_{31}}{b_{11}} \left[\frac{\sigma'_r - \sigma'_\theta}{1 - \xi - \frac{v_0}{v(1-\xi)}} + \frac{b_{11} - b_{12}}{\Delta(1-\xi)} \right] - \frac{b_{32} - b_{31}}{\Delta(1-\xi)} \quad (53c)$$

$$\frac{Dv}{D\xi} = \frac{v\Delta}{b_{11}} \left[\frac{\sigma'_r - \sigma'_\theta}{1 - \xi - \frac{v_0}{v(1-\xi)}} + \frac{b_{11} - b_{12}}{\Delta(1-\xi)} \right] \quad (53d)$$

which can be solved for any material particle r_x as an initial value problem with the independent variable starting at $\xi = \xi_0 = 0$. As in the undrained case, the initial conditions for stresses and specific volume are simply as follows

$$\sigma'_r(0) = \sigma'_{r0}, \quad \sigma'_\theta(0) = \sigma'_{\theta0}, \quad \sigma'_z(0) = \sigma'_{z0}, \quad v(0) = v_0 \quad (54)$$

Obviously, the above differential equations (53a)-(53d) are expressed with respect to the auxiliary variable ξ rather than the radial coordinate r . To complete the solutions, it is therefore necessary to establish a link between ξ and r , which can be found as (Chen and Abousleiman 2013)

$$\frac{r}{a} = e^{\int_{\xi(a)}^{\xi} \frac{d\xi}{1 - \xi - \frac{v_0}{v(\xi)(1-\xi)}}} \quad (55)$$

Results and discussions

The parameters of the bounding surface model used for both undrained and drained cavity contraction analyses are $R = 2.72$, $M = 1.05$, $\lambda = 0.14$, $\kappa = 0.05$, $\nu = 0.15$, $e_0 = 0.95$ ($v_0 = 1.95$), $m = 0.2$, and $h = 30$, as listed in Table 1. Four different values of $OCR = 1, 1.2, 2$, and 5 are considered to investigate the impact of overconsolidation ratio on the stress and pore pressure distributions around the cavity. The in situ effective stresses and pore pressure, as well as the relevant values of $\bar{p}'_{C,0}$ are also summarized in Table 1.

Fig. 3 shows the variations of the internal support pressure σ_a (compression positive) and excess pore pressure $\Delta u(a)$ at cavity wall $r = a$ with the normalized cavity radius $\frac{a_0}{a}$, generally known as the tunnel characteristic curve and wellbore closure curve, for the undrained case and for $OCR = 1, 1.2, 2$, and 5 . It is seen that the cavity pressure decreases significantly as OCR increases from 1 for normally consolidated geomaterial to 5 for heavily overconsolidated geomaterial. Similar trends can also be observed for the excess pore pressure build-up in Fig. 3a, where the negative result of Δu gives a clear indication of the decrease in total pore pressure.

Figs. 4-7 show the distributions of σ'_r , σ'_θ , and σ'_z and also of p' , q , and Δu along the radial distance corresponding to a reduced cavity pressure of $\sigma_a = 0$ for all the overconsolidation ratios in the range from 1 to 5, where the radial axis has been normalized with respect to the current contracted radius a and the results again presented for undrained case. It can be clearly observed that for the case of $OCR = 1$ (Fig. 4), all the radial, tangential, and vertical stresses as well as the mean effective stress and deviatoric stress remain unchanged in the vicinity of the cavity $\frac{r}{a} < 1.3$, indicating the occurrence of critical state failure for the geomaterial of this range. However, for larger values of $OCR = 1.2, 2$, and 5 , such an internal critical state zone vanishes and only

the plastic zone may be expected outside the cavity. In fact, for given initial stresses ($\sigma'_{r0} = \sigma'_{\theta0} = 11.25$ MPa, $\sigma'_{z0} = 15$ MPa, and $u_0 = 10$ MPa), the lower the value of OCR , the closer the stress state at the cavity wall will approach the critical state.

Figs. 4-7 also clearly show that the negative excess pore pressure increases rapidly with distance near the cavity. Especially, for normally consolidated geomaterial with $OCR = 1$, Δu varies linearly with the logarithm of $\frac{r}{a}$ in the critical state failure zone, an expected feature arising from Eq. (44) for constant σ'_r and σ'_θ . It is further interesting to note that the excess pore pressure distribution is only slightly influenced by the varying level of overconsolidation, which differs very much from the results in Chen and Abousleiman (2012) where significant impact of OCR has been observed on the pore pressure change for the Cam Clay model. Such a paradox occurs because under the same reduced cavity pressure of $\sigma_a = 0$, the calculated excess pore pressure shown in Figs. 4-7 for the four different values of OCR actually corresponds to steadily decreasing values of $\frac{a_0}{a} = 1.40, 1.31, 1.18$, and 1.10 , respectively. Referring now to Fig. 3b, it becomes obvious that the excess pore pressure developed indeed remains almost unchanged with these $\frac{a_0}{a}$ values. However, if the variation of Δu is alternatively presented with respect to a specified contracted cavity radius, say $\frac{a_0}{a} = 1.2$, one may foresee a considerable effect of OCR on the excess pore pressure distribution as similarly noted in Chen and Abousleiman (2012).

Figs. 8-11 further present the undrained effective stress path (ESP) followed in the $p' - q$ plane for a material particle at the cavity wall. For $OCR = 1$, recalling Eq. (39), the initial bounding surface must pass through the initial stress point (Fig. 8). In this case the stress path will start from $p'_0 = 12.5$ MPa and $q_0 = 3.75$ MPa, continue to move upper-left and finally stop at point F on the critical state line, which also lies on the failure bounding surface with $\bar{p}'_c =$

19.92 MPa (see Fig. 8). Since at point F , $\eta = M$ and $\delta = 0$ so $K^* = K_b^*$ while the latter is equal to zero following Eq. (22) [$f(M) = \frac{1}{R}$], the actual plastic modulus K^* must also vanish. This explains why the stress path terminates on the critical state line and remains stationary there. For OCR greater than 1, the initial stress point is always located within the initial bounding surface, as shown in Figs. 9-11. Note that in the overconsolidation cases ($OCR > 1$), the stress paths all initially cross the critical state line parallel to the q axis, which makes a subtle difference from the results when a regular plasticity model is used, e.g. the modified Cam Clay model (Chen 2012). As noted in Dafalias and Herrmann (1980), this is a property for any shape of bounding surface which has $n_p = 0$ at $\eta = M$. In fact, at these cross points, $\delta > 0$ and $K^* > K_b^* = 0$, so the material at this moment has not reached the failure state and therefore the stress paths may bend over and continue to move towards the critical state line. To see this point more clearly, in Fig. 11 for the case of $OCR = 5$, the stress path is extended from point B ($\sigma_a = 0$) to F which corresponds to a sufficiently contracted cavity radius of $\frac{a}{a_0} = 5$. Also included in this figure is the trajectory of the image stress ($\bar{p}', \bar{q}$) on the expanding/contracting bounding surface, represented by the curve $A'B'F'$. As expected, the two points F and F' coincide and both lie on the critical state line again. This implies that the material has truly reached the failure state and that failure can occur only if the loading surface merges with the bounding surface.

Note again that the use of the current version of bounding surface model excludes the existence of a region of purely elastic response. However, if the value of the model parameter h is set extremely high in the computation, the plastic modulus K^* [see Eq. (21)] will become so large that the overall response of the material inside the bounding surface will be almost fully elastic as in a classical yield surface formulation (Dafalias and Herrmann 1986). Fig. 12 shows the profound effects of varying h value on the support pressure versus cavity radius curve as well

as on the $p' - q$ stress path for the case of $OCR = 5$. In Fig. 12a, the cavity response becomes stiffer (less deformable) as the plasticity parameter h increases gradually from 30 to 300 and then to 3000, but the $\sigma_a - \frac{a_0}{a}$ curve, as anticipated, converges eventually to the limiting case of classical yield surface plasticity when h is sufficiently large equal to 30000. Such a feature is even more clearly observed from Fig. 12b, where the stress path for $h = 30000$ is nearly vertical before it touches the initial bounding surface. This corresponds to a constant mean effective stress p' and provides a clear indication of purely elastic deformation of the material, which is consistent with the finding in Chen and Abousleiman (2012) for the classical Cam Clay model. One may thus conceive that the “real” material response shall be somewhere between the above-mentioned two extremes of fully elasticity inside a classical yield surface and of always plasticity inside a bounding surface, which can be suitably treated by the bounding surface model having an elastic nucleus.

Figs. 13-20 show the calculation results of stresses and specific volume for the drained situation, corresponding again to a zero effective cavity pressure which is defined as $\sigma'_a = \sigma_a - u_0 = 0$. Throughout the range of OCR from 1 to 5, it is observed that (Fig. 13-16) the geomaterial undergoes dilation during the contraction process and the change in the specific volume decreases with increasing OCR . As regarding the $p' - q$ stress paths for different values of OCR , Figs. 17-20 illustrates that the material particles all first harden plastically from the wet side of the critical state line ($\eta < M$), the bounding surface expanding simultaneously to accommodate the new stress state. After hitting the critical state line, the geomaterials start to soften plastically in the dry side ($\eta > M$) and the bounding surface must decrease in size, the stress paths eventually ending at point B corresponding to $\sigma'_a = 0$. One may expect that for sufficiently small value of the cavity pressure, the stress path will be brought to the critical

failure state again at lower values of p' and q .

Fig. 17 also presents the Abaqus prediction of the $p' - q$ stress path at the cavity surface for $OCR = 1$, which has been conducted through the appropriate development and implementation of the user subroutine (UMAT) for the bounding surface plasticity (Chen 2012) that is not available in Abaqus finite element commercial program. It is clear that the numerical results are in excellent agreement with the drained analytical solutions. This verifies the accuracy of the proposed semi-analytical approach for the cavity contraction boundary value problem, and also indicates the validity and reliability of the UMAT code written for the currently employed bounding surface model.

Applications in tunnelling and wellbore drilling

This section examines the applications of the present semi-analytical solutions to the modelling of tunnel excavation and wellbore drilling. For illustration purpose, only the undrained condition will be considered and the emphasis will be given to the prediction of critical (minimum) support pressure required to stabilize the tunnel and wellbore.

The critical support pressure $\sigma_{a,cr}$ could be defined in three different ways depending on the stability criterion used for the tunnel/wellbore design (Yu 2000). The first one is based on the elastic theory which assumes that the tunnel/wellbore will approach collapse condition once the calculated elastic stress state anywhere around the cavity attains certain yielding criteria. With reference to this criterion the critical support pressure therefore must correspond to the cavity pressure for which the plastic deformation begins to take place at the tunnel/wellbore wall. The other two criteria require the analysis of the tunnel/wellbore stability as an elastoplastic problem (Charlez 1997; Yu 2000), as is elaborated in this paper. The tunnel and wellbore are regarded as

unstable either when their surfaces reach the failure/critical state, or when their inward displacements are too large to meet the allowable deformation requirement.

Recall now that the characteristic curves plotting the reduced support pressure as a function of the contracted cavity radius, corresponding to the undrained condition, have been presented in Fig. 3a. As a consequence, the critical support pressure $\sigma_{a,cr}$ can be directly obtained from these curves for the first and third (allowable displacement) stability criteria. It is noteworthy that for the bounding surface model, the material points around the cavity harden plastically immediately after the cavity pressure drops below the in situ horizontal stress σ_h , so the critical support pressure following the elastic analysis shall be $\sigma_{a,cr} = \sigma_h$. The value of $\sigma_{a,cr}$ corresponding to the second stability criterion (tunnel/wellbore wall reaching the critical state), however, needs to be determined from the associated bounding surface critical state condition.

Table 2 compares the calculated results of $\sigma_{a,cr}$ based on the three stability criteria mentioned above. The geomaterial parameters used in the analysis are the same as those listed in Table 1, and two strain values of $\varepsilon = \frac{a_0 - a}{a_0} = 2\%$ and 5% at the tunnel/wellbore surface have been identified as controlling thresholds. It is found that a very large surface strain (contraction) must have been mobilized before the tunnel/wellbore finally reach the critical state, and therefore the critical state-based stability criterion is not controlling. Of course, which stability criterion, i.e., tunnel/wellbore deformation exceeding the allowable limit or tunnel/wellbore stress reaching the critical state, will control the design will be dependent on the actual soil/rock properties. Table 2 also shows that the predicted $\sigma_{a,cr}$ in general decreases with the overconsolidation ratio, which is reasonably expected as the higher the value of OCR , the stiffer the geomaterials.

Conclusions

This paper is devoted to developing the rigorous semi-analytical solutions for the drained and undrained cavity contractions in geomaterials by using the bounding surface plasticity model. One of the important features of the bounding surface model is that yielding could occur as soon as loading commences in the stress space. Therefore, there is no purely elastic zone existing outside the cavity during the contraction process. Extensive parametric studies show that, for undrained problem, both the cavity pressure and the induced excess pore pressure decrease significantly as the overconsolidation ratio of the geomaterials increases. The lower the value of OCR , the closer the stress state at the cavity wall will approach the critical state. In the overconsolidation cases ($OCR > 1$), it is found that the undrained stress paths all initially cross the critical state line parallel to the q axis, then bend over and continue to move towards the critical state line until they truly reach the failure state. For the drained case, the geomaterial generally undergoes dilation during the cavity contraction and the change in the specific volume decreases with increasing OCR . The stress paths first head towards the critical state line from the wet side, so the material hardens plastically accompanied by progressive expansion of the bounding surface. After passing through the critical state line, the material however tends to soften in the dry side and the bounding surface gradually decreases in size. The validity and accuracy of the semi-analytical approach are justified by the finite element numerical modelling. The proposed solutions are finally applied for the practical tunnelling and wellbore drilling problems, to explore the critical support pressure that is required to maintain the tunnel/wellbore stability.

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Captions of table and figures

Table 1. Parameters used in example analyses with bounding surface model

Table 2. Predicted $\sigma_{a,cr}$ using three different stability criteria

Fig. 1. Schematic illustration of bounding surface and radial mapping rule in general stress (σ'_{ij}) space and p' - q space

Fig. 2. Geometry of cavity boundary value problem

Fig. 3. Variations of (a) cavity pressure; (b) excess pore pressure at cavity wall with normalized cavity radius for undrained case

Fig. 4. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 1, undrained case

Fig. 5. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 1.2, undrained case

Fig. 6. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 2, undrained case

- Fig. 7. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 5, undrained case
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- Fig. 20. p' - q stress path at cavity wall for OCR = 5, drained case

Table 1. Parameters used in example analyses with bounding surface model

$\sigma'_{r0} = \sigma'_{\theta0} = 11.25 \text{ MPa}, \sigma'_{z0} = 15 \text{ MPa}, u_0 = 10 \text{ MPa}, p'_0 = 12.5 \text{ MPa}, \text{ and } q_0 = 3.75 \text{ MPa}$											
R	M	λ	κ	ν	e_0	m	h	$\bar{p}'_{c,0} \text{ (MPa)}$			
								$OCR = 1$	1.2	2	5
2.72	1.05	0.14	0.05	0.15	0.95	0.2	30	14.80	17.76	29.60	74.00

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Table 2. Predicted $\sigma_{a,cr}$ using three different stability criteria

OCR	Elastic analysis	Allowable deformation $\varepsilon = 2\%$	Allowable deformation $\varepsilon = 5\%$	Cavity surface reaches critical state
1	21.25 MPa ($\varepsilon = 0$)	13.34 MPa	9.50 MPa	2.96 MPa ($\varepsilon = 18\%$)
1.2	21.25 MPa ($\varepsilon = 0$)	13.12 MPa	9.00 MPa	0.55 MPa ($\varepsilon = 25\%$)
2	21.25 MPa ($\varepsilon = 0$)	12.48 MPa	7.58 MPa	-7.46 MPa ($\varepsilon = 35\%$)
5	21.25 MPa ($\varepsilon = 0$)	11.32 MPa	5.10 MPa	-27.62 MPa ($\varepsilon = 53\%$)

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Fig. 1. Schematic illustration of bounding surface and radial mapping rule in general stress (σ'_{ij}) space and p' - q space

Fig. 2. Geometry of cavity boundary value problem

Fig. 3. Variations of (a) cavity pressure; (b) excess pore pressure at cavity wall with normalized cavity radius for undrained case

Fig. 4. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 1, undrained case

Fig. 5. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 1.2, undrained case

Fig. 6. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 2, undrained case

Fig. 7. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and excess pore pressure around the cavity for OCR = 5, undrained case

Fig. 8. p' - q stress path at cavity wall for OCR = 1, undrained case

Fig. 9. p'-q stress path at cavity wall for OCR = 1.2, undrained case

Fig. 10. p' - q stress path at cavity wall for OCR = 2, undrained case

Fig. 11. p' - q stress path at cavity wall for OCR = 5, undrained case

Fig. 12. Influences of plasticity parameter h on (a) support pressure versus cavity radius curve; (b) p' - q effective stress path with zero support pressure ($\sigma_a = 0$), undrained case and OCR = 5

Fig. 13. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and specific volume around the cavity for OCR = 1, drained case

Fig. 14. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and specific volume around the cavity for OCR = 1.2, drained case

Fig. 15. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and specific volume around the cavity for OCR = 2, drained case

Fig. 16. Distributions of (a) effective radial, tangential, and vertical stresses; (b) effective mean, deviatoric stresses and specific volume around the cavity for OCR = 5, drained case

Fig. 17. p' - q stress path at cavity wall for OCR = 1, drained case

Fig. 18. p' - q stress path at cavity wall for OCR = 1.2, drained case

Fig. 19. p' - q stress path at cavity wall for OCR = 2, drained case

Fig. 20. p' - q stress path at cavity wall for OCR = 5, drained case