HAIL FORMATION AND ITS EFFECT ON A MODEL UPDRAFT

by

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ABSTRACT

The simulation of hail growth and its effects on cloud liquid water and energy balance is achieved by injecting hail embryos at the freezing level of a model updraft. Steady-state one-dimensional numerical models in which hail is only ascending or where it is ascending and descending are considered. The problems associated with the hail accumulation zones of former similar models are overcome by introducing a size distribution of hail embryos.

Moderate concentrations of hail embryos, \( \frac{1}{2} \) to 2 per cubic meter, grow to deplete a significant fraction of the cloud's liquid water content. If this significant depletion of liquid water occurs, a hail accumulation zone is formed wherein the downward force on the cloud's updraft due to the hailstones is of the same order of magnitude as the upward buoyant force and further these zones are found to contain a maximum hail mass only three or four times greater than the wet-adiabatic liquid water content.

The growth of hailstones from 0.25 or 0.50 cm embryos to 3 cm while ascending or \( \frac{1}{2} \) to 6 cm, after ascending and descending takes about 10 min in the former case and 20-30 min in the latter. These growth rates are consistent with radar observations and they require only adiabatic liquid water contents (3.5 gm/m\(^3\) at maximum). Even larger
stones are grown when they are non-spherical. The growth of large hailstones is therefore not dependent on an elaborate model of hailstone recycling within the cloud.

Hail surface icing conditions and heat and mass exchange between the hailstones and the cloud air are found to be important to hailstone characteristics, the updraft's energy balance and liquid water content. Therefore, any future models of hailclouds should include the feedback mechanisms between growing hail and cloud air if moderate to heavy hail showers are to be numerically simulated.

Altering the number of hail embryos or the cloud's liquid water content for the purpose of weather modification can produce positive or negative effects on hail production which are highly dependent on the hailcloud's parameters. For this reason any methods used to suppress hail should be allied with a definitive model of hail formation.
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To S.T. Coleridge who may have inspired this study when in 1797 he wrote:

'A mighty fountain momently was forced:
Amid whose swift half-intermitted burst
Huge fragments vaulted like rebounding hail,'

the author gives belated thanks.

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LIST OF SYMBOLS

\( b \) : a subscript referring to the balance level

\( BE \) : potential increase of a rising cloud parcel's kinetic energy since it passed the hail input level. The vertical integral of the thermal buoyancy \( f_B \) plus the hydrometeor force \( f_{HR} \) (erg/gm).

\( C_{1,2} \) : a function of temperature in Appendix 3
(\( \text{cal} \ \text{°K}/\text{cm}^3/\text{mb} \))

\( C_A \) : vertical flux of cloud air (\( \text{gm/cm}^2/\text{s} \))

\( C_c \) : vertical flux of air when hailstone volume is accounted for (\( \text{gm/cm}^2/\text{s} \))

\( C_i \) : vertical flux of the \( i \)th size of hailstones in the size spectrum (\( \text{cm}^{-2} \ \text{s}^{-1} \))

\( C_M \) : vertical flux of liquid water and hail mass (\( \text{gm/cm}^2/\text{s} \))

\( C_N \) : flux of hailstones (\( \text{cm}^{-2} \ \text{s}^{-1} \))

\( C_P \) : specific heat of dry air (\( \text{cal/gm/°K} \))

\( C_{PV} \) : specific heat of water vapor (\( \text{cal/gm/°K} \))

\( C_w \) : specific heat of water (\( \text{cal/gm/°K} \))

\( \bar{C}_w \) : the mean specific heat of water between \( T \) and \( T_d \) (\( \text{cal/gm/°K} \)).
$D$: diameter of sphere or major dimension of oblate spheroid (cm).

$D_f$: diameter on returning to input level if $N_o = N_M$ (cm).

$D_{fo}$: diameter on returning to input level if $N_o = 0$ (cm).

$D_t$: major axis of a hailstone whose terminal velocity equals the updraft velocity (cm).

$D_{wa}$: diffusivity of water vapor in air (cm$^2$/s).

$d$: a subscript referring to descending hailstones.

$E$: collection efficiency of hailstone

$e_i$: saturation vapor pressure with respect to ice at hailstone's surface temperature (mb).

$e_w$: saturation vapor pressure of cloud air with respect to water (mb).

$f_B$: thermal buoyant force per gm of cloud air (cm/s$^2$).

$f_{HR}$: force per gm of cloud air due to its liquid water and hail content (cm/s$^2$).

$PB$: force per gm of cloud air due to thermal buoyancy and hydrometeors, $f_B + f_{HR}$ (cm/s$^2$).

$g$: the acceleration of gravity (cm/s$^2$).

$H$: height above ground (Km).
$I$ : proportion of accreting water which freezes.

$i$ : a subscript referring to the $i$th hailstone size in a size spectrum.

$k$ : the thermal conductivity of air (cal/cm/s/°K).

$KE$ : increase of cloud parcel's kinetic energy per gram since it passed the input level (erg/gm).

$L_f$ : latent heat of fusion of water (cal/gm).

$L_s$ : latent heat of sublimation of water (cal/gm).

$L_v$ : latent heat of vaporization of water (cal/gm).

$LWC$ : liquid water content (gm/m$^3$).

$M$ : the mass of a hailstone (gm).

$M_{ES}$ : rate of evaporation and sublimation from a hailstone (gm/s).

$N$ : number of hailstones per unit volume (cm$^{-3}$)

$\bar{N}_i$ : the mean number density of the $i$th size of hailstones in the region of their balance level (cm$^{-3}$).

$N_M$ : maximum number density of hail embryos for steady-state solution (cm$^{-3}$).

$N_T$ : the total number density of hailstones of all sizes at a given height (cm$^{-3}$).

$n$ : the number of sizes in the hail embryo size spectrum.
\( o \) : subscript referring to cloud's freezing level (the embryo injection level).

\( P \) : air pressure of updraft or cloud environment (mb).

\( Q_{cc} \) : rate of heat conducted or convected from a hailstone (cal/s) or (erg/s).

\( Q_E \) : rate of heat required to re-condense water vapor which has evaporated or sublimated from a hailstone's surface (cal/s) or (erg/s).

\( Q_{ES} \) : rate of heat required for evaporation and sublimation from a hailstone's spongy surface (cal/s) or (erg/s).

\( Q_{wk} \) : rate of viscous heat dissipation in a hailstone's wake (erg/s).

\( R \) : gas constant of dry air (mb cm\(^3\)/gm\(^°\)K) or (erg/gm\(^°\)K) or (cal/gm\(^°\)K).

\( R_w \) : gas constant of water vapor (mb cm\(^3\)/gm\(^°\)K) or (erg/gm\(^°\)K) or (cal/gm\(^°\)K).

\( r \) : cloud mixing ratio (gm liquid water per gm air).

\( r_b \) : liquid water at balance level if \( N_o = N_M \) (gm water per gm air).

\( r_s \) : saturation mixing ratio of air with respect to water (gm water vapor per gm air).

\( r_T \) : total liquid water and water vapor (=\( r + r_s \)) (gm water per gm cloud air).
\[ \frac{dr}{dz} \mid_{FZ} \] : rate at which cloud water freezes with height between \(-32^\circ C\) and \(-40^\circ C\) levels (gm water/gm air/cm).

\( t \) : time, usually of hail growth (s).

\( t_M \) : time for hailstone to return to input level when \( N_0 = N_M(s) \).

\( t_o \) : time for hailstone to return to input level when \( N_0 = 0 \) (s).

\( T \) : cloud temperature (°K).

\( T_d \) : surface temperature of hailstone (°K).

\( T_E \) : temperature of cloud's environment (°K).

\( T_v \) : cloud virtual temperature (°K).

\[ \frac{dT}{dz} \] : mean temperature lapse rate in cloud between \(-32^\circ C\) and \(-40^\circ C\) levels (°K/cm).

\( u \) : a subscript referring to ascending hailstones.

\( V_t \) : terminal velocity of spherical hailstone (cm/s).

\( V_{ta} \) : terminal velocity of spheroid (cm/s).

\( V_Z \) : vertical updraft velocity (cm/s).

\( W_f \) : liquid water content in (gm/m^3).

\( Z \) : vertical height above hailstone injection level (cm).
$z_{bm}$ : the balance level height when $N_o = N_M$ (cm).

$z_{bo}$ : the balance level height when $N_o = 0$ (cm).

$\alpha$ : ratio of minor to major axis of spheroid.

$\gamma$ : ratio of spheroid's area to that of a sphere with same major dimension.

$\Delta$ : refers to a finite change.

$\varepsilon$ : ratio of drag coefficient of a spheroid to a sphere with same major dimension.

$\eta$ : ratio of molecular weight of water to that of dry air.

$\Theta$ : roughness parameter for heat transfer.

$\nu$ : kinematic viscosity of air (cm$^2$/s).

$\rho$ : cloud air density (gm/cm$^3$).

$\rho_E$ : environment air density (gm/cm$^3$).

$\rho_I$ : density of hailstone (gm/cm$^3$).

$\chi$ : shape factor for determining Nusselt number.
CHAPTER I

INTRODUCTION

The last ten years have seen many researchers expanding our knowledge of cumulus clouds. Their achievements have been envied and assisted by all workers in the physical sciences; surface chemists, fluid dynamicists, crystallographers, and instrument designers alike. There is still, however, much to be done before these many fields of physics are brought together to form a nearly unquestionable model of a cumulonimbus cloud.

The practical need for this realistic model can be explained in a single phrase, 'weather modification'. The mixed blessings of plentiful rain and potentially as plentiful hail put thunderstorms in a position where their modification will require a compromise which must be based on the most accurate possible description of their physical mechanisms.

It is generally accepted from the results of internal probes that most mid-latitude thunderstorms have hail in them, even though this hail often does not reach the ground. Certain problems associated with describing the growth of hail in the updraft of these clouds must be studied before this ever present hail can be included in our description of a thundercloud.

Several notable models of hail growth in a predetermined updraft have been presented in the recent
literature by Atlas (1966); Srivistava and Atlas (1969); Browning (1963); and Musil (1969). Only one hailgrowth model to date, besides those associated with this author, has included the effects of growing hailstones on the cloud's water content. This model was presented by Iribarne and de Pena in 1960 and although it was a major step forward in hailgrowth theory it went largely unnoticed by the scientific community. Since then, much research has been done on the conservation of cloud water substance, most of which is summarized in a Meteorological Monograph by Kessler, 1969, but still very little on these aspects of hail formation. This thesis will present a model of hailgrowth where water conservation is taken into account.

Light hailstorms are common in nature, that is, those with hailstones of less than one centimeter in diameter and which yield only a fraction of an inch of precipitation, however, heavy hailfalls are not especially unusual and they are notorious for their destructiveness. When heavy hailstorms occur, a large proportion of the precipitation which falls in the 'hail streak' is hail, not rain, and further, it falls at a rate which would most certainly cause a substantial fraction of the cloud's liquid water to be depleted in the cloud's region of hail formation. The model presented will be applicable to both light and heavy hailfalls, however, its conclusions will be most interesting for moderate to heavy hail for more than sensational reasons.
When liquid water depletion is included in the model there are certain natural benefits. One can calculate the amount of hail per unit volume in the cloud's updraft and hence the force of the hail on the cloud parcels. This force's effects on the updraft's kinetic energy can then be found. The model leads to a study of heat and mass exchange between the hailstones and the cloud and then to turbulent heat dissipation in the hailstones' wakes. An attempt is then made to grow the spheroidal hailstones which are often observed in nature and also to evaluate their heat and mass exchange as well as their ice and water content. The discontinuities in number concentration which occur in the models of Iribarne and de Pena, 1960, and Atlas, 1966, are cleared up when the hail embryo source is assumed to be a number distribution of hailstones described by their diameters.

The model consists of injecting hail embryos near the freezing level of the updraft at a constant rate and allowing them to grow by accretion as they are carried aloft in the updraft. They either grow to a point where they are assumed to leave the updraft or they reach a maximum altitude, referred to as a 'balance level', and fall back down to the input level.

It should be stressed that the updraft velocity profile of this model is predetermined, i.e. it is not influenced by the forces due to the hailstones. The model is, therefore, not dynamic. Comparisons of available
buoyant energy to updraft kinetic energy are made only to determine whether the model is dynamically reasonable and hence compares well with normal atmospheric conditions. In the hailgrowth model the only major influence of the hailstones on the updraft is liquid water depletion. Two other main characteristics of this model are that it is horizontally homogeneous and all aspects are steady-state.

The description of the model which leads to its equations is presented fully in sections 2.1.1. through 2.3.1. of Chapter 2. These sections would probably give the reader sufficient background about the model for understanding any one of Chapters 3 to 6 on the numerical results. It is sufficient to state here that the following chapters describe many possibilities for hailgrowth, none of which may be exactly reproducible in nature, but all of which are shown to be important to a useful description of hail producing clouds.
2.1.1. Introduction to the Model

The cloud model will be characterized by discussing six fundamental points which are based on field observations, previous numerical models, and currently accepted descriptions of hail producing clouds. These basic outlines are essential for any conclusions about the hailstones' influence on the updraft region of these clouds.

2.1.2. Steady-State Concept

Current views on large hail producing clouds suggest that for periods in the order of twenty minutes the structure of the updraft region is essentially unchanged. Graphic pictures of 'supercells', some lasting up to three hours, can be drawn as in the case of the Wokingham hailstorm of July 9, 1959 which was presented in reports by Browning, Ludlam, and Macklin (1963) and Browning (1963). Further indication of this time scale is shown by radar echo persistance in many references on this subject, notably Donaldson (1965) and Wexler (1961).

With this time element in mind, the growth of hailstones can be calculated for periods of from 10 to 60 minutes with the updraft velocity held constant. After these periods one should look carefully at the effects which the hailstones would have had on the updraft velocity and liquid water
This approach should not be interpreted as a time dependent solution but rather as a justification for considering the calculations to be representative of actual cloud conditions for the time periods in question.

### 2.1.3. Space Dimension

Current models of large hail clouds which involve updrafts of from 15 to 40 meters per second show a minimum amount of divergence or convergence in the core region above the freezing level and below about the six or eight kilometer level where the tropopause is seen to decelerate the updraft. This can be seen in the numerical models of Orville (unpublished, 1970) and in the measurements and interpretations of Sulakvelidze (1967) and Browning (1963). This essentially non-divergent concept does not rule out an increase in the updraft with height since the air density decreases by fifty percent by moving from the freezing level to the -40°C level which results in a near doubling of the vertical velocity over this vertical extent. For a normal hailcloud this temperature drop of 40°C is achieved in about six kilometers. A careful examination of the models mentioned above shows that this is the approximate rate of vertical velocity increase in the region below the maximum updraft when large convective clouds are considered in their mature stage. Convective clouds which do not penetrate the tropopause and, as a result, have their maximum updraft below the -30°C level are not considered here since they
are relatively small and are generally not good hail producers. The importance of the region between -32°C and -40°C, generally just below the tropopause, is discussed in Section 2.1.5 with respect to the freezing of liquid water.

Horizontal wind sheer is not generally considered to be important in the region of strong updraft below the tropopause, but high level jet streams have been seen to act as a driving force on the updraft once the hailcloud has developed to that height (Das, 1962). Strong updrafts are not generally seen to slope more than 20 degrees from the vertical in the 700 to 300 mb region so that an updraft of one to two kilometers in width and six kilometers in vertical extent can, to a first approximation, be considered vertical.

The region of maximum vertical wind sheer is found (Sulakvelidze, 1967) to be located away from the updraft core leaving the central core updraft essentially constant for radii of up to about one kilometer.

2.1.4. Liquid Water Content

The liquid water content (LWC) can be assumed to be wet adiabatic because there is little mixing with dry environment air in large hailclouds. The cloud water is assumed to move with the updraft for the following reason: an updraft of 20 m/s would cause the cloud parcels to rise the 6 km extent of the updraft in 5 minutes leaving insufficient time within present rain formation theories to grow a significant
number of rain drops with terminal velocities comparable to the updraft speed. This is supported by the appearance of echo-free vaults, i.e. regions of low rain content, which appear in radar observations (Browning, 1965). Hydrometeors of significant terminal velocity will therefore be considered in this model to be hail embryos or hailstones, i.e., frozen or partially frozen.

### 2.1.5. Cloud Particle Freezing

It is assumed that by $-40^\circ$C all liquid water in the updraft is frozen. This is in agreement with Das (1962), and to the author's knowledge liquid water is generally not observed above the $-40^\circ$C level of clouds, reports to the contrary are not sufficiently substantiated. Das postulated that the liquid water in an updraft would begin to freeze out spontaneously at $-32^\circ$C and proceed in a linear manner between $-32^\circ$C and $-40^\circ$C. This model will assume that the only frozen water below the $-32^\circ$C level is in the form of hail. This liquid water freeze-out puts an upper boundary on the region of hail growth so that only the $0^\circ$C to $-40^\circ$C region need be considered for the growth of hail.

### 2.1.6. Depletion of Liquid Water by Growing Hailstones

Meso-scale observations of hailclouds and squall lines often refer to the efficiency of precipitation in the overall region of updraft to be about 10 to 50 percent. It is therefore reasonable that the updraft of a hailcloud is
9.
capable of precipitating a significant proportion of its liquid water in the form of hail. Observations of precipitation efficiencies are summarized by Kessler (1969, pp. 59).
To compare a hail precipitation rate to the amount of water substance ascending one can calculate the upward water substance flux and compare it to hail fall rates. A 20 m/s updraft with 6 gm/m³ of liquid and water vapor represents an upward flux of potential liquid water of 43 cm/hr or 17 inches/hr. This upward flux is not out of line with heavy hail rates where one inch can fall in 5 or 10 minutes during the peak hailfall periods. The foregoing observations lead one to believe that heavy hailfalls are associated with significant liquid water depletion in the region of hail formation and that liquid water depletion must be considered in the model, otherwise, the heavy hailfalls in nature would not represent descending water substance fluxes comparable to normally expected upward fluxes.

2.1.7. Injection of Hail Embryos

Investigations by List (1960) indicate that a large proportion of hailstones contain graupels as embryos. These graupels appear to be grown at temperatures just below freezing and are usually one to five millimeters in diameter. These embryos are often seen during unstable atmospheric conditions from high vantage points on mountains (personal communication with List) but they would not be expected to be seen at lower levels since they would melt quickly.
The source of hailstone embryos for the main updraft at the freezing level can also be attributed to what were referred to as 'feeder clouds' during Project Hailsworth meetings (Schleusener, 1966). These 'feeder clouds' are regions of instability near the major updraft region which appear to be amalgamated into the main updraft before they can develop significantly. These unstable regions would make reasonable graupel sources for the major updraft which could be expected to be maintained for several minutes.

Placing the embryo source near the cloud freezing level allows individual hailstones to grow for a long period of time because as the hailstones rise to regions of greater updraft their terminal velocity increases, thus keeping their overall ascent speed to a minimum. This embryo source model conveniently avoids the need to describe mechanisms by which they are created. The purpose of the model, to assess the effects of hailstones on an updraft, can therefore be fulfilled by requiring that there be a continuous source of hail embryos near the freezing level for all examples calculated.

2.2.1. The Basic Cloud

The hailcloud is based on Beckwith's (1960) soundings taken on days with hail in the Denver area. The environment temperature $T_E(\text{°K})$ is related to pressure (List, et al., 1965) according to:

$$T_E = 53 \ln (0.284P) \quad [1]$$
where the pressure $P(\text{mb})$ is given as a function of height $H (\text{km})$ above the ground as:

$$H = 2.76 \times 10^{-4} \left(9.0066 \times 10^4 - T_E^2\right) \quad [2]$$

The Denver soundings gave the lifting condensation level (LCL) as $H = 3.48 \text{km}$, $P = 670.8 \text{mb}$ and $T_E = 5^\circ C$.

The environment temperature is taken to be equal to the cloud temperature $T$ at the LCL. With the above information cloud temperature can be calculated as a function of cloud pressure by the wet adiabatic parcel method as is shown in Appendix 2. The saturation mixing ratio of the cloud air $r_s (\text{gm water vapor/gm air})$ can be calculated as a function of cloud pressure and temperature. The saturation mixing ratio at the LCL is $r_s = 0.822 \times 10^{-2} \text{gm/gm}$.

In order to determine the cloud temperature and pressure as a function of height it is sufficient to postulate that the cloud pressure is the same as that of the environment. This assumption will be used in all models and is quite reasonable for non-divergent updrafts.

On integrating the relationship between cloud temperature and pressure, (Eq. A2-4 of Appendix 2) up to the freezing level of the cloud, the pressure there is found to be $P = 595 \text{mb}$, the height $H$ according to Equation 2 is $4.44 \text{km}$, and the saturation mixing ratio $0.647 \times 10^{-2} \text{gm/gm}$ giving a liquid water content $r = 0.175 \times 10^{-2} \text{gm/gm}$. 

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The cloud model will have hail embryos injected at a constant rate at the cloud freezing level as discussed in Section 2.1.7. and defined above. The new vertical coordinate now gives the cloud pressure $P$(mb) as a function of height $Z$(cm) above the freezing level:

$$P = 3.521 \exp \left\{ \left( \frac{20.417 - Z \times 10^{-5}}{0.7753} \right)^{\frac{1}{2}} \right\} \text{ (mb)} \quad [3]$$

2.3.1. The Simple and Complex Models

At this point a distinction must be made between the need to describe all the cloud model parameters in detail or just part of them and the rest in general terms. This will lead to the simple and complex models whose virtues will be stressed according to the desired results and the effort required for calculation.

The simple model is defined as a one dimensional updraft where the mass flux of air is constant with height and where the saturation mixing ratio and cloud temperature are predetermined by integrating the wet adiabatic equation (Eq. A2-4 of Appendix 2) to the five km level. Further, the liquid water content of the cloud will be determined by the change in $r_s$ between the freezing level and the height $Z$ in question taking into account the water accreted on the hailstones. When the simple cloud model is used no account will be made of the freezing of liquid water at temperatures between $-32^\circ C(4.6km)$ and $-40^\circ C(5.5km)$, however, consideration
will be given to this phenomenon when the results are presented. The numerical approximations used with the simple cloud model will be discussed later in this chapter, however it should be stated that it is used whenever a number density of hailstones is introduced (Chapter 6) or when ascent and descent of hailstones is considered (Chapters 5 and 6).

The complex model will be used primarily to evaluate the shortcomings of the simple model. It will take into account the heat and mass exchange of the hailstones, the freezing of high level liquid water, and cloud air heating through the two previous effects and through viscous wake dissipation. It will also account for the volume occupied by the hailstones in maintaining one dimensionality. This model must evaluate the magnitude of the foregoing effects as compared to a model where they are ignored. The complex model will be used only when the growth of hailstones on ascent of a single embryo size is considered as in Chapters 3 and 4.

2.4.1: Icing Conditions of Hailstones

The icing conditions of a hailstone are characterized by its surface temperature $T_d$ or the fraction $I$ of accreting liquid water which freezes, in which case the surface temperature is 0°C. The most recent paper by List (List and Dussault, 1967) on the heat and mass exchange of hailstones is summarized in Appendix 3 where the
method of calculating the icing conditions, the rate of heat exchange between cloud air and hailstone per hailstone per unit time due to convection and conduction \( Q_{cc}^* \), evaporation and sublimation \( Q_{ES}^* \), and the rate of mass exchange between the hailstone and cloud air \( M_{ES}^* \) due to evaporation and sublimation are shown along with values of all physical constants. The paper by List and Dussault (1967) gives the icing conditions for spheroidal hailstones as does Appendix 3 which was included in this work so that a complete and brief method of calculating these icing conditions as they apply to this model would be available.

2.5.1. The Simple Model

The pressure as a function of height was given by Equation 3. The temperature is determined as a function of \( P \) by integration of the wet adiabatic Eq. A2-4 of Appendix 2 and converted to a function of height by Equation 3. The saturation mixing ratio \( r_s \) is determined from the cloud temperature and pressure as in Appendix 2.

Figure 1 shows the temperature \( T_{environment} \) temperature \( T_E \), and saturation mixing ratio \( r_s \) as functions of height. Figure 1 also shows the method of approximating \( r_s \) by two straight lines for ease of calculation which can be summarized as:

\[
\begin{align*}
    r_s &= (6.5 - 1.6 \times 10^{-5} Z) \times 10^{-3} \text{(c.g.s.) for } 0 < Z < 2.5 \text{km} \quad [4a] \\
    r_s &= (2.5 - 0.8 (Z - 2.5 \times 10^5) \times 10^{-3} \text{(c.g.s.) for } 2.5 \text{km} < Z < 5 \text{km} \quad [4b]
\end{align*}
\]
Figure 1 also gives the cloud air density $\rho$ (c.g.s.) as a function of height as determined by:

$$\rho = \frac{P}{RT}$$  \[5\]

where $P$ is in mb, $R = 2.87$ mb cm$^3$ gm$^{-1}$ °K$^{-1}$ and $T$ is in °K.

The virtual temperature does not replace the air temperature in Equation 5 because the air density $\rho$ is not needed for purposes of cloud buoyancy, only for determining the hailstones' terminal velocity. The density curve of Figure 1 is a linear approximation to the actual values which are shown as asterisks. The linear approximations for $\rho$ are:

$$\rho = (7.59 - 0.84 \times 10^{-5} z) \times 10^{-6} \text{ c.g.s. for } 0 < z < 1 \text{ km} \quad [6a]$$

$$\rho = (6.75 - 0.60 \times 10^{-5} (z-10^5)) \times 10^{-6} \text{ c.g.s. for } 1 \text{ km} < z < 5 \text{ km} \quad [6b]$$

The thermal buoyant force on a parcel of cloud air $f_B$ can be determined by the difference between cloud and environment air densities. For the calculation of these air densities the virtual temperature $T_v$ should be used, however, since the mixing ratio of the environment is not known the observed temperature will be used for the environment virtual temperature. In general, the buoyant force $f_B$ per gram of cloud air is given by:

$$f_B = g \frac{\rho_c - \rho}{\rho}$$  \[7a\]
where the cloud and environment densities \( \rho \) and \( \rho_E \) respectively are:

\[
\rho = \frac{P}{RT_v} = \frac{P}{R(T + 0.6078 r_s)} \quad [7b]
\]

and

\[
\rho_E = \frac{P}{RT_E} \quad [7c]
\]

\[
f_B = g \left( \frac{T - T_E}{T_E} \right) + g \times 0.6078 r_s \frac{T}{T_E} \quad [7d]
\]

Approximating \( T/T_E = 1 \) for the perturbation on \( f_B \) due to water vapor gives:

\[
f_B = g \left( \left( \frac{T - T_E}{T_E} \right) + 0.6078 r_s \right) \quad \text{c.g.s.} \quad [7e]
\]

The acceleration of gravity \( g \) is \( 980 \text{cm/s}^2 \).

Fig. 2a gives the term \( g \left( \frac{T - T_E}{T_E} \right) \) as a function of height above the freezing level \( \frac{T - T_E}{T_E} \) and as a three line linear approximation which can be summarized as:

\[
g \left( \frac{T - T_E}{T_E} \right) = 4.3 + \frac{6.7 \times Z}{1.75 \times 10^5} \quad \text{c.g.s.} \quad [7f]
\]

for \( 0 < Z < 1.75 \text{km} \).
\[ \frac{g}{T_E} \left( T - T_E \right) = 11.0 + \frac{1.4 \times (Z - 1.75 \times 10^5)}{1.5 \times 10^5} \quad \text{c.g.s.} \]

for \( 1.75 \text{km} < Z < 3.25 \text{km} \) \[ (7g) \]

\[ \frac{g}{T_E} \left( T - T_E \right) = 12.4 - \frac{2.9 \times (Z - 3.25 \times 10^5)}{1.75 \times 10^5} \quad \text{c.g.s.} \]

for \( 3.25 \text{km} < Z < 5.0 \text{km} \) \[ (7h) \]

The thermal buoyant force per gram of cloud air \( f_B \) is, therefore, determined by equations \( 4 \) and \( 7 \).

The simple model maintains a vertical velocity \( V_Z \) as a function of height as determined by the continuity equation:

\[ \rho V_Z = C_A \quad [8] \]

The constant \( C_A \) is determined by the updraft velocity \( V_{ZO} \) and the air density \( \rho_0 \) at the cloud freezing level.

Updraft and environment of the simple cloud are now complete.
2.5.2. Hail Growth in the Simple Cloud.

The simple model will describe the growth of spherical hailstones whose drag coefficient is taken to be $C_D = 0.5$, a generally accepted value for hailstones, and density $\rho_I$ is 0.915 gm/cm$^3$ as was argued by List (1965) and Cantin (1966). Equating the drag force to the hailstone's weight gives a terminal velocity $V_t$ of:

$$V_t = 48.8 \left( \frac{D}{\rho} \right)^{1/2} \text{ c.g.s.} \quad [9]$$

where $D$ is the diameter and $V_t$ is always positive.

Fig. 2b gives the hailstone diameter as a function of height which has a terminal velocity equal to the updraft's velocity for various input level updraft velocities.

List (1963) in hail tunnel experiments showed that the excess liquid water in growing hailstones is generally incorporated into the ice lattice to form 'spongy' ice. Therefore, this model will assume that all liquid water swept out by the hailstones is accreted and that the collection efficiency $E$ is equal to 1. The growth of a hailstone of diameter $D$ on rising, $\frac{dD}{dZ}$, is therefore given by:

$$\frac{dD}{dZ} = \frac{E V_t \rho}{2 \rho_I (V_Z - V_t)} \quad [10]$$
Since \( E \) and \( \rho_I \) are constants and \( V_Z, V_t \) and \( \rho \) are known functions of \( D \) and \( Z \) if the updraft at the input level \( V_{z0} \) is known (Eq. 7), Eq. 10 can be solved in the form \( D=D(Z) \) if the liquid water content \( r \) is known. To solve this model it remains necessary to write equations for conservation of water substance, be it solid (hail), liquid or vapor, and for conservation of hailstone numbers.

All models, as was described in Sections 2.1, are steady-state with constant and continuous sources of hail embryos at the freezing level; therefore the equations of conservation of water substance and hailstone numbers will be time independent. They need only describe the upward flux of water substance and hailstone numbers as being constant with height. The various forms of these fluxes will be described in Sections 2.7, 2.8 and 2.9 as they apply (i) to hailgrowth models in which a single size of hailstones is injected and in which they grow to a point where they return and fall out at the input level (Chapter 5), or (ii) to a model where a size distribution of hail embryos is injected and only their upward growth is considered or they return to fall out at the freezing level (Chapter 6). Sections 2.7, 2.8, and 2.9 will also describe the calculation of cloud energies, forces, and water content along with hailstone size distributions and total number densities.

The description of the simple cloud was placed at this point to avoid confusion as to what might be the
essential or overriding aspects of the complex cloud model which will next be described in Section 2.6.

2.6.1. The Complex Model

The complex model calculates the growth of spherical and spheroidal hailstones as they are injected in constant number density and grow on ascent to their balance level or to the five kilometer level where essentially all of the liquid water is frozen. Having reached the balance level the hailstones are removed artificially and are no longer considered. They may at this point be looked upon as being thrown out of the updraft due to its slanted nature or they could be considered to have been growing for the period since hail embryos were first injected which just brings the first hailstones to a balance level, i.e., no descent has started.

2.6.2. The Terminal Velocity of Spheroidal Hailstones

The terminal velocity $V_{t\alpha}$ of a spheroid can be shown to be:

$$V_{t\alpha} = V_t \left(\frac{\alpha}{\varepsilon}\right)^{1/2}$$

where $V_t$ is the terminal velocity of a sphere of equal density and a diameter equal to the major axis of the spheroid. $\alpha$ is the ratio of the minor to major axis $(\alpha \neq 1)$ and $\varepsilon$ is the ratio of the drag coefficient of a spheroid to that of a sphere with the same major dimension.
(e>1). Lis' and Dussault (1967) have summarized the drag coefficients as well as heat and mass exchange of spheroidal hailstones. Figure 1 of Appendix 3 gives the approximate formula:

\[
\left( \frac{\alpha}{\epsilon} \right)^{\frac{1}{2}} = 0.8\alpha + 0.2 \quad [12]
\]

If only hailstones of density \( \rho_1 = 0.915\text{gm/cc} \) and spherical drag coefficient of 0.5 are considered, the terminal velocity for spheroids is given by a combination of Equations 9, 11 and 12:

\[
V_{ta} = 48.8(D/\rho)^{\frac{1}{2}} \times (0.8\alpha + 0.2) \text{ c.g.s.} \quad [13]
\]

where \( D \) is the major axis.

2.6.3. The Equations of Growth for a Spheroid.

The axis ratio is assumed to be either constant or a function of the major axis only. This is discussed in Chapter 4.

The rate of change of mass \( M \) of a spheroid where the collection efficiency equals one is given by:

\[
\frac{dM}{dt} = -\pi D^2 V_{ta} \rho + M^* \quad [14a]
\]
where:

\[ M = \rho I \frac{\pi}{6} - D^3 \alpha \]  \hspace{1cm} [14b]

and \( M^* \text{ (gm/s)} \) is the rate of evaporation or sublimation of mass from the spheroid which can be determined from Appendix 3.

Since \( M \) is a function of \( \alpha \) and \( D \), and \( \alpha \) is, according to observations, assumed to be a function of \( D \) only in the form:

\[ \alpha = \alpha_0 + \frac{d\alpha}{dD} D \]  \hspace{1cm} [14c]

where \( \frac{d\alpha}{dD} \) is a constant.

one gets:

\[ \frac{dM}{dD} = \rho I \frac{\pi}{2} \left( D^2 \alpha + \frac{D^3}{3} \frac{d\alpha}{dD} \right) \]  \hspace{1cm} [14d]

Since the net rate of ascent of the spheroid is:

\[ \frac{dZ}{dt} = V_r - V_t \alpha \]  \hspace{1cm} [14e]

and \( \frac{dM}{dt} \) and \( \frac{dD}{dM} \) are given above, the growth with height

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is obtained according to:

\[
\frac{dD}{dZ} = \frac{dM}{dt} \frac{dt}{dZ} = \frac{2\left(\frac{\pi}{4} D^2 V_{t0} \rho + M_{ES}^*\right)}{\rho \pi (V_Z - V_{t0}) \left(D^2 a + \frac{D^3}{3} \frac{da}{dD}\right)}
\]  

[14f]

This then is the growth equation of an ellipsoid if it maintains terminal velocity and rises in an updraft of air density \( \rho \) and liquid water content \( r \) while losing mass at a rate \( M_{ES}^* \), providing that its axis ratio \( \alpha \) is determined by its major axis \( D \).

The acceleration time to reach terminal velocity is insignificant in cases where updraft accelerations are other than those which are normally considered turbulent as was shown by List, et al. (1969a).

2.6.4. Conservation of Hailstones

If a concentration of \( N \) hailstone embryos per unit volume is fed into the cloud freezing level it is necessary under steady-state conditions that the upward flux of hailstones is constant at all levels.

\[
N(V_Z - V_{t0}) = C_N
\]  

[15]

\( C_N \) (cm\(^{-2}\)s\(^{-1}\)) is the constant of hail number flux.
2.6.5. One Dimensional Updraft

To maintain a one dimensional model the volume occupied by the hailstones was added to the continuity equation of the simple model giving:

\[ NM(V_Z - v_{ta}) + \left( 1 - \frac{NM}{\rho_I} \right) \rho V_Z = C_c \]  \[ [16] \]

Equation 16 gives the mass flux of hailstones plus that of air corrected for the volume occupied by the hailstones \( 1 - \frac{NM}{\rho_I} \). The left-hand side of the equation equals a constant \( C_c \) for all levels which is determined by the input level conditions. The mass flux due to the liquid water was not considered since it is only about \( \frac{1}{4} \) percent of the air flux. Equation 16 was modified to include the hail's volume only for the possibility of high hail concentrations.

Eliminating \( N \) from Equation 16 with Equation 15 and solving for \( V_Z \) yields:

\[ V_Z = v_{ta} - 0.5 \left( \frac{MC_N - C_c}{\rho} + v_{ta} - \frac{CN^M}{\rho_I} \right) + \]

\[ + 0.5 \left[ \frac{MC_N - C_c}{\rho} + v_{ta} - \frac{CN^M}{\rho_I} \right]^2 + \frac{MC_N^M v_{ta}}{\rho_I} \]  \[ [17] \]
2.6.6 The Equation of Conservation of Water Substance

As a parcel of cloud air rises it loses liquid water hailstones, gains water from the reduction of saturation mixing ratio and from evaporation from the hailstones, and loses liquid water to ice crystals which freeze out above the -32°C level. This can be expressed by the following equation of steady-state:

\[
\frac{dr}{dZ} = -\frac{dr_s}{dZ} - \frac{\pi D^2 V \alpha r N}{4V_Z} + \frac{NM^*}{pV_Z} + \frac{dr}{dZ}
\]  

\[ [18] \]

\[\frac{dr_s}{dZ}\] is the change in the saturation mixing ratio with height.

\[-\frac{\pi D^2 V \alpha r N}{4V_Z}\] is the rate per unit height at which a spheroidal hailstone sweeps out liquid water from the air providing the collection efficiency equals one.

\[\frac{NM^*}{pV_Z}\] is the rate at which the hailstones act as mass sources through evaporation to the combined liquid water and water vapor \((r + r_s)\).

\[\frac{dr}{dZ}\] is the rate at which liquid water freezes out. This term is effective between -32°C and -40°C in the
This gives a freezing out of liquid water per unit height which is proportional to the liquid water \( r \) and inversely proportional to the difference between the cloud temperature \( T \) and \(-40^\circ C\). \( \frac{dT}{dZ} \) is the mean lapse rate between \(-32^\circ C\) and \(-40^\circ C\) approximated from the simple model to be \(-8.4^\circ K/km\). \( T \) is in °K in Equation 19 and \( \frac{dT}{dZ} \) is in c.g.s. when calculations are made. This formula is in close agreement with observations by Das (1962) as they were discussed in Section 2.1.5.

### 2.6.7. The Cloud Temperature as a Function of Height

The wet adiabatic equation which includes heat and mass exchange between hailstones and the cloud air and the heating due to the freezing out of liquid water at high levels is derived in Appendix 2. Since the pressure is determined by that of the environment (Equation 3), the temperature \( T \) is derived as a function of height \( Z \) in the following form:

\[
\frac{dT}{dZ} = \frac{A + B \left( \frac{dP}{dZ} \right)}{C} \tag{20a}
\]
where:

\[
\begin{align*}
A &= N^{-1} \rho^{-1} V^{-1} (-Q^*_{cc} - Q^*_{E} + MgV_{ta}) (1 + r_s) + \frac{dr}{dz} \bigg|_{FZ} (1 + r_s) L_f \tag{20b} \\
B &= RT (p - e_w)^{-1} + \eta e_v L_v (p - e_w)^{-2} \tag{20c} \\
C &= C_p r_s C_{pv} + (\eta L_v^2 e_v) \left[ (p - e_w) R_v T_v^2 \right]^{-1} (1 + e_v (p - e_w))^{-1} \tag{20d}
\end{align*}
\]

The values of the variables making up A, B, and C are given in Appendix 2, except those of the heat exchange rates between hailstones and the cloud \( Q^*_{cc} \) and the rate of heat exchange per hailstone required to recondense the water vapor which evaporates from it \( Q^*_{E} \). \( Q^*_{cc} \) and \( Q^*_{E} \) are given in Appendix 3. It is sufficient to state that \( L_f \) and \( L_v \) are the latent heats of fusion and vaporization of water, \( R \) and \( R_v \) are the gas constants of dry air and water vapor, \( \eta \) is the ratio of the molecular weight of water to the mean value for dry air, \( C_p \) and \( C_{pv} \) are the specific heats of dry air and water vapor, and \( e_v \) is the saturation vapor pressure.

The air density \( \rho \) of Equation 20 and for all other places where it is required in this model is calculated using the virtual temperature \( T_v \) which is defined in Appendix 2 and yields the equation:

\[
\rho = \frac{P}{RT_v} = \frac{P}{RT(0.6078r_s + 1)} \tag{21}
\]
2.6.8. The Solution to the Equations

To solve the foregoing equations the variable \( r_T \) was substituted for \( r + r_s \) so that the terms \( \frac{dr}{dZ} \) and \( \frac{dr_s}{dZ} \) were eliminated from Equation 22.

\[
r_T = r + r_s
\]  

[22]

There remain three first order differential equations with dependent variables \( D, r_T \) and \( T \) and independent variable \( Z \). Eliminating \( r \) from Eqs. 14, 18 and 20a with Equation 22, one gets:

\[
\frac{dD}{dZ} = \left( \frac{\pi D^2 V_t a (r_T - r_s) \rho}{4} + M_{ES} \right) \frac{2}{\rho I \pi (V_Z - V_t a) (D^2 a + \frac{D^3 \partial a}{3 \partial D})}
\]

[23]

\[
\frac{dr_T}{dZ} = - \frac{\pi D^2 V_t (r_T - r_s) C_n}{4V_Z (V_Z - V_t a)} + \frac{C_n M_{ES}^*}{(V_Z - V_t) V_Z \rho} + \frac{dr}{dZ} \bigg|_{FZ}
\]

[24]

\[
\frac{dT}{dZ} = \frac{A + B \frac{dP}{dZ}}{C}
\]

[25]

Equations 11 through 21 can be used along with the definitions given in Appendices 2 and 3 to characterize all of the variables in Equations 23, 24 and 25 in terms of...
D, rT, T, and Z. The values of D, VZ, α(D), rS, r and N at the input level where T=0°C and Z=ao are sufficient initial conditions to give a solution if the three simultaneous equations are integrated for increasing height Z. The end of the integration occurs when Z=5 km or \( \frac{dD}{dZ} \) approaches infinity in which case the injected hailstones reach a balance level.

A Runge-Kutta method of integration is used where the computational errors for D, rT and T after integrating 5km are about 0.001cm, 10^{-5}gm/gm, and .01°C respectively. Therefore, these parameters are far more accurately determined than the values of the constants used, but they help to maintain reliability for the purposes of comparing various examples.

2.6.9. The Results Obtained from the Complex Model

The complex model allows the calculation of diameter or major axis, liquid water content r(gm/gm) or \( W_f(gm/m^3) \), number of hailstones per unit volume N, hailstone icing conditions I or Td, and updraft velocity \( V_Z \) all as functions of height Z for given input level conditions.

Further information can be gained by calculating the time \( t \) required for the hailstones to ascend to a given level by integrating as follows:

\[
t = \int_0^Z \frac{dZ}{V_Z - V_t}
\]
The increase of kinetic energy per gram of cloud air KE between input level and a given height can be calculated by means of the equation:

\[
KE = \frac{1}{2} (v_Z^2 - v_{Z0}^2)
\]  

[27]

The subscript 'O' refers to input level values.

The kinetic energy of the air can be compared to the integrated upward force on a gram of cloud air FB due to the thermal buoyancy \( f_B \) as defined in Equation 7a plus the integrated force on a gram cloud air due to the weight of the hailstones and the liquid water \( f_{HR} \).

\[
f_{HR} = -\frac{gNM}{\rho} - gr
\]  

[28a]

The energy available to a gram of cloud air BE, i.e. the total buoyant energy, after ascending to a height Z is therefore:

\[
BE = \int_{0}^{Z} (f_B + f_{HR})dZ
\]  

[28b]

or

\[
BE = g \int_{0}^{Z} \left\{ \frac{(T-T_E)}{T_E} - 0.6078r_s - \frac{NM}{\rho} - r \right\} dZ
\]

By comparing KE and BE one can get an estimate of the updraft's energy balance, and by comparing \( f_B \) with \( f_{HR} \).
the thermal buoyancy can be compared to the force of the liquid water and hailstones.

Several fine details of the complex model can be assessed by taking advantage of its completeness. The cloud temperature can be calculated with and without the effects of thermal feedback by setting \( \frac{M^*}{E_S} \), \( Q_L^* \), and \( Q_{cc}^* \) and the wake heating term \( V_{kel} \) all equal to zero. This gives an estimate of how much heat is released through latent heat of fusion and wake heating from the hailstones. The rate of mass evaporation can be compared to that of accretion and the various terms in the heat exchange equations can also be compared. Examples of thermal feedback effects and other 'fine details' are given in Chapter 3.

2.7. Growth of Hailstones Ascending and Descending in Steady-State

The model for growth of hailstones on ascent of Section 2.6 leads to a need for a model where growth on descent is also considered. The cloud model will be the simple model of Section 2.5, where the saturation mixing ratio \( r_s \) (Eq. 4) and the air density \( \rho \) (Eq. 6) are known functions of height \( Z \) above the freezing level. Only spherical hailstones of a single embryo size are considered as they grow while ascending to their balance level and descend to the freezing level in steady-state. After descending below the freezing level the hailstones will have no effect on the model.
The set of equations in Sections 2.5 can be made complete by equations of hailstone conservation and water conservation: Let \( N_u \) be the number per unit volume of ascending hailstones of diameter \( D_u \) while \( N_d \) and \( D_d \) represent the respective values on descent. Similarly, \( V_{tu} \) and \( V_{td} \) are the terminal velocities. For conservation of hailstones in steady-state the flux of hailstones upward must equal the downward flux and both fluxes must be constant with height \( C_N \) below the balance level, therefore:

\[
N_u (V_Z - V_{tu}) = -N_d (V_Z - V_{td}) = C_N \tag{29}
\]

The vertical flux of liquid water, water vapor and hailstone mass must also be constant (=\( C_M \)) with height, even above the balance level, therefore:

\[
\rho (r + r_s) V_Z + M_u N_u (V_Z - V_{tu}) + M_d N_d (V_Z - V_{td}) = C_M \tag{30}
\]

\( C_M \) is essentially the vertical flux of liquid and water vapor above the balance level. \( M_u \) and \( M_d \) are the individual masses of ascending and descending hailstones respectively.

\[
M_u = \rho_l \frac{\pi}{6} D_u^3 \quad M_d = \rho_l \frac{\pi}{6} D_d^3 \tag{31a}
\]

and

\[
M_u = \rho_l \frac{\pi}{6} D_u^3 \quad M_d = \rho_l \frac{\pi}{6} D_d^3 \tag{31b}
\]
Using Eq. 8 for conservation of air flux, \( \rho v_Z = C_A \), and Eq. 29, Eq. 30 becomes:

\[
(r + r_s)C_A + (M_u - M_d)C_N = C_M \tag{32}
\]

Eq. 10 for hailstone growth on ascent becomes:

\[
\frac{dD_u}{dZ} = \frac{V_{tu} \rho}{2 \rho I (V_Z - v_{tu})} \tag{33a}
\]

\[
\frac{dD_d}{dZ} = \frac{V_{td} \rho}{2 \rho I (V_Z - v_{td})} \tag{33b}
\]

The terminal velocities are given by Eq. 9 as:

\[
V_{tu} = 48.8(D_u/\rho)^{\frac{1}{2}} \text{ c.g.s.} \tag{34a}
\]

\[
V_{td} = 48.8(D_d/\rho)^{\frac{1}{2}} \text{ c.g.s.} \tag{34b}
\]

\( r_s \) and \( \rho \) are given as functions of height by Eqs. 4 and 6. If Eqs. 31a, 31b and 32 are used to eliminate \( M_u \), \( M_d \) and \( r \), Eqs. 33a and 33b become two simultaneous first-order differential equations with independent variables \( D_u \) and \( D_d \) and dependent variable \( Z \). \( C_A \) and \( C_N \) are determined by the input level values of \( D_u, N_u \),
and \( V_Z \). The simple model fixes the input level liquid water content \( r \). The problem of finding an input level value of \( D_d \) which will cause the ascending and descending growth equations to approach the same balance level remains. If \( D_d \) at the freezing level is known, Equations 33a and 33b can be integrated by a standard Runge-Kutta routine to a height \( Z \) where either \( D_u \) or \( D_d \) approach a balance level, that is, one of their derivatives becomes infinite. This Runge-Kutta routine should, of course, allow a smaller and smaller vertical step size to be used as the balance level is approached. The correct input level value of \( D_d \) was found by iterating about values which cause the descending hailstones to have either a higher or a lower balance level than the ascending hailstones. When the correct input value of \( D_u \) is found to within 0.01cm the vertical integrations are stopped. The input level value of \( D_u \) will now, if changed by 0.01cm in either direction, cause the descending hailstones to have either a higher or lower balance level than their ascending counterparts whose growth on descent they now represent.

Providing that there is an ascent with descent solution for given input level \( D_u \) and \( V_Z \) when \( N=0 \), there is also a maximum value of \( N \) at the input level \( N_M \) which will allow a steady-state solution for these same input values of \( D_u \) and \( V_Z \). As these input values of \( D_u \) and \( V_Z \) are capable of giving a balance level in the ascent only case for numbers greater than \( N_M \), it is assumed that the
solutions for growth on ascent with descent are time
dependent for input level numbers greater than \( N_M \). No
attempt is made to calculate time dependent solutions. In
terms of the solving for \( D_{do} \), the values of input \( N \)
greater than \( N_M \) represent a region of \( D_{do} \) 's to the input
level which allow both the ascending and descending dia-
meters to be calculated to the five kilometer level.

The maximum input number \( N_M \) was calculated for
various embryo diameters and updraft velocities in
Chapter 5, see Figure 20. Growth times, hailstone icing
conditions, buoyancy, and comparisons of the cloud's
buoyant and kinetic energies are given in Chapter 5 for
several updraft velocities and input number densities with
an embryo diameter of 0.5cm.

2.8.1. Growth on Ascent of Hailstones Injected in a Size
Spectrum

It is desirable to consider a number distribution
(or size spectrum) of embryos being injected into the
cloud model so that one might avoid the infinities in
number density which are inherent in the balance level
conditions of the single input diameter model of Sections
2.6 and Chapters 3 and 4. A number density can be
defined by a number of distinct hailstone diameters \( D_i \)
each having a finite number density per unit volume \( N_i \)
at a given altitude. The subscript \( i \) refers to the \( i \)th
diameter. If this distribution is handled carefully it
can be used to smooth the effects of the infinities mentioned above and as a result more correctly describe the hailstone distributions in a real cloud.

\( n \) different diameters are fed continuously into the freezing level with diameters \( D_{oi} \) equally spaced, i.e., \( D_{oi+1} - D_{oi} = \) constant, and with initial number concentrations per unit volume \( N_{oi} \). The subscript 'o' refers to the input level. The hailstones are allowed to grow while ascending in steady-state until they reach their respective balance levels where they are assumed to disappear from the model. This consideration gives a fair estimate of the hail conditions in a cloud just before the effects of their subsequent descent are felt by the cloud model. These manifestations would be in the form of increased liquid water depletion and increased forces on the cloud updraft. This interpretation is made more reliable if the various hail sizes reach their balance level at approximately the same time after the first hailstones were injected.

2.8.2. The Equations and their Solution

The simple model of Sections 2.5 give the cloud saturation mixing ratio \( r_s \), the air density \( \rho \), and the updraft velocity \( V_z \) if the injection level value is known as functions of height \( Z \).
The conservation of upward water flux requires:

\[(r+r_s)pV_{Z} + \sum_{i=1,n} N_i M_i (V_{Zi} - V_{ti}) = r'\]

where

\[C_w = (r_0+r_s)pV_{Z0} + \sum_{i=1,n} N_{oi} M_{oi} (V_{Z0} - V_{toi})\]

\(C_w\) is a constant of water substance flux determined by the input level conditions. \(M_i\) and \(V_{ti}\) are the masses and terminal velocities of the \(i^{th}\) size of particles.

Although this model has the hailstones disappear when they reach their balance levels, i.e., become inactive with respect to growth and water depletion, it is desirable that they take their share of the liquid water with them while performing this rather unnatural task. This is accomplished by setting equal to their balance level values, those \(M_i\)'s of Eq. 5 which represent hailstones which have reached their balance level below the altitude under consideration. Equation 35 is used to solve for the liquid water content \(r\).

Conservation of hailstones as they ascend implies that for all heights:

\[N_i (V_{Zi} - V_{ti}) = c_i\]

where the \(c_i\) are constants determined by the number...
density and terminal velocities of the $i^{th}$ hailstone size group at the injection level.

The continuity equation for air as in Equation 8 requires for all levels:

$$ \rho V_z = C_A \quad [37] $$

where $C_A$ is determined by updraft velocity and air density at the input level.

Equations 36 and 37 can be used to eliminate $N$ and $V_z$ from Equation 35 to get

$$ (r+r_s)C_A + \sum_{i=1}^{n} C_i M_i = C_W \quad [38] $$

The growth of spherical hailstones of collection efficiency equal to 'one' from Equation 10 becomes:

$$ \frac{d}{dZ} \frac{D_i}{D_i} = \frac{\rho r}{2\rho_i} \left( \frac{C_A}{\rho V_{ti}} - 1 \right)^{-1} \quad [39a] $$

From Equation 9, one gets:

$$ V_{ti} = 48.8 \left( \frac{D_i}{\rho} \right)^{1/2} \text{ c.g.s.} \quad [40] $$

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The hailstone's mass referring to the $i^{th}$ diameter is:

$$M_i = \rho_i \frac{\pi}{6} D_i^3$$

By solving Equation 38 for $r$, then eliminating $r$ from the set of $n$ Equations 39a one gets:

$$\frac{d D_i}{d Z} = \frac{\rho(C_w C_A^{-1} - r_s - \sum_{i=1}^{n} M_i C_i C_A^{-1})}{2\rho_i (C_A \rho^{-1} V_{ti}^{-1} - 1)}$$

Since $r_s$ and $\rho$ are known functions of $Z$ from Equations 4 and 6 and $V_{ti}$ and $M_i$ are known functions of $D_i$ and $\rho$ from Equations 40 and 41, Equations 39b became $n$ simultaneous first order differential equations for $\frac{dD_i}{dZ}$ in terms of the $n$ dependent variables $D_i$ and the independent variable $Z$. The integration is carried out from the input level, $Z=0$, where the input values of $D_i$, $N_i$, $r$, and $V_Z$ are known.

A standard Runge-Kutta method of integrating is used where the vertical step size $\Delta Z$ gets smaller as the largest of the remaining $D_i$'s approaches its respective balance level. The integration from $Z=0$ is carried out until a diameter $D_i$ grows to within 0.01 cm. of the terminal diameter $D_t$ at height $Z$ which is determined by updraft velocity and air density at that height, i.e.:

$$D_t = \rho \left( \frac{V_Z}{48,8} \right)^2 \text{ c.g.s.}$$
At the altitude where $D_i$ satisfies the above condition the conservation of water flux, Eq. 38, is made valid by setting $M_1$, i.e. $D_i$, equal to the appropriate balance level value, i.e. Equation 41 for $M_i$ becomes:

$$M_i = \frac{\pi}{6} D_i^3$$

for all altitudes above the balance level.

This was explained by considering 35 to be the equation used for calculating the liquid water content, see just after Equation 35. For integration above the balance level of the $n^{th}$ diameter, $\frac{dN_n}{dZ}$ becomes equal to zero leaving $n-1$ equations in $\frac{dD_i}{dZ}$. This assumes that the $n^{th}$ diameter is always the biggest at the input level and hence reaches a lower balance level than the $n-1^{th}$ embryo. The integration continues to greater $Z$ until all of the hailstones have reached their balance level or until $Z=5\text{km}$ is reached, whereupon the freeze-out of liquid water is assumed to stop hail growth.

### 2.8.3 Representative Number Distributions at Various Levels

The total number of hailstones at a balance level can be found by summing the number per unit volume of those diameters which have not approached their balance level diameter and then adding the mean number $\bar{N}_i$ per unit volume of those which reach their balance level at that height.
The vertical extent used to calculate $\bar{N}_i$ near the balance level is determined by the balance level heights achieved by the next biggest and next smallest input hailstones to that group under consideration. If $Z_{b,i+1}$ and $Z_{b,i-1}$ are the altitudes at which $D_{i+1}$ and $D_{i-1}$ reach their balance levels, to a first order approximation there is a region of vertical extent equal to $\frac{1}{2} (Z_{b,i-1} - Z_{b,i+1})$ and centered about $Z_{b,i}$ where the $i^{th}$ diameter $D_i$ imposes a mean number density equivalent to the number flux into the bottom of that region, i.e. $C_i$, times the time $t_{bi}$ which $D_i$ took between entering the region and reaching its balance level diameter, all divided by the vertical extent of the region.

The time $t_{bi}$ can be calculated by integrating the inverse of the net upward velocity of the hailstone, $D_i$, from $(Z_{b,i} - 0.25(Z_{b,i-1} - Z_{b,i+1}))$ to $Z_{b,i}$, as follows:

$$
t_{bi} = \left\{ \begin{array}{ll} 
Z_{b,i} & \\
\left(\frac{V_Z}{V_{ti}}\right)^{-1} & \\
Z_{b,i} - 0.25(Z_{b,i-1} - Z_{b,i+1}) & 
\end{array} \right.
$$

This gives:

$$
\bar{N}_{ui} = \frac{C_i t_{bi}}{0.5(Z_{b,i-1} - Z_{b,i+1})}
$$

[44]
This procedure has eliminated the infinities in number densities encountered when only a single size of embryo was considered. Suitable \( (Z_{b_i} - Z_{b_{i+1}}) \) must be taken when calculating \( \bar{N}_{ui} \) if the \( i-1 \) diameter does not reach a balance level, i.e. it is carried to the top of the cloud. Namely, \( Z_{b,i-1} - Z_{b,i+1} \) is set equal to \( 2(Z_{b,i} - Z_{b,i+1}) \). If \( Z_{b,i+1} \) does not exist, i.e., \( i=n \), \( (Z_{b,i-1} - Z_{b,i+1}) \) is taken to be equal to \( 2(Z_{b,i-1} - Z_{b,i}) \) for calculating \( \bar{N}_{ui} \) and \( t_{b,i} \). There is, therefore, little or no difficulty encountered in interpretation of these calculated mean number densities if at least three of the embryo sizes reach balance levels in the cloud.

The mean hail mass per unit volume in the balance level regions can be calculated from the mean number densities. Hence, the buoyant force can be compared to the hail drag force in these regions.

In conclusion, the model considering growth of hailstones injected with a size spectrum gives a reasonable interpretation of what the hailstones' distributions at various heights would look like if the embryos had been injected continuously and allowed to grow for little enough time that a significant number had not begun to fall from their balance level. Chapter 6 describes the results of this model. Section 2.9 of Chapter 2 and Chapter 6 also describe the consequences of an injection of non-homogeneous sized embryos where the hailstones are allowed
to descend after reaching their balance level while
injection continues until steady-state growth is achieved.

2.9.1. Growth of Hailstones During Ascent and Descent, Starting from a Size Spectrum of Embryos.

After considering the growth of embryos on ascent, a model is developed where a size spectrum of embryos is injected continuously and allowed to grow and rise, reach terminal velocity and fall back down through the cloud until steady-state is achieved.

Let the rising hailstones at a given height be represented by diameters $D_{ui}$ in numbers per unit volume of $N_{ui}$ and the falling hailstones be represented similarly by $D_{di}$ and $N_{di}$. The subscripts $u$ and $d$ refer to upward moving and downward moving respectively in all equations of this model. Water substance and hail number conservation equations are needed to complete the simple cloud model of Sections 2.5. for this type of growth.

The conservation of upward flux of water is given by:

$$(r + r_s) p V_z + \sum_{i=1}^{n_{ui}} N_{ui} (V_z - V_{tui}) M_{ui} + N_{di} (V_z - V_{tdi}) M_{di} = C$$  \[45\]

This equation implies 'n' different hail embryo sizes. To be valid Eq. 45 must have $M_{di} = 0$ if the $i^{th}$ embryo group does not achieve a balance level or, if the equation is being solved for liquid water $r$ above the balance level.
of the $i^{th}$ size, both $M_{di}$ and $M_{ui}$ equal zero.

$C_M$ is the constant upward flux of water substance and is valid for all heights. $C_M$ can be determined if the input level values of $D_{ui}, D_{di}, N_{ui}, N_{di}$ and $V_Z$ are known. $r$ and $r_s$ are assumed to be given at the input level of the simple model.

The conservation of number flux of hailstones is given by:

$$C_i = N_{ui}(V_Z - V_{tui}) = -N_{di}(V_Z - V_{tdi}) \tag{46}$$

The second equivalence sign in Equation 46 does not hold if the $i^{th}$ group of embryos did not achieve a balance level, i.e., if they were carried above the 5km level. The constants $C_i$ are constant at all levels and are determined by the input level conditions of injected diameter, number density, and updraft velocity. It should be noticed that while $C_M$ is a function of the descending hailstones the $C_i$'s are dependent only upon the rising hailstones.

Using $\rho V_Z = C_A$, the equation of constant updraft air flux, and Equation 46, Equation 45 becomes:

$$(r+r_s)C_A + \sum_{i=1}^{n} C_i(M_{ui} - M_{di}) = C_M \tag{47}$$

Eliminating $r$ from Equation 39a by using Equation 47 and keeping in mind the need for up or down subscripts on...
Equations 49 constitute 2n equations for \( \frac{dD_{ui}}{dZ} \) and \( \frac{dD_{di}}{dZ} \). If the \( i \)th ascending diameter does not achieve a balance level, \( \frac{dD_{di}}{dZ} \) need not be considered just as \( M_{di} \) was set equal to zero in this case.

The 2n Equations 49 are not solvable by direct integration from the input level as 2n simultaneous first-order differential equations because the diameters of the descending hailstones are not known at \( Z=0 \). Further, Equations 49 give no relationship between \( D_{ui} \) and \( D_{di} \) to cause them to achieve the same balance levels. This condition can be achieved as follows. Dividing the \( i \)th Equation 49 for \( \frac{dD_{di}}{dZ} \) by that for \( \frac{dD_{ui}}{dZ} \) one gets:

\[
\frac{dD_{di}}{dD_{ui}} = \frac{\nu Z V_{tui}^{-1} - 1}{\nu Z V_{tdi}^{-1} - 1}
\]

If Equation 50 is to be considered over a small vertical extent where the air density \( \rho \) changes are small compared to diameters \( D_i \), i.e., near the balance level where growth of the hailstone with respect to height is large, one can write an approximation of the terminal velocity Equation 40
as follows:

\[
\frac{dD_i}{d} = d \left( \frac{\rho V_{ti}^2}{(48.8)^2} \right) = \frac{\rho}{(48.8)^2} d V_{ti}^2 \quad [51a]
\]

Putting the above into Equation 50 one gets for small vertical extents:

\[
\frac{d V_{tdi}}{d V_{tui}} = \frac{V_Z - V_{tui}}{V_Z - V_{tdi}} \quad [51b]
\]

Integrating the above equation from the balance level where \( V_{tdi} = V_{tui} = V_Z \) down to a level where air density changes from the balance level are small one gets:

\[
2V_Z = V_{tui} + V_{tdi} \quad \text{[near balance level]} \quad [52]
\]

This equation is valid near the balance level since it is seen in general from the calculations of Chapter 6 at a distance of 50 meters from the balance level that the terminal velocity has changed by 10% while the air density has changed by \( \frac{1}{2} \% \). A vertical integration step size of 50m is therefore appropriate for the calculation if the approximation of Equation 52 is to be used. Since Equation 52 gives the value of the descending hailstone diameter in terms of its ascending counterpart, we have as many new conditions as there are hailstone sizes descend-
ing at the input level. The hailstones sizes carried to the 5km level are assumed to not return as descending particles.

The equations are solved by converging on the solution in the following steps:

**Step 1:** Using Equation 49 for the growth of ascending hailstones while setting the descending hailstones' masses and diameters equal to zero, $C$, is calculated from Equation 45 and the growth of the ascending hailstones is determined by the integration of Equation 49 through a first order finite difference method. Step sizes $\Delta Z$ of 50 meters are used. The value of $D_{ui}$ at the top of a step is evaluated from the value of $D_{ui}$ at the bottom of the step plus $\frac{dD_{ui}}{dZ} \Delta Z$ where $\frac{dD_{ui}}{dZ}$ is Equation 49 evaluated at the bottom of the step. If after integrating upwards several vertical steps $\Delta Z$, the value of $D_{ui}$ becomes greater than the maximum diameter which could be supported by the updraft at that level, i.e. $D_{ui} > pV_z^2(48.8)^{-2}$, $D_{ui}$ is set equal to zero at that level and for all levels above. This integration is carried out to the 5km level.

**Step 2:** For the diameters $D_{ui}$ which reached their balance level their growth on descent is calculated 'ignoring' their depletion of liquid water. This downward growth is integrated from the top of the model down to the input level as follows. Writing Equation 50 in finite difference form
gives:

$$\Delta D_{di} = \frac{\Delta D_{ui}}{\frac{C_A \rho^{-1} V_{tu}^{-1} - 1}{(C_A \rho^{-1} V_{td}^{-1} - 1)}}$$ \[53\]

Since the $\Delta D_{ui}$ have been calculated as functions of height in step 1, $\Delta D_{di}$ can be calculated from the values of $\rho$, $D_{ui}$ and $D_{di}$ at the top of each vertical step. $D_{di}$ at the top of the first step (i.e. just below the balance level for the $i^{th}$ diameter) is obtained from Equation 52 and hence Equation 53 can be integrated for each descending $D_{di}$ stepwise downward to get an estimate of the $D_{di}$ at the input level. This use of the $\Delta D_{ui}$ calculated in Step 1 has given the descending particles no effect on the cloud liquid water content.

**Step 3:** Using the input level values of $D_{di}$ obtained in Step 2 and the known values of $D_{ui}$ at the input level Equations 49 are integrated upwards for both ascending and descending hailstones. Equation 47 is first evaluated at the input level to give a new value of $C_M$ which includes the descending hailstones. Equations 49 are integrated in a manner similar to the finite difference method of Step 1. As the integration continues upward the hailstone diameters are set equal to zero at and above the level where the terminal velocities of the ascending and descending hailstones become greater than or less than the updraft velocity respectively.
This integration has now taken into account the depletion of liquid water by both ascending and descending hailstones, however, since the values of $D_{ui}$ at the input level are estimates, the $i^{th}$ ascending and descending hailstones do not necessarily reach their balance levels at the same altitude.

**Step 4:** - Step 2 is carried out again with the new values of $\Delta D_{ui}$ and the new balance levels of the ascending hailstones obtained in Step 3. This gives new estimates of the input level $D_{di}$ and a new estimate of $C_w$ which can be used to carry out an upward integration similar to Step 3.

**Step 5:** - Steps 2 and 3 are carried out alternately for a finite number of times. If the solutions on alternate steps are sufficiently similar, i.e., the $i^{th}$ ascending and descending particles reach similar balance levels and alternate steps give the same values for $D_{di}$ at the input level, a steady-state solution is presumed to have been obtained. Between 10 and 20 upward integrations are sufficient to give the balance level values of $D_{ui}$ to within 0.01 cm on alternate steps.

If the conditions of Step 5 are not met the steady state solution is said to have not been obtained but nothing will be said about its existence since the calculations of this model are made only to display some reasonable examples and not to determine solutions for all possible embryo distributions. A lack of convergence is
met in this method of solution when the alternate integrations of Step 5 cannot determine whether $i$ or $i+1$ of the embryo sizes will grow to reach a balance level.

Chapter 5 on the solutions of the ascent and descent of a single size of hailstones will give insight into which steady-state solutions are unattainable physically and which are not with this numerical model.

Appendix 4 gives the fortran 'h' computer program used to calculate this model. A representative number distribution at various levels in the cloud can be calculated in a manner similar to that given in Section 2.8.3 for the growth on ascent model, the difference being that now the descending hailstones must be considered. Number distribution graphs showing the number of hailstones per unit diameter as a function of diameter can also be calculated for various levels in the cloud if some artistic license and physical intuition is permitted for the smoothing of the distribution when only five distinct embryo sizes are fed into the input level.

The mean force due to the hailstones' weights at each level can also be calculated and the energy available to a parcel of air due to thermal buoyancy and hail force can be obtained in a manner similar to Equation 28b. These calculations are made in the computer program of Appendix 4.
The hailstones' surface icing conditions are calculated in Appendix 4 by application of the equations in Appendix 3.

2.10.1 Conclusions Concerning the Equations of the Models

The computer programs and exact methods of solution were not given for the models of hail growth on ascent only or for the ascent and descent model with single embryo size at the input level. The author feels that a set of first-order differential equations with all dependent variables defined at some value of the independent variable can be solved with ease by using the standard Runge-Kutta subroutines available with today's scientific computers. The R-K subroutine must have a feature whereby one can stop the subroutine's computing, without printing a large number of error messages in the event that the subroutine has been asked to do an impossible integration. For instance, if it was asked to calculate a hailstone's diameter above its balance level. In this event, the program should automatically request integration to a lower level in the cloud, i.e., change the value of the independent variable where the dependent variables are to be evaluated.

In the case of the ascent and descent of embryos with a single diameter, a reasonable method of iterating to the output diameter which causes the ascending and descending hailstones to reach a similar balance level is
quite easily obtained since there is only one unknown input parameter.

The computer program for ascent and descent of a number distribution is given because it may be useful to others who wish to study the reliability of this method in future applications.
CHAPTER III

HAIL GROWTH ON ASCENT

3.1. New Considerations

This chapter is a review of work already published by the author and a discussion of several details which were not covered in those publications. Appendix 1 contains the complete paper as it appeared in 1968 in the Journal of Atmospheric Sciences, Vol. 25, under the authorship of R. List, R. Charlton and P. Butulis. That paper treated the growth on ascent of a single size of spherical embryos fed into a non-divergent updraft in constant number densities as does this chapter and Chapter 2, Section 6 which describes the complex model. The discussion in the previous study confines itself to the variation of hail growth and liquid water depletion with various input level diameters $D_0$, number densities $N_0$, and updraft velocities $V_{zo}$. The force of the liquid water and the hailstones on the updraft, the heat exchange between the hailstones and the cloud air, and the surface icing conditions were included in that work.

This chapter extends the model of Appendix 1 to include additional considerations and refinements.

The further considerations are:

1) The effect on hail growth of varying the input level liquid water content.

2) The energy required to accelerate the updraft air is
compared to the energy available from the thermal buoyant force and from the downward force of the hailstones and liquid water.

3) The hailstones in this model are allowed to become spheroidal (Chapter 4).

The Refinements to the Model are:

1) A Runge-Kutta method of integration replaces the second order finite difference method. Any practical computational accuracy can be obtained by this method.

2) The heating of the updraft by hailstones is included in the equation of state of this updraft (Eq. 20 Section 2.6). In the previous model (Appendix 1) these effects were calculated by a perturbation method after the hail growth had been determined.

3) The cloud heating by heat exchange with the hailstones is put on a more accurate basis by considering the rate of heat lost by convection and conduction per hailstone $Q_{cc}^*$ and the rate of heat exchange caused when the water evaporating from the hailstones $Q_{E}^*$ is required to recondense into the cloud air to maintain water conservation. In the previous model the heat exchange was approximated as the heat required to freeze the liquid water being accreted on the hailstone and the heating of
the updraft was found by a constant pressure heating process which formed a perturbation temperature. Both models include the heat dissipated by the hailstones' wakes.

4) The equation of conservation of water substance includes the evaporation from the hailstones in this model. (Eq. 18, Section 2.6.6).

5) The rate of freezing out of liquid water above the -32°C level is proportional to the amount of liquid water present rather than its being a function of height only. (Eq. 19, Section 2.6.6). The cloud air heating from this effect is also included in this model (Eqs. 20, Section 2.6.7).

The refinements above do not detract significantly from the conclusions of the paper given in Appendix 1.

The input level conditions and assumptions used to solve the complex model of Section 2.6 are as follows:

i) The collection efficiency of the hailstones is 100 percent, i.e., E=1.

ii) The density of the hailstones ρ is 0.915gm/cm³.

iii) The input level conditions are: pressure P = 595mb, cloud temperature T=0°C, saturation mixing ratio r₈ = 6.47 x 10⁻³ gm/gm and liquid water content r = 1.75 x 10⁻³ gm/gm or Wf = 1.334gm/m³ unless the example considers variable liquid water.
3.2. Hail Growth Dependence on Input Level Parameters

The input level parameters; updraft velocity, liquid water content, hail embryo diameter and hailstone concentration; will be varied about their respective central values of 22m/s, 1.334gm/m³, 0.5cm, and 1.0 -³ to form four sets of graphs (Figures 4, 5, 6 and 7) showing hail growth dependence on their variation. Two further examples (Figures 8 and 9) will have 0.25cm hail embryos growing for various number concentrations and input level liquid water contents so that some further conclusions can be made about number concentration, the parameter which most effects the cloud, and liquid water content, the parameter not considered in previous publications.

In the examples of Figures 4, 5, 6 and 7 the input level parameters are varied independently in the ranges: \( D_0 = 0.3 \) to \( 0.8 \) cm, \( N_0 = 0 \) to \( 10 \) m⁻³, \( V_{Z0} = 18 \) to \( 24 \) m/s, and \( W_{fo} = 0.5 \) to \( 2.5 \) gm/m³.

The example of Figures 8 has the number density \( N_0 \) varying between 0 and \( 5 \) m⁻³ while \( D_0 = 0.25 \) cm, \( V_{Z0} = 20 \) m/s and \( W_{fo} = 1.334 \) gm/m³.

Figure 9 has the input level liquid water content varying from \( 0.5 \) to \( 2.5 \) gm/m³ while \( D_0 = 0.25 \) cm, \( V_{Z0} = 18 \) m/s and \( N_0 = 5 \) m⁻³.

These values of input parameters were not chosen because they represented all possible conditions but rather
because they are quantities which could represent the conditions near a hailcloud's freezing level. They are varied only moderately to show the high dependence of hail growth and cloud liquid water on them.

The appropriateness of these freezing level conditions can be enhanced by a discussion of the cloud model's energy. Figure 3 shows the increase of kinetic energy with height per gram of cloud air with respect to the input level value. This kinetic energy increase is compared to the energy available per gram of cloud air BE due to the integrated thermal buoyancy and force due to liquid water for various freezing level liquid water contents. The values of BE are found by computing the complex model without hailstones and integrating the buoyant forces in a manner similar to Eq. 28b of Chapter 2. Fig. 3 indicates that the kinetic energy compares favorably with the buoyant energy for input level updrafts of 18 to 22 m/s when the Denver sounding's input level liquid water content of 1.334 gm/m³ is used. The available buoyant energy is seen to change in the order of 20% for input level liquid water content changes of 1.0 gm/m³. This indicates that for the hailcloud's energy to be effected by less than about 20% the mean mass of hailstones per unit volume must be kept below 1.0 gm/m³. The considerations above make the choice of a 22 m/s updraft and 1.0 hailstone per cubic meter reasonable from an energy point of view for these examples of growth on ascent. The appro-
priateness of these values will be confirmed in the description of the calculations.

Figures 4a through 7a show the hail growth curves as functions of height for various input parameters. Two types of growth curves are found, one is the sigmoid type where the hailstones leave through the top of the cloud and the second is where the hailstones reach their balance level where, according to the model, they are assumed to no longer influence the cloud. Considerable difference in growth is seen for variation of any of the four input parameters but only the two types of growth curves are seen.

Figures 4b through 7b gives the liquid water profiles corresponding to Figures 4a through 7a respectively. The liquid water content $W_f$ is seen to fall off rapidly with height just below the balance level where the concentration of hailstones is high.

Figures 4a and 4b, on the variation of embryo diameter, show the expected results; large embryos reach their balance level low in the updraft while the smallest sizes rise too quickly for significant growth and are hence thrown from the top of the cloud model without having appreciably depleted the liquid water. Hailstones which reach their balance level high in the cloud are most active in depleting liquid water and for this example represent embryos of 0.4 to 0.5cm.
Figures 5a and 5b show an important fact; to grow big hailstones on ascent the number concentration must be quite low and in the order of 0.5 to 2.0 per cubic meter at the source level, other base level parameters being held constant. Conditions for no liquid water depletion are shown in Figures 5a and 5b by the curves for $N=0$.

Figure 5e gives the number density of hailstones as a function of height for the variable number example. There is a build up of hailstones at the level where the terminal velocity approaches that of the updraft. If the particles, on approaching the balance level, sweep out sufficient liquid water they tend to create a high number concentration even if they do not attain a balance level. Due to this inability of the particles to grow above this 'near balance' level, as in the case of $N_0 = 1.8$ or $2.0/m^3$, there is a tendency for the updraft, which increases with height, to spread out (vertically) the particles above this level. These number density observations are clarified in Chapter 6 where a number distribution of embryos is introduced.

A somewhat different effect as it pertains to the variation of input number concentration is shown in Figures 8a and 8b where 0.25 cm. embryos are injected into a 20m/s updraft. The nil number density example shows hailstones reaching their balance level just below the region of liquid water freeze-out, 4.5km. If significant
numbers are injected into the cloud the hailstones are carried to the region of liquid water freeze-out and hence they do not achieve a balance level. This example illustrates the importance of the liquid water freeze-out to hailgrowth and justifies the cut-off of hail growth above the 5km. level in later studies which use the equations of the simple model. Liquid water freeze-out, therefore has a 'capping' effect on the model which allows one to limit the cloud's vertical extent for purposes of hailgrowth.

The surprisingly big effect of the updraft velocity on hailstone growth is shown in Figures 6a and 6b. It can be concluded that it is not the strongest updrafts which produce the biggest hailstones since big updrafts, barring big embryos, have a strong tendency to carry the hailstones to the top of the growth region before they can achieve large diameters or a balance level. Conversely, small updrafts do not have large balance level diameters and hence they produce lesser growth. Figure 6b also indicates that the fraction of liquid water depleted for cases where the balance level is achieved increases rapidly with increasing updraft.

Figures 7a, 7b, 9a and 9b show hailgrowth and liquid water as functions of height for 0.5 cm. and 0.25cm. embryos fed into 22m/s and 18m/s input level updrafts in input number densities of 1.0 and 5.0 per m³ respectively where the base level liquid water content is varied. The
variation of input level water from 0.5 to 2.5 gm/m³ in these two examples, the Denver standard cloud having 1.33 gm/m³ is equivalent to moving the lifting condensation level (LCL) from 0.6 km to 1.7 km below the freezing level. The Denver data gives 0.96 km below the freezing level as the LCL. This range corresponds to a change of about 8°C in wet bulb potential temperature and a change of 1.1 km in cloud base height.

The greatest hailgrowth and liquid water depletion is found to be attributed to the intermediate values of input liquid water which allow high balance levels, other parameters being kept constant.

In conclusion one sees that the depletion of liquid water by finite numbers of hailstones works in two ways. First, increased competition due to higher particle concentration causes lesser hailstone growth for a given vertical ascent from the input level. Second, these lower growth rates lead to larger absolute upward speeds of the particles allowing them to either reach higher balance levels and greater sizes or to be carried to the level of liquid water freeze-out.

3.3. Growth Times

The time for hailstones to reach a given height is shown by the fine solid lines in Figures 4a through 9a. The growth times required for the larger stones to reach their
balance levels are from 8 to 12 minutes which is consistent with radar echo observations. The only cases of long growth times occur when the liquid water is nearly all swept out leaving the hailstones growing slowly with time as they approach a 'near balance level'. This is seen in Figures 5a and 9a where times of 20 to 40 minutes occur. However, this possibility is refuted in Chapter 6 where a number distribution of embryos makes high liquid water depletion impossible.

3.4. Surface Icing Conditions of Hailstones

The fraction of the accreting liquid water which is freezing I or the temperature of the hailstone's surface $T_d$ when all the accreting water freezes is given in Figures 8a to 9a by the dashed isolines. These lines indicate that the biggest stones usually have a deposit temperature slightly below freezing since they reach large sizes in higher and colder regions of the cloud than do the smaller hailstones. For a given cloud height and liquid water content, however, larger hailstones imply more spongy growth. In the laboratory fast growth is generally associated with spongy growth, i.e., large liquid water contents and high relative velocities. An investigation by List et. al. (1969) showed that many outer shells of 4cm. hailstones are definitely grown non-spongy. Whether or not those hailstones grew in a cloud similar to this model is open to question.
The formation of 'shells' on hailstones due to changes in low-level liquid water and updraft velocity becomes very possible when considering the icing conditions dependence at higher levels on these parameters.

3.5. Forces and Energies in the Updraft

The foregoing sections of this chapter have shown that reasonable hail embryo numbers can cause significant liquid water depletion. It is the re-distribution of the cloud's liquid water into the hail phase which will change the available buoyant energy of the updraft.

Figure 3, which was discussed in Section 3.2 shows that a LWC of 0.5gm/m³ will allow the model to balance the available buoyant energy with the kinetic energy of a 20m/s base level updraft while a LWC of 2.5gm/m³ can accommodate a 17m/s updraft. Figure 3 shows that the available buoyant energy between the freezing and the 5km level decreases by nearly 40% when the base LWC is increased from 0.5 to 2.5gm/m³. If liquid water amounts of similar magnitude were able to increase their residence time in the updraft through the formation of hail it is apparent that the forces and available energy would change considerably.

The total force per gram of cloud air $F_B$ due to thermal buoyancy $f_B$, Equation 7a, and the weight of the hailstones and liquid water $f_{HR}$, Equation 28a, is given by:

$$F_B = f_B + f_{HR}$$  [54]
Integrating $FB$ from the freezing level to a given height $Z$ as in Equation 28b, gives the available buoyant energy per gram of cloud air $BE$ which can be compared to the kinetic energy increase $KE$ between those levels as given in Equation 27.

Figures 5c, 6c, 8c and 9c show values of the buoyant force $FB$ for the examples in which $D_o = 0.5 cm$ where $N_o$ and $V_{Zo}$ are varied and $D_o = 0.25 cm$ where $N_o$ and $W_fo$ are varied respectively. Figures 5c, 8c and 9c also include the non-thermal feedback buoyancy which is the force which would be available if the hailgrowth was computed leaving out the heat and mass exchange due to hailstones. Thermal feedback is discussed in Section 3.6 of this chapter.

Figures 5d, 6d, 8d and 9d show the available buoyancy energy $BE$ and the kinetic energy $KE$ of the cloud parcels as functions of height for the appropriate examples listed above.

Figures 5c and 5d showing the buoyant force and energies respectively indicate, when interpreted with the aid of the liquid water profiles of Figure 5b, that the greatest forces and energy changes are associated with the maximum depletions of liquid water.

These figures should be approached carefully if one wishes to point out input level number concentrations which significantly deplete liquid water. Anomalies appear in Figure 5d for the case of $N_o = 0.5, 1.0$ and $2.0$ per $m^3$.
where the doubling of the input number more than doubles the buoyant energy depletion and hence the depletion of liquid water. These anomalies are taken into consideration in Chapter 6 where the liquid water depletion is dependent on several input diameters injected simultaneously each in concentrations not great enough to individually deplete most of the liquid water. The forces and energies of these figures are, however, realistic for the indicated rates of depletion of liquid water with height.

Figures 6c and 6d show the total buoyant force and buoyant energies respectively as functions of height for the example where the updraft velocities are varied, the conclusions being that the forces, energies and liquid water depletion are greatly dependent on modest changes in updraft velocity.

Figures 8c and 8d give forces and energies associated with the growth of 0.25cm embryos injected into a 20m/s updraft in various number densities. The available buoyant energy BE does not vary greatly near the balance level because there is only moderate liquid water depletion and hence hail mass accumulation (Figure 8b).

Figures 9c and 9d show the total buoyant force FB and buoyant energies BE respectively as functions of height where the freezing level liquid water content is varied from 0.5 to 2.5gm/m³ while D₀ = 0.25cm, N₀ = 5 per m³, and VZ₀ = 18m/s. Figure 9d also includes the KE of the
air parcels and the buoyant energy BE for the case of
$W_f = 0.5\text{gm/m}^3$ and $2.5\text{gm/m}^3$ had there been no depletion.
The buoyant energies of these examples are seen to be more
compatible with the kinetic energy for the instances of
higher input liquid water. This is compatible with the
observation that the more liquid water depleted by hail-
stones, the greater is the effect on the updraft's forces
and energy since in this case the moderate values of input
liquid water led to the greatest depletion and energy
imbalance.

Figures 4a through 9a indicate the height on the
growth curves where the thermal buoyancy equals the force
due to the weight of water substance, FB = 0, by a star
*; where the available buoyant energy equals the kinetic
energy gained by the air parcel on ascending from the
input level, BE = KE, by a dot (●), and where the available
buoyant energy equals zero, BE = 0, by a circled dot (○).
Recall that the available buoyant energy at a given height
is the total buoyant force integrated from the input level
as given in Equation 28b. These symbols are useful in
interpreting the strong dependence of liquid water depletion
upon the input level parameters as it pertains to the forces
experienced by and energies available to the rising cloud
parcels.
3.6. Thermal Feedback and Fine Details

The previous section has shown that the depletion of liquid water by ascending hailstones plays an important role in the overall structure of the one dimensional model. It remains to be shown that the considerations which complicated the complex model with respect to the simple model can be assessed, Section 2.3.1.

To test this, the complex model is solved without the heat and mass exchange between hailstones and hailcloud, i.e. $Q^*_E = 0$, $Q^*_cc = 0$, $M^*_E = 0$, and without wake heating term, i.e. $g^M V^*_ta = 0$. This being done, it was found that the hail growth curves and liquid water depletion is changed very little by leaving out these terms, however, a small increase in cloud temperature due to thermal feedback, of the order of 0.5°C, can alter the Archimedean (thermal) buoyancy term considerably and hence change the available buoyant forces and energies.

Figures 5c, 8c and 9c show the 'adiabatic' buoyancy forces as 'dashed' lines along with their total thermodynamic counterparts which include 'thermal feedback' as 'solid' lines. These figures show that for moderate liquid water depletion the increase in buoyant force due to thermal feedback is of the order of 0.5 dynes/gm. For regions of high liquid water depletion resulting from a balance level or high hailstone number densities the additional buoyancy reaches 2 to 5 dynes/gm. It is important to note that,
even though the thermal feedback hailstones reach higher balance levels, about 0.25 \text{ km} higher, and hence achieve slightly bigger diameters while depleting more liquid water, the integrated buoyant force (from the input level) is greater for the thermal feedback case. Therefore, there is greater overall buoyant energy available to the cloud's updraft when thermal feedback is considered even though larger hailstones are grown.

Another method of assessment of thermal feedback can be carried out by comparing the increase in cloud temperature when depletion occurs with the temperature when no depletion occurred. For the example in Figures 5a to 5c where $D_0 = 0.5 \text{ cm}$, $V_{Z_0} = 22 \text{ m/s}$ and $N_0$ is varied from 0 to 10 \text{ per m$^3$}, a few examples of thermal feedback temperatures are as follows: for $N_0 = 0.5 \text{ per m$^3$}$ at the balance level the temperature excess $\Delta T$ is $0.38^\circ \text{ C}$ over normal cloud temperature; for $N_0 = 1 \text{ per m$^3$}$ at $Z = 3.5 \text{ km}$, $\Delta T = 0.28^\circ \text{ C}$ and at the balance level $\Delta T = 0.75^\circ \text{ C}$; for $N_0 = 2 \text{ per m$^3$}$ at $Z = 2.5 \text{ km}$, $\Delta T = 0.1^\circ \text{ C}$ and at 5.0 \text{ km}, $\Delta T = 1.72^\circ \text{ C}$; for $N_0 = 5.0 \text{ per m$^3$}$ at 2.5 \text{ km}, $\Delta T = 0.24^\circ \text{ C}$ at 5 \text{ km}, $\Delta T = 1.1^\circ \text{ K}$. In case of no hailstones the temperature excess in the cloud over that of the environment goes from $1.22^\circ \text{ C}$ at the freezing level to $3.16^\circ \text{ C}$ at 3.0 \text{ km} and reduces to $2.73^\circ \text{ C}$ at 5 \text{ km}. It is apparent, therefore, that increases in cloud temperature due to thermal feedback of about $0.5^\circ \text{ C}$ significantly effect the
If the effects of thermal feedback are to be taken into account in future models an estimate of the relative importance of the heat exchange per hailstone due to conduction and convection $Q_{cc}^*$, evaporation $Q_{E}^*$, and wake heating $Q_{wk}^* = gMV_t$ should be given. For most cases the value of $Q_{E}^*$ is about 20 percent greater than $Q_{cc}^*$. $Q_{wk}^*$ generally has a magnitude about 20 to 40 percent that of $Q_{cc}^*$. All three of these heating effects should therefore be considered together since their magnitudes are similar.

The rate of change of diameter due to evaporation ranges between 2 and 5 percent of that due to liquid water accretion and could for most models be left out in terms of hail growth. However, if most of the liquid water has been depleted from the cloud the growth due to evaporation can dominate. In spite of the unlikelihood of such high cloud water depletion, the hailstone mass loss due to evaporation within a saturated cloud is quite small and could be neglected.

The heat exchange due to hailstones can be summarized by stating that the partial freezing of the liquid water on the hailstones tends to increase the cloud temperature at a lower level than if the latent heat of fusion were left to be released at the level of spontaneous cloud water freezing ($-32^\circ C$ to $-40^\circ C$). The potential
Energy being approximately the integral with height of the cloud to environment temperature excess makes it advantageous to have this latent heat released in the lower levels of the cloud where it can offset the weight of the hailstones.

This chapter has shown that the growth of ascending hailstones, the liquid water depleted by hailstones, and the accumulation of hail mass are dependent on all input level parameters. One also sees that a detailed study of energies and forces in the updraft of a hailcloud should include the heat and mass exchange of the hailstones.
CHAPTER IV

SPHEROIDAL HAILSTONES

1.1. Introduction

Although most hailstones can be roughly described as spheres, a significant fraction of hailfalls contain stones which are conical, spheroidal or irregular.

Conical ice particles, which are usually small and opaque, are called 'graupels'. They are believed to be potential hail embryos since their outline is often found near the center of large hailstones (List, 1958).

Highly irregular hailstones are quite rare but those of lesser irregularity are rather common. The drag coefficients and heat exchange of irregular spheres has been studied in the laboratory by Schuepp (1967) and List, et. al. (1969b). It is not felt that examples of hail growth with increased drag coefficients and modified heat and mass exchange to simulate roughness would add significantly to this study.

Spheroidal hailstones with axis ratios of up to 2 to 1 are generally larger than one centimeter when they reach the ground. These hailstones are commonly seen in the shape of oblate spheroids or bi-axial ellipsoids, i.e. like squashed spheres. Considerable work has been done on the drag coefficients of spheroids by List (1959) and on their heat and mass exchange by List and Dussault (1967).
The formation of spheroidal hailstones has been attributed to 'spongy' growth by List (1960) and to melting after the growth phase by Macklin (1964). This study attributes the spheroids to spongy growth and further assumes that the orientation of the spheroids is that which presents the greatest drag coefficient.

The author has observed a hailfall where spheroidal hailstones of relatively regular major dimension, 1.5 cm, and minor dimension, 0.75 cm, fell for about five minutes leaving about 10 per square foot on the ground. All of these spheroids reached the ground before the major hailstorm and before significant rainfall. They all had a clear center with an outer doughnut of opaque ice. After a lull in precipitation of about five minutes the hailfall proper began and lasted for ten minutes leaving the ground surface fifty percent covered with spherical hailstones having diameters between 0.5 centimeters and 1.5 centimeters and opaque centers with clear outer shells. These observations lead one to expect that the portion of hail which reached the ground was formed through wet growth followed by dry growth in the case of the spheroids and dry followed by wet growth in the case of the spheres. The concept of wet growth-clear ice, and dry growth-opaque ice does not, however, always stand up to experiments in wind tunnels nor does the concept of cold cloud regions yielding dry growth as is indicated in Figures 4a to 9a of Chapter 3. In this case the spheroids' outer 'doughnuts' could have grown wet.
Observations did, of course, prove that ellipsoidal hailstones can exist in sufficient quantities to make up the majority of the hail falling at a given time, but I cannot say whether the spheroidicity of these hailstones was caused by melting or wet growth.

The equations used to describe the growth on ascent have been given in Sections 2.6.1 to 2.6.9. It is, therefore, necessary to only discuss the concepts which pertain to the ellipsoidal nature of these hailstones. The first examples will cover the growth of hailstones which maintain a constant axis ratio α. Later examples will show growth where the embryos are spherical but on growing their axis ratio increases linearly with the major axis until the major axis reaches 3 cm, whereupon the axis ratio is 1:2 and after which they maintain this 1:2 ratio for further growth. These later examples are consistent with observations that greater spheroidicity is associated with larger hailstones.

4.2. Growth for Constant Axis Ratio

Growth of spheroids with input level hail conditions similar to the examples of Figures 4 through 7 of Chapter 3, i.e. $D_0 = 0.5 \text{cm}$, $V_{zo} = 22 \text{m/s}$ and $N_0 = 1/\text{m}^3$, is shown in Figures 10a and 10b where the axis ratios are varied from 0.5 to 1.0. These examples show that even the modest axis ratios of 0.9 or 0.95 change the growth characteristics greatly from that of the spherical hailstones. The greatest
effect is that the smaller drag coefficients of ellipsoidal hailstones tend to carry them to the top of the cloud model with minimal growth. This example is not ideal since the hailstones of moderate axis ratio are given little opportunity to grow or to deplete liquid water. It is seen, however, that the general growth curves are much the same for the spherical examples of Chapter 2 and that the example with axis ratio 0.95 causes over 70 percent of the liquid water to be depleted and in that case considerable force would be applied to the cloud through accumulation of hail mass near the balance level. The symbol $\odot$ on that example indicates where the buoyant energy has become negative; it did not for the other axis ratios. The lines of constant ascent time of Figure 10a show that the ellipsoids tend to rise much faster than the spherical stones due partially to lesser input level terminal velocities and partially to lesser growth as they ascend.

Growth of spheroids in smaller updrafts is now considered since the former example showed most of the stones rising to the top of the cloud. Figures 11a, b and 4 show an example where the updraft velocity $V_{zo}$ is 18m/s at the input level and Figures 12a and 12b show one where $V_{zo} = 15$m/s.

Figures 11a and 11b for growth of 0.5cm spheroids with various axis ratios where $V_{zo} = 18$m/s and $N_o = 0$ or $2/m^3$ show the high dependence of growth on axis ratio as
well as the ability of modest embryo numbers to deplete a significant fraction of the liquid water if growth conditions are ideal as in the example where the axis ratio $\alpha$ is 0.75. One sees that a change of axis ratio from 1.0 to 0.75 results in the near doubling of the major dimension, 1.15 cm to 2.15 cm, at the balance level, as well as the depletion of so much liquid water with its consequent accumulation of mass that the energy balance of the cloud as seen in Figure 11 d breaks down. This example shows little change in growth with increase in number density if a balance level is not approached because the numbers are not sufficient to cause significant depletion of liquid water.

Figures 12a and 12b give the growth and liquid water curves for 0.25 cm embryos injected into a 15 m/s updraft at various axis ratios and concentrations of 0, 2 or 10 m$^{-3}$. These figures show that only under optimum axis ratios and concentrations are high balance levels, i.e., large hailstones, achieved. When these conditions are realized, as in the case of $N_0 = 0$ and $\alpha = 0.5$ a 3.25 cm hailstone can grow from a 0.25 cm embryo even in this relatively low velocity updraft. For this example only the case of $N_0 = 2$ m$^{-3}$ and $\alpha = 0.75$ is capable of applying sufficient force by depleting liquid water and accumulating mass to offset the energy balance. The point where the buoyant energy becomes less than the updrafts kinetic energy is indicated by the symbol \( \circ \). Figure 3 for kinetic and
buoyant energies without hailstones shows clearly that the available buoyant energy is considerably greater than the kinetic energy required for the 15m/s updraft when the input level liquid water content is 1.33gm/m$^3$.

In conclusion one sees that large spheroidal hailstones can be grown in considerably lower updrafts than are required for spherical hailstones of the same major dimension. The dominance of available buoyant energy over kinetic energy in these weaker updrafts makes this a good mechanism for the growth of large hail. Unfortunately, or maybe fortunately, this growth of large hail is highly dependent on the injection level parameters which have now been expanded to include the spheroidicity.

Hail growth times of Figures 10a and 11a give reasonable growth times for large hailstones, 10 to 15min. The surface icing conditions as shown in Figure 10a indicate that the hailstones surface temperature is generally just below freezing or just not spongy. More examples of surface icing are given in the next section on hail growth with the axis ratio varying with the major dimension.

### 4.3. Axis Ratio Varies with Major Axis

In this consideration the axis ratio $\alpha$ is varied as a function of the major axis $D$ for embryos of sizes $D_0 = 0.5\text{cm}$.

as follows:

$$\alpha = 1.1 - \frac{D}{5} \quad \text{for} \quad D[\text{cm}] < 3 \text{ cm.}$$

[55]
\[
\alpha = \begin{cases} 
0.5 & \text{for } D > 3 \text{ cm.} \\
\frac{23 - 2D}{22 - 11} & \text{for } D[\text{cm}] < 3 \text{ cm.}
\end{cases}
\]

When \( D_0 = 0.25 \text{cm} \):

\[
\alpha = \frac{23 - 2D}{22 - 11} \quad \text{for } D[\text{cm}] < 3 \text{ cm.}
\]

\[
\alpha = 0.5 \quad \text{for } D > 3 \text{ cm.}
\]

The examples of Figures 13 and 14 show the growth of 0.5cm embryos with variable updraft and number densities respectively where the central values are \( N_0 = 1/m^3 \) and \( V_{z_0} = 18 \text{m/s} \). Both examples reveal the high ability of this type of hailstone to deplete liquid water and upset the cloud's energy balance through their tendency to take considerable time to reach their balance levels or to linger near a balance level which is never quite achieved. Only in the case of the highest updrafts, \( V_{z_0} = 22 \text{m/s} \), is there some relief to the high liquid water depletion, Figures 13-b and 14-b. The icing condition isolines show a strong tendency for these large hailstones to grow spongy i.e., only a fraction of the surface water freezes, until they have swept out a significant proportion of the liquid water. The growth capabilities are apparent, large hailstones grow with ease under these conditions and their final diameters are modified much less by changes in up-
draft and number than is the case for spherical hailstones. However, the vertical height achieved is considerably effected by updraft and number density.

The remarks above for 0.5cm embryo size are also borne out for the 0.25cm embryos of Figures 15 and 16 where updraft and input number are varied respectively about central embryo concentrations of 2 per m³ and input level updrafts of 17m/s. Figures 13 through 16 all indicate high liquid water depletion when updrafts are between 16 and 20m/s and embryo numbers between 0.5 and 10/m³. Hailstones whose spheroidicity increases with diameter have therefore a greater ability to achieve large sizes than do their spherical counterparts during growth on ascent. As a consequence, spheroidal hailstones are more able to deplete liquid water and upset the cloud's energy balance. The varying of axis ratios has yielded a reasonable method for growing 3.5 to 4cm hailstones from 0.25 to 0.5cm embryos in 17 to 20m/s updrafts in a period of from 15 to 20 minutes. This consideration should, therefore, not be neglected in future hail cloud models unless proof can be given that this type of growth does not occur readily enough in natural clouds.
CHAPTER V

HAIL GROWTH ON ASCENT AND DESCENT

5.1. Introduction

Chapter 3's description of hail growth on ascent presents a need to investigate the growth of the hailstones as they descend after achieving their balance level. There is, of course, the possibility that the creation of a balance level by the ascending hailstones has upset the cloud's energy balance with the probable result that a region of horizontal cloud divergence has been created and the hailstones in that region would be spread out and thrown from the main part of the updraft. In the event that this does not occur during the time that the majority of the hailstones are ascending, the investigation of the steady-state solution where hail is ascending and descending will be profitable.

The hailstones are presumed to not deplete liquid water or effect the cloud after they have fallen below the input level. Ascent and descent will be abbreviated as A & D. Chapter 2, Section 7 gives this model's equations and their solution. Only spherical hailstones are considered with an emphasis on 0.5cm. embryos. They are injected at various number concentrations into several updraft velocities.
Before the solutions are described it should be noted from Figures 5 and 8 of Chapter 3 that for given input diameters, updraft velocity, and input level liquid water there is an upper limit on input number which will allow hailstones growing on ascent to reach a balance level. It is also apparent that the limiting embryo number \( N_M \) for the A&D growth of this chapter will be smaller than its 'ascent growth only' counterpart since greater liquid water depletion will occur in lower levels with this model. These remarks refer to the steady-state solution which is being sought. It is therefore assumed that between the limiting embryo number concentration for a balance level on ascent and the limiting number for A&D growth there is a region of input number concentrations which are incapable of achieving a steady-state solution. No attempt will be made to calculate these time dependent solutions, however, a description of this situation will be given in Chapter 6 where the embryo diameters are spread over a finite range.

5.2. Ascent and Descent of 0.5cm Embryos

Figures 17, 18 and 19 describe the growth of 0.5cm embryos at various input number densities for 18, 20, and 22m/s input level updrafts respectively. Figures 17a, 18a, and 19a show growth curves, growth times and the vertical heights \((i)\) where the force of the hailstones and liquid water equals the thermal buoyancy force \((FB=0)\).
(ii) where the kinetic energy increase from the input level per gram of cloud air becomes greater than the available buoyant energy \( (BE=KE) \) and (iii) where the available buoyant energy calculated from the input level upwards has become zero due to excessive hail mass accumulation \( (BE=0) \). Figures 17b, 18b, and 19b give the liquid water profiles for the appropriate hail growth curves.

The growth curves of these figures indicate that the maximum embryo number concentrations \( N_M \) compatible with a steady-state solution decreases from 2.35 to 0.1/m\(^3\) and the maximum output diameter \( D_r \) increases from 2.25 to 3.3cm as the updraft is increased from 18 to 22m/s. It is apparent that significant input numbers of hailstones in the A & D model are not necessarily of the one per cubic meter densities as they were in the ascent only model. These numbers are in concentrations of one hailstone per ten cubic meters for velocities of 22m/s. The output size of hailstones as they return to the input level is highly dependent on number concentration, however, the height at which the balance level is achieved is generally doubled by increasing the input number from zero to maximum values. Unlike the growth on ascent model, potential liquid water depletion is greater for low velocity updrafts since high embryo numbers cannot be accepted for high velocities. The 22m/s example, Figures 19a and 19b, allows growth above the five kilometer level because the simple model does not include liquid water freeze-out above the \(-32^\circ C\) level. The
maximum number density for the 22m/s would be smaller still if this freeze-out was included.

For $V_{zo} = 18m/s$ and $N_o = 1/m^3$, Figure 17a, the kinetic energy $KE$ is greater than the buoyant energy $BE$ to the 0.3km. level. Increasing $N_o$ to $2/m^3$ makes the available buoyant energy negative at that level (0.3km.). For $V_{zo} = 20m/s$ only the maximum number $N_M = 0.48/m^3$ gives a negative available buoyant energy, but the kinetic energy required to accelerate the updraft is greater than the available buoyant energy at very low levels for input concentrations between 0.1 and 0.3/m$^3$. Recalling that Figure 3 showed the available buoyant energy to be only slightly greater than the kinetic energy for $V_{zo} = 22m/s$, in fact $BE=KE$ at 3.5km. without hail, the energies of the cloud are not badly upset for the 22m/s example since the total buoyancy energy does not become negative even for the highest number density possible (Figure 19a). Figure 19b shows that liquid water depletion can still reach about 50% for the low $N_o$ of 0.1/m$^3$. In the case of $V_{zo} = 22m/s$, the cloud could, from an energy point of view, approach the maximum number possible for steady-state A&D.

An alternative to this A&D model would be to inject a size distribution of embryos into the cloud. As the larger hailstones in the size distribution first reach their balance level in a manner similar to the ascent only model they will begin to descend causing their ascending
counterparts to grow in a region of lower water content than they had experienced. If these descending hailstones deplete enough liquid water some of the embryo sizes which grew to a balance level in the ascent only model, i.e. during the early stages of hail growth, will no longer be able to reach a balance level and hence they will be carried to the top of the cloud. This filtering will be better described in the size distribution models of Chapter 6 but it should be kept in mind when assessing Figures 17 to 19 as well as when assessing the overall validity of this model.

For a given size embryo the maximum diameter achievable in the A&D model is highly dependent on the updraft velocity as is and the maximum number possible for steady-state solutions with all input hailstones descending back to the input level.

Growth times as indicated by the fine lines on Figures 17a, 18a, and 19a for no depletion cases are reasonable, 15 to 20 minutes, but they increase to 40 minutes when a high proportion of liquid water is depleted, compare Figures 17a and 17b. For moderate depletion, $V_{zo} = 22\text{m/s}$, there is only a fifty percent time increase to 30 minutes for the maximum number case. These growth times for maximum input numbers $N_M$ are approaching the maximum conceivable period during which one could hope to describe an updraft as steady-state (30 to 40 min.).
The surface icing conditions, temperature $T_d$ or fractional icing $I$, are indicated on Figures 17a, 18a and 19a. They indicate that a hailstone growing in this model would be almost entirely composed of spongy ice with the exception of a shell about midway between the center and outside composed of dry growth type ice. This dry growth occurs mostly on the ascending part of the growth curves since the larger descending hailstones tend to grow spongy, other conditions being equal. Dry growth in the mid-levels of the model is seen to only occur if there is moderate liquid water depletion. Smaller embryos at these mid-levels would grow dry but they would not reach balance levels. In conclusion, one would not expect hail samples which grew by this method in a real cloud to show much dry growth ice between the center and outer shells but the dominance and distribution of wet growth ice in a natural sample could lead one to conclude that growth on ascent and descent had occurred.

5.3. A General Solution to A & D Growth

For a given embryo diameter and input level updraft a maximum input number $N_M$ for A&D growth has been found to exist. $N_M$ has been calculated for several input diameters and updraft velocities and summarized in Figure 20 where the abscissa is input diameter and the ordinate is input level updraft velocity. Lines of constant output parameters are given in Figure 20 to about 10% accuracy.
for ease of display. The solid lines of Figure 20 are lines of constant maximum output diameter \( D_f \) as determined when \( N_0 = N_M \). These lines also represent constant output diameter \( D_{f0} \) as determined by zero input number \( (N_0=0) \) except in the case of the lower updraft velocities where \( D_{f0} \) is given by 'crossed' lines since these lines were not sufficiently parallel to the lines of constant \( D_f \). The dashed lines of Figure 20 are lines of constant maximum input number \( N_M \), balance level liquid water content in grams of water per kilogram of air \( r_b \) when \( N_0 \) is \( N_M \), balance level height \( Z_{bM} \) when \( N_0=N_M \), and balance level height \( Z_{bo} \) for \( N_0=0 \), and times for growth from embryo to fall-out \( t_M \) for \( N_0=N_M \) and \( t_0 \) for \( N_0=0 \). These dashed lines are labelled at the right hand side of Figure 20 and their labelling is clarified by the 'key'.

Many interesting observations can be made from Figure 20 if one keeps a real hail cloud in mind. For cases where \( N_M \) is greater than \( 2/m^3 \) nearly all of the liquid water can be depleted at the balance level but the final diameters \( D_f \) are less than \( 2.5cm \). if the embryos are \( 0.25cm \). or \( 0.50cm \). These cases where \( N_M \) is large and there is high depletion, besides not being useful for describing the growth of large hail, are probably unrealistic due to their large growth times. As larger updrafts are considered, the maximum number decreases to a reasonable value and the overall growth increases dramatically to
yield 3 to 5 cm. hailstones when the balance level is below 5 km. and even 6 cm. hailstones if 6 km. is allowed. As in Figures 17, 18 and 19, an embryo can grow much larger in higher updrafts but increasing the updraft decreases the potential liquid water depletion. The growth times are approximately proportional to the updraft velocity for the case of $N_0=0$, however, as $N_0$ approaches the maximum value $M$, the growth times are very high for low updrafts, i.e. high depletion, but only 25 to 50% greater for high updrafts where only moderate depletion is possible. Figure 20 shows that the input level updraft velocities cannot be more than about twice the embryo terminal velocity without the hailstones being carried to the top of the cloud whereupon overall growth is restricted.

5.4. Uses and Conclusions

The concept of maximum embryo number density will be useful for estimating reasonable size distributions of embryos for the examples of Chapter 6. This model can also assess the accuracy of the calculated growth curves of the number distribution model of Chapter 6 since that model uses a fixed-step finite-difference method while this model used a Runge-Kutta method where the step size is varied to achieve the desired accuracy.

Finally one concludes that a steady-state solution of the A&D model depends highly on input level parameters.
Once the ranges of these parameters which give steady-state are established, the overall hail growth and liquid water depletion depends mainly on updraft velocity.
CHAPTER VI

NUMBER DISTRIBUTIONS OF EMBRYOS

6.1. Introduction

The need to calculate the growth of a size distribution of hail embryos has been elaborated in Chapter 2, Sections 8 and 9. These sections give the method of calculating hail growth of various distributions introduced at finite embryo diameter intervals along with the calculation of growth times, energy and force considerations, surface icing conditions, and number distributions at various levels in the cloud. Section 8 of Chapter 2 pertains to growth on ascent only while Section 9 treats ascending and descending hailstones, which will be referred to as the A&D model.

It is quite obvious from the growth curves of Figures 4, 5, 6, and 7 in Chapter 2 that hail growth on ascent from single sized embryos is causing very high liquid water depletion along with, given the correct input level conditions, high accumulations of hailstone mass near the balance levels. These accumulation zones result in a great imbalance of forces and energy which would upset any steady-state nature the cloud might have had when the embryos first entered the updraft. The A&D growth of Chapter 5 showed that the higher embryo numbers, i.e., those which caused the previously mentioned breakdown of the cloud, are unable
to yield a steady-state solution for growth with A&D. It was shown in Chapter 5 that these high numbers of particles depleted too much liquid water to allow all of the injected hailstones to pass through their balance level and then grow on descent to the input level. This reference to all of the hailstones being able to reach a balance level under steady-state conditions, whether it be for growth on ascent or A&D, can be cleared up by continuously injecting particles of various sizes where no size group is in sufficient numbers to deplete more than a 'small fraction' of the liquid water during their growth period.

Section 2 of this chapter will attempt to interpret the high water depletion caused by growth on ascent in the examples of Figure 5 in Chapter 2 as well as estimate the magnitude of a 'small fraction' as it pertains to the above discussion.

Sections 3, 4 and 5 deal with A&D growth of various embryo size distributions and the growth which occurred while the injected hailstones were only ascending. The time dependent situation between the steady-state growth on ascent and steady-state A&D growth will not be calculated but some interpretation will be given as to the hailgrowth model's ability to evolve between these two solutions.

An example where the total embryo number is greater than that capable of reaching steady-state, as was determined by the A&D model of Chapter 5, is considered in Section 4.
Section 5 treats a case where the smaller embryos are in greater numbers than the large ones which is probably the situation in a real cloud. This example will show the importance of small embryos in large quantities relative to large embryos in small quantities as it pertains to the principle that the correct embryo size in the correct quantities yields maximum effects on the updraft and hailgrowth.

6.2. Examples of Chapter 2 with Embryo Size Distributions

This section assesses the dependence of hailgrowth, water depletion, and energy balance on embryo diameter size spread when the total embryo concentration is kept constant. This is done by considering examples of Chapter 2 which have high and moderate liquid water depletion, examples of no depletion being trivial. The situations to be recalculated are represented in Fig. 5 where $V_{z_0} = 22\text{m/s}$, $D_0 = 0.5\text{cm}$, and $N_{t_0} = 1$ or $2/\text{m}^3$. The $N_{t_0} = 1$ case leads to 45% liquid water depletion; the $N_{t_0} = 2$ case to 90%. Only growth on ascent will be calculated.

Figs. 22a and 22b show the growth of hailstones and liquid water profiles for the case where $N_{t_0} = 1/\text{m}^3$. The embryos are injected in five diameters each having equal number concentrations. The diameter difference $\Delta D$ between each of the hail embryo's five diameters is either 0.02, 0.005, or 0.0cm, which for these examples will be called the small, very small, and no spread cases respectively. The embryo sizes for the small spread case are
0.46, 0.48, 0.50, 0.52, and 0.54 cm each with concentrations of 0.2/m^3, and for the very small spread 0.49, 0.495, 0.5, 0.505, and 0.51 cm each with 0.2/m^3. For the no spread case 0.5 cm embryos are injected at N_0=1/m^3.

Figs. 22a and 22b give similar examples but with twice as many embryos being injected, i.e., the total number of embryos is 2/m^3.

Only the growth curves for the largest and the smallest embryos are given in Fig. 21a but the growth of the medium size embryos is not difficult to visualize. Fig. 22a shows the growth of all 5 embryos for the small spread case but only the envelope for the very small spread example. Figs. 21d and 22d give the energy available from buoyancy and cloud parcel's kinetic energy for the two examples.

For the N_t_0 = 1/m^3 case, Figs. 21a, b and d, the growth curves, liquid water depletion and buoyancy energy do not depend highly on the spread of embryo diameters. For the case of N_t_0 = 2/m^3 where high depletion occurs, 90% with a single embryo size, considerably less liquid water is depleted as the wider embryo spreads are considered. The N_t_0 = 2/m^3 case also indicates that the buoyancy energy and the growth curves are greatly changed by wider spreads in embryo size. One is forced to conclude that the examples of Chapter 3, where growth of single embryo sizes depletes a high percentage of liquid water, are not realistic since...
the spreading of these embryos over several closely spaced sizes would result in the smaller embryos being carried to the top of the cloud even in the case where the spread ΔD is 0.025 cm. The examples of Chapter 3 which give 50% water depletion are reasonable, however, since the spreading of embryos over 0.1 cm., Fig. 21a, does not result in significant changes in liquid water content or energy balance.

In Fig. 4 of Chapter 3 one sees that 0.4 cm. embryos at $N_0 = 1/m^3$ are carried quickly to the top of the cloud and 0.6 cm. embryos are caught at a low balance level. Both of these examples show considerably less liquid water depletion and energy imbalance than does the $D_0 = 0.5 cm.$ case. One therefore concludes that the embryos with diameters somewhere between 0.4 and 0.6 cm. have the greatest growth and liquid water depletion potentials and that these embryos in reasonable numbers ($N_0 = \text{about } 1/m^3$) can be represented by $1/m^3$ of 0.5 cm. embryos. However, if one wishes a greater total concentration of embryos, say $2/m^3$, to be injected between these sizes which have been deemed significant for this updraft (0.4 to 0.6 cm.) it is seen that more than one embryo size will be necessary to get a reasonable picture of hail growth and water depletion.

Two conclusions can be shown from the above observations. First, there is a natural protection against high liquid water depletion in this model but depletion of about 50% by ascending hailstones is feasible since a natural
concentration of one embryo per cubic meter between 0.45 and 0.55 cm is not unreasonable. Secondly, one concludes that for maximum depletion and significant hail mass accumulation zones, one would require a moderate number of embryos in the correct diameter range for balance level to be achieved high in the cloud. This conclusion comes naturally from the observation that small embryos play only a minimal role as they are carried to the top of the cloud while large embryos, which are not likely to appear in high concentrations, reach low balance levels and do not grow as large on ascent or deplete as much liquid water per hailstone as do those ideal embryos which reach high balance levels.

It should be noted, that the embryo concentrations which deplete significant liquid water while growing on ascent are capable of disrupting the updraft's energy balance and further that they are in numbers too great to set up a steady-state A&D condition as determined in the A&D single embryo size model of Chapter 5 (Fig. 20).

6.3. Ascent and Descent Growth as Function of Embryo Size Spread.

The updrafts in the rest of this chapter are reduced to $V_{z0} = 18 m/s$ since A&D growth for $V_{z0} = 22 m/s$ and $D_0 = 0.5 cm$ has been shown in Chapter 5 to require balance level heights of greater than 4 km and to give low depletion even in the case of the maximum input number. Better examples are
therefore possible with the more easily depleted cases of \( V_{zo} = 18 \text{ m/s} \) and \( D_o = .5 \text{ cm} \) (Figs. 17 and 19).

The first example (Fig. 23) compares growth of 5 embryo diameters injected over a diameter range of what will be called a _medium spread_, \( \Delta D = 0.1 \text{ cm} \) between successive embryo sizes. These particles will be injected in total numbers of 0.0 or 0.2/m³ at each size. In future examples a _small spread_ will refer to \( \Delta D = 0.02 \text{ cm} \) and a _large spread_, \( \Delta D = 0.2 \text{ cm} \). This section considers only equal numbers of embryos at each embryo diameter.

Fig. 23a shows the growth on ascent and growth for A&D for the example above. While the hailstones are only ascending there is little difference between the growth curves of the \( N_{To} = 0.0 \) case and the \( N_{To} = 5 \times 0.2/\text{m}^3 \) case. There is, however, considerable growth difference in these cases when steady-state A&D is reached. The liquid water depletion (Fig. 23b) is high only for A&D growth. Fig. 23d for energies shows that the available buoyant energy drops by 50% once A&D growth has been established. This example is interesting from an energy point of view since the kinetic energy KE compares favorably with the buoyant energy BE. The buoyant energy is greater than the kinetic energy during ascent only growth but less when A&D is established (Fig. 23d). Growth times (Fig. 23a) for the finite input number case are reasonable but perhaps a little long in the case of the 0.3 cm embryos (23 min.).
Fig. 23e gives the hailstone number distributions at the various balance and input levels for ascent growth and for A&D growth. For the first 5 to 10 min. while ascent only is occurring there is a general spreading of the distributions with height towards higher diameters while the number in the input diameter range decreases with height. In the final steady-state A&D, which should be nearly achieved in 25 to 40 min. after the first injection of embryos, higher hailstone concentrations exist above 2.0 km than in the ascent only case. For the A&D case the number distributions have a low spot in the region between ascending and descending diameters which becomes less pronounced at high levels and finally disappears between 1.6 and 2.5 km. The number distributions are seen to be highly dependent on the height in the cloud and on the development of hail growth.

The examples of Figs. 23 have shown that small numbers of hail embryos spread over the medium range in diameters $\Delta D = 0.1 \text{cm}$ can affect the updraft considerably when A&D growth occurs.

Figs. 24 show the results if the same total embryo number, $N_{To} = \text{1/m}^3$, is injected for a wide spread of input diameters, $D_o = .1, .3, .5, .7, \text{and} .9 \text{cm}$, and for a small spread, $D_o = .46, .48, .50, .52, \text{and} .54 \text{cm}$. Figs. 24a and 24b give growth curves and liquid water contents for both the wide and small spread examples. For growth on ascent.
there is a large difference in growth since the 'wide spread' examples form their balance levels at all levels in the cloud while the 'small spread' example has its balance levels between 1.0 and 1.5km. Liquid water depletion is low in both cases of growth on ascent. For A&D growth both examples deplete about 30% of the liquid water, whereas for the moderate spread case of Fig. 23b, 40% was depleted. For a given total number of embryos with a fixed central diameter there is, in this example, an optimum spread of diameters for maximum liquid water depletion. This can be understood if the need for some of the hailstones to reach high balance levels for high depletion is weighed against the possibility that the small hailstones, which have potentially high balance levels, may be carried to the top of the cloud's growth region. This would not be true for the very high depletion examples of Chapter 5, Fig. 20, where a single embryo size, i.e., a zero diameter spread, can deplete all of the liquid water, but it would generally be true for lesser total input numbers which result in moderate depletion. The foregoing observations express the difficulties met in trying to generalize the hail growth encountered when size distributions of embryos are considered.

Figs. 24c and 24e give the forces due to liquid water and hail and the total hailstone numbers respectively as functions of height for the wide and small spread during either ascent only or A&D growth. The small spread case
forms a classic accumulation zone near 1.5km in which both number concentrations and total water substance forces are high for both ascent and A&D growth. The wide spread forms no such mass or hail number accumulation zones (Figs. 24c and 24e). Accumulation zones, in the sense of mass accumulation or hailstone number accumulation, are therefore, more likely in cases where the embryos are spread over a small range of diameters. Only in the A&D cases of these examples are there sufficient body forces on the updraft due to the hailstones and liquid water to significantly change the available buoyant energy (Fig. 24d). The overall effect on the buoyant energy is nearly the same for both the wide and small spread examples, except that in the small spread case it is about 40% lower in the accumulation zone region (at 1.5km) but it becomes equal to the wide spread buoyant energy by 3km.

Fig. 24f gives growth curves for the wide spread A&D case along with isolines of growth times and icing conditions. The growth times for A&D are reasonable, i.e., 10min. for 1.5cm output from .7cm embryos and 17min. for 3.1cm output from .4cm embryos. Since this example depletes only about 30% of the liquid water one would expect this steady-state growth to be nearly reached in about twice the maximum steady-state growth time, say 30min. As was the case in the moderate spread examples, Figs. 23, the time for A&D steady-state growth to be established is getting quite long for a real hailcloud to

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to remain in steady-state. It is not unreasonable, however, to expect a condition approaching this final state to be realized, since about 15 min. would be required for a significant number of embryos to ascend and descend back to below the one kilometer level. As this state is approached a considerable force, the order of which was not achievable while ascent only was taking place, would be applied by the large descending hailstones.

The hail surface icing conditions of Fig. 24f show the tendency of large hailstones to grow spongy. They also indicate that at a given level, the surface icing conditions can vary from -5°C to 50% fractional icing for small and large stones respectively. Further, hailstones over two centimeters would have difficulty growing within 1 km of the freezing level since the fractional icing falls to 25%. This would result in considerable shedding of surplus water. The icing conditions indicate the difficulty of growing large stones as they descend through the hail cloud's updraft. The low fractional icing of the embryos in the first half to 1 km. is not bothersome since little significant growth has occurred in that region and, further, raising the input level or reducing the cloud's temperatures would not detract greatly from the conclusions already made about hail growth and water depletion. The problem of highly spongy growth for large hailstones within 1 km of the freezing level does remain. Investigations of freshly fallen natural hailstones may help to determine the level at which
they left the growth zone of the updraft.

6.4. A Gaussian Type Embryo Number Distribution.

The dominance of the embryo size which enters the updraft in the greatest numbers will now be assessed with the proviso that these sizes reach a reasonably high balance level. It will also be interesting to inject this number distribution in total numbers which are greater than the maximum, \( N_T = 2.35/m^3 \), for a steady-state solution as was evaluated in Chapter 5 for input level conditions \( D_0 = 0.5 \) and \( V_{zo} = 18m/s \). The steady-state A&D solution for this number distribution example should show the smaller embryos being carried to the top of the cloud.

The example, shown in Fig. 25, has the following embryo characteristics: five diameters of .3, .4, .5, .6, and .7cm., previously called the moderate spread, with input level concentrations of .2, .6, 1.4, .6 and .2/m^3 respectively, giving a total embryo number \( N_T \) of 3/m^3 at the input level.

For growth on ascent (Fig. 25a) all embryo sizes reach a balance level between 0.5km. and 3km. with the maximum liquid water being swept out as the hailstones of highest number concentration reach their balance levels; Fig. 25b gives liquid water curves. For A&D growth, where about 70% of the liquid water is swept out, the smaller embryos are carried to the top of the model cloud as the
high number embryos deplete the liquid water in mid-levels (Fig. 25a). The balance level heights rise considerably for embryos of diameter greater than 0.4cm. when growth on descent is included. Total growth in the A&D case is quite dependent on embryo diameter since 0.7cm. embryos achieve 1.6cm. while 0.5cm. embryos achieve 2.4cm. The A&D growth times shown by fine lines on Fig. 25a are reasonable, 10 to 25 min., but one sees from Fig. 25c that the forces involved in this example would not allow steady-state since the downward force of the hailstones and liquid water is greater than the upward thermal buoyant force from the input level up to 2km. While ascent only is occur hydrometeor forces are not large, so approaching the A&D solution should be possible from an energy and force point of view. These remarks are confirmed in Fig. 25d where the buoyant energy of the two cases is compared to the updraft's kinetic energy.

Fig. 25e gives the hailstone number density distributions for various heights, mostly at balance levels, and the total number of hailstones as a function of height. For the ascent only case a weak maximum in the total number concentration is indicated where the most populous embryos reach a balance level, 1.25km. The effect of this maximum (Fig. 25c) is to apply a hydrometeor force only about forty percent greater than it would have been in the case of no depletion.
where only liquid water is considered (compare dashed and crossed line). Above this region of liquid water depletion the force due to hail and liquid water was actually decreased by the inclusion of hail in the updraft. For the A&D case the total number is about \(5/m^3\) from the input level to 2.25km. (Fig. 25e) and the force due to hailstones and liquid water reaches a maximum at 1km. In the A&D case there was no accumulation zone of either mass or number except in the sense that from the input level to 2km. the hydrometeor mass was sufficient to totally offset the thermal buoyant force. Above 2km. in the A&D case the hydrometeor force and the liquid water content fall off leaving the smaller embryos to rise to the top of the model. This is the effect which was sought when the example was presented; the embryos with central diameters, which were in greatest quantities, have the largest influence on the cloud, leaving the smaller embryos insufficient liquid water for growth and the large embryos low balance levels and minimal growth.

From this example, and from the previous cases of Figs. 23 and 24 one sees that, for A&D growth, it is possible to concentrate the cloud destroying forces into the lower levels of the updraft, contrary to what occurred in the ascent only examples. The destruction of an updraft due to hailgrowth can occur from the top or from the bottom providing, still, that significant liquid water is depleted.
The number density distributions for various altitudes of Fig. 25e, are best summarized by stating that, from the time the hailstones are first injected until steady-state A&D is achieved, there are in all probability no unusual hailstone number distributions anywhere in the updraft. All of the steady-state distributions can be drawn as smooth curves from five points with little trouble provided an estimate of the growth of the diameters at the edge of the distribution is known (as it was by calculating the growth of two appropriate embryo diameters with zero concentrations along with the five with finite concentrations).

A comparison should be made between this example, Fig. 25, and Fig. 17 where A&D growth was shown for a single embryo diameter with similar input level conditions. There is, just as in the case for ascent growth, a tendency for less liquid water depletion when the embryos are spread out; providing the example for a single embryo size had high liquid water depletion (compare Figs. 17a and 17b) for \( N_0 = 2.35/m^3 \) to Figs. 25a and 25b where \( N_T = 3.0/m^3 \).

6.5. **High Concentrations of Small Embryos**

This final example demonstrates the effect of injecting large numbers of small hail embryos along with small numbers of larger embryos. These embryos grow on ascent or A&D in steady-state. This example should more closely approximate reality than previous ones since smaller embryos are likely to be the most numerous in hailclouds.
Embryos are injected with $D_0$ of 0.1, 0.2, 0.3, 0.4, and 0.5 cm, with the respective concentrations $N_0$ of 2, 1, 0.25, 0.15, and $0.10/m^3$ into an input level updraft of 18 m/s.

Figs. 26a and 26b give the hail growth curves and liquid water profiles respectively for ascent only growth and for A&D growth. While the hailstones are ascending the 0.2 cm embryos are able to reach their balance level and their being in moderately high quantities allows them to deplete about 25% of the liquid water above 3.5 km. By the time steady-state for A&D growth is achieved the 0.2 cm embryos can no longer reach a balance level due to low-level liquid water depletion by the descending hailstones. In some ways this reaffirms the high dependence of ascent only growth on the available liquid water at the input level as was established in Chapter 3. This also stresses the need for high level liquid water if high balance levels are to be formed. Fig. 20 of Chapter 5 showed that the maximum number of 0.25 cm embryos which can form steady-state A&D growth in an 18 m/s updraft is about $0.1/m^3$. This maximum number would be somewhat less for the 0.2 cm embryos of this example so one would not expect them to reach A&D steady-state since their number here is $1.0/m^3$.

The life cycle of hail growth in this example would probably go as follows, providing, of course, that the source of embryos is continuous: Little hail force or liquid water depletion would be experienced by the updraft
until the high quantities of 0.2 cm embryos begin to deplete high level liquid water. If this high level force (Fig. 26c) was not sufficient to upset the updraft's energy balance (Fig. 26d), as it probably wouldn't be, these high balance levels would soon be terminated for lack of liquid water as the low balance level hailstones begin accreting on descent. Fig. 26b shows that for A&D growth the liquid water depletion caused by 0.3, 0.4, and 0.5 cm embryos is much greater than it was during ascent only. As steady-state A&D is approached (Fig. 26c) there is a considerable increase in hydrometeor force on the low and mid-levels of the updraft's hailgrowth region. This reduces the updraft's available buoyant energy BE (Fig. 26d) to less than the kinetic energy KE. The total number of hailstones $N_T$ is nearly constant with height for both cases (Fig. 26e).

This example indicates that the high level mass accumulation attributed to the numerous small embryos during ascent only growth is highly dependent on a liquid water source. When the less numerous hailstones which grew from large embryos begin their descent they shut off this water source and soon dominate the model, leaving the numerous small embryos to ascend ineffectively to the top of the growth region.

6.6. Summary of Size Distributions

Hailstone size distributions have shown many
possibilities for depletion of liquid water by embryos injected into low levels of the updraft. Some overall conclusions can be made:

1) High fractional depletion of the cloud's liquid water is unlikely;

2) The mass density of hailstones at a given height in the cloud is unlikely to exceed more than 3 or 4 times the mass of liquid water which would be at that level if there was no accretion by hail, i.e., 4 times 3gm. of liquid water per m³ equals 12 grams of hail per m³ is probably a good upper limit;

3) Comparing the thermal buoyant force of Fig. 25c to the force due to natural liquid water one concludes that hail mass densities in the order of 2 to 4 times that of natural cloud liquid water are sufficient to upset the force and energy balance of the cloud;

4) These hail mass accumulation zones are most likely to appear near the top of the growth region soon, 10 to 15min., after embryos first enter, but, if the embryos fall back down through the updraft the mass accumulation can appear near the bottom 20 to 25min. after injection began.

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CHAPTER VII

CONCLUSIONS

The author realizes that these models are only possible approximations to the many types of naturally observed hailclouds. However, the inexplicable state at which physical theories on entrainment, cloud water budgets, dynamic pressure effects, time dependence and precipitation formation currently stand could easily cause a more complex model of two or three dimensions to obscure, in a sea of variables and hypotheses, the conclusions which can be made from this study. Between these bounds of realistic description and model complexity some very worthwhile conclusions can be made about the stated goal of this model: to assess the probable feedback effects between hail formation and hailclouds. The artificial nature of this model leaves much opportunity for the further studies which must be carried out if our knowledge of hailclouds is to be applied to weather modification. The models have shown that hail growth from embryos injected near the freezing level is highly dependent on all of the cloud variables studied; that is, updraft velocity, freezing level liquid water content, embryo diameter, hailstone spheroidicity, embryo number, and the range of embryo sizes. As relatively small embryo concentrations, 0.2 to 2.0/m³, were capable of depleting a significant fraction of liquid cloud water, it becomes necessary to carefully consider the water conservation mechanisms associated with hail formation.
It should be stressed that in this discussion a hail accumulation zone means a region of the cloud where the mass density of the hail is of the same order of magnitude as the liquid water content, i.e. it is not of smaller magnitude. This is a reasonable description for natural clouds as well as cloud models.

Several specific conclusions will now be given. Although they are not necessarily in logical order as they pertain to some aspects of the model; they are in the order of their arising during the model's development with appropriate modifications from later chapters. The conclusions include applications to the description of natural phenomena as did the discussion of the text's examples. The conclusions are as follows:

1) For given updraft parameters, only selected hail embryo sizes injected near the freezing level can grow from a fraction of a centimeter to 2 or 3 cm. on ascent or up to 4 or 5 cm. if they fall back through the updraft. Maximum growth is attained when balance levels are reached high in the updraft, that is, where the balance level hailstone diameter and updraft velocity are the greatest. The smallest embryos are carried above the region of growth by accretion while the large embryos reach low balance levels. In real clouds this could be the selection process which causes a substantial amount of some hail-falls to consist of roughly single sized hailstones.
2) Besides the correct embryo size for a given updraft, maximum growth and liquid water depletion is dependent on optimising the available embryo number density for both growth on ascent and growth with ascent and descent.

3) All examples indicate that the depletion of liquid water is synonymous with forming hail mass accumulation zones. These 'zones' are most easily formed in upper regions for growth during ascent because an accumulation zone at low levels requires larger numbers of larger embryos while one at upper levels requires the more likely case of smaller numbers of smaller embryos. When hail is ascending and descending in the model these accumulation zones form in lower levels of the cloud. A study of the region of high intensity radar echoes in hailclouds could possibly determine which of these two types of growth was being observed.

4) For a given embryo size and number, liquid water depletion is greatest for high updraft velocities providing that a balance level is achieved. This is true for both 'ascent' and 'ascent with descent' growth, however, the latter tends to resist steady-state growth solutions for strong updrafts and moderate embryo numbers.

5) Considering the greatly enhanced ability of a given small number of embryos to deplete liquid water with increasing updraft velocity and with the change from 'ascent' growth to 'ascent with descent' growth, it is most
likely that strong updrafts will have high level accumulation zones and that weaker updrafts will have the low level accumulation zones. This is, of course, assuming that natural embryos near a cloud's freezing level are more likely to occur in small quantities; that is; embryos of from 0.25 to 0.50cm. in quantities of from 0.5 to 2.0/m³.

6) Spheroidal hailstones of constant axis ratio show growth which is highly dependent on their oblateness. By allowing the hailstones to increase their axis ratio with size, they readily grow large with 3 to 4 cm. hailstones resulting from growth on ascent in updrafts of 16 to 18m/s at the input level. The ability of hailstones with variable oblateness to deplete liquid water is not highly dependent on embryo number as it was in the case of spherical hailstones, however, their balance level height is still highly dependent on embryo number. The tendency of spheroidal hailstones to grow 'spongy' when they are large provides a practical reason for increasing their axis ratio with size.

7) The considerable variety of surface icing conditions in the model indicates that a more explicit knowledge of growth conditions as determined by laboratory studies of hail samples would be invaluable in determining a hailstone's trajectory. In general, hailstones which grew large on ascent and were then thrown from the updraft would have much more dry growth ice than those which fell down through the updraft.
Hail surface icing conditions are quite dependent on liquid water content, the variation of which could be the explanation of the shells observed in natural hail samples. Hailstones of greater than two centimeters grow very 'spongy' in the lower 1.5km of the model's growth region where they would be expected to shed a substantial fraction of the accreting cloud water. This tends to justify not stressing hailgrowth as the hailstones return to the vicinity of the input level and fall past it while descending.

8) Thermal feedback is worth considering whenever a significant fraction of liquid water was depleted by hail. On the average this effect, though not highly effective in changing hail growth, increases the cloud temperature by \(\frac{2}{3}\)°K and the thermal buoyant force by about 20% in the region of mass accumulation where the additional buoyancy is needed to help offset the weight of the hail. It is not possible, however, to totally offset the hail's weight with thermal feedback.

9) The examples with size distributions of embryos showed that only selected embryo sizes have a significant effect on the cloud's liquid water content. Large embryos see little growth or water depletion as do the smaller ones which are carried to the cloud's frozen region. When ascent and descent growth begins to occur that embryo size which causes the greatest depletion is larger than it was when
111.

'ascent only' growth occurred.

10) A number distribution can be represented by those hailstones which grow from a finite number of embryo sizes providing that no single size of embryos is in sufficient quantities to deplete more than about forty percent of the cloud's liquid water. Chapter 6 on size distributions affirms the examples of Chapters 2 to 5 which meet this depletion criterion. In general, five sizes of embryos were found to be sufficient to describe the effects of hailstones on the cloud.

11) The distribution of hailstones at any height can be represented by a smooth curve of number density vs. diameter. One concludes that no unusual hail mass accumulations occur in nature and that a practical hail accumulation zone would have mass densities of no more than three or four times the liquid water mass density which would naturally have been at that height if no hailstones were present.

12) Hail accumulation zones; all of which, by definition, deplete a significant proportion of the cloud's liquid water; are capable of applying a force to the updraft which will upset its steady-state nature.

13) From the time of the first hail embryo injection, accumulation zones would be most likely to first form at high levels as a consequence of 'ascent only' growth. If no high level accumulation zone forms and the high level
hailstones are not thrown from the updraft, the updraft could continue in steady-state while 'ascent with descent' growth becomes established. This second type of growth forms accumulation zones in the low to middle levels of the cloud and requires fewer embryos for accumulation zones.

14) Growth times for 2 to 3 cm. hailstones are reasonable, 6 to 15 minutes, for 'ascent only' cases but growth times for the larger hailstones of 'ascent with descent' examples are somewhat longer so that the total time to establish steady-state in the latter case would in most cases not be available but it could be approached.

15) The models discussed in this thesis give a reasonable method of growing large hailstones within the five kilometers of vertical updraft between the freezing level and the 'freeze-out' level. Elaborate recycling or time dependent updrafts are therefore not necessary to describe hail formation.

16) Since moderate to heavy hailfalls require a downward mass flux of hail comparable with a hailcloud's local upward mass flux of liquid water, the examples of this study describe a mechanism by which this liquid water to hail the 'thermal buoyant', 'liquid'

17) Since the 'thermal buoyant', 'liquid water', and 'hail accumulation zone' forces are of the same order of magnitude, they can all contribute to the updraft velocity field.
This must be accounted for when proposals to modify hail formation by effecting the hail embryo number are presented. Changing the embryo concentration or partially freezing the liquid cloud water through seeding would not necessarily lead to the desirable effects of smaller less numerous hailstones and stronger updrafts for rain formation. Indeed, the artificial introduction of hail embryos near the main updrafts freezing level by direct injection or by stimulating their growth in 'feeder clouds' through ice nucleant seeding could cause any of the effects described in the models. The larger hailstones which can result from reduction of liquid water also casts some doubt on the desirability of injecting icing nuclei into the main updraft. Better field measurements of the updraft's characteristics would be invaluable in solving these problems and, no doubt, will be made soon.
REFERENCES


LIST, R.J., 1958: *Smithsonian Meteorological Tables.*
Smithsonian Institute, Washington, D.C.

Meteorological Monogr. 5, No. 27, 1-30.

Q.J.R.M.S., 89, 360-370.

Q.J.R.M.S., 90, 84-190.

South Dakota School of Mines and Technology, Rapid City, S.D., Report 69-11.

ORVILLE, H.D., 1970: Report to be published; South Dakota School of Mines and Technology, Rapid City, S.D.

Prepared for National Sci. Foundation by South Dakota School of Mines and Technology, Rapid City, S.D., 3 volumes.


APPENDIX I

A Publication on Growth of Ascending Hail

The paper published in the Journal of Atmospheric Sciences by R. List, R.B. Charlton and P. Buttuls in November 1968, Vol. 25, pp. 1061-1074 follows. Although it is somewhat redundant to Chapters 2 and 3 of this thesis, this appendix expands and clarifies the growth of spherical hailstones during ascent. The inclusion of these observations in the main body of the thesis would have overemphasized this type of growth with respect to the growth of size distributions of hailstones.
A Numerical Experiment on the Growth and Feedback Mechanisms of Hailstones in a One-Dimensional Steady-State Model Cloud

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ABSTRACT

Calculations are made on the growth of hailstone embryos of given size and concentration which are injected into a one-dimensional steady-state updraft, and grow while ascending, the updraft obeying the condition that \( \rho V_s - \text{constant} \). The growth was found to have a considerable effect on the free water content of the cloud due to depletion by the growing particles. The hailstones of this model generally reach biggest sizes if their concentration is low and if the embryos are as big as possible. Embryos of 5 mm diameter can grow to 2.5-3.0 cm in diameter within 8-12 min if the conditions are right.

It is further shown that thermal feedback is of great importance in calculating the cloud temperature since it greatly affects buoyancy and icing conditions; in this case, the frictional heating of the falling hydrometeors has to be included along with the heat of fusion. The buoyancy is investigated because it is necessary to decide which set of input parameters for the growth curves and the free water contents distributions is reasonable. For those hailclouds where hailstones grow while ascending, it may be concluded that the biggest updrafts do not necessarily produce the biggest hailstones. The icing conditions of the growing particles turned out to be such that the outermost layers of the biggest stones always grow non-spongy.

1. Introduction

A big step forward in the theoretical study of the growth of hailstones by Iribarne and de Pena (1962) remained nearly unnoticed by the scientific community. However, application of the same type of conservation equations for water substances in rain clouds by Kessler (1967) made it imperative to investigate the depletion of the liquid water content in a hailcloud by the growing hailstones. It seemed necessary, however, to extend Iribarne and de Pena's range of parameters, and to consider the variation of temperature and pressure vs height and the replenishing of liquid water by condensation according to pseudo-adiabatic lifting. Further, the scope of the previous work was considerably enlarged by studies of thermal feedback and buoyancy effects. Growth time and icing conditions at various growth states are also included in this study.

Since the Toronto cloud physics group is mainly laboratory oriented, the microphysics is stressed rather than rigorous dynamic modeling of storms. The work was done in order to demonstrate that the growth of typical hailstones can be explained without complex assumptions such as hailstone oscillation or rain accumulation.

Although the model as such is intended to be fairly simple, it may be useful for recognizing the important parameters and the simplifying assumptions which will be used in more complex models.

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The calculations are based on the assumption that the effects of an ensemble of hailstones are equal to the sum of the effects of its single particles, i.e., that there is no interference between neighboring hailstones by way of wakes. This is not unreasonable as has been shown by List and Hand (1969) for raindrops, unless we consider conditions near or within balance levels where the concentration of hailstones increases considerably.

2. The cloud model

The selection of a proper hail-producing model cloud is rather difficult and somewhat arbitrary since reliable field observations on hail growth are not available. From the mathematical point of view it is easiest to assume a steady-state model, i.e., a model with all variables constant at any point in space. It is also felt that before three-dimensional studies are undertaken certain basic problems should be investigated in depth; for instance, the aerodynamic feedback of the precipitation particles to the updraft (List and Lozowski, 1968) and the concepts of entrainment. Therefore, further simplification is introduced by treating the cloud as one-dimensional; that is, all parameters are functions of height \( z \) only. Hence, the updraft is represented by the continuity equation for air, i.e.,

\[ \rho V_s = \text{constant}, \]  

where \( \rho \) is the air density and \( V_s \), the updraft speed at the level \( z \). Eq. (1) implies that the updraft is increasing with height as the density decreases.
From investigations of natural hailstones it is known that a big portion of these hydrometeors contains graupel as embryos. It is assumed now that, in nature, graupel or frozen raindrops are formed outside the main updraft column; they are, therefore, injected into the model updraft at a certain level. This is quite convenient since difficulties involved in formulating hailstone-embryo growth are avoided. We further assume, for reasons of simplicity, that the hailstones grow only while ascending. If they reach a level where their terminal speed equals the updraft speed, the hailstones will be removed. Other models could certainly also be justified. No claim is made that the one described here is representative of all hailstorms, but it could closely represent clouds with tilted updrafts where the embryo particles are sucked into the updraft at a low level and where the grown hailstones fall out at the balance level; it could also represent vertical updrafts where graupel or frozen drops are collected in converging regions and are thrown out of the funnel as hailstones in upper regions where there is high divergence.

It is also assumed that the graupel source is located at the 0°C level and that it is dispersing embryos of only one size at an initial concentration given by \( N_0 \) (particles m\(^{-3}\)). In this one-dimensional model the number flux of hailstones must be constant at all levels; this implies that

\[
N(V_s-V_i)=\text{constant},
\]

where \( V_s \) is the terminal speed of the particles. The cloud in which the hailstones grow is based on Beckwith's (1960) soundings taken on days with hail in the Denver area. Environment temperature \( T_E \) (°K) is related to pressure \( P \) (mb) according to

\[
T_E=53 \ln(0.284P),
\]

where \( P \) is the pressure (mb), and is given as a function of height by

\[
H=2.76 \times 10^{-4}(9.006 \times 10^4-T_E^2),
\]

where \( H \) is the height in km above the ground (Fig. 1). The cloud base is assumed to be at the 5C level and at 670.8 mb, the pressure in the cloud being assumed equal to the pressure of the environment.

Because the hailstones grow inside the cloud where temperature and density are different from the values in the environment, the pseudo-adiabatic cloud temperature will be calculated as a first approximation. Starting with a parcel of saturated air containing 1 gm dry air and \( r_w \) gm water vapor at a given pressure, the parcel is lifted a small height interval, resulting in a change of the total pressure of \( \Delta P \) (dyn cm\(^{-2}\)), a change in temperature \( \Delta T \) (°K) and an amount of water \( \Delta \omega \) (gm) condensing out, the latent heat release being \(-L \cdot \Delta \omega\) heat units. With the assumption of no heat exchange with the surroundings, the first law of thermodynamics leads to

\[
-I_n \Delta \omega=C_p \Delta T-RT \Delta \ln(p_\phi),
\]

where \( L \) is the latent heat of condensation per gram of water vapor; \( C_p \) and \( C_{pw} \) are the specific heats of dry air and water vapor, respectively; \( R \) and \( r_w \) the universal gas constants for dry air and water vapor, respectively; \( p_\phi \) (dyn cm\(^{-2}\)) the saturation water vapor pressure at temperature \( T \); and \( r_w \) the saturation mixing ratio.

Employing the Clausius-Clapeyron equation and replacing \( r_w \) by \( R R_w \cdot e_s (P-e_s)^{-1} \), the expression (5) can be rewritten as

\[
\Delta T=\frac{[RT(P-e_s)^{-1}+RR_w^{-1}I_n \cdot e_s(P-e_s)^{-2}] \Delta P}{C_p+RR_w^{-1}e_s(P-e_s)^{-1}C_{pw}+RR_w^{-1}I_n \cdot e_s[(P-e_s)R_p T]^{-1}[1+e_s(P-e_s)^{-1}]}.
\]

A step-by-step integration of Eq. (6) based on height steps of 0.1 km is now performed by making use of (3) and (4) to obtain the pseudo-adiabatic cloud temperature; the calculations are started at the environment freezing level, which was found to correspond to a cloud temperature of 0.8°C and are carried out up to a height of 5 km above the freezing level.

The increase in pseudo-adiabatic free water \( r_{w0} \) \([(gm \text{ water}) (gm \text{ cloud air})^{-1}] \) of the cloud parcel is equal the decrease in \( r_w \) since the free water is assumed to ascend at the speed of the updraft. The pseudo-adiabatic temperature thus obtained is shown as a function of height above the environment freezing level in Fig. 1. In these calculations the heat of fusion of freezing cloud water is not considered since droplet freezing is assumed to start at the -32°C level
Fig. 1. Environment (A) and cloud (B, C) temperature variations with height, curve B being calculated solely on the basis of pseudo-adiabatic lifting of an air parcel, while C includes heat sources induced by hailstones.

Fig. 2. Diameter of hailstones having terminal speeds equal to updrafts (balance level conditions) as function of height, for different 0°C level updraft speeds $V_u$. 

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Fig. 3. Growth curves (a, b) and liquid water distributions (c) for 0.5-cm hailstones injected at the OC level at various concentrations $N_a$ into an updraft of 22 m sec$^{-1}$. Isolines are also shown for buoyancy $b$ and cumulative buoyancy $2b$ in (a), for growth time in (a), and for icing conditions $t_0, I$ in (b).
fore only be obtained if the updrafts are high enough. No statement can be made as yet about what input parameters, such as embryo size and embryo concentration, will lead to such big stones.

3. The growth of hailstones and their effects on liquid water content

As already mentioned, the main feature of these calculations will be the particle growth and the depletion of the liquid water content by the growing stones. The free water content at the 0°C level, the level of the embryo source, is known. An input embryo diameter $D_0$ (cm) can be assumed as well as the number concentration $N_e$ ($m^{-3}$). The embryos and the hailstones shall be spherical with a drag coefficient of 0.5. These conditions yield a terminal speed $V_t$ of

$$V_t = \frac{48.8(D/\rho)^{1/2}}{D},$$

where $D$ is the diameter.

The growth of the hailstones as they rise in the updraft from one level to another is given by

$$\frac{\Delta D}{\Delta z} = \frac{V_t \rho}{2 \rho (V_t - V)}.$$  \hspace{1cm} (8)

The difference $(V_t - V)$ between updraft and particle speed appears here because the time spent by the hailstones in a layer of thickness $\Delta z$ is inversely proportional to its speed relative to the ground.

This growth equation assumes a collection efficiency of unity; that is, all of the liquid water swept out by the hailstone is accreted. The hailstone density $\rho_t$ is assumed to be 0.915 gm cm$^{-3}$.

The expression for the conservation of liquid water $r$ [gm water (gm air)$^{-1}$] in an updraft parcel is given by

$$\frac{\Delta r}{\Delta z} = \frac{\pi D^2 V_t \rho N}{4 V},$$ \hspace{1cm} (9)

where $\Delta r$ is the change of the liquid water content due to condensation. The liquid water, which is swept out by the $N$ hailstones per unit volume as the air ascends, is given by the second term on the right side of the equation. The depletion term is proportional to $V_t^{-1}$; that is, to the time spent by the air in traveling from the lower to the upper boundary of a height interval.

Using Eqs. (1)–(7) and the equation of state for the cloud air converts Eqs. (8) and (9) into two first-order simultaneous differential equations in $D$, $r$ and $z$. They were solved for various values of the input parameters $N_e$, $V_t$, and $D_0$ using finite height steps $\Delta z$ of 100 m. The solution for $D$ and $r$ as functions of $z$ was accomplished by using second-order approximations to all variables. Values for diameter and free water have a computational error of less than 5% after integration has been carried out to the top of the cloud (5 km above the base).

Growth curves and free water contents were calculated for embryo diameters of 0.25, 0.50, 0.75 and 1.0 cm and for velocities varying in steps of 1 m sec$^{-1}$ for a range of 18–28 m sec$^{-1}$. A selection of these calculations is shown in Figs. 3–5. The hailstone number density was chosen in such a manner that the buoyancy at the input level was big enough to sustain the weight of the hailstones at this level. The criterion led to conditions represented in Fig. 6, which displays input number density vs diameter for zero buoyancy. This criterion turned out quite reasonable as will be shown in Section 5.

Figs. 3–5 show the existence of two types of growth curves. The first is a sigmoid type with an inflection point between the 0°C and the 5-km level and describes the growth of hailstones which are thrown out on top of the cloud, whereas the second type is representative of hailstones reaching a balance level at a height <5 km. According to the model, the hailstones are removed at such a level and the calculations are stopped. The free water content is seen to drop off rapidly just below the balance level because the number density increases rapidly in this region of the model.

Figs. 3–5 show one very important fact: In order to obtain big hailstones, given the embryo diameter and the updraft speed, the input number concentration has to be quite low and of the order of 0.5–2 m$^{-3}$.

The conditions which would be obtained if depletion is not considered are shown by the curves for number concentrations $N_e=0$. No depletion results in the same free water content distributions, whereas, depending on input parameters, depletion leads to considerable variation of free water profiles. This finding is of great importance because icing conditions which are dependent on free water content could change very easily at one level and help to explain the shell growth of hailstones without resorting to theories involving recycling of hailstones or accumulation of raindrops. In this case the authors are mainly thinking of variations in updraft speed.

The surprisingly big effect of the updraft velocity on the hailstone growth is demonstrated in Fig. 7 for $D=0.5$ cm and $\rho_t=2$ m$^{-3}$. It can also be concluded from Fig. 7 that it is not the biggest updraft which produces the biggest hailstones. If the updraft is big, the growing hailstones don't stay in the cloud long enough to grow considerably; on the other hand, if the updraft is relatively small, the hailstones grow fast and reach only a low balance level. The most favorable updrafts are the intermediary ones which lift the hailstones to the levels where the highest terminal velocities coincide with the highest updraft speeds. This statement holds for all cases where hailstones grow while ascending.

The depletion works in two ways: (i) increased competition due to higher particle concentration leads to smaller growth rates; and (ii) the lower growth rates
Fig. 4. Same as Fig. 3 except for updraft of 23 m sec$^{-1}$.
Fig. 5. Same as Fig. 3 except for updraft of 24 m sec$^{-1}$. 

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lead to larger absolute upward speeds of the particles and, as a result, the hailstones pass more quickly through the cloud—an additional reason for lesser growth.

The variation of the number density with height is displayed in Fig. 8. High number densities can be attributed to balance levels or to conditions which nearly lead to balance levels, a balance level being defined by the level for which the terminal particle speed equals the updraft velocity (Atlas, 1966).

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4. Thermal feedback

In Section 3 it was shown that hailstone feedback in form of depletion of the liquid water content can no longer remain neglected. This is reason enough to search for other feedback mechanisms. The most important one for a one-dimensional model seems to be the thermal feedback. Growing hailstones represent a heat source due to the release of latent heat of fusion $L_f$ according to

$$Q_r^* = (\pi/4)D_V V_f E_w L_f$$

where $Q_r^*$ is the rate of heat liberated by the accreted and partially or completely frozen water droplets, $E_w$ the collection efficiency ($= 1$), and $w_f = w_p$ the free water content (gm cm$^2$). The fraction $I$ of accreted water which freezes is determined by the heat balance equation (List, 1963), while $L_f$ is a function of the hailstone surface temperature $t_p$. As described more completely in Section 6, $I$ can assume values between 0 and 1 if $t_p = 0\,^\circ C$. It is further assumed that all the heat released on the hailstone surface is given off to the surrounding air and is not used for heating or cooling of the hailstone.

There is also a second heat source: frictional heating of the air by the falling hailstones, a factor never previously considered, but necessary on the basis of physical arguments. The rate at which heat is produced by one hydrometeor is given by the product of particle weight $W$ and terminal speed $V_t$. (This term is also applicable to models involving raindrops or rain formation.)

To see what effect these additional heat sources have on the cloud temperature which is given by the pseudo-adiabatic values, the above mentioned two terms must be introduced into Eq. (6). The heat given off per unit time by thermal feedback to $(1 + w_p)$ gm of cloud air is represented by $N \rho (Q_r^* + W Y_c) / (1 + r_p)$. When this expression is included in (6), one obtains

$$\Delta T = \frac{N \rho (Q_r^* + W Y_c)}{C_p + R R_w - \varepsilon_w (1 - \varepsilon_w) - \varepsilon_w R_c} - \frac{R T (p - \varepsilon_p) - R R_w - \varepsilon_w (1 - \varepsilon_w) - \varepsilon_w R_c} {C_p + R R_w - \varepsilon_w (1 - \varepsilon_w) - \varepsilon_w R_c} \Delta T$$

This equation was solved for temperature as function of height using values of $D$ and $w_f$ obtained from different combinations of input parameters. One of the cases which showed the most marked effect of thermal feedback is illustrated in curve C of Fig. 1, for $V_t = 25$ m sec$^{-1}$, $D_h = 1.0$ cm and $X = 2.0$ m$^{-1}$. For this case there is very little difference between the thermal feedback temperature $T_{ff}$ and the pseudo-adiabatic cloud temperature $T_{ad}$ in the lower part of the cloud, but at the 5 km level the difference between $T_{ff}$ and

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\( T_B \) is about the same (\( \sim 2.2^\circ C \)) as the difference between \( T_B \) and \( T_E \), the temperature of the environment.

The ratio \((T_{TF} - T_B)/(T_{TF} - T_E)\) for the small embryo sizes \((D = 0.25 \text{ cm})\) increases with increasing concentration \( N_0 \); however, for large \( D_0 \) this ratio shows decreasing tendencies. The above ratio increases for bigger particles \((2.06 \text{ at the 5-km level for } N_0 = 1.5 \text{ m}^{-2}, D_0 = 1 \text{ cm})\), where the heating by thermal feedback might even be twice as high as the effect due to pseudo-adiabatic lifting.

Since the sum of two terms, \( Q_F^* \) and \( W V_I \), is contributing to the thermal feedback, the relative importance of the single terms will be looked into by forming their ratio, \( Q_F^*/W V_I \). If \( Q_F^* \) is replaced by \( D^2 w_I V_I \) and \( W V_I \) by \( D^2 V_I \), neglecting constants, the ratio becomes \( w_I D^{-4} \). Since \( w_I \) generally decreases at the higher levels while \( D \) increases, the frictional heat term \( W V_I \) is of greater relative importance at these heights than at lower levels. In the cases examined \( W V_I \) was found to be one order of magnitude smaller than \( Q_F^* \) in the lowest levels, but became equal to or greater than \( Q_F^* \) higher up as the free water fell off.

Including the thermal feedback term, the first term in the numerator of Eq. (11), in the pseudoadiabatic equation (6) considerably increases the computational chores. Growth curves and free water content distributions were therefore evaluated on the basis of the pseudo-adiabatic cloud temperature. The effect of thermal feedback was assumed to be quite small in this respect as an air density change in (7) and (8) of 2\% would affect the growth rates and free water profiles by about the same amount.

Performing computations of some growth curves and \( w_I \) distributions including thermal feedback demonstrated, indeed, that growth and depletion are insensitive to changes in air density introduced by feedback. The biggest effects on \( D = D(z) \) and \( w_I = w_I(z) \) were less than about 3\%. However, thermal feedback will produce considerable effects if buoyancy is treated because buoyancy depends upon density differences.

5. Buoyancy

In the previous sections no attempt was made to link the updraft with forces exerted and created by the growing hailstones. One of the necessary steps toward a dynamic treatment of a model cloud containing growing hailstones is the assessment of buoyancy. Therefore, properties of our one-dimensional updraft model will be compared with the environment which is described by expressions (3) and (4). As defined in this paper, buoyancy consists of two parts, the first contribution resulting from the difference in density between the environment and the cloud, and the second a negative contribution due to the downward drag forces exerted on the air parcel by the water droplets and the hailstones. Since terminal speed is assumed, the drag forces are represented by the particle weights. Such a treatment is generally applied by others (e.g., Srivastava, 1967).

The buoyancy force per unit volume due to air density difference alone is given by

\[
\nabla' = g(\rho_E - \rho), \quad (12)
\]

where \( g \) is the acceleration due to gravity; the subscript \( E \) stands for environment. Since the pressures in the cloud and the environment are assumed to be the same at a given level, (12) can be rewritten as

\[
\nabla' = g(T T_E^{-1} - 1). \quad (13)
\]

Because the cloud temperature is higher than that of the environment, \( \nabla' \) is always positive. Negative contributions are represented by the weight of the cloud droplets, \( \omega g \), and the weight of the hailstones, \( \nabla \chi \). Thus, the buoyancy force per volume becomes

\[
\nabla = g(T T_E^{-1} - 1) - (\nabla \chi + \omega g). \quad (14)
\]

To demonstrate the large effect of thermal feedback, the next two figures illustrate buoyancies both with and without thermal feedback. Fig. 9 gives a good indication of how these buoyancies vary with height and updraft speed for hailstones which do not reach balance level. The trend for the buoyancy to increase just above the freezing level and to decrease at higher levels is quite general. The first increase is due to the fact that the first term in (13) becomes bigger with height because

![Buoyancy of growing hailstones as a function of height](image)
of the increasing difference between the cloud and the environment temperature; however, above a certain level the second term of the same equation becomes more important due to the increasing weight of the hailstones and the cloud droplets. This latter effect diminishes again in most cases as the upper regions of the updraft are reached.

It can be observed in Fig. 9 that the inclusion of thermal feedback does not alter the buoyancy in the first 1.5 km; however, between levels 3 and 5 km, its effect (as could be expected from Fig. 1) is quite pronounced. For an updraft of 23 m sec⁻¹, inclusion of thermal feedback keeps the buoyancy positive throughout the whole height range, whereas considerable negative buoyancy results without the direct and indirect heating of the cloud by the hailstones.

Fig. 10 shows similar buoyancy-height relationships for three different values of \( N_0 \). Again the effect of feedback is considerable, particularly at the higher levels where it might not be so obvious due to the different scale of the negative part of the abscissa.

It is obvious from Figs. 9 and 10 that the thermal feedback has to be considered in any buoyancy calculations. All the subsequent calculations, therefore, do include heating effects by release of latent heat of fusion and by friction. If buoyancy is mentioned later on, it includes thermal feedback.

For zero particle concentration the buoyancy is always positive (Fig. 10). If the input concentration is 2 m⁻³ with \( D_p = 0.5 \) cm and \( V_e = 22 \) m sec⁻¹, the buoyancy reaches very large negative values, caused by high hailstone concentrations or small absolute upward speeds of these hydrometeors. Higher embryo input concentrations reduce the maximum positive and negative buoyancies considerably.

The variation of buoyancy with updraft velocity is further explored in Fig. 11, where single cases correspond to both types of growth curves, the sigmoid and the balance level. The break in the trend of the curves while passing through zero buoyancy is caused by the compression of the negative scale. Higher zero-level updrafts lead to greater buoyancies for two reasons: strong updrafts cause lesser hailstone accumulation or smaller concentrations, and faster particle growth at low levels in weak updrafts increases the magnitude of the negative buoyancy term.

Since the cloud model treated in this paper is principally one-dimensional and since no entrainment is considered, the buoyancy force of one layer is acting upon the neighboring layers. If one assumes no vertical mixing or compression, caused by a nonhydrostatic pressure force on the updraft due to the weight of the hailstones, a new expression, the cumulative buoyancy, \( \Delta B \), is defined as the sum of the buoyancies of the layers starting from the 0°C level. The argument for the introduction of \( \Delta B \) is that the motion of the layer is informed by the buoyant forces above and below, and air parcels with negative buoyancy may, as a result, be forced to move upward. In a one-dimensional model the cumulative

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buoyancy represents a driving force which should be, in the end, closely related to the updraft strength. The level at which this sum becomes zero has some special significance because at this point the net force upon a vertical air column between these levels is zero. If this situation occurs at the cloud top, we have a realistic situation for a steady-state hail cloud because the net force on a column of air within the whole of the cloud will be zero; thus, with no friction and other countervailing forces in the flow system, no accelerations or decelerations occur. On the other hand, if this level of zero cumulative buoyancy occurs lower down in the cloud, then in order to obtain a non-negative net force, \( \Sigma b \) has to be compensated by a non-hydrostatic vertical pressure gradient; such a situation is unrealistic for steady-state hail clouds unless some strong external mechanism (high upper winds) exists for their development. The combination of parameters giving rise to strong negative \( \Sigma b \) may therefore be rejected as far as this model is concerned. On the other hand, if the cumulative buoyancy is positive at the top of the cloud and no other reactive forces could be accounted for, the updraft would have to increase, i.e., the model would not be steady-state.

The level at which the cumulative buoyancy \( \Sigma b \) becomes zero was determined for all the combinations of parameters given in Section 3. The selection of Figs. 3-5 was particularly based on realistic values for buoyancy and cumulative buoyancy. Figs. 3a, 4a, and 5a contain isolines of \( \Sigma b = 0 \) as well as isolines at which the buoyancy changes from positive to negative values, i.e., \( b = 0 \). For all the figures shown the spread between the two isolines becomes smaller with decreasing input embryo concentration \( N_o \). This is in accordance with Fig. 10 which shows the rapid drop of the buoyancy with height for the smaller \( N_o \). The successive Figs. 3a, 4a and 5a for \( D_o = 0.5 \) cm demonstrate how the height of both isolines rises for increasing \( V_u \). Furthermore, this rise in height is more pronounced for the higher \( N_o \), causing the gradual change of slope of both lines with increasing \( V_u \). The greatest change in the \( \Sigma b = 0 \) level for \( D_o = 0.5 \) cm occurs between \( V_u \) values of 21 and 22 m sec\(^{-1}\). Here, for \( N_o \) values of 5 and 10 cm\(^{-3}\), this level changes from about 3.5 km to somewhere above 5 km, while for lower values of \( N_o \) the change is fairly small. For \( V_u = 23 \) m sec\(^{-1}\) (Fig. 4a) the \( \Sigma b = 0 \) isoline has already moved above the 5-km level; the same is true for \( V_u = 24 \) m sec\(^{-1}\).

The remarkable fact of these considerations is that buoyancy and cumulative buoyancy are extremely sensitive to changes in the updraft speed. In other words, dynamic relationships between updraft speed and buoyancy have to be quite accurate (\( \pm 5\% \) in velocity) to predict the right cloud conditions. No attempt in this direction will be made now.

6. Icing conditions

In the previous sections we explored the growth of hailstones, the depletion of the liquid water content...
they cause, the heat source they represent, and the buoyancy which can be attributed to the updraft air. In order to make the picture more complete, we now investigate the icing conditions of the hailstones, i.e., their surface temperature \( t_D \) or the sponginess of the deposits growing at certain levels, the latter represented by the fraction \( I \) of accreted water which freezes.

The surface temperature and the sponginess can be obtained from a heat balance equation (List, 1963) in which the sum of the heat transferred to a growing hailstone by conduction and convection, by evaporation or sublimation, and by accretion of supercooled cloud droplets is used to freeze the accreted water either partially \( (I < 1; t_D = 0 \text{C}) \) or completely \( (I = 1; t_D \leq 0 \text{C}) \). This balance equation can be solved for either \( t_D \) or \( I \); thus,

\[
Y \left[ L_f(t_D) + \dot{C}_w A \right] + 1.68k t_A - \left[ e_{sh}(t_D) - e_w(t_A) \right] C_{1.5} D_{oa} T_A^{-1} \cdot 1.68k + \dot{C}_w Y \]

\[
I = \frac{-1.68k t_A + \left[ e_{sh}(0 \text{C}) - e_w(t_A) \right] C_{1.5} D_{oa} T_A^{-1} - \dot{C}_w Y}{Y L_f(0 \text{C})}, \tag{16}
\]

with

\[
Y = 5.48b \rho D^{-1} E R^{3} \rho W^{1.5}, \tag{17}
\]

where \( t_D \) (C) is the hailstone temperature, \( t_A \) (C) and \( T_A \) (K) the air temperature, \( \dot{C}_w \) the specific heat of water averaged over the temperature range \((t_D, t_A)\), \( k \) the thermal conductivity of air, \( e_{sh}(t_D) \) the saturation vapor pressure over the hailstone surface, and \( e_w(t_A) \) the saturation vapor pressure over water at air temperature; \( C_{1.5} \) has a value of 0.235 cal (°C) cm⁻² mb⁻¹ at −20°C or lower and varies linearly with temperature from −20°C to a value of 0.207 cal (°C) cm⁻² mb⁻¹ at 0°C; \( D_{oa} \) is the diffusivity of water vapor in air, \( L_f \) the latent heat of fusion, \( \nu \) the kinematic viscosity, \( E \) the collection efficiency (=1), and \( \theta \) a roughness factor; \( E R = 1 \) is 0.675 for conditions of spongy growth explored by List in 1960. Eq. (15) was solved by an iteration process leading to a computation accuracy of \( t_D \) of ±0.02°C. Values for \( t_D \) or \( I \) were obtained in this way at intervals of 0.2 km for all the cases for which the buoyancies were determined.

The results are shown in Figs. 3b, 4b, and 5b in the form of dashed isolines, giving the hailstone surface temperature \( t_D \) or \( I \), the fraction of the accreted water which freezes. In this calculation the heat stored in the hailstones was neglected, i.e., their temperature was assumed to be the equilibrium temperature of deposits growing at a given level. As long as spongy ice is growing, no error is introduced by such a procedure; at \( t_D < 0 \text{C} \), the hailstone surface temperatures would, for our model, be somewhat higher than the calculated values due to heat flow from the warmer interior of the rising hailstone.

In most cases the isolines of \( t_D \) and \( I \) slope from the lower left to the upper right. At constant height, \( t_D \) increases and \( I \) decreases with higher values of \( \dot{C}_w \) and \( D \) [see Eqs. (15) and (16)]. Since the liquid water content and the diameter at a given height usually increase with decreasing \( \dot{C}_w \), the sloping of the isolines is sufficiently explained. Exceptions which can be observed, but not in the data shown here, can be explained by close examination of the liquid water content distributions.

The most important conclusion from the calculated icing conditions is that the biggest hailstones usually have a deposit or surface temperature slightly below 0°C, i.e., their growth is usually not spongy in the outer shell as might be expected on the basis of laboratory measurements. The reason for this result can be understood by looking at the growth curves and the velocity profiles. Fast growth normally means spongy growth; however, in our model fast growth leads to low balance levels and dropout at smaller sizes. It is interesting to learn from a thorough investigation of two samples of hailstones (List et al., 1969) that many outer shells of 4-cm hydrometeors were definitely grown non-spongy. Whether or not these stones grew in a cloud similar to our model is naturally still an open question.

The idea, mentioned in Section 3, that shells could grow upon hailstones by changes in input parameters, particularly of updraft speed (or the level of the cloud base) becomes even more attractive when considering the remarkable dependence of the icing conditions on these factors. However, extensive studies would have to be undertaken to clarify this point beyond any doubt.

7. Time of growth

Last, but not least, the time taken by the hailstones to reach a certain level in the cloud, as they are carried upward by the updraft, were obtained from

\[
l = \int_0^l \left( \frac{1}{V_s - V_i} \right) dz, \tag{18}
\]

where \( z \) is the vertical distance measured from the input level. For the growth curves that terminate at a balance level, the calculation was stopped at the preceding level.

The growth times obtained are shown in Figs. 3a, 4a, and 5a as solid isolines intersecting the growth curves. For a given updraft and at a given height, smaller concentrations \( X_s \) normally mean larger diameters.
which, in turn, results in a higher terminal velocity and thus a longer growth time. The longest growth times at the 5-km level occur for the smallest concentration of particles which reach this level.

In general, it can be seen that the growth times of relatively big stones, i.e., stones which reach a balance level below or at the 5-km level, are of the order of 8-12 min. These values are consistent with radar observations.

8. Discussion

The authors recognize that the simplifications of the model studied in this paper are sometimes quite artificial and that more thorough studies are needed to improve and expand the model or to make it more realistic. Nevertheless, this study was quite successful in demonstrating that “normal” spherical hailstones can grow to 2-4 cm in size without having to resort to recycling or water accumulation zones. They may well exist, but they do not seem to be necessary.

If ellipsoidal particles are considered, they may grow to even larger sizes in a steady-state model. Bigger spherical particles could grow in updrafts which increase in strength with time.

A one-dimensional model is only a first step toward a three-dimensional one. However, the available concepts of entrainment often defy realism and three-dimensional aerodynamic feedback (List and Lozowski, 1968) is so completely unexplored that more basic studies have to be undertaken to make the step to three dimensions safer. It has to be realized that non-hydrostatic pressure terms are essential and that buoyancy can be partly compensated or over-compensated by such terms and not only by momentum changes. The magnitude of the resistance to a thunderstorm circulation should also be better known so that it might be compared to the driving forces already mentioned.

In summary, it can be said that the study of a one-dimensional model of the growth of hailstones has improved our insight into the mechanics of hail formation and many conclusions might, with caution, also be applied to certain natural hailstorms. Of particular importance is the conclusion that the degree and the variation of free water depletion as functions of updraft variations and particle parameters can account for the growth of hailstone shells.

Hail prevention. The calculations on hailstone growth might also be of some importance for preventing hail fall from clouds for which this model is representative. First it is seen that relatively big stones can only be expected for particle concentrations at the 0°C level of 0.5-2.0 m⁻³, a range which might also be representative in nature. If the concentrations are increased to about 10 m⁻³ or higher, the final size is such that complete or nearly complete melting will occur while the hailstones fall toward the ground. Hail damage might therefore be prevented by increasing the embryo concentrations to about 10 m⁻³.

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REFERENCES


Thermodynamic Equations of the Updraft

For a reversible process involving 1 gm of dry air and $r_g$ gm of water vapor which ignores the heat required to keep the condensed liquid water $r_s$ at the mixture's temperature (pseudo-adiabatic by definition) the first law of thermodynamics can be written as:

$$- L_v A r_s + \frac{dQ}{dp} (1 + r_s) \Delta P$$

$$= C_p \Delta T - RT \Delta \ln(P - e_w) + r_s C_p \Delta T - r_s R \Delta \ln e_w$$

The symbols of A2-1 are defined in Section 2.6.7 of Chapter 2 and given numerical values later in this derivation.

The heat sources of the left side of A2-1 are considered reversible since they can be written as functions of temperature and pressure. $- L_v A r_s$ represents the heat of condensation, and $\frac{dQ}{dp}$ is the heat added per unit pressure change by wake heating and heat exchange between the hailstones and the air parcel plus the latent heat of fusion caused when the cloud's liquid water freezes out spontaneously between the -32°C and -40°C levels.

Using the following two relationships:

$$r_s = \frac{R}{R_w} \frac{e_w}{P - e_w}$$

[A2-2]
and [A2-1] one gets:

\[
\frac{dT}{dP} = \frac{dQ}{dP} \left(1+r_s\right) \left[RT(P-e_w)^{-1} + \frac{n \Delta L_v}{e_w} (P-e_w)^{-2}\right]
\]

\[
dP C_p + r_s C_{pv} + \left[n \Delta L_v^2 e_w\right] (P-e_w)R_v T^2 \left[1+e_w (P-e_w)^{-1}\right]^{-1}
\]

[A2-4]

For use in the simple model of Chapter 2, Eq. A2-4 can be integrated directly to give the cloud temperature \( T \) as a function of pressure \( P \) since then \( \frac{dQ}{dP} \) equals zero and \( e_w \) is a known function of temperature (Eq. A2-8).

The above equations do not consider a change in \( r_s \) due to evaporation from the hailstones because this evaporating water goes directly into cloud liquid water \( r \), see Eq. 18 of Chapter 2. This necessitates that the heat of condensation of the water vapor which is evaporating from the hailstone \(-Q^*_E\) be directed into the cloud air through the term \( \frac{dQ}{dP} \). The heat required to cool the evaporating water from the hailstone's surface temperature to the cloud temperature is ignored. The rates at which heat is added to the cloud air per hailstone due to evaporation and sublimation, and convection and conduction from the hailstone's surface are symbolized by \(-Q^*_E\) and \(-Q^*_cc\) respectively, and they are evaluated in Appendix 3.
The negative signs of $Q_E^*$ and $Q_{cc}^*$ are due to the way they are defined in Appendix 3. The rate of heating due to turbulent wake dissipation per hailstone is $gMV_t \alpha$.

The rate of energy given off per hailstone is therefore:

$$-Q_E^* - Q_{cc}^* + gMV_t \alpha$$

[units of power, dependent on constants used in Eq. A2-4.]

If there are $N$ hailstones per unit volume, each of mass $M$ in air of density $\rho$ the rate of energy gain per gram of cloud air due to hailstones is:

$$N\rho^{-1}(-Q_E^* - Q_{cc}^* + gMV_t \alpha)$$

If the air parcel is ascending at velocity $V_Z$ the rate of energy released per gram of cloud air from these hailstones per unit height is:

$$NV_Z^{-1} \rho^{-1}(-Q_E^* - Q_{cc}^* + gMV_t \alpha)$$

The freezing out of liquid water $r$ between the $-32^\circ C$ and $-40^\circ C$ levels $\frac{dr}{dZ}$ is given in Eq. 19 of Chapter 2. The rate of energy released per unit height per gram of cloud air to the ascending cloud air is
therefore:

\[
\frac{dr}{dZ} \bigg|_{FZ} \left( L_f \right)
\]

where \( L_f \) is the latent heat of fusion of water.

The rate of energy added to the ascending cloud parcels per gram of cloud air per unit height due to both hailstone heating and high level freeze-out is therefore:

\[
NVZ^{-1} \rho^{-1} (-Q_E^* - Q_{cc}^* + gMV\alpha) + \frac{\frac{dr}{dZ}}{FZ} \left( L_f \right)
\]

Since pressure \( P \) is a function of height \( Z \) only (Eq. 3, Chapter 2) the energy per gram of cloud air per unit pressure change \( \frac{dQ}{dP} \) due to the above effects is given as:

\[
\frac{dQ}{dP} = \left( \frac{\frac{dP}{dZ}}{dZ} \right)^{-1} \left[ NVZ^{-1} \rho^{-1} (-Q_E^* - Q_{cc}^* + gMV\alpha) + \frac{\frac{dr}{dZ}}{FZ} \left( L_f \right) \right]
\]

\[\text{[A2-5]}\]

Pressure being a function of height only allows that:

\[
\frac{dT}{dZ} = \frac{dT}{dP} \left( \frac{dP}{dZ} \right)
\]

\[\text{[A2-6]}\]
Using (A2-5) and (A2-6) in (A2-4) and rearranging yields Eq. 20a of Chapter 2 for \( \frac{dT}{dZ} \). The first goal of this appendix is therefore reached; only a definition of Eq. 20a's variables is needed.

To integrate the three simultaneous first-order differential equations of cloud temperature change \( \frac{dT}{dZ} \) (Eq. 20a), hailstone growth \( \frac{dD}{dZ} \) (Eq. 23), and liquid plus vapor water conservation \( \frac{dT}{dZ} \) (Eq. 24) one needs the values of \( Q^*_{cc} \) and \( Q^*_E \) given in Appendix 3 and the following values of the variables and constants of Eqs. 20a, 23, and 24.

Differentiating the pressure \( P \) with respect to height \( Z \) (Eq. 3) gives:

\[
\frac{dP}{dZ} \text{ [mb/cm]} = -\frac{P \text{[mb]}}{1.5506 \times 10^5} \left( \frac{20.417 - Z \text{[cm]} \times 10^{-5}}{0.7753} \right)^{-\frac{1}{2}}
\]

[A2-7]

The saturation vapor pressure with respect to water \( e_w \) from Godson (1958) is given by:

\[
\log_{10} e_w \text{[mb]} = 24.0068 - (2956.3/T[°K]) - 5.0687 \log_{10} T[°K]
\]

[A2-8]

The ratio of the molecular weight of water vapor to that of dry air \( \eta \) is given by:

\[
\eta = \frac{R}{R_w}
\]

[A2-9]
where the gas constant for dry air $R$ and water vapor $R_w$ are given by:

\[ R = 2.8705 \times 10^3 [\text{mb cm}^3 \text{ gm}^{-1} \text{ oK}^{-1}] \]  \[\text{[A2-10]}\]

\[ = 2.8705 \times 10^6 [\text{erg gm}^{-1} \text{ oK}^{-1}] \]

\[ = 0.06856 [\text{cal gm}^{-1} \text{ oK}^{-1}] \]

\[ R_w = 4.6152 \times 10^3 [\text{mb cm}^3 \text{ gm}^{-1} \text{ oK}^{-1}] \]  \[\text{[A2-11]}\]

\[ = 0.11023 \text{ cal gm}^{-1} \text{ oK}^{-1} \]

The specific heats of dry air $C_p$ and water vapor $C_{pv}$ are:

\[ C_p = 0.240 \text{ cal gm}^{-1} \text{ oC}^{-1} \]  \[\text{[A2-12]}\]

\[ C_{pv} = 0.4436 \text{ cal gm}^{-1} \text{ oC}^{-1} \]  \[\text{[A2-13]}\]

A linear approximation to the latent heat of vaporization of water $L_v$ which fits the values given in Godson (1958) at $0^\circ C$ and $-40^\circ C$ is:

\[ L_v [\text{cal/gm}] = 597.26 - 0.610 (T[^\circ K] - 273.15) \]  \[\text{[A2-14]}\]
The mean latent heat of sublimation of water $L_s$ between $0^\circ C$ and $-40^\circ C$ from Godson (1958) is:

$$L_s = 677.5 \text{ cal/gm} \quad [\text{A2-15}]$$

$L_s$ varies by no more than 0.5 cal/gm in that temperature range.

The latent heat of fusion of water $L_f = L_s - L_v$ using A2-14 and A2-15 is:

$$L_f [\text{cal/gm}] = 80.2 + 0.61 (T[\degree K] - 273.15) \quad [\text{A2-16}]$$

The development of Eq. 20a of Chapter 2 for $\frac{dT}{dZ}$ is now fully described as are the definitions of the thermodynamic constants required for the calculation of $\frac{dT}{dZ}$. Appendix 3 will now give the equations for the heat and mass exchange of an accreting hailstone for calculating $Q_E^*$ and $Q_{cc}^*$. 

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APPENDIX 3

The Heat and Mass Exchange of an Accreting Hailstone

This appendix gives a method of calculating the quasi-steady-state heat and mass exchange of spheroidal hailstones as given by List and Dussault (1967).

The fraction of accreting liquid water I which is being frozen on the stone's surface is given by:

\[
I = \frac{(1.68k + \gamma_v)(273.15 - T) + (e_i[0°C] - e_v[T])C_{1,2}}{Y L_f[0°C] D_wa T^{-1}}
\]

where:

\[
Y [gm/cm^2 s^{-1}] = 5.483 \rho \frac{\gamma_v}{\partial \partial T} E_{0^{-1}} D_{\frac{\gamma_v}{4}} \frac{\rho}{r} \left( \frac{\alpha}{\epsilon} \right)^{\gamma_v} (Y_v)^{-1}
\]

If I is calculated above and it turns out to be greater than 'one', then I is set equal to 'one' and the temperature of the depositing water \( T_d \) is no longer assumed to be 0°C as it was in (A3-1) but it is given by:

\[
T_d[^{o}K] = T + \frac{YL_f[T_d] - C_{1,2} D_{w a \left( e_i[T_d] - e_v[T] \right)}}{1.68k + \gamma_v Y}
\]

The constant 5.483 is in \([gm/cm^2 s^{-1}]\).
Several of the parameters in (A3-3) are functions of the deposit temperature $T_d$ which must, therefore, be solved by an iteration process. (A3-3) converges satisfactorily if the initial value of $T_d$ is set equal to 273.15°K for purposes of calculating the second estimate of $T_d$. Previous estimates are used to calculate a newer value of $T_d$ until $T_d$ changes by less than 0.1°K on successive iterations.

The variables in the foregoing are as follows:
The saturation vapor pressure with respect to water $e_w$ (mb) was given in Appendix 2, Eq. (A2-8) and is evaluated here at the air temperature $T (°K)$.

The saturation vapor pressure with respect to ice $e_i$ evaluated at $T_d$ from Godson (1958) is:

$$
\log_{10} e_i (\text{mb}) = 2.07039 - 248.9 T_d^{-1} + 3.5665 \log_{10} T_d - 3.2099 \times 10^{-3} T_d
$$

The thermal conductivity of air $k$ is:

$$
k(\text{cal cm}^{-1} \text{sec}^{-1} \text{°K}^{-1}) = (5.8 + 0.0175(T - 273.15)) \times 10^{-5}
$$
The diffusivity of water in air $D_{WA}$ is:

$$D_{WA} (\text{cm}^2/\text{s}) = 9.02 \times 10^{-3} T^{1.8} \rho^{-1} \quad [A3-6]$$

The kinematic viscosity of air $\nu$ is:

$$\nu (\text{cm}^2/\text{s}) = 5.54 \times 10^{-3} T^{1.8} \rho^{-1} \quad [A3-7]$$

$K$, $D_{WA}$, and $\nu$ are from the Smithsonian Meteorological tables.

The mean specific heat of water $C_w$ between $T$ and $T_d$ is found by integrating a first order polynomial for $C_w$ which fits the Smithsonian Meteorological tables at $-10^\circ C$ and $-40^\circ C$ between $T$ and $T_d$. This gives:

$$C_w (\text{cal gm}^{-1} ^\circ \text{K}^{-1}) \quad [A3-8]$$

$$= 1.0074 - 2.875 \times 10^{-4} (T_d + T - 546.3)$$

$$+ 2.283 \times 10^{-5} [(T_d + T - 546.3)^2 - (T_d - 273.15)(T - 273.15)]$$

The latent heat of fusion of water $L_f[T_d]$ evaluated at $T_d$ is found by evaluating $(A2-16)$ of Appendix 2 at $T_d$. 

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List and Dussault (1967) give the value of $C_{1,2}$ as:

For $T > 253.15^\circ K$

$$C_{1,2} \text{ (cal deg cm}^{-3} \text{mb}^{-1}) = 0.207 - 0.0014(T - 273.15)$$  \[A3-9\]

For $T < 253.15^\circ K$

$$C_{1,2} \text{ (cal deg cm}^{-3} \text{mb}^{-1}) = 0.235$$  \[A3-10\]

In Eqs. A3-1 through A3-3 the air density $\rho$, the hailstone's major dimension $D$ and the liquid water mixing ratio $r$ are in c.g.s. units.

$E \theta^{-1}$ is the collection efficiency $E$ divided by the roughness parameter for heat transfer $\theta$. Since $E=1$ and $E \theta^{-1} = 0.675$ from List and Dussault (1967),

$$\theta^{-1} = E \theta^{-1} = 0.675$$  \[A3-11\]

$\alpha$ = is the ratio of the minor axis to major axis

$\gamma$ = is the ratio of the area of the spheroid to that of a sphere whose diameter equals the major axis (dependent on $\alpha$).
Source of linear approximations to the coefficients which determine heat exchange and terminal velocity of spheroidal hailstones.
$\chi = $ is the shape factor determining the Nusselt number of a spheroid.

$\varepsilon = $ is the ratio of the drag coefficient of a spheroid to that of a sphere whose diameter equals the spheroid's major dimension.

Experimental values of $\varepsilon$ and $\chi$ as functions of $\alpha$ were obtained from Dussault (1966) who got the values of $\chi$ from Macklin (1963). These experimental values were used to plot $\left( \frac{\alpha}{\varepsilon} \right)^{\frac{1}{2}}$ [note power $\frac{1}{2}$] or $\gamma \chi$ as functions of $\alpha$ in Fig. (A3-1) where the following linear approximations from a 'best straight line fit' were obtained:

$$\left( \frac{\alpha}{\varepsilon} \right)^{\frac{1}{2}} = 0.8 \alpha + 0.2 \quad \text{(A3-12)}$$

$$\gamma \chi = 0.376 \alpha + 0.62 \quad \text{(A3-13)}$$

From List and Dussault (1967) the rate of heat exchange per hailstone due to evaporation and sublimation $Q_{ES}^*$, and convection and conduction $Q_{cc}^*$ are:

$$Q_{ES}^* \text{[cal/s]} =$$

$$- C_{12} D w_a T^{-1} \theta v^{-\frac{1}{2}} V_{ta} \frac{1}{2} D^{3/2} \left( e_i [T_d] - e_w [T] \right) \gamma \chi \quad \text{(A3-14)}$$
\[ Q_{cc}^{*}[\text{cal/s}] = -1.68 \times 10^{-1} \theta v^{-12} a^{12} D T (T_d - T) Y \times \quad \text{(A3-15)} \]

The constant 1.68 is non-dimensional.

\[ C_{1,2} \text{(Eq. A3-9) from Dussault (1966) is a function of the latent heat of evaporation and sublimation from 0^\circ C \text{ to } -20^\circ C \text{ and is proportional to the latent heat of sublimation } L_s \text{ below } -20^\circ C \text{ (Eq. A3-10). Therefore, the rate of mass exchange } M_{ES}^{*} \text{ can be found by dividing Eq. (A3-10) by } L_s = -677.5 \text{ cal/gm, defined in Appendix 2, to give:} \]

\[ M_{ES}^{*}[\text{gm/s}] = 
\]

\[ 3.468 \times 10^{-1} D_{wa} T^{-1} \theta v^{-12} a^{12} D^{12} (e_{1}[T_d] - e_{s}[T]) Y \times \quad \text{(A3-16)} \]

The constant '3.468 \times 10^{-1}' is in units of \([\text{gm } K^{-1} \text{ cm}^{-3} \text{ mb}^{-1}]\).

The rate of heat added to the cloud air per hailstone as the water evaporating and sublimating from the stone re-condenses to form cloud liquid water is given by the product of\(-M_{ES}^{*}\) and the latent heat of condensation \(L_v\) evaluated at the cloud temperature, Appendix 2, Eq. A2-1h, and is symbolized by \(Q_E^{*}\).
\[ Q_E^* \text{[cal/s]} = -L_v M_{ES}^* \quad [A3-17] \]

Note that \( Q_E^* \) does not equal \( Q_{ES}^* \) because \( Q_E^* \) represents only condensation while \( Q_{ES}^* \) is an approximation to the heat of evaporation and sublimation from the mixed liquid and frozen surface of the hailstone.

The terminal velocity \( V_{t\alpha} \) as a function of \( D \), \( \rho \) and \( \alpha \) is given in Chapter 2, Eq. 11 or 12.

It should be noted that when Eq. 20a of Chapter 2 is used to calculate the cloud temperature gradient \( \frac{dT}{dz} \), the units of \( Q_E^* \) and \( Q_{cc}^* \) should be carefully considered since the values given here are in (cal/s) and should be in (erg/s) in Eq. 20a where c.g.s. units are used.
APPENDIX 4

Program for A&D Size Distributions

The following is the computer program used to calculate the growth of hailstones injected in size distributions. The hail ascends and descends in steady-state. Calculation of forces, energies, hail surface icing conditions, and hail number distributions is included. The language is Fortran IV.
C A + D HAIL GROWTH WITH SIZE DISTRIBUTION

DIMENSION CN(10),DU(10,101),DDU(10,101),DDZ(10,101),
   1AN(10),Z(101),R(101),VT(101),NIT(10,101),RU(101),
   WR(101),
   5,ISTOP(10),TIME(10),EN(10),ZDELTA(10),TIMEUV(10),T(10,101),T1(10,101),
   201,ENNU(10,101),ENND(10,101),ANI(10),ANTUT(101),ALICE(10,101),
   3AIR(10,101),G(3),SM(101),
   4,TASS(11),FORCE(11),FMI(101),ENG(Y101),ENGY(101),FBE(101),ENGY(11)

C READ VZ0,LIQ+VAPOR WATER,AND NUMBER OF SIZES
99 READ(5,101) VT(1),RWAT0,N
   WRITE(6,101) VT(1),RWAT0,N
101 FORMAT(1X,2E12.4,1X,112)

C READ EMBRYO SIZES
READ(5,102)(DU(I,1),I=1,N)
   WRITE(6,102)(DU(I,1),I=1,N)
102 FORMAT(1X,10F7.2)

C READ EMBRYO CONCENTRATIONS
READ(5,103)(AN(I),I=1,N)
   WRITE(6,103)(AN(I),I=1,N)
103 FORMAT(1X,10E10.2)

C CONSTANTS
RO1=0.915
PI=3.14159
A=48.8
RU0=7.59E-04

C AIR FLUX
C2=ROO*VT(1)
C3=C2/A
DELZ=5.0E+03
BOL=1.0

C IS UPWARDS
DO 300 J=2,101
   C IS I TH SIZE GROUP
   DO 301 I=1,N
      DU(I,J)=0.0
      DD(I,J)=0.0
   301 CONTINUE
   DO 3 1=1,N
      CN(I)=AN(I)*(VT(1)-ASSORT(DU(I,1)/RUU))/C2*PI/12.
      DD(I,1)=0.0
      RSKXU,AND VT AS FCNS OF Z
      DU 199 J=1,101
      (Z(J)=FLONAT(J-1)*DELZ
      RU(J)=(6.5-1.6E-05*(Z(J)-1.0E05))/1.0E-05
      IF(Z(J),GT,2.5E05) RU(J)=(6.5-1.6E-05*(Z(J)-2.5E05))/1.0E-05
      IF(Z(J),GT,1.0E05) RU(J)=(7.59-0.84E-05*(Z(J)-1.0E05))/1.0E-05
      VT(J)=C2/RU(J)

C LIP IS COUNTER OF ITERATIONS
LIP=1

C POINT WHERE ITERATIONS BEGIN
   1 C=0.0
   C AT TOP OF CLOUD I=100 AND DO(1)=0
   DU 200 I=1,N
   IF(DU(I,100),GT,0.1E-20) DU(I,1)=0.0
200 CONTINUE
LIP = LIP + 1
DO 2 I = 1, N
2 C = C + (DU(I, 1)**3 - DD(I, 1)**3) * CN(I)
C ALFA IS MODIFIED WATER SUBSTANCE FLUX
ALFA = C * RWAT/2, / RO/J + R(I)
C FIRST ESTIMATE OF HAIL GROWTH BETWEEN HERE AND 4 CUNI.
C INTEGRATION IS UPWARD
DO 4 J = 1, 100
SUM = 0.0
DO 5 I = 1, N
5 SUM = SUM + CN(I) * (DD(I, J)**3 - DU(I, J)**3) * RUL
DO 6 I = 1, N
6 NIT(I, J) = 1
IF (DU(I, J) .LT. 1.0E-20) GO TO 7
DUDZ(I, J) = R/J(J) * (ALFA + SUM - R(J)) / (C3 / SQRT(RU(J) * DU(I, J)) - 1.)
IF (DUDZ(I, J) .LT. 0.0) GO TO 7
C NO -VE DS ALLOWED / DDDZ = 0.0
IF (DD(I, J) .LT. 1.0E-20) GO TO 77
DDDZ(I, J) = DUDZ(I, J) * (C3 / SQRT(RU(J) * DU(I, J)) - 1.)
IF (DDDZ(I, J) .GT. 0.0) GO TO 7
GET DU AT NEXT LEVEL
7 DU(I, J + 1) = DU(I, J) + DELZ * DUDZ(I, J)
V = A * SQRT(DU(I, J + 1) / R/J(J + 1))
C IF DU AT BL, GO TO 44
IF (V .GT. VT(J + 1)) GO TO 44
C IF DD .LT. 0.0 DD = 0.0 AT NEXT LEVEL
IF (DD(I, J) .LT. 1.0E-20) GO TO 8
DD(I, J + 1) = DD(I, J) + DELZ * DDDZ(I, J)
C NO -VE DDS ALLOWED
IF (DD(I, J + 1) .LT. 0.1E-20) GO TO 44
V = A * SQRT(DD(I, J + 1) / R/J(J + 1))
C IF DD AT BL GO TO 44
IF (V .LT. VT(J + 1)) GO TO 44
GO TO 9
C GO TO 7 IF DU = 0.0
7 DUDZ(I, J) = 0.0
C GO TO 44 WHEN ABOVE A HL
44 DU(I, J + 1) = 0.0
DDDZ(I, J + 1) = 0.0
GO TO 9
C GO TO 7 IF DU = 0.0
7 DUDZ(I, J) = 0.0
C CONTINUE FOR ALL SIZES
CONTINUE
4 CONTINUE
C GET NIT = 2 AT DDS HL AND ABOVE
IF (DU(I, J) .LT. 0.1E-20) NIT(I, J) = 2
IF (DU(I, J) .LT. 0.1E-20) NIT(I, J - 1) = 2
C CONTINUE FOR ALL SIZES
6 CONTINUE
C CALC. LWC AT LEVEL J
W(J) = ALFA + SUM - R(J) / RU(J) = RU/J = 2.
C CONTINUE TO NEXT LV.
4 CONTINUE
C IF ITERATION AT 1ST STEP NIT = 2, WRITE OUT. IF LAST STEP WRITE OUT
IF (LIP .EQ. 2) GO TO 900
900 IF (LIP .LT. 20) GO TO 197

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$JOB CHARLTON R B VPP085SPRECMPH 55 20 $ A+U = DISTN.

900 CONTINUE
WRITE(6,22)
22 FORMAT(1X,///)
WRITE(6,21) LIP
21 FORMAT(1X,5HLIP= ,112)
WRITE(6,20)(Z(J), (DD(I,J), I=1,N), (DU(I,J), I=1,N), WF(J), J=1,100)
20 FORMAT(1X,1E10.2,10F8.4,1E11.3)
IF(LIP.EQ.2) GO TO 197
C 700 CALCS ADDITIONAL PARAMETERS AT ITERATIONS END
GO TO 700
197 CONTINUE
C SET DDS=0.0
DO 13 J=1,101
DU 14 I=1,N
14 DD(I,J)=0.0
13 CONTINUE
C CALC. NEW DDS FROM BL S OF DUS DOWN TO INPUT LVL
DO 10 J=1,100
DO 11 K = 1, N
NJ=100—J+1
NK=N—K+1
IF(NIT(NK,NJ).EQ.2) GO TO 12
C IF NIT=2 AND DD=0.0, THEN NJ IS AT BL
V=ASORT(DU(NK,NJ)/RO(NJ))
IF(DD(NK,NJ),LT.1.0E-20) V=2.*VT(NJ)–V
C DD AT BL OF DU
IF(DD(NK,NJ),LT.1.0E-20)DD(NK,NJ)=V**2*RO(NJ)/A**2
IF(NJ,EQ.1) GO TO 15
DDDZ(NK,NJ)=DUDZ(NK,NJ)*(C3/SORT(RU(NJ)=DU(NK,NJ))–1.)/(C3/SORT(RO(NJ)))*DD(NK,NJ))–1.)
DDDZ(NK,NJ)=DDDZ(NK,NJ)—DDDZ(NK,NJ)*DELZ
12 CONTINUE
IF(DU(NK,100),GT.0.1E-20) DD(NK,NJ—1)=0.0
15 IF(DU(NK,100),GT.0.1E-20) DD(NK,NJ)=0.0
11 CONTINUE
10 CONTINUE
C 198 CONTINUE
WRITE(6,22)
198 CONTINUE
BUL=1.0
C IF LIP=20 AN EXAMPLE IS COMPLETE. 99 READS NEW DATA
IF(LIP.EQ.20) GO TO 99
C DO NEW ITERATION FROM STATEMENT 1
GO TO 1
C GO TO 700 FROM JUST BEFORE (197 CONTINUE)
700 CONTINUE
C CALCULATE VARIOUS PARAMETERS
DU 201 I=1,N
ISTOP(I)=0
DU 202 J=1,100
ISTOP(I)=ISTOP(I)+1
IF(NIT(I,J),EQ.2) GO TO 201
C ISTOP IS NUMBER OF LVLS WHERE U(I,J) IS FINITE
202 CONTINUE
201 CONTINUE
DU 203 I=1,N

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IF(I.EQ.1) GO TO 204
IF(I.EQ.N) GO TO 205
IF(ISTOP(I).EQ.100) GO TO 203

C ZDELTA IS DISTANCE TWIX BL'S OF I-1 + I+1
ZDELTA(I)=FLOAT(ISTOP(I-1)-ISTUP(I+1))*DELTZ/2.
GO TO 203

204 ZDELTA(I)=FLOAT(ISTOP(I)-ISTUP(I+1))*DELTZ
GO TO 203

205 ZDELTA(I)=FLOAT(ISTOP(I-1)-ISTUP(I))*DELTZ

203 CONTINUE
D 206 I=1,N
C IF NO ITH BL, GO TO 299
IF(ISTOP(I).EQ.100) GO TO 206
IF(I.EQ.N) GO TO 207

C BOTTOM OF ITH BL REGION IS AT INTUP AND INTDW LVLS
C ROUNDOFF SMOOTHED FOR ASCENDING AND DESCENDING STUNES
INTUP=ISTOP(I)-INT((FLOAT(ISTOP(I))-ISTUP(I+1))/0.1)*2.
INTDW=ISTOP(I)-INT((FLOAT(ISTOP(I))-ISTUP(I+1))/2.0)+0.6
GO TO 208

207 INTUP=ISTOP(I)-INT((FLOAT(ISTOP(I-1)-ISTUP(I)))/0.1)*2.
INTDW=ISTOP(I)-INT((FLOAT(ISTOP(I-1)-ISTUP(I)))/2.0)+0.6

208 CONTINUE
IF(INTUP.LT.1) INTUP=1
IF(INTDW.LT.1) INTDW=1

C NUMBER OF STEPS FROM BOTTOM OF BL REGION TU BL
IUPDIF=ISTOP(I)-INTUP
IDWDIF=ISTOP(I)-INTDW

TIME(I)=0.0
C TIME IS TOTAL TIME IN BL REGION
C CALC TIME FOR DU TO GO FROM ENTERING BL REGION TILL REACHING BL
DO 209 L=1,IUPDIF
IRR=L+INTUP-1
TIME(I)=TIME(I)+DELTZ/(VT(IIRR)-ASSORT(DU(I,IRR)/RO(IIRR)))
209 CONTINUE

C ADD TIME FOR DD TO DESCEND FROM BL AND LEAVE BL REGION
DO 260 L=1,IDWDIF
ISS=L+INTDW-1
260 TIME(I)=TIME(I)+DELTZ/(-VT(ISS)-ASSORT(DD(I,ISS)/RO(ISS)))
CTime(I)=TIME(I)

C FIND TIME TO GROW FROM DU TO DD AT BL
TIMEOVI(I)=(DD(I,NNN)-DU(I,NNN))/WP(NNN)/VT(NNN)/(2.*ROI)

C TOTAL TIME IN BL REGION
C MEAN NUMBER DENSITY OF ITH SIZE IN ITH BL REGION
EN(I)=CN(I)/PI*12.*C2*TIME(I)/ZDELTA(I)
GO TO 206

299 EN(I)=0.0
C NOW DU FOR NEXT BL
206 CONTINUE
C TO GET NUMBER DENSITY AND GROWTH TIMES AFTER INJECTION
C SET T,TD,AND NUMBERS=0.0
DU 215 I=1,N
DU 215 J=1,100
T(I,J)=0.0
TD(I,J)=0.0
ENNU(I,J)=0.0
215 ENND(I,J)=0.0
DO 210 I=1,N
T(I,1)=0.0
DO211 J=1,100
C
IF(NIT(I,J).EQ.2) GO TO 250
C GROWTH TIME ON ASCENT
T(I,J+1)=T(I,J)+DELZ/(VT(J)-A*SQRT(DU(I,J)/RO(J)))/60.
211 CONTINUE
250 IF(J.EQ.100) GO TO 210
TD(I,J)=T(I,J)+TIMEOV(I)/60.
C TD IS TIME WHEN DD BEGINS DESCENT
NICE=ISTOP(I)-1
DO 212 J=1,NICE
ILL=ISTOP(I)-J
ILL2=ISTOP(I)-J+1
C TDS NOW CALCULATED FROM BLS DOWNWARD
TD(I,ILL)=TD(I,ILL2)+DELZ/(-VT(ILL2)+ASSORT(DD(I,ILL2)/KU(ILL2)))-1/60.
212 CONTINUE
C DO NEXT SIZE TIMES
210 CONTINUE
C CALCULATE NUMBERS (ENNU,ENND) AT ALL LVLS AND SIZES
DO 213 I=1,N
CNN=AN(I)*(VT(I)-A*SQRT(DU(I,1)/KU(I)))
NNN=ISTOP(I)
DO 214 J=1,NNN
ENNUI(I,J)=CNN/(VT(I,J)-A*SQRT(DU(I,J)/RO(J)))
214 CONTINUE
213 CONTINUE
C GET FORCE OF WATER+HAIL AT BLS
DO 216 I=1,N
FORCE(I)=0.0
TASS(I)=0.0
ANI(I)=0.0
IF(ISTOP(I).EQ.100) GO TO 216
DO 217 L=1,N
LL=ISTOP(I)
IF(L.EQ.1) GO TO 217
ANI(I)=ENNU(L,LL)+ENND(L,LL)+A*DU(I,LL)/KU(LL)
IFI.I.EQ.1) GO TO 217
TASS(I)=PI/6.*ROI*(ENNU(L,LL)*ROI(LL)-ENND(L,LL)*ROI(LL))*ROI(LL)+TASS(I)
217 CONTINUE
C TASS(I) IS MEAN MASS OF HAIL PER VOLUME OF CLOUD AT ITH HL
C ANI(I) IS NUMBER OF HAILSTONES PER UNIT VOLUME AT ITH HL
C ADD FORCE OF WATER TO HAIL
FORCE(I)=TASS(I)/KU(LL)-4*ANI(I)*ROI(LLL)/KU(LL)
C FORCE(I) IS FORCE PER UNIT MASS OF CLOUD AT ITH HL ON HAIL + HAIL
216 CONTINUE
C ENGR(I)=0.0
C GET THERMAL BUOYANCY FORCE AND ENERGY
DU:350 J=1,100
FS(I,J)=11.4*(Z(J)-1.7*0.5)/1.250
IFI(Z(J).GT.3.25E05) FS(I,J)=12.4-2.9*(Z(J)-3.25E05)/1.250
IFI(Z(J).LT.1.75E05) FS(I,J)=4.3+6.7*(Z(J)/1.750)
$JOB CHARLTON R B YFPP0855PRECPH 55 20 $ A+1 = DISTN.

FB(J) = FB(J) + 980.6 * 6078 * R(J) ** 2 * KUI
ENGYB(J+1) = ENGYB(J) + FB(J) * DELZ
ENGYA(J) = 0.5 * (VT(J) ** 2 - VT(1) ** 2)

GET TOTAL NO. OF STONES AT EACH LVL
DU 218 J = 1, 100
ANTOT(J) = 0.0
DU 219 I = 1, N
219 CONTINUE

GET Icing CONDITIONS (AICEU OR AICEU) AT ALL LVLS AND SIZES
DU 221 J = 1, 100
DU 222 I = 1, N
IF(DU(I, J) * LT. 0.1E-20) GO TO 223
X = Z(J)
Y(2) = DU(I, J)

WFF = WF(J)
RUA = R0(J)
CALL FCN GIVES AICE IN FRACTION FROZEN OR SFC TEMP
CALL FCN(X, Y, WFF, AICE, RUA)
AICEU(I, J) = AICE
GO TO 224

223 AICEU(I, J) = 0.0
224 IF(DD(I, J) * LT. 0.1E-20) GO TO 225
Y(2) = DD(I, J)

CALL FCN FOR DESCENDING HAIL
CALL FCN(X, Y, WFF, AICE, RUA)
AICED(I, J) = AICE
GO TO 222

225 AICED(I, J) = 0.0
222 CONTINUE
221 CONTINUE

C SILL IS INTEGRATED FORCE OF HAIL AND LWC FROM INPUT LVL
SILL = 0.0
DU 227 J = 1, 100
SM(J) = 0.0
FM(J) = 0.0
DU 228 I = 1, N
SM(J) = ROI * PI / 6.0 * (DD(I, J) ** 3 * ENNU(I, J) + DU(I, J) ** 3 * ENNU(I, J)) + SM(J)
228 CONTINUE

FM(J) IS FORCE AT EACH LVL DUE TO HAIL, NOW ADD LWC FORCE
FM(J) = FM(J) + 980.6 * WFI(J) / R0(J)
SILL = SILL - FM(J) * DELZ

CALL FCN FOR THERMAL ENERGY TO GET NEW ENGY
ENGYB(J+1) = ENGYB(J+1) + SILL
227 CONTINUE

C NOW HAVE ENERGIES BUT NEED GOOD VALUES NEAR HLS
C NTH SIZE HAS LOWEST HL
INTUP = ISTOP(N) - INT ((FLOAT(ISTOP(N)-1) - ISTUP(N)) + 0.1) / 2.0
C INTUP IS BOTTOM OF REGION REPRESENTING LOWEST HL
IF(INTUP.LT.2) INTUP = 2
C NOW HAVE ENERGY AT INTUP
ENGY(N) = ENGY(INTUP)
C GET MEAN AT HLS BY INTEGRATING MEAN FM+LWC IN HL TIMES 71+LTA
OF BL RGN.
DU 351 I = 1, N
LL = N + 1 + 1
IF(LL.EQ.N) ENGY(N) = ENGY(N) + FM(J) ** 71 * LTA(N)
IF(LL.EQ.N) GO TO 351
ENGY(LL) = ENGY(LL+1) + FORCE(LL) * DELTA(LL)

351 CONTINUE

C NOW HAVE BL ENERGIES

WRITE (6, 701) (Z(J), (TO(I,J), I=1,N), (TI(I,J), I=1,N), J=1,100,1)

701 FORMAT (1X, 1E10.2, 10F8.2)

WRITE (6, 22)

C WRITE NUMBER DENSITIES

WRITE (6, 702) (Z(J), (ENNO(I,J), I=1,N), (ENNU(I,J), I=1,N), J=1,100,1)

702 FORMAT (1X, 11E10.2)

WRITE (6, 22)

C WRITE, BL STEP, MEAN NUMBER OF THOSE STONES REACHING BL,
C TOTAL NUMBER AT BL, HAIL MASS PER UNIT VUL, FORCE OF HAIL AND WATER,
C TOTAL BUOY ENERGY AT ALL BLs

WRITE (6, 703) (ISTOP(I), EN(I), ANI(I), TIMEOVI(I), ZDELT(A(I), TASS(I),
  IFORCE(I), ENGY(I), I=1,N)

703 FORMAT (1X, 1I3, 2X, 7E11.3)

WRITE (6, 22)

C WRITE ICING CONDITIONS

WRITE (6, 704) (Z(J), (AICEQD(I,J), I=1,N), (AICEU(I,J), I=1,N), J=1,100,5)

704 FORMAT (1X, 1E10.2, 10F7.2)

C WRITE HAILMASS, NUMBER, FORCE OF HAIL AND HAIL, WATER, AND THERMAL BUOY

C TOTAL BUOY ENERGY, KINETIC ENERGY, ALL AT ALL LVLS.

WRITE (6, 705) (Z(J), SM(J), ANTUT(J), FM(J), FB(J),
  ENGYB(J), ENGYA(J), J=1,100)

705 FORMAT (1X, 7E10.2)

C LEADS TO READING NEW EXAMPLE AT STATEMENT 99

GU TO 197

C END OF MAIN PROGRAM

END

SUBROUTINE FOR ICING CONDITIONS

C AS FUNCTION OF HEIGHT X, DIAMETER Y(2), LWC WFF, AND AIR DENSITY RUA.

SUBROUTINE FCN (X, Y(2), LWC, WFF, AICE, RUA)

DIMENSION Y(3)

C GAS CONSTANT
R = 2.8705E+03

C ICE DENSITY
KUL = 0.915

C ROUGHNESS FACTOR
THETA = 1.0/0.675

C COLLECTION EFFICIENCY
E = 1.0

C MOLECULAR WT. RATIO
EPS = 0.62197

PI = 3.141593

KUL = RUA

C SET AXIS RATIO = 1, THIS SUBROUTINE COULD BE MODIFIED FOR SPHERICS
ALFA = 1.0

C CON = (20.4174 - 1E+05) / 0.775284

C P IS PRESSURE
P = 3.5211E+05

C Y(3) IS CLOUD TEMP
Y(3) = P/R/NO

C THERMAL CONDUCTIVITY OF AIR
AK = (5.8 + 0.0175 * (Y(3) - 273.15)) * 1.0E-05

C DIFFUSIVITY OF WATER IN AIR
DWA = 9.02E-03 * Y(3) ** 1.0/P

C KINEMATIC VISCOSITY

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SJUB CHARLTON R 6 YFPP0855PRECPH---------------^--------20

A N U = D W A * 5.54 / 9.02

C CONSTANT OF FRACTIONAL ICINGS EFFECT ON LATENT HEATS

C 12 = 0.207 - 0.0014*(Y(3)-273.15)

C E (C 12 . GT .0.235) C12=0.235

LATENT HEAT OF VAPORIZATION

ALV = 597.31 - 0.61 * (Y(3) - 273.15)

SAT. VAPOR PRESSURE OVER WATER

EW = 10.0 * (24.0068 - 2956.3 / Y(3) - 5.0887 * ALUG10(Y(3)))

SAT MIX RATIO

KPS=EPS*EW /(P-EW)

Y(1) IS SUM OF LIQUID AND VAPOR WATER

Y(1)=WPS+WFF/RO

TERMINAL VELOCITY

VT = 48.8 * SQRT (ABS (Y(2)) / RO) * (0.8*ALFA+0.2)

CONSTANT Y  OF APPENDIX 3

Y Y = 5.483 * ANU ** 0.5 / RO ** 0.25*E / THETA*ABS (Y(2))**0.75*<Y(I)-RPS)*RO

l / (0.376 * ALFA + 0.624)

I TTERAT ION COUNTERS

NCC = 1

NC= 1

1ST ESTIMATE OF SFC TEMP.

TD = TDQ

NNN= 1

FIRST ESTIMATE OF DIFF. TWIX AIR AND STONE TEMP

DEL T=(Y(3)-273.15) / 4.0

SPECIFIC HEAT OF WATER


SAT. VAPOR PRESSURE OVER ICE

EI = 10.0 * 0.07387 - 2488.9 / TD + 3.5656 * ALUG10(TO)

LATENT HEAT OF FUSION

AF = 80.19 + 0.16*(TD-273.15)

IF NC = 2 » AI = 1

I F(NC •EQ .2) GU TO 9

FRACTIONAL ICING.

AI = (1.68*AK +YY*CW) *(TD-Y(3))+(EI-EW)*C12*DWA/Y(3))/(YY*APL)

NO FREEZING

___ IF (AI .LT .0.0 ) GO TO 15 ___

D R Y G ROWTH

IF(AI.GT.1.0) GO TO 7

48.8 * SQRT (ABS (Y(2)) / RO) * (0.8*ALFA+0.2)

IF ((TD - TDO) .GT .0.0) GO Tu 99

TRY A COLDER TDU TILL TO IS WARMER

TO = TDO + DELT
$JOB CHARLTON R B YKPPOB55PRECPH 50 20 $ A+O = DISTN.

TDU=TD
C ANOTHER ITERATION
GO TO 11
C NOW TD IS WARMER THAN TDU SO SET TDU WARMER BY -DELT/2
99 DELET=DELT/2.0
C ALSO HALF DELT FOR FUTURE ESTIMATES WHEN TD BECOMES LT TDU
TD=TDU-DELT
TDU=TD
GO TO 11

15 TD=273.15
NNN=2
A1=0.0
8 CONTINUE
C AICE IS FRACTION ICING OR SFC TEMP. IN DEG C
IF(A1.LT.0.9999E+00) AICE=A1
IF(A1.LT.0.9999E+00)GO TO 100
AICE=TD=273.15
100 CONTINUE
RETURN
END

$DATA

| 0.1800E 04 | 0.1750E-02 | 5 |
| 0.46      | 0.48      | 0.50 | 0.52 | 0.54 |
| 0.00E-06  | 0.00E-06  | 0.00E-06 | 0.00E-06 | 0.00E-06 |
| 0.1800E 4  | 0.1750E-02 | 5 |
| 0.46      | 0.48      | 0.50 | 0.52 | 0.54 |
| 0.20E-06  | 0.20E-06  | 0.20E-06 | 0.20E-06 | 0.20E-06 |
| 0.1800E 4  | 0.1750E-02 | 5 |
| 0.30      | 0.40      | 0.50 | 0.60 | 0.70 |
| 0.20E-06  | 0.20E-06  | 0.20E-06 | 0.20E-06 | 0.20E-06 |
| 0.1800E 4  | 0.1750E-02 | 5 |
| 0.10      | 0.30      | 0.50 | 0.70 | 0.90 |
| 0.20E-06  | 0.20E-06  | 0.20E-06 | 0.20E-06 | 0.20E-06 |
| 0.1800E 4  | 0.1750E-02 | 5 |
| 0.30      | 0.40      | 0.50 | 0.60 | 0.70 |
| 0.00E-06  | 0.00E-06  | 0.00E-06 | 0.00E-06 | 0.00E-06 |
| 0.1800E 4  | 0.1750E-02 | 5 |
| 0.30      | 0.40      | 0.50 | 0.60 | 0.70 |
| 0.20E-06  | 0.20E-06  | 1.00E-06 | 0.60E-06 | 0.20E-06 |
| 0.1800E 4  | 0.1750E-02 | 5 |
| 0.10      | 0.20      | 0.30 | 0.40 | 0.50 |
| 2.00E-06  | 1.00E-06  | 0.25E-06 | 0.15E-06 | 0.10E-06 |
FIGURES 1 - 26

The caption for each figure includes the page, in brackets, on which each aspect of the figure is discussed in the body of this thesis.
Cloud model parameters as functions of height $Z$ above freezing level. Updraft air density $\rho$ and saturation mixing ratio $r_s$ show linear approximations (pp. 14).
Thermal buoyant force on cloud air (pp. 16).
Balance level diameters $D$ of spheres for various input level updraft velocities $V_{z0}$ as functions of height $Z$ (pp. 18).
Increase of kinetic energy KE with height for various updrafts $V_{z0}$ and increase of potential buoyant energy BE with height when no hailstones are injected (pp. 37).
Figs. 4a & 4b: Hailgrowth and liquid water curves for growth on ascent. (pp. 56, 58, 88, 92). Variable is embryo diameter $D_e$. Surface icing conditions (pp. 62), growth times (pp. 61) and, buoyant force and energy balances (pp. 66), are shown in Fig. 4a.
**Figs. 5a & 5b**: Like Figs. 4a and 4b but with variable embryo concentration $N_0$. (pp. 56, 59, 88, 89)

**Fig. 5c**: Buoyant force with and without thermal feedback. (pp. 63-67).

**Fig. 5d**: Potential buoyant energy with thermal feedback BE compared to increase of air parcel's kinetic energy KE. (pp. 63).
Hailstone concentration $N$ for various embryo concentrations (pp. 59).
Figs. 6a & 6b: Like Figs. 4a & 4b but with variable updraft velocity $V_{20}$. (pp. 56, 60).

Fig. 6c: Buoyant force on cloud parcels $FB$ for variable $V_{20}$. (pp. 63).

Fig. 6d: Buoyant $BE$ and kinetic energies $KE$ like 5d but with variable $V_{20}$. 

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$6a$ $6b$ $6c$ $6d$
Figs. 7a & 7b: Growth curves and liquid water contents like Figs. 4a & 4b but with variable input level liquid water $W_{fo}$ (pp. 56, 60).
Figs. 8a & 8b: Hailgrowth and liquid water curves like Figs. 4a & 4b but with $D_0=0.25$ cm, $V_{Z0}=20$ m/s and variable embryo concentration $N_0$ (pp. 56, 59).

Fig. 8c: Buoyant forces, like Fig. 5c but with $D_0=.25$ cm and $V_{Z0}=20$ m/s (pp. 63, 67).

Fig. 8d: Buoyant BE and kinetic KE energies like Fig. 5d but with $D_0=.25$ cm and $V_{Z0}=20$ m/s (pp. 63).
Figs. 9a & 9b: Growth and liquid water curves with $D_0 = 0.25\, \text{cm}$, $V_0 = 18\, \text{m/s}$, $N_0 = 5\, \text{m}^3$, and variable $W_{fo}$. Like Figs. 4a & 4b (pp. 56, 60).

Fig. 9c: Buoyant forces like Fig. 5d (pp. 63-67).

Fig. 9d: Potential BE and kinetic KE energies. Like Fig. 5d. (pp. 63)
Figs. 10a and 10b: Hailgrowth and liquid water curves for spheroidal hailstones of various axis ratios $a$. Compare to Figs. 4a & 4b which also have $D_0=0.5\text{cm}$, $V_{z0}=22\text{m/s}$, and $N_0=1/\text{m}^3$. (pp. 71, 73).
**Figs. 11a & 11b:** Hailgrowth and liquid water curves for spheroidal hailstones. Like Figs. 10a and 10b but with $V_z = 18 \text{ m/s}$ and $N_0 = 0$ or $2 \text{ m}^3$. (pp. 71, 73).

**Fig. 11d:** Potential buoyant energy $BE$ for various axis ratios $\alpha$ compared to kinetic energy $KE$ of updraft. Like Fig. 5d (pp. 73).
Figs. 12a & 12b: Hailgrowth and liquid water curves for spheroidal hailstones where input diameter $D_0 = 0.25$ cm. Like Fig. 11a and 11b but with $V_0 = 15$ m/s and $N_0 = 0.2$ or $10$/m$^3$. (pp. 71, 73).
Growth of 0.5 cm embryos with axis ratio \( \alpha \) increasing with major dimension D. Growth and liquid water curves are given for examples with variable updraft \( V_{z0} \) and constant \( N_o = 1/m^3 \) (Figs. 13) and with variable embryo concentration \( N_o \) and constant updraft \( V_{z0} = 18 m/s \) (Figs. 14). (pp. 76-78).
Figs. 15a, 15b: Growth of 0.25cm embryos with axis ratio $\alpha$ increasing with major dimension $D$. Growth and liquid water curves are given for examples with variable updraft $V_{20}$ and constant $N_0=2/m^3$ (Figs. 15) and with variable embryo concentration $N_0$ and constant updraft $V_{20}=17m/s$ (pp. 76-78).
Hailgrowth and liquid water curves for ascent with descent (A&D) growth where $D_0 = 0.5\, \text{cm}$, $V = 18\, \text{m/s}$ (Figs. 17) or $20\, \text{m/s}$ (Figs. 18) and the embryo concentration $N_0$ is varied. (pp. 80, 102). Figs. 17a and 17b also give hail surface icing conditions $I$, $T$, (pp. 84), growth times $t$ (pp. 83), and buoyant force $FB$ and energy balances $BE=KE$ (pp. 82).
Figs. 19a and 19b: A&D growth of 0.5 cm embryos and liquid water curves for various embryo concentrations $N_o$. Figs. 19 are like Figs. 17 and 18 but with $V_{Z_0} = 22 m/s$. 

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Various output quantities of A&D growth as functions of 

\( D_0 \) and \( V_{Z0} \). \( D^* \) is maximum output diameter when maximum embryo number \( N_M \) is injected and \( D_{f0} \) is output diameter when \( N_o=0 \) (See solid and 'crossed' lines). Dashed lines are lines of constant \( N_M \), liquid water at balance level 

\( r_o \) when \( N_o=N_M \), balance level heights \( Z_{BM} \) and \( Z_{bo} \) when \( N_o =N_M \) or \( 0/m^3 \), and growth times \( t_n \) and \( t_o \) when \( N_o=N_M \) and \( 0/m^3 \). (pp. 35, 85, 93).
Figs. 21a & 21b: Hailgrowth on ascent and liquid water curves for various embryo size spreads. Central embryo diameter is 0.5 cm, $V_{Z0} = 22$ m/s, and total embryo number $N_T = 1/m^3$. (pp. 90)

Fig. 21d: Potential buoyant energy $BE$ compared to cloud parcel kinetic energy $KE$ (pp. 91).
Figs. 22a, 22b & 22d: Hail growth (22a), liquid water (22b) and energy (22d) curves for growth on ascent of three embryo size spreads. Same as Figs. 21 except $N_T = 2/m^3$, i.e. twice as many embryos are injected (pp. 90, 91)
Figs. 23a & 23b: Hailgrowth (Fig. 23a) and liquid water (Fig. 23b) curves of a size spread of embryos centered on $D_0=0.5$ cm. Values for 'ascent only' and 'A&D' growth are given. $V_0=1.8$m/s and total input number $N_T=0$ or $1/m^3$ (pp. 93).

Fig. 23d: Potential buoyant energy BE compared to cloud parcel kinetic energy KE for 'ascent only' and 'A&D' growth (pp. 94).
Fig. 23e

Smoothed number distributions at various heights for 'ascent only' and 'A&D' growth.

Heights are at balance levels (pp. 95).

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Figs. 24a & 24b: Hailgrowth and liquid water curves for example of Figs. 23 with wider and smaller embryo size spreads. (pp. 93, 95).

Figs. 24c: Comparing hydrometeor forces to thermal buoyant forces and to force of liquid water when no depletion by hail occurs (pp. 96).

Fig. 24d: Potential buoyant energy is compared to updraft kinetic energy (pp. 96).

Fig. 24e: Total number of hailstones for 'ascent only' and 'A&D' growth with wide or small embryo size spread. (pp. 97)
Hailgrowth curves of widespread A&D case showing surface icing conditions $I$, $T_d$, and growth times. $I$ is fractional icing and $T_d$ is surface deposit temperature in °C. (pp. 97).
Growth curves and cloud parameters for an example where a 'Gaussian' type size distribution of embryos is injected into the updraft. Fig. 25a shows growth curves for 'ascend only' and 'A&D' cases. The other figures are similar to those of Figs. 24. Fig. 25e is continued. Note that the total embryo number $N_m = 3/m^3$ in this example and that the greatest number of embryos is at $D_0 = 0.5 cm$. (pp. 99, 100)
Smoothed hailstone number density distributions at various heights, mostly at balance levels (pp. 100, 102).
Figs. 26a - 26e: An example where the large embryos are in low number while the small ones are numerous. Both 'ascent only' and 'A&D' cases are given. The description of the figures is similar to Figs. 24 or 25. (pp. 102-104).