Centralized Portfolio Optimization in the Presence of Decentralized Decision Making

by

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A thesis submitted in conformity with the requirements for the degree of Masters of Applied Science
Graduate Department of Mechanical & Industrial Engineering
University of Toronto

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Abstract

We study an asset allocation problem for a multi-asset fund where multiple decentralized managers implement investment strategies in separate asset classes. To control for portfolio measures at the fund, it is a common practice to follow a traditional asset allocation methodology to obtain optimal target weights and force the investment managers to stay close to the target. However, the sophistication and high level of specialization involved in alternative investment strategies create uncertainty in investment managers achieving the target weight; hence causing misallocations. Therefore, we develop two asset allocation models that provide a range of portfolio allocations (i.e. bounds) for individual investment managers to operate in order to maintain desirable firm wise portfolio measures while accounting for potential misallocations. Also, the proposed optimization problems have potentially large number of constraints, so we suggest a procedure to reduce the number of constraints before the optimization.
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Contents

1 Introduction ................................................................................................................................ 1

1.1 Purpose & Motivation ......................................................................................................... 1

2 Decentralized Asset Allocation .......................................................................................... 5

3 Asset Range Allocation ....................................................................................................... 9

3.1 Problem Formulation ...................................................................................................... 9

3.2 Computational Results ...................................................................................................... 13

3.2.1 Application in Forward Looking Asset Allocation ................................................... 14

3.2.2 Application in Strategic Asset Allocation .................................................................... 16

4 Vertex Elimination Procedure (VEP) .................................................................................. 20

4.1 Computational Challenge ............................................................................................... 20

4.2 Identifying Critical Vertices ............................................................................................ 20

4.3 Computational Results ..................................................................................................... 23

5 Robust Optimization ........................................................................................................... 24

5.1 Purpose & Motivation ..................................................................................................... 24

5.2 Uncertainty in Mean Return .......................................................................................... 24

5.2.1 Error Covariance Matrix ............................................................................................ 26

5.3 Computation Studies ...................................................................................................... 26

6 Conclusion & Future Work .................................................................................................. 30
List of Tables

Table 1 Percentage of incidents where the portfolios become infeasible when the investment managers miss their target weight. The data used for this test is given in Table 2. .......................... 10

Table 2 Data used for the numerical studies. (In-sample period: Jan 1995 – July 2004) Returns and volatilities are computed using the average of the historical returns. The illiquidity column consists of Boolean variables that indicate if the asset are liquid or illiquid. We assume minimum leverage of 5% at the fund level using the G7 Bonds. The 5% lower bounds for alternative investment strategies are to guarantee existence of the strategies. * indicates absolute bounds and are not violated in our numerical studies. ................................................................. 14

Table 3 Constraints used in the numerical studies ........................................................................ 14

Table 4 Computational time and memory required to solve $n$ programs. The model was run on TOMLAB in MATLAB with 16GB RAM and Intel i7-3770 processor. ......................................... 20

Table 5 Computational time required with and without the vertex elimination procedure (VEP) for problem (1) .............................................................................................................. 23
List of Figures

Figure 1 Target portfolio weights obtained from MVOA, ARA with uncertainty parameters from Table 2. The (0.5x), (1x) and (2x) indicates the multiple of the uncertainty parameter $\delta$ defined in Table 2. This is to show how the increase or decrease in the uncertainty parameter impact the target portfolio weights and the relationship across DAA, ARA, and MVOA. ........................... 15

Figure 2 Out-of-sample performance of simulated returns. It is 3 years of monthly returns. (a) Portfolio mean return (ARA mean = 11.1% and MVOA mean = 11.3%), (b) Portfolio standard deviation, (c) Sharpe ratio. The total number of simulations is 1,000,000. (1,000 for misallocations and 1,000 for simulated returns) ................................................................. 17

Figure 3 Out-of-sample performance of real returns. (Out of sampled period: Aug 2004 – July 2009): (a) Mean portfolio return; (b) portfolio standard deviation; and (c) Sharpe ratio ............ 18

Figure 4 Evaluating Out-of-Sample Performance of Problem (12) relative to Problem (2) from 2007 to 2014 ................................................................................................................................... 27

Figure 5 Distribution of the Cumulative Returns at the End of the Horizon (t = March 2014) .......................... 28

Figure 6 Evaluating Out-of-Sample Performance of Problem (12) relative to Problem (2) from 2007 to 2014 with different constraint limits......................................................... 28

Figure 7 Distribution of the Cumulative Returns at the End of the Horizon (t = March 2014) with different constraint limits.................................................................................................................. 29
Chapter 1

1 Introduction

1.1 Purpose & Motivation

Multi-asset funds, such as the Canada Pension Plan Investment Board (CPPIB), often decentralize or delegate investment decisions in specific areas to managers of adequate specialties. The decentralized investment management, defined as a practice of having multiple managers implement investment strategies in different asset classes, has been a common industry practice due to high level of specialization required to achieve superior returns in a specific asset class[1]. For instance, investments in infrastructure assets require a specialized deal team to properly capitalize on the acquired information to make investment decisions.

Decentralized investment management, however, inevitably suffers from inefficient control of total portfolio characteristics and loss of diversification since decentralized managers generally do not account for the correlation of their returns with returns of other managers at the fund (Blake et al, 2013). In an effort to mitigate the inefficiency, the firms have central managers (e.g. Chief Investment Officer) that employ estimates of return, risk and correlation among the investment managers to set optimal target allocations. Coupled with the long-term perspective of the firm, the central manager sets long-term target weights and the investment managers periodically rebalance their portfolio to stay near the allocation targets (i.e. strategic asset allocation).

The traditional strategic asset allocation is contingent on the assumption that the investment strategies can be easily rebalanced to the target weights. However, for multi-asset funds like
CPPIB, pressuring the investment managers to rebalance exactly to the target weight may not be preferable due to the following reasons:

1. **Illiquidity**: Some alternative assets such as private and real estate investments are illiquid in nature, and require considerable time, efforts and costs to buy or dispose of the assets.

2. **Asset Selection**: The managers of different asset classes likely have a special process for asset selection. Therefore, forcing the investment managers to add or remove assets to reside near a target weight may deteriorate their performance. This contradicts the objective of the decentralized managers, as in practice they are mostly incentivized to maximize their individual performance.

3. **Market Timing**: Some investment strategies are executed only when the market is in a desired condition of the managers. The firm should delegate any assessments on market timing of the specific asset class to the expertise of the asset class.

As a result, the firm wishes to provide and guide the investment managers with ranges of allocations (i.e. bounds) such that the managers can invest independently with minimal pressure to stay close to a specific target, but the resulting total fund remains within the acceptable limits of desired total portfolio characteristics, such as portfolio return, risk and liquidity. We propose the extension of the Decentralized Asset Allocation (DAA) model in chapter 2.

However, DAA may not be attractive to many investment institutions as it does not optimize the firm's economic utility (e.g. find the best tradeoff between return and risk). Construction of economically (utility maximizing) optimal portfolio requires a centralized investment structure which, in contrast to DAA, forces the investment managers to stay close to a specific allocation target. However, the traditional asset allocation models are limited in our application because it ignores the illiquid and large chunk investment (e.g. some investments are $1.0 Billion per investment) nature of the alternative investment strategies. In addition, with the same

---

1 “CPPIB believes that the “asset allocation” approach to investing tends to create pressure, possibly at inopportune times, to buy or dispose of illiquid investments in order to stay close to allocation targets” from [http://www.cppib.com/en/public-media/faq.html](http://www.cppib.com/en/public-media/faq.html)
motivations outlined for DAA, some investment strategies may require a degree of
decentralization to allow investment managers flexibility and independence in asset selection,
choice in investment horizons and market timing.

These features of alternative investment strategies create considerable possibility that the
investment decisions made by alternative investment managers operating in a decentralized
environment deviate from firm wide target portfolio allocation (weightings). In the event that an
investment manager misses the target allocation, it may require an ad hoc adjustment to increase
or decrease the positions of other investment strategies to maintain feasibility. This may be
undesirable or unrealistic from the point of view of a central manager as ad hoc increase or
decrease in positions can move risk, return, and liquidity from ideal levels. Therefore, in chapter
3 we propose the Asset Range Allocation (ARA) model that combines the traditional asset
allocation model and DAA to optimize for a fund's economic utility while explicitly accounting
for some degree of flexibility (uncertainty) in allocation. This results in an optimization problem
that is a combination of the features of centralized and decentralized investment structures. It is
similar to robust optimization as it is based on worst case optimization that maintains feasibility
for all possible deviation from a target portfolio which we call misallocations within a specified
range. An uncertainty parameter indicates the degree of flexibility (uncertainty), which also
controls the degree of centralization. The end result in solving ARA is a target portfolio and a
range of allocations from the target allocation that all satisfy firm wide portfolio measures in risk
and liquidity. The range of allocations can provide individual investment managers lower and
upper bounds on their respective investments such that operating within these bounds always
maintains firm wide portfolio measures and therefore reduces the complexity involved in
achieving nearness to target allocations.

In the second part of chapter 3 we discuss two types of applications. The first application is the
forward looking target portfolio. Alternative investments take time to grow their investments;
therefore, it is common practice to have target weights in the future and to rebalance to the target
weights over time. Several values of the uncertainty parameters will be tested to show the
relationship across three models: ARA, DAA and traditional asset allocation model. The second
application is strategic asset allocation, which sets a target portfolio and periodically rebalances
the portfolio to the target weights when the weights are skewed due to investment returns.
Numerical results using both simulation and out of sample testing show that target portfolios
produce by ARA out performed MVOA allocations (i.e. return, volatility and Sharpe ratio) and are more robust to deviations from target allocations than MVOA.

The Decentralized Asset Allocation (DAA) and Asset Range Allocation (ARA) problem requires higher computational effort. We will show that the computational time and memory required to solve these problems grow exponentially in the number of investment strategies. In our computational tests, intractability of the problem occurred with a moderate number of investment strategies (20 or more), which may be the problem size of concern for large investment firms. Therefore, in chapter 5 we propose a procedure to pre-process the constraints to reduce the complexity of the model.

Finally, we address the issue inherent in general portfolio optimization. Many practitioners have problems applying portfolio optimization due to its sensitivity to the inputs (e.g. expected returns of assets and their covariance). As a result, it is important to minimize estimation errors in the inputs to make the portfolio selection process robust. We have seen many attempts to address the limitation such as Bayes-Stein Estimator [2], and Black and Litterman [3], and various robust optimization algorithms [4]. In this research we choose robust optimization to explicitly account for the estimation errors in our optimization.

The main contribution of the research is two-fold. First, we propose two new types of optimization problem to maximize the investment range in decentralized asset allocation or maximize return given some flexibility parameter for the investment managers, and second we develop a pre-processing algorithm to reduce the complexity of the problem we propose. We highlight the value of our extension by allowing investment managers to operate independently while achieving desirable characteristics at the total portfolio level.

The paper is organized as follows. Chapter 2 presents the concept behind the Decentralized Asset Allocation (DAA) model, followed by Chapter 3 combining the traditional asset allocation and Decentralized Asset Allocation (DAA) model. Chapter 4 proposes a procedure that reduces the complexity of the model. Finally, Chapter 5 incorporates robust optimization to DAA and compares the results to the original DAA.
Chapter 2

2 Decentralized Asset Allocation

The model that we propose in this section is constructed with three assumptions in mind. First, the managers are independent in their investment activities meaning the investment activity of one manager should not limit or pressure other managers to adjust their position. In a decentralized investment structure, there are usually minimal information flows across the investment managers at the fund. Second, the central manager has complete transparency in all investment strategies; hence, the expected returns and covariance of returns across investment strategies can be estimated. Third, the residual of capital after the allocations to investment strategies is invested into a market portfolio (e.g. G7 Index).

We begin our model construction by redefining the input parameters to reflect the third assumption mentioned above. Let $w$ ($n \times 1$) vector of weights represent the allocations by $n$ number of investment managers. Then the amount of allocation to the market portfolio $w_M$ is:

$$w_M = 1 - 1^T w.$$ 

where $1$ is $(n \times 1)$ vector of $1$’s. Then the total expected return of the portfolio can be simplified as follows:

$$R_{total} = r_{IM} w + (1 - 1^T w)r_M$$

$$= [r_{IM} - 1^T r_M]w + r_M$$

(1)

where $r_{IM}$ ($1 \times n$) and $r_M$ ($1 \times 1$) are the vector of expected returns of the investment managers and the expected return of the market portfolio, respectively. It is easy to show that the first term (i.e. $[r_{IM} - 1^T r_M]w$) in equation (1) is the excess return over the market portfolio, so the expected return vector can be constructed as follows:
\[ \mu = \left( \begin{bmatrix} r_{IM}^T - 1^T r_M \end{bmatrix} \right). \]

Similarly, the covariance matrix is simplified as follows:

\[ Q = \text{Cov} \left( \begin{bmatrix} r_{IM}^T - 1^T r_M \end{bmatrix} \right). \]

This representation simplifies the problem formulation by not explicitly accounting for the budget constraint. Finally, we reflect the difficulty and preference of investment managers achieving a specific target in our objective function by maximizing the width of investment ranges for all investment managers. We propose the following convex optimization problem to maximize the width of investment ranges \((u_i - l_i), i = 1, 2, \ldots, n\) (i.e. flexibility) and preserves complete autonomy across all investment managers while guaranteeing desired total portfolio return, risk and liquidity:

\[
\begin{align*}
\text{max} & \quad \prod_{i=1}^{n}(u_i - l_i)^{k_i} \\
\text{s.t.} & \quad \mu^T x \geq R_{\text{min}} \\
& \quad x^T Q x \leq \sigma^2 \\
& \quad 1 - \gamma^T x \geq \tau \\
& \quad L_i \leq l_i, u_i \leq U_i \forall i = 1, \ldots, n \\
& \quad u_i - l_i \geq \xi_i \forall i = 1, \ldots, n \\
& \quad x \in [V; 1]
\end{align*}
\]

where \(u_i\) and \(l_i\) are upper and lower bound variables, respectively. In addition, we have exogenously set parameters:

- \(k_i\) (> 0): Illiquidity penalty of the investment strategy \(i\)
- \(\xi_i\) (> 0): Minimum width of the bounds
- \(\gamma^T \in \{0,1\}^n\): Indicator of illiquid investment strategies
- \(R_{\text{min}}\): Minimum total portfolio return
- \(\sigma^2\): Maximum variance of the portfolio
- \(U_i\) and \(L_i\): Absolute lower and upper bounds
It is important to distinguish the liquidity constraint $\tau$ and the illiquidity penalty $k_i$. The illiquidity penalty $k_i$ acts to differentiate the level of difficulty and preference of investment managers achieving a specific weight, whereas the liquidity constraint acts to limit the overall amount of capital allocated to illiquid investment strategies. A large illiquidity penalty will result in a wider investment range for investment strategy $i$ and large $\tau$ will result in decrease in amount of capital allocated to illiquid investment strategies. Finally, $V$ is a set of all possible combinations of upper and lower bound variables depicted by the Cartesian product of $V_i$'s:

$$V = V_1 \times V_2 \times \ldots \times V_n = \{(v_1, v_2, \ldots, v_n) | v_i \in V_i \ \forall \ i \in \{1, \ldots, n\}\}$$

where $V_i = \{l_i, u_i\} \quad i = 1, \ldots, n$. (3)

Then $x$ is all possible combinations of upper and lower bound variables with a scalar 1 to include the return and risk of the market portfolio. For example, if there are three investment strategies, the set $V$ is defined as follows:

$$V = V_1 \times V_2 \times V_3 = \{(u_1, u_2, u_3), (u_1, l_2, u_3), (u_1, l_2, l_3), (u_1, u_2, l_3), (l_1, u_2, u_3), (l_1, l_2, u_3), (l_1, u_2, l_3), (l_1, l_2, l_3)\}^T$$

and

$$x \in \{(u_1, u_2, u_3, 1), (u_1, l_2, u_3, 1), (u_1, l_2, l_3, 1), (u_1, u_2, l_3, 1), (l_1, u_2, u_3, 1), (l_1, l_2, u_3, 1), (l_1, u_2, l_3, 1), (l_1, l_2, l_3, 1)\}^T.$$

The model (2) seeks to maximize the investment ranges across investment managers subject to minimum return (first constraint), risk limit (second constraint, liquidity requirement (third constraint), absolute upper and lower bounds (fourth constraint) and a minimum width constraint (fifth constraint). The optimal solutions from the model (2) expressed as $l^*_i$ and $u^*_i$, suggest that an allocation in investment strategy $i$ can range from $l^*_i$ to $u^*_i$ without violating the overall portfolio characteristics of return, risk and liquidity. The solutions from our model (2) allow the individual managers to take complete autonomy and maximized flexibility in their investment activities.

Geometrically, we are maximizing the weighted volume of a hyper-rectangle enclosed by the absolute bounds and total portfolio constraints. The purpose of the maximization of the volume
is to maximize the opportunity set across the fund, meaning we maximize the number of possible portfolios resulting from investment managers’ allocations. The vertices of the hyper-rectangle are the bounds we defined in problem (2) and sides of the hyper-rectangle are weighted by the illiquidity penalty $k_l$. For a convex problem, it is sufficient to check the vertices of the hyper-rectangle (i.e. bounds) for feasibility as any points in between two feasible vertices are guaranteed to be feasible from the definition of convexity. Therefore, it is sufficient that the optimization problem checks only the vertices ($V$) of the hyper-rectangle for feasibility and optimality.
Chapter 3

3 Asset Range Allocation

3.1 Problem Formulation

The traditional asset allocation model is based on mean-variance optimization [5] with additional constraints (MVOA) specific to an investor’s needs. We begin our formulation of MVOA as follows:

\[
\begin{align*}
\max_{x} & \mu^T x \\
\text{s.t.} & \quad x^T Q x \leq \sigma^2 \\
& \quad 1 - \beta^T x \geq \tau \\
& \quad x_i \geq L_i, i = 1, ..., n \\
& \quad x_i \leq U_i, i = 1, ..., n \\
& \quad 1^T x = 1
\end{align*}
\]

(4)

where the decision vector \(x\) is the target portfolio weights for \(n\) investment alternatives, \(\mu\) is a \((n \times 1)\) vector of expected returns of the investment alternatives, \(Q\) is a \((n \times n)\) covariance matrix of the returns. We also have exogenously set parameters: \(\beta^T \in \{0,1\}^n\) is a \((n \times 1)\) vector to indicate illiquid investment strategies among the \(n\) investment alternatives, \(\sigma^2\) is the maximum variance of the portfolio, \(U_i\) and \(L_i\) for the absolute lower and upper bounds on investment \(i\), respectively, and \(1\) is \((n \times 1)\) vector of 1’s. MVOA seeks to maximize expected return subject to a risk limit (first constraint), liquidity requirement (second constraint), upper and lower bounds on each investment alternative (third and fourth constraints, respectively), and a budget constraint (fifth constraint). The result of solving an MVOA model is a set of portfolio weights that give the best expected return subject to risk and other constraints.

However, it is difficult and non-preferable to force investment managers to rebalance to specific target weights. We conduct a simple test to show the implication on portfolio characteristics
when the investment managers miss their targets. First we solve problem (4) to obtain the optimal target weights $x^*$. We then randomly sample the portfolio weights within $\delta_i$ from $x_i^*$ capped and floored at the absolute upper and lower bounds, respectively. Finally, we count the number of infeasible portfolios with respect to each constraint as shown in Table 1.

| Table 1 Percentage of incidents where the portfolios become infeasible when the investment managers miss their target weight. The data used for this test is given in Table 2. |
|-----------------|------------------|
| Violated Constraint | Percentage of Infeasible Incidents |
| Volatility       | 70.80%            |
| Liquidity        | 37.01%            |

The result shown in Table 1 arises from the fact that the optimal solutions lie at the edge of the feasible region. So if the managers miss the target, it is likely to violate some constraints. Therefore, we propose the Asset Range Allocation (ARA) model (5) to explicitly account for the uncertainty in target portfolio weights and ensure that all misallocations are feasible within a certain tolerance. The model is constructed as follows:

\[
\begin{align*}
\max_{x \in X} & \min_{x \in X} \mu^T x \\
\text{s.t.} & \max_{x \in X} x^T Q x \leq \sigma^2 \\
& 1 - \max_{x \in X} \beta^T x \geq \tau \\
& \min_{x \in X} x_i \geq b_i, i = 1, \ldots, n \\
& \max_{x \in X} x_i \leq U_i, i = 1, \ldots, n \\
& 1^T x = 1
\end{align*}
\]

(5)

where $X$ represents the uncertainty set of portfolio weights:

\[
X = X_\delta(\hat{x}) = \{x||x_i - \hat{x}_i| \leq \delta_i, i = 1, \ldots, N\}
\]

(6)

The uncertainty set $X$ in (6) includes every portfolio within $\delta_i$ from the target weight $\hat{x}_i$ where $\delta_i$ is specified exogenously. The first constraint in (5) ensures that the maximum risk (portfolio variance) of any portfolio in an uncertainty is less than or equal to $\sigma^2$ which represents the fund's limit on portfolio risk. The second constraint in (5) ensures that any portfolio in an uncertainty set must meet a minimum level of liquidity. The second and third constraints ensure that any portfolio in an uncertainty set must meet absolute lower and upper bounds in each investment alternative. Finally, the last constraint ensures that a budget constraint is met.
ARA is a max-min problem which seeks to find a target portfolio \( \hat{x} \) such that the worst possible expected return is maximized over portfolios \( x \) in the uncertainty set \( X \) associated with \( \hat{x} \). Any such target portfolio \( \hat{x} \) and portfolio \( x \) in the associated uncertainty set will meet portfolio characteristics in terms of risk, liquidity, and upper/lower bounds. Observe that the uncertainty set \( X \) is not determined definitively until the optimal target portfolio \( \hat{x} \) is found. Thus, ARA seeks to find a optimal target allocation \( \hat{x} \) and an uncertainty set \( X \) or equivalently a range of portfolio allocations within \( \delta_i \) of \( \hat{x}_i \) that meet minimal overall portfolio performance measures (e.g. in risk and liquidity), but whose worst case return in the uncertainty set is maximized. Then, this information can be used to guide individual investment managers to suggest that an allocation in investment alternative \( i \) can range from \( x_i - \delta_i \) to \( x_i + \delta_i \) without affecting the overall portfolio characteristics of risk and liquidity. As long as an investment manager stays within these suggested bounds than she will not have to be concerned with how her investment decision will effect overall portfolio characteristics and therefore will not have to be concerned with the effects on other alternative investments.

The idea of ARA is similar to robust optimization as it is a worst case optimization [4], [6], and [7]). Robust optimization explicitly accounts for uncertainty in input parameters within the optimization such that the solution is “good” for all constraints and uncertainty sets of the parameters. On the other hand, ARA incorporates uncertainty in the optimal solution, such that our portfolio measures are robust to neighborhood of solutions around the optimal target portfolio.

The \( \delta_i \) defined in equation (6) controls the degree of flexibility (uncertainty). A small \( \delta_i \) implies that the central managers are confident and would prefer the investment managers to rebalance their portfolio exactly to the target. Also as \( \delta_i \to 0 \), ARA approaches the centralized investment structure MVOA defined in problem (4). A large \( \delta_i \) implies that the investment strategy is difficult and not preferred to be rebalanced towards the target. Similarly, as \( \delta_i \to M_i \) for all \( i \), where \( M_i \) is an investment width that would be obtained from DAA for investment alternative \( i \), we approach the decentralized investment structure. As a result, ARA behaves to trade-off the features of centralized and decentralized investment structures. For the purpose of this paper, choice of the uncertainty parameter, \( \delta_i \), will not be discussed in depth as it is a decision of the
fund, $\delta_i$, however, should account for illiquidity of the investment strategy and investment managers’ desire to rebalance towards the target (i.e. flexibility).

In order to solve problem (5), we can reformulate similar to DAA. Consider any convex constraint $f(x) \leq 0$, and the uncertainty set $X$ given in equation (6). Given that the decision vector $x$ is within this uncertainty set, the constraints have to satisfy the following constraint as given in problem (5):

$$\max_{x \in X} f(x) \leq 0$$

In other words, we are looking for the worst case value of the function $f(x)$ given by:

$$\max f(x)$$

$$|x_i - \hat{x}_i| \leq \delta_i, \quad i = 1, \ldots, n$$

The objective function $f(x)$ is convex; therefore, the optimal solution $x^*$ lies on the boundary of the feasible set, i.e. $|x_i - \hat{x}_i| = \delta_i$ for $i = 1, \ldots, n$. Following this result, we define

$$x_i = \begin{cases} \hat{x}_i + \delta_i, & \text{if } x_i \geq \hat{x}_i \\ \hat{x}_i - \delta_i, & \text{otherwise} \end{cases}$$

Set $\hat{x}_i + \delta_i = u_i$ and $\hat{x}_i - \delta_i = l_i$. Then it follows that $u_i - l_i = 2\delta_i$ and $x_i \in \{l_i, u_i\}$ and problem (1) can be reformulated to

$$\max f(x)$$

$$u_i - l_i = 2\delta_i$$

$$x_i \in \{l_i, u_i\} \quad \forall \ i = 1, \ldots, n$$

Or

$$\max f(x)$$

$$u_i - l_i = 2\delta_i$$

$$x \in V$$

Where

$$V = V_1 \times V_2 \times \ldots \times V_n = \{(v_1, v_2, \ldots, v_n) | v_i \in V_i \forall i \in \{1, \ldots, n\}\}$$

where $V_i = \{l_i, u_i\}, \ i = 1, \ldots, n$. 

Therefore, problem (5) can be reformulated as follows:

\[
\begin{align*}
\max t \\
\text{s.t.} \mu^T x &\geq t \\
x^T Q x &\leq \sigma^2 \\
1 - y^T x &\geq \tau \\
L_i &\leq l_i, u_i \leq U_i \forall i = 1, \ldots, n \\
u_i - l_i = \delta_i \forall i = 1, \ldots, n \\
x &\in [V; 1]
\end{align*}
\]

(7)

where everything is exactly the same and \( t \) is an auxiliary variable. Geometrically, we are allocating the hyper-rectangle with widths \( u_i - l_i = \delta_i \), such that the minimum return inside the hyper-rectangle is maximized.

### 3.2 Computational Results

We conduct numerical studies using data given in Table 2, which is calculated from monthly time series of returns from Jan 1995 to July 2004. 20,000 allocations are sampled randomly around the solutions of MVOA and the solutions of ARA. The misallocations are limited by the absolute lower and upper bounds shown in Table 2. The uncertainty parameter \( \delta_i \) are chosen randomly on a relative measure. Notice that, the choice of \( \delta_i \) is a management decision and hence deemed exogenous in this paper. In this paper, we assign 2.5% to very illiquid, 1% to moderately illiquid, and 0% to liquid assets to show how the uncertainty parameters affect our portfolio weights and the implication of the uncertainty parameter to the forward looking target portfolio.

In the second part of numerical studies (section 3.2), we compare the out-of-sample portfolio measures, such as return and risk when we apply ARA and MVOA to strategic asset allocation. The fund will be penalized if the misallocations cause violation in leverage constraints (i.e. negative position in G7 Bonds). For simplicity, we penalize additional 5% of return for excess leverage required. There will be no penalty for the violation of liquidity constraint, although we have seen that it is often violated (e.g. Table 1).
Table 2 Data used for the numerical studies. (In-sample period: Jan 1995 – July 2004) Returns and volatilities are computed using the average of the historical returns. The illiquidity column consists of Boolean variables that indicate if the asset are liquid or illiquid. We assume minimum leverage of 5% at the fund level using the G7 Bonds. The 5% lower bounds for alternative investment strategies are to guarantee existence of the strategies. * indicates absolute bounds and are not violated in our numerical studies.

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Mean Return</th>
<th>Volatility</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Illiquidity ($\beta$)</th>
<th>Uncertainty Parameter ($\delta_1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Funds of funds</td>
<td>11.6%</td>
<td>22.0%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>2.5%</td>
</tr>
<tr>
<td>Private Equity</td>
<td>13.1%</td>
<td>24.2%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>2.5%</td>
</tr>
<tr>
<td>Private Equity Natural Resource</td>
<td>14.8%</td>
<td>26.0%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>2.5%</td>
</tr>
<tr>
<td>Infrastructure</td>
<td>9.30%</td>
<td>13.2%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>2.5%</td>
</tr>
<tr>
<td>Private Debt</td>
<td>6.50%</td>
<td>4.40%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>1%</td>
</tr>
<tr>
<td>Agriculture</td>
<td>1.70%</td>
<td>7.60%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>2.5%</td>
</tr>
<tr>
<td>Real Estate Debt</td>
<td>7.60%</td>
<td>3.20%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>1%</td>
</tr>
<tr>
<td>REITs</td>
<td>13.1%</td>
<td>13.0%</td>
<td>5%*</td>
<td>100%*</td>
<td>1</td>
<td>1%</td>
</tr>
<tr>
<td>Developed Equity</td>
<td>9.60%</td>
<td>15.0%</td>
<td>0%*</td>
<td>100%*</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>G7 Bonds</td>
<td>7.54%</td>
<td>3.07%</td>
<td>-5%</td>
<td>100%</td>
<td>0</td>
<td>0%</td>
</tr>
</tbody>
</table>

Table 3 Constraints used in the numerical studies

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max Volatility</td>
<td>14%</td>
</tr>
<tr>
<td>Min Liquid Assets</td>
<td>0%</td>
</tr>
</tbody>
</table>

3.2.1 Application in Forward Looking Asset Allocation

As alternative investments take time to grow in size, it is a common practice to set a medium-term target weight (e.g. 3 years from now) to guide the investment managers towards the target. The medium term target portfolio will incorporate long-term views on return and risk of the investment strategies. However, different investment managers favor dissimilar market conditions; therefore, it is difficult to project which manager will be aggressive or passive in their investment execution between now and the target horizon. In addition, some investment strategies are illiquid and have different investment horizons which may take more time to rebalance to the target weight.

Both ARA and MVOA can be used to compute the medium-term target portfolio. Figure 1 shows the optimal portfolio weights for ARA with various $\delta_1$ and MVOA. As mentioned before, as
As $\delta_l \to 0$, the target portfolio weights approach the MVOA’s optimal weights, which is a centralized asset allocation, and as $\delta_l \to M_l$, where $M_l$ is a big number, then it moves away from MVOA to a more decentralized investment structure (i.e. DAA in Lee et. al Working paper). The portfolios in Figure 1, except MVOA, guarantee feasibility for some $\delta_l$ around the optimal weights. As such, $\delta_l$ can be controlled by the fund to: 1) increase or decrease flexibility of investment managers, 2) account for illiquidity of some investment strategies, and 3) size of average investments. In addition, for forward looking target portfolio, the $\delta_l$ will decrease as the target horizon (i.e. time at which we want to reach the target weights) reduces, and $\delta_l$ will increase as the target horizon increases, to account for uncertainty in market conditions in between.

**Figure 1** Target portfolio weights obtained from MVOA, ARA with uncertainty parameters from Table 2. The (0.5x), (1x) and (2x) indicates the multiple of the uncertainty parameter $\delta$ defined in Table 2. This is to show how the increase or decrease in the uncertainty parameter impact the target portfolio weights and the relationship across DAA, ARA, and MVOA.
3.2.2 Application in Strategic Asset Allocation

Under strategic asset allocation, a firm (fund) sets a target portfolio and periodically rebalances the portfolio towards the target whenever the investment returns skew the portfolio weights. Strategic asset allocation takes a long-term view on the portfolio characteristics, such as return and risk. In this study, we compare the out of sample results of ARA and MVOA when misallocations are present in quarterly rebalancing to the target portfolio. They are rebalanced if they outside of $\delta_l$ from the target weight at quarter-mark. This is with the assumption that if the investment strategies are within the $\delta_l$ from the target weight, there is no value in rebalancing as the investment managers will still miss the target. If the deviation is more than $\delta_l$, then the strategy is rebalanced randomly within $\delta_l$ as we are uncertain which specific weight the investment manager could rebalance to. Again, we penalize additional 5% of return for excess leverage (i.e. negative position in G7 Bonds) required. For simplicity, we will ignore transaction costs.

3.2.2.1 Monte-Carlo Simulation

The objective of this study is to compare MVOA and ARA when the estimation of the inputs drive the out-of-sample performance. We assume a normal distribution of returns with the mean returns and covariance as shown in Table 2. We run 1000 simulations of 3 years of monthly returns and 1,000 random weights around the target weights.
Figure 2 shows that ARA produces target portfolios more robust than MVOA to incidents of misallocations. Figure 2. (a) shows MVOA and ARA having similar mean returns, but Figure 2. (b) shows that the volatility of returns of MVOA are significantly higher than of ARA’s, as well as that MVOA standard deviations are more spread than ARA standard deviations, meaning the violations with respect to standard deviation constraint can be significant. Finally, Figure 2.(c) shows that ARA’s Sharpe ratios are superior to the Sharpe ratios of MVOA; hence the portfolios
around the optimal solution of ARA has better risk adjusted returns. Therefore, we can conclude that the portfolio performance is more robust to the misallocations in ARA than MVOA.

3.2.2.2 Actual Data

The purpose of this test is to show how ARA and MVOA behave with respect to actual realization of returns, which may be significantly different from the estimates in Table 2. The out of sample data is a monthly time series from August 2004 to July 2009.

Figure 3 Out-of-sample performance of real returns. (Out of sampled period: Aug 2004 – July 2009): (a) Mean portfolio return; (b) portfolio standard deviation; and (c) Sharpe ratio

Again, similar results are present in Figure 3: the portfolio measures are more robust to misallocations in ARA than MVOA. Here, not only did ARA outperform MVOA in portfolio
returns, but all three portfolio measures: return, standard deviation, and Sharpe ratio of ARA are less volatile than of MVOA’s. The result shown in Figure 3 further enhances the argument of robustness of ARA’s portfolio measures with respect to potential misallocations.
Chapter 4

4 Vertex Elimination Procedure (VEP)

4.1 Computational Challenge

As shown in Section 2, \( x \) in problem (2) and problem (7) consists of all possible combinations of lower and upper bound variables, resulting in \( 2^n \) vertices of the hyper-rectangle to be checked per constraint. Problems with 20 or more alternative investments cannot be solved due to lack of memory as shown in Table 4, which summarizes the computational time and memory usages for \( n \) number of alternative investments.

Table 4 Computational time and memory required to solve \( n \) programs. The model was run on TOMLAB in MATLAB with 16GB RAM and Intel i7-3770 processor.

<table>
<thead>
<tr>
<th>Number of Investment Managers</th>
<th>Time</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.3s</td>
<td>3.2 MB</td>
</tr>
<tr>
<td>15</td>
<td>9.7s</td>
<td>1,023 MB</td>
</tr>
<tr>
<td>16</td>
<td>57.5s</td>
<td>2,048 MB</td>
</tr>
<tr>
<td>18</td>
<td>117.2s</td>
<td>8,193 MB</td>
</tr>
<tr>
<td>20</td>
<td>N/A</td>
<td>Out of Memory</td>
</tr>
<tr>
<td>22</td>
<td>N/A</td>
<td>Out of Memory</td>
</tr>
</tbody>
</table>

4.2 Identifying Critical Vertices

The vertex elimination procedure (VEP), an algorithm proposed in this paper to improve the computational efficiency of Problem (2) and Problem (7), is motivated by the fact that not every vertex of Problem (2 & 7) is binding (or critical in our term). Therefore, computational efficiency can be gained by eliminating non-critical vertices before solving the problem. More specifically, we want to choose to eliminate either the upper or lower bound variable per investment strategy in problem (2 & 7).
Consider a convex constraint \( f(y) \leq 0 \), then finding the binding (or critical) vertices is equivalent to solving the following optimization problem:

\[
\begin{align*}
\max f(y) \\
\text{s.t. } y \in V
\end{align*}
\]

\( (8) \)

where \( V \) is defined as in equation (3). From the notion of convexity, solutions of \( \max f(y) \) lie on the boundary of the feasible set. From the KKT conditions of problem (8) we get the following:

1. \( \nabla f(y) = \lambda - \pi \), where \( \lambda = [\lambda_1, \lambda_2, ..., \lambda_n]^T \) and \( \pi = [\pi_1, \pi_2, ..., \pi_n]^T \),
2. \( \lambda_i(y_i - u_i) = 0 \) and \( \pi_i(l_i - y_i) = 0 \), for \( \forall i = 1, ..., n \),
3. \( \lambda_i \geq 0 \) and \( \pi_i \geq 0 \), for \( \forall i = 1, ..., n \).

These conditions lead to the following results:

1. If \( \frac{\partial f}{\partial y_i} > 0 \), then \( y_i = u_i > l_i \), which results in \( \delta_i = 0 \) and \( \frac{\partial f}{\partial y_i} = \bar{\delta}_i > 0 \).
2. If \( \frac{\partial f}{\partial y_i} < 0 \) then \( y_i = l_i < u_i \), which results in \( \lambda_i = 0 \) and \( \frac{\partial f}{\partial y_i} = -\pi_i < 0 \).

Intuitively, \( \frac{\partial f}{\partial y_i} > (<) 0 \) implies that \( f \) is strictly increasing (decreasing) in \( y_i \) and \( f \) at \( u_i(l_i) \) will be greater than \( f \) at \( l_i(u_i) \). Therefore, if \( f \) at \( u_i(l_i) \) binding implies \( f \) at \( l_i(u_i) \) is feasible and \( l_i(u_i) \) could be removed from the set \( V_i \) defined in equation (3).

Let us consider a special case of the convex constraint \( f(y) \leq 0 \) such as the following linear inequality constraints:

\( Ay - b \leq 0 \)

where \( A = [a_1, a_2, ..., a_n] \) is a coefficient matrix and \( y = [y_1, y_2, ..., y_n]^T \) is the decision vector. The gradient of the function \( Ay - b \) is

\[
\nabla (Ay - b) = A^T = \left[ \frac{\partial f}{\partial y_1} \quad ... \quad \frac{\partial f}{\partial y_n} \right]^T = [a_1 \quad ... \quad a_n]^T.
\]
As the partial derivatives are constant for linear constraints, we remove either the upper or lower bound variable from the set \( V_i \), depending on the sign of the coefficient \( a_i = \frac{\partial f}{\partial y_1} \) as follows:

\[
V_i^* = \begin{cases} 
V_i - \{l_i\}, & \text{if } a_i \geq 0 \\
V_i - \{u_i\}, & \text{if } a_i < 0 
\end{cases}
\]

and the set \( V \) is redefined as follows:

\[
V = V_1^* \times V_2^* \times ... V_n^* = \{(v_1, v_2, ..., v_n) | v_i \in V_i^* \forall i \in \{1, ..., n\}\}
\]

Note that the reduction from \( V_i = \{l_i, u_i\} \) to \( V_i^* = \{u_i\} \) (or \( \{l_i\} \)) means the elimination of vertices with \( v_i = l_i \) (or \( u_i \)), in which case the cardinality of \( V \) is halved.

In case of nonlinear constraints, it is less intuitive since the partial derivatives are functions of the decision variables. Consider the following quadratic constraint \( y^T Q y \leq 0 \) as an example, in which we get:

\[
\nabla (y^T Q y) = 2Qy = [2Q_1y \ldots 2Q_ny]^T
\]

where the gradient is a linear function of \( y \). We then find the minimum and the maximum \( \frac{\partial f}{\partial y_1} \) in \( S \) to determine if \( \frac{\partial f}{\partial y_1} \) is strictly positive, negative or changes signs within a region \( S \) defined as follows:

\[
S = \left\{ \begin{array}{l}
\text{All Linear Constraints} \\
y_i \in V_i \forall i = 1, ..., n \\
u_i - l_i \geq \xi_i \forall i = 1, ..., n
\end{array} \right. 
\]

The following pseudo code displays the vertex elimination process for quadratic constraints:

Step 1: Start with \( i = 1 \)

Step 2: If \( \min_{x \in S} Q_i x > 0 \), then \( V_i^* = V_i - \{l_i\} = \{u_i\} \)

Step 3: Else if \( \min_{x \in S} Q_i x < 0 \), then check \( \max_{x \in S} Q_i x \)

a. If \( \max_{x \in S} Q_i x < 0 \), then \( V_i^* = V_i - \{u_i\} = \{l_i\} \)

b. Else if \( \max_{x \in S} Q_i x > 0 \), then \( V_i^* = V_i \)

Step 4: If \( i = n \) go to step 5, otherwise \( i = i + 1 \) and go to step 2
Step 5: Redefine set $V$ in problem (2):

$$V = V_1^* \times V_2^* \times \ldots \times V_n^* = \{(v_1, v_2, \ldots, v_n) | v_i \in V_i^* \forall i \in \{1, \ldots, n\}\}$$

Finally, we solve problem (2) with the new $V$ defined in step 5. Note that for every elimination of $u_i$ or $l_i$, the size of set $V$ decrease by a factor of 2.

The VEP eliminates a large number of vertices, although in the worst case it may not eliminate any. We have seen that the VEP eliminated all the vertices but two in a problem with 24 alternative investments (the number of vertices is $2^{24}$).

### 4.3 Computational Results

Table 5 shows the computational times required to solve problem (2) with and without VEP. The advantage of the constraint elimination procedure becomes apparent with 15 or more strategies. Also notice that the computation time would not increase dramatically in the number of alternative investments.

<table>
<thead>
<tr>
<th>Number of Investment Managers</th>
<th>Time (With VEP)</th>
<th>Time (Without VEP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.3s</td>
<td>0.3s</td>
</tr>
<tr>
<td>15</td>
<td>0.7s</td>
<td>9.7s</td>
</tr>
<tr>
<td>16</td>
<td>2.2s</td>
<td>57.5s</td>
</tr>
<tr>
<td>18</td>
<td>2.5s</td>
<td>117.2s</td>
</tr>
<tr>
<td>20</td>
<td>5.2s</td>
<td>Out of Memory</td>
</tr>
<tr>
<td>22</td>
<td>5.3s</td>
<td>Out of Memory</td>
</tr>
<tr>
<td>24</td>
<td>6.2s</td>
<td>Out of Memory</td>
</tr>
</tbody>
</table>

Table 5 Computational time required with and without the vertex elimination procedure (VEP) for problem (1)
Chapter 5

5 Robust Optimization

5.1 Purpose & Motivation

Many fund managers are cautious when it comes to practical applications of portfolio optimization because it is very sensitive to the inputs (e.g., expected returns of assets and their covariance). Therefore, it is critical that the inputs are robust to different sources of risk, mainly estimation risk. Several robust statistical estimates methodologies have been proposed to better estimate the parameters, such as the Black-Litterman [3] and Bayes-Stein Estimator [17].

However, even these robust estimation approaches may be limited in our portfolio construction because alternative investments potentially have higher estimation risk than the traditional assets (e.g. stocks and bonds) due to less number of observations and poor data quality. Therefore, we apply robust optimization methodology to achieve increased level of robustness in our model with respect to errors in the estimation of the input parameters. A number of papers show the benefit of robust optimization methodologies in various portfolio construction applications such as Goldfarb and Iyengar [6], which incorporates robustness to factor models of returns, and Chen and Kwon [8], which proposes a robust portfolio selection methodology for index tracking.

5.2 Uncertainty in Mean Return

In this paper, we only account for estimation errors in expected returns as the sensitivity to the error in expected return estimates has been found to be much greater than that to the error in variances [9]. We follow Ceria and Stubbs [7] to model the uncertainty sets for expected return as an ellipsoid. The uncertainty set for $\mu$ is expressed as follows:

$$U_x(\hat{\mu}) = \{(\mu - \hat{\mu})^T \Sigma_{\mu}^{-1} (\mu - \hat{\mu}) \leq \rho^2\}$$

(10)
It captures the idea that the investor would like to be protected in instances in which the total scaled deviation of the realized average returns from the estimated returns is within \( \rho \). Also, \( \hat{\mu} \) is an estimator of expected return \( \mu \), and \( \Sigma_\mu \) is the estimation error covariance matrix of expected returns. For example, if we assume that the true expected return \( \mu \) is normally distributed \( \mu^2 = \kappa^2 \), where \( \kappa^2 = \chi_n^2(1 - \zeta) \) is related to a \( \zeta \) percentile of chi-square distribution with \( n \) degrees of freedom. In other words, the true expected returns lies inside the confidence region (10) with probability \( \zeta \).

Together with problem (2), the robust form of the model can be formulated as follows:

\[
\begin{align*}
\max & \quad \prod_{i=1}^{n} (u_i - l_i)^{k_i} \\
\text{s.t.} & \quad \min_{(\mu-\bar{\mu})^T\Sigma^{-1}(\mu-\bar{\mu})\leq \rho^2} \mu^T x \geq R_{\min} \\
& \quad x^T Q x \leq \sigma^2 \\
& \quad 1 - \delta^T x \geq \tau \\
& \quad L_i \leq l_i, u_i \leq U_i \quad \forall \ i = 1, \ldots, n \\
& \quad u_i - l_i \geq \xi_i \quad \forall \ i = 1, \ldots, n \\
& \quad x \in [V; 1]
\end{align*}
\]

(11)

From Ceria and Stubbs [7] the following formulation is equivalent to problem (11):

\[
\begin{align*}
\max & \quad \prod_{i=1}^{n} (u_i - l_i)^{k_i} \\
\text{s.t.} & \quad \hat{\mu}^T x - \rho \sqrt{x^T \Sigma x} \geq R_{\min} \\
& \quad x^T Q x \leq \sigma^2 \\
& \quad 1 - \delta^T x \geq \tau \\
& \quad L_i \leq l_i, u_i \leq U_i \quad \forall \ i = 1, \ldots, n \\
& \quad u_i - l_i \geq \xi_i \quad \forall \ i = 1, \ldots, n \\
& \quad x \in [V; 1]
\end{align*}
\]

(12)

Intuitively, problem (12) aims to maximize the weighted volume of alternative investments while maintaining feasibility for possible realizations of the worst returns. Incorporating uncertainty to our model will produce ranges that are smaller than the non-robust model. This implies that the investment strategies may be required to stay within smaller ranges in order to account for the worst realizations. However, the benefit of flexibility can only be observed qualitatively.

Similarly, the robust ARA looks as follows:
\[ \begin{align*} 
& \text{max } t \\
& \text{s.t. } \hat{\mu}^T x - \rho \sqrt{x^T \Sigma_{\mu} x} \geq t \\
& \quad x^T Q x \leq \sigma^2 \\
& \quad 1 - \delta^T x \geq \tau \\
& \quad L_i \leq l_i, u_i \leq U_i \forall i = 1, \ldots, n \\
& \quad u_i - l_i \geq \xi_i \forall i = 1, \ldots, n \\
& \quad x \in [V; 1] 
\end{align*} \] 

\[ (13) \]

5.2.1 Error Covariance Matrix

The estimation of the estimation error covariance matrix $\Sigma_{\mu}$ used in equation (10), is not immediately obvious. In theory, if the returns are independently and identically distributed with mean vector $\mu$ and covariance matrix $\Sigma$, then $\Sigma_{\mu} = T^{-1} \Sigma$, where $T$ is the number of observation data. However, this may not be the best method in practice due to the following reasons: 1) This approach applies only in a world in which returns are stationary, 2) We cannot guarantee that the estimate of the asset covariance matrix, $\Sigma$, itself is reliable as computing a meaningful covariance matrix requires a large number of observations [4].

Several robust statistical approaches for estimating expected returns, such as the Black-Litterman method and the James-Stein method, can estimate the estimation error covariance matrix in the process of generating the estimate of the expected returns. Readers can refer to Fabozzi et al [4] which describes how the estimation error covariance matrix are obtained in the process of using Black-Litterman and James-Stein Estimator to estimate expected returns.

5.3 Computation Studies

We conduct numerical studies similar to Asl and Etula [10] to examine how the robust version of our model copes with errors in expected return estimates. We perturb the expected returns and compare the out-of-sample performance. To this end, we use the average of historical returns to estimate the expected returns and covariance for seven active investment strategies, using monthly proxy data for alternative assets from January 1995 to September 2007. We then generate 1,000 perturbations of these expected returns by randomly sampling from a multivariate normal distribution with average of historical return as the mean and error covariance matrix. We
feed these perturbed expected returns and the asset covariance matrix into our robust model (12) and the original model (2) to arrive at respective portfolio ranges for each of the 1,000 perturbations. Finally we randomly allocate within the range and evaluate the performance of the portfolios between September 2007 and December 2014 as presented in Figure 4-7. Figures 1 and 2 are constructed from different constraint limits from Figures 3 and 4, but using the same input and out-of-sample data.

The results in Figure 5 and Figure 7 are consistent with the findings in Asl and Etula [10], where the cumulative returns generated by the robust model (12) are less dispersed than those generated by the original model. This means that the portfolio is more robust to perturbation of expected return; therefore, having more consistent cumulative returns over expected returns scenarios. Moreover, we observe in Figure 4 that the robust portfolio outperforms the original model from time step = 0 to time step = 20. This shows that robust optimization protects the portfolio in periods of market downturn, leading to a larger number of greater cumulative returns at the end of the horizon. However, it can be noted that the observation in Figure 4 is an instance of particular set of constraint limits and robust model may not always outperform as shown in Figure 6 and Figure 7.

Figure 4 Evaluating Out-of-Sample Performance of Problem (12) relative to Problem (2) from 2007 to 2014
Figure 5 Distribution of the Cumulative Returns at the End of the Horizon (t = March 2014)

Figure 6 Evaluating Out-of-Sample Performance of Problem (12) relative to Problem (2) from 2007 to 2014 with different constraint limits
Figure 7 Distribution of the Cumulative Returns at the End of the Horizon (t = March 2014) with different constraint limits
6 Conclusion & Future Work

We developed two extensions to the strategic asset allocation for multi-asset fund consisting of alternative investments such as private equity, real estate and infrastructure, and robustified the extension to account for uncertainty in the estimation of expected returns. The first extension maximizes the flexibility of allocation of the alternative investment strategies rather than the traditional characteristics of portfolio such as return and risk. This aims to minimize the incidents where individual investment managers are pressured to buy or dispose their assets at non-preferable time just to reside near a target weights provided by the fund. The second extension maintains a flexibility feature of the first extension, while optimizing for economic utility, specifically portfolio return. Furthermore, we tackled the robust version of the problem to explicitly account for the uncertainty in estimation of expected returns. We believe that this is essential addition to our problem as the alternative assets potentially have higher chances of estimation errors due to less observation data and lower data quality. Finally, we reduced the complexity of the model such that the problem can be solved for moderate number of alternative investment strategies (20 or more).

There are many potential extensions to our model. First, we could quantify the trade-off between flexibility of allocation of investment strategies and fund’s economic utility, such as risk and return. Second, lack of data could lead to greater estimation errors in the covariance matrix. Although the error sensitivity in expected return estimates has been found to be approximately one order of magnitude greater than the error sensitivity in variances [9], examining the effect of uncertainty in the covariance matrix could be of interest. Interested readers on this issue can refer to Goldfarb and Iyengar[6], which imposes the uncertainty of return and covariance matrix on top of factor models for returns or Tutuncu and Koenig[11], which considers confidence intervals for the individual covariance matrix entries. These may require a development of a more general algorithm for our model. Third, the choice of the uncertainty parameters could be
studied more in depth by incorporating views on market conditions. In this paper, we assumed that the fund does not have any views on the market condition; hence, the size of $\delta$ were consistent across the investment managers. By having some views on the market condition, we can specify which investment strategies will have better turnover rate, which makes it easier for the investment managers to rebalance their portfolio. Finally, the choice of uncertainty set of the investment managers could be studied by taking into account of correlations in investment managers’ behavior. For example, the uncertainty set $X$ defined in equation (3) can be a form of an ellipsoid to account for correlated investment executions across all investment managers.
Bibliography


