ON THE INTERACTION OF BAROCLINIC INSTABILITY
AND CUMULUS CONVECTION IN POLAR AIR STREAM CYCLONES

by

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ABSTRACT

A linearized stability analysis of some idealized and observed atmospheric jet flows has been performed. The semigeostrophic approximation was used and a cumulus parameterization based on low-level moisture convergence included. Calculations using idealized basic states showed the nature of the instability i.e. whether it is dominated by baroclinic instability or by release of latent heat through conditional instability of the second kind, is determined by the constant of proportionality between heating and convergence. For disturbances dominated by baroclinic processes, the effects of diabatic heating are equivalent to a reduction in static stability. It is possible for latent heat to be the dominant source of disturbance energy even when the structure is that of a baroclinic wave, although in this case the mode will have significant amplitude only in the region where heat is released. The effects of heating were found to be greater in regions where the background static stability was small. Thus the low vertical stability found on the cold-air side of the jets where comma clouds typically form gives an environment where heating is especially effective in producing small scale disturbances. Calculations of the instabilities of two observed jets showed similar behavior. The wavelength and structure of the modes agreed well with those of the comma clouds which were observed to form.
ACKNOWLEDGEMENTS

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This work is dedicated to my parents, Alan and Grace Craig, who made the person who made the thesis.
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CHAPTER 1

Introduction and One-Dimensional Model

1.1 Introduction

In recent years there has been considerable interest in the synoptic and mesoscale meteorology of weather systems in which cumulus convection plays an important or even dominant role. Among these systems are the small scale cyclones that form in cold air masses poleward of the main frontal zone. In this thesis we will refer to these collectively as polar air stream cyclones. The term encompasses the systems referred to in the literature as polar lows, comma clouds, arctic lows, and polar air depressions. These disturbances range in size from a few hundred kilometers to over 1000 km in diameter. They tend to be rapidly developing but vary considerably in maximum intensity. The wide variety of structures has led to the proposal of several different mechanisms as being responsible for their formation. The most frequently discussed are baroclinic instability and conditional instability of the second kind (CISK), where disturbance growth is driven by release of latent heat in cumulus convection.

In this thesis the interaction between baroclinic instability and convective processes will be examined, using linear models which include both mechanisms. An initial study, summarized in this chapter, showed a range of
disturbances could occur, from purely baroclinic instability to conditional instability of the second kind. This range, including intermediate types, is observed among polar air stream cyclones (Businger, 1987). The nature of the system is determined by conditions in the environment where it forms. The main body of this work describes an investigation of how the environmental properties influence the two instability mechanisms in a common type of polar air stream cyclogenesis. This will involve the analysis of an idealized model followed by consideration of two case studies.

The remainder of this chapter is divided into two sections. The first contains a brief review of observational and theoretical studies of polar air stream cyclones. The second section describes a simple model where both baroclinic instability and cumulus heating can occur. The results of this model are then compared to some observed weather systems. Subsequent chapters describe models based on the same physics but with more realistic initial conditions.

1.2 Polar Air Stream Cyclones

a) Observational Studies

Due to their small size and formation over the data-sparse oceans, polar air stream cyclones have proven difficult to study and little work was done until the introduction of satellite imaging (Businger and Reed, 1989).
In recent years, improved observational data sets and field experiments such as the Arctic Cyclones Expedition (Shapiro et al., 1987) have stimulated a great deal of research. Rather than attempt a comprehensive review of this literature, only a few important investigations will be described here. More complete reviews are given by Rasmussen (1983, 1989), and Businger and Reed (1989).

The study by Reed (1979) described a number of small synoptic or subsynoptic disturbances which formed over the eastern Pacific Ocean. These systems resembled the usual synoptic scale frontal cyclones, except in their smaller size, with cloud patterns less than 1000 km across, and being spaced at intervals of 1000 - 1500 km when occurring in multiple form. They occurred most often over the ocean in winter months on the poleward side of the jet stream. The region of cyclogenesis thus featured substantial baroclinicity and cyclonic vorticity. Additionally there tended to be reduced vertical stability in the lower levels of the atmosphere due to surface fluxes, and conditional instability through a substantial depth of the troposphere. Reed and Blier (1986a) note that the system generally first appears as a patch of convective cloud, probably associated with an upper level vorticity maximum although the data is not always sufficient to detect this feature. The surface development is often weak in comparison with synoptic-scale cyclones, although the more intense cases show pronounced surface lows and often a trough with many features of a cold
front (Locatelli et al., 1982).

The systems studied by Reed (1979) are referred to as comma clouds in the literature although this is perhaps an unfortunate name, since most frontal cyclones also produce a comma-shaped cloud pattern. Reed concluded that these disturbances were baroclinic in nature but that conditional instability of the second kind and barotropic instability were possible additional influences. Climatological studies suggest that comma clouds are the most common type of polar air stream cyclone, accounting for approximately half of the observed systems (Forbes and Lottes, 1985; Carleton, 1985; Carleton and Carpenter, 1989).

Rasmussen (1979) also studied a number of cases, this time over the north-eastern Atlantic Ocean. The systems he described were quite small in size and formed well to the north of the jet stream baroclinic zone. The cloud patterns often appeared as isolated spiraliform vortices, resembling tropical cyclones. These systems are commonly called polar lows although the term has also been applied to other types of polar air stream cyclones. Rasmussen proposed that these polar lows could result from the same process as had been suggested by Charney and Eliassen (1964) and Ooyama (1964) for tropical cyclones, Conditional Instability of the Second Kind (CISK). In this theory, the circulation is driven by release of latent heat in cumulus convection. The heat causes rising motion which produces convergence and lifting
of moist air in the lower levels, driving further convection. The result is an almost circular disturbance with spiral bands of convective cloud. Some polar lows even feature a clear "eye" as found in tropical cyclones (Rasmussen, 1981). The environments where this type of system form are similar to regions of comma cloud cyclogenesis in that they are characterized by low static stability and high positive vorticity, as well as large surface fluxes of latent and sensible heat (Businger, 1985, 1987). These systems are found most frequently in at very high latitudes near the polar ice edge (Carleton, 1985; Carleton and Carpenter, 1989).

As early as the work of Locatelli et al. (1982) it was realized that the studies of Reed (1979) and Rasmussen (1979) treated different types of weather systems. The classification scheme of Businger and Reed (1989) is typical of recent attempts to identify distinct types of polar air stream cyclones in that it includes a short wave/jet streak type, which describes the systems documented by Reed (1979), as well as a cold-low type which encompasses the more convective disturbances of Rasmussen (1979). An additional class defined by Businger and Reed (1989) is the arctic front type which includes small low-level systems which occur in shallow baroclinic zones often resulting from differential surface heating near the polar ice edge. These systems are sometimes described as reversed-shear lows since the frontal zones where they form often have wind speeds at
a maximum at the surface and decreasing with height. Furthermore, Businger and Reed note that disturbances are frequently observed that are intermediate in character between the principal types.

b) Theoretical Studies

Most of the recent attempts to explain cyclogenesis in polar air streams have focused on two mechanisms, baroclinic instability and CISK. Early studies (Mansfield, 1974; Duncan, 1977, 1978) used conventional quasi-geostrophic models of baroclinic instability. Disturbances with short wavelengths were produced in these models by assuming the static stability in the lowest levels to be much reduced in comparison to values in the mid and upper troposphere. More recent models (Moore and Peltier, 1987; Schär and Davies, 1990; Tully and Thorpe, 1990) using more sophisticated physics have also found short wave disturbances on frontal zones with regions of reduced Richardson number or potential vorticity in low levels. Moore and Peltier (1989) computed the stability of an observed arctic front (Reed and Duncan, 1987) and found good agreement in wavelength and growth rate between the fastest growing mode and the observed wavetrain of disturbances. It was noted by Reed (1979) however, that these shallow modes do not agree so well with the larger deeper systems that he described.

The CISK theory was applied to polar lows by Rasmussen (1979), who used a quasi-geostrophic model with the simple
cumulus parameterization introduced by Charney and Eliassen (1964) and Ooyama (1964) which assumes heating in a column to be directly proportional to convergence of moisture in the lowest model level. Reasonable agreement was obtained with observed system sizes and growth rates, but only after boundary layer moisture content was adjusted to take into account surface fluxes of heat and moisture. If the system were forced to rely on convergence of water vapor already in the atmosphere, it would grow slowly or not at all. The potential of polar and tropical air masses to support CISK was compared by Wang (1987a) who found that although the atmospheric moisture content was much lower in polar regions than in the tropical atmosphere, this was offset by reduced stability and lower cloud tops. This latter factor serves to confine heating to the lower levels where it is more efficient in producing convergence and further convection.

An alternative to the CISK theory, which does not use a convergence-based cumulus parameterization was proposed by Emanuel (1986). It was suggested that convective clouds serve to rapidly adjust the atmosphere to a state of moist symmetric neutrality. The atmosphere thus contains no convective available potential energy to drive the growth of a weather system. The growth comes solely through surface fluxes of heat and moisture, giving rise to an air-sea interaction instability. The idea was applied to polar lows by Emanuel and Rotunno (1989) who carried out numerical simulations of an idealized system with an axisymmetric,
cloud-resolving model. While this theory is conceptually different from CISK in its treatment of cumulus convection, it is not clear that they are mutually exclusive since conventional CISK models will not produce instability unless surface fluxes are taken into account (Rasmussen, 1979; Craig and Cho, 1988). It is an advantage of the air-sea interaction hypothesis that this constraint is an intrinsic part of the theory rather than a result of appropriately chosen parameter values. The convergence-based parameterization, on the other hand, provides an adequate representation of cumulus effects in a simple mathematical form which is particularly convenient for use in linearized models.

While the calculations described above each produce good results for a certain class of disturbance, it is clear that models including more than one instability mechanism are required to account for the range of observed systems. One of the first studies of this type was by Mak (1982) who modified the classic Eady model of baroclinic instability (Gill, 1982) to include a parameterization of cumulus heating similar to that used in the CISK models above. It was found that as the boundary layer moisture content and thus the heating rate increased, the wavelength of the model disturbance was decreased and the growth rate increased. These trends did not continue for large heating rates, however. It will be shown later in this chapter that the change in behavior is associated with a transition from
baroclinic instability to CISK. Later calculations by Wang and Barcilon (1986) and Bannon (1986) confirmed the ability of heating to enhance baroclinic instability, but CISK was not identified.

A combined baroclinic-CISK model was applied to polar air stream cyclogenesis by Sardie and Warner (1983) who added Ekman friction and parameterizations for release of latent heat in both cumulus convection and stable ascent to a two-layer quasi-geostrophic model. The results of the calculation for various combinations of model physics were compared with several observed case studies and in most cases reasonable agreement was found for some combination. In some cases however the model equations became invalid, probably due to the unrealistic representation of the vertical distribution of latent heat release in a two-layer atmosphere (Pedersen and Rasmussen, 1985; Wang, 1987a).

Numerical modelling of polar air stream cyclones is difficult due to the lack of data with adequate resolution to effectively initialize the models and to verify the results. However, reasonably successful simulations of particular systems have been performed recently by Blier (1989), Grønas et al. (1987), and others.

1.3 The One-Dimensional Model

The remainder of this chapter is devoted to a summary of a first investigation of the interaction of baroclinic
instability and CISK. This work has been described in detail elsewhere (Craig and Cho, 1988) and only a brief summary will be included here. We examine a minimal model similar to those of Mak (1982), Wang and Barcilon (1986), and Bannon (1986), which includes both mechanisms. The primary purpose is to identify the types of instability that will occur when both baroclinicity and cumulus convection are present, and to determine through comparison with case studies whether these types correspond to different types of observed weather systems. This will provide a background for work in subsequent chapters where we will consider in greater detail how the environment where the system forms influences this interaction.

a) Formulation

We make use of the semigeostrophic Eady model and include a simple parameterization of cumulus heating. The Eady model considers perturbations to a basic state with constant static stability, constant vertical wind shear and a rigid lid at the upper boundary. The amount of cumulus convection is assumed to be controlled by low-level convergence of moisture. If it is then assumed that static stability, moisture content and other properties of the ambient atmosphere do not change too rapidly, the heating rate is simply proportional to convergence below a fixed cloud base level. The heating is distributed vertically according to a given function, resulting in the following
form for the heating rate $E = \frac{d\theta}{dt}$:

$$E = \left[ \frac{\theta_o Q}{g f} \right] \varepsilon h(z) w(z_B)$$  \hspace{1cm} (1.1)

where $\varepsilon$ is a constant of proportionality which will be referred to as the heating parameter, $h(z)$ is the normalized vertical heating profile depicted in Fig. 1.1, and $w(z_B)$ is the vertical velocity at cloud base level $z_B$. The group of constants in square brackets serves to nondimensionalize $\varepsilon$. $Q$ is the potential vorticity which is a measure of stability in the semigeostrophic system, $\theta_o$ is a fixed reference value of potential temperature, $g$ is the gravitational acceleration and $f$ is the Coriolis parameter. Using this representation for the diabatic heating and writing in terms of geostrophic coordinates $X$, $Y$, $Z$, and $T$ (Hoskins, 1975), the linearized governing equation is

$$\left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \left[ \frac{1}{f^2} \left( \frac{\partial^2 \Phi'}{\partial X^2} + \frac{\partial^2 \Phi'}{\partial Y^2} \right) + \frac{f}{Q} \frac{\partial^2 \Phi'}{\partial Z^2} \right]$$

$$= - \frac{f}{Q} \varepsilon h_z (Z) \left[ \left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \frac{\partial \Phi'}{\partial Z} - U_z \frac{\partial \Phi'}{\partial X} \right]_{Z=Z_B}$$

$$= - \frac{f}{Q} \varepsilon h_z (Z) \left[ \left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \frac{\partial \Phi'}{\partial Z} - U_z \frac{\partial \Phi'}{\partial X} \right]_{Z=Z_B}$$

$$= - \frac{f}{Q} \varepsilon h_z (Z) \left[ \left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \frac{\partial \Phi'}{\partial Z} - U_z \frac{\partial \Phi'}{\partial X} \right]_{Z=Z_B}$$

$$= - \frac{f}{Q} \varepsilon h_z (Z) \left[ \left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \frac{\partial \Phi'}{\partial Z} - U_z \frac{\partial \Phi'}{\partial X} \right]_{Z=Z_B}$$

$$= - \frac{f}{Q} \varepsilon h_z (Z) \left[ \left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \frac{\partial \Phi'}{\partial Z} - U_z \frac{\partial \Phi'}{\partial X} \right]_{Z=Z_B}$$

$$= - \frac{f}{Q} \varepsilon h_z (Z) \left[ \left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \frac{\partial \Phi'}{\partial Z} - U_z \frac{\partial \Phi'}{\partial X} \right]_{Z=Z_B}$$

where $\Phi'$ is perturbation geopotential, $U$ is the basic state wind, and $U_z$ is the basic state wind shear. This equation and its boundary conditions define an eigenvalue problem which is solved according to the method described by Craig and Cho (1988).

b) Results
Figure 1.1. The prescribed vertical distribution of heating. Except where noted we will use the values 
$$(z_B, z_L, z_T) = (.1H, .4H, .7H)$$. $h(z)$ is normalized so that its integral from 0 to $H$ is unity.
The behavior of the solutions is summarized in Fig. 1.2, which shows the evolution of the wavenumber of the fastest growing mode as the heating parameter $\epsilon$ is varied. The vertical heating profile was as shown in Fig. 1.1 ($z_B = .1H$, $z_L = .4H$, $z_T = .7H$). The plot can be divided into two regions: for small values of $\epsilon$ there is a single maximum with wavenumber increasing (wavelength decreasing) as heating increases, while for larger values of $\epsilon$ the trend in wavenumber reverses and two secondary maxima appear. These local maxima correspond to waves with much smaller growth rates than the main branch and will not be discussed further.

The nature of the disturbance in these two regions is apparent in the vertical structure of the perturbation geopotential fields. Fig. 1.3a (for $\epsilon = 0.3$) is typical of the structure of disturbances with values of the heating parameter in the lower region, and is dominated by a westward-tilting trough, indicative of baroclinic energy conversion. In fact, the structure is almost identical to that of a purely baroclinic wave in the Eady model (eg. Gill, 1982; Fig. 13.4). The disturbance shown in Fig. 1.3a can thus be characterized as being caused predominantly by baroclinic instability. For values of $\epsilon$ in the upper region the geopotential field (eg. Fig. 1.3b, for $\epsilon = 2.1$) shows the typical features of a CISK mode: a low-level cyclone (negative geopotential anomaly) beneath an upper-level anticyclone (positive geopotential anomaly). Unlike the more
Figure 1.2. Summary of instability characteristics for the combined baroclinic-CISK model. Solid lines indicate wavenumber of fastest growing normal mode as a function of heating parameter $\varepsilon$. Dashed lines are boundaries of the unstable region.
Figure 1.3. X-Z cross-sections of perturbation geopotential for the fastest growing wave with a) $\varepsilon = 0.3$ and b) $\varepsilon = 2.1$. 
baroclinic type of waves, the amplitude is greatest in the heating region in the lower and mid-troposphere, which tends to confirm the dominant role of CISK for this disturbance.

We see then that both CISK and baroclinic instability can occur depending on the amount of heating. The occurrence of CISK only for sufficiently high heating rates corresponds to the behavior of CISK models without baroclinicity where instability is possible only if the heating exceeds a certain threshold value (Craig and Cho, 1988). In the present case, while a threshold level could be defined in the vicinity of $\varepsilon = 1.5$ (for the heating profile in Fig. 1.1), there is a smooth transition between the two types of behavior as $\varepsilon$ is varied, giving rise to the possibility of a range of intermediate disturbances where the two mechanisms cooperate. Even well into the CISK region (Fig. 1.3b) the structure of the disturbance shows traces of the westward-tilting trough of the baroclinic wave.

c) Comparison with Case Studies

In order to explore how the model results correspond to observed systems, estimates of the nondimensional wavenumber $m$ and heating parameter $\varepsilon$ were calculated for six systems which have been previously documented in the literature. Three of these systems were designated by the authors of the original studies as comma clouds (Reed and Blier, 1986a,b; the land case of Wakimoto and Durkee, 1987), and three as the more convective type of polar low (Rasmussen, 1985;
Rabbe, 1987; Shapiro et al., 1987). The following formula derived by Craig and Cho (1988) was used to evaluate $\varepsilon$,

$$\varepsilon = \frac{g L_c}{\theta_o c_p} \left[ \frac{1}{H} \right] \left[ \frac{f}{Q} \right] q_m (1 + r + \eta) (1 - b) \quad (1.3)$$

The interpretation of this relationship is that the heating induced by a given boundary layer convergence rate is determined by the moisture content of the sub-cloud layer $q_m$, with corrections for surface fluxes of moisture, water vapour going to moisten the air column rather than being condensed out, and moisture entrained in the cloud layer, denoted by $r$, $b$ and $\eta$, respectively. $L_c$ is the latent heat of condensation for water and $c_p$ the specific heat capacity of air at constant pressure. $\varepsilon$ is also influenced by the static stability, as measured by $(Q/f)^{1/2}$, and by the depth of the troposphere, $H$. The values of these parameters for the six case studies are listed in Table 1.1. A detailed discussion of the data and methods used to obtain these estimates and of the accuracy of the results is given by Craig and Cho (1988). The uncertainty in the final values of $\varepsilon$ is typically on the order of $\pm 50\%$.

The estimates of the heating parameter and observed wavenumber for the six cases are plotted in Fig. 1.4 along with curves showing model predictions for a variety of vertical heating profiles. Two of the polar lows (Rasmussen, 1985; and Rabbe, 1987) can be fairly unambiguously identified as the results of CISK, while two of the comma clouds (Reed and Blier, 1986a,b) appear to
Table 1.1. Estimates of parameters used to calculate $\varepsilon$ according to Eq. (1.3) for six case studies of polar lows and comma clouds. The resulting value of $\varepsilon$ and an estimate of the wavelength of the observed systems are given in the final two columns. The quantity $(Q/f)^{1/2}$ is listed in units of $10^{-3}$ s$^{-1}$, $H$ and $\lambda$ are in km, and $q_m$ is in (g kg$^{-1}$).

<table>
<thead>
<tr>
<th>$(Q/f)^{1/2}$</th>
<th>$H$</th>
<th>$q_m$</th>
<th>$r$</th>
<th>$\eta$</th>
<th>$b$</th>
<th>$\varepsilon$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rasmussen (1985)</td>
<td>5.5</td>
<td>7.0</td>
<td>2.0</td>
<td>1.4</td>
<td>0.7</td>
<td>0.16</td>
<td>2.0</td>
</tr>
<tr>
<td>Rabbe (1987)</td>
<td>5.5</td>
<td>8.0</td>
<td>4.5</td>
<td>0.7</td>
<td>0.6</td>
<td>0.07</td>
<td>2.7</td>
</tr>
<tr>
<td>Shapiro et al.  (1987)</td>
<td>7.5</td>
<td>7.0</td>
<td>2.3</td>
<td>1.2</td>
<td>0.8</td>
<td>0.16</td>
<td>1.1</td>
</tr>
<tr>
<td>Wakimoto and Durkee (1987)</td>
<td>10.8</td>
<td>9.0</td>
<td>11.0</td>
<td>0.0</td>
<td>1.1</td>
<td>0.28</td>
<td>1.1</td>
</tr>
<tr>
<td>Reed and Blier (1986b)</td>
<td>11.3</td>
<td>9.0</td>
<td>5.0</td>
<td>1.3</td>
<td>0.8</td>
<td>0.15</td>
<td>0.9</td>
</tr>
<tr>
<td>Reed and Blier (1986b)</td>
<td>10.5</td>
<td>8.0</td>
<td>4.0</td>
<td>1.1</td>
<td>0.8</td>
<td>0.15</td>
<td>0.7</td>
</tr>
</tbody>
</table>
Figure 1.4. Observed values of wavenumber and heating parameter for the polar lows studied by i) Rasmussen (1985), ii) Rabbe (1987), and iii) Shapiro et al. (1987), and the comma clouds studied by iv) Wakimoto and Durkee (1987), v) Reed and Blier (1986b) and vi) Reed and Blier (1986a). Error bars give an estimate of the uncertainty in values derived from observations, as discussed by Craig and Cho (1988). Also shown are curves of the predicted wavenumber of the fastest growing mode as a function of $\varepsilon$ for several levels of maximum heating $z_L$. 

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correspond to short wavelength baroclinic systems. The remaining two cases, one polar low and one comma cloud, cannot be clearly assigned to either region and are more likely of a transitional nature. It would seem that the classic polar low and comma cloud can be identified as resulting primarily from CISK and baroclinic instability, respectively. However a continuous spectrum of intermediate systems can and probably does occur.

In this simple model it is the nondimensional heating parameter which determines the dynamical nature of the resulting system. It is interesting to consider what physical quantities contributed to making η higher for the CISK disturbances discussed above. As can be seen from the values in Table 1.1, the dominant factor is the static stability of the environment where the system formed: \((Q/f)^{1/2}\). Perhaps surprisingly, the boundary layer moisture content does not play a major role in determining whether CISK will occur. In fact, \(q_m\) is higher for the comma cloud systems which formed further to the south where the air and sea surface temperatures were warmer. Surface fluxes of moisture were important in all cases except that of Wakimoto and Durkee (1987) which formed over land, but there was no systematic difference between the polar low and comma cloud systems.

It should also be noted that the trend towards increasing nondimensional wavelength with increasing heating
in the CISK region (Fig. 1.4) does not imply longer dimensional wavelengths for the observed CISK disturbances (Table 1.1). The wavelength is nondimensionalized using the Rossby radius of deformation, \( L = (Q/f)^{1/2} H/f \), which is much smaller for the low stability environments where polar lows form.

d) Summary

Within the context of the linear model used here, the interaction of baroclinic and convective processes may be understood as follows. If the heating rate is not too great a baroclinic wave will grow in the usual manner. Since cumulus heating is controlled by low level convergence, it occurs where convergence and rising motion already exist and merely intensifies the existing circulation. The effects of convective heating are thus essentially the same as a reduction in static stability, which results in a faster growing disturbance with shorter wavelength.

CISK occurs when the circulation induced by the heating is sufficient to supply the same or greater amount of heating. It is thus only possible when the heating parameter, which is a measure of the efficiency of the heating in resupplying itself, is above a certain threshold. The contribution of baroclinicity can be seen from the fact that the transition to CISK takes place at a lower value of \( \varepsilon \) than the threshold of instability in the corresponding pure CISK model (Craig and Cho, 1988), due to the
contribution of convergence forced by baroclinic instability. Since the convective and baroclinic processes cooperate at all stages, the onset of CISK is through a smooth transition and there is no precise point where one instability mechanism ceases to dominate and the other starts.

The six case studies of polar air mass disturbances listed in Table 1.1 showed the full range of behavior discussed above. The systems called polar lows tended to be CISK-dominated while the comma clouds tended to be primarily baroclinic, however there were systems of a transitional nature given either name in the literature. In the rapidly changing air masses where polar lows often form it is even conceivable that a system could change character over time. For example, an initially baroclinic system could become CISK-dominated as surface fluxes of heat and moisture decrease the stability of the atmosphere and enhance cumulus convection.
CHAPTER 2

Moist Stability of
A Constant Potential Vorticity Jet

2.1 Introduction

The results of the one-dimensional model suggest that the combination of baroclinic instability and CISK can account for some of the basic properties of two major classes of polar air stream cyclones, as well as transitional disturbances where both mechanisms are of comparable importance. However these same physical processes are also active in most midlatitude wintertime cyclones (Tracton, 1973). This raises the question of \( \omega_x \), the range of behaviors found in polar air streams is not found more generally. The answer is easily found by computing a typical value for the heating parameter for the midlatitude environment. Consider an atmosphere with stability given by \( N = Q^{1/2}/f = 10^{-2}s^{-1} \), depth of troposphere \( H = 10 \text{ km} \), and boundary layer moisture content \( q_m = 5 \text{ gkg}^{-1} \) which corresponds to a dew point temperature of about 5°C. If we ignore the correction for surface fluxes, which will be small in the absence of a large air-sea temperature difference, and the correction factors for convergence above the boundary layer and moisture not condensed out which largely cancel each other (Craig and Cho, 1988), equation (1.3) gives a value of \( \varepsilon \) less than 0.5.
The heating parameters for the polar air stream cyclones in Table 1.1 are considerably larger, in some cases many times this value.

The two factors most responsible for the extreme values of heating parameter in these systems are surface fluxes of moisture and low static stabilities. In the restricted context of the simple cumulus parameterization used here, surface fluxes are treated by directly increasing $\varepsilon$ by an appropriate factor, which is similar in most of the cases listed in Table 1.1. The static stability is the main distinguishing factor among the different systems. As discussed below, the role of reduced stability in enhancing the growth of baroclinic and convective systems is well known, however the environments where polar air stream cyclones form are also characterized by high vorticity (Businger, 1985; Reed, 1979; Mullen, 1979), which can have a stabilizing effect. In this chapter we will consider in detail the effects of a low stability high vorticity environment on the combined baroclinic-CISK system.

The precise nature of the environment where polar air stream cyclones form is highly variable (Businger and Reed, 1989). This study will focus on one particular situation; the formation of comma cloud disturbances in the cyclonic shear region on the cold air side of a baroclinic jet. This situation is of interest since comma clouds are probably the most common type of polar air stream cyclone, and also
because there is a typical genesis environment (described in the conceptual model of Reed and Blier, 1986a) which presents a well-defined problem for further study. The genesis environment is characterized by cyclonic shear, low vertical stability and conditional instability, which is produced both by surface fluxes of latent and sensible heat and by advection of cold air at upper levels (Reed and Blier, 1986b). The model described in this chapter will be used to explore how these conditions provide a favorable environment for the growth of comma cloud weather systems.

Some inferences can be made about the behavior of the combined baroclinic-convective instability in complex environments from previous studies of each mechanism in isolation. It is well known that reduced vertical stability increases growth rates of baroclinic systems while reducing their wavelength (e.g. Pedlosky, 1987). On the other hand, shear vorticity, which corresponds to increased inertial stability, is unfavorable for baroclinic growth and contracts the scale of the system in the direction perpendicular to the flow. This effect has been termed the "barotropic governor" by James (1987). At the level of semigeostrophic theory, Hoskins (1976) has shown that the baroclinic instability problem depends on the potential vorticity, a combined measure of vertical and inertial stabilities, rather than on either factor separately.

Low static stability is also favorable for CISK (Wang,
1987a). This instability depends on the ability of the ensemble of convective clouds to produce enough low level moisture convergence to maintain itself. In a low stability environment, greater vertical motions and thus greater convergence rates are produced by the same heating rate. A second effect, which is only implicitly included in many CISK models, is the role of reduced stability in generating conditional instability, necessary for any convective heating. This is accounted for in the arbitrarily specified parameters of the cumulus parameterization, rather than modeled in a realistic fashion. The influence of vorticity on CISK is less well known, and will be discussed further below.

The model and basic state to be used in this study are described in section 2.2. The basic results of the stability analysis are shown in section 2.3, with emphasis on identifying the dynamical mechanisms of the unstable modes. Section 2.4 considers how the properties of the cold air side of the baroclinic jet make it especially favorable for the growth of small scale modes such as may be responsible for polar lows and comma clouds. The results are summarized in section 2.5.

2.2 Model Formulation

We wish to consider the moist stability of an idealized baroclinic jet which resembles the typical genesis region for the comma cloud systems discussed in the introduction.
To represent the large horizontal wind shears and varying static stabilities found in these regions, it is desirable to employ the semigeostrophic equations (Hoskins, 1975) rather than quasi-geostrophic theory, as the quasi-geostrophic scaling requires that the relative vorticity be small in comparison to the planetary vorticity and that the vertical stability be horizontally uniform in the reference state (Pedlosky, 1987).

While these restrictions are relaxed in semigeostrophic theory, simulations of the lifecycles of baroclinic disturbances by Peltier et al. (1990) using the primitive equations have shown many features that similar semigeostrophic models have been unable to produce (Hoskins and West, 1978; Schar and Davies, 1990). Recent analyses of the linear baroclinic instability problem by Nakamura (1988), Moore and Peltier (1989), and Bannon (1989) show that the semigeostrophic approximation becomes inaccurate as the Richardson number becomes small. In particular, it was found that in this limit the fastest growing modes had longer wavelengths and faster growth rates than the corresponding primitive equation solutions. While exact comparison of the present analysis with these results is difficult due to the different basic states, a simple comparison of scales suggests the possibility that errors of 10-15% could occur. This level of approximation is acceptable for present purposes, however Bannon (1989) points out that diabatic heating may create a very low
effective Richardson number which could result in larger errors. Since the semigeostrophic equations have frequently been used for this type of study (Emanuel et al., 1987; Joly and Thorpe, 1990), it is clear that their validity should be further examined.

For this first attempt at meridionally varying flow, a basic state with constant potential vorticity will be employed. As noted above, this implies that the effects of the vorticity and stability variations on baroclinic instability will exactly cancel each other. The choice of constant potential vorticity allows realistic horizontal wind shears and static stability distributions in the lower troposphere, however it is a rather crude approximation. In the cross sections of Reed and Blier (1986a,b), values on the cold air side of the jet are as much as 50% to 100% larger in the upper troposphere than those found on the warm air side (see Figs. 3.2b, 3.3b). This variation is not likely to be of primary importance for comma clouds however, since these disturbances are most prominent at lower levels. More realistic basic states will be considered in the next chapter.

a) Governing Equations

We consider disturbances to a basic state with zonal flow \( U(Y,Z) \) and constant potential vorticity, \( Q = fN^2 \), where \( N \) is a reference value of the Brunt-Vaisala frequency. The governing equation is the linearized semigeostrophic
potential vorticity equation:

$$\left[ \frac{\partial}{\partial T} + U_0^2 \frac{\partial}{\partial X} \right] q' = \frac{q}{\theta_0} \zeta E' Z$$

(2.1)

where the perturbation potential vorticity is given by

$$q' = \frac{\zeta}{f} \left[ \frac{1}{2} \left( \phi'_{XX} + \phi'_{YY} \right) + \frac{f}{Q} \phi' ZZ \right].$$

(2.2)

The quantity $\zeta$ is the basic state absolute vorticity, $E'$ is the diabatic heating rate, $\theta_0$ a reference value of potential temperature, and $\phi'$ the perturbation value of the transformed geopotential associated with the geostrophic coordinates $X, Y, Z,$ and $T$ (Hoskins, 1976; Craig and Cho, 1988). The domain is taken to be infinite in $X$, and periodic in $Y$ with period $2\pi L/\lambda$, where $\lambda$ is a dimensionless parameter and $L$ is the Rossby radius of deformation given by $L = NH/f$. The region is bounded by rigid surfaces at $Z = 0, H$, where $H$ is the depth of the domain. The boundary conditions of no vertical motion at $Z = 0, H$ can be expressed in terms of geopotential using the thermodynamic equation:

$$\left( \frac{\partial}{\partial T} + U_0^2 \frac{\partial}{\partial X} \right) \phi' - U_0^2 \phi' X = 0, \text{ at } Z = 0, H.$$  

(2.3)

The diabatic heating rate $E'$ is given as in chapter 1 by a simple cumulus parameterization which presumes the heating to be proportional to the convergence of moisture below a fixed cloud base level $Z_B$, and distributed in the vertical according to a prescribed function $h(Z)$. Thus
\[ E'(X, Y, Z) = \frac{\theta_o Q}{gT} \varepsilon(Y) h(Z) w(X, Y, Z_B) \]  

(2.4)

where \( \varepsilon(Y) \) is a nondimensional function of \( Y \) described below, and \( h(Z) \) is given by

\[
h(Z) = \begin{cases} 
0, & Z > Z_T \\
\frac{Z - Z_T}{(Z_L - Z_T)}, & Z_T > Z > Z_L \\
\frac{Z - Z_B}{(Z_L - Z_B)}, & Z_L > Z > Z_B \\
0, & Z_B > Z.
\end{cases}
\]  

(2.5)

where \( Z_L \) and \( Z_T \) are the level of maximum heating and cloud-top height respectively. This is a piecewise linear function with its maximum value in the midtroposphere. The dependence of the instability on the choice of vertical heating distribution is discussed in Craig and Cho (1988). The values \((Z_B, Z_L, Z_T) = (.1H, .4H, .7H)\) will be employed in the present study, except where noted. This treatment of convection, sometimes referred to as the wave-CISK parameterization, has been widely employed in analytic models.

Two problems have been associated with the use of a simple convergence-based parameterization in linear studies. The first is that depending on the vertical distribution of heating and the heating parameter, the model may predict growth rates that diverge to infinity as the wavelength goes to zero. This can occur since the heating is exactly in phase with the vertical velocity at cloud base level \( Z_B \), and
if it is sufficient to overcome the static stability there, the atmosphere is conditionally unstable in the ordinary sense and small scale growth is predicted (Wang, 1987a,b). This short wave cutoff problem arose frequently in early two-layer models where large heating rates at cloud base could only be avoided by specifying that most of the heat be released in the upper level (Pedersen and Rasmussen, 1985). In the present model with high vertical resolution, a more realistic heating profile that goes to zero at cloud base level is used (Eq. 2.5). The effective static stability is thus never negative and instability at the smallest scales will not occur by this mechanism.

A second problem with the formula (2.4) is that in the region where $w(z_B)$ is negative, there will be diabatic cooling when in fact the heat source should be zero. This situation is unavoidable if one is to assume a wavelike perturbation rather than use a numerical model to resolve the $x$-dependence, however caution is required in interpreting the results. Studies comparing the instabilities found with and without cooling in the downdraft regions are reviewed by Wang (1987a). It has generally been found that the wavelength of the most unstable mode was somewhat longer when heating was allowed only in the updraft, since only this region was contracted by diabatic effects. The growth rates of the fastest growing modes were not found to change significantly.
Emanuel et al. (1987) were able to find an analytic solution for a two layer model of baroclinic instability with latent heat release included by using a reduced moist potential vorticity in the updraft region. Comparing their results for the width of the updraft with the half-width of the most unstable wave, calculated using the reduced potential vorticity everywhere, agreement is found to better than 10% accuracy for all values of moist potential vorticity, ranging from the dry value to zero. The width of the downdraft region remained approximately constant at the same scale as in the adiabatic mode. However the growth rates of the fastest growing modes when heating was allowed only in the updraft region were significantly smaller than those found in the simpler calculation. This might be associated with differences between the parameterization of heating in this model and those described by Wang (1987a).

It is clear that while it is desirable for ease of solution to allow unrealistic cooling in downdraft regions, care must be taken when interpreting the results. It would appear that the half-wavelength of the most unstable mode provides a useful predictor for the scale of the updraft region, however the growth rate may be overestimated.

The function $\varepsilon(Y)$ represents the constant of proportionality between the amount of latent heat released in a column and the rate of boundary layer convergence. It will be nonzero only in the presence of conditional
instability, and its magnitude will be influenced by a number of factors including the moisture content of the air and sea surface fluxes. These dependencies were discussed in detail in Craig and Cho (1988), where values were estimated from observations of the genesis environments of a number of comma clouds. In general, $\varepsilon$ will be highly variable, but certain features are typical for the specific circumstances of comma cloud formation. Due to the large variations in vorticity across the polar front, one expects relatively small values of vertical stability in the cold air mass, if potential vorticity is relatively uniform. The conditional instability in this case is largely confined to the cold air side of the jet, and increases into the cold air mass, where cold air aloft and surface fluxes of heat and moisture have destabilized the atmosphere (Reed, 1979; Reed and Blier, 1986b). An idealized distribution function which includes these features is given by

$$
\varepsilon (Y) = \varepsilon_0 \begin{cases} 
0, & 0 \leq Y/L \leq \pi/l \\
-\cos(\pi Y/2L), & \pi/l \leq Y/L \leq 2\pi/l 
\end{cases} \tag{2.6}
$$

where the magnitude $\varepsilon_0$ will be referred to as the heating parameter. For some calculations, an alternative distribution function

$$
\varepsilon (Y) = \varepsilon_0 \begin{cases} 
0, & 0 \leq Y/L \leq \pi/l \\
-\sin(\pi Y/L), & \pi/L \leq Y/L \leq 2\pi/l 
\end{cases} \tag{2.7}
$$

will be used. It will be shown in section 2.3c that this change produces only minor modifications to the results. Some calculations using other distribution functions are
presented in section 2.4.

Using the expression (2.4) for heating, the governing equation can be written
\[
\left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \left[ \frac{1}{f^2} \left( \Phi'_{XX} + \Phi'_{YY} \right) + \frac{f}{Q} \Phi'_{ZZ} \right]
\]
\[
= \varepsilon (Y) h_Z (Z) \left[ \left( \frac{1}{Q} \right) \left[ \left( \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right) \Phi'_{Z} - U_Z \Phi'_{X} \right] \right] \bigg|_{Z=Z_B},
\]
with boundary conditions:
\[
\left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \Phi'_{Z} - U_Z \Phi'_{X} = 0, \quad \text{at } Z = 0, H. \quad (2.9)
\]

These equations are nondimensionalized by defining nondimensional variables (denoted by stars) according to
\[
(X, Y, Z, T, U, \zeta, \Phi) = (LX^*, LY^*, HZ^*, \frac{L}{U_0} T^*, \frac{U_0 U^*, f^*}{N^2 H^2 \Phi^*}).
\]

(2.10)

Typical dimensional values are listed in Table 1. The resulting equation and boundary conditions are
\[
\left[ \frac{d}{dT^*} + U^* \frac{d}{dX^*} \right] \left[ \frac{1}{f^2} \left( \Phi^*_{XX^*} + \Phi^*_{YY^*} + \Phi^*_{ZZ^*} \right) \right]
\]
\[
= -\varepsilon (Y^*) h^* (Z^*) \left[ \left( \frac{1}{Q^*} \right) \left[ \left( \frac{d}{dT^*} + U^* \frac{d}{dX^*} \right) \Phi^*_{Z^*} - \Phi^*_{X^*} \right] \right] \bigg|_{Z^*=Z_B^*},
\]
\[
\left[ \frac{d}{dT^*} + U^* \frac{d}{dX^*} \right] \Phi^*_{Z^*} - \Phi^*_{X^*} = 0 \quad \text{at } Z^* = 0, 1. \quad (2.11)
\]

The corresponding vorticity is given by \( \zeta^* = (1 + RU^* Y^*)^{-1} \), where \( R = U_0 / f L \). It is straightforward to verify that in the limit \( R \to 0 \), the governing equations revert to the quasi-geostrophic form. This result will prove useful for
Table 2.1. Typical dimensional parameters and some derived quantities for comparison of model results with observations. Derived quantities calculated using $R = 1$, $\lambda = 1.3$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coriolis parameter</td>
<td>$f \quad 10^{-4} \text{s}^{-1}$</td>
</tr>
<tr>
<td>Height of lid</td>
<td>$H \quad 8 \text{ km}$</td>
</tr>
<tr>
<td>Buoyancy frequency</td>
<td>$N \quad 10^{-2} \text{s}^{-1}$</td>
</tr>
<tr>
<td>Radius of deformation $L=NH/f$</td>
<td>$800 \text{ km}$</td>
</tr>
<tr>
<td>Wind speed</td>
<td>$U_0 = RfL \quad 80 \text{ ms}^{-1}$</td>
</tr>
<tr>
<td>Width of domain</td>
<td>$2\pi L/\lambda \quad 3867 \text{ km}$</td>
</tr>
<tr>
<td>Potential vorticity</td>
<td>$Q = fN^2 \quad 10^{-8} \text{s}^{-3}$</td>
</tr>
</tbody>
</table>
comparison between the semigeostrophic and quasi-geostrophic representations of the problem.

b) Energy Equations

To derive the energy budget equations for the perturbation, it is convenient to start from the geostrophic momentum equations in physical coordinates. The linearized geostrophic momentum equations are:

\begin{align}
\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial y} + w' \frac{\partial U}{\partial z} - f v' + \frac{\partial \phi'}{\partial x} &= 0 \quad (2.12) \\
\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} + u' \frac{\partial U}{\partial y} + w' \frac{\partial U}{\partial z} + f u' + \frac{\partial \phi'}{\partial y} &= 0 \quad (2.13) \\
\frac{\partial \phi'}{\partial z} &= \frac{\partial \phi'}{\partial z} \quad (2.14) \\
\frac{\partial \theta'}{\partial t} + U \frac{\partial \theta'}{\partial x} + v' \frac{\partial \theta}{\partial y} + w' \frac{\partial \theta}{\partial z} &= E' \quad (2.15) \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0 \quad , \quad (2.16)
\end{align}

where \( U(y,z) \) and \( \theta(y,z) \) are basic state wind and potential temperature, and the perturbation geostrophic wind components are given by \( u_g' = -f^{-1} \frac{\partial \phi'}{\partial y} \) and \( v_g' = f^{-1} \frac{\partial \phi'}{\partial x} \). These are combined in the usual way to give equations for the perturbation kinetic energy

\begin{align}
\frac{\partial}{\partial t} \left( \frac{1}{2} u_g'^2 + v_g'^2 \right) &= -u_y v' u_g' \\
&\quad - u_z w' u_g' + w' \frac{\partial \phi'}{\partial z} \quad , \quad (2.17)
\end{align}

and perturbation available potential energy.
\[ \frac{\partial}{\partial t} \left[ \frac{1}{2g\Theta_z} \left( \frac{\delta^2}{\delta z} \right)^2 \right] = \frac{\tau \Theta}{g\Theta_z} \frac{U}{V} \frac{\partial^2}{\partial z} + \frac{1}{\Theta_z} \frac{\partial^2}{\partial z} \]  

where the average is over the entire domain

\[ \left( \ldots \right) = \int_{0}^{H} \int_{0}^{2\pi L} \int_{0}^{2\pi L/k} \left( \ldots \right) dx dy dz \]

\[ = \int_{0}^{H} \int_{0}^{2\pi L} \int_{0}^{2\pi L/k} \left( \ldots \right) \frac{1}{\xi} dx dy dz . \]  

Since the eigenfunctions are calculated in geostrophic coordinates, it is necessary to reexpress these equations in terms of the transformed variables. After nondimensionalizing, they take the form

\[ \frac{\partial}{\partial t^*} \left[ \frac{1}{2} \left( \Phi^* \frac{Y^*}{Y^*} + \Phi^* \frac{X^*}{X^*} \right) \right] = \zeta U \frac{Y^*}{Y^*} \frac{\Phi^*}{\Phi^*} \frac{X^*}{X^*} + \varphi \frac{\Phi^*}{\Phi^*} \]

HRS VRS VHF

and

(2.20)
\[
\frac{\partial}{\partial t^*} \left( \frac{1}{2} (\zeta^{*-1} + R_k^* U_*^2) \frac{\Phi^*}{Z_*^*} - \Phi^* \right) = \frac{U_*^*}{Z_*^*} (\zeta^{*-1} + R_k^* U_*^2) \frac{\Phi^*}{Z_*^*} - \frac{1}{\nu^*} \frac{\Phi^*}{Z_*^*} \\
\\
\sqrt{H^F} \\
\\
\frac{w^* \frac{\Phi^*}{Z_*^*}}{Z_*^*} + (\zeta^{*-1} + R_k^* U_*^2) \frac{\Phi^*}{Z_*^*}
\]

\begin{align}
V^F \quad L^H \quad (2.21)
\end{align}

where \( \Phi^* \) is the nondimensional perturbation geopotential determined from the eigenfunction calculation. The other perturbation quantities needed to calculate the terms in the energy equations are obtained as follows. Heating is determined from the definition (2.4)

\[
E^* = \varepsilon (Y^*) h(Z^*) \left[ \zeta^* \left( - \frac{\partial}{\partial T^*} + U_*^* \frac{\partial}{\partial X^*} \right) \frac{\Phi^*}{Z_*^*} + U_*^* \frac{\Phi^*}{Z_*^*} \right] = (2.22)
\]

and \( w^* \) and \( v^* \) from the thermodynamic and zonal momentum equations, respectively,

\[
w^* = \zeta^* \left[ - \left( \frac{\partial}{\partial T^*} + U_*^* \frac{\partial}{\partial X^*} \right) \frac{\Phi^*}{Z_*^*} + U_*^* \frac{\Phi^*}{Z_*^*} \right] \quad (2.23)
\]

\[
v^* = R \left( \frac{\partial}{\partial T^*} + U_*^* \frac{\partial}{\partial X^*} \right) \frac{\Phi^*}{Y^*} + \zeta^* \frac{\Phi^*}{X^*} + R^2 \frac{w^*}{U_*^*} \quad (2.24)
\]

The stars will be omitted from this point on. The terms in the energy equations (2.20) and (2.21) are interpreted as a vertical Reynolds stress (VRS) and horizontal Reynolds stress (HRS), which add to give the total Reynolds stress (RS = VRS + HRS), representing conversion from mean flow
kinetic energy to eddy kinetic energy; vertical heat flux (VHF), which is conversion from eddy available potential energy to eddy kinetic energy; horizontal heat flux (HHF), representing conversion from mean flow available potential energy to eddy potential energy; and a source of eddy potential energy due to release of latent heat (LH).

c) Basic State

For a constant potential vorticity basic state, Hoskins and West (1979) showed that the nondimensional zonal wind field must be a solution of Laplace's equation:

\[ U_{yy} + U_{zz} = 0. \]  \hspace{1cm} (2.25)

The basic state wind to be used in this study, which satisfies this condition, is specified by an analytic function in the transformed coordinates,

\[ U(Y,Z) = Z - \frac{Y}{Z} \left[ Z + \frac{(1-\alpha)\sinh Z + \alpha\cosh^2 Z}{(1-\alpha)\sinh + \alpha\cosh^2} \cos \gamma \right], \]  \hspace{1cm} (2.26)

where \( \alpha \) and \( \gamma \) are constants. The zonal flow used by Hoskins and West (1979), which corresponds to the case \( \alpha = 0 \), is inadequate for the present study since the baroclinic jets associated with comma cloud formation tend to be relatively narrow (large \( l \)). Under these circumstances, the flow defined by (2.26) with \( \alpha = 0 \) becomes concentrated near the upper boundary and the horizontal wind shears in the lower troposphere are weak.

The basic state wind field is plotted in Fig. 2.1 in geostrophic and physical coordinates, using parameter values
Figure 2.1. Nondimensional basic state wind field plotted in (a) transformed and (b) physical coordinates. Typical dimensional values for maximum wind speed and domain size from Table 1.1 are $U_\infty = 80 \text{ ms}^{-1}$, depth 8 km, and meridional width 3870 km.
\[ \alpha = 0.5, \gamma = 1.1, \ell = 1.3 \text{ and } R = 1. \] These values will be used throughout this study, except where noted. The effect of the transformation back to physical coordinates is a visible contraction of the region of positive relative vorticity and stretching of the negative vorticity region (Fig. 2.1b). This asymmetry is also apparent in the observed cross sections obtained by Reed and Blier (1986a,b), and reproduced here in Fig. 3.1. The relative vorticity in the lower troposphere is approximately \( 0.2f - 0.3f \) on the cyclonic shear side, with similar negative values on the anticyclonic side. The corresponding temperature field has a reduction in static stability in the cyclonic shear region of 20% - 30% in comparison with the jet axis. These values are consistent with the observations of Mullen (1979) and Reed and Blier (1986a,b). However, the wind speed at the upper boundary of 80 m/s (obtained from the dimensional values in Table 2.1) is unrealistically large. This occurs due to the lack of a sloping tropopause which would produce a potential vorticity gradient in the interior of the domain. The high winds at the top of the domain do not appear to affect the flow in the lower troposphere to any great extent, but disturbances which have large amplitudes near the upper boundary must be regarded with suspicion.

d) Numerical Solution of the Eigenvalue Problem

To solve the linearized potential vorticity equation
(2.11), we look for solutions of the form

$$\Phi(x,y,z,t) = \phi(y,z) \exp ik(x-ct)$$

(2.27)

where $k$ is the wavenumber in the $x$ direction and $c$ is the complex phase speed. The growth rate of the disturbance is then given by $\sigma = k \text{Im}(c)$. The result is a two-dimensional nonseparable eigenvalue problem in $y$ and $z$.

The equation and boundary conditions are discretized in the vertical using second order finite differences at $N+1$ levels numbered from 0 (surface) to $N$ (lid). Uncentered second order differences were used in the boundary conditions following Brevdo (1987), who emphasized the inadequacy of the simpler first order formulation. The resulting equations are

$$\frac{1}{\zeta B} (U_i - c) \left[ \frac{N^2}{H^2} \phi_{i+1} + \left[ -k^2 + \frac{\partial^2}{\partial y^2} - \frac{2N^2}{H^2} \right] \phi_i + \frac{N^2}{H^2} \phi_{i-1} \right]$$

$$= \varepsilon \left[ \frac{\partial h}{\partial z} \right] \left[ (U_{B_i} - c) \frac{N}{2H} (\phi_{B_{i+1}} - \phi_{B_i}) \right] - (U_Z)_{B_i} \phi_B,$$

$$i = 1,2,\ldots,N-1$$

$$(U_0 - c) \frac{N}{2H} \left[ -3\phi_0 + 4\phi_1 - \phi_2 \right] - (U_Z) \phi_0 = 0$$

$$(U_N - c) \frac{N}{2H} \left[ \phi_{N-2} - 4\phi_{N-1} + 3\phi_N \right] - (U_Z) \phi_N = 0$$

(2.28)

where the subscript $B$ denotes function values at height $Z_B$ which is assumed to coincide with one of the $N+1$ levels. The equations can be written in matrix form,

$$D_{ij} \phi_j = cE_{ij} \phi_j$$

(2.29)

where the elements of $D$ and $E$ are functions of basic state variables and $Y$ derivative operators.
The reduction to discrete form in the Y direction is then accomplished by expanding the eigenfunctions and matrix elements in finite Fourier series (following Hoskins and West, 1979)

\[ \phi_j = \sum_{v=-M}^{M} (\phi_j)_v e^{i\nu Y} \]

\[ [D_{ij}, E_{ij}] = \sum_{v=-M}^{M} \left( (D_{ij})_v, (E_{ij})_v \right) e^{i\nu Y}. \tag{2.30} \]

The result can be expressed in the form

\[ (A_{ij})_{\mu v} (\phi_j)_v = c (B_{ij})_{\mu v} (\phi_j)_v \tag{2.31} \]

where

\[ A_{ij\mu v} = \begin{cases} 
(D_{ij})_{\mu v} & -M \leq \mu - v \leq M \\
0 & \text{otherwise} 
\end{cases} \tag{2.32} \]

\[ B_{ij\mu v} = \begin{cases} 
(E_{ij})_{\mu v} & -M \leq \mu - v \leq M \\
0 & \text{otherwise.} 
\end{cases} \tag{2.33} \]

The elements of A and B are listed in Appendix A.

The four-dimensional eigenvalue problem defined above can be mapped onto a standard two-dimensional problem,

\[ A_{rs} \phi_s = c B_{rs} \phi_s, \tag{2.34} \]

using

\[ r(i,\mu) = (N-i)(2M+1) + \mu + M+1 \]

\[ s(j,\nu) = (N-j)(2M+1) + \nu + M+1. \tag{2.35} \]

\[ A_{rs} \text{ and } B_{rs} \text{ are complex matrices of order } (2M+1)(N+1). \]
Discussion of the resolution required for accurate solution of the problem will be deferred to the next section, after the primary eigenmodes have been identified.

The eigenvalues were found using the LZ algorithm of Kaufman (1974). For such large matrices this method occasionally failed to converge, and it was generally incapable of producing accurate eigenvectors. The convergence problem could easily be solved by perturbing the data, in particular by evaluating the problem for a slightly different wavenumber, for example $k = 4.01$ rather than $4.00$. The eigenvectors were found by substituting the known eigenvalue into (2.34) and solving the resulting system of linear equations. The eigenvalue-eigenvector pairs were checked by inserting the values into the original equation and computing the performance index described by Garbow et al. (1977). In the case of the problem (2.34), the index is defined as

$$\mu = \frac{||A_{rs}\phi_s - cB_{rs}\phi_s||}{\epsilon_M(||A_{rs}|| + ||c||||B_{rs}||)||\phi_s||} \quad (2.36)$$

where $||\phi_s||$ is the norm given by

$$||\phi_s|| = \sum_s (|\text{Re}\phi_s| + |\text{Im}\phi_s|) \quad (2.37)$$

and $\epsilon_M$ is the machine precision. For all the results to be shown here, the index was less than 100 which indicates accurate solution of the matrix eigenvalue problem.

A: alternative strategy for solving (2.34) is to
express it in terms of real matrices of twice the degree and use the QZ algorithm of Moler and Stewart (1973). This method proved to be more robust than the LZ algorithm but was slower by a factor of about eight which rendered it prohibitively expensive.

2.3 Results

a) Unstable Modes With No Cumulus Heating

For comparison purposes, we look first at the stability of the jet in the absence of diabatic heating (i.e. $\epsilon = 0$ in (2.8)). Fig. 2.2 shows curves of growth rate and phase speed as a function of wavenumber for the four fastest growing unstable modes. The spectrum is dominated by two main branches. The first of these is associated with the standard baroclinic instability mechanism and has a maximum growth rate of 0.14 at wavenumber 1.5. These values correspond to a dimensional e-folding time of 0.83 days and a wavelength of 3350 km. The results are similar to those obtained by Hoskins and West (1979), although not identical due to the differences in basic state. The identification of this mode with the Eady model instability is supported by the disturbance structure and the energy budget. Fig. 2.3a shows the amplitude of the complex eigenfunction. As in the basic Eady model the disturbance has maximum amplitude at the upper and lower boundaries. The energy budget (Fig. 2.4a) shows that the mode is driven by baroclinic energy conversions (horizontal heat fluxes), with a slight negative
Figure 2.2. Nondimensional (a) phase speed and (b) growth rate as a function of zonal wavenumber $k$ with no heating ($\varepsilon_o = 0$). The first four meridional harmonics are labelled. Curves are plotted from calculations at $k$ intervals of 0.125.
Figure 2.3. Meridional \((y-z)\) cross sections of perturbation geopotential for the fastest growing modes at (a) \(k = 1.5\) and (b) \(k = 0.75\) for \( \varepsilon _{0} = 0 \). The plots have been normalized to a maximum amplitude of one.
Figure 2.4. Energy budget diagrams for the fastest growing modes at (a) $k = 1.5$ and (b) $k = 0.75$ for $\xi \equiv 0$. The abbreviations used to label the terms are defined in eqs. (2.20) and (2.21). The magnitude of the fluxes has been normalized so that $HHF = 1.00$. The negative value for $RS$ in (a) indicates that energy is transferred from eddy to zonal kinetic energy.
contribution from barotropic processes (Reynolds stress terms).

Large growth rates are also found for a separate branch at longer wavelengths. The fastest growing mode of this branch has a growth rate of 0.14 and wavenumber 0.65, corresponding to dimensional e-folding time .83 days and wavelength 7730 km. The energy budget (Fig. 2.4b) shows this mode to be driven primarily by barotropic instability. The structure of the eigenfunction is also substantially different from that of an Eady wave (Fig. 2.3b), with largest amplitude in upper levels on the anticyclonic shear side of the wind maximum. The growth rate of this mode is strongly dependent upon the horizontal shear of the basic state flow. Fig. 2.5 shows growth rate spectra of the main baroclinic and barotropic modes for three values of the meridional scale parameter \( \lambda \) which correspond to jet half-widths of 3590 km, 2500 km and 1930 km, and maximum horizontal wind shears of \( 2.8 \times 10^{-5} \) s\(^{-1} \), \( 3.5 \times 10^{-5} \) s\(^{-1} \) and \( 4.2 \times 10^{-5} \) s\(^{-1} \). The growth rate of the long wave mode increases roughly in proportion to the horizontal shear while the effect on the baroclinic mode is small and negative. These results are similar to those of Brown (1969) and Mudrick (1974) who examined the stability of a number of jet flows with varying half-widths. While the large growth rates of the barotropic mode contrast with the instabilities of broader jets such as those studied by Hoskins and West (1978), they are typical of the narrower
Figure 2.5. Growth rate curves as a function of wavenumber for the primary baroclinic (solid) and barotropic (dashed) modes at three values of the domain width parameter \( l \). Calculations were made at \( k \) intervals of 0.125.
jets where comma clouds form. This will be shown in the next chapter.

The resolution required for accurate solution of the eigenvalue problem depends on the structure of the mode being solved for. Table 2.2 lists the growth rates and phase speeds as a function of horizontal resolution, of the five fastest growing waves for a single wavenumber corresponding to the largest growth rate of the baroclinic branch. For \( M = 5 \) and 9, only four eigenvalues with significant growth rates are found, and even at \( M = 13 \) the eigenvalue for the fifth mode has apparently not begun to converge to an accurate solution. This reflects the more rapidly varying meridional structure of the corresponding eigenfunctions, as seen in Fig. 2.6 which shows the surface geopotential perturbation for the first three eigenmodes. The error for the main baroclinic mode, on the other hand, is less than 1\% for \( M \geq 11 \). Table 2.2 also shows results for varying resolution in the vertical. For the Eady modes, which extend through the depth of the domain, even 10 layers is sufficient to provide accurate results for all modes except those that were not well-resolved in the horizontal. Note that the requirements will become more stringent when heating is introduced, as discussed below. The resolution used for the results in this chapter, \( N = 20 \) and \( M = 11 \), is sufficient to provide good accuracy for the first few harmonics, which are physically most important. For this choice of values, the stability matrices in (2.34) have
Table 2.2. Nondimensional phase speed and growth rate (Re(c),\sigma) for the five fastest growing eigenvalues at \varepsilon_0=0, k=1.5. i; Vertical resolution N=20, while horizontal resolution M takes the values shown. b) M=11 while N takes the values shown.

* indicates growth rate is less than 10^{-3}.

a)  
<table>
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<td>(.2849, .0429)</td>
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</table>

b)  
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<th>3</th>
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<th>5</th>
</tr>
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<td>(.2782, .0500)</td>
<td>(.1930, .0257)</td>
<td>(.2412, .0198)</td>
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<tr>
<td>20</td>
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<td>(.1928, .0259)</td>
<td>(.2401, .0214)</td>
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<tr>
<td>30</td>
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<td>(.3360, .0522)</td>
<td>(.2781, .0501)</td>
<td>(.1928, .0255)</td>
<td>(.2398, .0212)</td>
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</tbody>
</table>
Figure 2.6. Horizontal (x-y) plots of geopotential perturbation at $z = 0$ in physical coordinates for the three fastest growing modes at $k = 1.5$ with $\epsilon_0 = 0$ are shown in (a) - (c) respectively. The plots have been normalized so that the maximum amplitude is one. Using the dimensional values in Table 1, the zonal wavelength of these modes is 3350 km.
order 483.

b) Unstable Modes With Cumulus Heating

Parameterized cumulus heating is introduced with the horizontal and vertical distributions defined by (2.5) and (2.6). Curves of growth rate vs. wavenumber are shown in Fig. 2.7 for several values of the heating parameter $\epsilon_0$. This parameter determines the overall proportionality between heating rate and low-level convergence. For $\epsilon_0$ up to about 1.0, the wavelength of the fastest growing mode is slightly decreased and the growth rate increased, but there is little substantial change to the spectrum of unstable modes. However, as $\epsilon_0$ becomes larger, there is a dramatic destabilization at short wavelengths, and for $\epsilon_0$ greater than about 1.6, the short waves have faster growth rates than the original long wave mode. It is perhaps surprising that large growth rates are found at short wavelengths in the barotropic branch, however the energy budget (Fig. 2.4b) shows that baroclinic processes are also significant in the growth of the disturbance, so that the motion will not be entirely barotropic. For very long wavelengths the growth rates of all modes, including the peak associated with barotropic instability, are virtually unaffected by heating.

Fig. 2.8 shows the geopotential perturbation at the lowest model level for the fastest growing waves at $\epsilon_0 = 1.5$ and 2.0. The structure of the wave with lower values of heating (Fig. 2.8a) is similar to the adiabatic mode (Fig.
Figure 2.7. Nondimensional growth rate as function of wavenumber for the first three harmonics for four values of the heating parameter: (a) $\varepsilon_0 = 1.0$, (b) $\varepsilon_0 = 1.5$, (c) $\varepsilon_0 = 1.7$, and (d) $\varepsilon_0 = 2.0$. The horizontal distribution of heating is given by (2.6). Calculations were made at $k$ intervals of 0.25.
Figure 2.8. Horizontal (x-y) plots of geopotential perturbation at $z = 0$ in physical coordinates for the fastest growing mode for heating distribution (2.6) with heating parameter, wavenumber, growth rate and phase speed given by (a) $\varepsilon_0 = 1.5$, $k = 1.75$, $\sigma = 0.17$, $c = 0.41$, and (b) $\varepsilon_0 = 2.0$, $k = 5.25$, $\sigma = 0.41$, $c = 0.35$. The plots have been normalized so that the maximum amplitude is one. Using the dimensional values in Table 2.1, the wavelengths of these modes are 2870 km and 960 km, respectively.
2.6a). There is some displacement of the wave into the cold air mass, towards the region where the distribution function (2.6) allows greater heating rates. On the other hand, the structure of the most unstable mode at $\varepsilon_0 = 2.0$ (Fig. 2.8b) differs substantially from the no-heating case. In addition to the shorter wavelength along the jet, the disturbance amplitude is largely confined to the cyclonic shear region. The energy budgets for the two modes are shown in Fig. 2.9. While heating contributes to the growth of the long wave mode, the baroclinic term is much larger. For the short wave mode this is reversed with the latent heat term having twice the magnitude. Barotropic processes are of negligible importance in both of these cases.

The location of the short wave mode in the cyclonic shear region is similar to that of observed comma clouds. This mode is the fastest growing for $\varepsilon_0$ larger than about 1.6. If the corresponding value of $\varepsilon(Y)$ at the meridional position of the disturbance center is computed, it can be compared to the values of the heating parameter estimated for the genesis environments of observed systems in chapter 1. For $\varepsilon_0 = 1.7$, the fastest growing mode is centered at $Y = 0.65(2\pi/\ell)$. In this case from (2.6) we have $\varepsilon(Y) = 0.8$. This is comparable to the values 0.9±0.3 and 0.7±0.3 estimated in chapter 1 for the two case studies of Reed and Blier (1986a, b). The error estimates are taken from Craig and Cho (1988). The dimensional wavelength of the fastest growing mode for $\varepsilon_0 = 1.7$ is about 1100 km, which is
Figure 2.9. As in Fig. 2.4 but for the fastest growing modes at (a) $\varepsilon_0 = 1.5$ and $\varepsilon_0 = 2.0$ using the heating distribution (2.6).
somewhat smaller than the 1400 km and 2000 km estimated for the two observed systems but within the range usually given for comma clouds (Reed, 1979). As will be seen, the wavelength of the short wave mode varies depending on the choice of parameters for the background flow and heating distribution.

The smaller scale structure of the short-wave disturbances imposes more stringent requirements on the model resolution than the adiabatic results discussed earlier. It is apparent in Table 2.3 that while a horizontal resolution of $M = 11$ is sufficient for convergence, the growth rate of the short wave at $N = 20$ differs by about 10% from the value obtained with $N = 30$. Nevertheless, it appears that by $N = 20$ the eigenvalue is converging towards the accurate solution, albeit slowly, and so the results should be qualitatively correct. Given the idealizations in the basic state and cumulus parameterization, exact quantitative results can not be expected, thus an accuracy of 10% should be adequate for present purposes. The slow convergence as the number of layers is increased is probably due to the discontinuous derivatives of the vertical heating distribution (defined in (2.5)). This form was chosen for consistency with the analytic model of chapter 1, and will be replaced by a smoother form in the next chapter.

c) The Dynamical Nature of the Instabilities
Table 2.3. Nondimensional phase speed and growth rate for the fastest growing mode at $\alpha_0=2.0$, $\kappa=5.0$, using the horizontal heating distribution defined in (2.6). a) Vertical resolution $N=20$, while horizontal resolution $M$ takes the values indicated. b) $M=11$ while $N$ is varied.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N$</th>
</tr>
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<tbody>
<tr>
<td>7</td>
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<tr>
<td></td>
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<td>(.3528, .4067)</td>
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In the model of the previous chapter, which did not include meridional variations, both heating and baroclinic processes contributed to the growth of the instability in varying proportions, depending on the value of the heating parameter. The resulting disturbances could be divided into those driven primarily by baroclinic instability and those dominated by CISK, although there was a region of continuous transition between the two types as $\varepsilon_0$ was varied.

We now present results for the present model over a wide range of values of the heating parameter in order to explore the interaction of the two mechanisms on a meridionally varying flow. To do this, it is convenient to use the alternative horizontal heating distribution given by (2.7). Fig. 2.10 shows growth rate curves for several values of $\varepsilon_0$. The behavior of the most unstable mode as the heating parameter is increased is similar to that seen in Fig. 2.7, with decreasing wavelengths, increasing growth rates, and rapid growth at very short wavelengths for higher values of $\varepsilon_0$. Unlike the results for the previous heating distribution, the rapid growth at short wavelengths is associated with the main baroclinic mode, rather than other branches. This distinction is not of physical importance since the nature of the short wave instability is the same in each case, as shown below. However, it is more convenient for the purpose of displaying results to show only a single growth rate curve for each value of the heating parameter.
Figure 2.10. As in Fig. 2.7 but for the heating distribution (2.7) with (a) $\varepsilon_o = 0.8$, (b) $\varepsilon_o = 1.0$, and (c) $\varepsilon_o = 1.2$. 
The transition of the fastest growing mode to shorter wavelengths occurs near $\varepsilon_0 = 1.1$, which is smaller than the value found for the previous distribution of heating. However if we consider the corresponding value of $\varepsilon(Y)$ at the disturbance center, we again obtain $\varepsilon(Y) = 0.8$. The structure of the disturbance is also substantially unchanged by the use of the alternative heating distribution. The plot of surface geopotential perturbation for the fastest growing mode at $\varepsilon_0 = 1.2$ in Fig. 2.11 is similar to the corresponding structure obtained previously (Fig. 2.8b), although the center of the disturbance is located somewhat closer to the jet axis, as might be expected from comparison of the functions (2.6) and (2.7).

Fig. 2.12 shows growth rates for the heating distribution $\varepsilon(Y)$ defined in (2.7) but including higher values of $\varepsilon_0$ than were shown in Fig. 2.10. Fastest growth occurs at short wavelengths for $\varepsilon_0$ greater than about 1.1. However the scale of the most unstable mode continues to decrease with increasing $\varepsilon_0$, up to about $\varepsilon_0 = 1.5$. Beyond this point the wavelength increases as the heating parameter increases.

A similar reversal in the dependence of wavelength on heating parameter was identified in chapter 1 as being associated with the transition to CISK. This identification was supported by examination of the structure of the eigenfunctions. In the region where baroclinic processes
Figure 2.11. As in Fig. 2.8 but for the heating distribution (2.7) and $\epsilon_0 = 1.2$, $k = 4.25$, $\sigma = 0.30$, and $c = 0.39$. Using the dimensional values in Table 2.1, the zonal wavelength is 1180 km.
Figure 2.12. Nondimensional growth rate as a function of wavenumber for the fastest growing mode with horizontal heating distribution (2.7) and $\varepsilon_0$ taking the values shown.
dominated, the geopotential perturbation was characterized by a trough tilting westward with height, while systems dominated by convective heating featured a region of low geopotential in the lower troposphere almost directly beneath a region of high geopotential.

These characteristics are also found for the eigenfunctions in the present case. Fig. 2.13 shows X-Z cross sections of geopotential perturbation through the disturbance center for the fastest growing mode at three values of $\epsilon_0$. For small heating rates (eg. $\epsilon_0 = .8$, Fig. 2.13a), the structure of the wave is not greatly affected by heating. As discussed in section 1.3d, the location of the heating in this case is determined primarily by the convergence associated with the baroclinic instability. Thus the latent heat is released where rising motion is already occurring, and the net result is similar to the effects of a reduction in vertical stability. The wave shown in Fig. 2.13b ($\epsilon_0 = 1.2$) corresponds to heating slightly above the transition where the short wave becomes dominant. Although there is a tendency for the amplitude of the perturbation to be confined in the vertical to the region where heating occurs, it can still be characterized as a westward-tilting trough with the upper level geopotential minimum lagging the surface low by less than 90°. At $\epsilon_0 = 1.7$, in the region where the wavelength of the fastest growing mode increases with increasing heating, the perturbation shows a more CISK-like configuration with the
Figure 2.13. Zonal (X-Z) cross sections of perturbation geopotential for the fastest growing modes with (a) $\varepsilon_0 = 0.8$, $k = 1.75$, $\sigma = 0.17$, $c = 0.41$, (b) $\varepsilon_0 = 1.2$, $k = 4.25$, $\sigma = 0.30$, $c = 0.39$, and (c) $\varepsilon_0 = 1.7$, $k = 4.5$, $\sigma = 0.65$, $c = 0.49$, using the alternative heating distribution function (2.7). Cross-sections are taken through the disturbance centers which occur at $Y$ values of $(0.55)\pi/l$, $(0.65)\pi/l$, and $(0.7)\pi/l$, respectively.
low level geopotential minimum almost $180^\circ$ out of phase with
the upper level low (Fig. 2.13c).

The energy conversion terms associated with these modes
are summarized in Table 2.4. The transition to short wave
meridionally confined disturbances, which occurs around $\varepsilon_0 =
1.1$, appears to coincide with the point where latent heating
surpasses baroclinic instability as a source of eddy
available potential energy (LH > HHF). The transition to
CISK, determined from the structure of the eigenfunctions,
occurs at much larger $\varepsilon$, i.e., this case around $\varepsilon_0 = 1.5$. The
Reynolds stress term becomes important for $\varepsilon_0 = 1.0$, just as
the shift of the fastest growing mode to shorter wavelengths
begins. It represents as much as a third of the source of
eddy kinetic energy for short wave baroclinic disturbances,
but becomes negative again above the CISK threshold.

While the structure of the short wave (Fig. 2.13b)
indicates that it is dominated by baroclinic processes, the
amplitude of the perturbation decays above the heating
region rather than extending to the upper boundary as in the
Eady wave structure in Fig. 2.13a. A similar structure was
noted in Craig and Cho (1988) (Fig. 11 of that paper) in a
case when the vertical distribution of heating was confined
to low levels. This was interpreted using the observation
that for systems dominated by baroclinic processes, the
effect of the parameterized convective heating was similar
to a reduction of the effective static stability in the
Table 2.4. Nondimensional wavenumber, growth rate, and energy conversion terms (defined in eqs. 2.20 and 2.21) for the fastest growing mode for varying values of the heating parameter $\varepsilon_0$. Values were calculated using the alternative distribution of heating, eq. 2.7.

<table>
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<th>HHF</th>
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region where the heat was released. This analogous situation, where baroclinic instability occurs in an atmosphere with two layers of differing stability, was considered by Blumen (1979) and Weng and Barcilon (1987). These authors found that the fastest growing mode would have amplitude largely confined to the lower layer, provided that two conditions were met. These were that there be a sufficiently large difference in stability between the two layers, and that the lower layer be sufficiently shallow.

In the present case the vertically confined structure in Fig. 2.13b appears only for a sufficiently large value of the heating parameter, and thus a sufficiently reduced effective stability in the heating region. To determine whether this structure is only found for a sufficiently shallow heating region, a subset of the calculations was repeated using $Z_B = .1$, $Z_L = .5$, and $Z_T = .9$ in (2.5). This choice increases the depth of the layer of reduced effective stability and should tend to favor deeper modes. The growth rate curves plotted in Fig. 2.14 show that the wavelength of the most unstable mode is again reduced as $e_0$ increases. As found in Craig and Cho (1988), when the heating is released higher in the troposphere, larger values of the heating parameter are required for short wavelength disturbances. As anticipated, the amplitude of the perturbation extends to the upper boundary in this case (Fig. 2.15). The horizontal structure is again confined to the cyclonic shear region (Fig. 2.16).
Figure 2.14. As in Fig. 2.12 but for a deeper vertical distribution of heating with \((Z_B', Z_L', Z_T') = (0.1, 0.5, 0.9)\).
Figure 2.15. As in Fig. 2.13 but for the deeper vertical heating distribution $(Z_B, Z_L, Z_M) = (.1, .5, .9)$, with $\varepsilon_0 = 1.7$, $k = 3.25$, $\sigma = 0.45$, and $c = 0.26$. The cross-section is taken at $Y = (0.65)2\pi/\lambda$. 
Figure 2.16. Horizontal cross-section of geopotential perturbation at \( z = 0 \) for the mode shown in Fig. 2.15. Using the dimensional values in Table 2.1, the zonal wavelength of this mode is 1550 km.
In summary, when heating is allowed in the cold air mass, three types of disturbances may be found, depending on the value of the heating parameter. If the heating rate is not too large, the effect of the cumulus convection is similar to a reduction in static stability. The result is a baroclinic instability with larger growth rates and shorter wavelengths, but with the structure of the eigenfunction largely unaffected. For larger values of $\epsilon_0$, the system becomes confined to the heating region away from the jet axis. However, the vertical structure of the disturbance is still determined primarily by baroclinic instability. Finally, if the heating parameter is sufficiently large, the threshold for CISK is exceeded and the structure is influenced predominantly by convective processes. There are continuous transitions between each of these regimes, as was found for the baroclinic instability to CISK transition in chapter 1.

2.4 The Cyclonic Shear Region as Favorable Environment

The results of the previous section show that the cyclonic shear region north of the polar front is a favored region for rapid development of small scale disturbances similar in structure to observed polar lows and comma clouds. There appear to be two related processes responsible for this finding: the presence of conditional instability in this region, and the large positive vorticity and reduced static stability which enhance disturbance
growth. Both of these factors appear to be necessary for the development of small scale disturbances. To demonstrate this, tests were done to examine each effect separately.

First, the effects of increased vorticity and reduced static stability are tested by placing the heating on the anticyclonic shear side of the jet. Thus conditional instability is assumed only to exist in the region of reduced vorticity and increased stability on the south side. The horizontal distribution is similar to the original form (2.6), and is now given by

\[
\varepsilon(Y) = \varepsilon_0 \begin{cases} 
\cos(\lambda Y/2L), & 0 \leq Y/L \leq \pi/\lambda \\
0, & \pi/\lambda \leq Y/L \leq 2\pi/\lambda
\end{cases} \tag{2.38}
\]

The curves in Fig. 2.17 show that in this case the short wave modes do not attain faster growth rates than the long waves until the heating parameter is in excess of \(\varepsilon_0 = 2.5\). Even greater heating rates would be required to produce the large growth rates found for disturbances on the other side of the jet. At \(\varepsilon_0 = 2.5\) the disturbance center for the fastest growing short wave mode is located at \(Y = 0.15(2\pi/\lambda)\), and thus \(\varepsilon(Y)\) at the location of cyclogenesis is approximately 2.2. This is much greater than the corresponding threshold value of 0.8 when heating occurs in the cyclonic shear region (section 2.3b). Furthermore, this value is more than twice as large as the figures shown in Table 1.1 for the comma cloud cases of Reed and Blier (1986a,b). A value of the heating parameter higher than 2.2 was only obtained for the most extreme convective polar low.
Figure 2.17. As in Fig. 2.7 but with heating on the warm-air side of the jet ($\varepsilon(Y)$ given by (38)) for (a) $\varepsilon_0 = 1.7$, (b) $\varepsilon_0 = 2.0$, and (c) $\varepsilon_0 = 2.5$. 
Comparison of the two experiments with conditional instability on the two sides of the jet shows that short wave modes will only have large growth rates at physically reasonable values of the heating parameter when the system forms in the region of high vorticity and low stability. The causes of this interaction are complex. As noted in the introduction, at constant potential vorticity the growth rate of baroclinic instabilities is unaffected by vorticity and stability variations since their effects are equal and opposite. One way of looking at this is to note that reduced static stability produces enhanced vertical circulation and thus enhanced conversion of eddy potential to eddy kinetic energy through warm air rising and cool air sinking. On the other hand positive vorticity, which corresponds to high inertial stability, inhibits conversion of potential energy from the mean flow to the eddy by resisting meridional heat transports. The situation is different for the feedback between convective heating and low level convergence which drives the CISK mechanism. CISK is enhanced by reduced vertical stability since greater convergence is produced by a given heating rate. However this effect is not opposed by increased inertial stability which produces only a reduction in the scale of the disturbance. These processes are illustrated by a simple calculation of the threshold of instability for self-sustaining CISK which is described in Appendix B. Barotropic instability is not a dominant mechanism for any
of the modes considered here although it contributes to the growth of the short wave baroclinic disturbances. One would expect that this mechanism would be most dependent on the vorticity of the basic state. The examination of barotropic processes and the relation to CISK will therefore be left to future work where the potential vorticity is not assumed to be constant and vorticity and static stability can be varied independently.

Another interesting question in this context is to compare the results to a quasi-geostrophic analysis where meridional variations are assumed to be small. The equations of section 2.2 reduce to the quasi-geostrophic system in the limit \( R \to 0 \). The growth rate spectrum of the fastest growing mode using the alternative horizontal heating distribution (2.7) is shown in Fig. 2.18. Comparing these results with the semigeostrophic analysis in Fig. 2.12 shows that the threshold value of \( \varepsilon_0 \) for short wave baroclinic waves is increased by about 30%. Although the quasi-geostrophic results are qualitatively similar, the instability of the cyclonic shear region is significantly underestimated.

To test the importance of the localization of conditional instability in the cyclonic shear region, we now consider a situation where the heating parameter is constant, that is \( \varepsilon(Y) = \varepsilon_0 \). From Fig. 2.19 it is apparent that as the heating increases, the fastest growing mode
Figure 2.18. As in Fig. 2.12 but calculated in the quasi-geostrophic approximation.
Figure 2.19. As in Fig. 2.12 but with constant heating parameter $\varepsilon(Y) = \varepsilon_0$. 

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moves to short wavelengths comparable to those found previously. In fact the transition occurs at lower values of the heating parameter. However, a plot of the horizontal structure of the geopotential perturbation (Fig. 2.20) shows that the disturbance remains centered near the jet axis. In this case, the location of the heating is determined solely by low level convergence, and unless the heating parameter is large enough for CISK to occur, the location of the convergence is controlled by baroclinic instability. The existence of large amounts of conditional instability in the region of maximum baroclinicity allows an optimal cooperation between convective and baroclinic processes, thus allowing rapid growth rates and short wavelengths at lower values of the heating parameter than were possible when the heating only occurred away from the jet axis.

The possibility of modes centered on the jet is not unrealistic since the ability of latent heating to enhance the growth of large scale cyclones is well known. However the very large values of heating parameter required to produce short wavelength disturbances occur preferentially to the north of the jet axis where cold advection aloft combines with surface heating and moistening to give a particularly unstable atmosphere (Businger and Reed, 1989).

2.5 Chapter Summary

The investigation described in this chapter was motivated by observations that polar air stream cyclones
Figure 2.20. As in Fig. 2.8 but for the fastest growing mode with constant heating distribution and $\varepsilon_O = 1.2$, $k = 4.75$, $\sigma = 0.59$, and $c = 0.52$. Using the dimensional values in Table 2.1, the zonal wavelength of this disturbance is 1060 km.
form preferentially in regions of high positive vorticity and low static stability. Even the more baroclinic comma clouds tend to form on the flank of the baroclinic zone where these features are present rather than in the region of maximum baroclinicity. A semigeostrophic linear stability analysis of a constant potential vorticity zonal jet was performed with a cumulus parameterization based on low level convergence. The basic state featured realistic meridional variations of stability, vorticity and conditional instability. In comparison to the jet axis, the cyclonic shear region on the north side had high relative vorticity, low vertical stability, and enhanced conditional instability.

For parameter values comparable to those found in the genesis region of observed comma clouds, the fastest growing mode was a short wavelength disturbance confined in the meridional direction to the cyclonic shear region. Analysis of the structure of the eigenfunction showed it to be dominated by baroclinic processes, although the largest source of disturbance energy was release of latent heat. These features are in agreement with observations of Eastern Pacific comma clouds (Reed, 1979; Mullen, 1979; Reed and Blier, 1986a,b). For very low values of heating the fastest growing mode was similar to a dry baroclinic wave with slightly reduced wavelength and increased growth rate. For large heating rates, the instability was dominated by CIISK. It is important to note that there was a continuous
transition between each of these behaviors as the parameters of the background state were varied. Under appropriate conditions the disturbance could be of a transitional nature or even change its characteristics in response to changes in the large scale flow.

The cyclonic shear region in this model possessed three features which could contribute to the preferred growth of small scale systems; the existence of conditional instability in that region, low vertical stability, and high positive vorticity. It was found that the localized region of heating was necessary for the existence of short wave meridionally confined disturbances. In the absence of heating or when heating was allowed over the entire domain, such modes did not occur. However the mere presence of heating in a given region does not ensure the existence of small scale systems. It was found that such disturbances would not occur for reasonable parameter values in the absence of reduced stability and enhanced vorticity. It appears that low static stability is the primary factor since this enhances feedback between heating and low level convergence and thus promotes CISK. For baroclinic instability, the effects of low stability are offset by the high vorticity. High vorticity may provide a positive contribution, however, through barotropic instability. Calculations of energy budgets show that Reynolds stress conversions contribute to the growth of short wave meridionally confined baroclinic waves, although they are
detrimental to both jet-centered baroclinic modes and CISK disturbances.

The relative importance of baroclinity, horizontal shears, vertical stability and heating depends upon the details of the basic state being considered. While the values in this study were chosen to be typical of the comma cloud genesis environment, the parameters can be expected to vary considerably from case to case. In order to determine the relevance of the processes described here, it is necessary to consider individual systems. This will be attempted in the next chapter.
CHAPTER 3

Moist Stability of TwoObserved Jets

3.1 Introduction

In the previous chapter it was shown that the combination of baroclinic instability and convection in the cold air mass could produce a short wavelength disturbance confined to the cyclonic shear region of a jet flow. This structure is similar to that of observed comma cloud systems. It was further shown that the low vertical stability in this region produced a favorable environment for such disturbances due to its interaction with convective processes, even though the effects on baroclinic instability were offset by high inertial stability. It was suggested that the reduction in stability might be crucial for the formation of comma clouds in environments with physically reasonable values of baroclinicity and heating.

The precise balance between the effects of static stability and vorticity on baroclinic instability in a constant potential vorticity environment will not necessarily occur in the real atmosphere. It is therefore desirable to consider more realistic flows to determine whether the interactions found to be important in the previous model are the most relevant ones in observed comma cloud formation. This chapter will employ a generalized model to consider initial conditions taken from observations.
of two cases of comma cloud cyclogenesis. More sophisticated numerical methods will be used in order to efficiently obtain solutions when the more complex basic states are used.

The case studies to be examined here are the two classic comma clouds of Reed and Alier (1986a,b). Both of these disturbances formed on the cyclonic shear side of an upper level jet streak in the confluent region on the west side of a planetary wave trough. Thus the genesis environments of both systems have strong narrow jets with high vorticity values. Furthermore, southward advection of cold air over relatively warm ocean waters results in reduced vertical stability in the lower troposphere and increased moisture for convection. These cases, which are typical of many observed systems, are thus qualitatively similar to basic states examined in chapter 2.

The outline of the chapter is as follows. Section 3.2 describes the model equations and the numerical techniques used to solve them. In addition, the analysis and preparation of the basic state fields are discussed. In section 3.3, the solutions are presented and interpreted in terms of the results from chapter 2. The calculated disturbances are then compared to observations in section 3.4 and the results are summarized briefly in section 3.5.

3.2 Model Description
a) Equations and Numerical Methods

As in the previous chapter, we consider the stability of small perturbations to a parallel flow in thermal wind balance. The basic state wind is denoted by $U$ and the potential temperature by $\Theta$. The vorticity is then given by $\zeta = f/(1 + U_Y/f)$, and the potential vorticity by $Q = \zeta (g/\Theta_O) \Theta_Z$, where $Y$ and $Z$ are the geostrophic meridional and vertical coordinates and $\Theta_O$ is a reference value of potential temperature. $Y$ is referred to as the meridional direction for convenience, but may correspond to any geographical direction on an $f$-plane. The domain is a channel bounded by rigid walls at $Y = -L$, $L$ and $Z = 0$, $H$. It will be required that the basic state wind vanish at the lateral boundaries, which implies that the dimensions of the region will be the same in physical coordinates.

The governing equation is the linearized potential vorticity equation

$$\left(\frac{\partial}{\partial T} + U \frac{\partial}{\partial X}\right) \left[\frac{1}{f^2} \left(\Phi''_{XX} + \Phi''_{YY}\right) + \left(\frac{f}{Q} \Phi'_Z\right)_Z\right]$$

$$+ S \Phi'_X = \frac{g}{\Theta_O} \left(\frac{f}{Q} E'_Z\right)_Z$$

(3.1)

where $\Phi'$ is the perturbation semigeostrophic geopotential and $E'$ the perturbation heating. The quantity $S$, defined by Hoskins (1976), is the meridional derivative of potential vorticity on isentropic surfaces.
\[ S = \frac{1}{\zeta \theta} \left[ Q_y + \frac{\theta_x}{\theta_y} Q_z \right]. \] (3.2)

At the top and bottom boundaries, the vertical velocity must vanish, resulting in the following condition (cf. 2.3),

\[ \left[ \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right] \Phi' Z - U_z \Phi' X = 0, \quad Z = 0, \quad H. \] (3.3)

The normal velocity at the lateral walls must also vanish, which for this linear problem is equivalent to

\[ \Phi' = 0, \quad Y = -L, \quad L. \] (3.4)

The heating is assumed to be proportional to the boundary layer convergence rate,

\[ E'(X,Y,Z) = E_0(Y,Z)w'(X,Y,Z_B) \]

\[ = E_0 \left[ -\frac{\xi}{Q} \left( \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right) \Phi' X - \frac{\Theta_x}{\Theta_y} \Phi' X \right] \text{at } z = z_B \] (3.5)

where \( z_B \) is the boundary layer depth. The function \( E_0(Y,Z) \) is nonzero only in the presence of conditional instability and is determined primarily by boundary layer moisture content. In the notation of the previous chapter, \( E_0 = \epsilon(Y)h(Z) \). The data sources and methods of analysis for the basic state dynamical fields and heating functions will be described in a subsequent section. The energy equations are as listed in the previous chapter (equations 2.17 and 2.18), since the assumption of constant potential vorticity was not used in the derivation.

For the numerical solution of the problem, it is convenient to have the domain extend from -1 to 1 in both the lateral and vertical directions. The independent
variables will thus be nondimensionalized as follows:

\[(X, Y, Z, T) = (LX^*, LY^*, H'(Z^*+1), (L/U_0)T^*) \quad (3.6)\]

where stars denote nondimensional quantities and \(H' = H/2\).

The basic state quantities are nondimensionalized by

\[(U, \Theta, \zeta, Q, S) = (U_0U^*, (\Theta_0/g)N^2H'\Theta^*, f\zeta^*, \epsilon N^2Q^*, (lNH')^{-1}S^*) \quad (3.7)\]

where \(l = Lf/NH'\) is the ratio of the domain half-width to the radius of deformation. Lastly, the perturbation quantities are nondimensionalized by

\[(\Phi', \varepsilon', w') = (N^2H'^2\Phi^*, (\Theta_0/g)N^2H'(U_0/L)E^*, (U_0H'/L)\omega^*) \quad (3.8)\]

The nondimensional equations are (dropping stars)

\[
\left[\frac{\partial}{\partial T} + U_0\frac{\partial}{\partial X}\right] \left[\frac{1}{l^2}\left(\Phi_{XX} + \Phi_{YY}\right) + \left(\frac{1}{Q}\Phi_Z\right)\right] + \frac{1}{lR}\Phi X = \left[\frac{1}{Q}\right] E Z \quad (3.9)
\]

with boundary conditions

\[\Phi = 0, \quad Y = -1, 1 \quad (3.10)\]

and

\[
\left[\frac{\partial}{\partial T} + U_0\frac{\partial}{\partial X}\right] \Phi Z - \frac{1}{lR}\Theta_Y \Phi_X = 0, \quad Z = -1, 1. \quad (3.11)
\]

A nondimensional parameter \(R = U_0/NH\) has been introduced.

We look for solutions which are wavelike in the \(X\) direction, that is

\[\Phi(X,Y,Z) = \phi(Y,Z)\exp[ik_z - cT] \quad (3.12)\]

where \(k\) is wavenumber and \(c\) is the complex phase speed. The
resulting equation is

\[(U-c) \left[ -k^2 \phi + \phi_{YY} + \frac{k^2}{Q^2} \phi_{ZZ} - \frac{k^2}{Q^2} \phi_z \right] + \frac{\lambda}{R} \phi = 0\]

\[= \left[ E_{0Z} - \frac{Q_{zz}}{Q_{zz}} \right] + \frac{1}{Q} \left[ -\frac{1}{\theta_{zz}} \left( U_{zz} + \frac{\theta_{zy}}{\theta_{zy}} - C_{zz} \right) \right]_{z=Z_B} \quad (3.13)\]

with

\[\phi = 0, \quad Y = -1, 1 \quad (3.14)\]

and

\[(U-c) \phi_z + (\theta_{zy}/\lambda R) \phi = 0, \quad Z = -1, 1. \quad (3.15)\]

This equation is solved as a matrix eigenvalue problem. To achieve high accuracy, a spectral method is employed with collocation weighting (Gottlieb and Orszag, 1977; Canuto et al., 1988). The eigenfunction \(\phi\) is expanded in series of orthogonal functions and equations for the coefficients are obtained by requiring that the series exactly satisfy the governing equation at a set of collocation points. Therefore we write

\[\phi(Y,Z) = \phi_{j} P_{\beta}(Y) T_{j}(Z). \quad (3.16)\]

The boundary conditions largely determine the choice of basis functions \(P_{\beta}\) and \(T_{j}\), which in turn determine the collocation points. The simple lateral boundary conditions (3.14) allow the use of a Fourier sine expansion in the \(Y\) direction,

\[P_{\beta} = \sin((1+Y)(\beta-1)\pi/2), \quad \beta = 2...M. \quad (3.17)\]

The values \(\beta = 0, 1\) are omitted since the basis functions
satisfy the boundary conditions at the endpoints. The collocation points for this expansion are equally spaced (Gottlieb and Orszag, 1977),

\[ Y_\alpha = 1 - 2\alpha/M, \quad \alpha = 1 \ldots M-1. \quad (3.18) \]

In the vertical direction it is not possible to choose basis functions which satisfy the boundary conditions (3.15) since they contain the unknown eigenvalue. In this case we use an expansion in Chebyshev polynomials, which can take arbitrary values at -1 and 1, and include the endpoints as collocation points, applying the boundary condition rather than the interior equation. The Chebyshev polynomials are defined as

\[ T_j(Z) = \cos(n \arccos Z), \quad j = 0 \ldots N \quad (3.19) \]

but are usually evaluated from a recurrence relation, which is given by Boyd (1978), for example, who also provides recurrence relations for the first and second derivatives. From the definition (3.19) it can be seen that the polynomials of quadratic and higher order are identical to the sine series under a mapping which stretches the central region of the domain and compresses the regions near the boundaries. The optimal set of collocation points for use with these basis functions are the Gauss-Lobatto quadrature points

\[ Z_i = \cos(\pi i/N), \quad i = 0 \ldots N \quad (3.20) \]

which are equal to the evenly spaced points (3.18) under the
same transformation (Canuto et al., 1988).

It was sometimes suggested in the early literature that the increased density of points near the boundary resulted in greater accuracy in these regions than would be obtained with a Fourier expansion and equally spaced collocation points. In fact the additional points serve only to offset the greater tendency of the polynomials to oscillate (Solomonoff and Turkel, 1989). The number of points required for a given level of accuracy is roughly proportional to the widest spacing between points (the center of the domain), implying that a Chebyshev series would require approximately \( \pi/2 \) more terms than a Fourier series. This relation seemed to hold in the present model when results using Fourier and Chebyshev basis functions in \( Y \) were compared.

Substituting the expansion (3.16) into the equation and boundary conditions and evaluating at the grid of collocation points \((Y_{\alpha}, Z_i)\) defined by (3.18) and (3.20) gives a real generalized matrix eigenvalue problem

\[
A_{\alpha i \beta j} = c B_{\alpha i \beta j}
\]

(3.21)

with matrix elements given by

\[
A_{\alpha i \beta j} = U(Y_{\alpha}, Z_i) \left[ -k^2 P_{\beta} (Y_{\alpha}) T_j(Z_i) + P'' \beta (Y_{\alpha}) T_j(Z_i) \right]
\]
\[ + \frac{k^2}{Q(Y_{\alpha}, Z_i)} P_\beta (Y_{\alpha}) T''_{j}(Z_i) - \frac{k^2 Q_{Z}(Y_{\alpha}, Z_i)}{Q^2(Y_{\alpha}, Z_i)} P_\beta (Y_{\alpha}) T'_{j}(Z_i) \]

\[ + \left[ E_{oZ}(Y_{\alpha}, Z_i) - \frac{Q_{Z}(Y_{\alpha}, Z_i)}{Q(Y_{\alpha}, Z_i)} E_{o}(Y_{\alpha}, Z_i) \right] \frac{k^2}{Q(Y_{\alpha}, Z_i) \Theta_{Z}(Y_{\alpha}, Z_B)} \]

\[ \left[ U(Y_{\alpha}, Z_i) P_\beta (Y_{\alpha}) T''_{j}(Z_B) + \frac{\Theta_{Y}(Y_{\alpha}, Z_B)}{\ell_R} P_\beta (Y_{\alpha}) T'_{j}(Z_B) \right] \] (3.22)

and

\[ B_{\alpha i \beta j} = -k^2 P_\beta (Y_{\alpha}) T_{j}(Z_i) + P''_{\beta} (Y_{\alpha}) T'_{j}(Z_i) \]

\[ + \frac{k^2}{Q(Y_{\alpha}, Z_i)} P_\beta (Y_{\alpha}) T''_{j}(Z_i) - \frac{k^2 Q_{Z}(Y_{\alpha}, Z_i)}{Q^2(Y_{\alpha}, Z_i)} P_\beta (Y_{\alpha}) T'_{j}(Z_i) \]

\[ + \left[ E_{oZ}(Y_{\alpha}, Z_i) - \frac{Q_{Z}(Y_{\alpha}, Z_i)}{Q(Y_{\alpha}, Z_i)} E_{o}(Y_{\alpha}, Z_i) \right] \]

\[ \frac{k^2}{Q(Y_{\alpha}, Z_i) \Theta_{Z}(Y_{\alpha}, Z_B)} P_\beta (Y_{\alpha}) T''_{j}(Z_B) \] (3.23)

for \( i = 1 \ldots N-1, \ j = 0 \ldots N, \ \alpha = 1 \ldots M-1, \) and \( \beta = 2 \ldots M. \) The boundary terms are given by

\[ A_{\alpha i \beta j} = U(Y_{\alpha}, Z_i) P_\beta (Y_{\alpha}) T''_{j}(Z_i) \]

\[ + \frac{\Theta_{Y}(Y_{\alpha}, Z_i)}{\ell_R} P_\beta (Y_{\alpha}) T'_{j}(Z_i) \] (3.24)

and

\[ B_{\alpha i \beta j} = P_\beta (Y_{\alpha}) T''_{j}(Z_i) \] (3.25)

for \( i = 0, N, \ j = 0 \ldots N, \ \alpha = 1 \ldots M-1, \) and \( \beta = 2 \ldots M. \)

The four-dimensional matrices are mapped into standard two-dimensional form.
\[ A_{rs} \phi_s = cB_{rs} \phi_s \]  

(3.26)

with the transformation

\[ r = \alpha + (M-1)i \]

\[ s = \beta - 1 + (M-1)j. \]  

(3.27)

Unlike the band matrices obtained from finite difference methods, those resulting from spectral collocation methods are dense and frequently ill-conditioned. There has been much work recently on pre-conditioning methods for such matrices but no additional treatment was required for the present model. The eigenvalues were found using the QZ algorithm of Moler and Stewart (1973), as implemented in EISPACK (Garbow et al., 1977). The eigenfunctions were then found by using the known eigenvalues and solving the resulting system of linear equations. This method is much faster than using the QZ algorithm when only a few eigenfunctions are required. The accuracy of the eigenvalue-eigenfunction pairs as solutions of the matrix problem was verified using the performance index defined by Garbow et al. (1977)

\[ \mu = \frac{||A_{rs} \phi_s - cB_{rs} \phi_s||}{\epsilon_M (||A_{rs}|| + ||c|| ||B_{rs}||) ||\phi_s||} \]  

(3.28)

where \( ||\phi_s|| \) is the norm given by

\[ ||\phi_s|| = \sum_s ||\phi_s|| \]  

(3.29)

and \( \epsilon_M \) is the machine precision. In all cases, the index was less than 100 w.h., indicating an accurate solution.
Convergence was tested by varying the horizontal and vertical resolution M and N separately (Table 3.1). Calculations were made for both a long-wave mode without heating and a short wave mode, but results are shown only for the short wave since the convergence was much faster for the long wave mode, as was the case in chapter 2. The values in Table 3.1 show that a resolution of M = N = 21 is sufficient to produce eigenvalues with error less than 1%. The matrix size for the spectral method is thus similar to that used in the previous model but produces better accuracy for a more difficult problem (one with greater finer structure in the basic state).

b) Basic State Fields

This study will focus on the genesis environments of the two comma clouds studied by Reed and Blier (1986a,b). These were chosen primarily because of the availability of relatively good data sets through Reed and Blier's analysis. The two systems were small-scale cyclones (1000-2000 km in wavelength) which formed over the eastern Pacific Ocean in March and November of 1982. Each system formed beneath the cyclonic shear region of a jet streak and thus appears to correspond to the scenario treated in the idealized model of the previous chapter.

The stability analysis is based on the cross-sections of the flow constructed by Reed and Blier (1986a,b) which
Table 3.1. Nondimensional phase speed and growth rate of the fastest growing mode for the March case at wavenumber $k = 8.0$. Horizontal and vertical resolution take the values shown. For comparison purposes, the solution correct to four significant figures (computed with $M = N = 30$) is $(0.5066, 0.3947)$, and the solution at the resolution used in this study ($M = N = 21$) is $(0.5080, 0.3956)$.

<table>
<thead>
<tr>
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<tr>
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<td>15</td>
<td>(0.5060, 0.3662)</td>
</tr>
<tr>
<td>16</td>
<td>(0.4902, 0.3889)</td>
<td>16</td>
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<td>(0.5072, 0.3875)</td>
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<td>(0.5069, 0.3802)</td>
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<td>25</td>
<td>(0.5066, 0.3950)</td>
<td>25</td>
<td>(0.5071, 0.3820)</td>
</tr>
</tbody>
</table>
are reproduced here in Fig. 3.1. They were prepared using the National Meteorological Center (NMC) analysis of wind and geopotential height on standard pressure levels. Bogus temperature "soundings" were constructed at positions indicated by the vertical lines in Fig. 3.1. The wind field is largely determined by the geostrophic wind except near the surface and jet stream levels where ship and aircraft observations were available. The reader is referred to the original papers for a complete description of the analysis procedure and a discussion of the reliability of the results.

An attempt was made for the present study to construct a cross-section of geopotential height by objective interpolation directly from the NMC charts, and to use this to determine geostrophically balanced wind and temperature fields. However, the data at the surface and five standard pressure levels were not sufficient to produce realistic vertical structures, especially near the tropopause. It was therefore decided to base the calculations on Reed and Blier's subjectively corrected temperature analysis.

Two modifications were made to the temperature profiles shown in Fig. 3.1. First, the convectively neutral boundary layer on the cold air side of the jet in the lowest 100 mb was eliminated by reploting the bogus soundings and extrapolating the temperature curves downwards. Secondly, a reanalysis of the temperature field on the 700 and 850 mb
Figure 3.1. Synthetic cross sections for (a) 0000 GMT 16 March 1982 (reproduced from Reed and Blier, 1986a), and (b) 1200 GMT 8 November 1982. The reader is referred to original references for location of sections. Isotachs normal to section (ms⁻¹), solid lines; isotherms (°C), dashed lines. Stippling in (b) depicts clouds; heavy lines indicate regions of positive buoyancy for lifted surface parcels. The letters T and H in (a) and A and B in (b) show positions of particular cloud masses; see text for details. The width of the cross sections are (a) 1720 km, and (b) 1680 km.
surfaces in the cold air mass was carried out for the March case. It can be seen in Fig. 3.1b, that the wind and temperature fields in Reed and Blier's analysis in this region deviate substantially from thermal wind balance. The reanalysis resulted in modifications of the temperatures on the two bogus soundings at the far right of Fig. 3.1b, which brought the calculated geostrophic wind much closer to that observed. The maximum change in temperature was 1.5 K, and in most cases less than one degree. This is probably less than the error in the observations in this data-sparse region.

Cubic spline interpolants were constructed for the temperature data which were used to calculate geostrophic wind by integrating the thermal wind equation. Observed wind speeds at the 400 mb level were used as initial conditions in this integration. The other basic state quantities were then calculated, moved to the transformed space, and nondimensionalized.

The fields resulting from this analysis procedure were rather noisy. While the small scale noise is not of physical importance, the spectral method used in the stability analysis will only provide accurate solutions if the background state is smooth (Boyd, 1978). It was therefore necessary to filter the basic state data. This was accomplished by representing the fields as truncated two-dimensional Chebyshev series,
\[ \chi(Y,Z) = \sum_{i=0}^{M} \sum_{j=0}^{N} \chi_{ij} T_{i}(Y) T_{j}(Z) \] (3.30)

where \( \chi \) represents \( U \) or any other basic state quantity. A simple truncation would suffice to remove small-scale noise, but when the data contains large gradients in the field quantities, as is the case here, oscillations are produced which contaminate the field well away from the feature which caused them. The oscillations can be suppressed by multiplying the coefficients by a smoothly decreasing function. The present study employed a modification of the raised cosine filter described by Canuto et al. (1988),

\[ \chi'_{ij} = \alpha_{i} \beta_{j} \chi_{ij} \] (3.31)

with

\[
\alpha_{i} = \begin{cases} 
1, & 0 \leq i \leq (1-s)M \\
(1 + \cos[(i-(1-s)M)\pi/sM])/2, & (1-s)M \leq i \leq M \\
0, & i \geq M 
\end{cases}
\]

and

\[
\beta_{j} = \begin{cases} 
1, & 0 \leq j \leq (1-s)N \\
(1 + \cos[(j-(1-s)N)\pi/sN])/2, & (1-s)N \leq j \leq N \\
0, & j \geq N.
\end{cases}
\]

The value of the parameter \( s = 0.6 \) was chosen after some experimentation as offering the minimum amount of smoothing necessary to suppress the undesired oscillations. The resolution \( M = N = 15 \) for the basic state expansions was used in all the experiments to be discussed here. A limited number of calculations were performed with \( M = N = 20 \), but no significant differences were detected in either the
appearance of the basic state or the results of the stability analysis.

Figs. 3.2 and 3.3 show the resulting basic state fields. For the March case (Fig. 3.2), the potential vorticity in the troposphere is largely uniform except in the upper levels where it decreases to the south. The vorticity at cloud base level on the cyclonic shear side of the jet in the lower troposphere is about 1.3f while the vertical stability (or more precisely $\Theta_z$) is reduced by about 30% from the values at the jet axis. These values are similar to those used in the idealized basic state in the previous chapter.

In the November case (Fig. 3.3) there is some evidence of a tongue of high potential vorticity such as might be associated with a tropopause fold, although the data is insufficient to fully resolve such a feature. The wind field shows a minimum in baroclinicity in the cyclonic shear region. The vorticity and stability patterns are qualitatively similar to those of the March case but there is a more pronounced minimum of stability in the cold air mass, possibly due to prior convective activity.

c) Parameters for Cumulus Heating

The highly idealized nature of the cumulus parameterization used in this model makes the calculation of parameter values for observed cases problematic. The
Figure 3.2. Basic state fields based on cross section for March case: (a) wind speed (ms$^{-1}$), solid lines; potential temperature (K), dashed, (b) potential vorticity (contour interval 0.2 PVU = 10$^{-6}$Ks$^{-1}$kg$^{-1}$m$^{-2}$ for Q < 1 PVU, 0.5 PVU for Q > 1 PVU), (c) absolute vorticity (10$^{-6}$s$^{-1}$), (d) $\partial \Theta / \partial Z$ (10$^{-4}$K/m). The depth of the domain is H = 10.6 km, and the half-width is L = 1346 km.
Figure 3.2, continued.
Figure 3.3. Basic state fields based on cross section for November case: (a) wind speed (ms$^{-1}$), solid lines; potential temperature (K), dashed; (b) potential vorticity (contour interval 0.2 PVU = 10$^{-6}$Ks$^{-1}$kg$^{-1}$m$^2$ for Q < 1 PVU, 0.5 PVU for Q > 1 PVU), (c) absolute vorticity (10$^{-6}$s$^{-1}$), (d) $\partial \theta / \partial z$ (10$^{-4}$K/m). The depth of the domain is $H = 10.6$ km, and the half-width is $L = 1440$ km.
Figure 3.3, continued.
formula (3.5), widely used in analytic models, can be regarded as a simplification of the parameterization of Kuo (1974). In this scheme the total heating in an atmospheric column is determined from the total moisture convergence including surface fluxes, less a fraction which remains as water vapour in the column. The heat is then distributed in the vertical according to some function derived from cloud and environment properties. If surface fluxes, convergence of moisture above the boundary layer, and loss of water vapour to moisten the air column are neglected, and the vertical distribution of heating is simply specified, (3.5) is obtained. In this case

$$E_0 = \frac{L_C}{c_p} q_m \left( \frac{p_0}{p} \right)^k h(z)$$  \hspace{1cm} (3.32)

where $L_C$ is the latent heat of evaporation for water, $c_p$ the heat capacity of air at constant temperature and $q_m$ is the vertically averaged boundary layer water vapour mixing ratio. The factor $(p_0/p)^k$ converts the heating rate from rate of change of temperature to rate of change of potential temperature. It is approximately equal to one in the troposphere and will be absorbed into the definition of the vertical distribution function $h(z)$.

It has been shown that the factors neglected in this simple formulation, especially surface fluxes, are not generally negligible for polar air stream cyclones (Rasmussen, 1979; Reed and Blier, 1989b; Craig and Cho, 1988). Therefore the present study will employ the modified
formula introduced in the earlier work which includes corrections for these factors

\[ E_0 = \frac{L_c}{c_p} \left( \frac{p_0}{p} \right)^{x} (1 - b)q_m(1 + r + \eta) h(z). \] (3.33)

Here, \( b \) is the parameter defined by Kuo (1974) which represents the fraction of converged moisture which remains in the column as water vapour, \( r \) is a correction for surface fluxes of moisture, and \( \eta \) for convergence of moisture above the boundary layer.

The factor \( r \) is defined as the ratio of surface evaporation, determined from the bulk aerodynamic formula, to boundary layer moisture convergence,

\[ r = \frac{C_D V (q_s - q_m)}{q_m W_B} \] (3.34)

where the drag coefficient \( C_D = 1.5 \times 10^{-3} \), \( q_s \) is the saturation mixing ratio at the sea surface temperature, and \( V \) and \( W_B \) are typical values of the surface wind speed and vertical velocity at cloud base, respectively. The values for \( V \) and \( W_B \) were estimated in Craig and Cho (1988) to be 14 m/s and 0.020 m/s respectively for the March case and 15 m/s and 0.021 m/s for the November case. The mixing ratios \( q_m \) and \( q_s \) were obtained from surface charts given by Reed and Blier (1986a,b) and are shown in Table 3.2 for the two cases.

The estimates of \( b \) and \( \eta \) require knowledge of the moisture field in mid levels. Unfortunately no direct
Table 3.2. Parameters used in calculation of cumulus heating. See text for definitions. The numbers identifying each bogus sounding correspond to the vertical lines in the cross sections of Reed and Blier (1986a,b) shown in Fig. 3.1, numbered from left to right.

<table>
<thead>
<tr>
<th>&quot;sounding&quot;</th>
<th>November</th>
<th>March</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>( q_m ) (g/kg)</td>
<td>5.5</td>
<td>4.8</td>
</tr>
<tr>
<td>( q_s ) (g/kg)</td>
<td>10.0</td>
<td>10.6</td>
</tr>
<tr>
<td>( r )</td>
<td>0.88</td>
<td>1.29</td>
</tr>
<tr>
<td>cloud class</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>( 1-b )</td>
<td>0.74</td>
<td>0.90</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0.83</td>
<td>0.81</td>
</tr>
<tr>
<td>( (1-b)q_m(1+r+\eta) )</td>
<td>11.0</td>
<td>13.4</td>
</tr>
<tr>
<td>cloud top height (mb)</td>
<td>550</td>
<td>510</td>
</tr>
</tbody>
</table>
measurements of these quantities exist. As a substitute, relative humidity profiles were inferred using a technique developed for the initialization of numerical models (NMC, 1972). In this scheme, satellite imagery is used to divide the cloud regions into 14 classes for which typical relative humidity profiles have been measured. The cloud class was identified for each bogus sounding and the appropriate humidity profiles used to calculate \( b \) and \( \eta \). The factor \( b \) is estimated from the empirical formula of Anthes (1977)

\[
b = \begin{cases} 
   \left( \frac{1 - RH}{1 - RH_c} \right)^n, & \text{RH} \geq RH_c \\
   1, & \text{RH} < RH_c 
\end{cases}
\]  

(3.35)

where \( RH \) is the average relative humidity below 400 mb and \( RH_c \) and \( n \) are constants, which are given the values \( RH_c = 0.25 \) and \( n = 2 \), consistent with the estimates of Kuo and Anthes (1984). The correction for convergence above the boundary layer is given by

\[
\eta = \frac{1}{q_m w(z_B)} \int_{z_B}^{H} q \frac{\partial w}{\partial z} \, dz.
\]  

(3.36)

As a rough approximation, it is assumed for the purposes of this calculation that \( w \) has the form

\[
w(z) = w_{\max} \sin \frac{\pi z}{z_0}
\]  

(3.37)

where \( z_0 = 8 \) km. The values of \( b \) and \( \eta \) are listed in Table 3.2.
The detailed structure that the vertical profile of heating should have is not well known, although most studies suggest a distribution with level of maximum heating in the mid-troposphere and decreasing to zero below cloud base and above cloud top (Johnson, 1974; Yanai et al., 1973). Here we will use a cubic polynomial distribution

\[
\left( \frac{p_0}{p} \right)^k h(z) = \begin{cases} 
0, & z < z_B \\
A(z - z_B)(z - z_T)(z - C), & z_B < z < z_T \\
0, & z_T < z
\end{cases} \tag{3.38}
\]

where

\[
C = \frac{3z_L^2 - 2z_L(z_B + z_T) + z_Bz_T}{2z_L - z_T - z_B}
\]

with cloud base \( z_B \), cloud top \( z_T \) and \( z_L \) being the level of maximum heating. The function is normalized so that the integral from \( z_B \) to \( z_T \) is unity by setting

\[
A = \left[ (z_T^4 - z_B^4)/4 - (z_B + z_T + C)(z_T^3 - z_B^3)/3 
+ (z_Bz_T + Cz_B + Cz_T)(z_T^2 - z_B^2)/2 - Cz_Bz_T(z_T - z_B) \right]
\]

At each of the bogus soundings indicated by vertical lines in Fig. 3.1, the cloud top is determined from the level of zero buoyancy for lifted surface parcels. The cloud base is fixed at 900 mb. The level of maximum heating is set at \( (z_B + z_T)/2 \). The dependence of the parameterization on this value is discussed by Craig and Cho (1988).

Cumulus heating is nonzero only where conditional instability exists. In the March case, this was only true
for the two most northerly verticals in Fig. 3.1a. In particular, there was no convective instability at the adjacent two verticals in the cyclonic shear region, just to the north of the jet axis where the comma cloud was observed to form. This is probably due to the cross-section being taken about 200 km upstream of where the disturbance was observed to develop. At the cross section Reed and Blier calculate lifted indices of +2 or +3 on the cyclonic shear side of the jet, but they note that further downstream at the incipient disturbance the lifted index was reduced to between +1 and 0. Indeed satellite images showed strong convective activity in this region (Reed and Blier, 1986a). In view of this, the heating parameters were recalculated using values for a cross-section which passed through the developing system. The original fields of temperature and other dynamical variables were used since they did not vary greatly above the boundary layer. The primary difference between this and the data at the position of the original section is a greater boundary layer moisture content, presumably due to greater travel over warm water. This implies a greater boundary layer equivalent potential temperature which leads to conditional instability to a considerable depth. The change to the corrected value of $E_0$ defined in (3.33) is actually quite small since the larger $q_m$ is almost entirely offset by a reduction in the surface flux correction. This is due to a smaller contrast between $q_m$ and the saturated surface value $q_s$. 
The heating function $E_o(y,z) = \epsilon(y) h(z)$ was smoothed as described for the other basic state fields, and is shown in Fig. 3.4 for the two systems. In both cases there is heating throughout the cold air mass, terminating near the axis of the upper level jet. This is in agreement with the observations of Reed and Blier (1986a,b) who identified the jet axis as the dividing line between deep convection on the cold air side and low level stratiform cloud in the warm air. The heating rates tend to increase towards the jet axis where warmer air temperatures allow a greater atmospheric moisture content.

3.3 Results

Results of the stability analysis will now be presented in turn for each of the cases described in the previous section. The discussion in this section will focus on the dynamical character of the disturbances. A comparison of the model instabilities with the observed systems will follow in section 3.4.

a) March Case

We consider first the stability of the jet excluding cumulus heating. The curves of growth rate and phase speed, shown in Fig. 3.5, are dominated by two unstable modes, as found in chapter 2 and the previous work of Brown (1969). The fastest growing mode, with a wavelength of 2820 km, is driven primarily by barotropic energy conversions, although
Figure 3.4. Constant of proportionality ($E_o$) between heating rate and cloud base vertical velocity ($10^{-4}$ Km$^{-1}$), for (a) March case, (b) November case.
Figure 3.5. (a) Nondimensional phase speed $c$ and (b) growth rate $\sigma$ as functions of nondimensional wavenumber $k$, for March case with no cumulus heating.
baroclinic instability also contributes to the growth of the system (Fig. 3.6b). The structure of the eigenfunction, plotted in Fig. 3.6a, shows that the disturbance extends across the width of the jet and through the depth of the troposphere with largest amplitude on the anticyclonic shear side of the jet maximum. These features are similar to those of the long wave barotropic mode seen in the model of the previous chapter (Figs. 2.3b, 2.4b). The primary differences in structure are due to the presence of a tropopause in this case.

The second mode, with structure and energetics shown in Fig. 3.7, is a baroclinic instability with wavelength 2250 km. The contribution of barotropic processes is negative for this mode, as for the baroclinic instability in the idealized model (Figs. 2.3a, 2.4a). The comparison of growth rates of the two instabilities for different jet widths (Fig. 2.5) suggests that the reason for the relative dominance of the more barotropic mode in this case is the narrowness of the observed jet. The maximum relative vorticity seen in Fig. 3.2c is approximately equal to the planetary vorticity $f$, and is twice as large as the largest value in Fig. 2.5. This is consistent with the disturbance energy budgets (Figs. 3.6b, 3.7b) where the Reynolds stress terms (RS) are approximately twice as large in comparison to the horizontal heat flux (HHF) as for the corresponding mode in chapter 2 (Fig. 2.4).
Figure 3.6. (a) Cross section of amplitude of eigenfunction for the fastest growing mode in Fig. 3.5 with wavenumber $k = 3.0$. The function has been normalized to a maximum amplitude of one. (b) Energy budget of mode shown in (a). Abbreviations are defined in section 2.2b. Fluxes are normalized so that horizontal heat flux (HHF) is one unit.
Figure 3.7. (a) Cross section of amplitude of eigenfunction for the fastest growing mode in Fig. 3.5 with wavenumber \( k = 4.25 \). The function has been normalized to a maximum amplitude of one. (b) Energy budget of mode shown in (a). Abbreviations are defined in section 2.2b. Fluxes are normalized so that horizontal heat flux (HHF) is one unit.
The amplitude structure of the more baroclinic and barotropic instabilities of the observed flow are very similar. The different signs of the barotropic energy conversion term are associated with different horizontal phase tilts at the level of the jet axis (not shown). The relative weakness of the more baroclinic wave at the surface is probably due to a combination of low static stability in the upper troposphere in the center of the domain and weaker baroclinicity in the lowest levels.

When cumulus heating, with the distribution plotted in Fig. 3.4a, is included, the growth rate curves take the form shown in Fig. 3.8. The large-scale barotropic mode is essentially unmodified by diabatic effects, but there is a large destabilization of the more baroclinic branch at short wavelengths as was found in chapter 2. The fastest growing wave has a wavelength of 1210 km, although the scale selection is very weak due to a pronounced shoulder in the curve near 1990 km at the location of the baroclinic mode in the adiabatic spectrum (Fig. 3.5). We will examine the structure and energetics at both of these wavelengths. The mode at the short wave peak grows rapidly with an e-folding time of 0.64 days. As for the short-wave meridionally confined modes studied in the previous chapter, this disturbance has significant amplitude only in the heating region (Fig. 3.9a), and is driven by a combination of baroclinic and convective processes, with heating providing the largest contribution (Fig. 3.9b). The eigenfunction
Figure 3.8. (a) Nondimensional phase speed $c$ and (b) growth rate $\sigma$ as functions of nondimensional wavenumber $k$, for March case with cumulus heating.
Figure 3.9. (a) Cross section of amplitude of eigenfunction for the fastest growing mode in Fig. 3.8 with wavenumber $k = 7.0$. The function has been normalized to a maximum amplitude of one. (b) Energy budget of mode shown in (a). Abbreviations are defined in section 2.2b. Fluxes are normalized so that horizontal heat flux (HHF) is one unit.
also shows a northward phase tilt with height, reflecting the sloping of the absolute vorticity vector in the basic state wind field. A cross section of the eigenfunction in the X-Z plane reveals a westward-tilting trough structure (Fig. 3.10).

The longer wave (Fig. 3.11) is similar in location and structure but extends with significant amplitude to the tropopause and is less closely confined in the meridional direction. Like the shorter wave, the energy budget indicates that this mode is a combined baroclinic-convective disturbance, but with baroclinic instability dominant in this case. The near equality of the growth rates at these two wavelengths suggests that the heating rates found in this case study are very close to the short wave transition discussed in section 2.3c.

This assertion was tested by arbitrarily reducing the heating rates to 0.8 and 0.6 of the original values. The growth rate spectra for these calculations (Fig. 3.12) show that even a 20% reduction in heating is more than sufficient to reduce the growth rate of the shorter wavelengths to the point where the fastest growing mode is approximately 2000 km. The structure and energetics of this disturbance (not shown) are similar to those depicted in Fig. 3.11. The abrupt transition in wavelength is associated with the change in vertical scale from the tropopause height to the height of the heating region. Indeed the ratio of
Figure 3.10. Cross section in X-Z plane at $Y = 0.08$, parallel to jet axis, of mode shown in Fig. 3.9.
Figure 3.11. (a) Cross section of amplitude of eigenfunction for the fastest growing mode in Fig. 3.8 with wavenumber $k = 4.25$. The function has been normalized to a maximum amplitude of one. (b) Energy budget of mode shown in (a). Abbreviations are defined in section 2.2b. Fluxes are normalized so that horizontal heat flux (HHF) is one unit.
Figure 3.12. (a) Nondimensional phase speed $c$ and (b) growth rate $\sigma$ as functions of nondimensional wavenumber $k$, for March case with cumulus heating, but with heating rates reduced to 0.8 of their original values. (c) and (d) as in (a) and (b) but with heating rates reduced to 0.6 of the original values.
Figure 3.12, continued.
wavelengths of the two modes, 4.25/7.0, is very similar to the ratio of vertical scales, approximately 0.5/0.8.

b) November Case

The results of the stability analysis for the November case are for the most part analogous to what was found for the March case. In the absence of heating the only mode with significant growth rates (Fig. 3.13) is a long wave barotropic mode, here with wavelength 2590 km. The structure and energetics (Fig. 3.14) are similar to those of the corresponding mode in the March case (Fig. 3.6). A baroclinic mode as found for the March case may also exist here, but with maximum growth rate less than 0.05 making it difficult to identify. The lack of large growth rates may be due to the very weak surface temperature gradient; there is only a 5.4°C change across the entire domain.

With heating, the growth rate curves in Fig. 3.15 show the barotropic mode to be unaffected as before. A second mode also appears, with maximum growth rate at 1030 km. The structure and energy budgets of this mode (Fig. 3.16) show it to be the same short wave, meridionally confined, baroclinic-convective disturbance that was found in the previous cases.

A third eigenmode has also been plotted in Fig. 3.15, which has an e-folding time of 4.4 days at its wavelength of maximum instability. Modes with such small growth rates
Figure 3.13. (a) Nondimensional phase speed $c$ and (b) growth rate $\sigma$ as functions of nondimensional wavenumber $k$, for November case with no cumulus heating.
Figure 3.14. (a) Cross section of amplitude of eigenfunction for the fastest growing mode in Fig. 3.13 with wavenumber $k = 3.5$. The function has been normalized to a maximum amplitude of one. (b) Energy budget of mode shown in (a). Abbreviations are defined in section 2.2b. Fluxes are normalized so that horizontal heat flux (HHF) is one unit.
Figure 3.15. (a) Nondimensional phase speed $c$ and (b) growth rate $\sigma$ as functions of nondimensional wavenumber $k$, for November case with cumulus heating.
Figure 3.16. (a) Cross section of amplitude of eigenfunction for the fastest growing mode in Fig. 3.15 with wavenumber $k = 8.75$. The function has been normalized to a maximum amplitude of one. (b) Energy budget of mode shown in (a). Abbreviations are defined in section 2.2b. Fluxes are normalized so that horizontal heat flux (HHF) is one unit.
have been ignored in most of the analysis since they tend to be less well resolved and of little physical relevance when modes with much more rapid growth also exist. However the amplitude of the eigenfunction (Fig. 3.17) shows it to be located in the north of the domain in the region where the wind is decreasing with height. Since none of the faster-growing modes have significant amplitude in this area, it is possible that this mode might be observable. A longitudinal cross section (Fig. 3.18) shows the phase of the disturbance to tilt eastward with height, appropriately for the reversed shear of the basic state, and opposite to the conventional baroclinic wave (Fig. 3.10).

3.4 Comparison With Observations

In contrast to a time-dependent numerical simulation, a linear stability analysis offers a limited number of factors which can be potentially compared to observations. From the eigenvalue spectrum one can predict the wavelength of the fastest growing mode, as well as the phase speed and growth rate. The structure of the eigenfunctions indicates the location and extent of the unstable mode in the meridional direction. As noted in chapter 2, caution is required in these comparisons because the model allows unrealistic cooling in the downdraft region of the wave. Consistent with the earlier discussion, we will compare only the half-wavelength of the mode, that is the width of region of rising motion, to the corresponding portion of the observed
Figure 3.17. (a) Cross section of amplitude of eigenfunction for the reversed-shear mode (see text) in Fig. 3.15 with wavenumber $k = 7.25$. The function has been normalized to a maximum amplitude of one. *(r)*; Energy budget of mode shown in (a). Abbreviations are defined in section 2.2b. Fluxes are normalized so that horizontal heat flux (HHF) is one unit.
Figure 3.18. Cross section in X-Z plane at $Y = 0.4$, parallel to jet axis, of mode shown in Fig. 3.17.
disturbance. Direct comparisons of growth rates will be avoided since the predictions may not be accurate, and the values are difficult to determine from the available observations. It is not known how useful the model predictions of phase speed are, but this comparison may not be a very stringent test since in the model they are largely determined by the steering flow. These factors will be considered for each case in turn.

a) March Case

The comma cloud of March 1982 was relatively large in size for this type of system. Reed (1979) stated that the wavelength of comma cloud wavetrains tended to be in the range 1000-1500 km, whereas Craig and Cho (1988) estimated the wavelength of the March case to be 2000 ± 500 km. This disturbance occurred as an isolated system and the estimated wavelength represents twice the size of the updraft region as determined from the size of the cloud pattern on satellite imagery. A precise determination of this quantity is difficult and the error term indicates the range of values which would be consistent with the observations. An estimate of the phase speed can be obtained from the position of the cloud mass on satellite photographs at different times. The average speed of the disturbance for the twelve hour period from 2345 GMT 15 March 1982 to 1145 GMT 16 March was 23 ms$^{-1}$. A similar value was obtained for the preceding twelve hour period. The meridional position
of the disturbance at the time of the initial data is indicated on the cross section in Fig. 3.1a. The letter H marks the northern tip of the comma cloud head, while the T indicates the point where the cross section intersects the center of the comma tail. The southern limit of the cloud pattern is more difficult to define, since the convectively active region merges with more stratiform cloud on the warm air side of the jet. The positions H and T correspond to $y/L = 0.60$ and $0.15$ respectively in the model domain.

It was shown in section 3.3 that the values of the heating parameter estimated for this case study are very close to the transition between a short wave instability confined to the heating region and longer wavelength deeper mode. Since the range of heating values over which the transition occurs is smaller than the uncertainty in the estimates, it is not immediately clear which mode should be expected to appear in reality. The observed wavelength of 2000 km is very close to the 1990 km value for the longer wave and does not agree well with the shorter 1210 km mode. The phase speed for both instabilities is similar at approximately 31 ms$^{-1}$, which is somewhat larger than the observed value. This difference could be due either to a poor prediction by the model or to the estimation procedure which considers only the average speed over a twelve hour interval and may not be representative of the linear phase of the development. The position of the northern limit of the system is the same for both the long and short
wavelengths (Figs. 3.9a, 3.11a) and agrees well with the observed value of y/L = 0.6. Both eigenfunctions are decaying south of y/L = 0.15 with the longer wave extending further into the warm air. Because of the uncertainty defining the southern limit of the cloud, it is difficult to say which mode shows better agreement for this value. In conclusion, the longer wave shows reasonable agreement with the observed system in wavelength and structure. In addition the greater importance of baroclinic instability over latent heating in this mode agrees with the assessment of Reed and Blier (1986a) based on the observed development.

b) November Case

At the time of the cross section in Fig. 3.1b, there were two cloud clusters in the cold air mass, in the positions labelled A and B in the figure. Cloud cluster B grew rapidly and eventually formed the comma cloud system. Cluster A grew more slowly and eventually dissipated as B grew larger. The wavelength estimated for cluster B is typical for comma clouds at 1400 ± 400 km. The average speed of the disturbance from 1145 GMT 8 November 1982 to 1945 GMT was 20 ms⁻¹. At later times the system slowed dramatically to a speed of less than 10 ms⁻¹. The position of the endpoints of the cloud pattern correspond to y/L = 0.06 and 0.34 in the model domain.

The wavelength of the short wave instability on this basic state, 1030 km, is somewhat less than the observed
size. It should be noted however that the scale selection is rather weak, so it is possible that the size of the upper level short wave which triggered the development may have played a significant role in determining the size of the system which developed. The phase speed of the mode is 19 ms\(^{-1}\), which agrees well with the observed value. The growth rate, with an e-folding time of 1.3 days, is smaller than that found for the March case but still quite rapid. The long wavelength barotropic mode also has a faster growth rate but is unlikely to be physically relevant since its scale is larger than the comma cloud, or even of the baroclinic zone on which it formed. The meridional position and extent of the eigenfunction agree very well with observations, with both the north and south endpoints well predicted.

Cloud cluster A was an amorphous mass of convective activity closer to the trough axis than the developing comma cloud. It existed along side cluster B but moved more slowly and eventually wrapped behind the other system before decaying away. From the available data it is not possible to determine if the cloud pattern in fact indicates an organized weather system, however the results of Blier's (1989) numerical simulations suggest that this might be the case. He found that the moisture field in a 12 hr forecast, while not reproducing the observed pattern precisely, gave a better representation of the cloud region than was present in the initial data, suggesting that the convection was
under the control of the large scale dynamics. With these
provisos in mind, we will compare the position and size of
this cluster to the slowly growing eigenmode described in
section 3.3b. The wavelength estimated from twice the size
of the cloud pattern is $2000 \pm 1000$ km. The large
uncertainty is due to the difficulty in distinguishing the
edge of the organized cloud mass from scattered convection
which occurred throughout the cold air. The average speed
of the disturbance over the same interval used for cloud
cluster B was about $10 \text{ ms}^{-1}$.

The wavelength of the mode in Fig. 3.17 of 1248 km is
within the uncertainty of the observed value however, as in
the previous case, the scale selection is very weak. The
phase speed is $5 \text{ ms}^{-1}$ which is less than the observed speed,
but given the uncertainty in the values this difference may
not be significant. As in the previous case, the position
and extent of the eigenfunction in the meridional domain
corresponds very closely to the observed values. While the
e-folding time of this mode of 4.3 days is much longer than
for the instability closer to the jet axis, the
eigenfunctions do not overlap significantly so it is
possible that both might have the opportunity to grow
simultaneously. It is also probable that this growth rate
is underestimated since the baroclinicity in the reversed
shear region is reduced by a factor of two in the model
basic state (compare Figs. 3.1b, 3.3a). The wind field at
the edge of Reed and Blier’s analysis region was matched to
zero at the boundary of the model domain. When smoothing was applied, the amplitude of the wind near the matching region was reduced.

For both comma cloud cases, the instability calculation was able to produce reasonable values for the wavelength and disturbance structure. In particular the position and extent of the disturbances in the meridional plane were very well predicted. While it is not clear that cloud cluster A in the November case resulted from a dynamical instability, an instability with some appropriate characteristics was found.

3.5 Chapter Summary

In this chapter a semigeostrophic model with parameterized cumulus heating was used to examine the instabilities which form on two baroclinic jets. The basic state fields were taken from the two case studies of comma cloud formation of Reed and Blier (1986a,b). The motivation for this analysis was to verify that the dependencies of the instability processes on their environment that were observed in the constant potential vorticity model of the previous chapter are relevant to the systems that occur in nature.

The range of behaviors exhibited by the instabilities of the observed flows was similar to that seen in the idealized model. For the March case the dry instability
spectrum featured a predominantly barotropic and a baroclinic mode. When heating was added the model showed the same transition to a short wave spatially confined mode that was found previously. In the November case, the barotropic instability was the only mode with significant growth rates in the absence of heating. However when cumulus heating was included, a short wave mode also appears. An interesting feature of the November basic state is the presence of a region of negative vertical wind shear at the north end of the analysis region which gives rise to a weakly growing instability localized there.

As with the simple one-dimensional model described in chapter 1, the more realistic calculation is able to produce reasonable values for the zonal scale of the two observed systems. In addition, the meridional location and extent of the disturbances is well reproduced. It is perhaps an indication of the greater difficulty in defining observed values of the heating parameters that the northern limit, which must be defined by the dynamical fields since the heating is relatively uniform in those regions, is better predicted than the southern extent which tends to be associated with the boundary of the heating region.

It is reasonable to suggest from these results that the combination of baroclinic instability and cumulus heating is sufficient to explain several aspects of the observed comma cloud systems. In addition, the effect of the environment
on these mechanisms appears similar to that found in the constant potential vorticity atmosphere. An exact balance no longer exists between inertial and static stability in their effects on baroclinic instability, however the potential vorticity is sufficiently uniform that there is no tendency for the dry instabilities of the jet to become confined to the region of reduced static stability. Only when heating is included does this confinement occur. The implication from the energetics of the eigenfunctions that baroclinic instability is more important in the March case, while convection is relatively more important for the November system, supports Reed and Blier's interpretation of the observations.
CHAPTER 4

Conclusions

This thesis describes the use of a model of baroclinic instability with cumulus convection to investigate the effects of the genesis environment on the formation of comma clouds, a common type of polar air stream cyclone. These systems typically form on the cold air side of a baroclinic jet, in a region characterized by low static stability and high positive vorticity, features associated with most cases of polar air stream cyclogenesis. Comma clouds are considered to be a result of baroclinic instability but with behavior strongly modified by cumulus heating, therefore to determine how their environment affects the development, it is necessary to consider both processes simultaneously. The analysis was carried out using the semigeostrophic equations in a linear stability analysis of two-dimensional jet flows, with cumulus heating parameterized by setting the heating rate proportional to low level moisture convergence.

A constant potential vorticity basic state was considered first, with heating in the cold air mass, as is typically the case for comma clouds. It was possible to distinguish three different regimes among the unstable modes. The control parameter determining the nature of the disturbance is the nondimensionalized constant of
proportionality between heating and low level convergence. If the heating parameter is sufficiently large, the fastest growing mode was characterized by conditional instability of the second kind (CISK). This instability occurs due to a feedback process where heating produces a sufficient convergent circulation to supply increased moisture convergence and increased heating at subsequent times. The structure of the modes in this regime were typical of CISK, with a low level cyclone beneath an upper level anticyclone, and disturbance amplitude confined in the meridional and vertical directions to the region where heating occurs.

For small values of the heating parameter, the fastest growing mode was a baroclinic instability only slightly modified by the heating. The horizontal heat flux was a greater source of disturbance energy than latent heating in this regime, and the eigenfunction was centered in the region of greatest baroclinicity at the jet axis. At higher values of the heating parameter, the mode underwent a transition to a strongly modified baroclinic instability. The structure was still characterized by a westward-tilting trough, but the latent heat term in the energy budget now exceeded the horizontal heat flux. This transition was accompanied by a dramatic shift of the maximum growth rate to shorter wavelengths as the vertical extent of the disturbance changes from the rigid lid to the top of the heating region. The shift was either abrupt or gradual, depending on the proximity of the top of the heating region.
to the lid. At the same time the eigenfunction amplitude became confined to the heating region in the cross-jet direction.

A long wavelength barotropic mode was also found with this basic state, however it was largely unaffected by heating. The changes between the three regimes occurred smoothly as the heating parameter was varied. The importance of the heating parameter is that it provides a measure of the ability of the convection to resupply itself and to produce instability independent of baroclinic processes. The values where the transitions occurred were also affected by the vertical distribution of heating, since release of latent heat in low levels favors the CISK feedback process.

Experiments varying the meridional distribution of heating showed the cold-air side of the jet to be favorable for the occurrence of short wavelength, meridionally confined disturbances, since they could be produced for physically reasonable values of the heating parameter. With constant potential vorticity, the stabilizing effect of high vorticity offsets the destabilization due to low static stability for baroclinic instability, but this relationship was not found to hold for convective processes. A diagnostic calculation of the threshold for self-sustaining CISK showed it to be lowered by reduced static stability, which enhances the necessary feedback. However, it was not
affected by vorticity since the scale of the disturbance merely contracts in proportion to the reduced radius of deformation.

It was concluded from these results that the important factor in the formation of short wavelength comma clouds confined to the cold air side of the jet was the influence of low static stability in enhancing convective effects. A generalized model was then used to examine the stability of two observed basic states where the particular balances that hold in a constant potential vorticity atmosphere do not occur. While it is more difficult to isolate the mechanisms at work in such a flow, the instabilities of the observed basic states showed similar behavior to that found in the more idealized model, suggesting that similar interactions were occurring. In the March case the expected baroclinic and barotropic instabilities were found, with a transition of the baroclinic mode to the strongly modified regime when heating was included. The November case did not have a strong baroclinic mode in the absence of heating, but when the convective parameterization was turned on, a short-wave mode, as found in the constant potential vorticity calculation, appeared.

The meridional structure of the appropriate modes for both cases agreed very well with observations. The wavelengths were also reproduced to within the estimated error of the values determined from the data. It is perhaps
surprising that the wavelength of the dry baroclinic mode in the March case also corresponded closely to the observed value, despite the structure of the mode which was confined principally to the upper levels. This result is in fact in accord with the numerical study of Blier (1989), in which the adiabatic simulation of this system produced an upper level development of the correct size in the correct location but underestimated the surface perturbation. The fact that adiabatic processes alone were unable to produce the disturbance in the cold air mass in either case shows the importance of the heating in those regions, and supports the hypothesis that the same interactions between the environment and instabilities found for the constant potential vorticity basic state also controlled the development in these cases.

A number of limitations of the present analysis suggest possibilities for future work. Clearly it would be desirable to eliminate some of the simplifications associated with the parameterization of cumulus effects, particularly the unrealistic cooling in the downdraft portion of the wave. However, most cumulus parameterizations will still involve a number of free parameters which can be difficult to specify with confidence. The best alternative is to explicitly resolve the convective clouds in the model. While computationally intensive, such calculations are now quite feasible in two dimensions and will soon become possible in three
dimensions. The complexity of the dynamics in such models, where many scales of motion are explicitly resolved, will require carefully designed experiments to obtain unambiguous results. The relegation of surface fluxes of heat and moisture in the present study to a role of preconditioning the basic state and enhancing existing cumulus convection is oversimplified, especially in light of the recent work on air-sea interaction instability by Emanuel and Rotunno (1989). A realistic treatment of the interactions of surface fluxes and associated boundary layer processes with the large scale flow would probably require a nonlinear time-dependent model, even if those processes were parameterized.

It would also be interesting to explore the effects of different initial conditions other than the normal mode structure. Polar lows are usually initiated by some pre-existing disturbance at upper levels, and it is not immediately obvious what the effects of the initial perturbation on the development of the system would be. The results of Joly and Thorpe (1989) show a complex series of interactions between the initial perturbation and the normal mode structure when latent heating is included.

A final concern is the possible breakdown of semigeostrophic theory in strongly heated flows that was mentioned in chapter 2. The problem has not yet been studied sufficiently to draw conclusions about the range of
validity of this approximation or through what mechanism the breakdown occurs. Further investigation is clearly required.
Matrix Elements for Stability Calculations

The elements of the matrices $A_{ij\mu\nu}$ and $B_{ij\mu\nu}$ in equation (2.31) are equal to zero except as follows. For $1 \leq i, j \leq N-1, -M \leq \mu - \nu \leq M$:

$$
A_{ij\mu\nu} = \begin{cases} 
-\left[ \frac{U}{\ell_B} \right]_{i,\mu-\nu} (k^2 + v^2 \ell^2 + 2iZ^2) & j=i \\
\left[ \frac{U}{\ell_B} \right]_{i,\mu-\nu} (1/\Delta Z^2) & j=i+1, i-1
\end{cases} \quad (A.1)
$$

$$
B_{ij\mu\nu} = \begin{cases} 
-\left[ \frac{1}{\ell_B} \right]_{i,\mu-\nu} (k^2 + v^2 \ell^2 + 2iZ^2) & j=i \\
\left[ \frac{1}{\ell_B} \right]_{i,\mu-\nu} (1/\Delta Z^2) & j=i+1, i-1
\end{cases} \quad (A.2)
$$

For $i=0$,

$$
A_{ij\mu\nu} = \begin{cases} 
-\frac{3}{2} (U)_{0,\mu-\nu}/\Delta Z - (\partial U/\partial Z)_{0,\mu-\nu} & j=0 \\
(U)_{0,\mu-\nu}/\Delta Z & j=1 \\
-(U)_{0,\mu-\nu}/(2\Delta Z) & j=2
\end{cases} \quad (A.3)
$$

$$
B_{ij\mu\nu} = \begin{cases} 
-3/(2\Delta Z) & j=0 \\
2/\Delta Z & j=1 \\
-1/(2\Delta Z) & j=2
\end{cases} \quad (A.4)
$$

and for $i=N$, 

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\[ A_{ij\mu\nu} = \begin{cases} \frac{3}{2}(U)_{N,\mu-\nu}/\Delta Z - (\partial U/\partial Z)_{N,\mu-\nu} & j=0 \\ -(U)_{N,\mu-\nu}^{2}/\Delta Z & j=N-1 \\ (U)_{N,\mu-\nu}/(2\Delta Z) & j=N-2 \end{cases} \quad (A.5) \]

\[ B_{ij\mu\nu} = \begin{cases} 3/(2\Delta Z) & j=N \\ -2/\Delta Z & j=N-1 \\ 1/(2\Delta Z) & j=N-2 \end{cases} \quad (A.6) \]

The following terms associated with the heat source are added to the above expressions; for \( B<i<N \), where \( B \) is the index number of the level where \( Z_B \) occurs,

\[ A_{ij\mu\nu} = A_{ij\mu\nu} + \begin{cases} (\varepsilon U)_{B,\mu-\nu}(\partial h/\partial Z)_{i}/(2\Delta Z) & j=B+1 \\ -(\varepsilon U)_{B,\mu-\nu}(\partial h/\partial Z)_{i} & j=B \\ -(\varepsilon U)_{B,\mu-\nu}(\partial h/\partial Z)_{i}/(2\Delta Z) & j=B-1 \end{cases} \quad (A.7) \]

\[ B_{ij\mu\nu} = B_{ij\mu\nu} + \begin{cases} (\varepsilon)_{\mu-\nu}(\partial h/\partial Z)_{i}/(2\Delta Z) & j=B+1 \\ -(\varepsilon)_{\mu-\nu}(\partial h/\partial Z)_{i}/(2\Delta Z) & j=B-1 \end{cases} \quad (A.8) \]

In the above expressions, the Fourier components of the basic state quantities are given by zero except for,

\[ (U)_{i,p} = \begin{cases} (1-b)Z_i & p=0 \\ -ba(Z_i)/(2a(Z_N)) & p=\pm 1 \end{cases} \quad (A.9) \]

\[ (\partial U/\partial Z)_{i,p} = \begin{cases} (1-b) & p=0 \\ -ba'(Z_i)/(2a(Z_N)) & p=\pm 1 \end{cases} \quad (A.10) \]

\[ \left[ \frac{1}{\zeta_B} \right]_{i,p} = \begin{cases} 1 & p=0 \\ -\pi bR_\alpha(Z_B)/(2a(Z_N)) & p=\pm 1 \end{cases} \quad (A.11) \]
\[ \begin{align*}
\left[ \frac{U}{\xi_B} \right]_{i,p} &= \begin{cases} 
(1-b)Z_i & p=0 \\
(-a(Z_i) - \pi R Z_i (1-b) a(Z_i) l) b / (2a(Z_N)) & p=\pm 1 \\
\frac{\pi}{8} \left( \frac{b}{a(Z_N)} \right)^2 \text{Ra}(Z_i) a(Z_B) l & p=\pm 2 
\end{cases} 
\end{align*} \tag{A.12} \]

where \( a(Z) = (1-a) \sinh l Z + a \cosh l Z \). The expression \( (dh/dZ)_i \) refers to the derivative of the function \( h(Z) \) defined in (2.5) evaluated at level \( Z_i \). For \(-M \leq p \leq M, 0 < i < N\),

\[ (\xi)^p = A_p \tag{A.13} \]

\[ (\xi U)_{i,p} = (1-b) Z_i A_p - b a(Z_i) B_p / a(Z_N) \tag{A.14} \]

\[ \left[ \xi \frac{dU}{dZ} \right]_{i,p} = (1-b) A_p - b a'(Z_i) B_p / a(Z_N) \tag{A.15} \]

For the horizontal distribution of heating defined by (2.6),

\[ A_p = \frac{(-1)^ {P-2} i p}{\pi (1-4p^2)} \tag{A.16} \]

\[ B_p = \frac{1}{2} \left[ A_p - \frac{3 (-1)^ {P+2} i p}{\pi (9-4p^2)} \right] \tag{A.17} \]

and for the alternative distribution (2.7),

\[ A_p = \begin{cases} 
i p/4 & p=-1,1 \\
\frac{1+(-1)^{P}}{2\pi (1-p^2)} & \text{otherwise} 
\end{cases} \tag{A.18} \]

\[ B_p = \begin{cases} 
i p/16 & p=-2,2 \\
\frac{1+(-1)^{P}}{2\pi (4-p^2)} & \text{otherwise.} 
\end{cases} \tag{A.19} \]
APPENDIX B

Effects of Vorticity and Static Stability on CISK Threshold

In this appendix we use a highly simplified model to consider the effects of vorticity and static stability on the ability of a disturbance driven by cumulus convection to maintain itself. The effects of vorticity on the growth of convectively driven disturbances is not clear from previous results. Hack and Schubert (1986) found that the growth rate of a vortex spun up by a fixed heat source increased as the relative vorticity of the system increased. They showed that the increased efficiency in converting potential energy generated by heating into kinetic energy of the vortex was associated with a decrease in the convergent circulation, due to greater inertial stability. But if an interactive treatment of heating were used, where the heating rates are proportional to low level convergence, this process would reduce the heating at subsequent times and inhibit the growth of the system. This is in contrast to the results of Okland (1987) who performed a linear stability analysis of a vortex in solid body rotation with interactive heating and found the growth rates to be increased by increasing basic state vorticity.

The present analysis will employ the balanced vortex equations of Eliassen (1951) for axisymmetric flows in
gradient balance, and the same parameterization for cumulus heating as before. Rather than solve a time-dependent problem, a diagnostic calculation will be made for the threshold of instability. This is the value of the heating parameter for which the convergence produced by a heat source is exactly sufficient to induce that same rate of heating. The dependence of the threshold on vorticity and static stability will provide a crude measure of the ability of a convective circulation to maintain itself in such environments, and thus of the ability of the heating to influence baroclinic systems discussed in the body of the paper.

If a sufficiently simple basic state is employed, it is possible to obtain the solutions analytically. We therefore consider perturbations to a basic state in solid body rotation. This flow has constant relative vorticity. It will also be assumed that the basic state static stability is constant, as well as the horizontal distribution of heating.

The present development will employ the notation of Schubert and Hack (1982). For a Boussinesq atmosphere on a f-plane, the vertical and radial circulation induced by a heat source can be diagnosed by defining a streamfunction \( \psi \) such that

\[
    w = \frac{\partial \psi}{\partial \theta}, \quad \text{and} \quad u = \frac{\partial \psi}{\partial z},
\]

and solving an elliptic equation,
\[
\frac{\partial}{\partial r}\left[\rho \frac{\partial \psi}{\partial r} + B \frac{\partial \psi}{\partial z}\right] + \frac{\partial}{\partial z}\left[\rho \frac{\partial \psi}{\partial r} + C \frac{\partial \psi}{\partial z}\right] = \frac{g}{\theta_0} \frac{\partial E}{\partial r}
\]  \hspace{1cm} (B.2)

where

\[
A = \frac{g}{\theta_0} \frac{\partial \theta}{\partial z}, \hspace{1cm} (B.3)
\]

\[
B = -\frac{g \partial \theta}{\theta_0 \partial r}, \hspace{1cm} (B.4)
\]

and

\[
C = \left[ f + \frac{\partial \psi}{\partial r} \right] \left[ f + \frac{2 v}{r} \right]. \hspace{1cm} (B.5)
\]

In these equations, \( E \) is the heating rate, \( v \) the azimuthal velocity, \( \theta \) the potential temperature, and \( \theta_0 \) a fixed reference value of \( \theta \). A coordinate transformation analogous to the geostrophic coordinates used with the semigeostrophic equations has been defined for this system (Schubert and Hack, 1983) but will not be used here. The flow will be assumed to be frictionless and bounded by a rigid lid at \( z = H \). As noted by Eliassen (1951) the boundary must be a streamline, so the boundary conditions may be taken to be \( \psi = 0 \) at \( z = 0, H \) and \( r = 0, r^{-\infty} \).

The equation is linearized about a basic state consisting of an atmosphere with constant static stability \( \theta_z \) and vorticity \( \zeta \). In the basic state, the three parameters in (B.3) - (B.5) are given by \( A = \frac{g}{\theta_0} \theta_z \), \( B = 0 \), and \( C = \zeta^2 \). The linearized streamfunction equation is then

\[
\frac{g}{\theta_0} \theta_z \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) + \zeta^2 \frac{\partial^2 \psi}{\partial z^2} = \frac{g \partial E}{\theta_0 \partial r}
\]  \hspace{1cm} (B.6)

where \( \psi \) and \( E \) are perturbation quantities.
The cumulus heating that the resulting circulation induces is determined from the cumulus parameterization defined in (2.4). Using (B.1), the heating is given by

\[ E = \frac{\theta_o Q}{gT} \varepsilon h(z) \left[ \frac{\partial (rw)}{\partial x} \right]_{z=z_B}. \]  

(B.7)

The threshold condition is obtained by substituting this expression into the streamfunction equation (B.6).

The solution proceeds through separation of variables by considering solutions of the form

\[ \psi(r,z) = \Psi(z) J_1(\lambda r), \]  

(B.8)

where \( J_1(\lambda r) \) is a Bessel function of order 1 and the separation constant \( \lambda \), is a horizontal scale parameter. The resulting equation for \( \Psi \) is

\[ \frac{d^2 \Psi}{dz^2} - m^2 \Psi = -m^2 \frac{\theta_o Q}{gT} \varepsilon h(z) \Psi(z_B) \]  

(B.9)

where \( m^2 = g\theta_z \lambda^2 / (\theta_o \zeta^2) \). This defines an eigenvalue problem for the threshold value of the heating parameter \( \varepsilon \). The solution can be obtained by the Green's function method of Craig and Cho (1988) and is given by

\[ \varepsilon^{-1} = -\frac{2\theta_o Q}{gT} \frac{H}{\theta_z} \frac{\sinh m_B}{m \sinh m_H} \left( \frac{\sinh m(z_L-H) - \sinh m(z_L-H)}{(z_T-z_B)(z_L-z_B)} \right) \]

\[ + \frac{\sinh m(z_L-H) - \sinh m(z_T-H)}{(z_T-z_B)(z_L-z_T)}, \]  

(B.10)

The threshold value of \( \varepsilon \) is a function of the horizontal scale, through \( m \), and the parameters which define the vertical distribution of heating.
When examining the results it is convenient to relate the horizontal scale parameter to a physical disturbance radius. Since \( w = \partial (r \psi) / \partial r \), from (B.8) it can be seen that \( w = \lambda \psi(z) J_0(\lambda r) \) and thus the region of rising motion will be bounded by the first zero of \( J_0 \); thus \( \lambda r_o = 2.4048 \) or \( r_o = (2.4048/m) \left( g \frac{\Theta_z}{\Theta_0} \right)^{1/2} \).

The curve of marginal stability (\( \varepsilon \) as a function of \( r_o \)) is plotted in Fig. B.1 for several values of static stability. It can be seen that unless \( \varepsilon \) exceeds a certain minimum value, the threshold will not be exceeded for any \( r_o \), and disturbances cannot grow. Above that minimum, growth is possible for a limited range of horizontal scales. As the stability varies, two effects are observed. First the minimum value of the threshold \( \varepsilon \) is reduced in proportion to \( \Theta_z \). This occurs since a given rate of low level convergence can be produced with less heating, and the process can become self-sustaining with lesser boundary layer moisture content. Second there is a reduction in the horizontal scale of the unstable region which is proportional to the reduction in the local radius of deformation. Both these effects are as seen in the quasi-geostrophic model of Wang (1987a).

The threshold curves for disturbances on basic states with values of absolute vorticity greater than the planetary vorticity \( f \) are plotted in Fig. B.2. In contrast to the dependence on vertical stability, the minimum value of the
Figure B.1. Threshold value of the heating parameter (see text) as a function of dimensional disturbance radius for $\xi = f$ and static stability given by the values of $N^2 = (g/\theta_o)(\partial \theta/\partial z)$ shown.
Figure B.2. Threshold value of the heating parameter as a function of dimensional disturbance radius for static stability \( N^2 = (g/\theta_o) (\partial \Theta/\partial z) = 10^{-4} \text{s}^{-2} \) and vorticity taking the values shown.
threshold $\epsilon$ does not depend on inertial stability, although the scale of the disturbance is again reduced in proportion to the reduction in the radius of deformation. Unlike baroclinic instability which depends on horizontal heat transports and thus on motion across angular momentum contours, CISK depends only on the convergent part of the circulation. Thus the ability of convection to resupply itself is not affected by inertial stability, although the same contraction of horizontal scale associated with reduction in the local radius of deformation occurs.

The result that the basic state vorticity does not affect the minimum value of the threshold of instability does not contradict the results of Hack and Schubert (1986) or Okland (1987) mentioned at the beginning of the appendix. Disturbances with fixed heating parameter and fixed horizontal size are below the threshold for sufficiently large basic state vorticities since the unstable range moves to smaller vortex size. The analysis of Hack and Schubert used a heat source of fixed size rather than comparing vorticities of different sizes at each value of inertial stability. In Okland's stability analysis, the growth rates were proportional to the basic state vorticity, whereas the present calculation only determines the threshold of instability. It is probable that the growth rates of CISK disturbances in the time dependent model in chapter 2 are similarly influenced, although because of the presence of baroclinic processes it is not possible to isolate the
effect.
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