**Bianchi Type VI Cosmological Model with Electromagnetic Filed in Lyra Geometry**

<table>
<thead>
<tr>
<th>Journal:</th>
<th><em>Canadian Journal of Physics</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript ID</td>
<td>cjp-2016-0274.R1</td>
</tr>
<tr>
<td>Manuscript Type:</td>
<td>Article</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>19-Jun-2016</td>
</tr>
<tr>
<td>Complete List of Authors:</td>
<td>Abdel-Megied, M.; Minia University Faculty of Science, Mathematics Hegazy, E.; Minia University Faculty of Science</td>
</tr>
<tr>
<td>Keyword:</td>
<td>Lyra geometry, Einstein field equations, Bianchi type VI, Cosmological Models, Expansion scalar</td>
</tr>
</tbody>
</table>

https://mc06.manuscriptcentral.com/cjp-pubs
Bianchi Type VI Cosmological Model with Electromagnetic Field in Lyra Geometry

M. Abdel-Megied\(^1\) and E. A. Hegazy\(^2\)
Mathematics Department, Faculty of Science, Minia University, 61915 El-Minia, EGYPT.

Abstract
Bianchi type VI cosmological model in the presence of electromagnetic field with variable magnetic permeability in the framework of Lyra geometry is presented. An exact solution is introduced by considering that the eigenvalue \( \sigma_3 \) of the shear tensor \( \sigma_j \) is proportional to the scalar expansion \( \Theta \) of the model, that is \( C = (AB)^{L} \), where \( A, B \) and \( C \) are the coefficients of the metric and \( L \) is a constant. Bianchi type V, III and I cosmological models are given as special cases of Bianchi type VI. Physical and geometrical properties of the models are discussed.

1 Introduction
In general relativity, Einstein [1] succeeded in geometrizing gravitation by identifying the metric tensor with gravitational potentials. Weyl [2] developed a more general theory based on a generalization of Riemannian geometry in order to geometrize gravitation and electromagnetism. He assumed in addition to coordinate transformation, there is a gauge transformation, which means that the metric tensor \( g_{ij} \) is changed not only under coordinate transformation but also under a gauge transformation where there is a vector \( \varphi_i \) which is identified with the potential vector describing the electromagnetic field. If a vector is carried from one point to another point, its length will, in general depend on the path between the two points and depend on \( \varphi_i \), which show that length is not integrable. Non-integrability of length transfer has been criticized at that time by Einstein because it implies that the frequency of spectral lines emitted by atoms would not remain constant but would depend on their past histories, which is in a contradiction to the observed uniformity of their properties [3].

Lyra [4] suggested a modification of Riemannian geometry by introducing a gauge function into the structure manifold, this modification bears a remarkable resemblance to Weyl geometry but in Lyra geometry, unlike that of Weyl, the connection is metric preserving as in

---

\(^1\) Email Address: mmegied@gmail.com;
\(^2\) Email Address: sayed00ali@gmail.com; elsayed.mahmoud@mu.edu.eg;

---

https://mc06.manuscriptcentral.com/cjp-pubs
Riemannian; in other words, length transfer is integrable. Lyra introduced a gauge theory which in the normal gauge the curvature scalar is identical with that of Weyl.

Sen [5], Sen and Dunn [6] proposed a theory of gravitation based on Lyra [4], using a modified Riemannian geometry in which a gauge function has been introduced into the structure less manifold as a result of which the cosmological constant arises naturally from the geometry.

Halford [7] has pointed out that the constancy of \( \phi_i \) (displacement vector field) in Lyra's geometry plays the same role of cosmological constant \( \Lambda \) in the normal general relativistic treatment. It is shown by Halford [8] that the scalar-tensor treatment based on Lyra's geometry predicts the same effects within observational limits as the Einstein's theory. Several authors [9] have studied cosmological models based on Lyra's manifold with a constant displacement field vector. However, this restriction of the displacement field to be constant is merely one for convenience and there is no a priori reason for it. Beesham [10] considered FRW models with time dependent displacement field. Singh and Singh [11], Singh and Desikan [12] have studied Bianchi-type I, III, Kantowski-Sachs and a new class of cosmological models with time dependent displacement field and have made a comparative study of Robertson Walker models with constant deceleration parameter in Einstein's theory with cosmological term and in the cosmological theory based on Lyra's geometry. Soleng [13] has pointed out that the cosmologies based on Lyra's manifold with constant gauge vector will either include a creation field and are equal to Hoyle's creation field cosmology [14] or contain a special vacuum field, which together with the gauge vector term, may be considered as a cosmological term. In the latter case the solutions are equal to the general relativistic cosmologies with a cosmological term.

Moreover: Pradhan et al. [15], Casama et al. [16], Rahaman et al. [17] Bali and Chandnani [18], Kumar and Singh [19], Yadav et al. [20], Rao et al. [21], Pradhan [22] and Singh and Kale [23] have studied cosmological models based on Lyra's geometry in various contexts.

In the present paper, the Bianchi type VI cosmological model in the presence of electromagnetic field with variable magnetic permeability based in Lyra geometry has been studied for the time varying displacement field vector. In section 2 we derived the field equations for the Bianchi type VI cosmological model with suitable choice for the magnetic permeability. Exact solution for the field equations is given in section 3. Some physical and geometrical properties of the model are discussed in subsection (3.1). Sections (4, 5, 6) are devoted for study some special cases.
2 The metric and field equations

A spatially homogenous space time of Bianchi type VI can be written in a Synchronous coordinates in the form:

\[ ds^2 = dt^2 - A^2 e^{-2\alpha} dx^2 - B^2 e^{-2\beta} dy^2 - C^2 dz^2 \]  
\[ (x^0 = t, x^1 = x, x^2 = y, x^3 = z), \]

where \( A, B \) and \( C \) are functions of \( t \) only.

The volume element of the model (2.1) is given by

\[ V = \sqrt{-g} = ABC e^{(\alpha-\beta)z}. \]

Let us assume that the coordinates to be co-moving so that

\[ u^0 = 1, \quad u^1 = u^2 = u^3 = 0. \]

The expansion scalar \( \Theta \) for the model is given by:

\[ \Theta = u^i_j = \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \]

The shear tensor read as:

\[ \sigma^2 = \frac{1}{2} \sigma^i_j \sigma^i_j, \]
\[ \sigma^i_j = \frac{1}{2} \left[ u_{ij} A^j + u^j_i A^i \right] - \frac{1}{3} \Theta A^i, \]

where the projection vector \( A^i_j \):

\[ A^2 = A, \quad A^i_j = g^i_j - u^i u^j, \quad A^i_j = \delta^i_j - u^j u_i. \]

The non vanishing components of the shear tensor \( \sigma^i_j \) are given by:

\[ \sigma^1 = \frac{1}{3} \left( \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \]
\[ \sigma^2 = \frac{1}{3} \left( \frac{2\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right), \]
\[ \sigma^3 = \frac{1}{3} \left( \frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right). \]

The shear \( \sigma \) is given by:

\[ \sigma^2 = \frac{1}{2} \left[ (\sigma^0)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \right], \]

that is

\[ \sigma^2 = \frac{1}{3} \left( \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} \right). \]

Field equations based in Lyra geometry as obtained by Sen [5] may be written as:

\[ G^i_j + \frac{3}{2} \phi_i \phi_j - \frac{3}{4} g^i_j \phi_k \phi^k = - \chi T^i_j, \]
where \( \phi \) is a displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Timelike displacement vector \( \phi \) in (2.9) takes the form:

\[
\phi_i = (\beta(t), 0, 0, 0).
\]  

(2.10)

\( T_o \) is the energy momentum tensor given by

\[
T_{ij} = (\rho + p)u_iu_j - pg_{ij} + E_{ij},
\]

(2.11)

where \( E_{ij} \) is the electro-magnetic field given by [Lichnerowicz [24]]:

\[
E_{ij} = \bar{\mu} [h_ih_j (u_iu_j - \frac{1}{2}g_{ij}) + h_ih_j].
\]

(2.12)

Here \( \rho \) and \( p \) are the energy density and isotropic pressure respectively and \( \bar{\mu} \) is the magnetic permeability and \( h_j \) the magnetic flux vector defined by:

\[
h_j = \frac{\sqrt{-g}}{2\bar{\mu}} \varepsilon_{ijkl} F^{kli}u^l,
\]

(2.13)

\( F_{ij} \) is the electromagnetic field tensor and \( \varepsilon_{ijkl} \) is the Levi-Civita tensor density. If we consider that the current flow along \( z \)-axis, then \( F_{12} \) is the only non-vanishing component of \( F_{ij} \).

The Maxwell’s equations

\[
F_{ij;k} + F_{jki} + F_{kij} = 0,
\]

(2.14)

and

\[
\frac{1}{\bar{\mu}} [F^{ij}]_j = 0,
\]

(2.15)

are satisfied with \( F_{12} = \text{constant (} K \text{ say)} \).

Equation (2.13) leads to

\[
h_3 = \frac{CK}{\bar{\mu}AB} e^{(m-n)z},
\]

(2.16)

then

\[
h_i h_i = h_3 h_3 = g^{33} h_3^3 = \frac{K^2}{\bar{\mu}^2 A^2 B^2} e^{2(m-n)z}.
\]

(2.17)

Using (2.16) and (2.17) in (2.12) we get:

\[
E^i_1 = \frac{-K^2}{2\bar{\mu} A^2 B^2} e^{2(m-n)z} = E^2_2 = E_3^3 = -E^0_0,
\]

(2.18)

from (2.11) and (2.18) we get:
\[ T_1' = -\left[p + \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right], \quad T_2' = -\left[p + \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right], \]
\[ T_3' = -\left[p - \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right], \quad T_0' = \rho + \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}. \]

For the metric (2.1) the field equations (2.9) become:
\[ \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{n^2}{C^2} + \frac{3}{4}\beta^2 = -\chi \left[p + \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right], \]
(2.20)
\[ \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{C^2} + \frac{3}{4}\beta^2 = -\chi \left[p + \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right], \]
(2.21)
\[ \frac{\ddot{A}}{AB} + \frac{\dot{A}\dot{B}}{BC} + \frac{mn}{C^2} + \frac{3}{4}\beta^2 = -\chi \left[p - \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right], \]
(2.22)
\[ m \left[\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right] + n \left[\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right] = 0, \quad (2.23) \]
\[ \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} + \frac{1}{C^2} [mn-m^2-n^2] - \frac{3}{4}\beta^2 = \chi \left[\rho + \frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right], \]
(2.24)
where dots means ordinary differentiation with respect to \( t \).

From the conservation of the energy momentum tensor \( T_{ij} = 0 \) we get:
\[ \dot{\rho} + (\rho + p) \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) \left[\frac{K^2}{2\mu a^2 B^2} e^{2(m-n)z}\right] + 2(m-n)p + \frac{K^2}{2\mu a^2 B^2} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) e^{2(m-n)z} = 0 \]
(2.25)

The left hand side of equations (2.20) - (2.24) depends on \( t \) alone but the right hand side depends on \( z \) and \( t \), then, equations (2.20) - (2.24) are inconsistent for a Banchi type VI in the presence of electromagnetic field in general. To restore the consistency of equations (2.20) - (2.24) we take the magnetic permeability \( \mu \) as a function of \( z \) and \( t \) in the form:
\[ \mu = f(t) e^{2(m-n)z}, \]
(2.26)
where \( f(t) \) is unknown function of \( t \).

Applying the conservation condition for the left hand side of equation (2.9) we get:
\[ \beta (\dot{\beta} + \beta \left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right]) = 0. \]
(2.27)

Solutions of equation (2.27) are given by:
\[ \beta = 0, \]
(2.28)
and
\[ \beta = \frac{M}{ABC}, \]
(2.29)
where $M$ is a constant of integration.
Solutions of the field equations (2.20)- (2.24) with the additional term $\beta$ introduced by Lyra is equal zero are solution of the Einstein equations in general relativity.

3 Solution of the field equations

To determine the six unknown functions $A, B, C, \rho, p$ and $f(t)$ we have only five independent equations, then we can add another condition between the coefficients of the metric tensor by considering that the expansion $\Theta$ of the model is proportional to the eigenvalue $\sigma_3^3$ of the shear tensor $\sigma_i^j$ [Thorne [25]], that is

$$\frac{1}{3} \left( \frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = \alpha \left( \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right),$$

(3.30)

where $\alpha$ is a constant. Equation (3.30) can be rewritten as

$$\frac{\dot{C}}{C} = \left( \frac{3\alpha + 1}{3} \right) \frac{\dot{ABC} + \dot{BAC} + \dot{CAB}}{ABC},$$

(3.31)

by integration we get:

$$C = (AB)^L,$$

(3.32)

where $L = \frac{3\alpha + 1}{2 - 3\alpha}$.

From (2.23) and (3.32) we have:

$$A = c_1 B^a,$$

(3.33)

where $c_1$ is a constant of integration and $a = \frac{n(L - 1) - mL}{m(L - 1) - nL}$.

Equations (2.20) and (2.21) lead to:

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}}{B} - \frac{\dot{B}\dot{C}}{BC} + \frac{n^2 - m^2}{C^2} = 0.$$  

(3.34)

From (3.32) and (3.33), equation (3.34) becomes:

$$\frac{\dot{B}^2}{B} + [a + L(a + 1)] \frac{\dot{B}^2}{B^2} = bB^{2(a+1)L},$$  

(3.35)

where $b = \frac{m^2 - n^2}{c_1^2(a - 1)}$.

Equation (3.35) has a solution in the form:

$$B(t) = \left[ L \sqrt{b(a + 1)} t + c_2 \right]^{\frac{1}{L(a + 1)}},$$

(3.36)

where $c_2$ is a constant of integration.

From (3.33), we get:

$$A(t) = c_1 \left[ L \sqrt{b(a + 1)} t + c_2 \right]^{\frac{a}{L(a + 1)}}.$$  

(3.37)
Equation (3.32) gives:
\[ C(t) = c_1^L \left[ L \sqrt{b(a+1)} \ t + c_2 \right]. \]  
(3.38)

The line element (2.1) read as:
\[ ds^2 = dt^2 - c_1^2 T L^{(a+1)} e^{-2mz} dx^2 - T L^{(a+1)} e^{2mz} dy^2 - c_1^2 T^2 dz^2, \]  
(3.39)

where
\[ T = L \sqrt{b(a+1)} \ t + c_2. \]

### 3.1 physical and geometrical properties of the model

From (2.29) the additional term \( \beta \) introduced by Lyra is given by:
\[ \beta = \frac{M}{c_1 L^{a+1}} T \frac{L^{a+1}}{L}. \]  
(3.40)

From (2.21) and (2.22) we get:
\[ f(t) = -\frac{\chi K^2}{c_1^2 H} T^{2(\frac{1}{L}-1)}, \]  
(3.41)

where \( H \) is a constant given by:
\[ H = aLb - \frac{m^2}{c_1^2 L} \frac{b(1-L(a+1))}{(a+1)} - \frac{ab}{(a+1)} - \frac{mn}{c_1^2 L}. \]

From (2.26), the magnetic permeability \( \mu \) read as:
\[ \mu = -\frac{\chi K^2}{c_1^2 H} T^{2(\frac{1}{L}-1)} e^{2(n-m)z}. \]  
(3.42)

From (2.20) and (2.22) the pressure \( p \) is given by:
\[ p = H_1 T^{-2} - \frac{3M^2}{4\chi c_1^{2(L+1)}} T^{-2(\frac{1}{L}+1)}, \]  
(3.43)

where
\[ H_1 = \frac{-1}{2\chi(a+1)} [b(2 + a + a^2) - Lb(a+1)^2 + (a+1)c_1^{-2L}(mn-n^2)]. \]

The density \( \rho \) can be given from equation (2.24) in the form:
\[ \rho = H_2 T^{-2} - \frac{3M^2}{4\chi c_1^{2(L+1)}} T^{-2(\frac{1}{L})}, \]  
(3.44)

where
\[ H_2 = \frac{1}{\chi(a+1)} [ab + Lb(a+1)^2 + (a+1)c_1^{-2L}(mn-m^2-n^2) + \frac{H(a+1)}{2}]. \]

The reality conditions [Ellis [26]]
\[ (i) \rho + p > 0, \quad (ii) \rho + 3p > 0, \]

lead to:
\[(H_1 + H_2)T^{-2} > \frac{3}{2\chi} \beta^2(t), \quad (3.45)\]

and
\[(3H_1 + H_2)T^{-2} > \frac{3}{\chi} \beta^2(t), \quad (3.46)\]

respectively.
The dominant energy conditions [Hawking and Ellis [27]]

\[\rho - p \geq 0, \quad (ii) \rho + p \geq 0,\]

lead to
\[(H_2 - H_1)T^{-2} \geq 0, \quad (3.47)\]

and
\[(H_1 + H_2)T^{-2} \geq \frac{3}{2\chi} \beta^2(t), \quad (3.48)\]

respectively. Equations (3.45), (3.46) and (3.48) imposes some restriction on \(\beta(t)\).
The spatial volume \(V\) is given by:
\[V = c_L^{1+L} T^{L+1} e^{(a-m)z}. \quad (3.49)\]
The expansion scalar \(\Theta\) of the model read as:
\[\Theta = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = (1 + L) \sqrt{b(a+1)} T^{-1}. \quad (3.50)\]

Shear \(\sigma\) is given by:
\[\sigma^2 = \frac{1}{2} \left[ (\sigma_0)^2 + (\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2 \right], \quad (3.51)\]

where
\[\sigma_1 = \frac{1}{3} \left( \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) = \left( \frac{2a - 1 - (a+1)L}{3} \right) \frac{b}{a+1} T^{-1}, \quad (3.52)\]
\[\sigma_2 = \frac{1}{3} \left( \frac{2\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) = \left( \frac{2 - a - (a+1)L}{3} \right) \frac{b}{a+1} T^{-1}, \quad (3.53)\]
\[\sigma_3 = \frac{1}{3} \left( \frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = \left( \frac{2L - 1}{3} \right) \sqrt{b(a+1)} T^{-1}, \quad (3.54)\]

then
\[\sigma^2 = \frac{bT^{-2}}{18(a+1)} \left[ (2a - 1 - (a+1)L)^2 + (2 - a - (a+1)L)^2 \right.\]
\[\left. + (2L - 1)^2 (a+1)^2 \right]. \quad (3.55)\]

Since \(\frac{\sigma}{\Theta}\) = constant, this model dose not approach isotropy for large value of \(T\). The model starts expanding at \(T > 0\) and it stop expanding as \(T \rightarrow \infty\). The parameters \(\rho, p, \Theta\) and \(\sigma^2\).
tend to infinity at \( T = 0 \), that is the universe starts from initial singularity with infinite energy, infinite internal pressure, infinite rate of shear and expansion. Moreover \( \Theta, \sigma^2, \rho \) and \( p \) decreasing toward a non-zero finite quantity as \( 0 < T < T_0 \) which require that \( L \) to be positive. The displacement field vector is increasing as \( T \) decreasing which require that \( L < -1 \) and it tends to constant value if \( L = -1 \). Additional condition between the constants can be obtained from equation (2.25). In figure 1: The behaviors of \( \Theta, \sigma^2 \) and \( \beta \) with \( T \) are shown as we indicated. Dash line represented \( \Theta \), dotted line for \( \sigma^2 \) and thick line for \( \beta^2 \).

![Figure 1](image)

Now we discuss the following cases:

### 4 Case 1: \( m = -n \)

If \( m = -n \), then the metric (2.1) reduced to Bianchi type V:

\[
ds^2 = dt^2 - A^2 e^{2\nu} dx^2 - B^2 e^{2\nu} dy^2 - C^2 dz^2,
\]

where \( A, B \) and \( C \) are functions of \( t \) only. Equations (3.33) and (3.35) become:

\[
A = c_4 B^{-1},
\]

and

\[
\frac{\ddot{B}}{B} = \frac{\dot{B}^2}{B^2}.
\]

Equation (4.3) has a solution in the form:

\[
B(t) = c_5 e^{\frac{\epsilon}{6} \cdot t}.
\]

From (4.2) we get:
\[ A(t) = c_7 e^{-c_6 t}, \]  
\[ \text{where } c_4, c_5, c_6 \text{ and } c_7 \text{ are constants. Equation (3.32) gives:} \]
\[ C(t) = c_8 \quad \text{(constant)}. \]  
\[ \text{where } c_8 = c_4^L. \]

The line element (4.1) read as:
\[ ds^2 = dt^2 - c_7^2 e^{-2c_6 t} e^{2n z} dx^2 - c_5^2 e^{2c_6 t} e^{2n z} dy^2 - c_8^2 dz^2. \]  

### 4.1 physical and geometrical properties of the model

From (2.29) the additional term \( \beta \) introduced by Lyra read as:
\[ \beta = \frac{M}{c_5 c_6 c_8} = \beta_0 \quad \text{(constant)}. \]  
\[ \text{From (2.21) and (2.22) we have:} \]
\[ f(t) \to \infty. \]  
\[ \text{From (2.26), the magnetic permeability } \mu \to \infty. \]

For the model (4.1) the density \( \rho \) and the pressure \( p \) are given by:
\[ p = -\frac{1}{\chi} [c_6^2 - n^2 + \frac{3}{4} \beta_0^2], \]  
\[ \rho = -\frac{1}{\chi} (c_6^2 + \frac{3n^2}{c_8^2} - \frac{3}{4} \beta_0^2). \]

The reality conditions [Ellis [26]]

\( (i) \rho + p > 0, \quad (ii) \rho + 3p > 0, \)

lead to:
\[ c_6^2 + \frac{n^2}{c_8^2} > 0, \]  
\[ \beta_0^2 > \frac{8}{3} c_6^2, \]  

respectively.

The dominant energy conditions [Hawking and Ellis [27]]

\( (i) \rho - p \geq 0, \quad (ii) \rho + p \geq 0, \)

lead to
\[
\beta_0^2 \leq \frac{8n^2}{3c_8^2}.
\]

and

\[
c_6^2 + \frac{n^2}{c_8^2} \geq 0, \quad (4.15)
\]

respectively.

From (4.13) and (4.14) we get the following restriction on the constant \( \beta_0 \):

\[
\frac{8c_6^2}{3} < \beta_0^2 \leq \frac{8n^2}{3c_8^2}.
\]

The scalar expansion \( \Theta \) of the model is given by:

\[
\Theta = \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0.
\]

Shear \( \sigma^2 \) is given by:

\[
\sigma^2 = \frac{1}{2} \left[ (\sigma_0^0)^2 + (\sigma_1^1)^2 + (\sigma_2^2)^2 + (\sigma_3^3)^2 \right],
\]

where \( \sigma_1^1 = -c_6 \), \( \sigma_2^2 = c_6 \), \( \sigma_0^0 = \sigma_3^3 = 0 \),

then

\[
\sigma^2 = c_6^2.
\]

This model is not expanding and the shear has a constant value.

5 Case 2: \( n = 0 \)

If \( n = 0 \), the line element (2.1) reduced to Bianchi type III, that is:

\[
ds^2 = dt^2 - A^2 e^{-2nz} dx^2 - B^2 dy^2 - C^2 dz^2,
\]

where \( A, B \) and \( C \) are functions of \( t \) only.

Equation (3.33) becomes:

\[
A = \alpha_1 B_L^b,
\]

where \( \alpha_1 \) is a constant of integration and \( b = \frac{L}{1-L} \).

Equation (3.35) reduces to:

\[
\frac{\dot{B}}{B} + \left[ b + L(b + 1) \right] \frac{\dot{B}}{B^2} = NB^{-2(b+1)L},
\]

5 Case 2: \( n = 0 \)

If \( n = 0 \), the line element (2.1) reduced to Bianchi type III, that is:

\[
ds^2 = dt^2 - A^2 e^{-2nz} dx^2 - B^2 dy^2 - C^2 dz^2,
\]

where \( A, B \) and \( C \) are functions of \( t \) only.

Equation (3.33) becomes:

\[
A = \alpha_1 B_L^b,
\]

where \( \alpha_1 \) is a constant of integration and \( b = \frac{L}{1-L} \).

Equation (3.35) reduces to:

\[
\frac{\dot{B}}{B} + \left[ b + L(b + 1) \right] \frac{\dot{B}}{B^2} = NB^{-2(b+1)L},
\]

This model is not expanding and the shear has a constant value.
where \( N = \frac{m^2}{\alpha_1^{2L} (b_1 - 1)} \) is a constant.

Equation (5.3) has a solution in the form:

\[
B(t) = \left[ L \sqrt{N(b_1 + 1)} \ t + \alpha_2 \right]^{1 \over L(b_1 + 1)}, \tag{5.4}
\]

where \( \alpha_2 \) is a constant of integration.

From (5.2) we get:

\[
A(t) = \alpha_1 \left[ L \sqrt{N(b_1 + 1)} \ t + \alpha_2 \right]^{b_1 \over L(b_1 + 1)}. \tag{5.5}
\]

Equation (3.32) gives:

\[
C(t) = \alpha_1^T \left[ L \sqrt{N(b_1 + 1)} \ t + \alpha_2 \right]. \tag{5.6}
\]

From (5.4), (5.5) and (5.6), the line element (5.1) takes the form:

\[
d s^2 = d t^2 - \alpha_1 T_1^{L(b_1 + 1)} e^{-2mz} d x^2 - T_1^{L(b_1 + 1)} d y^2 - \alpha_1^{2L} T_1 d z^2, \tag{5.7}
\]

where

\[
T_1 = L \sqrt{N(b_1 + 1)} \ t + \alpha_2.
\]

### 5.1 Physical and Geometrical Properties of the Model

From (2.29) the additional term \( \beta(t) \) read as:

\[
\beta(t) = \frac{M}{\alpha_1^{L+1}} \frac{T_{L+1}}{T}. \tag{5.8}
\]

Two equations (2.21) and (2.22) lead to:

\[
f(t) = - \frac{\chi K^2}{\alpha_1^2 H_3} T_1^{2(1-L)} \tag{5.9}
\]

where

\[
H_3 = b_1 LN - \frac{m^2}{\alpha_1^{2L}} - \frac{N b_1}{(b_1 + 1)} - \frac{N (1 - L(b_1 + 1))}{(b_1 + 1)}.
\]

From (2.26), the magnetic permeability \( \mu \) read as:

\[
\mu = - \frac{\chi K^2}{\alpha_1^2 H_3} T_1^{2(1-L)} e^{2mz} \tag{5.10}
\]

For the model (5.1), the density \( \rho \) and the pressure \( p \) are given by:

\[
p = \frac{H_4 T_1^{-2}}{4 \chi \alpha_1^{2(1-L)}} T_1^{-2(1-L)} \tag{5.11}
\]
where \[ H_4 = -\frac{1}{2\chi} \left[ \frac{2N(1-L(b_1+1))}{b_1+1} + \frac{b_1N[b_1-L(b_1+1)]}{(b_1+1)} + LN + \frac{b_1N}{b_1+1}, \right] \]
and
\[ \rho = H_5 T_1^{-2} - \frac{3M^2}{4\chi \alpha_1^{2(1+z)}} T_1^{-2(1+z/2)}, \] (5.12)

where \[ H_5 = \frac{1}{\chi} \left[ \frac{b_1N}{b_1+1} + b_1NL + LN - \frac{m_1}{\alpha_1^{2L}} + \frac{1}{2} H_3 \right]. \]

The reality conditions [Ellis [26]]

\((i)\rho + p > 0, \quad (ii)\rho + 3p > 0,\]

lead to:

\[(H_4 + H_5) T_1^{-2} > \frac{3}{2\chi} \beta^2(t), \] (5.13)

and

\[(3H_4 + H_5) T_1^{-2} > \frac{3}{\chi} \beta^2(t), \] (5.14)

respectively.

The dominant energy conditions [Hawking and Ellis [27]]

\((i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0,\]

lead to

\[(H_5 - H_4) T_1^{-2} \geq 0 \] (5.15)

and

\[(H_4 + H_5) T_1^{-2} \geq \frac{3}{2\chi} \beta^2(t), \] (5.16)

respectively. Equations (5.13), (5.14) and (5.16) imposes some restriction on \( \beta(t) \).

The expansion scalar \( \Theta \) of the model is given by:

\[ \Theta = \sqrt{\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}} = (1+L)\sqrt{N(b_1+1)} T_1^{-1}. \] (5.17)

Shear \( \sigma \) is given by:

\[ \sigma^2 = \frac{1}{2} [(\sigma_0^0)^2 + (\sigma_1^1)^2 + (\sigma_2^2)^2 + (\sigma_3^3)^2], \] (5.18)

where

\[ \sigma_1^1 = \frac{1}{3} \left( \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) = \frac{b_1(2-L)-(L+1)}{3} \sqrt{\frac{N}{b_1+1}} T_1^{-1}, \] (5.19)

\[ \sigma_2^2 = \frac{1}{3} \left( \frac{2\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) = \frac{(2-L)-b_1(L+1)}{3} \sqrt{\frac{N}{b_1+1}} T_1^{-1}, \] (5.20)
\[ \sigma_1^3 = \frac{1}{3} \left( \frac{2\dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = \left( \frac{2L-1}{3} \right) \sqrt{N(b_1 + 1)} T_1^{-1}, \]  

(5.21)

then

\[ \sigma^2 = \frac{T_1^{-2} N}{18(b_1 + 1)} \left[ (b_1(2 - L) - (1 + L))^2 + ((2 - L) - b_1(1 + L))^2 
+ (2L - 1)^2 N(b_1 + 1)^2 \right]. \]  

(5.22)

Since \( \frac{\sigma}{\Theta} \) is constant, this model does not approach isotropy for large value of \( T_1 \). The model starts expanding at \( T_1 > 0 \) and it stops expanding as \( T_1 \to \infty \). The parameters \( \rho, p, \Theta \) and \( \sigma^2 \) tend to infinity at \( T_1 = 0 \), that is the universe starts from initial singularity with infinite energy, infinite internal pressure, infinite rate of shear and expansion. Moreover \( \rho, p, \Theta \) and \( \sigma^2 \) decreasing toward a non-zero finite quantities as \( 0 < T_1 < T_0 \) which require that \( L \) to be positive. The displacement field vector is increasing as \( T_1 \) decreasing and it tends to constant value if \( L = -1 \). **Figure 2 shows the behaviors of** \( \Theta, \sigma^2 \) and \( \beta^2 \) **with** \( T_1 \) **as we indicated.**

Dash line represents \( \Theta \), dotted line for \( \sigma^2 \) and thick line for \( \beta^2 \).

**Figure 2**

6 **Case 3: \( m = n = 0 \)**
If $n = m = 0$, the line element (2.1) reduced to Bianchi type I model:

\[ ds^2 = dt^2 - A^2 dx^2 - B^2 dy^2 - C^2 dz^2. \]  

(6.1)

where $A$, $B$ and $C$ are functions of $t$ only.

For the metric (6.1) field equations (2.9) become:

\[ \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{B}C}{BC} + \frac{3}{4} \beta^2 = -\chi [p + \frac{K^2}{2\mu A^2 B^2}], \]

(6.2)

\[ \frac{\dot{A}}{A} + \frac{\dot{C}}{C} + \frac{\dot{A}C}{AC} + \frac{3}{4} \beta^2 = -\chi [p + \frac{K^2}{2\mu A^2 B^2}], \]

(6.3)

\[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A}B}{AB} + \frac{3}{4} \beta^2 = -\chi [p - \frac{K^2}{2\mu A^2 B^2}], \]

(6.4)

\[ \frac{\dot{A}B}{AB} + \frac{\dot{A}C}{AC} + \frac{\dot{B}C}{BC} - \frac{3}{4} \beta^2 = \chi [\rho + \frac{K^2}{2\mu A^2 B^2}], \]

(6.5)

where dots means ordinary differentiation with respect to $t$.

### 6.1 Solution of the field equations

Since we have six unknown with four independent equations so we need some condition for determinations of the unknowns. Let us assume the following relations between the coefficient of the metric:

\[ \frac{\dot{A}}{A} = \frac{\dot{C}}{C}, \quad \frac{\dot{A}}{A} = \frac{\dot{C}}{C} \]  

(6.6)

with equation (3.32):

\[ C = (AB)^L. \]  

(6.7)

Two equations (6.6) and (6.7) lead to

\[ B = N_1 C^{-\frac{1-L}{L}}, \]  

(6.8)

where $N_1$ is a constant of integration.

From (6.2) and (6.3) we get:

\[ \frac{\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \left( \frac{\dot{B}}{B} + \frac{\dot{A}}{A} \right) = 0. \]  

(6.9)

Using (6.8) and (6.6) in (6.9) then:

\[ \frac{\dot{C}}{C} + \frac{1}{L} \frac{\dot{C}^2}{C^2} = 0. \]  

(6.10)

Equation (6.10) has solution in the form:

\[ C(t) = \left( \frac{1+L}{L} \right) N_2 t + \left( \frac{1+L}{L} \right) N_3 \left(\frac{1}{t} \right)^{\frac{L}{1-L}}, \]  

(6.11)
where \( N_2 \) and \( N_3 \) are constants of integration.

From (6.8) we get:

\[
B(t) = N_1 \left[ \left( \frac{1 + L}{L} \right) N_2 t + \left( \frac{1 + L}{L} \right) N_3 \right]^{1-L}.
\]  

(6.12)

Equation (6.7) gives:

\[
A(t) = \frac{1}{N_1} \left[ \left( \frac{1 + L}{L} \right) N_2 t + \left( \frac{1 + L}{L} \right) N_3 \right]^{L}.
\]  

(6.13)

The line element (6.1) read as:

\[
ds^2 = dt^2 - \frac{1}{N_1^2} T_2^{2L} dx^2 - N_1^2 T_2^{2(1-L)} dy^2 - T_2^{2L} dz^2,
\]  

(6.14)

where

\[
T_2 = \left( \frac{1 + L}{L} \right) N_2 t + \left( \frac{1 + L}{L} \right) N_3.
\]  

(6.15)

### 6.2 physical and geometrical properties of the model

From (2.29) the additional term \( \beta(t) \) read as:

\[
\beta(t) = MT_2^{-1}.
\]  

(6.16)

The two equations (6.3) and (6.4) lead to:

\[
f(t) \to \infty.
\]  

(6.17)

The magnetic permeability \( \mu \to \infty \)

For the model (6.1), the density \( \rho \) and the pressure \( p \) are given by:

\[
p = \frac{1}{\chi} \left[ 3N_2^2 - \frac{3M^2}{4} T_2^{-2} \right],
\]  

(6.18)

and

\[
\rho = \frac{1}{\chi} \left[ N_2^2 \left( \frac{2}{L} - 1 \right) - \frac{3M^2}{4} T_2^{-2} \right].
\]  

(6.19)

The reality conditions [Ellis [26]]

\((i) \rho + p > 0, \quad (ii) p + 3 \rho > 0,\)

lead to:

\[
M^2 < \frac{4N_2^2}{3} \left( 1 + \frac{1}{L} \right),
\]  

(6.20)

and
respectively.
The dominant energy conditions [Hawking and Ellis [27]]

\[(i) \rho - p \geq 0, \quad \rho + p \geq 0,\]

lead to

\[2N_2^2 \geq \frac{L}{1-2L}, \quad (6.22)\]

and

\[M^2 \leq \frac{4N_2^2}{3}(1 + \frac{1}{L}), \quad (6.23)\]

respectively.
The expansion scalar \( \Theta \) of the model is given by:

\[\Theta = (\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) = \frac{N_2(1-L)}{L} T^{-1}. \quad (6.24)\]

Shear \( \sigma \) is given by:

\[\sigma^2 = \frac{1}{2} [(\sigma_0^0)^2 + (\sigma_1^1)^2 + (\sigma_2^2)^2 + (\sigma_3^3)^2], \quad (6.25)\]

where

\[\sigma_1^1 = \frac{1}{3} \left( \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) = \frac{N_2(2L-1)}{3} T^{-1}, \quad (6.26)\]

\[\sigma_2^2 = \frac{2N_2(1-2L)}{3L} T^{-1}, \quad (6.27)\]

\[\sigma_3^3 = \frac{2N_2(2L-1)}{3L} T^{-1}, \quad (6.28)\]

then

\[\sigma^2 = \frac{N_2^2(2L-1)^2}{3} T^{-2}. \quad (6.29)\]

Since \( \frac{\sigma}{\Theta} \) = constant, this model dose not approach isotropy for large value of \( T_2 \). The model starts expanding at \( T_2 > 0 \) and it stop expanding as \( T_2 \to \infty \). The parameters \( \rho, p, \Theta, \sigma^2 \) and \( \beta \) tend to infinity at \( T_2 = 0 \), that is the universe starts from initial singularity with infinite energy, infinite internal pressure, infinite rate of shear and expansion. Moreover \( \Theta, \sigma^2, \rho, p \) and \( \beta \) decreasing as \( T_2 \) increasing and tend to a finite quantity as \( T_2 = T_0 < \infty \).

In figure 3 below, we show the behaviors of \( p \) and \( \rho \) and \( \beta^2 \) with \( T_2 \) as we indicated. Dash line represent \( p \), dotted line for \( \beta^2 \) and thick line for \( \rho \).
ACKNOWLEDGEMENT

The authors are thankful to the referee for his valuable comments.

References

   (Translated from the Russian by Julian B. Barbooue) Basel; Boston; Berlin: Birkhäuser.


