BARYONS IN A RELATIVIZED QUARK MODEL WITH QCD

by

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ABSTRACT

The three quark system is studied in a 'relativized' version of the quark model which incorporates the important features of quantum chromodynamics. These include a universal one-gluon exchange potential and a string picture confining potential. A good description is obtained of the masses and compositions of all the known baryons from the nucleon to the Σ⁺, with parameters similar to those of an analogous study of the q̅q̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅̅...
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To Hilary, who stuck with me through thick and thin, and to my parents who started the ball rolling.
"It seems probable to me, that God in the Beginning form'd Matter into solid, massy, hard, impenetrable, moveable Particles, of such Sizes and Figures, and with such other Properties, and in such Proportion to Space, as most conduced to the End for which he form'd them."

From "Opticks" by Sir Isaac Newton.
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CHAPTER I: INTRODUCTION

The Hadron Zoo and the Eight-Fold Way

After the discovery of the $\pi$ mesons, it was not long before the, long lived $K$ mesons and hyperons were discovered, which curiously were produced only in pairs. This observation led Gell-Mann and Nishijima\(^1\) to propose a new quantum number called 'strangeness', which was conserved by the strong interactions. This quantum number was violated by the weak interactions, however, which explained the relatively long lifetimes of these particles. As more and more strongly interacting particles (hadrons) were discovered, in the early 1960's Gell-Mann and Ne'eman\(^2\) introduced what Gell-Mann called the 'Eight-fold Way', a classification scheme which gave the strong interactions an approximate SU(3) symmetry, and grouped the hadrons into representations of this group. As this was an approximate symmetry members of a given multiplet had only approximately the same mass. It became obvious that mesons and baryons could all be fit into this classification scheme, although doing so was not always easy. One of the great successes of this model was the prediction of the existence of the $\Omega^-$ baryon to complete the $J^P = \frac{3}{2}^+$ baryon decuplet made up of the various charge states of the $\Delta(1232)$, the $\Sigma^+(1384)$ and the $\Xi(1533)$.

Quarks

The success of the classification scheme based on SU(3) (hereafter referred to as SU(3)$_f$, for flavor) led Zweig\(^3\) and Gell-Mann\(^4\) to propose that hadrons were made up of constituents, which Zweig called p, n, and $\lambda$, which comprised the fundamental representation 3 of SU(3)$_f$, an idea which was not popular with some of the particle physics community at the time\(^5\). These constituents were spin-$\frac{1}{2}$ particles with fractional charge and baryon number. Baryon representations of SU(3)$_f$ were built up from three of these objects, and meson representations from a member of the 3 and from one of their anti-particles in the conjugate representation, which transformed as $\bar{3}$. Zweig called these particles aces, and Gell-Mann called them quarks after a line in James Joyce's Finnegans Wake, and they are now known as the quarks up, down and strange ($u$, $d$ and $s$). The approximate nature of the SU(3)$_f$ symmetry was ascribed to the mass difference between the $u$ and $d$ quarks.
and the $s$ quark. Isospin symmetry was a useful idea to the extent that the $u$ and $d$ quarks had equal mass.

The naive quark model described above was then able to successfully account for the quantum numbers of all of the hadrons. Baryons are states of the form $|qqq\rangle$ with the quark spins combining to $\frac{1}{2}$ or $\frac{3}{2}$, and this total spin combining with the total orbital angular momentum $\vec{L}$ to give total angular momentum $\vec{J} = \vec{L} + \vec{S}$ which is half-integral, and a parity of $(-1)^L$. Mesons are states of the form $|q\bar{q}\rangle$ with the $q$ and $\bar{q}$ spins combining to 0 or 1, and this combining with the angular momentum $\vec{L}$ to give integral $J$. The parity is the product of the intrinsic parity of the $q$ and $\bar{q}$ and the orbital parity, $(-1)^{L+1}$. The quark model explanation of the quantum numbers of some of the lowest lying states of the different flavors of baryons is shown in Table I.

The naive quark model was also successful in explaining the magnetic moments of hadrons, and explained the mass differences between hadrons with similar properties but different quark content in terms of the mass difference between the quarks. A further level of understanding of the baryons could be attained if one assumed that the strong interactions were not only approximately independent of a quark's flavor but also of whether its spin was up or down. Then one could group the quarks into a $\mathbf{6}$ of SU(6) (and the antiquarks into a $\overline{\mathbf{6}}$) and then baryons should appear in SU(6) "supermultiplets" made up from $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6}$. This idea was successful in explaining the fact that all of the $J^P = \frac{1}{2}^+$ and $\frac{3}{2}^+$ baryons which had been discovered seemed to fit into a multiplet of fifty-six states (counting spin multiplicity) with $L^P = 0^+$, i.e. a decuplet of spin-$\frac{3}{2}$ states, and an octet of spin-$\frac{1}{2}$ states. Such a supermultiplet, labelled $[56, 0^+]$, had a natural explanation in terms of SU(6), since $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{20} \oplus \mathbf{56} \oplus \mathbf{70} \oplus \mathbf{70}$ contained a $\mathbf{56}$. Similar observations could be made for the mesons.

Despite all of these successes, the quark model still had its problems, such as the lack of experimental evidence for free quarks and the difficulty with applying the Pauli principle to some states. It was also not understood why quarks only appeared in states of the form $|qqq\rangle$ and $|q\bar{q}\rangle$.

Color

The naive quark model description of the $\Delta^{++}(1232)$ is that it is made up of three $u$ quarks, and since it is the lowest lying isospin-$\frac{3}{2}$ state then it should have $L^P = 0^+$, which strongly suggests a spatial wavefunction which is symmetric under permutation of the three $u$ quarks. Since it is a $J^P = \frac{3}{2}^+$ state its spin should be $\frac{3}{2}$,
Table I: Some low lying baryon states

<table>
<thead>
<tr>
<th>State</th>
<th>Mass(MeV)</th>
<th>Quark content</th>
<th>(J^P)</th>
<th>(2S^+1L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>938</td>
<td>(uud) (\frac{1}{2}^+)</td>
<td></td>
<td>(2S)</td>
</tr>
<tr>
<td>n</td>
<td>940</td>
<td>(udd) (\frac{1}{2}^+)</td>
<td></td>
<td>(2S)</td>
</tr>
<tr>
<td>(\Delta^{++})</td>
<td>1232</td>
<td>(uuu) (\frac{3}{2}^+)</td>
<td></td>
<td>(4S)</td>
</tr>
<tr>
<td>(\Delta^0)</td>
<td>1116</td>
<td>(uds) (\frac{1}{2}^+)</td>
<td></td>
<td>(2S)</td>
</tr>
<tr>
<td>(\Sigma^+)</td>
<td>1193</td>
<td>(uus) (\frac{1}{2}^+)</td>
<td></td>
<td>(2S)</td>
</tr>
<tr>
<td>(\Xi^0)</td>
<td>1318</td>
<td>(uss) (\frac{1}{2}^+)</td>
<td></td>
<td>(2S)</td>
</tr>
<tr>
<td>(\Omega^-)</td>
<td>1672</td>
<td>(sss) (\frac{3}{2}^+)</td>
<td></td>
<td>(4S)</td>
</tr>
<tr>
<td>(\Lambda_c^+)</td>
<td>2282</td>
<td>(udc) (\frac{1}{2}^+)</td>
<td></td>
<td>(2S)</td>
</tr>
</tbody>
</table>
which implies that its spin wavefunction is also symmetric, i.e. it has the spin/flavor wavefunction $|u_u u_u\rangle$. Then the overall space/spin/flavor wavefunction must be symmetric under the interchange of the identical quarks,

$$|\Delta^{++}\rangle = \Psi_S |u_u u_u\rangle$$

which violates the Pauli principle. This led Greenberg$^7$ and Han and Nambu$^8$ to propose that quarks had an additional three valued internal degree of freedom now called color, in which the baryon wavefunction was totally antisymmetric. The $\Delta^{++}$ now has the space/spin/flavor/color wavefunction

$$|\Delta^{++}\rangle = \epsilon_{ijk} \Psi_S |u_i u_j u_k\rangle$$

where the indices $i$, $j$ and $k$ run over the three colors red, blue and yellow. A quark therefore transformed as the fundamental representation $3$ of the group SU(3)$_c$ and an anti-quark as the $\bar{3}$. One could then account for the absence of states other than $|qqq\rangle$ and $|q\bar{q}\rangle$ by making the hypothesis that only color singlet states appeared in nature; note that both $3 \otimes \bar{3}$ and $3 \otimes 3 \otimes 3$ contain the singlet representation $1$. In support of this hypothesis, the color singlet in the $3 \otimes 3 \otimes 3$ is totally antisymmetric under the exchange of the quarks, which is exactly what was needed to explain the baryon spin statistics problem. The color degree of freedom also explained the apparent discrepancy of almost a factor of ten in the calculated and observed electrodynamic decay rate for $\pi^0 \rightarrow 2\gamma$. The calculation found the decay amplitude proportional to the number of different species of quark which could run around a loop, and with color that number is larger by a factor of three, making the decay rate larger by a factor of nine$^9$. Other experimental evidence for the existence of color is in the behaviour of the ratio

$$\frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \propto \sum \left(\frac{\xi_i}{e}\right)^2$$

where the sum runs over all flavors and colors of quarks. This ratio, when plotted as a function of the centre of mass momentum $s$ of the $e^+e^-$ pair, is seen to reach a constant below the threshold for production of a new flavor of quark, and this constant value is three times what it would be in the absence of color.

Quantum Chromodynamics

The success of the color hypothesis pointed to a tantalizing possibility; could it be that color-dependent forces between quarks$^{10}$ were responsible for binding
them together into hadrons? Yang and Mills\textsuperscript{11} had established that one could have a Lagrangian field theory of particles with such an internal SU(3) symmetry made into a local symmetry, so that one could make a local gauge theory out of the SU(3)\textsubscript{c} interactions. Since SU(3) is a non-Abelian group, i.e. its generators do not commute with each other, this theory is known as a non-Abelian gauge theory. 't Hooft and Veltmann\textsuperscript{12} and Lee and Zinn-Justin\textsuperscript{13} had subsequently shown that non-Abelian gauge theories were renormalizable. This theory is known as Quantum Chromodynamics\textsuperscript{14}, or QCD, in analogy to Quantum Electrodynamics. The forces between the colored quarks are mediated by intermediate vector bosons called gluons, which are analogous to the photon. However because of the non-Abelian nature of the theory, gluons couple to each other, unlike photons in QED.

The QCD Lagrangian is of the form

\begin{equation}
L_{QCD} = \bar{q}_f \gamma^\mu \left( i \gamma_5 \delta_{\alpha \beta} \partial_\mu + g_0 \frac{\lambda^i_{\alpha \beta}}{2} A_\mu^i \right) q_f - \frac{i}{4} F_{\mu \nu}^i F^{i \mu \nu} \tag{1}
\end{equation}

where

\begin{equation}
F_{\mu \nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g_0 f^{ijk} A_\mu^j A_\nu^k \tag{2}
\end{equation}

and the \(\lambda^i\) are the three dimensional Gell-Mann representation of the generators of SU(3)\textsubscript{c}, which satisfy the relations

\begin{equation}
\begin{pmatrix}
\lambda^i \\
\lambda^j
\end{pmatrix}
= \frac{1}{2} \epsilon_{ijk} \frac{\lambda^k}{2}
\end{equation}

\begin{equation}
\begin{pmatrix}
\frac{\lambda^i}{2} \\
\frac{\lambda^j}{2}
\end{pmatrix}
= \frac{1}{3} \delta^{ij} + d^{ijk} \frac{\lambda^k}{2},
\end{equation}

and the \(f^{ijk}\) and \(d^{ijk}\) are the structure constants of SU(3)\textsubscript{c}. The fields \(q_f\) are the different flavors of quark, and the fields \(A^i\) are the eight gluons. The term \(-\frac{1}{4} F_{\mu \nu}^i F^{i \mu \nu}\) has in it the gluon cubic and quartic vertices which lead to the gluon self coupling.

In the study of QED it is useful to define an effective coupling constant \(\alpha(Q^2)\), which gives the \(Q^2\) dependence of the renormalized vertex function. The contributions to this function all come from vacuum polarization graphs which describe the electron loop corrections to the photon propagator, with the result that the effective coupling (\(\mu^2\) is the renormalization point)

\begin{equation}
\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{\Delta S} \ln \left( \frac{Q^2}{\mu^2} \right)} + O(\alpha^2) \tag{4}
\end{equation}
increases with $Q^2$. In QCD\textsuperscript{15} this situation is complicated by the gluon self-coupling, however the leading logarithm contributions to the effective quark-gluon vertex can be summed to all orders in $g$, the renormalised coupling, by using the renormalisation group equation\textsuperscript{16}. This technique allows us to calculate the variation of the effective coupling constant $\bar{g}$ with some scale parameter $\lambda$, by solving a differential equation

$$\frac{d\bar{g}}{d(ln\lambda)} = \beta(\bar{g}) ,$$

(5)

where $\beta$ is defined by $\beta(g) \equiv \lim_{\Lambda \to \infty} \frac{\partial}{\partial \mu} g(g_0, \Lambda/\mu)$ ($\Lambda$ is a high momentum cut-off used to regulate divergent loop integrals).

Now providing there exists a renormalisation point $\mu^2$ for which $\alpha_s \equiv g^2 / 4\pi \ll 1$, then we can evaluate $\beta(g)$ perturbatively from the Feynman diagrams contributing in lowest non-trivial order to the basic Green functions of the theory. The result is (see, for example, Ref.16)

$$\beta(g) = -\frac{1}{16\pi^2} \left( \frac{33 - 2N_f}{3} \right) g^3$$

(6)

We see that $\beta(g)$ is negative if $N_f$, the number of flavors of quark, is less than sixteen. The negative term in (6) is entirely due to the three-gluon vertex contributions to the Green functions. Solving (5) for $\bar{g}$ yields

$$\bar{g}^2(ln\lambda) = \frac{g^2}{1 + \frac{1}{24\pi^2} (33 - 2N_f) g^2 ln\lambda}$$

(7)

If we choose as our scale parameter $\sqrt{Q^2/\mu^2}$, we have

$$\bar{g}^2(Q^2) = \frac{g^2(\mu^2)}{1 + \frac{1}{48\pi^2} (33 - 2N_f) \bar{g}^2(\mu^2) ln(Q^2/\mu^2)}$$

$$= \frac{1}{48\pi^2} (33 - 2N_f) ln(Q^2/\Lambda_{QCD}^2) ,$$

(8)

where $\Lambda_{QCD}^2 \equiv \frac{\mu^2 e^{-48\pi^2}/(33 - 2N_f) \bar{g}^2(\mu^2)}{ln(Q^2/\Lambda_{QCD}^2)}$. In terms of $\alpha_s$ this gives

$$\alpha_s(Q^2) \equiv \frac{\bar{g}^2(Q^2)}{4\pi} = \frac{12\pi}{(33 - 2N_f) ln(Q^2/\Lambda_{QCD}^2)}$$

(9)

QCD is now entirely defined if we know $\Lambda_{QCD}$, which we can find from (9) by performing an experiment and measuring $\alpha_s$ and $Q^2$. It is generally accepted that $\Lambda_{QCD}$ has a value $\approx 200$ $MeV$. 

6
We see from (9) that at large $Q^2$, or at short distances, the effective quark gluon coupling goes to zero. This property of QCD is called asymptotic freedom, and it implies that at short distance, or in processes with large $Q^2$, the quarks behave as free particles. This has been verified by the approximate Bjorken scaling observed in deep inelastic scattering of electrons from the quarks in hadrons. Also as $Q^2 \to 0$ the running coupling constant diverges logarithmically. This signals that the perturbation theory used to get this result has broken down, and that at low $Q^2$ the theory becomes strongly interacting, a feature which is described as "infrared slavery". This suggests a possible explanation for the idea that quarks only appear in color singlets; colored quarks may be confined in this configuration by a strong interaction which grows stronger at larger distances. It is one of the outstanding problems of modern theoretical physics that this non-perturbative property of QCD has not yet been established.\(^{17}\) Numerical studies of QCD on a lattice\(^ {18}\) have, however, demonstrated that static quarks have a linearly rising potential between them at large distance of the form $\sqrt{\sigma r}$, and that in the strong coupling limit QCD has a confining phase. A consequence of our inability to calculate with QCD in the low $Q^2$ regime is that it makes it necessary to use phenomenological models of hadron structure based on our expectations of the way low energy QCD behaves.

**Soft QCD**

In particular this means that we must impose confinement on our low energy theory by assuming a confining potential between quarks, or by only allowing quarks to move within a cavity in space filled with perturbative vacuum, as in the MIT bag model.\(^ {19}\) Soon after a fourth quark called charm was discovered\(^ {20}\) (its existence had been predicted earlier by Glashow, Iliopoulos and Maiani\(^ {21}\) as necessary to remove difficulties in the theory of the weak interactions of quarks) in the $J/\psi$ particle in 1974, calculations of the energy spectrum of the $c\bar{c}$ system in a confining potential with a color Coulomb interaction at short distance were made.\(^ {22}\) The ordinary Schrödinger equation was solved with this potential, in much the same way as the spectrum of positronium had been derived. The justification for using a non-relativistic calculation was that the large mass of the charmed quark would compensate for the much stronger binding of the color forces. This approach gained acceptance as the spectrum of the $c\bar{c}$ system was explored and these calculations were verified. de Rujula, Georgi and Glashow\(^ {23}\) took this idea further by suggesting that the short distance (or large $Q^2$) behaviour would be dominated by perturbative one gluon exchange between the quarks. This would lead to a short distance
interaction between quarks analogous to the Breit-Fermi interaction used to find the effective potential between the electron and positron in positronium. The resulting potential had spin-spin, spin-orbit and color Coulomb terms, and its spin and mass dependences were used to give mass relations between the hadrons. A similar model of the short distance interactions was proposed by De Grand, Jaffe, Johnson and Kiskis in the MIT bag model.
1.2 BARYON SPECTROSCOPY

Dalitz\textsuperscript{25} and Greenberg\textsuperscript{26} and their collaborators had calculated the masses of non-strange negative parity baryons in a potential model with Breit-Fermi type terms as early as 1964, but their work differed from that of de Rujula, Georgi and Glashow and the MIT group in that the relationship between the strengths of these terms was not understood on the basis of a non-relativistic reduction of the one-gluon exchange amplitude.

Isgur and Karl\textsuperscript{27} put these two approaches to hadron spectroscopy together. The spectrum of the low lying negative parity baryons\textsuperscript{28,29} was investigated by solving the three body problem in a harmonic long range confining potential with a one-gluon exchange potential at short range. Both the contact and tensor parts of the spin-spin forces were included, but the spin-orbit forces were left out, contrary to Dalitz and Greenberg's emphasis. The zero-th order wavefunctions used were only symmetrized in the equal mass u and d quarks and not between u, d and s quarks\textsuperscript{30}, which made explicit the breaking\textsuperscript{31} of SU(3)\textsubscript{f} in the strange baryons before any mixings were considered. The mass dependence of the Hamiltonian was solely responsible for breakings of SU(3)\textsubscript{f}. The results for the masses of the negative parity baryons were in impressive agreement with experiment, and the mixings in particular explained an empirical proposal made earlier by Petersen and Rosner\textsuperscript{32} and Faiman\textsuperscript{33} about the decoupling of some states from the K\bar{N} channel.

The success of this analysis answered, and led to, some important questions. It verified that the short distance behaviour of quarks in baryons is dominated by one-gluon exchange effects. It also established that the confining potential between a pair of quarks is independent of their flavors and therefore of their masses. But why were spin-spin forces so important and spin-orbit forces negligible, when the Breit-Fermi reduction yielded roughly equal strengths for these interactions? Was the value of \(\alpha_s\) found to best reproduce the data consistent with other independent determinations? A partial answer to the former question was given by Isgur and Karl\textsuperscript{34} by noting that one-gluon exchange was not the only source of spin-orbit coupling, but it could also arise from the Thomas precession of the quark spins in the confining potential. They showed that these two contributions came in with the opposite sign and that it was possible to have almost complete cancellation of the two-body spin-orbit terms. However, certain "three-body" terms, whose origin was not completely understood at the time\textsuperscript{35}, did not cancel.)
Isogur and Karl\textsuperscript{36} then went on to extend their analysis to the low lying positive parity baryons\textsuperscript{37}. In doing so they introduced non-harmonic terms into their potential to correspond to the expected Coulomb plus linear nature of confinement. The masses and mixings of a large number of ground and excited states\textsuperscript{38} were confronted with the data in a largely successful application of their earlier ideas. In particular the model was able to predict the masses of the first few states for every flavor of baryon and for every total J, and also the predicted mixings were able to account for the $\bar{K}N$ channel decouplings and the absence of some states in photoproduction noted earlier\textsuperscript{39} in experiments. The model was also successfully applied to the charmed baryons $\Lambda_c$ and $\Sigma_c$ by Copley, Isogur and Karl\textsuperscript{40}, to the strangeness $-2$ baryons $\Xi$ and the strangeness $-3$ baryons $\Omega$ by Chao, Isogur and Karl\textsuperscript{41}, as well as to the ground state baryon magnetic moments\textsuperscript{42} and to the neutron charge form factor\textsuperscript{43} by Isogur, Karl and Sprung. In their paper on D waves in the nucleon, Isogur, Karl and Koniuk\textsuperscript{44} demonstrated that violations of the Becchi-Morpurgo\textsuperscript{45} selection rule (that the decay $\Delta \rightarrow N\gamma$ is pure $M1$) result from tensor mixing of D waves into the nucleon and $\Delta$, confirming the validity of the one-gluon exchange picture of the short range interactions in hadrons.

Although the Isogur-Karl model showed by this point an impressive agreement with the data on the low lying states which could be uniquely identified with states in these analyses, a study of the couplings of these states to various formation and decay channels was needed to sort out the many excited states predicted to exist by the model. Koniuk and Isogur\textsuperscript{46} carried out an extensive analysis of the couplings of all baryons, which demonstrated that the 'missing' resonances were those which this model predicted to decouple from the partial wave analyses, as in the case of $\bar{K}N$ and photoproduction above, and that the masses and decay rates of those states which did couple agreed well with the data.

Despite all of the successes of this model, a few important questions remained, beyond the obvious problem with the apparent lack of spin-orbit forces. These analyses had been performed within first order perturbation theory, with the result that the first order splittings within a given oscillator level were comparable to the level splittings themselves\textsuperscript{47}, which suggests that one should solve the Schrödinger equation non-perturbatively. Another possible problem was that the value of $\alpha_s$ needed to produce these splittings seemed quite large compared to the value determined by applying perturbative QCD to scattering processes at large momentum transfer. Finally there is the worry that these calculations are non-relativistic when one expects relativistic behaviour in a system as strongly bound as the baryons.
J.3 BEYOND THE ISGUR-KARL MODEL

Since the introduction of the Isgur-Karl model various authors have attempted to refine the model by using potentials closer to that expected from QCD, and to go beyond first order perturbation theory. Böhm calculated the masses of the non-strange baryon resonances from a quark model with a linear plus Coulomb spin-independent potential (with the non-relativistic form of the kinetic energy) and a one-gluon exchange hyperfine interaction. He treated the non-relativistic problem with the aid of a variational method. This was accomplished by adjusting the oscillator constants of each harmonic oscillator band to minimize the expectation value of the spin-independent potential. This was then fitted to the centre of gravity of the bands by adjusting the slope and intercept of the linear potential and the strong coupling constant. However the resulting best fit to the spectrum had a large \( \alpha_s \) (2.0), and a very small string tension \( (0.02 \text{ GeV}^2) \).

The resulting \( \alpha_s \) was then used to determine a spin-spin contact and tensor interaction which was then diagonalised within the harmonic oscillator supermultiplets. One further parameter, the constituent quark mass, was used in the prediction of the non-strange baryon masses. This procedure obtained good fits to the excited \([70, 1^-]\) and \([56, 0^+]\) multiplets. However the \( \Delta - N \) mass difference was predicted to be twice as big as it should be. The author suggests that this is because relativistic effects in the ground states cannot be treated by perturbation theory (since they are so small they have a larger \( \frac{E}{m} \); We feel that the large value of \( \alpha_s \) used may also have contributed to this problem.

Richard and Taxis analysed the properties of the ground state baryons in the non-relativistic quark model with various two-body potentials of the power law type, and the three-body \( \Upsilon \) shaped string potential, and solved the three-body problem by the hyper-spherical expansion. They arrived at the following conclusions: they found that the quark-diquark picture was not supported by the data (or necessary); smooth confinement with a power law potential \( r^\beta \) with \( \beta \leq 1 \) was preferred to harmonic confinement; the \( \Upsilon \) shaped potential yielded almost the same results as a pairwise linear one; and finally they found hyperfine splittings analogous to Breit-Fermi ones, but with a large \( \alpha_s \), as other authors have found, which is difficult to reconcile with the charmonium value.

Isigur, and later Bhaduri, Cohler and Nogami, pointed out that if you solve (exactly) the Schrödinger equation with the zero range attractive contact potential
the system will collapse, with no lower bound in energy, and so in a non-perturbative calculation one should replace the delta function contact interaction with some short range function. They also pointed out that one cannot use perturbation theory for such a short range interaction and expect to get accurate results. Stanley and Robson \textsuperscript{61} applied this idea in an attempt to unify meson and baryon spectroscopy by extending their meson analysis to baryons. Their Hamiltonian had half of the meson pairwise linear potentials, and half of the one-gluon exchange potentials, as expected from the different expectation values of $\vec{A}_i \cdot \vec{A}_j$ in the meson and baryon colour wavefunctions. They used smeared quarks (with Yukawa-type form factors) to avoid singularities due to the contact interaction, which was necessary since they performed a variational calculation of the baryon masses by expanding the baryon wavefunction in a large harmonic oscillator basis. They calculated only the ground state baryon masses, and so they ignored tensor and spin-orbit terms, which are expected to have small expectation values in the ground states.

The analysis of baryon resonances within the context of the Isgur-Karl model was extended to states in the $N=3$ harmonic oscillator band by Corvi\textsuperscript{62}, who examined the masses and compositions of the $\Delta^{*\frac{3}{2}+}$, $\Delta^{*\frac{5}{2}+}$ and $\Delta^{*\frac{1}{2}-}$ resonances, and later by Forsyth and Cutkosky\textsuperscript{63}, who calculated the masses and widths of the $S = 0$ baryons in the $[70,1^-]$ and $[56,1^-]$ multiplets. Forsyth and Cutkosky examined three models for the decay of these and other states, and found that the usual spectator model for the decays had to be modified. They found no positive evidence for a tensor force, and conflicting evidence for a three-body spin-orbit force. They also found that the experimental $[56,1^-]$ band masses were lower than predicted.

In a later paper\textsuperscript{64} Forsyth and Cutkosky extended their calculation to include all states in the $N=0$, 1, 2 and 3 harmonic oscillator bands. The resulting picture of baryon spectroscopy and decays led to similar conclusions as above, and they also found many of the resonances in the $N=2$ and $N=3$ bands decouple, supporting the Koniuk and Isgur picture. Within their model they found evidence that the contact interaction strength varies with band.

Carlson, Kogut and Pandharipande\textsuperscript{65}, in the first of two papers dating from 1983, calculated the spectrum of $S = 0$ baryons in a semi-relativistic (i.e. they used the relativistic form of the kinetic energy) quark model with a $Y$ shaped string confining potential\textsuperscript{49,66} with a conventional value of string tension ($1$ GeV/fm), and a color-Coulomb potential with an $\alpha_s$ of 0.375 consistent with charmonium spectroscopy. They solved the three-body problem using a variational method for the spin-averaged baryon masses up to the $N = 2$ band.
In their second paper they put colour magnetic interactions into above model, and calculated meson and $S = 0$ baryon masses simultaneously. The $\rho - \pi$ splitting was fit to determine the strength of the contact interaction and this was then used to predict the $N - \Delta$ splitting. They had problems with the spin-orbit splittings in the P-wave baryons which were too large; this was probably the result of ignoring the scalar spin-orbit forces (resulting from Thomas-precession of the quarks in the confining potential). These authors used a factor of \( \frac{1}{(E_i)(E_j)} \) multiplying the spin-orbit interaction between quarks \( i \) and \( j \), noting that for light quarks this is the natural replacement for the factor of \( \frac{1}{m_i m_j} \) of the usual Breit-Fermi reduction of the one-gluon exchange amplitude. They also used a form factor for the quark-gluon vertex (which has the effect of regulating the contact interaction and the \( \frac{1}{\mu^2} \) spin-orbit potential). The masses of radial excitations of mesons and baryons in this calculation were found to be in error by 100-150 MeV. For example, the mass of the Roper resonance \( N(1440) \) was predicted 130 MeV too high; this discrepancy is, however, much less of a problem than in conventional non-relativistic quark models. The authors suggested that this might be because their string tension of 1 GeV/fm was too high. (Bhaduri, Jennings and Waddington, and Murthy, Dey, Dey and Bhaduri explain the low position of the Roper resonance by taking the \( N \) and \( \Delta \) spectrum as the rotational spectrum of a deformed state on top of a harmonic oscillator spectrum).

Carlson, Kogut and Pandharipande also found good evidence for the one gluon exchange tensor interaction in both mesons and baryons. The charge radii of the states predicted by their model were found to be too small; the authors suggest that in fact experiment may not measure the size of the valence quark distribution, but that of the pion clouds surrounding the valence quarks. They found, as expected, that their fitted (constant) \( \alpha_s \) decreased as masses of quarks increases. Also the radius of their quark-gluon form factor decreased as the quark mass increased, as we might expect from the uncertainty principle, and from the mass dependences of the Feynman graphs for quarks coupling to gluons.

This approach was also adopted later by Sartor and Stancu for the non-strange baryon masses in the \( N=0, 1 \) and 2 harmonic oscillator bands.

The above analyses had gone beyond the harmonic oscillator potential (at least for the \( S=0 \) baryons), had dealt with the problems of applying first order perturbation theory and had made an attempt at introducing relativistic corrections into the model. However none had provided a satisfactory explanation of the mysterious lack of spin-orbit interactions in baryons. Gromes defined the spin-orbit problem
in baryons\textsuperscript{34}; he proposed to solve this problem by allowing non-local terms in the
confining potential. These introduce further spin-orbit terms into the Hamiltonian
which sufficiently suppressed the spin-orbit effects in the P-wave baryons, and im-
proved the splitting of the singlet \(\Delta\) resonances. However this splitting was still
too small, putting the \(\Lambda^{*}\frac{1}{2}^{-}\) at just below threshold for decay to \(\bar{K}N\), and Gromes
points out, following Isgur and Kar\textsuperscript{28,29}, that it is reasonable to assume that virtual
mixing of the \(\bar{K}N\) channel into this \(\Lambda\) provides the rest of the splitting.

Our Analysis

In our analysis we calculate the masses of all baryons using a Hamiltonian
calculated from a Lorentz vector one-gluon exchange short range interaction and
the \(Y\) shaped string-picture confining potential. All of the one-gluon terms are
kept, including spin-orbit effects, and spin-orbit effects from Thomas precession in
the confining potential are also included. Some of the effects of higher order in
\(\alpha_s\) corrections are taken into account by using a ‘running’ strong coupling con-
stant. Furthermore, the exact vector exchange T-matrix element is used to suggest
relativistic corrections to the one-gluon exchange potential. A similar analysis is
performed for the confining potential (which Gromes\textsuperscript{61} has shown is equivalent to
Lorentz-scalar exchange) to suggest relativistic corrections to the confining potential
spin-orbit interaction. Finally we use the relativistic expression for the quark kinetic
energies. This ‘relativized Schrödinger equation’ is then solved non-perturbatively
for the masses of all baryons by expanding the wavefunction in a complete set of
basis states for the bound state three-body problem. This model of quark-quark
interactions in baryons is the essentially unique generalisation of a similar model of
mesons due to Godfrey and Isgur\textsuperscript{82}.
CHAPTER II: THE BARYON HAMILTONIAN

The Hamiltonian that we use for the baryon system is of the form

$$H = H_0 + V_{oge} + V_{conf},$$  \hspace{1cm} (10)

where $H_0$ is a fully relativistic kinetic energy term,

$$H_0 = \sum_{i=1}^{3} \sqrt{p_i^2 + m_i^2},$$  \hspace{1cm} (11)

$V_{oge}$ is a one gluon exchange potential which in the non-relativistic limit goes to the usual Breit-Fermi interaction, and $V_{conf}$ consists of a string picture potential and a spin-orbit term arising from this potential via Thomas precession. In this Chapter we will begin by explaining the origin of these terms in detail. In particular the origin of the terms in the one-gluon exchange potential will be used to introduce the prescription for making relativistic corrections to the usual non-relativistic baryon Hamiltonian.

We will then go on to outline the details of the baryon wavefunctions, the variational calculation and the procedure used to give estimates of the baryon masses; in Appendix A we will outline the techniques used to reduce matrix elements of the various terms in the Hamiltonian to simple integrals.
II.1 THE ONE-GLUON EXCHANGE EFFECTIVE POTENTIAL

Philosophy

We will begin our discussion of the one-gluon exchange terms in the Hamiltonian by outlining in detail how one arrives at the usual Breit-Fermi effective potential for quarks. We will not, however, make the usual non relativistic reduction of the $T$-matrix element, in order to get a feeling for the kinds of relativistic corrections that one expects to see in the baryon Hamiltonian. We would like to stress at this point that we do not expect that the one-gluon exchange approximation will yield the exact form of these corrections, even if we could deal exactly with the (complicated) momentum dependences which arise, for reasons that will become apparent below. We will instead simply model the relativistic corrections to the Breit-Fermi Hamiltonian with what we feel is a natural ansatz based on the exact $T$-matrix element.

If we make the (incorrect) assumption that one-gluon exchange will yield the exact effective potential for quarks in baryons, how do we extract the effective potential? We follow Berestetskii, Lifshitz and Pitaevskiï\textsuperscript{63} and Barnes and Ghandour\textsuperscript{64}, i.e. we calculate the effective quark-quark potential by starting from the one-gluon exchange Feynman diagram evaluated in a frame where $Q^0 = 0$, where $Q$ is the four-momentum transfer $(Q^0, \vec{Q})$, so that the resulting potential is instantaneous. The absence of $Q^0$ in the expression for the Feynman diagram avoids ambiguities which would otherwise arise on Fourier transforming to find the potential. Evaluating in this frame also allows for a natural way to define the ordering of operators in the potential itself. We define a potential operator in position space to be that function $U_{op}$ of $\vec{r}_1, \vec{r}_2, \vec{r}_1, \vec{r}_2$ and $\vec{r}_2$ which yields the correct momentum space scattering amplitude when inserted between non-relativistic states,

$$S_{fi} = -i(2\pi)^4 \delta^4(p_f - p_i)T_{fi}$$

$$= -2\pi i \delta(E_f - E_i) \int d^3 \vec{x}_1 \, \int d^3 \vec{x}_2 e^{-i(\vec{p}_1 \cdot \vec{x}_1 + \vec{p}_2 \cdot \vec{x}_2)}$$

$$\chi_{s_1}^{\dagger} \chi_{s_2}^{\dagger} U_{op}(\vec{r}_{12}, \vec{r}_1, \vec{r}_2, \vec{r}_2) \chi_{s_1} \chi_{s_2} e^{i(\vec{p}_1 \cdot \vec{x}_1 + \vec{p}_2 \cdot \vec{x}_2)},$$  \hfill (12)

where $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$, and the $\chi$'s are the Pauli spinors representing the spin wavefunctions of the incoming and outgoing quarks. $T_{fi}$ is to be evaluated in the frame where the interaction is instantaneous, that is where $Q^0 = 0$, and is to be calculated in terms of $\vec{P}_1 = \frac{1}{2}(\vec{p}_1 + \vec{p}_1), \vec{P}_2 = \frac{1}{2}(\vec{p}_2 + \vec{p}_2)$, and $\vec{Q} = (\vec{p}_1 - \vec{p}_1) = (\vec{p}_2 - \vec{p}_2)$; then

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the order of the derivatives and functions of \( \tilde{r}_{12} \) are unambiguous. If we inverse Fourier transform (12) we find that

\[
\chi_{s_2}^\dagger \chi_{s_1}^\dagger U_{op}(\tilde{r}_{12}, \vec{\nabla}_1, \vec{\nabla}_2, \vec{\nabla}_2) \chi_{s_1} \chi_{s_2} = \frac{1}{(2\pi)^3} \int d^3 \bar{Q} e^{i \vec{Q} \cdot \tilde{r}_{12}} T_{fi} \tag{13}
\]

so that, for example, a term such as \( \vec{F}_1 \cdot \vec{F}(\bar{Q}) \) appearing in \( T_{fi} \) would be interpreted as the operator

\[
\frac{1}{2} (\vec{p}_1' + \vec{p}_1) \cdot \vec{F}(\bar{Q}) \to \vec{F}(\tilde{r}_{12}) \cdot \left( -\frac{i}{2} \vec{\nabla}_1 \right) + \left( \frac{i}{2} \vec{\nabla}_1 \right) \cdot \vec{F}(\tilde{r}_{12}), \tag{14}
\]

in the potential, where \( \vec{F}(\tilde{r}_{12}) \) and \( \vec{F}(\bar{Q}) \) are Fourier transforms of each other.

**The Exact One-Gluon Exchange T-Matrix Element**

The exact one-gluon exchange T-matrix element in the \( Q^0 = 0 \) frame is given by

\[
T_{fi} = \frac{m_1 m_2}{E_1 E_2} \bar{u}(\vec{p}_1', s'_1) \gamma^\mu u(\vec{p}_1, s_1) g_{\mu\nu} G(Q^2) \bar{u}(\vec{p}_2', s'_2) \gamma^\nu u(\vec{p}_2, s_2), \tag{15}
\]

where, in this frame, \( G(Q^2) \) (with \( Q^2 = \bar{Q} \cdot \bar{Q} \)) is the 'dressed' gluon propagator, and we have that \( E_1 = E'_1 \) and \( E_2 = E'_2 \). If we expand the four-spinors and the \( \gamma \) matrices in terms of large and small components we find

\[
\bar{u}(\vec{p}_1', s'_1) \gamma^\mu u(\vec{p}_1, s_1) = \frac{E_1 + m_1}{2m_1} \left[ \chi_{s_1}^\dagger \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right] \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{c} \bar{\sigma} \\ \bar{\sigma}_1 \cdot \vec{p}_1 \end{array} \right]. \tag{16}
\]

This reduces (after some algebra) to

\[
\bar{u}(\vec{p}_1', s'_1) \gamma^\mu u(\vec{p}_1, s_1) = \chi_{s_1}^\dagger \left[ \frac{E_1}{m_1} - \frac{Q^2}{4m_1(E_1 + m_1)} + \frac{i}{2m_1(E_1 + m_1)} \left( \vec{p}'_1 \times \vec{p}_1 \right) \cdot \vec{\sigma}_1 \right] \left( \frac{\vec{p}_1 + \vec{p}'_1}{2m_1} - i \bar{Q} \times \vec{\sigma}_1 \right) \chi_{s_1}, \tag{17}
\]

and if we write this entirely in terms of \( \vec{P}_1 \) and \( \bar{Q} \) we find

\[
\frac{m_1}{E_1} \bar{u}(\vec{p}_1', s'_1) \gamma^\mu u(\vec{p}_1, s_1) = \chi_{s_1}^\dagger \left[ 1 - \frac{Q^2/2 + i(\bar{Q} \times \vec{P}_1) \cdot \vec{\sigma}_1}{2\bar{Q} \cdot \vec{P}_1 - i \bar{Q} \times \vec{\sigma}_1} \right] \chi_{s_1}. \tag{18}
\]
where we have made the identification
\[ \mathcal{E}_1 = \left( \mathcal{P}_d^2 + Q^2/4 + m_1^2 \right)^{\frac{1}{2}}. \]  
(19)

A similar equation holds for \( \frac{m_2}{E_2} \psi(\mathcal{P}_e^2, s_2) \gamma^\nu u(\mathcal{P}_2, s_2) \) with \( 1 \to 2 \) and \( \mathcal{Q} \to -\mathcal{Q} \), and (15) becomes
\[
T_f = \chi_{s_2}^+ \chi_{s_1}^+ \left\{ 1 + \frac{-Q^2/2 + i(\mathcal{Q} \times \mathcal{P}_1) \cdot \mathcal{\bar{\sigma}}_1}{2 \mathcal{E}_1(\mathcal{E}_1 + m_1)} \right\} G(Q^2) \left[ 1 + \frac{-Q^2/2 - i(\mathcal{Q} \times \mathcal{P}_2) \cdot \mathcal{\bar{\sigma}}_2}{2 \mathcal{E}_2(\mathcal{E}_2 + m_2)} \right] 
- \left( \frac{2 \mathcal{P}_1 - i \mathcal{Q} \times \mathcal{\bar{\sigma}}_1}{2 \mathcal{E}_1} \right) G(Q^2) \left( \frac{2 \mathcal{P}_2 + i \mathcal{Q} \times \mathcal{\bar{\sigma}}_2}{2 \mathcal{E}_2} \right) \right\} \chi_{s_1} \chi_{s_2}. \]  
(20)

**Implications for the Inter-Quark Potential**

The Fourier transform (13) of the non-relativistic limit (keeping terms up to order \((v/c)^2\)) of (20) is the Breit-Fermi interaction between the quarks which was used in the original treatment of the baryon spectrum. As it stands (20) is the correct T-matrix element for on-shell scattering of (free) quarks by exchange of a single gluon. However there are various reasons to not take this expression as it stands and use it to 'derive' a fully relativistic one-gluon exchange effective potential for the quarks in a baryon. In such a strongly bound system as baryons we can expect further relativistic momentum dependence due to the quarks being off-shell. We also know that in general we have states with gluonic excitations and \( q \bar{q} \) pairs mixing into the baryons, and we are neglecting such configurations and their effect on the potential. Furthermore, even though the above prescription gives a unique way to define the effective potential in terms of derivatives acting on the quark coordinates, we can see that it would yield an extremely complicated potential even for the simplest of terms, due to its non-linear dependences on \( \mathcal{P}_1^2, \mathcal{P}_2^2 \) and \( Q^2 \). So we adopt a more realistic approach and use the above as a guide to the kinds of relativistic effects we expect to see in the potential.

We first note that the inter-quark potentials have \( Q^2 \) dependence not only through the gluon propagator but also from the spinors. When we carry out the Fourier transform (13) this will lead to spatial dependence of the potential arising from sources other than the dressed propagator. Examine, for example, the spin-independent term (the analogue of the Coulomb potential)
\[
\chi_{s_2}^+ \chi_{s_1}^+ \chi_{s_1} \chi_{s_2} G(Q^2) \left\{ 1 - \frac{Q^2}{4 \mathcal{E}_1(\mathcal{E}_1 + m_1)} \right\} \left\{ 1 - \frac{Q^2}{4 \mathcal{E}_2(\mathcal{E}_2 + m_2)} \right\} - \frac{\mathcal{P}_1 \cdot \mathcal{P}_2}{\mathcal{E}_1 \mathcal{E}_2}, \]  
(21)
which has explicit $Q^2$ dependence (including that of the $\mathcal{E}$ factors) outside of the propagator $G$. This $Q^2$ dependence is only important when $|\vec{Q}| \approx O(m_q)$. It changes the shape of the potential resulting from Fourier transforming (21), smearing the potential out over distances of order $(1/m_q)$.

There is also another way in which our Hamiltonian can be expected to depend on $Q^2$. In QCD the mass of a quark will run as a function of $Q^2$, in analogy to the $Q^2$ dependence of the coupling constant which we have examined in Chapter I. This is an example of scale dependence, which is a general property of any interacting field theory which requires an ultraviolet cut-off. Indeed our picture of baryons, that of three quarks bound together by a string-like tube of chromoelectric flux (see below in Section II.4), breaks down if one chooses to describe it on a fine distance scale (which corresponds to using a large momentum cut-off). On such a scale the quarks, through the dependence of their mass on $Q^2$, become light 'current' quarks so that the vacuum inside a baryon is full of $q\bar{q}$ pairs. The flux tube can pick up complicated topologies with branching flux lines, and the vacuum may contain disconnected fluctuations of localized flux. The quark model is based on choosing the largest distance scale which can still accurately describe a hadron, which for the light quarks is about 0.1 fm.

On this coarse scale we can expect our (light) quarks to behave like point particles only outside of a radius of the order of 0.1 fm. The effective mass of this 'constituent' quark is the current mass plus the nearby gluonic energy, about 200 $MeV$, which is the mass we adopt for our light quarks (see Table III in Chapter IV). A system which consists of only heavy quarks has a size of the order of $1/\alpha_s m_q$, and can be cut-off at a distance scale of order $1/m_q$. Therefore in the $m_q \to \infty$ limit the effective mass of a quark will approach its current quark mass.

As a result of this scale dependence we should give the constituent quarks a form factor, with a radius of about 0.1 fm for the light quarks and of $O(1/m_q)$ for the heavy quarks. This corresponds to smearing our potentials, or equivalently giving them further $Q^2$ dependence. We will for simplicity assume that all of the $Q^2$ dependence of the potentials can be absorbed into a universal (i.e. the same for all of the two-body potentials) smearing function

$$\rho_{12}(Q^2) = \frac{1}{\pi^{\frac{3}{2}} \sigma_{12}^2} e^{-Q^2/\sigma_{12}^2}$$

(22)

which will multiply the propagator $G(Q^2)$, so that

$$\tilde{G}(Q^2) = \rho_{12}(Q^2) G(Q^2)$$

(23)
The smearing parameters $\sigma_{ij}$ are, in terms of the masses of the quarks,

$$\sigma_{ij}^2 = \sigma_0^2 \left[ \frac{1}{2} + \frac{1}{2} \left( \frac{4m_im_j}{(m_i + m_j)^2} \right)^4 \right] + s^2 \left( \frac{2m_im_j}{m_i + m_j} \right)^2,$$

(24)

(where $\sigma_0$ and $s$ are the smearing parameters given in Table III). A simpler form would have sufficed for baryons, but the meson spectrum is more sensitive to the form of the mass dependence of the $\sigma_{ij}$ than the baryon spectrum, and so we adopt this form for consistency with the Godfrey and Isgur analysis of the meson spectrum and decays. Note that this makes explicit the scale dependence of the quark form factor noted above, i.e. when $m = m_i = m_j \to \infty$, $\sigma_{ij}$ is proportional to $m$, and when $m \to 0$, $\sigma_{ij}$ goes to a constant. If $G(r_{12})$ is the Fourier transform of the propagator $G(Q^2)$, then the spatial potential corresponding to $\tilde{G}(Q^2)$ is

$$\tilde{G}(r_{12}) = \int d^3\tilde{r}' \rho_{12}(\tilde{r}_{12} - \tilde{r}') G(r'),$$

(25)

where

$$\rho_{12}(\tilde{r}_{12} - \tilde{r}') = \frac{\sigma_{12}^3}{\pi^2} e^{-\sigma_{12}^2(\tilde{r}_{12} - \tilde{r}')^2}.$$

(26)

We see from (20) that, apart from extra dependence on $Q^2$, we should also expect dependence of the terms in our one-gluon exchange Hamiltonian on the momenta of the quarks. Since we intend to use the same smearing prescription (23) to represent all of the extra $Q^2$ dependence of our one-gluon exchange terms, we examine (20) in the $Q^2 \to 0$ limit. The Coulomb term (21) becomes

$$\chi_{s_2}^\dagger \chi_{s_1}^\dagger \chi_{s_1} \chi_{s_2} \tilde{G}(Q^2) \left\{ 1 - \frac{\vec{F}_1 \cdot \vec{F}_2}{(\vec{P}_1^2 + m_1^2)^{\frac{1}{2}} (\vec{P}_2^2 + m_2^2)^{\frac{1}{2}}} \right\},$$

(27)

and we see that in this limit there is a relativistic factor which modifies the strength of the interaction, which depends on the energies of the quarks.

The same is roughly true of the other terms in (20) in the $Q^2 \to 0$ limit; the one-gluon exchange terms become modified by strength factors like $\frac{1}{E_1 E_2}$, which in the non-relativistic limit go over to the $\frac{1}{m_1 m_2}$ factors of the Breit-Fermi interaction. There is a term which yields the contact and tensor parts of the hyperfine interaction

$$-\chi_{s_2}^\dagger \chi_{s_1}^\dagger \left( \frac{m_1 m_2}{E_1 E_2} \right) \frac{\vec{Q} \times \vec{S}_1}{m_1} \vec{Q} \times \vec{S}_2 \frac{\tilde{G}(Q^2)}{m_2} \chi_{s_1} \chi_{s_2},$$

(28)

where $\vec{S}_i = \frac{1}{2} \vec{s}_i$, and where $E_i$ is the $Q^2 \to 0$ limit of (19), $(\vec{P}_i^2 + m_i^2)^{\frac{1}{2}}$. Note that if we carry out the Fourier transform (13) above we may write this interaction in terms of derivatives of the spatial potential (without reference to $\tilde{Q}$);

$$\chi_{s_2}^\dagger \chi_{s_1}^\dagger \left( \frac{m_1 m_2}{E_1 E_2} \right) (\delta_{rs} \delta_{uv} - \delta_{ru} \delta_{sv}) \frac{S_{1r} S_{2s}}{m_1 m_2} \nabla_u \nabla_v \tilde{G}(r_{12}) \chi_{s_1} \chi_{s_2}.$$

(29)
There is also a group of terms which are spin-orbit interactions,

\[
\chi_{s_2}^\dagger \chi_{s_1}^\dagger \left\{ \left( \frac{2m_1^2}{E_1(E_1 + m_1)} \right) \frac{i(\vec{Q} \times \vec{P}_1) \cdot \vec{S}_1}{2m_1^2} - \left( \frac{2m_2^2}{E_2(E_2 + m_2)} \right) \frac{i(\vec{Q} \times \vec{P}_2) \cdot \vec{S}_2}{2m_2^2} \right. \\
- \left( \frac{m_1m_2}{E_1E_2} \right) \left( \frac{i(\vec{Q} \times \vec{P}_1) \cdot \vec{S}_1 - i(\vec{Q} \times \vec{P}_1) \cdot \vec{S}_2}{m_1m_2} \right) \right\} \tilde{G}(Q^2) \chi_{s_1} \chi_{s_2} ,
\]

where we have written these terms in a symmetric manner with the aid of the identity \((\vec{Q} \times \vec{S}_1) \cdot \vec{P}_2 = -(\vec{Q} \times \vec{P}_2) \cdot \vec{S}_1\), and a similar identity for the last term. Note that the momentum dependent factors all go to unity in the non-relativistic limit \(p_i/m_i \to 0\). Note also that in this method the Thomas precession terms, which provide the famous factor of \(\frac{1}{2}\) reduction in the first two terms of (30), emerge automatically. Carrying out the Fourier transform (13) yields

\[
\chi_{s_2}^\dagger \chi_{s_1}^\dagger \frac{1}{r_{12}} \frac{d\tilde{G}(r_{12})}{dr_{12}} \left\{ \left( \frac{2m_1^2}{E_1(E_1 + m_1)} \right) \frac{(\vec{r}_{12} \times \vec{P}_1) \cdot \vec{S}_1}{2m_1^2} \right. \\
- \left( \frac{2m_2^2}{E_2(E_2 + m_2)} \right) \frac{(\vec{r}_{12} \times \vec{P}_2) \cdot \vec{S}_2}{2m_2^2} \\
- \left( \frac{m_1m_2}{E_1E_2} \right) \left( \frac{(\vec{r}_{12} \times \vec{P}_1) \cdot \vec{S}_1 - (\vec{r}_{12} \times \vec{P}_1) \cdot \vec{S}_2}{m_1m_2} \right) \right\} \chi_{s_1} \chi_{s_2} .
\]

Finally there are also “second order spin-orbit” terms in (20) of the form \((\vec{Q} \times \vec{P}_1) \cdot \vec{S}_1 \tilde{G}(Q^2)(\vec{Q} \times \vec{P}_2) \cdot \vec{S}_2\) which we have ignored for simplicity since the first order terms will already prove to be quite small.

In Section II.3 we describe how we model, based on (27), (29) and (31), this dependence of the potentials on the energy of the quarks. Before doing so, however, we first explain how we take into account the \(Q^2\) dependence of the coupling constant in our dressed gluon propagator.
II.2 THE RUNNING COUPLING CONSTANT

\[ \alpha_s \text{ and the Dressed Gluon Propagator} \]

The Feynman rules resulting from \( L_{QCD} \) tell us that the dressed gluon propagator should have the form (in the Coulomb gauge)

\[ G(Q^2) = -\frac{g^2}{Q^2} \sum_{i=1}^{8} \left( \frac{\lambda_i^1}{2} \right) \left( \frac{\lambda_i}{2} \right), \tag{32} \]

where the two Gell-Mann \( \lambda \) matrices act separately on each of the colour wavefunctions of the quarks being scattered, and where \( g \) is the renormalised coupling constant. For the colour wavefunction of a baryon the sum in (32) has the expectation value \( \frac{2}{3} \), so that

\[ G(Q^2) = -\frac{2}{3} \alpha_s \frac{4\pi}{Q^2}, \tag{33} \]

However we saw in the Chapter I that the leading log contributions to the effective quark gluon vertex can be summed to all orders, with the result that \( \alpha_s \) picks up a dependence

\[ \alpha_s(Q^2) = \frac{12\pi}{(33 - 2N_f) \ln(\frac{Q^2}{\Lambda^2_{QCD}})}, \tag{34} \]

where \( \Lambda^2_{QCD} \) is the scale at which QCD becomes strong, determined by deep inelastic scattering experiments to be of the order of 200 \( MeV \). Note that in low energy (and so necessarily low \( Q^2 \)) processes in which only the u and d quarks participate we should use \( N_f = 2 \), and as \( Q^2 \) increases, \( N_f \) should increase once the threshold for production of a pair of new quarks is passed. This formula cannot be used as it is to give a formula for the gluon propagator, since it blows up at a low \( Q^2 \) or large distance, indicating the onset of confinement. Since our string confining potential already takes into account this effect, we replace the formula above by one that saturates to some value \( \alpha_s(0) \) as \( Q^2 \rightarrow 0 \). The form we use is

\[ \alpha_s(Q^2) = \sum_{k=1}^{3} \alpha_k e^{-Q^2/\gamma_k^2}, \tag{35} \]

where \( \sum_k \alpha_k \) is used to fix the saturation point \( \alpha_s(0) \), and the rest of the freedom in (35) is used to fit to the formula (34) (see Fig. 1). One of the reasons for taking
Figure 1: the leading order formula (34) for the effective coupling constant with $\Lambda_{QCD} = 200$ MeV and our fit $
abla_s(Q^2) = 0.25 e^{-Q^2} + 0.15 e^{-Q^2/10} + 0.20 e^{-Q^2/1000}$ ($Q^2$ in $GeV^2$).
this Gaussian form is because it is easy to Fourier transform, and another is that it is easy to integrate against our harmonic oscillator wavefunctions. The resulting dressed gluon propagator is modified from (33) to

$$G(Q^2) = -\frac{2}{3} \sum_{k=1}^{3} \alpha_k e^{-Q^2/4} \frac{4\pi}{Q^2}. \tag{36}$$
II.3 ONE-GLUON TERMS IN THE HAMILTONIAN

The Relativized Coulomb Potential Energy

In the following we see how the relativistic corrections described in Section II.1 above are applied to the various one-gluon exchange potentials to come up with our final form for the (coordinate and momentum-dependent) potentials. Let us examine the spatial part of the colour Coulomb interaction, and for the moment ignore the momentum dependent factors in (27). The Fourier transform of the dressed gluon propagator (36) is the function

$$G(r_{ij}) = -\frac{2\alpha_s(r_{ij})}{3r_{ij}} = -\sum_k \frac{2\alpha_k}{3r_{ij}} \text{erf}(\gamma_k r_{ij})$$  \hspace{1cm} (37)

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$  \hspace{1cm} (38)

In order to take into account the relativistic smearing mentioned above we have to convolute $G(r_{ij})$ with the smearing (26). The resulting colour Coulomb potential is

$$\sum_{i<j} \tilde{G}(r_{ij}) = -\sum_{i<j} \sum_k \frac{2\alpha_k}{3r_{ij}} \text{erf}(\sigma_{kij} r_{ij})$$  \hspace{1cm} (39)

where

$$\sigma_{kij}^2 = \frac{\gamma_k^2 \sigma_{ij}^2}{\gamma_k^2 + \sigma_{ij}^2}$$  \hspace{1cm} (40)

We see from (27) that away from the non-relativistic limit we should expect the strength of the colour Coulomb interaction to be altered by a factor which goes like

$$1 - \frac{\vec{P}_i \cdot \vec{P}_j}{(P_i^2 + m_i^2)^{1/2} (P_j^2 + m_j^2)^{1/2}}$$  \hspace{1cm} (41)

where $\vec{P}_i = (\vec{p}_i + \vec{p}_i')/2$. Rather than take this form too seriously (for the various reasons described above, and also for reasons of calculational simplicity), we replace the above factor with

$$(\beta_{ij})^{1+2\varepsilon_{coul}} = \left(1 + \frac{p_{ij(cm)}^2}{(p_{ij(cm)}^2 + m_i^2)^{1/2} (p_{ij(cm)}^2 + m_j^2)^{1/2}}\right)^{1+2\varepsilon_{coul}}$$  \hspace{1cm} (42)
where $p_{ij}(cm) = |\vec{p}_{ij}(cm)| = \frac{1}{2} (\vec{p}_i - \vec{p}_j)$ is the magnitude of the momentum of either quark $i$ or $j$ in their centre of momentum frame (note that in this frame we have $\vec{F}_i = -\vec{F}_j$, so that $\vec{F}_i \cdot \vec{F}_j = -p_{ij}(cm)$. The parameter $\epsilon_{coul}$ reflects our ignorance of the exact form of the relativistic dependence of the effective potential, and is introduced to allow the spectrum to determine the amount of enhancement at a given momentum $p_{ij}(cm)$. To avoid any possible ordering ambiguities with these momentum dependent operators multiplying the coordinate dependent potential $\tilde{G}(r_{ij})$, we take the final form of our Coulomb potential to be

$$\tilde{V}_{coul} = \sum_{i<j} \tilde{V}_{coul}^{ij} = \sum_{i<j} (\beta_{ij})^{\frac{1}{2}+\epsilon_{coul}} \tilde{G}(r_{ij}) (\beta_{ij})^{\frac{1}{2}+\epsilon_{coul}}$$

(43)

**The Relativized Hyperfine Interaction**

We see from (29) above that the coordinate dependent part of our hyperfine energy operator takes the form

$$\sum_{i<j} (\delta_{r_0} \delta_{sv} - \delta_{rv} \delta_{su}) \frac{S_{ir} S_{js}}{m_i m_j} \nabla_u \nabla_v \tilde{G}(r_{ij})$$

(44)

which, using the identity

$$\nabla_u \nabla_v \tilde{G}(r) = \frac{1}{3} \delta_{uv} \nabla^2 \tilde{G}(r) + \left[ \frac{1}{3} \delta_{uv} - \frac{r_u r_v}{r^2} \right] \left[ \frac{1}{r} \frac{d\tilde{G}(r)}{dr} - \frac{d^2 \tilde{G}(r)}{dr^2} \right]$$

(45)

yields

$$\sum_{i<j} \frac{2 \vec{S}_i \cdot \vec{S}_i}{3 m_i m_j} \nabla^2 \tilde{G}(r_{ij})$$

$$+ \sum_{i<j} \frac{1}{3 m_i m_j} \left[ \frac{3 \vec{S}_i \cdot \vec{r}_{ij} \vec{S}_j \cdot \vec{r}_{ij}}{r_{ij}^2} - \vec{S}_i \cdot \vec{S}_j \right] \left( \frac{1}{r_{ij}} \frac{d \tilde{G}(r_{ij})}{dr_{ij}} - \frac{d^2 \tilde{G}(r_{ij})}{dr_{ij}^2} \right)$$

(46)

Our model of the momentum dependent factor suggested by (29) is similar to that of the factor in the colour Coulomb interaction. We adopt a momentum dependent factor of the form

$$(\delta_{ij})^{1+2\epsilon} = \left( \frac{m_i m_j}{(p_{ij}(cm) + m_i^2)^{\frac{1}{2}} (p_{ij}(cm) + m_j^2)^{\frac{1}{2}}} \right)^{1+2\epsilon}$$

(47)

where $\epsilon$ is one of $\epsilon_{cont}$ and $\epsilon_{coul}$. Finally with the ordering prescription (43) we arrive at the contact interaction

$$\tilde{V}_{cont} = \sum_{i<j} \tilde{V}_{cont}^{ij} = \sum_{i<j} (\delta_{ij})^{\frac{1}{2}+\epsilon_{cont}} \frac{2 \vec{S}_i \cdot \vec{S}_j}{3 m_i m_j} \nabla^2 \tilde{G}(r_{ij}) (\delta_{ij})^{\frac{1}{2}+\epsilon_{cont}}$$

(48)
and the tensor interaction

\[ \tilde{V}_{\text{tens}} = \sum_{i<j} \tilde{V}_{ij}^{\text{tens}} = \sum_{i<j} (\delta_{ij})^{\frac{1}{2}+\epsilon_{\text{tens}}} \left[ \frac{3\vec{s}_i \cdot \vec{r}_{ij} \vec{s}_j \cdot \vec{r}_{ij}}{r_{ij}^2} - \vec{s}_i \cdot \vec{s}_j \right] \]

\[ \left( \frac{1}{r_{ij}} \frac{d\tilde{G}(r_{ij})}{dr_{ij}} - \frac{d^2\tilde{G}(r_{ij})}{dr_{ij}^2} \right) (\delta_{ij})^{\frac{1}{2}+\epsilon_{\text{tens}}} \]  

(49)

The Relativized One-Gluon Spin-Orbit Interaction

We have seen above that the spin-orbit potential arising from the one gluon exchange amplitude includes a colour magnetic piece and a Thomas precession piece and has the form

\[ \sum_{i<j} \frac{1}{r_{ij}} \frac{d\tilde{G}(r_{ij})}{dr_{ij}} \left\{ \left[ \frac{\vec{r}_{ij} \times \vec{p}_i \cdot \vec{s}_i}{2m_i^2} - \frac{\vec{r}_{ij} \times \vec{p}_j \cdot \vec{s}_j}{2m_j^2} \right] - \left[ \frac{\vec{r}_{ij} \times \vec{p}_j \cdot \vec{s}_i - \vec{r}_{ij} \times \vec{p}_i \cdot \vec{s}_j}{m_im_j} \right] \right\} \]

(50)

There are three different types of relativistic strength factors associated with the potentials in (31), which we model in a similar way as those for the Coulomb and hyperfine interactions: the resulting one-gluon exchange spin-orbit potential that we adopt is

\[ \tilde{V}_{\text{uso}} = \sum_{i<j} \tilde{V}_{ij}^{\text{uso}} = \sum_{i<j} \frac{1}{r_{ij}} \frac{d\tilde{G}(r_{ij})}{dr_{ij}} \left\{ (\delta_{ii})^{\frac{1}{2}+\epsilon_{\text{uso}}} \frac{\vec{r}_{ij} \times \vec{p}_i \cdot \vec{s}_i}{2m_i^2} (\delta_{ii})^{\frac{1}{2}+\epsilon_{\text{uso}}} \right. 

\[ - (\delta_{jj})^{\frac{1}{2}+\epsilon_{\text{uso}}} \frac{\vec{r}_{ij} \times \vec{p}_j \cdot \vec{s}_j}{2m_j^2} (\delta_{jj})^{\frac{1}{2}+\epsilon_{\text{uso}}} 

\[ - (\delta_{ij})^{\frac{1}{2}+\epsilon_{\text{uso}}} \left[ \frac{\vec{r}_{ij} \times \vec{p}_j \cdot \vec{s}_i - \vec{r}_{ij} \times \vec{p}_i \cdot \vec{s}_j}{m_im_j} \right] (\delta_{ij})^{\frac{1}{2}+\epsilon_{\text{uso}}} \} \]

(51)
II.4 THE CONFINING POTENTIAL

The String Energy

Our $V_{\text{conf}}$ consists of the three body adiabatic potential $V_{\text{string}}$ generated by the quantum ground state of the gauge invariant $\mathcal{Y}$ string configuration (see Fig. 2), and a spin-orbit potential generated by Thomas precession. We first examine the string potential energy.

The string picture implies that, up to an overall constant, the confining potential is the adiabatic potential given by the energy of the minimum length $\mathcal{Y}$ shaped string. The adiabatic approximation is the assumption that the string attains its minimum energy (and so minimum length) configuration quickly in response to the motion of the quarks. It has been shown that this approximation is justified\(^{66}\) in the low energy baryons. There is a rule for finding the junction point of the strings to attain the minimum length, which is as follows: if any one of the angles in the triangle made by connecting the quarks is greater than 120° then the junction point is on top of the quark which lies at the vertex of that angle (see Fig. 2); otherwise the junction is at the unique point which makes the arms of the $\mathcal{Y}$ shaped string at 120° to each other\(^{66}\). The string length then has a simple formula in terms of $\rho$, $\lambda$ and $\cos \theta = \frac{F \lambda}{\rho \lambda}$, i.e.

$$V_{\text{string}} = C_{\bar{q}qq} + \sqrt{\sigma}$$

\[ \begin{cases} \sqrt{\frac{3}{2}} (\rho^2 + \lambda^2 + 2\rho \lambda \sqrt{1 - \cos^2 \theta})^{\frac{1}{2}} & \text{if all angles } \theta_{ijk} < 120^\circ, \\ r_{12} + r_{13} & \theta_{312} > 120^\circ, \\ r_{12} + r_{23} & \theta_{123} > 120^\circ, \\ r_{13} + r_{23} & \theta_{123} > 120^\circ. \end{cases} \]  

(52)

where

$$r_{12} = \sqrt{2}\rho$$

$$r_{13} = \frac{1}{\sqrt{2}} \left( \rho^2 + 3\lambda^2 + 2\sqrt{3}\rho\lambda\cos \theta \right)^\frac{1}{2}$$

(53)

$$r_{23} = \frac{1}{\sqrt{2}} \left( \rho^2 + 3\lambda^2 - 2\sqrt{3}\rho\lambda\cos \theta \right)^\frac{1}{2},$$

and $C_{\bar{q}qq}$ is an overall constant energy shift (not at present calculable) which arises from the vacuum modifications due to the presence of the colored fields in the baryon.
Figure 2: the gauge invariant string configuration.
We have found it useful to break $V_{\text{string}}$ up into an effective two-body piece and a three-body piece

$$ V_{\text{string}} = C_{qqq} + f\sqrt{\sigma} \sum_{i<j} r_{ij} + V_{3b} \quad (54) $$

where

$$ V_{3b} = \sqrt{\sigma} \left( \sum_{i=1}^{3} |\vec{r}_i - \vec{r}_{\text{function}}| - f \sum_{i<j} r_{ij} \right), \quad (55) $$

with $f = 0.5493$ chosen to minimize the size of the expectation value of $V_{3b}$ in the harmonic oscillator ground state of the baryon system. We can then include the two-body part of $V_{\text{string}}$ in our variational calculation and compute the (small) corrections due to $V_{3b}$ perturbatively.

Smearing of the Confining Potential and Spin-Orbit Terms

We now wish to treat the confining potential relativistic corrections in a similar way to the treatment of the one-gluon exchange terms. However we cannot write down the corrections to the confining potential by a direct analogy with those which we found by examining the one-gluon exchange T-matrix element, since the latter has a Lorentz vector $\gamma^\mu \times \gamma^\nu$ structure and the confining potential has a Lorentz scalar $1 \times 1$ structure. Gromes\textsuperscript{58} has shown that in leading order the spin dependent terms arising from the confining potential in QCD have the same form as those coming from the exchange of a Lorentz scalar particle. If we then proceed in analogy to the above and examine the T-matrix element for the exchange of a scalar (whose propagator is the Fourier transform of the effective two-body linear potential) between quarks $i$ and $j$, we again find additional $Q^2$ dependence introduced into the confining potential, which we model by adopting the universal smearing function $\rho_{ij}(Q^2)$ for the two-body terms in our confining potential (we ignore the effects of smearing on the small $V_{3b}$ term). The resulting smeared two-body potential is

$$ \tilde{V}_{\text{string}} = \sum_{i<j} f\sqrt{\sigma} \left( r_{ij} \rho_{ij}(\sigma_{ij} r_{ij}) \left( 1 + \frac{1}{2\sigma_{ij}^2} \right) + \frac{1}{\sqrt{\pi} \sigma_{ij}} e^{-\sigma_{ij} r_{ij}} \right) \quad (56) $$

If we now examine the $Q^2 \to 0$ limit of the scalar exchange T-matrix element, we find that it suggests that the (two-body) confining potential itself is modified by a strength factor $\frac{m_i m_j}{E_i E_j}$. Since the correspondence between scalar exchange and the string picture is only valid in lowest order, and since studies of a simpler field theory
which confines, QED in 1+1 dimensions, have shown that such corrections are in fact not present, we ignore these possible modifications. The spin-orbit potential from the scalar exchange T-matrix element is

\[
\sum_{i<j} \frac{1}{(2\pi)^3} \int d^3 \vec{Q} e^{i\vec{Q} \cdot \vec{r}_{ij}} \tilde{V}_{\text{string}} \left[ \frac{i(\vec{Q} \times \vec{P}_i) \cdot \vec{\sigma}_i}{2E_i(E_i + m_i)} + \frac{i(\vec{Q} \times \vec{P}_j) \cdot \vec{\sigma}_j}{2E_j(E_j + m_j)} \right].
\]  

(57)

In analogy to our treatment of the one-gluon exchange terms, (57) suggests that we adopt the coordinate and momentum dependent spin-orbit potential

\[
\tilde{V}_{\text{asso}} = \sum_{i<j} \tilde{V}_{\text{asso}}^{ij} = \sum_{i<j} \frac{1}{r_{ij}} \frac{\partial \tilde{V}_{\text{string}}}{\partial r_{ij}} \left\{ (\delta_{ii})^{1/2+\epsilon_{\text{asso}}} \frac{\vec{r}_{ij} \times \vec{p}_i \cdot \vec{S}_i}{2m_i^2} (\delta_{ii})^{1/2+\epsilon_{\text{asso}}} \\
- (\delta_{jj})^{1/2+\epsilon_{\text{asso}}} \frac{\vec{r}_{ij} \times \vec{p}_j \cdot \vec{S}_j}{2m_j^2} (\delta_{jj})^{1/2+\epsilon_{\text{asso}}} \right\}.
\]  

(58)

Note that we have ignored the small term \( V_{3b} \) when calculating the spin-orbit potentials.
CHAPTER III: METHODS

Our task is to solve the equation

\[ H |\Psi\rangle = E |\Psi\rangle, \tag{59} \]

where \( H \) is the baryon Hamiltonian (10). It is impossible to solve (59) directly for the energies and wavefunctions of the baryons, because of the complicated nature of \( H \). We instead adopt the variational method, where we expand the baryon wavefunction in a large harmonic oscillator basis. The most convenient method for carrying out the variational calculation is to diagonalize (10) in this basis as a function of a single variational parameter \( \alpha \) (see below), which corresponds to a distance scale. The harmonic oscillator basis is chosen because it spans the space of bound state wavefunctions, and because it is easily Fourier transformed to momentum space. This greatly simplifies the task of evaluating the many momentum dependent operators in our Hamiltonian (10).

We will, in this Chapter, outline the non-trivial process of constructing the allowed states of baryons (with a given set of quantum numbers) from the available flavour, spin and orbital wavefunctions. We will then describe briefly the variational calculation and some important techniques for simplifying the calculation of the Hamiltonian matrix in this basis. We have relegated further details of the calculation of the various matrix elements to Appendix A. Finally, we outline the prescription we use to extract the masses of the baryons from examining the way that the oscillator basis eigenvalues are tending toward the (infinitely) large basis limit.
III.1 THE WAVEFUNCTIONS

The wavefunction of a baryon is made up of four factors: a colour wavefunction, a flavour wavefunction \( \Phi \), a spin wavefunction \( \chi \) and a spatial wavefunction \( \Psi \). The colour wavefunction is found by the rules of \( SU(3) \) for \( 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8' \oplus 10 \) to be the totally antisymmetric singlet combination

\[
C_A = \sum_{i,j,k=1}^{3} \frac{1}{\sqrt{6}} \epsilon_{ijk} q_1^i q_2^j q_3^k .
\]

The usual procedure for baryons made up from \( u, d \) and \( s \) quarks is to construct totally symmetric states from direct products of the irreducible representations \( \Phi, \chi \) and \( \Psi \) of the permutation group \( S_3 \). This is what is meant by working in an \( SU(6) \) basis, which is useful because of the permutational symmetry of the flavour-spin-space wavefunctions, even though \( SU(3)_f \) is broken. The \( uds \) basis of Isgur and Kar127 allows for explicit breaking of this permutational symmetry by the strange quark mass, and only symmetrizes \( \Phi \chi \Psi \) in the \( u \) and \( d \) quarks. We generalise this basis by only symmetrizing \( \Phi \chi \Psi \) in \textit{identical} quarks. The reason for doing so is to avoid having to carry out the symmetrization in the spatial wavefunction between the quarks \( u \) and \( d \), for reasons that will become clear.

The flavour wavefunction \( \Phi \) which we adopt, therefore, is one of the combinations displayed in Table II. The states are displayed in order of decreasing electric charge, and are either symmetric or antisymmetric under exchange of quarks one and two. The spin of three spin-\( \frac{1}{2} \) particles can be either \( \frac{1}{2} \) or \( \frac{3}{2} \), the spin-wavefunctions being one of (only top states are displayed, the others follow from the Condon-Shortley convention)

\[
\begin{align*}
\chi^{S}_{\frac{3}{2}} &= | \uparrow \uparrow \uparrow \rangle \\
\chi^{M}_{\frac{1}{2}} &= \frac{1}{\sqrt{2}} (| \uparrow \downarrow \uparrow \rangle - | \downarrow \uparrow \uparrow \rangle) \\
\chi^{MA}_{\frac{1}{2}} &= -\frac{1}{\sqrt{6}} (| \uparrow \downarrow \uparrow \rangle + | \downarrow \uparrow \uparrow \rangle - 2 | \uparrow \uparrow \downarrow \rangle),
\end{align*}
\]

which are again either symmetric or antisymmetric under interchange of the first two quarks.
Table II: The Baryon Flavour Wavefunctions

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<thead>
<tr>
<th>State</th>
<th>++</th>
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<tr>
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<td>uud</td>
<td>ddu</td>
<td>ddd</td>
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<tr>
<td>Σ</td>
<td>—</td>
<td>uus</td>
<td>$\frac{1}{\sqrt{2}}(ud + du)s$</td>
<td></td>
</tr>
<tr>
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<td>—</td>
<td>—</td>
<td>sss</td>
<td>ssd</td>
</tr>
<tr>
<td>Ω</td>
<td>—</td>
<td>—</td>
<td>sss</td>
<td></td>
</tr>
<tr>
<td>Λc</td>
<td>—</td>
<td>—</td>
<td>$\frac{1}{\sqrt{2}}(ud - du)c$</td>
<td></td>
</tr>
<tr>
<td>Σc</td>
<td>uuc</td>
<td>$\frac{1}{\sqrt{2}}(ud + du)c$</td>
<td>ddc</td>
<td></td>
</tr>
<tr>
<td>Λb</td>
<td>—</td>
<td>—</td>
<td>$\frac{1}{\sqrt{2}}(ud - du)b$</td>
<td></td>
</tr>
<tr>
<td>Σb</td>
<td>uub</td>
<td>$\frac{1}{\sqrt{2}}(ud + du)b$</td>
<td>ddb</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3: the relative coordinates $\vec{p}$ and $\vec{\lambda}$.
Finally the spatial wavefunctions are taken to be made up of functions with definite total \( \vec{L} = \vec{l}_p + \vec{l}_\lambda \) made from a Clebsch-Gordan sum of harmonic oscillator wavefunctions in the two relative coordinates

\[
\vec{\rho} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2)
\]

\[
\vec{\lambda} = \frac{1}{\sqrt{6}} (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)
\]

of the three-body problem (see Fig. 3). These are

\[
\Psi_{LMn_p,l_pn_\lambda l_\lambda} = \alpha^3 \sum_m C(l_p l_\lambda m M - m; LM) \mathcal{N}_{nl_p}(\alpha \rho)^{l_p} e^{-\frac{1}{3} \alpha^2 \rho^2} L^{l_p + \frac{1}{2}}_{n_p}(\alpha \rho) Y_{lm}(\Omega_p) \mathcal{N}_{n_\lambda l_\lambda}(\alpha \lambda)^{l_\lambda} e^{-\frac{1}{3} \alpha^2 \lambda^2} L^{l_\lambda + \frac{1}{2}}_{n_\lambda}(\alpha \lambda) Y_{l_\lambda M - m}(\Omega_\lambda),
\]

(63)

where the \( L^{l + \frac{1}{2}}_n(x) \) are the associated Laguerre polynomials

\[
L^{l + \frac{1}{2}}_n(x) = \sum_{m=0}^{n} (-1)^m \left( \frac{n + l + \frac{1}{2}}{n - m} \right) \frac{x^{2m}}{m!}
\]

(64)

(half-integral factorials are defined by the \( \Gamma \) function), and the normalisation coefficient \( \mathcal{N}_{nl} \) is defined by

\[
\mathcal{N}_{nl} = \sqrt{\frac{2n!}{\Gamma(n + 1/2)}}
\]

(65)

We then expand the wavefunction in a set of states of the form

\[
|\alpha \rangle = C_{\alpha} \Phi \sum_{ML} C(L S M_L J - M_L; J M) \Psi_{LMn_p,l_pn_\lambda l_\lambda} X_S J - M_L.
\]

(66)

Note that the entire wavefunction is now only explicitly antisymmetrized under exchange of quarks one and two. For a given sector we expand in a restricted set of these states. For example, the proton, with flavour wavefunction \( uud \) and \( J^P = \frac{1}{2}^+ \), must have a (spin-space) wavefunction which is symmetric under exchange of quarks one and two. Furthermore the sum over the \( |\alpha \rangle \) is restricted to states with \( J = \frac{1}{2} \) and with parity \( (-1)^{l_p + l_\lambda} \) positive.

There is, however, a penalty for not symmetrizing in the quarks \( u \) and \( d \). The state we have described, which is the most general state consistent with the Pauli principle for the two \( u \) quarks in \( uud \), can also be a \( \Delta^+ \) (see Table II) with \( J^P = \frac{1}{2}^+ \). This is not a problem, however, because the Hamiltonian we use cannot distinguish the quarks \( u \) and \( d \) (their masses and interactions are the same—we have
ignored isospin violation) and so automatically separates the eigenstates into two non-communicating blocks of states with isospin $I = \frac{1}{2}$ and $I = \frac{3}{2}$. We then have to identify which eigenstates are $N$'s (like the proton) and which are $\Delta$'s. Note that for the $sss$ states one must separate out the true $\Omega$ baryons from spurious states which are not fully antisymmetrized. This slight inconvenience is far preferable to having to construct totally symmetrized states up to $8\hbar\Omega$ in the harmonic oscillator spectrum.
III.2 THE VARIATIONAL CALCULATION

Our programme is to calculate the energy eigenstates of the Hamiltonian of Chapter II in the basis of harmonic oscillator wavefunctions (66), subject to the restrictions noted above. This is accomplished by forming a matrix of the Hamiltonian operator, and diagonalising this matrix. The harmonic oscillator wavefunctions have only one parameter, $\alpha$ (see (63)), and in principle if we expand in an infinitely large basis set, our results would be independent of this parameter. However with a truncated set, there exists an $\alpha$ for which the ground state energy is minimized, one for which the first excited state energy is minimized, etc. The Hylleraas-Undheim Theorem states that each minimum is separately an upper bound for each energy eigenvalue, i.e. that we may minimize each eigenvalue independently as a function of $\alpha$. Of course if we want to have a set of approximate (orthogonal) wavefunctions we have to choose one $\alpha$ (in practice the ground state energy does not vary as strongly as a function of $\alpha$ as the excited states, so one should choose the $\alpha$ which minimizes the energy of the highest energy state of interest) but for spectroscopy we can do better.

Our problem is now reduced to one of calculating the matrix elements

$$H_{\alpha\beta} = \langle \alpha | H | \beta \rangle.$$  

(67)

There are two important details which we should explore here, before going on to describe the calculation of the matrix elements of the various terms in the Hamiltonian, which is is outlined in Appendix A. The first is that most of the terms in the Hamiltonian (10) are of the form $\sum_{i<j} H_{ij}(r_{ij})$, and so we have to integrate functions of $r_{12}, r_{13}$ and $r_{23}$, where

$$r_{12} = \sqrt{2}\rho$$

$$r_{13} = \frac{1}{\sqrt{2}} \sqrt{\rho^2 + \sqrt{3\rho} \cdot \bar{\lambda} + 3\lambda^2}$$

$$r_{23} = \frac{1}{\sqrt{2}} \sqrt{\rho^2 - \sqrt{3\rho} \cdot \bar{\lambda} + 3\lambda^2}.$$  

(68)

Since the wavefunction (66) is always antisymmetric under exchange of quarks one and two, then

$$\langle \alpha | H_{13} | \beta \rangle = \langle \alpha | H_{23} | \beta \rangle,$$  

(69)
so that
\[
\left\langle \alpha \right| \sum_{i<j} H_{ij}(r_{ij}) \left| \beta \right\rangle = \left\langle \alpha \right| H_{12} + 2H_{13} \left| \beta \right\rangle.
\]
(70)

Calculation of the \(H_{12}(r_{12})\) matrix elements are straightforward because \(r_{12} = \sqrt{2}\rho\). However the \(H_{13}(r_{13})\) matrix elements will be harder to calculate as \(r_{12}\) and \(r_{13}\) are more complicated functions of \(\tilde{\rho}\) and \(\tilde{\lambda}\). To perform the \(r_{13}\) integration we change variables to
\[
\tilde{\rho}' = \frac{1}{\sqrt{2}} (\tilde{r}_1 - \tilde{r}_3),
\]
\[
\tilde{\lambda}' = \frac{1}{\sqrt{6}} (\tilde{r}_1 + \tilde{r}_3 - 2\tilde{r}_2),
\]
(71)

which is accomplished by applying the transformation
\[
\tilde{\rho} \rightarrow \left( \frac{\tilde{\rho}'}{2} + \frac{\sqrt{3}}{2} \tilde{\lambda}' \right)
\]
\[
\tilde{\lambda} \rightarrow \left( \frac{\sqrt{3}}{2} \tilde{\rho}' - \frac{\tilde{\lambda}'}{2} \right).
\]
(72)

to the wavefunctions. Then in this basis, which we will denote by \(\left| \alpha' \right\rangle\), the calculation of the \(H_{13}\) part of the Hamiltonian becomes identical to that of the \(H_{12}\) part in the usual basis. In practice this transformation is carried out by first forming the matrix of expectation values of the \(H_{13}\) terms in the primed basis, and then transforming this matrix back into the usual basis by the matrix transformation
\[
\left\langle \alpha \left| H_{13} \right| \beta \right\rangle = \sum_{\alpha' \beta'} \left\langle \alpha \left| \alpha' \right\rangle \left\langle \alpha' \left| H_{13} \right| \beta' \right\rangle \left\langle \beta' \left| \beta \right\rangle \right\},
\]
(73)

where the matrix
\[
T_{\alpha \alpha'} = \left\langle \alpha \left| \alpha' \right\rangle \right\}
\]
(74)
gives us the transformation between the two bases. From this point onwards we will specialize our discussion to techniques for the calculation of the \(H_{12}\) terms, since by the above the calculation of the \(H_{13}\) terms (in the primed basis) follows immediately upon replacing \(m_2\) by \(m_3\).

The second problem we have to deal with is the momentum dependent factors in the Hamiltonian, which are combined with spatially dependent potentials. These are dealt with by inserting 'complete' sets (in practice the largest set of wavefunctions \(\{| \alpha ' \}\) available—see below in Section III.3) of harmonic oscillator wavefunctions.
between the two types of operator. For example, the contact interaction $V_{12}^{\text{cont}}$ has the matrix element

$$
\langle \alpha | (\delta_{ij})^{\frac{1}{2} + \epsilon_{\text{cont}}} V_{12}^{\text{cont}} (r_{12}) (\delta_{ij})^{\frac{1}{2} + \epsilon_{\text{cont}}} | \beta \rangle = \sum_{\gamma \delta} \langle \alpha | (\delta_{ij})^{\frac{1}{2} + \epsilon_{\text{cont}}} | \gamma \rangle \langle \gamma | V_{12}^{\text{cont}} (r_{12}) | \delta \rangle \langle \delta | (\delta_{ij})^{\frac{1}{2} + \epsilon_{\text{cont}}} | \beta \rangle .
$$

(75)

The momentum expectation values are then evaluated in the Fourier transform basis, and the potential expectation values in the usual basis. One of the advantages of the harmonic oscillator basis is that the wavefunctions are form invariant (up to a phase) under Fourier transformation,

$$
\Phi_{n \ell m} (\alpha; \vec{p}) = \frac{-1}{(2\pi)^{\frac{3}{2}}} \int d^{3} \vec{r} e^{-i \vec{p} \cdot \vec{r}} \Psi_{n \ell m} (\alpha; \vec{r}) = (-i)^{2n+\ell} \Psi_{n \ell m} (\frac{1}{\alpha}; \vec{p}) ,
$$

(76)

where

$$
\Psi_{n \ell m}(\alpha; \vec{r}) = \alpha^{\frac{\ell}{2}} N_{n \ell}(ar)^{l} Y_{\ell m}(\Omega) \int_{n}^{\ell + \frac{1}{2}} (ar) e^{-\frac{1}{2} \alpha^{2} r^{2}} .
$$

(77)

Further details of these calculations and the resulting matrix elements (of all the terms in the Hamiltonian) are given in Appendix A. In the next Section we describe how we predict the masses of the various states from the eigenvalues of the Hamiltonian matrix diagonalized in a finite sized basis.
III.3 CONVERGENCE CONSIDERATIONS

A given wavefunction set \( \{|\alpha\rangle\}_{N_{max}} \) is taken to be large enough to include all wavefunctions (subject to the conditions outlined above) up to a given harmonic oscillator \( N = 2(n_p + n_\lambda) + l_p + l_\lambda \), which we label \( N_{max} \). \( N_{max} \) is then increased from the minimum required to describe the wavefunction of the state of interest, until the energy eigenvalues appear to be approaching a large \( N_{max} \) limit. Our predictions for the baryon energies are determined by plotting the energy eigenvalue of a state against the inverse of the number of oscillator levels included in its expansion (with the 'complete' sets always the largest possible), and then making a linear extrapolation to an infinite number of levels. Let's illustrate this with an example.

The proton, with \( J^P = \frac{1}{2}^+ \) and with flavour wavefunction \( uud \), has one harmonic oscillator wavefunction at \( N = 0 \), which, we may denote \( |uud\frac{1}{2}^+(1)\rangle \), which is of the form (see (66))

\[
|uud\frac{1}{2}^+(1)\rangle = C_A \ uud \ \Psi_{OO0000} \ \chi_{\frac{1}{2} \frac{1}{2}}.
\]  

(78)

Parity rules out all \( N = 1 \) wavefunctions, and at \( N = 2 \) there are six wavefunctions which have the correct spin-parity and (12) exchange symmetry. Two linear combinations of these six have their spin-space wavefunctions completely symmetric under the permutation group \( S_3 \), and so represent \( \Delta \)'s and not \( N \)'s, and so diagonalising the \( uud \) Hamiltonian in the \( N \leq 2 \) basis will yield five Nucleon states and two Delta states. The six \( N = 2 \) wavefunctions are

\[
|uud\frac{1}{2}^+(2)\rangle = C_A \ uud \ \Psi_{OO1000} \ \chi_{\frac{1}{2} \frac{1}{2}}
\]

\[
|uud\frac{1}{2}^+(3)\rangle = C_A \ uud \ \Psi_{OO0100} \ \chi_{\frac{1}{2} \frac{1}{2}}
\]

\[
|uud\frac{1}{2}^+(4)\rangle = C_A \ uud \ \Psi_{OO0010} \ \chi_{\frac{1}{2} \frac{1}{2}}
\]

\[
|uud\frac{1}{2}^+(5)\rangle = C_A \ uud \ \sum_M C(1/2 M 1/2 - M; 1/2 1/2) \Psi_{1M0100} \ \chi^p_{\frac{1}{2} \frac{1}{2} - M}
\]  

(79)

\[
|uud\frac{1}{2}^+(6)\rangle = C_A \ uud \ \sum_M C(2/3 M 1/2 - M; 1/2 1/2) \Psi_{2M0000} \ \chi^S_{\frac{1}{2} \frac{1}{2} - M}
\]

\[
|uud\frac{1}{2}^+(7)\rangle = C_A \ uud \ \sum_M C(2/3 M 1/2 - M; 1/2 1/2) \Psi_{2M0000} \ \chi^S_{\frac{1}{2} \frac{1}{2} - M}
\]

Similarly, there are fifteen wavefunctions (either \( N \) or \( \Delta \)) with the correct properties at \( N = 4 \), twenty-eight at \( N = 6 \) and forty-five at \( N = 8 \). Or equivalently we need
1, 7, 22, 50 or 95 wavefunctions to expand the wavefunction (or the complete sets) up to \( N = 0, N \leq 2, N \leq 4, N \leq 6 \) or \( N \leq 8 \).

To extract an energy for the first eigenvalue, representing the proton energy, we first find the matrix of the Hamiltonian in the largest \( N_{\text{max}} \) basis and find the minimum of its lowest eigenvalue with respect to our one variational parameter, \( \alpha \). We then diagonalize smaller and smaller sub-matrices of this larger matrix, which correspond to an \( N \leq 6, N \leq 4, N \leq 2 \) and \( N = 0 \) expansion of the wavefunction but always an \( N \leq 8 \) expansion of the 'complete' set of wavefunctions inserted between the coordinate and momentum-space operators. The minimum with respect to \( \alpha \) of the lowest eigenvalue of each of these matrices is then plotted against the inverse of the number of oscillator levels included in the expansion of this state. A linear regression analysis is used to extrapolate to infinite \( N_{\text{max}} \) and estimate the error in this extrapolated value; since the higher \( N_{\text{max}} \) eigenvalues are more significant, the regression includes weights which increase linearly with the number of levels included (and which add to the number of degrees of freedom to preserve the error analysis). In some cases, including that of the proton, the point with only one level (and in this case only one state) in the expansion of the eigenstate is discarded, since in practice removing this point decreases the \( \chi^2 \) per degree of freedom.

The masses of the excited states of the proton (and the \( \Delta^* \frac{1}{2}^+ \) states) are represented by the higher eigenvalues of this Hamiltonian matrix, and of course don't appear until the wavefunction is expanded up to \( N \leq 2 \). Following the procedure above we may then estimate the \( N_{\text{max}} = \infty \) limit of these eigenvalues, with of course a slightly larger error since there will be fewer points in the extrapolation plots for a given \( N_{\text{max}} \).

These calculations were carried out up to \( N \leq 8 \) for the \( J^P = \frac{1}{2}^+ \) system (for all flavours), and up to \( N \leq 6 \) for the other positive parity states with \( J^P \) from \( \frac{3}{2}^+ \) to \( \frac{11}{2}^+ \). All negative parity states (\( J^P \) from \( \frac{3}{2}^- \) to \( \frac{11}{2}^- \)) were expanded up to \( N \leq 7 \). This meant that for the states with \( J^P \) of \( \frac{7}{2}^-, \frac{9}{2}^+ \) and \( \frac{11}{2}^\pm \), we have only two levels included in the expansion at \( N = N_{\text{max}} \). With these high \( J \) states we expect that the convergence should be quite rapid since the region where the wavefunction is large is well away from the origin, and the parts of the Hamiltonian with rapid curvature are all concentrated around \( p = 0 \) or \( \lambda = 0 \). This turns out to be the case, with the energy eigenvalues dropping only slightly between bases with one and two oscillator levels included. In these cases, rather than put a straight line through these two points (which would be rather artificial), we take the splittings (at \( N = 6 \)
or 7) between these states and their nearest like-parity neighbour with three points in its extrapolation, and assume that they remain the same at \( N_{\text{max}} = \infty \).

We will discuss the actual errors associated with these methods in the next Chapter, where we turn to a discussion of our results.
CHAPTER IV: RESULTS AND DISCUSSION

IV.1 PRELIMINARIES

The Parameters of the Model

The calculation outlined above has various parameters which are listed in Table III. We should point out here that with the exception of the string tension $\sqrt{\sigma}$ and the three relativistic parameters $\epsilon_{\pi\pi\pi}$, $\epsilon_{\pi\pi\sigma}$ and $\epsilon_{\pi\sigma\sigma}$, the parameters in Table III are exactly those of the meson analysis of Godfrey and Isgur\textsuperscript{62}. We will give a more detailed discussion of the correspondence of these two models in the next Chapter. $C_{qqq}$ is the only parameter in our model which does not appear in Ref. 62, since it has no analogue in the meson system. $C_{qqq}$ is fixed by normalising the entire baryon spectrum to one state, which is taken to be the $\Delta(1232)$, since our calculation for this state converges rapidly and the $\Delta$ is experimentally very well known.

Couplings to Formation Channels

The calculation as it is described above predicts many more resonances than are seen experimentally. This is because, as we have seen in Chapter I, there are many resonances which do not couple to their formation channels. Rather than carry out an extensive decay analysis using the wavefunctions generated by the spectroscopy, we instead truncated our Hamiltonian matrices at the $N=2$ band (as explained in Chapter III) to correspond to the calculations of Isgur and Karl. We then recalculated the hadronic couplings of our states to discover, using the criteria of Koniuk and Isgur in Ref. 46, which of the many predicted $\Delta$ and $N$ states ought to have been seen in $\pi N$ scattering, and which of the many predicted $\Lambda$ and $\Sigma$ states ought to have been seen in $KN$ scattering. States with more than one strange quark, such as the $\Xi$'s and $\Omega$'s, have no formation channel, and so it is not necessary to carry out such an analysis in their case. Similar remarks hold for the charmed and beauty baryons which we calculate.

Although our calculation predicts energies for states in higher $N$ oscillator bands (i.e. with their expansion starting at a higher minimum $N$), it is necessary to perform a decay analysis to determine if such states will be seen experimentally. For this reason we will not extensively discuss our results for the states in higher bands.
<table>
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<th>Table III: The parameters of the model</th>
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</table>

† Note: we ignore isospin violation.
IV.2 THE BARYON SPECTRUM.

The Ground State Baryons of SU(3)

In Table IV and Figure 4 we have shown the predicted masses of the ground state baryons made up of u, d and s quarks calculated in our model, and their experimental masses taken from the Particle Data Group\textsuperscript{67}. Note that the calculational error in our (extrapolated) baryon masses is usually about 20 MeV; this comment applies to all of the figures and tables which follow. The predicted masses have therefore been rounded to the nearest 5 MeV.

Since the results in this sector are practically identical if we use exactly the same parameters as Godfrey and Isgur, it is apparent from these results that the same contact interaction which they used to predict the $\rho - \pi$ splitting is responsible for the $\Delta - N$ and other ground state hyperfine splittings. It might appear from Table IV and Figure 4 that we require a slightly higher strange quark mass in the ground state baryons, since the error in our predictions for the strange baryons increases with the number of strange quarks included. However, when we examine the excited states of these strange baryons, we will see that the Godfrey-Isgur meson analysis strange mass minimises the error of the theoretical predictions over a wide variety of states.

Our results also confirm the observation\textsuperscript{23} that the $\Sigma - \Lambda$ mass difference is due to the hyperfine interaction. Our results give 75 $\pm$ 15 MeV for this mass difference (the theoretical error on differences in masses is smaller than the error on the masses) as opposed to the experimental value of $\simeq$ 80 MeV. Similarly we find $\Sigma^* - \Sigma = 180 \pm 15$ MeV and $\Xi^* - \Xi = 200 \pm 15$ MeV compared to their experimental values of 190 MeV and 215 MeV. Obviously our model, like the usual non-relativistic models, obtains good spectroscopic results for the ground state baryons.

The S=0 and S=-1 Negative Parity Baryons of the N=1 Band

In Table V and Figure 5 we have displayed the predicted masses for the negative parity excited S=0 N and $\Delta$ baryons in the N=1 band, and their experimental values. Note that we have listed in our Tables all of the states predicted by the model but have indicated with an arrow which of the predicted states couple to their formation channels. Where the Particle Data Group give a range of experimental
Table IV: The Ground State Baryons

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Mass (MeV)</th>
<th>Experiment (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{1/2}^+$</td>
<td>960</td>
<td>939</td>
</tr>
<tr>
<td>$\Delta_{3/2}^+$</td>
<td>1230</td>
<td>1232</td>
</tr>
<tr>
<td>$\Delta_{1/2}^+$</td>
<td>1115</td>
<td>1116</td>
</tr>
<tr>
<td>$\Sigma_{1/2}^+$</td>
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<td>1193</td>
</tr>
<tr>
<td>$\Sigma_{3/2}^+$</td>
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<td>1384</td>
</tr>
<tr>
<td>$\Xi_{1/2}^+$</td>
<td>1305</td>
<td>1318</td>
</tr>
<tr>
<td>$\Xi_{3/2}^+$</td>
<td>1505</td>
<td>1533</td>
</tr>
<tr>
<td>$\Omega_{3/2}^+$</td>
<td>1635</td>
<td>1672</td>
</tr>
</tbody>
</table>

Figure 4: The Ground State Baryons.
Table V: The Negative Parity $S = 0$ Excited Baryons

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Mass (MeV)</th>
<th>Experiment (MeV)</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^{*\frac{1}{2}}^-$</td>
<td>1460 ←</td>
<td>1520-1560</td>
<td>****</td>
</tr>
<tr>
<td></td>
<td>1535 ←</td>
<td>1620-1680</td>
<td>****</td>
</tr>
<tr>
<td>$\Delta^{*\frac{1}{2}}^-$</td>
<td>1555 ←</td>
<td>1600-1650</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*\frac{3}{2}}^-$</td>
<td>1495 ←</td>
<td>1510-1530</td>
<td>****</td>
</tr>
<tr>
<td></td>
<td>1625 ←</td>
<td>1670-1730</td>
<td>****</td>
</tr>
<tr>
<td>$\Delta^{*\frac{3}{2}}^-$</td>
<td>1620 ←</td>
<td>1630-1740</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*\frac{5}{2}}^-$</td>
<td>1630 ←</td>
<td>1660-1690</td>
<td>****</td>
</tr>
</tbody>
</table>

Figure 5: The Negative Parity $S = 0$ Excited Baryons: boxes show the experimental regions in which the resonances lie, bars show the predictions of the model for states that should be coupled, with $\Delta M = +50$ MeV.
values for the masses we have also given this range. For an explanation of the experimental status of a state, which we have indicated in these tables and figures, see Ref. 67, but the notation is that states with more stars are better established than those with less. Finally, note that in Figure 5 we have applied an overall shift $\Delta M$ corresponding to the error in our model's prediction for the centre of gravity of the band, in order to display more clearly our predictions for the splittings.

Our model differs from that of Isgur and Karl and their collaborators\textsuperscript{27,36,40–43} in that it attempts to predict the centre of gravity of the bands, whereas in their model this was essentially a free parameter; for these states this is predicted too low by about 50 $MeV$. Apart from this the agreement with experiment for these states is comparable to that of Ref. 27. One significant improvement is that we predict $\Delta^{*\frac{1}{2}^{-}} - \Delta^{*\frac{3}{2}^{-}} \simeq 70$ $MeV$ in agreement with experiment, whereas the usual non-relativistic model without spin-orbit interactions predicts these states to be degenerate.

In Table VI we show our predictions for the negative parity $S=1$ baryons in the $N=1$ band, and in Figure 6 we show those states which are predicted to couple appreciably to $\bar{K}N \rightarrow (\Sigma^{*}, \Delta^{*}) \rightarrow \bar{K}N, \Sigma\pi, \Delta\pi$. We have shifted the centre of gravity of the band upward by 50 $MeV$, as in Figure 5. The results here are again similar to the non-relativistic model, except for the $\Delta^{*\frac{1}{2}^{-}}$ state, which in our model is predicted to be almost 150 $MeV$ too high (i.e. degenerate with the $\Delta^{*\frac{3}{2}^{-}}$, instead of 110 $MeV$ lighter) as opposed to 80 $MeV$ in Ref. 27. The compositions of these states give quite good results for their decay amplitudes. In particular the lowest $\Lambda^{*\frac{1}{2}^{-}}$ state associated with the $\Lambda(1405)$ is predicted to be dominantly $\chi^0\psi^\Lambda$ which is what is necessary\textsuperscript{68} to explain its decays. It may be that the restriction of our model to only include states of three quarks is the source of this discrepancy. This state has an unusually large virtual coupling to the $\bar{K}N$ channel in which it is an $S$-wave resonance. This coupling can result in large mixings of the predicted state to the virtual decay channel which would have the effect of lowering its mass toward the $\bar{K}N$ threshold at 1440 $MeV$.

The $S=0$ and $S=1$ Positive Parity Baryons of the $N=2$ Band

In Table VII and Figure 7 we show our results for the positive parity excited $N$ and $\Delta$ baryons in the $N=2$ band. The centre of gravity of this band is predicted to be 40 $MeV$ too high, and again we have applied this shift to the predictions displayed in Figure 7. From Table VII we can see that there are many states predicted by the model which do not couple to their $\pi N$ formation channel, and those states
Table VI: The Negative Parity $S = -1$ Excited Baryons

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Mass (MeV)</th>
<th>Experiment (MeV)</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda^*_{\frac{1}{2}^-}$</td>
<td>1550 ← ← 1400-1410</td>
<td>****</td>
<td></td>
</tr>
<tr>
<td>1615 ← ← 1660-1680</td>
<td>****</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1675 ← ← 1720-1850</td>
<td>****</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma^*_{\frac{1}{2}^-}$</td>
<td>1630 ← ← 1610-1635</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>1675 ← ← 1730-1800</td>
<td>****</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1695</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Lambda^*_{\frac{3}{2}^-}$</td>
<td>1545 ← ← 1520</td>
<td>****</td>
<td></td>
</tr>
<tr>
<td>1645 ← ← 1685-1695</td>
<td>****</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1770</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma^*_{\frac{3}{2}^-}$</td>
<td>1655 ← ← 1665-1685</td>
<td>****</td>
<td></td>
</tr>
<tr>
<td>1750 ← ← 1900-1950</td>
<td>****</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1755</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Lambda^*_{\frac{5}{2}^-}$</td>
<td>1775 ← ← 1810-1830</td>
<td>****</td>
<td></td>
</tr>
<tr>
<td>$\Sigma^*_{\frac{5}{2}^-}$</td>
<td>1755 ← ← 1770-1780</td>
<td>****</td>
<td></td>
</tr>
</tbody>
</table>

Figure 6: The Negative Parity $S = -1$ Excited Baryons: legend as in Fig. 5.
**Table VII: The Positive Parity $S = 0$ Excited Baryons**

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Mass (MeV)</th>
<th>Experiment (MeV)</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^*_{\frac{1}{2}^+}$</td>
<td>1540 ← 1400-1480</td>
<td>1680-1740</td>
<td>****</td>
</tr>
<tr>
<td></td>
<td>1770 ← 1880</td>
<td>1975</td>
<td>***</td>
</tr>
<tr>
<td>$\Delta^*_{\frac{1}{2}^+}$</td>
<td>1835</td>
<td>1875 ← 1850-1950</td>
<td>****</td>
</tr>
<tr>
<td>$N^*_{\frac{3}{2}^+}$</td>
<td>1795 ← 1690-1800</td>
<td>1870</td>
<td>***</td>
</tr>
<tr>
<td></td>
<td>1910</td>
<td>1950</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2030</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta^*_{\frac{3}{2}^+}$</td>
<td>1795 ← 1520-1690</td>
<td>1915 ← 1860-2060</td>
<td>***</td>
</tr>
<tr>
<td></td>
<td>1985</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N^*_{\frac{5}{2}^+}$</td>
<td>1770 ← 1670-1690</td>
<td>1980</td>
<td>****</td>
</tr>
<tr>
<td></td>
<td>1995</td>
<td>1880-2030</td>
<td>**</td>
</tr>
<tr>
<td>$\Delta^*_{\frac{5}{2}^+}$</td>
<td>1910 ← 1890-1920</td>
<td>1990</td>
<td>****</td>
</tr>
<tr>
<td>$N^*_{\frac{7}{2}^+}$</td>
<td>2000 ← 1970-2020</td>
<td></td>
<td>**</td>
</tr>
<tr>
<td>$\Delta^*_{\frac{7}{2}^+}$</td>
<td>1940 ← 1910-1960</td>
<td></td>
<td>****</td>
</tr>
</tbody>
</table>
Figure 7: The Positive Parity $S = 0$ Excited Baryons: legend as in Fig. 5 but with $\Delta M = -40 \text{ MeV}$. 

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which do couple correspond neatly to those observed experimentally. The quality of the predictions in this sector is higher than that of the usual non-relativistic model. If we consider, for example, the \( J^P = \frac{1}{2}^+ \) Nucleon states, the lower of the two excited Nucleons which couple to \( N\pi \) now couples more strongly than the higher (as opposed to the other way around with the Isgur-Karl compositions), which corresponds to the experimental situation. The \( \Delta^*\frac{1}{2}^+ \)'s predicted by our model have better \( N\pi \) couplings (the higher mass state now couples strongly and the lower decouples convincingly; only one state is seen), although they do have slightly worse \( \Delta\pi \) couplings. The F-wave and P-wave \( \Delta\pi \) decays of the \( \Delta^*\frac{3}{2}^+ \) now have a large ratio, whereas in the calculation using the Isgur-Karl compositions this ratio was of order unity. The experimental P-wave amplitude is too small to be measured\(^{69}\).

Note that the Roper resonance \( N^*\frac{1}{2}^+ (1440) \) is not particularly problematical in our model. Figure 6 shows that, apart from the fact that the entire \( N=2 \) band is predicted to be 40 \( MeV \) too high, the Roper resonance seems to fit quite well into the pattern of states.

In Table VIII we show our predictions for the positive parity excited \( S=-1 \) and barvons in the \( N=2 \) band. Figure 8 shows those states which are predicted to couple to their formation channel, again shifted by the same amount as the \( S=0 \) \( N=2 \) band states. We again find the picture of “missing” resonances of Koniuk and Isgur is slightly improved: an example is that in their analysis only one of the two \( \Delta^*\frac{3}{2}^+ \) states couples strongly to \( \bar{K}N \), the second coupling weakly, whereas we find that both states are well coupled in agreement with experiment in this sector.

Some High Spin \( S=0 \) Baryons

In order to demonstrate that our model has a reasonable picture of the long range (confining) forces between quarks, we show the results of our calculations for the lowest lying state of the \( N \) or \( \Delta \) baryons of either parity with spins from \( \frac{1}{2} \) to \( \frac{5}{2} \). The results are displayed in Table IX and Figure 9. Until a decay analysis is done for the the \( N=3 \) and \( N=4 \) band states we cannot be sure which of the states predicted for the \( N^*\frac{3}{2}^+, \Delta^*\frac{3}{2}^+, N^*\frac{1}{2}^-, N^*\frac{3}{2}^- \) and \( N^*\frac{3}{2}^- \) are the ones seen experimentally. However in the \( N^*\frac{5}{2}^+ \) sector we expect that the state which couples is the one state which we predict to be considerably lower in energy than the (eight) others. For the \( N^*\frac{5}{2}^- \) it is not obvious which of the states in this region couple, so we show them all. For the \( N^*\frac{3}{2}^- \) we expect one of the four states shown to couple; the \( \Delta^*\frac{1}{2}^+ \) and \( N^*\frac{5}{2}^- \) are unique.
Table VIII: The Positive Parity $S = -1$ Excited Baryons

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Mass (MeV)</th>
<th>Experiment (MeV)</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda^{*\frac{1}{2}+}$</td>
<td>1680 ← 1560-1700</td>
<td>1750-1850 ***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1830 ← 1800</td>
<td>2010</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1910 ← 1850</td>
<td>2105</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2120</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma^{*\frac{1}{2}+}$</td>
<td>1720 ← 1630-1690</td>
<td>1830-1985 **</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1915 ← 1850</td>
<td>2005</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1970 ← 1900</td>
<td>2030</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2105</td>
<td>2120</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2160</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Lambda^{*\frac{3}{2}+}$</td>
<td>1900 ← 1850-1910</td>
<td>2010</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1960</td>
<td>2045</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1995</td>
<td>2085</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2050</td>
<td>2115</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2080</td>
<td>2155</td>
<td></td>
</tr>
<tr>
<td>$\Sigma^{*\frac{3}{2}+}$</td>
<td>1920 ← 1800-1925</td>
<td>2070-2140 **</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1970</td>
<td>2030</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2010 ← 2070</td>
<td>2045</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2085</td>
<td>2115</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2115</td>
<td>2180</td>
<td></td>
</tr>
<tr>
<td>$\Lambda^{*\frac{5}{2}+}$</td>
<td>1890 ← 1815-1825</td>
<td>2090-2140 ***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2035 ← 2025</td>
<td>2115</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2115</td>
<td>2180</td>
<td></td>
</tr>
<tr>
<td>$\Sigma^{*\frac{5}{2}+}$</td>
<td>1955 ← 1900-1935</td>
<td>2010-2110</td>
<td></td>
</tr>
</tbody>
</table>
| | 2030 ← 2010-2110 | 2095 ← 2010-2110 | *
| | 2110 | 2180 |
| | 2130 | 2180 |
| $\Lambda^{*\frac{7}{2}+}$ | 2120 ← 2020-2120 | | |
| $\Sigma^{*\frac{7}{2}+}$ | 2060 ← 2025-2040 | 2125 | **** |
Figure 8: The Positive Parity $S = -1$ Excited Baryons: legend as in Fig. 7.
Table IX: The $S = 0$ Baryons plotted against spin-parity.

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Mass (MeV)</th>
<th>Experiment (MeV)</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{\frac{1}{2}}^{+}$</td>
<td>960</td>
<td>939</td>
<td>****</td>
</tr>
<tr>
<td>$\Delta_{\frac{3}{2}}^{+}$</td>
<td>1230</td>
<td>1232</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*}_{\frac{3}{2}}^{+}$</td>
<td>1770</td>
<td>1670-1690</td>
<td>****</td>
</tr>
<tr>
<td>$\Delta^{*}_{\frac{3}{2}}^{+}$</td>
<td>1940</td>
<td>1910-1960</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*}_{\frac{1}{2}}^{+}$</td>
<td>2345</td>
<td>2150-2300</td>
<td>****</td>
</tr>
<tr>
<td>$\Delta^{*}_{\frac{1}{2}}^{+}$</td>
<td>2450</td>
<td>2380-2450</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*}_{\frac{1}{2}}^{-}$</td>
<td>1460</td>
<td>1520-1560</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*}_{\frac{3}{2}}^{-}$</td>
<td>1495</td>
<td>1510-1530</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*}_{\frac{5}{2}}^{-}$</td>
<td>1630</td>
<td>1660-1690</td>
<td>****</td>
</tr>
<tr>
<td>$N^{*}_{\frac{5}{2}}^{-}$</td>
<td>2090</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2205</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N^{*}_{\frac{7}{2}}^{-}$</td>
<td>2255</td>
<td>2120-2230</td>
<td>****</td>
</tr>
<tr>
<td></td>
<td>2305</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2355</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N^{*}_{\frac{7}{2}}^{-}$</td>
<td>2215</td>
<td>2130-2270</td>
<td>****</td>
</tr>
<tr>
<td></td>
<td>2600</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N^{*}_{\frac{9}{2}}^{-}$</td>
<td>2670</td>
<td>2580-2700</td>
<td>***</td>
</tr>
<tr>
<td></td>
<td>2700</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2770</td>
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<td></td>
</tr>
</tbody>
</table>
Figure 9: The $S = 0$ Baryons plotted against spin-parity: boxes show the experimental regions in which the lowest energy resonances lie, bars show the states predicted by the model for each spin-parity.
The $S=-2$ $\Xi$ and $S=-3$ $\Omega$ Baryons up to $N=2$.

Although there are only a few well established $\Xi$ states and only one $\Omega$, it is worthwhile listing here our predictions for the masses of the $N=1$ band and $N=2$ band states (along with the ground states shown above). Table X and Figure 10 show our predictions for the $\Xi$ states, while Table XI and Figure 11 show our predictions for the $\Omega$ states. Note that in both of these Figures we have applied a shift of $+50\;MeV$ to the negative parity excited states, and of $-40\;MeV$ to the positive parity excited states, to correct for the known error of the model in predicting the centre of gravity of the bands. Since there is little experimental data on these states these results are mainly predictions.

Heavy Quark Baryons

Recently there have been discovered$^{57}$ two baryon states which contain one charmed quark, called the $\Lambda_c$ and the $\Sigma_c$ in analogy to the corresponding states with one strange quark. We have found that the charmed quark mass taken from the Godfrey and Isgur model of mesons allows us to predict the mass of the $\Lambda_c^{*}\frac{1}{2}^+$ to within $20\;MeV$. Our model also predicts $\Sigma_c-\Lambda_c=175\pm 15\;MeV$, to be compared with the experimental result of $166\pm 1\;MeV$.

We present in Table XII and Figures 12 and 13 our results for states with one charmed quark up to $N=2$. Note that for the $\Lambda_c^{*}\frac{1}{2}^+$ (and for the $\Lambda_c^{*}\frac{1}{2}^-\frac{1}{2}$ states) there is a state from the $N=4$ band which has the same energy as the highest energy states in the $N=2$ band, and so we have listed its energy along with the other states. Again we have applied shifts to the predictions for the $N=1$ and $N=2$ bands in Figures 12 and 13 to account for our model's known error in predicting the centre of gravity of these bands.

There is also a controversial claim$^{71}$ that a $\Lambda_b$ state has been seen at a mass of $5425^{+175}_{-75}\;MeV$. We have calculated the masses of the beauty baryons up to $N=2$, and the results appear in Table XIII and Figures 14 and 15. Note that our $\Lambda_b$ prediction is consistent with the above claim.

Other Baryons

The experimental states shown in Figures 4 to 15 are nearly all of the known baryons. Our methods are not suited to describe the charmed strange baryons$^{72}$, although we hope to be able to report on these, and baryons with more heavy quarks.
Table X: The Ξ Baryons.

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Masses (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Xi_{1/2}^+$</td>
<td>1305</td>
</tr>
<tr>
<td>$\Xi_{3/2}^+$</td>
<td>1505</td>
</tr>
<tr>
<td>$\Xi_{3/2}^{*-}$</td>
<td>1755 1810 1835</td>
</tr>
<tr>
<td>$\Xi_{5/2}^{*-}$</td>
<td>1785 1880 1895</td>
</tr>
<tr>
<td>$\Xi_{7/2}^{*-}$</td>
<td>1900</td>
</tr>
<tr>
<td>$\Xi_{5/2}^{*+}$</td>
<td>1840 2040 2100 2130 2150 2230</td>
</tr>
<tr>
<td>$\Xi_{7/2}^{*+}$</td>
<td>2045 2065 2115 2165 2170 2210 2230 2275</td>
</tr>
<tr>
<td>$\Xi_{9/2}^{*+}$</td>
<td>2045 2165 2230 2230 2240</td>
</tr>
<tr>
<td>$\Xi_{9/2}^{*-}$</td>
<td>2180 2240</td>
</tr>
</tbody>
</table>

Figure 10: The Ξ Baryons: we have applied $\Delta M = +50$ MeV to the negative parity excited states and $\Delta M = -40$ MeV to the positive parity excited states. Note that the $\Xi(1820)$ is reported$^{67}$ to have spin-$\frac{3}{2}$ but its parity is unknown. Since it fits naturally into the spectrum if we assign a negative parity, we assume it has $J^P = \frac{3}{2}^-$.  

58
Table XI: The $\Omega$ Baryons.

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Masses (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_{\frac{3}{2}}^+$</td>
<td>1635</td>
</tr>
<tr>
<td>$\Omega_{\frac{1}{2}}^- $</td>
<td>1950</td>
</tr>
<tr>
<td>$\Omega_{\frac{1}{2}}^- $</td>
<td>2000</td>
</tr>
<tr>
<td>$\Omega_{\frac{3}{2}}^+$</td>
<td>2220 2255</td>
</tr>
<tr>
<td>$\Omega_{\frac{5}{2}}^+$</td>
<td>2165 2280 2345</td>
</tr>
<tr>
<td>$\Omega_{\frac{3}{2}}^+$</td>
<td>2280 2345</td>
</tr>
<tr>
<td>$\Omega_{\frac{5}{2}}^+$</td>
<td>2295</td>
</tr>
</tbody>
</table>

Figure 11: The $\Omega$ Baryons: legend as in Fig. 10.
Table XII: The $\Lambda_c$ and $\Sigma_c$ Baryons.

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Masses (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_c^{1/2^+}$</td>
<td>2265</td>
</tr>
<tr>
<td>$\Lambda_c^{3/2^-}$</td>
<td>2630 2780 2830</td>
</tr>
<tr>
<td>$\Lambda_c^{1/2^-}$</td>
<td>2640 2840 2885</td>
</tr>
<tr>
<td>$\Lambda_c^{3/2^-}$</td>
<td>2900</td>
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<tr>
<td>$\Lambda_c^{1/2^+}$</td>
<td>2775 2970 3015 3075 3170 3185 3200</td>
</tr>
<tr>
<td>$\Lambda_c^{3/2^+}$</td>
<td>2910 3035 3080 3145 3190 3200 3220</td>
</tr>
<tr>
<td>$\Lambda_c^{5/2^+}$</td>
<td>2910 3140 3165 3225 3230</td>
</tr>
<tr>
<td>$\Lambda_c^{7/2^+}$</td>
<td>3175</td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
<td>$\Sigma_c^{7/2^+}$</td>
<td>3090 3230</td>
</tr>
</tbody>
</table>
Figure 12: The $\Lambda_c$ Baryons: legend as in Fig. 10.

Figure 13: The $\Sigma_c$ Baryons: legend as in Fig. 10.
Table XIII: The $\Lambda_b$ and $\Sigma_b$ Baryons.

<table>
<thead>
<tr>
<th>State, $J^P$</th>
<th>Predicted Masses (MeV)</th>
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<tr>
<td>$\Lambda_b \frac{1}{2}^+$</td>
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</tr>
<tr>
<td>$\Lambda_b \frac{3}{2}^-$</td>
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</tr>
<tr>
<td>$\Lambda_b \frac{1}{2}^-$</td>
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</tr>
<tr>
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<td>$\Lambda_b \frac{1}{2}^+$</td>
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<td>$\Lambda_b \frac{3}{2}^+$</td>
<td>6145 6305 6355 6425 6470 6515 6515</td>
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<tr>
<td>$\Lambda_b \frac{1}{2}^+$</td>
<td>6165 6425 6440 6510 6510</td>
</tr>
<tr>
<td>$\Lambda_b \frac{3}{2}^+$</td>
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<tr>
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</tr>
<tr>
<td>$\Sigma_b \frac{5}{2}^+$</td>
<td>6340 6535</td>
</tr>
</tbody>
</table>
Figure 14: The $\Lambda_b$ Baryons: legend as in Fig. 10.

Figure 15: The $\Sigma_b$ Baryons: legend as in Fig. 10.
in them, in the future. There are also states made up of $u$, $d$, and $s$ quarks which are in higher $N$ bands for which we have not completed a coupling analysis, so we delay reporting on these for now. From our spectroscopic results for these states (which we have not fully described here, but see Table IX and Fig. 9) we expect that most will fit into our model. However, there is a state in the $N=3$ band, a $\Delta^*_{\frac{3}{2}^-}$ with mass $1930 \pm 30 \text{ MeV}$, which deserves special mention. This state has a controversial history; it appears to be too low in mass to belong to the $N=3$ band, and a suggestion was made that it was a hybrid baryon. Later it was noted that the $[56, 1^-] \text{SU}(6)$ multiplet containing this state naturally breaks off from the $N=3$ band, in the same way as the $[56, 0^+] \text{SU}(6)$ multiplet whose lowest member is the $N^*_{\frac{3}{2}^+}(1440)$ breaks off from the $N=2$ band. It was therefore argued that this was an ordinary $[56, 1^-]$ state. Our calculations do not resolve this controversy, since they predict the state to be at $2030 \text{ MeV}$. However we note that while our model is good at predicting the splittings within a given band, it is less accurate in predicting the absolute mass of a band; it misses each of the $N=1$ and $N=2$ bands by about $50 \text{ MeV}$. It would help to examine the couplings and decays of this state to see if our predictions match those seen experimentally.

We tentatively conclude that the known baryons all fit reasonably well into our model. In particular, we see no compelling evidence for any hybrid baryons (with excited glue) in the spectrum. For this reason it would appear that such states lie above $2 \text{ GeV}$, as expected in some models. If this is so it will be very difficult to find such states, since there are very many ordinary $qqq$ states in the $N=3$ and $N=4$ bands, and the hybrid baryons are not expected to have exotic quantum numbers.
IV.3 DISCUSSION

Relativistic Corrections and the Spin-Orbit Puzzle

The apparent absence of spin-orbit interactions in the baryon spectrum contrary to the expectation of the non-relativistic potential models has been a problem in baryon spectroscopy. We have found that this flaw is an example of a defect which cannot be absorbed into a re-definition of the parameters of the low energy theory (for example the quark masses or the strong coupling constant) but rather requires a model which treats realistically the expected relativistic behaviour of the quark Hamiltonian.

To see how this comes about, consider first that introducing relativistic effects into the Hamiltonian has the effect of regulating singular interactions like the hyperfine contact energy, as we have seen above. In previous calculations this interaction was treated in first order perturbation theory, so that even though a delta-function operator is illegal in the Schrödinger equation, finite results were obtained. However if higher order perturbation theory had been applied to the problem, the ground state expectation value of this operator would have decreased without bound. Even if the smeared delta-function used in this work had been used; first order perturbation theory would have underestimated the effect of the hyperfine contact interaction. Since the contact interaction is responsible for important effects in the spectrum, to compensate for the small expectation value yielded by first order perturbation theory a larger $\alpha_s$ was required to obtain, for example, the correct $\Delta - N$ splitting. This in turn increased the expectation values of other terms in the Hamiltonian proportional to $\alpha_s$, like the one-gluon exchange spin-orbit interaction, which did not have this singular behaviour, so that when this interaction was included into the baryon Hamiltonian, it spoiled the good spectroscopic results obtained from the hyperfine interaction.

In this work we have found that the effects on the spectrum of the spin-orbit interactions expected in the Hamiltonian are small in comparison to those of the contact interaction. This is partly due to $\epsilon_{so}$ being smaller than $\epsilon_{con}$, and partly (as has also been found by other authors in the past) that a partial cancellation of the confining potential and one-gluon exchange potential spin-orbit interactions occurs. However the strongest reduction in the spin-orbit strength comes about because we are calculating the hyperfine interaction non-perturbatively, and so are
using a significantly smaller $\alpha_s$ than has been used in the past.

Another interesting consequence of the use of the relativistic corrections, which were described in Chapter II, is the extent to which the baryon spectrum is insensitive to the parameters of the model. An example is the insensitivity of the contact interaction to $\alpha_s$ and the smearing parameters $\sigma_{ij}$. If we increase $\alpha_s$ the contact interaction potential-well that the proton sits in becomes deeper. Similarly if we increase $\sigma_{ij}$ the contact interaction, which is proportional to $\sigma_{ij}^3 e^{-\sigma_{ij}^2 r^2}$, becomes deeper and more localised around the origin. The variational calculation allows the wavefunction to adjust its size so that it has a larger probability of sitting over this well near the origin, and so lowers its energy. As a consequence it becomes more localised and so the average quark momentum increases. This in turn decreases the factor $\left(\frac{1}{E_i E_j}\right)^{1+2E_i E_j}$ which decreases the strength of the contact interaction. The resulting change in the proton contact energy is only a fraction of the change which would have appeared if the relativistic factors had not been present.

The Unification of Meson and Baryon Spectroscopy

When the parameters of the Godfrey-Isgur analysis of meson spectroscopy were introduced into our model we found a remarkable agreement between the predicted spectrum and experiment. It appears that the main features of the meson and baryon spectrum have a similar origin. The same contact interaction yielded the correct $\Delta - N$ and $\rho - \pi$ splittings. Similarly the same colour Coulomb plus linear spin-independent potential yielded the correct splittings $N^*_{\frac{5}{2}} - (1520) - N_{\frac{1}{2}} + (940)$ and $A_2 - \rho$ between states with one unit of orbital excitation and the ground state. The same quark masses were responsible for the correct $\Sigma - \Lambda$ splitting and for the correct ratio between $K^* - K$ and $\rho - \pi$, via the mass dependence of the hyperfine splittings, and for the correct $A_{\frac{5}{2}} - \Sigma_{\frac{5}{2}}$ splitting and the correct ratio between $f' - A_2$ and $\phi - \rho$ via the mass dependence of the orbital excitation energies.

We then asked if there was a set of parameters which could improve on the spectroscopy. We found that the baryon spectrum supports constituent masses identical to those required for the meson spectrum. However the baryon spectrum, especially the spectrum of the negative parity $N$ and $\Delta$ states, needs some $\epsilon$ parameters which are slightly different from those of Godfrey and Isgur. Indeed, in our best fit only our parameters $\epsilon_{cont}$ and $\epsilon_{coul}$ are the same as those in Ref. 62 (Godfrey and Isgur did not allow for an $\epsilon_{coul}$, but this is equivalent to our observation that $\epsilon_{coul} = 0$). However, it has been verified that adopting the new set of $\epsilon$'s does not adversely affect meson spectroscopy. The reason for this is that in baryons tensor
and spin-orbit effects can cause mixing between states like the two \( L = 2 \Delta \frac{1}{2} \) states with \( S = \frac{1}{2} \) and \( S = \frac{3}{2} \) which are degenerate in the absence of spin-dependent effects. When we have two states that are degenerate split by small perturbations, we can have large mixing angles which show up when we look at the compositions of the states. In mesons, however, such mixings are almost always small since the corresponding states are split by orbital excitation energy. Furthermore baryon spectroscopy seems to require the same \( \epsilon_{\Delta} \) and \( \epsilon_{\Sigma} \), which is natural since these two terms arise from the same term in the one-gluon exchange \( T \)-matrix element.

We have found, on the other hand, that the string tension used in the meson analysis of Ref. 62 consistently overestimates the masses of the baryons in the higher \( N \) bands. As a result we have adopted a string tension of \( 0.15 \text{ GeV}^2 \) as opposed to the \( 0.18 \text{ GeV}^2 \) of Ref. 62. This difference is a significant one, as shown by Godfrey's analysis of high spin mesons.\(^7\) It is however not fatal to our plan of unifying meson and baryon spectroscopy. We have mentioned above that our confining potential is generated by assuming that the \( Y \) shaped string finds its minimum energy quickly in a given quark configuration, so that the quarks move adiabatically in the potential generated by the string. One of the effects that we have ignored in our discussion of the confining potential is mixing of excited states of the string into this lowest energy string configuration. Whether or not this is a good approximation depends on the splitting between the lowest and first excited adiabatic surfaces of the string. Similarly we have neglected the mixing of \( q\bar{q} \) pairs into our states. It is clear that such effects may result in different effective string tensions in mesons and baryons.

The other parameters in Table III which describe the potentials (with the exception of \( C_{qqq} \) which has no analogue in the meson system) and the relativistic smearing parameters \( \sigma_0 \) and \( \sigma \) are the same as those used in Ref. 62. We conclude that with identical physics and with parameter sets that are as identical as could be expected given the simplifications inherent in the quark model, it is now possible to satisfactorily describe all of conventional meson and baryon spectroscopy.
CHAPTER V: CONCLUSIONS AND COMMENTS

One drawback of this work is that it lacks a thorough study of the strong couplings of these states, preferably in a flux tube breaking\textsuperscript{78} or naive quark pair creation model\textsuperscript{79}. It would also be useful to study the (relativized) photodecay amplitudes, weak current matrix elements, charge radii, etc., to see if these would improve the model, as indicated in Ref.\textsuperscript{62} and in Ref.\textsuperscript{80}. We hope to report on these topics in the future. Another problem is that we have had to resort to a phenomenological description of the cutoff and momentum dependence of the Hamiltonian by introducing the parameters $\sigma_0$, $s$ and the $\epsilon$'s, when in principle this dependence should be calculable. We would like to be able to estimate the effects of $q\bar{q}$ mixing into our states, and the effects of going beyond the lowest adiabatic surface approximation for the glue.

Despite these reservations this model has had considerable success in describing the properties of all baryons in a unified treatment. We have found that the good results of the usual non-relativistic model without spin-orbit effects have been retained, and in some sectors improved upon, in a much more tightly constrained treatment of all of the baryons. Other models may provide a more accurate description of some subset of this work, but we feel it is very important that we use a consistent model for all the baryons. Another important constraint that we have imposed on ourselves from the outset is that we work within the confines of a model which is capable of describing the properties of mesons.

The model described here is not QCD, however we have tried within our simplified model to retain many of the features of QCD. Our strong coupling constant evolves with $Q^2$ at short distance as QCD says it should, and with the accepted value of $\Lambda_{QCD}$. Our long distance potential is the one which one expects in the strong coupling limit of QCD, if one adopts the adiabatic approximation and ignores the effects of mixing of excitations of the glue (and $q\bar{q}$ pairs) into these states. We have also made a rough parameterization of relativistic effects by adopting the relativistic form of the kinetic energy, by allowing the quarks to have a mass dependent form factor, and by allowing for momentum dependence in the factors which multiply the one-gluon exchange terms and confining potential spin-orbit terms, as suggested by a relativistic treatment of the appropriate T-matrix elements. In spite of these simplifications we feel that the results presented here demonstrate that a successful quark model can be built which incorporates the effects expected from
QCD and from a relativistic treatment of the quark dynamics.
APPENDIX A: EVALUATION OF THE MATRIX ELEMENTS

In Chapter III we discussed the general outline of the calculation of matrix elements. We also discussed techniques for dealing with the $H_{13}$ and $H_{23}$ matrix elements, and with the momentum dependent factors in the Hamiltonian. Here we show how to obtain matrix elements of the various interactions in terms of simple radial integrals.

The Coulomb- and Two-Body String Energies

The Coulomb and two-body string energies (see (43) and (56)) are terms which are scalar in space and independent of spin. Recall that, for the reasons discussed in Chapter III, we have specialised our discussion to the calculation of $\tilde{V}_{12}(r_{12})$ matrix elements, so we evaluate $\langle \alpha | \tilde{G}(r_{12}) | \beta \rangle$, with $|\alpha\rangle$ and $|\beta\rangle$ two states with the same $JM$ given by equation (66). We also postpone discussion of the expectation value of the momentum dependent factors multiplying this and the other spatial potentials until later in this Appendix. Since $\tilde{G}(r_{12})$ only depends on $r_{12} = \sqrt{2}\rho$, the $d^3\vec{r}$ integration is zero, unless the two spatial wavefunctions $\Psi^\alpha$ and $\Psi^\beta$ have identical $\vec{r}$ oscillator wavefunctions, and the $d^3\vec{r}$ integration is zero unless the two spatial wavefunctions have identical $I_\rho$ values. Furthermore, since the interaction is an overall spin and space scalar, the entire matrix element is zero unless the spin wavefunctions are identical, and unless the total orbital angular momenta are equal. If we adopt the notation

$$|nl\rangle = \alpha^{\frac{1}{2}} N_{nl}(\alpha r) e^{-\frac{1}{2}\alpha^2 r^2} L_n^{l+\frac{1}{2}}(\alpha r) \quad A(1)$$

and

$$\langle n_{\rho a} l_{\rho a} | V(\sqrt{2}\rho) | n_{\rho b} l_{\rho b}\rangle = N_{n_{\rho a} l_{\rho a}} N_{n_{\rho b} l_{\rho b}} \alpha^2 \int_0^\infty \rho^2 d\rho e^{-\alpha^2 \rho^2} (\alpha \rho)^{l_{\rho a} + l_{\rho b} + \frac{1}{2}} (\sqrt{2}\alpha \rho)V(\sqrt{2}\rho) L_n^{l_{\rho a} + l_{\rho b} + \frac{1}{2}}(\sqrt{2}\alpha \rho) , \quad A(2)$$

then we may write

$$\langle \alpha | \tilde{G}(r_{12}) | \beta \rangle = \delta_{\alpha \beta} \delta_{L_a L_b} \delta_{l_{\rho a} l_{\rho b}} \delta_{n_{\lambda a} n_{\lambda b}} \delta_{i_{\lambda a} i_{\lambda b}} \langle n_{\rho a} l_{\rho a} | \tilde{G}(\sqrt{2}\rho) | n_{\rho b} l_{\rho b}\rangle \quad A(3)$$

Note that $S_\alpha$ represents both the type and total spin of the spin wavefunction of the state $|\alpha\rangle$. The analogous equation holds for the matrix element of the effective two-body string potential with $\tilde{G}$ replaced by $\tilde{V}_{\text{string}}$.  

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The Hyperfine Contact Energy

The hyperfine contact energy $\tilde{V}_{ij}^{\text{cont}}(r_{12})$ (see (48)) is an example of a scalar operator which is simultaneously scalar in both space and spin. Its expectation value breaks up into a product of a radial integral and a spin matrix element,

$$\langle \alpha | \tilde{V}_{ij}^{\text{cont}}(r_{12}) | \beta \rangle = \delta_{L_\alpha L_\beta} \delta_{l_{1\alpha} l_{1\beta}} \delta_{n_{1\alpha} n_{1\beta}} \delta_{l_{\lambda\alpha} l_{\lambda\beta}} \langle n_{\rho \lambda} l_{\rho \lambda} | \sum_k \frac{2 \sqrt{3}}{3 m_1 m_2} \frac{8 \pi \sigma_{k12}^3}{\pi^3} e^{-2 \sigma_{k12}^2 \rho^2} | n_{\rho \lambda} l_{\rho \lambda} \rangle \langle S_{\alpha m} | S_1 \cdot S_2 | S_{\rho m} \rangle, \quad A(4)$$

where the above holds for any $m$, and again we have ignored the momentum dependent factors in (48) for the time being. The spin matrix elements are very simple, the only non-zero matrix elements being

$$\langle x^{M_\alpha}_{\lambda m} | S_1 \cdot S_2 | x^{M_\beta}_{\lambda m} \rangle = \frac{-3}{4}$$
$$\langle x^{M_\alpha}_{\lambda m} | S_1 \cdot S_2 | x^{M_\beta}_{\lambda m} \rangle = \frac{1}{4}$$
$$\langle x^{S_{\lambda m}}_{\lambda m} | S_1 \cdot S_2 | x^{S_{\lambda m}}_{\lambda m} \rangle = \frac{1}{4}$$

The Hyperfine Tensor Interaction

The tensor terms (49) are evaluated with the aid of the Wigner-Eckhart Theorem, which is applied twice: once to the scalar product of the spin tensor and spatial tensor operators which make up this term in the Hamiltonian, and then again to the $L = 2$ tensor operator which is to be evaluated in a basis made up from coupling $I_\alpha$ and $I_\lambda$ to give $L$. Firstly let us write the tensor term $\tilde{V}_{12}^{\text{tens}}$ in terms of a scalar product of two operators,

$$\tilde{V}_{12}^{\text{tens}} = \tilde{V}^t(\sqrt{2} \rho) R_2(12) \cdot S_2(12), \quad A(6)$$

where

$$R_2(12) = \begin{pmatrix} \frac{3 \sqrt{f_0}}{2} & -\sqrt{3} f_0 \\ \frac{\sqrt{3} f_0}{2} & 3 f_0 \end{pmatrix}$$

$$S_2(12) = \begin{pmatrix} \frac{3}{2} S_{1-} S_{2-} \\ \frac{3}{2} (S_{1+} S_{2+} + S_{1-} S_{2+}) \end{pmatrix}$$

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and where
\[ \tilde{V}^i(\sqrt{2}\rho) = \sum_k \frac{2\alpha_k}{3m_1m_2} \left( \frac{erf(\sigma_{k12}\sqrt{2}\rho)}{2\sqrt{2}\rho^3} - \frac{4\pi \sigma_{k12}^3 e^{-\sigma_{k12}^2\rho^2}}{3\pi^{\frac{3}{2}}} \left( 1 + \frac{3}{4\sigma_{k12}^2\rho^2} \right) \right). \]  

Then we apply the Wigner-Eckhart Theorem to the tensor product \( \tilde{V}_{12}^{\text{tens}} \) to obtain
\[ \langle \alpha| \tilde{V}_{12}^{\text{tens}} | \beta \rangle = (-1)^{l_{\alpha} + l_{\beta} - l_{\lambda}} W(L\alpha L\beta S\alpha S\beta; 2J) \sqrt{2L\alpha + 1}\sqrt{2S\alpha + 1} \]
\[ \langle L\alpha n_{\rho_{\alpha}} l_{\rho_{\alpha}} n_{\lambda_{\alpha}} l_{\lambda_{\alpha}} | \tilde{V}^i(\sqrt{2}\rho) R_2(12) | L\beta n_{\rho_{\beta}} l_{\rho_{\beta}} n_{\lambda_{\beta}} l_{\lambda_{\beta}} \rangle \]
\[ \langle S\alpha || S_2(12) || S\beta \rangle. \]  

A(9)

The theorem is applied again to the spatial reduced matrix element
\[ \langle L\alpha n_{\rho_{\alpha}} l_{\rho_{\alpha}} n_{\lambda_{\alpha}} l_{\lambda_{\alpha}} | \tilde{V}^i(\sqrt{2}\rho) R_2(12) | L\beta n_{\rho_{\beta}} l_{\rho_{\beta}} n_{\lambda_{\beta}} l_{\lambda_{\beta}} \rangle = \]
\[ (-1)^{l_{\alpha} + l_{\beta} - l_{\lambda}} W(l_{\rho_{\alpha}} l_{\rho_{\beta}} L\alpha L\beta; 2l_{\lambda}) \]
\[ \langle n_{\rho_{\alpha}} l_{\rho_{\alpha}} | \tilde{V}^i(\sqrt{2}\rho) R_2(12) | n_{\rho_{\beta}} l_{\rho_{\beta}} \delta_{\lambda_{\alpha}, \lambda_{\beta}} \delta_{l_{\lambda_{\alpha}}, l_{\lambda_{\beta}}} \rangle. \]  

A(10)

The spin and space reduced matrix elements which appear in A(9) and A(10) respectively are
\[ \langle S\alpha || S_3(12) || S\beta \rangle = \begin{pmatrix} \sqrt{\frac{5}{8}} & 0 & \sqrt{\frac{5}{8}} \\ 0 & 0 & 0 \\ -\sqrt{\frac{3}{2}} & 0 & 0 \end{pmatrix}, \]  

A(11)

where the \( ij \)th entry in the matrix corresponds to \( \langle \chi_i || S_2(12) || \chi_j \rangle \), with \( \chi_1 = \chi^S_3 \), \( \chi_2 = \chi^O_\frac{1}{2} \) and \( \chi_3 = \chi^S_\frac{1}{2} \), and
\[ \langle n_{\rho_{\alpha}} l_{\rho_{\alpha}} | \tilde{V}^i(\sqrt{2}\rho) R_2(12) | n_{\rho_{\beta}} l_{\rho_{\beta}} \rangle = C(l_{\rho_{\beta}} 200; l_{\rho_{\alpha}} 0) \frac{\sqrt{2(2l_{\rho_{\beta}} + 1)}}{2l_{\rho_{\alpha}} + 1} \]
\[ \langle n_{\rho_{\alpha}} l_{\rho_{\alpha}} | \tilde{V}^i(\sqrt{2}\rho) | n_{\rho_{\beta}} l_{\rho_{\beta}} \rangle. \]  

A(12)

The Spin-Orbit Interactions

If we apply the baryon centre-of-momentum frame identities
\[ \vec{p}_1 = \frac{1}{\sqrt{2}} \vec{p}_\rho + \frac{1}{\sqrt{6}} \vec{p}_\lambda \]
\[ \vec{p}_2 = -\frac{1}{\sqrt{2}} \vec{p}_\rho + \frac{1}{\sqrt{6}} \vec{p}_\lambda \]
\[ \vec{p}_3 = -\frac{2}{\sqrt{6}} \vec{p}_\lambda \]

A(13)
to the spin-orbit potentials (51) and (58), we obtain (note that because of the
unusual dependence of the spin-orbit interactions on the quark masses, we show
both the (12) and (13) potentials)

\[ V_{12}^{\pm} = \frac{1}{2\sqrt{2\rho}} \frac{d\tilde{V}}{dr_{12}} \left( \frac{4g - 1}{m_1^2} (\vec{S}_1 + \vec{S}_2) \cdot \vec{r}_\rho - \frac{1}{m_1^2} (\vec{S}_1 - \vec{S}_2) \cdot \frac{1}{\sqrt{3}} \vec{p} \times \vec{p}_\rho \right) \]

\[ V_{13}^{\pm} = \frac{1}{2\sqrt{2\rho'}} \frac{d\tilde{V}}{dr_{13}} \left[ \frac{(2g - 1)}{m_2^2} \left( \vec{S}_1 \cdot \vec{r}_\rho' + \vec{S}_1 \cdot \frac{1}{\sqrt{3}} \vec{p}' \times \vec{p}_\rho' \right) \right.

\[ + \frac{(2g - 1)}{m_3^2} \left( \vec{S}_3 \cdot \vec{r}_\rho' - \vec{S}_3 \cdot \frac{1}{\sqrt{3}} \vec{p}' \times \vec{p}_\rho' \right) \]

\[ \left. + \frac{2g}{m_1 m_3} \left( (\vec{S}_1 + \vec{S}_3) \cdot \vec{r}_\rho' - (\vec{S}_1 - \vec{S}_3) \cdot \frac{1}{\sqrt{3}} \vec{p}' \times \vec{p}_\rho' \right) \right] \quad (14) \]

where \( g = 1 \) and \( \tilde{V} = \tilde{G} \) for the one-gluon exchange spin-orbit potential, and \( g = 0 \)
and \( \tilde{V} = \tilde{V}_{\text{string}} \) for the spin-orbit potential from confinement.

The terms proportional to \( \vec{p} \times \vec{p}_\rho \) and \( \vec{p}' \times \vec{p}_\rho' \) are the three-body spin-orbit
potentials. These can be understood as consequences of the Wigner rotation
involved with boosting from the centre of mass frame of the baryon to the centre
of mass frame of quarks \( i \) and \( j \). They must be dealt with specially, because they
are of a completely different form than the rest of the operators dealt with so far.
Let us write a general (12) spin-orbit potential (a similar form holds with different
mass dependences for the (13) potentials in the primed frame) in the form

\[ \frac{1}{2\sqrt{2m_2}} \frac{F(\rho)}{\rho} \left[ c_1 \vec{S}_{2b} \cdot \vec{r}_\rho + c_2 \vec{S}_{3b} \cdot \frac{1}{\sqrt{3}} \vec{p} \times \vec{p}_\rho \right] \quad (15) \]

where

\[ F(\rho) = \begin{cases} \frac{d\tilde{V}_{\text{string}}}{dr_{12}} \quad \text{(one-gluon exchange)} \\ \frac{d\tilde{V}}{dr_{12}} \quad \text{(confinement)} \end{cases} \quad (16) \]

\( c_1 \) and \( c_2 \) are constants, and where \( S_{2b} \) and \( S_{3b} \) are of the form \( a\vec{S}_1 + b\vec{S}_2 \). Then we
may apply the Wigner-Eckhart theorem to the spin-space scalar product to obtain

\[ \langle \alpha | \tilde{V}_{12}^{\pm} | \beta \rangle = \frac{1}{2\sqrt{2m_1}} (-1)^{J-L_\rho} W(L_\alpha L_\beta S_\alpha S_\beta; 1J) \sqrt{2L_\alpha + 1} \sqrt{2S_\alpha + 1} \left[ c_1 \langle L_\alpha n_{\rho a} l_\rho n_\lambda \lambda_\alpha l_\lambda \frac{F(\rho)}{\rho} \frac{1}{\sqrt{3}} \vec{p} \times \vec{p}_\rho \| L_\beta n_{\rho b} l_\rho n_\lambda \lambda_\beta (S_\alpha \| \vec{S}_{2b} \| S_\beta) \right. \]

\[ + \frac{c_2}{\sqrt{3}} \langle L_\alpha n_{\rho a} l_\rho n_\lambda \lambda_\alpha \| \frac{F(\rho)}{\rho} \frac{1}{\sqrt{3}} \vec{p} \times \vec{p}_\rho \| L_\beta n_{\rho b} l_\rho n_\lambda \lambda_\beta (S_\alpha \| \vec{S}_{3b} \| S_\beta) \right] \quad (17) \]
We may then apply the theorem again to the spatial reduced matrix elements
\[
(L_\alpha l_{\rho\alpha} n_{\lambda\alpha} l_{\lambda\alpha} \frac{F(\rho)}{\rho} l^\rho || L_\beta n_{\rho\beta} l_{\rho\beta} n_{\lambda\beta} l_{\lambda\beta}) = (-1)^{l_{\rho\alpha} + l_{\rho\beta}} \frac{1}{l_{\rho\alpha} + 1} \sqrt{2L_\beta + 1} l_{\rho\beta} \left( \begin{array}{ccc} L_\alpha & L_\beta & 1 \\ l_{\rho\alpha} & l_{\rho\beta} & 1 \\ l_{\lambda\alpha} & l_{\lambda\beta} & 1 \end{array} \right)
\]
and
\[
(L_\alpha l_{\rho\alpha} n_{\lambda\alpha} l_{\lambda\alpha} || \frac{F(\rho)}{\rho} \bar{\rho} \times \bar{\rho} || L_\beta n_{\rho\beta} l_{\rho\beta} n_{\lambda\beta} l_{\lambda\beta}) = \sqrt{(2L_\beta + 1)(2l_{\rho\alpha} + 1)(2l_{\lambda\alpha} + 1)} X \left( \begin{array}{ccc} L_\alpha & L_\beta & 1 \\ l_{\rho\alpha} & l_{\rho\beta} & 1 \\ l_{\lambda\alpha} & l_{\lambda\beta} & 1 \end{array} \right)
C(l_{\rho\alpha} 100; l_{\rho\beta} 0) \langle n_{\rho\alpha} l_{\rho\alpha} | F(\rho) | n_{\rho\beta} l_{\rho\beta} \rangle
C(l_{\lambda\alpha} 100; l_{\lambda\beta} 0) \langle n_{\lambda\alpha} l_{\lambda\alpha} | l_{\lambda\beta} l_{\lambda\beta} \rangle
\]
where \( X \) is a 9-j coefficient. The spin-reduced matrix elements appearing in A(17) are
\[
\langle S_\alpha || a\bar{s}_1 + b\bar{s}_2 || S_\beta \rangle = \left( \begin{array}{ccc} \frac{1}{2} \sqrt{3} (a + b) & \frac{1}{2} (a - b) & -\frac{1}{2\sqrt{3}} (a + b) \\ -\frac{1}{\sqrt{3}} (a - b) & 0 & -\frac{1}{2} (a - b) \\ -\frac{1}{\sqrt{6}} (a + b) & -\frac{1}{2} (a - b) & \frac{1}{\sqrt{3}} (a + b) \end{array} \right)
\]
where the \( ij \)th entry in the matrix corresponds to \( \langle \chi_i || a\bar{s}_1 + b\bar{s}_2 || \chi_j \rangle \), with the \( \chi_i \) as in A(11).

The Three-Body String Energy

We saw above that \( V_{\text{string}} (52) \) can be simply expressed in terms of \( \rho, \lambda \) and \( \cos \theta = \frac{E_X}{E_{\text{sc}}} \). Similarly the effective two body string potential can be written down in terms of these variables. We now form the function \( V_{3\beta}(\rho, \lambda, \cos \theta) \) from \( V_{\text{string}} \) by subtracting \( f\sqrt{\sigma} \sum_{i<j} r_{ij} \) (see (55)). Our task is to find the matrix element \( \langle \alpha || V_{3\beta} || \beta \rangle \). First we evaluate the angular integral (the Clebsch-Gordan sums required to couple the \( \rho \) and \( \lambda \) angular momenta are suppressed for simplicity)
\[
\int d\Omega_\rho d\Omega_\lambda Y_{l_{\rho\alpha}}^* m_{\rho\alpha}(\Omega_\rho) Y_{l_{\lambda\alpha}}^* m_{\lambda\alpha}(\Omega_\lambda) V_{3\beta}(\rho, \lambda, \cos \theta) Y_{l_{\rho\beta}} m_{\rho\beta}(\Omega_\rho) Y_{l_{\lambda\beta}} m_{\lambda\beta}(\Omega_\lambda)
\]
This is accomplished by re-writing the product of the two \( Y(\Omega_\rho) \)'s as a sum over a single \( Y_{L_{\rho}M_{\rho}}(\Omega_\rho) \), and similarly for the \( Y(\Omega_\lambda) \)'s. Since the angle \( \theta \) is the angle
of the $\tilde{\lambda}$ vector with respect to the $\tilde{\rho}$ vector, we may perform the $d\Omega_\lambda$ integral if we first rotate the $Y(\Omega_\lambda)$ into the coordinate frame where $\tilde{\rho}$ lies along the $z$-axis, using the relation

$$Y_{L_\lambda M_\lambda}(\Omega_\lambda) = \sum_N Y_{L_\lambda N}(\Omega_\lambda^c) D_{N M_\lambda}^{L_\lambda}(-R), \quad A(23)$$

where $D$ is a rotation matrix, $R$ is the rotation with Euler angles $\phi_\rho$, $\theta_\rho$ and $-\phi_\rho$, and $\Omega_\lambda^c = (\theta, \phi_\lambda^c)$ is the angular position of $\tilde{\lambda}$ with respect to $\tilde{\rho}$. The resulting $d\Omega_\lambda$ integral is proportional to

$$\delta_{N0} V_{L_\lambda}(\rho, \lambda) = \delta_{N0} \int_0^1 d\cos\theta P_{L_\lambda}(\cos\theta)V_{3b}(\rho, \lambda, \cos\theta), \quad A(23)$$

where $P_{L_\lambda}$ is a Laguerre polynomial. Carrying out the sum over $N$, we find that the $D$ matrix turns into a $Y(\Omega_\rho)$, which may then be integrated against the remaining $Y_{L_\rho M_\rho}(\Omega_\rho)$ to yield

$$(-1)^{l_\rho - \lambda_\rho + l_\rho - l_\lambda}(2L + 1)\sqrt{(2l_\rho + 1)(2l_\rho + 1)(2l_\lambda + 1)(2l_\lambda + 1)}$$

$$\sum_{m_\rho m_\lambda} (-1)^{m_\rho + m_\lambda} \begin{pmatrix} l_\rho & l_\lambda & L \\ m_\rho & m_\lambda & M \end{pmatrix} \begin{pmatrix} l_\rho & l_\lambda & L \\ m_\rho & m_\lambda & M \end{pmatrix}$$

$$\sum_{L_\rho M_\rho} (2L_\rho + 1)(-1)^{M_\rho} \begin{pmatrix} l_\rho & l_\rho & L_\rho \\ -m_\rho & m_\rho & M_\rho \end{pmatrix} \begin{pmatrix} l_\rho & l_\rho & L_\rho \\ -m_\rho & m_\rho & M_\rho \end{pmatrix}$$

$$\left( \begin{array}{ccc} l_\rho & l_\rho & L_\rho \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_\lambda & l_\lambda & L_\rho \\ 0 & 0 & 0 \end{array} \right) V_{L_\rho}(\rho, \lambda), \quad A(24)$$

where $LM$ is the total angular momentum of both $|\alpha\rangle$ and $|\beta\rangle$, and we have written all Clebsh-Gordan coefficients in terms of the $3\text{-}j$ coefficients for clarity. The next step is to identify the sum over the magnetic quantum numbers as a Racah coefficient and restore the radial integrations which results in (after some algebra)

$$\langle \alpha|V_{3b}|\beta \rangle = \delta_{S_\alpha S_\beta} \delta_{L_\alpha L_\beta} \sqrt{(2l_\rho + 1)(2l_\rho + 1)(2l_\lambda + 1)(2l_\lambda + 1)(-1)^{L + l_\lambda + l_\rho}}$$

$$\sum_{L_\rho} C(l_\rho, 0 l_\rho; L_\rho 0) C(l_\lambda, 0 l_\lambda; L_\rho 0) W(l_\rho, l_\rho, l_\lambda, l_\lambda; L_\rho L)$$

$$\langle n_\rho, l_\rho |n_\lambda, l_\lambda |V_{L_\rho}(\rho, \lambda) |n_\lambda, l_\lambda |n_\rho, l_\rho \rangle, \quad A(26)$$

where the last line is the obvious generalisation of the radial integral in A(2). The $\cos(\theta)$ integral in A(23) and the $\rho$ and $\lambda$ integrals in A(26) can be carried out numerically.
Momentum Dependent Terms

It remains to discuss the evaluation of the matrix elements of the momentum dependent terms, which consist of the kinetic energy (11), and the momentum dependent factors (which are to be evaluated separately from the spatial potentials they modify, as in (75)). The centre-of-momentum frame identities A(13) indicate that the way to proceed with the kinetic energy is to evaluate the matrix element \( \langle \alpha | (p_1^2 + m_1^2)^{\frac{3}{2}} | \beta \rangle \) in the \( \vec{p}, \vec{\lambda} \) basis, since there \( \vec{p}_3 \) is just proportional to \( \vec{p}_\lambda \). Since \(|\alpha\rangle\) and \(|\beta\rangle\) are antisymmetric under exchange of quarks one and two,

\[
\langle \alpha | (p_1^2 + m_1^2)^{\frac{3}{2}} | \beta \rangle = \langle \alpha | (p_2^2 + m_2^2)^{\frac{3}{2}} | \beta \rangle ,
\]

and these matrix elements are best evaluated in the \( \vec{p}', \vec{\lambda}' \) basis, where \( \vec{p}_2 = -\frac{2}{\sqrt{6}} \vec{p}_\lambda' \). Then we have that

\[
\langle \alpha | (p_3^2 + m_3^2)^{\frac{3}{2}} | \beta \rangle = \delta_{S_\alpha S_\beta} \delta_{L_\alpha L_\beta} \delta_{\rho_{\alpha \rho_\beta}} \delta_{l_{\alpha \lambda_\beta}} \delta_{\lambda_{\alpha \lambda_\beta}} \langle n_{\lambda_\alpha} l_{\lambda_\beta} | (\frac{3}{2} p_3^2 + m_3^2)^{\frac{3}{2}} | n_{\lambda_\beta} l_{\lambda_\beta} \rangle .
\]

Using the relation between the Fourier transform of a harmonic oscillator wavefunction and itself (see (76)), we find

\[
\langle n_{\lambda_\alpha} l_{\lambda_\beta} | (\frac{3}{2} p_3^2 + m_3^2)^{\frac{3}{2}} | n_{\lambda_\beta} l_{\lambda_\beta} \rangle = (-i)^{-N_\alpha + N_\beta} \langle n_{\lambda_\alpha} l_{\lambda_\beta} | (\frac{3}{2} \lambda^2 + m_3^2)^{\frac{3}{2}} | n_{\lambda_\beta} l_{\lambda_\beta} \rangle ,
\]

where

\[
\langle n_{\lambda_\alpha} l_{\lambda_\beta} | V(\lambda) | n_{\lambda_\beta} l_{\lambda_\beta} \rangle = N_{n_{\lambda_\alpha} l_{\lambda_\beta}} N_{n_{\lambda_\beta} l_{\lambda_\beta}} \frac{1}{\alpha^3} \int_0^\infty \lambda^2 d\lambda e^{-\lambda^2/\alpha^2} (\lambda/\alpha)^{l_{\alpha} + l_{\beta}} L_{n_{\lambda_\alpha}}^{l_{\alpha} + \frac{1}{2}} (\sqrt{2}\lambda/\alpha) V(\lambda) L_{n_{\lambda_\beta}}^{l_{\lambda_\beta} + \frac{1}{2}} (\sqrt{2}\lambda/\alpha) .
\]

The calculation of the matrix elements of the momentum dependent factors \( \vec{p}_{ij} \) to \( \vec{p}_{ij (cm)} \) is made simpler by the ansatz that the momentum appearing in the \( (ij) \) factors is \( \vec{p}_{ij (cm)} = (\vec{p}_i - \vec{p}_j)/2 \), since the relations A(13) tell us that

\[
\frac{1}{2} (\vec{p}_1 - \vec{p}_2) = \frac{1}{\sqrt{2}} \vec{p}/, \quad \quad \quad A(31)
\]

and the obvious corollary

\[
\frac{1}{2} (\vec{p}_1 - \vec{p}_3) = \frac{1}{\sqrt{2}} \vec{p}' ./ 
\]

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Then, for example, the factor appearing in $\tilde{V}_{12}^{coul}$ (see (42) and (43)) has the matrix element

$$
\langle \alpha | (\beta_{ij})^{\frac{1}{2}+\epsilon_{coul}} | \beta \rangle = \delta_{\sigma_{a}, \sigma_{\beta}} \delta_{L_{a}, L_{\beta}} \delta_{I_{p_{a}}, I_{p_{\beta}}} \delta_{n_{\lambda_{a}}, n_{\lambda_{\beta}}} \delta_{l_{\lambda_{a}}, l_{\lambda_{\beta}}}
$$

$$
\langle n_{\lambda_{a}} l_{\lambda_{a}} | \left( 1 + \frac{1}{2} \frac{p_{p}^2}{\frac{1}{2} p_{p}^2 + m_{1}^2} \right)^{\frac{1}{2}+\epsilon_{coul}} | n_{\lambda_{\beta}} l_{\lambda_{\beta}} \rangle, \quad A(33)
$$

where

$$
\langle n_{\lambda_{a}} l_{\lambda_{a}} | \left( 1 + \frac{1}{2} \frac{p_{p}^2}{\frac{1}{2} p_{p}^2 + m_{1}^2} \right)^{\frac{1}{2}+\epsilon_{coul}} | n_{\lambda_{\beta}} l_{\lambda_{\beta}} \rangle = \frac{(-i)^{-N_{a}+N_{\beta}}}{(n_{p_{a}} l_{p_{a}})} \langle n_{p_{a}} l_{p_{a}} | \left( 1 + \frac{1}{2} \frac{p_{p}^2}{\frac{1}{2} p_{p}^2 + m_{1}^2} \right)^{\frac{1}{2}+\epsilon_{coul}} | n_{p_{\beta}} l_{p_{\beta}} \rangle, \quad A(34)
$$

Similar expressions hold for the other (12) momentum dependent factor (47); the (13) momentum dependent terms, are calculated in much the same way except in their case there are two masses involved. The integrals $A(30)$ (and their $d\rho$ equivalents) can be evaluated numerically.

**Evaluation of the Integrals**

All of the radial and angular integrations necessary to form the matrix elements listed above are carried out by an algebraic method on the computer. The basic procedure for the radial integrals is as follows: the integral of each power of the radial coordinate against the integrand (which includes the potential and the Gaussian factors in the wavefunctions) is evaluated from an analytic formula and stored in an array; the Laguerre polynomials representing the radial wavefunctions are multiplied algebraically and stored in an array representing the product polynomial, and then the integral is evaluated by adding up the contributions due to each term in the product polynomial.

In some cases mentioned above the integrand is a function whose integral against the polynomial times Gaussian wavefunctions is not known analytically, and for these integrals we evaluate the integral against each power of the radial coordinate numerically, using Gaussian quadrature.
References


31] SU(3)_f breaking in the negative parity hyperons was considered by U. Ellwanger, Nucl. Phys. B139, 422 (1978).
66] See for example Ref. 55.
69] See, for example, Ref. 46.
70] At the recent (1985) European Conference on High Energy Physics held in Bari, Italy, there was reported the possible discovery of an excited Ω state. A spin parity analysis has apparently not been made.
71] See the references on page S136 of Ref. 67.
75] N. Isgur and G. Karl, Ref. 27; D. Gromes, Ref. 58; L.J. Reinders in Ref. 5.