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A Real Options Based Approach for Valuation of Government Guarantees in Public-Private-Partnerships

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Abstract

A fast and computationally efficient valuation tool assists governments involved in Public-Private-Partnership (P3) projects to examine many contractual configurations and design a guarantee which minimizes cost and reasonably mitigates the risk. This paper presents a continuous stochastic process derived from the risk factor forecast, thereby providing a more realistic and flexible model. A new valuation approach is developed using a finite difference method based on this continuous stochastic process. In a numerical example with one risk factor, it is shown that this new valuation tool is 100 times faster than the existing simulation based approach. Its superior speed presents the opportunity to examine different contractual configurations, and as a result, design a more cost effective guarantee contract. Exercise strategies are derived for a multiple-exercise (Australian) guarantees structure. This new approach can be used by a government to reserve budget for the guarantees. Finally, the continuous underlying random process and exercise strategy enable this method to value more complex guarantee structures.

Key words: Public-Private-Partnership (P3), Infrastructure Finance, Government Guarantee, Real Options, Finite Difference Method, Monte Carlo Simulation.

Introduction

Civil infrastructure is essential to the well being of an economy and plays an important role in supporting a nation's socio-economic stability, fostering prosperity, and attracting foreign investment (Ng and Loosemore 2007; Martini and Lee 1996; Threadgold 1996). The necessity of investing in infrastructure on the one hand and respecting fiscal budgetary constraints on the other has lead governments to seek partnerships private firms in the development of such projects. To participate in the project, the private parties must be comfortable with the risks inherent in the project. If the impacts of such risk exceed their tolerance, they will withdraw from the project. To incentivize the private parties to

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participate in such cases, governments usually provide financial support of which guarantees constitute the most common form. Guarantees however, place a financial burden on government budgets while providing value for the private party. Designing a guarantee contract that is cost effective for the government and at the same time mitigates the project risk to an acceptable level for the private party is of importance. The first step to achieve this goal, however, is to develop an accurate and computationally efficient valuation framework for the guarantees. This will be necessary during the guarantee design stage where potentially thousands of configurations of the guarantee contracts will be valued and compared to select the best guarantee structure.

Chiara et al (2007) proposed an elegant guarantee framework for infrastructure projects, known as the Australian guarantee, in which the concession period is only partially covered by the government guarantees. The smaller number of guarantees in this framework as compared to the classic full-coverage framework causes fewer financial commitments for the government. On the other hand, the Australian framework also provides value for the private party as it offers flexibility to choose the time of redeeming the guarantees. In a numerical example, it was shown that this new framework could be worth 99% of the traditional full-coverage guarantee structure with as few as half the number of guarantees.

Another novelty of their research was the fact that the underlying random process for the risk factor was derived from the forecasted values for the risk. However, the discrete nature of the underlying random process eliminates the possibility that a partial differential equation (PDE) approach could be adopted for guarantee valuation and leaves approaches based on Monte Carlo simulation as the only feasible option. Although very versatile, Monte Carlo simulation approaches (MC) are slow to converge and their valuations are subject to statistical variance (Glasserman 2004:3). MC methods only provide a confidence interval for where the value of the guarantee lies. To tighten this estimate and make the result closer to the true value, many more simulation runs are required which makes the MC comparably slow.

The slowness and the variation in MC results could cause difficulties in designing guarantee contracts as well as in investigating the sensitivity of the guarantee value to forecast parameters. In designing a guarantee, the parties involved need to value many configurations of the contract provisions with many different underlying assumptions. As such, a slow valuation tool would be a hindrance in designing an optimal contract. Analyzing the sensitivity of the guarantee value to changes in forecast parameters is important because the project has not been built and no data has been observed for accurate parameter estimation. However, MC has a poor performance when it comes to sensitivity analysis (Glasserman and Zhao 1999). Therefore, the necessity for a faster and more accurate valuation tool is obvious.

This paper presents a novel approach for guarantee valuation that is computationally more efficient than the existing Monte Carlo based approach. It is also capable of handling complex contracts. The approach is based on solving a PDE using a finite difference method (FDM). To achieve this, a continuous stochastic differential equation (SDE), which is better suited to model risk factors in long-term than methods the current literature

suggests, is derived from the forecasted risk factors. In addition to its greatly enhanced computational speed, once the FDM calculation is performed, it can return the value of the guarantee with different initial conditions. In contrast, commonly used Monte Carlo simulation based approaches require a new calculation for each new initial condition. These characteristics make FDM approach a superior valuation tool for the purpose of designing guarantee contracts and performing sensitivity analysis. Although the benefits of FDM diminish when solving higher dimensional problems (4 or more) and MC based approaches become more justifiable, for the existing low dimensional problems in the literature, the FDM approach is shown to be a superior tool. A benefit of keeping the number of risk factors to a minimum is that data will be required for each risk factor to calibrate the model, and these data may not be readily available.

Background

Guarantee Frameworks

Three guarantee frameworks exist in the literature and they can be distinguished by the number of guarantees (M) and the number of times the guarantees can be redeemed (N). The framework used by the Private Sector Advisory Service Department at the World Bank (Irwin 2003) assumes that the guaranteed party has the right to exercise the guarantee at one specific point in time (for example, at the end of year 5) within the concession period i.e. $M=N=1$.

The second framework provides full coverage with multiple, regular exercise dates throughout the concession period of the P3 project, and $M=N$ = the number of concession periods. Hence, at each exercise date, one guarantee is redeemed if its payoff is positive. Many scholars adopted this full coverage setting in their work (Brandao and Saraiva 2008, Huang and Chou 2006, Dailami et al. 1999) and treated this full coverage guarantee as a stream of European put options. In other words, they obtained the value of the guarantee contract by summation of the values of the N European put options whose maturities are at the end of each operational year.

The third framework provides M guarantees redeemable at N exercise dates such that $M \leq N$ (Chiara et al. 2007). In this so called Australian guarantee, the company exercises the guarantees at its discretion and must determine the optimal policy for exercising these guarantees in the presence of future uncertainty. The first two frameworks are special cases of the Australian guarantee i.e. the first framework is $N=M=1$ and the second framework is $M=N$ (=number of concession years). Valuation is significantly more complicated if $M < N$ since the company must decide whether to exercise a guarantee or to save it for a future year. The valuation tool presented in this paper is based on the Australian guarantee; therefore, it is capable of valuing the other frameworks as well.

Real Option

Guarantees of these types exhibit mathematical similarities to options contracts, which are financial securities traded in the market. In a revenue guarantee, for example, the government promises to compensate the company for a shortfall in revenue, Rev , from the guaranteed level, K , during a certain period (say, one year). In other words, the government is liable for the $\max(K - S, 0)$ at the end of each guaranteed period, which resembles a European put option payoff. The European put option (Hull 2006) is a contract that gives the buyer the right, without obligation, to sell an underlying security (e.g. stock) at a predetermined price (called a strike price, K'), at a specific maturity date. After the maturity date, the option expires worthless. The payoff of the European put option at the maturity date is $\max(K' - S, 0)$ where S is the actual security price at that time. These similarities sparked the idea of using the options valuation techniques in project finance. As a result, several scholars (Brandao and Saraiva 2008; Cui et al. 2008; Cheah and Liu 2006; Huang and Chou 2006; Garvin and Cheah 2004; Zhao et al. 2004; Ho and Liu 2002) implemented real options techniques to value infrastructure project guarantees, although none of these references employed a finite difference approach.

Modeling Underlying Risk

Both options and guarantees derive their value from stochastic random processes. Hence, having a model for the underlying risk factor is the starting point for the valuation of guarantees and options. For valuing option on a stock, a common assumption is that the stock price follows a Geometric Brownian Motion (GBM) stochastic process (Hull 2006). Being the classic model for valuing options, GBM was used for P3 valuation as well (Brandao and Saraiva 2008; Huang and Chou 2006; Zhao et al.'s 2004; Ho and Liu 2002). GBM, however, may not always be the appropriate underlying random process to model the demand for an infrastructure or the revenue generated by it because this form of stochastic model is only effective for problems of short lifespan (Chiara 2006). For example, if GBM is used to model highway traffic, it is quite probable that the traffic in the 10th year of the concession would be greater than the capacity of the highway.

A new model was recently presented that could accommodate demand forecasting and preserve Markovian property in the realization of the demand (Chiara and Garvin 2008). Although elegant, its discrete nature limits its application for methods such as finite difference. Therefore, developing a new continuous stochastic model is imperative as the basis for the development of more computationally efficient finite difference methods.

A Continuous Model for Risk Factor

Although having a model for the underlying risk is important for valuing P3 projects, often little historical data is available to model project uncertainties. Assuming that all information that is available about the project risk has been reflected in the forecasted curve of the risk, e.g. traffic demand for each of the next ten years, employing the forecast could be the initial step toward developing an appropriate model. It is also important that the risk factor model possesses the Markovian property (Chiara 2006); in order to ensure that the resulting stochastic dynamic optimization and valuation problem is finite dimensional.

Based on the premises presented, a continuous-time stochastic process model is herein proposed for P3 risk factor. Let's define:

$F(t, T)$: the forecasted value of the risk factor at time T , estimated from information available at time $t < T$.

$R(t)$: the value of the risk factor that is realized (observed) at time t .

By this definition, $F(t, t) = R(t)$. In the context of a toll road, $R(t)$ and $F(t, T)$ are referred to as instantaneous (spot) demand and forecasted demand respectively.

Using the rationale deployed by Clewlow and Strickland (1999) in their development of a stochastic differential equation (SDE) for energy forward price is the starting point. Assuming the analysts use the available information at time t to produce the forecasted values, this forecast has to be unbiased and, as a result, the expected change in the forecast should be zero. This research employs Wiener process (Hull 2006:265) for modeling the random changes in forecasted values. To obtain a Markovian spot demand process, the volatilities of the forecasted demands must have a negative exponential form (Carverhill 1994). Therefore, the SDE for the forecasted demand is:

$$\frac{dF(t, T)}{F(t, T)} = \sigma e^{-\alpha(T-t)} dW_{(t)} \quad \text{Eq. 1}$$

where W follows a Wiener process, σ determines the level of the spot and forecasted demand volatility, α indicates the rate at which the volatility of the longer dated forecasted demand declines. This is the same SDE that Clewlow and Strickland proposed but for energy forward price.

If forecasted values obey the evolution process described in Eq.1, the spot values will have the following SDE (Clewlow and Strickland 1999):

$$dR_{(t)} = \left[\frac{\partial \ln(F_{(0,t)})}{\partial t} + \alpha (\ln(F_{(0,t)}) - \ln(R_{(t)})) + \frac{\sigma^2}{4} (1 - e^{-2\alpha t}) \right] R_{(t)} dt + \sigma R_{(t)} dW_{(t)} \quad \text{Eq. 2}$$

It then follows that $R(t)$ is log-normally distributed (Clewlow and Strickland 1999):

$$\ln(R(T)) \sim N \left[\ln(F(0, T)) - \frac{\sigma^2}{4\alpha} [1 - e^{-2\alpha T}], \frac{\sigma^2}{2\alpha} [1 - e^{-2\alpha T}] \right] \quad \text{Eq. 3}$$

The evolution model suggested by Eq. 2 is a Markov process. It incorporates the initial forecast, which reflects all of the information available.

Explicit Finite Difference Method

PDE Derivation

The evolution process in Eq. 2 facilitates deriving a partial differential equation (PDE) that governs the present value of the guarantees. $V(R, t, m)$, in short V , is defined to be the

value of the guarantees at time t and when the value of the risk factor is R with m number of Australian guarantees remaining. The PDE governing V is (derivation is provided in the Appendix):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial Z^2} + \left[\frac{\partial \ln(F_{(0,t)})}{\partial t} + \alpha(\ln(F_{(0,t)}) - Z) - \frac{\sigma^2}{4}(1 + e^{-2\alpha t}) \right] \frac{\partial V}{\partial Z} - rV = 0 \quad \text{Eq. 4}$$

where Z is a change of variable such that $Z = \ln R_{(t)}$ and r is the risk free rate. $F_{(0,t)}$ is the risk adjusted forecast of the risk factor. Capital asset pricing model (CAPM) can be used to perform such adjustment (Brandao and Saraiva 2008). Knowing the risk premium per unit of the risk, λ , is important in implementing CAPM. The users of this valuation framework should know their premium for taking risk. In the absence of such estimate, one may use market price of risk.

Finite Difference Implementation for Government Guarantee

FDM is a numerical method that is used for solving the PDE. For the convenience of the reader, an explanation of FDM in the context of government guarantee valuation is provided. For more details on FDM, one can refer to Duffy (2006) and Wilmott (2006). FDM is a backward algorithm in time; that means for this application of government guarantee valuation, it starts from the end of the concession period when the value of the guarantee is certain for a given level of risk. Then it works backward one time step at a time and finds the value of the guarantee using values for different levels of risk factor at the end. This process continues to the present time.

Figure 1 illustrates how the finite difference method works. Changes in risk factor Z and time t are discretized into units of δZ and δt respectively. For the k^{th} time step and i^{th} spatial step, the risk factor and time are calculated as (the notations are adapted from Wilmott (2006)):

$$Z_i = Z_{\min} + i \cdot \delta Z \quad \text{Eq. 5}$$

$$t = T - k \cdot \delta t \quad \text{Eq. 6}$$

Where $0 \leq i \leq I_{\max}$ and $0 \leq k \leq K$. Note that $Z = \ln R$ and $Z_{\min}$ and $Z_{\max}$ correspond to minimum and maximum level of R that with high level of confidence, R would not pass these levels. The distance between $Z_{\min}$ and $Z_{\max}$ is equally discretized.

Each node on the grid represents a guarantee value. The following notation is used for representation of the guarantee value at time step k , with risk factor level of Z_i when m number of Australian guarantee is left.

$$V_{i,m}^k = V(Z_{\min} + i \cdot \delta Z, T - k \cdot \delta t, m) \quad \text{Eq. 7}$$

The value of the government guarantees can be obtained from Eq. 4, where V is the short form of $V(Z, t, m)$. Considering the general PDE form:

$$\frac{\partial V}{\partial t} + a(Z,t) \frac{\partial^2 V}{\partial Z^2} + b(Z,t) \frac{\partial V}{\partial Z} + c(Z,t)V = 0 \quad \text{Eq. 8}$$

Figure 1: Illustration of Finite Difference Method

The coefficient of derivative terms will be:

$$a(Z,t) = \frac{\sigma^2}{2} \quad \text{Eq. 9}$$

$$b(Z,t) = \left[\frac{\partial \ln(F_{(0,t)})}{\partial t} + \alpha (\ln(F_{(0,t)}) - Z) - \frac{\sigma^2}{4} (1 + e^{-2\alpha t}) \right] \quad \text{Eq. 10}$$

$$c(Z,t) = -r \quad \text{Eq. 11}$$

Each derivative term in Eq. 8 can be approximated by (Wilmott 2006):

$$\theta: \quad \frac{\partial V}{\partial t}(Z,t,m) = \frac{V_{i,m}^k - V_{i,m}^{k+1}}{\delta t} + O(\delta t) \quad \text{Eq. 12}$$

$$\Delta: \quad \frac{\partial V}{\partial Z}(Z,t) = \frac{V_{i+1,m}^k - V_{i-1,m}^k}{2\delta Z} + O(\delta Z^2) \quad \text{Eq. 13}$$

$$\Gamma: \quad \frac{\partial^2 V}{\partial Z^2}(Z,t) = \frac{V_{i+1,m}^k - 2V_{i,m}^k + V_{i-1,m}^k}{\delta Z^2} + O(\delta Z^2) \quad \text{Eq. 14}$$

Using θ , Δ , and Γ , the value of the guarantees at every non-exercise date can be obtained from Eq. 15:

$$V_{i,m}^{k+1} = V_{i,m}^k + \delta t [a_i^k (\frac{V_{i+1,m}^k - 2V_{i,m}^k + V_{i-1,m}^k}{\delta Z^2}) + b_i^k (\frac{V_{i+1,m}^k - V_{i-1,m}^k}{2\delta Z}) + c_i^k V_{i,m}^k] + O(\delta t, \delta Z^2) \quad \text{Eq. 15}$$

In every time step, there are k+1 variables but only k-1 equations are available. The other two equations come from the boundary conditions. The boundary condition in Eq. 16 can be imposed at $i=0$ and $i=I_{max}$. This means that at very high or very low prices, the value of an option varies linearly.

$$\frac{\partial^2 V}{\partial Z^2} = 0 \quad \text{Eq. 16}$$

Using a similar approximation as shown in Eq. 14 for the second derivative will result in:

$$Z_{Imax} = 2Z_{Imax-1} - Z_{Imax-2} \quad \text{Eq. 17}$$

$$Z_0 = 2Z_1 - Z_2 \quad \text{Eq. 18}$$

Let's define t_e is an exercise date such that

$$t_e = T - k_e \cdot \delta t \quad \text{Eq. 19}$$

T is the end of the concession and $V(Z, t_e^+, m)$ is the expected present value of the guarantees at time t_e^+ (slightly before the decision about redeeming a guarantee is made); the level of risk factor is R ($Z = \ln R_{(te)}$) with m Australian guarantees remaining. As FDM is a backward algorithm, going back from the end of concession ($k=0$) to $t=0$ ($k=K$), t_e^+ indicates the time slightly before reaching k_e th time step when the decision about exercising a guarantee is made. The value of the government guarantee when one Australian guarantee is redeemed:

$$\text{Redemption value} = V(Z, t_e^+, m-1) + \text{payoff}(Z, t_e) \quad \text{Eq. 20}$$

where $\text{payoff}(Z, t_e)$ is a predetermined contractual agreement between the government and the private party. The agreement specifies how much the government will reimburse the private party if the level of risk factor is R ($Z = \ln R$) at time t_e upon redeeming one guarantee. For example, one payoff structure could be that the government compensates the private party if the earning falls below a constant guaranteed level, E. Mathematical notification of this payoff structure would be:

$$\text{payoff}(Z, t_e) = \max(E - R, 0) = \max(E - e^Z, 0) \quad \text{Eq. 21}$$

The value of not exercising a guarantee (continuation value) is:

$$\text{Continuation value} = V(Z, t_e^+, m) \quad \text{Eq. 22}$$

The project company picks the alternative that creates greater value such that:

$$V(Z, t_e, m) = \max \left\{ V(Z, t_e^+, m), V(Z, t_e^+, m-1) + \text{payoff}(Z, t_e) \right\} \quad \text{Eq. 23}$$

If $m=0$, then the guarantee value $V(Z, t_e, m) = 0$. At the last exercise date when $t_e = T$ ($k_e = 0$), there is no continuation value because the concession ends. Therefore, if any Australian guarantees remain at the end of the concession, the decision is to exercise; Thus, the value of the government guarantee contract is:

$$\begin{aligned} V(Z, T, m) &= \text{payoff}(Z, T) & \text{for } m \geq 1 \\ V(Z, T, 0) &= 0 & \text{for } m = 0 \end{aligned} \quad \text{Eq. 24}$$

Now, the algorithm is complete.

QUICK Algorithm:

1. starting from time T , Eq. 24 is used to obtain the initial value of the guarantee contract
2. go one time step backward in time
3. calculate the $V(Z, t, m)$ using Eq. 15
4. if t is an exercise date (i.e. $t = t_e$), then:
 - 4.1. set $U(Z, t_e, m) = V(Z, t_e, m)$ (the $V(Z, t_e^+, m)$ defined in this section is essentially the $V(Z, t_e, m)$ calculated up to this point by this algorithm; the $U(Z, t_e, m)$ will be used in the next section when exercise strategy derivation is explained)
 - 4.2. decide whether or not to redeem an Australian guarantee and continue with $m-1$ remaining number of guarantees by comparing the results of Eq. 20 and Eq. 22 based on the maximization function presented in Eq. 23
 - 4.3. set $V(Z, t_e, m)$ equal to value of Eq. 23
5. if $t \neq 0$, then go to step 2; if $t = 0$, then the algorithm ends

VALUE Algorithm:

Once the algorithm is preformed, the value of the government guarantees is known at every time step, at any level of R (or Z), and with any number of remaining guarantees. Suppose that the value of the guarantee at the level of R ($Z = \ln R$), time t ($t = T - k \cdot \delta t$), and with m Australian guarantees remaining is of interest. The steps to be taken are:

1. Identify the 2 nearest grid-points on the k^{th} discretization of the time (Suppose that these two point are i^{th} and $(i+1)^{\text{th}}$ discretization of Z such that $Z_i \leq Z < Z_{i+1}$ as illustrated in Figure 2).
2. Interpolate between the values of $V(Z_i, t_k, m)$ and $V(Z_{i+1}, t_k, m)$ to find $V(Z, t_k, m)$; that is:

$$V(Z, t_k, m) = \frac{(Z_{i+1} - Z)V(Z_i, t_k, m) + (Z - Z_i)V(Z_{i+1}, t_k, m)}{\delta Z} \quad \text{Eq. 25}$$

Figure 2: Identify 2 Nearest Grid-points

EXERCISE Algorithm:

One novel feature of this research is the introduction of exercise strategies for government guarantees within the flexible Australian framework. Exercise strategy is a guideline that prescribes the manner in which the company will choose to exercise a guarantee or save it for future worse situation. The logic for the derivation of such strategies is similar to what was presented previously. Suppose the project company needs to know whether they should redeem one Australian guarantee on exercise date t_e , when the level of risk factor is R ($Z = \ln R$) with m Australian guarantees remaining. If they know $U(Z, t_e, m)$, the expected present value of the cash flow generated by guarantees, then they can answer the question of whether to exercise or not using logic similar to that of comparing the continuation and redemption values presented previously. The values of $U(Z_i, t_e, m)$ for all grid points on the exercise dates are stored after performing the QUICK algorithm; therefore, $U(Z, t_e, m)$ can be retrieved using the VALUE algorithm. However, instead of $V(Z_i, t_e, m)$, $U(Z_i, t_e, m)$ must be used in Eq. 39. Therefore, the steps to identify whether or not to exercise a guarantee are:

1. Find the value of $U(Z, t_e, m)$ and $U(Z, t_e, m-1)$:

- 1.1. Identify the 2 nearest grid-points on the discretization corresponds to t_e (Suppose that these two point are i^{th} and $(i+1)^{\text{th}}$ discretization of Z such that $Z_i \leq Z < Z_{i+1}$)

$$1.2. U(Z, t_e, m) = \frac{(Z_{i+1} - Z)U(Z_i, t_e, m) + (Z - Z_i)U(Z_{i+1}, t_e, m)}{\delta Z} \quad \text{Eq. 26}$$

$$1.3. U(Z, t_e, m-1) = \frac{(Z_{i+1} - Z)U(Z_i, t_e, m-1) + (Z - Z_i)U(Z_{i+1}, t_e, m-1)}{\delta Z} \quad \text{Eq. 27}$$

2. If $U(Z, t_e, m-1) + \text{payoff}(Z, t_e) > U(Z, t_e, m)$, redeem one guarantee; otherwise do not.

Verifying Exercise Strategies by Simulation

By solving the PDE in Eq. 4, FDM provides two pieces of information. First, it returns optimal exercise strategies; second, it indicates the value of the government guarantee if the exercise strategies are followed. One way to verify the FDM value is to check if the same value is obtained when these strategies are followed in simulated paths. If the FDM grid is weakly constructed and it is not converging, the FDM value and the value obtained from employing the exercise strategies in simulated paths would not match. These two values would be consistent only if the FDM converges to the right value.

To perform this validation, several paths are simulated according to the evolution process of R . The EXERCISE algorithm can help determine at every exercise date along each simulated path whether a guarantee should be redeemed. If a guarantee is redeemed, its payoff is obtained at that date and then discounted to $t = 0$. Adding all the discounted payoffs for a single simulated path will give the present value of the cash flow generated by Australian guarantees for that scenario. Similar computations must be performed on the other paths as well. If the present values of the cashflows for all simulated paths are averaged, the average value should asymptotically converge to the value of the Australian guarantee obtained by FDM.

In summary, the algorithm for applying exercise strategies on simulated paths is:

1. simulate a path for R until the end of the concession (T) using the SDE for R
2. starting from the beginning ($t = 0$),
 - 2.1. go to the next exercise date (t_e) and find the realization of R at t_e
 - 2.2. Using the Exercise algorithm, determine whether or not at t_e with m guarantees remaining, a guarantee should be redeemed
 - 2.3. if a guarantee should be redeemed,
 - 2.3.1. reduce one from the guarantees remaining ($m=m-1$)
 - 2.3.2. calculate the payoff of the guarantee at t_e
 - 2.3.3. discount the payoff to $t=0$
 - 2.4. if $t_e = T$, go to step 3; otherwise go to step 2.1
3. add all the discounted payoff value for this path. This sum is referred to as CF_i for the i^{th} path
4. if enough paths have been simulated go to step 5, otherwise go back to step 1

5. take the average of all CF_i

Estimation of the Probability of a Budget Shortfall

It is strongly recommended that governments value the liabilities arising from infrastructure guarantees and budget for them annually (Dailami and Klein 1997). If the government reserves an amount for these contingent liabilities in the fiscal budget, the important question is whether this amount is sufficient to meet the liability arising from the guarantee at the end of that year. In other words, what is the probability that the amount reserved in the budget fails to meet the contingent liability at the end of that year.

Budget shortfalls occur when, 1) the project company decides to redeem one guarantee; and, 2) the payoff of the guarantee upon being exercised is greater than the amount reserved in the budget.

For guarantees with put-style payoff structure (payoff = $\max(E - R(t_e), 0)$ where E is the guaranteed level), $R_{E(t_e, m)}$ is the level of R at exercise time t_e and with m remaining Australian guarantees below which the sponsor will redeem an Australian guarantee. $R_{E(t_e, m)}$ can be obtained from the exercise strategies derived from FDM explained previously. It is worth notifying that E is always greater than $R_{E(t_e, m)}$ for guarantees with put-style payoff.

If B is defined as the budget reserved by the government to meet the contingent liability of the guarantee, then the probability of budget shortfall can be formulated as:

$$\text{Prob}(\text{Budget Shortfall}) = \text{Prob}(\{R < R_{E(t_e, m)}\} \cap \{B < E - R\}) \quad \text{Eq. 28}$$

If the right hand side of the Equation is rearranged

$$\text{Prob}(\text{Budget Shortfall}) = \text{Prob}(\{R < R_{E(t_e, m)}\} \cap \{R < E - B\}) \quad \text{Eq. 29}$$

where $E - B$ is the minimum level of R that government has budgeted for. Rearranging the right hand side of Eq. 29 will result:

$$\text{Prob}(\text{Budget Shortfall}) = \text{Prob}(\{R < \min(R_{E(t_e, m)}, E - B)\}) \quad \text{Eq. 30}$$

As previously mentioned, R is following a Log-normal distribution (Eq. 3); if

$$MR = \min(R_{E(t_e, m)}, E - B) \quad \text{Eq. 31}$$

then the theoretical probability of Budget Shortfall will be:

$$\text{Prob}(\text{Budget Shortfall}) = N \left[z < \frac{\ln(MR) - \left[\ln(F(0, t_e)) - \frac{\sigma^2}{4\alpha} [1 - e^{-2\alpha t_e}] \right]}{\frac{\sigma^2}{2\alpha} [1 - e^{-2\alpha t_e}]} \right] \quad \text{Eq. 32}$$

where z indicates a standard normal random variable and $N[z < \Phi]$ is a normal cumulative density function.

This theoretical probability can be verified by simulation. In the simulation approach, N simulated paths are generated. Then using the EXERCISE algorithm, it is determined whether a guarantee should be redeemed. Then the number of the paths, n , in which the payoff of the guarantee is greater than the budget reserved is counted. The ratio of $\frac{n}{N}$ indicates the probability of budget shortfall.

Numerical Example

The FDM approach can be compared with the Least-Square Monte Carlo simulation method for the valuation of Australian Guarantees. The average of the implementing exercise strategies on simulated paths, hereafter referred to as exercise strategy result (ESR), is compared to the FDM result as well.

Suppose that in this example, Australian guarantees are granted to the project company to cover any shortfall of risk factor R below the guaranteed level of $E = \$19.459$ million. The Australian guarantees can be exercised at the end of each year for 10 years. Therefore, the payoff structure of the guarantee at the exercise date of t_e is $\text{payoff}(R(t_e)) = \max(E - R(t_e), 0)$. The observed level of R at time $t=0$ is $\$16.094$ million ($R(t=0) = \$16.094$ mil). The risk adjusted forecasted curve of $R(F_{(0,t)})$ for the next 10 years is shown in Figure 3. With risk free interest rate $r = 5\%$, the NPV is $\$190.3$ million.

Figure 3: Forecasted Curve of R

The parameters of α and σ in Eq. 4 are $\alpha=0.02$ and $\sigma=0.1$. The values of the Australian guarantees obtained from both LSM-based method and the FDM-based method are shown in Table 1. Figure 4 displays that the FDM values and LSM values are consistent with each other. The great advantage of this new method is, however, that FDM obtains the results in 1 second whereas LSM obtained the results in 120 seconds on the same computer (CPU of Core2 1.86 GHz, Ram of 1.99Gb); that means, FDM is almost 100 times faster than LSM on this machine for this one factor problem.

Figure 4: Comparing Australian Guarantee Values Obtained From LSM and FDM

In addition to the significant improvement in the calculation speed, a single run of FDM returns the value of government guarantees at different levels of risk factor through out concession period. Using LSM approach, however, the simulation and calculations must be performed again if the value of guarantees at a new level of risk factor or at a different time is required. Therefore in situations where multiple scenarios are to be examined, FDM would be significantly more computationally efficient than LSM.

Table 1: Comparing Australian Guarantee Values Obtained From LSM and FDM

# of Granted Aust. Gua	Lower bound	FDM (\$M)	LSM (\$M)	Upper bound
1	1.711	1.722	1.729	1.748
2	2.776	2.795	2.809	2.842
3	3.531	3.554	3.578	3.624
4	4.094	4.130	4.152	4.210
5	4.537	4.582	4.606	4.674
6	4.883	4.941	4.961	5.039
7	5.150	5.227	5.236	5.322
8	5.364	5.453	5.458	5.551
9	5.526	5.624	5.626	5.726
10	5.639	5.743	5.744	5.849

The lower bound and upper bound values in Figure 4 and Table 1 indicate the interval that the true value of the guarantee lies on with 95% confidence. The values of lower and upper bounds are calculated using the simulated paths generated for LSM. The realized value of the guarantee under each simulated scenario is calculated. Then, the values are used to obtain the confidence interval for a normal mean when the variance is unknown (Ross 2004: 246).

Table 2 and Figure 5 compare the values of Australian guarantees obtained from FDM and those obtained implementing exercise strategies on simulated paths (ESR). The fact that FDM values match with ESR values is further evidence for the validation of the FDM approach. This also confirms the correctness of the exercise strategies derived using the FDM approach.

Table 2: Comparing Australian Guarantee Values Obtained From FDM and ESR

# of Granted Aust. Gua	Lower bound	FDM (\$M)	ESR (\$M)	Upper bound
1	1.694	1.722	1.718	1.743
2	2.738	2.795	2.782	2.825
3	3.476	3.554	3.537	3.597
4	4.033	4.130	4.109	4.185
5	4.468	4.582	4.558	4.648
6	4.817	4.941	4.920	5.022
7	5.095	5.227	5.209	5.322
8	5.312	5.453	5.435	5.559
9	5.474	5.624	5.606	5.739
10	5.589	5.743	5.728	5.867

Figure 5: Comparing Australian Guarantee Values Obtained From ESR and FDM

The theoretical probability of budget shortfall and the probability obtained from simulation are also compared using this numerical example. As demonstrated in Figure 6, when reserved budget is \$1 million, the probability of budget shortfall calculated using both methods match with each other.

Figure 6: Comparing the Realized and Theoretical Probabilities of Budget Shortfall

Conclusion

A fast and computationally efficient valuation method is a necessity for designing guarantee contracts in P3 projects. Existing valuation tools in the literature either use simulation-based approaches, or assume a GBM process for the underlying risk and obtain the value analytically. The former is known to be a computationally inefficient method that often produces unreliable sensitivity estimates; and the latter is based on the incorrect assumption of GBM. Nevertheless, once the guarantee takes the form of an Australian guarantee, its value can not be obtained analytically even with the assumption that the underlying risk obeys a GBM process.

In this paper, concepts from financial engineering literature were employed to develop a new finite difference based approach for valuation of government guarantees while taking the forecasted project risk into consideration. This resulted in a more computationally efficient valuation tool. Using a simple one factor numerical example, it was shown that while the results of both LSM and FDM based approaches match with each other, the FDM approach was 100 time faster than LSM. The speed supremacy of FDM is important when designing guarantee contracts that involve potentially thousands of scenarios and contract structures to be examined. It is also important when the sensitivities to the many underlying assumptions and contract features are to be understood and explored.

To achieve this, a new continuous stochastic differential equation (SDE) was derived from the forecast for project risk. Using the SDE, the partial differential equation (PDE) that governs the value of government guarantee was obtained and then solved with the finite difference method. Another contribution of this research was deriving strategies for exercising the Australian guarantees, which ultimately enables this valuation tool to accommodate more complex guarantee contracts. Additional application of these strategies is in calculating the probability of a budget shortfall for the government offering the guarantees. As a result, this valuation method can advise the government as to how much budget to allocate in each year to meet their future guarantee obligations within a given confidence interval.

The superior performance of FDM diminishes when solving problems with 4 or more risk factors, so analysis should focus on the most critical risk factors. Identifying the key risk factors in PPP projects and estimating their associated parameters is an important area for future research.

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Appendix

PDE Derivation

Let's define:

R : the current value of the risk factor.

t : the current time.

m : the remaining number of Australian guarantees.

$T_1 < T_2 < \dots < T_N$: the potential exercise. (Note: only one guarantee may be exercised at each exercise date and $m \leq N$).

$G(R(T_i), t)$: the present value of the guarantee payoff at time T_i discounted to time t .

$V(R, t, m)$: the expected present value of the guarantees at time t and when the value of the risk factor is R with m number of Australian guarantees remaining.

With the above definition, $V(R, t, m)$ can be mathematically represented

$$V(R, t, m) = E \left[\sum_{i=1}^N G(R(T_i), t) \mid R, t, m \right] \quad \text{Eq. A1}$$

Suppose that R has the following general form of SDE:

$$dR = a(R, t) dt + b(R, t) dW \quad \text{Eq. A2}$$

where W follows a Wiener process (Hull 2006:265). According to Ito's lemma (Wilmott 2006: 80):

$$dV = \left[\frac{\partial V}{\partial t} + \frac{1}{2} b^2(R, t) \frac{\partial^2 V}{\partial R^2} + a(R, t) \frac{\partial V}{\partial R} \right] dt + b(R, t) \frac{\partial V}{\partial R} dW \quad \text{Eq. A3}$$

Taking the expectation of dV conditional on R, t, m

$$\begin{aligned} E[dV] = E \left[\left(\frac{\partial V}{\partial t} + \frac{1}{2} b^2(R, t) \frac{\partial^2 V}{\partial R^2} + a(R, t) \frac{\partial V}{\partial R} \right) dt \mid R, t, m \right] \\ + E \left[b(R, t) \frac{\partial V}{\partial R} dW \mid R, t, m \right] \end{aligned} \quad \text{Eq. A4}$$

The terms inside the first bracket on the right hand side are deterministic; therefore, their expectation would be themselves. The second bracket on the right hand side would be simplified to:

$$E\left[b(R,t)\frac{\partial V}{\partial R}dW \mid R,t,m\right] = b(R,t)\frac{\partial V}{\partial R}E[dW] \quad \text{Eq. A5}$$

Because W is a Wiener process, $E[dW] = 0$. Therefore, the second bracket on the right hand side of the Eq. A4 equals zero and Eq.A4 becomes:

$$E[dV] = \left(\frac{\partial V}{\partial t} + \frac{1}{2}b^2(R,t)\frac{\partial^2 V}{\partial R^2} + a(R,t)\frac{\partial V}{\partial R} \right) dt \quad \text{Eq. A6}$$

Since V is the expectation of the present value of future cashflow, the expectation of a change in V over a small time interval dt would be:

$$E[dV] = rV dt \quad \text{Eq. A7}$$

where r is the interest rate assuming that the forecast curve is risk adjusted. Eq. A6 and Eq. A7 results in:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}b^2(R,t)\frac{\partial^2 V}{\partial R^2} + a(R,t)\frac{\partial V}{\partial R} \right) dt = rV dt \quad \text{Eq. A8}$$

After canceling dt , this becomes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2(R,t)\frac{\partial^2 V}{\partial R^2} + a(R,t)\frac{\partial V}{\partial R} - rV = 0 \quad \text{Eq. A9}$$

If the evolution process proposed in Eq. 2 is employed, the partial differential equation (PDE) in Eq. A9 becomes:

$$\begin{aligned} & \frac{\partial V}{\partial t} + \frac{1}{2}(\sigma \cdot R_{(t)})^2 \frac{\partial^2 V}{\partial R^2} + \left[\frac{\partial \ln(F_{(0,t)})}{\partial t} + \alpha(\ln(F_{(0,t)}) - \ln(R_{(t)})) \right. \\ & \left. + \frac{\sigma^2}{4}(1 - e^{-2\alpha t}) \right] R_{(t)} \frac{\partial V}{\partial R} - rV = 0 \end{aligned} \quad \text{Eq. A10}$$

To obtain stable results using the Finite Difference method, dealing with this PDE could be computationally expensive because the step size for time has to be smaller than a certain limit (Wilmott 2006). By substituting $Z = \ln R_{(t)}$, this constraint on the size of the time step is relaxed. Employing Ito's lemma, the evolution process for Z becomes:

$$dZ = \left[\frac{\partial \ln(F_{(0,t)})}{\partial t} + \alpha(\ln(F_{(0,t)}) - Z) - \frac{\sigma^2}{4}(1 + e^{-2\alpha t}) \right] dt + \sigma dW_{(t)} \quad \text{Eq. A11}$$

Assuming the following general format:

$$dZ = a(Z,t) dt + b(Z,t) dW \quad \text{Eq. A12}$$

Because of the relationship between R and Z ($Z = \ln R_{(t)}$)

$$V(R, t, m) = E \left[\sum_{i=1}^N G(R_{(Ti)}, t) | R, t, m \right] = E \left[\sum_{i=1}^N G(e^{Z(Ti)}, t) | Z, t, m \right] = V(Z, t, m) \quad \text{Eq. A13}$$

The remainder of the proof and the logic is similar to what presented earlier, therefore:

$$\frac{\partial V}{\partial t} + \frac{1}{2} b^2(Z, t) \frac{\partial^2 V}{\partial Z^2} + a(Z, t) \frac{\partial V}{\partial Z} - r V = 0 \quad \text{Eq. A14}$$

Replacing the $a(Z, t)$ and $b(Z, t)$ from the SDE presented in Eq. A11 will result:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial Z^2} + \left[\frac{\partial \ln(F_{(0,t)})}{\partial t} + \alpha (\ln(F_{(0,t)}) - Z) - \frac{\sigma^2}{4} (1 + e^{-2\alpha t}) \right] \frac{\partial V}{\partial Z} - r V = 0 \quad \text{Eq. A15}$$