The physics of galactic spin

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<th>Journal:</th>
<th>Canadian Journal of Physics</th>
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<tr>
<td>Manuscript ID</td>
<td>cjp-2016-0625.R1</td>
</tr>
<tr>
<td>Manuscript Type:</td>
<td>Article</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>20-Oct-2016</td>
</tr>
<tr>
<td>Complete List of Authors:</td>
<td>Hofmeister, Anne; Washington University, EPS Criss, Robert; Washington University, EPS</td>
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<tr>
<td>Keyword:</td>
<td>Virial theorem, galactic rotation, non-Newtonian forces, dark matter, spin</td>
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The physics of galactic spin

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PACS No: 98.52.Nr; 98.62.Ck; 98.62.Dm; 04.50.Kd

Revised Version: November 2, 2016: For Canadian Journal of Physics

Word count: 181 = abstract; 9239 = text + 2 tables + captions + 50 refs; 8 Figures

The authors declare no conflict of interest.

Abstract: Spiral galaxies are discrete spheroidal objects with highly organized, circular internal motions about a special axis. Available models do not describe these key features or explain why the central regions rotate like a solid body. Practically all previous models describe galaxies as a collection of orbiting test particles, utilize numerous fitting parameters, and require either copious amounts of surrounding dark matter that have not been detected, or modifications to Newton’s law, to fit the observed dependence of equatorial velocity on radius. Our paper probes the reasonable alternative that galaxies are discrete, spinning objects. Our analytical forward models, constructed by applying the Virial theorem and Newton’s law to Maclaurin’s spinning spheroids with varying internal density, explain why galactic rotation is organized into this 3-dimensional shape. Without invoking dark matter, our spin model explains why the outermost rotational velocities are nearly constant, yet depend on galaxy size, and, with no free parameters, provides masses of 14 important galaxies consistent with their luminosities. We show that proposed modifications to Newton’s law compensate for the dynamical differences between a flattened, spinning, Newtonian spheroid, and a collection of orbits.

Keywords: Virial theorem; galactic rotation; dark matter; non-Newtonian forces; spin
1. Introduction

Despite decades of intensive searches, non-baryonic particles have not been directly detected [1, 2]. Observational evidence for dark matter is thus limited to forward (fitting) models of galactic rotation curves (RCs), derived from Doppler shifts that define the dependence of equatorial velocity ($v$) on cylindrical radius ($r$) [3]. These Newtonian orbital models (NOMs) originally compared forces, whereas Poisson’s equation is now mostly used [4, 5, 6]. Figure 1a shows RC behavior typical of non-interacting galaxies [7]. Figure 2 shows RC data at very large $r$ [8]; key characteristics are in Table 1.

Early (~1979) interpretations, that the observed, nearly constant rotational velocities at large radius (Figs. 1, 2) require copious amounts of non-baryonic matter surrounding the galaxy, evoked concerns [3]. This led Milgrom [9] to propose that Newton’s law may not hold under certain circumstances. Subsequent application of Milgrom’s MOdified Newtonian Dynamics (MOND) formulation demonstrated that dark matter is not needed for diverse galaxy types [10, 11]. Concurrently, NOMs became more sophisticated, but with two, rare exceptions [2,12], provide large amounts of dark matter, ~75% for the Milky Way and many others [4,13,14] and up to 99.9% for dwarf galaxies [15]. Because these ambiguous forward models provide dark matter exceeding cosmological estimates of ~27% [16], this component is arbitrarily minimized during fitting [3, 5, 6,17].

Whether dark matter exists or MONDian physics is correct is unlikely to be resolved through RC fits because constraints are lacking. For example, density functions are assumed. Although mass-light discrepancies [18] could arise from the model dependence of luminosity-mass relationships [4,17], this proposal does not address why RC are flat at large radius. The impasse has motivated new theories [19] and prompted explanations rooted in relativistic effects [e.g. 20].

An unexplored possibility is that NOMs are incorrectly posited, so that large amounts of dark matter are an artifact of their mathematical physics constructs. NOMs treat the galaxy as a collection of orbits of test particles around an enclosed mass (e.g., [13, 21]). However, orbits are independent of interior rotational motions: e.g., Earth’s orbit is unrelated to the Sun’s spin. Neither the mass nor the density of the satellite is relevant and retrograde motions and polar circuits are permitted (Fig. 1b).
velocity distributions of spinning and orbital systems differ for the same energy (Fig. 1c), and hence will produce different mass from the same RC. NOMs cannot explain galactic shapes (Fig. 3) or why motions inside spiral galaxies are highly organized in terms of their direction and axis of rotation. We refer to this phenomenon as dynamical coherence. Galactic motions resemble spin of stars or gas giants, but the high density in small objects obscures their interior and degrades their spin. RC behavior at any radius is unlike that of orbits (Fig. 1a). In particular, that centers of spiral galaxies spin like tops is unexplained [3], although this trend can be reproduced with fitting parameters. Lastly, recent force fitting models [2,12] that depart from mainstream NOMs by accounting for effects of mass outside the orbit of interest, do not require dark matter.

Orbits have been the focus, rather than spin, because the former phenomenon is familiar and important to astronomy. Although spin is relevant to discrete objects and was historically of great interest [22], it has subsequently received scant attention, as evidenced by only one model explaining axial spin in our Solar System [23, 24]. That galaxies spin is evident in images (e.g., spiral arms) and particularly in their oblate spheroid shapes (Fig. 3). Maclaurin proved that this was the Newtonian shape for large spinning bodies, as is confirmed by the known oblate shapes of numerous spinning stars and planets. This spheroidal shape was once considered to describe galaxies [25, 26], but has been replaced by an emphasis on disks [27]. Table 2 quantifies differences between spin and orbit models.

Section 2 discusses problems in NOMs, and explains why MOND gives reasonable fits to RC. To discern the first order properties of spinning galaxies, we apply the Virial Theorem of Clausius to rigid bodies with internally varying density (Sect. 3). We provide Newtonian forward models of solid bodies as a first approximation to galactic spin, given Fig. 1a and historic and modern treatments of stars, nebulae, and rain clouds [28], also see [23]. Non-rigid bodies are addressed in a companion paper\(^1\) using an inverse approach and properties of Newton’s homeoids. The focus here is the important effects of shape and the difference between the self-gravitational potential and the external potential [29]. Dark matter is unnecessary because spin and flattening each reduce the total (dynamical) mass by a factor of \(\sim \frac{2}{3}\) from

\(^1\) R.E. Criss and A.M. Hofmeister, in review, New Astronomy.
that required by NOMs (Sect. 4). Our model explains key features and trends observed in galaxies and provides mass within the visible edge that agrees with luminosity data (Sects. 4 and 5).

2 Existing formulations and problems in implementation

2.1 Essential mathematics of the MONDian formulation

Considering force balance permits us to elucidate the interplay of MONDian models and fits to RC data. Milgrom [9] originally proposed that the force experienced during rotation of a galaxy is:

\[ F_{\text{MOND}} = \mu F_{\text{Newton}} (s) = \mu(x) \times \frac{GMm}{s^2} \]  

(1)

where \( s \) = radius in spherical coordinates, \( G \) is the gravitational constant, \( M \) is galaxy mass, and \( m \) is test mass. The function \( \mu(x) \) depends on acceleration \( a \) through the ratio \( x = a/a_0 \) where \( a_0 \) is a constant. Two forms were proposed:

\[ \mu(x) \approx x = \frac{a}{a_0} < 1 \quad \text{or} \quad \mu(x) = \frac{x}{\sqrt{1 + x^2}} < 1. \]  

(2)

Acceleration during galactic rotation is centripetal. Because the rotational motions appear to be circular, velocities are provided only as a function of \( s \). Therefore, \( a = v^2/s \) and (1) simplifies to:

\[ F_{\text{MOND}} = q(s) \times \frac{GM(s)m}{b_0s^2} = \frac{G M_{\text{in}}(s)m}{Qs^2}. \]  

(3)

The middle term introduces the new constant \( b_0 \) which is related to \( a_0 \) but has different units depending on the functionality of \( q(s) \). With these two modifications \( (b_0 \) and \( q) \) to the force law, the MOND approach can reproduce flat RC at high \( s \) (or \( r \)) with a mass that is compatible with estimates based on luminosity.

Because orbits are being considered, \( m \) in (3) describes an exterior orbiting body (i.e. a point test mass or ring) and \( M \) describes the galactic mass interior to \( s \). Hence, \( M = M_{\text{in}} \) depends on \( s \), and first grows with \( s \), but then terminates at some large value of \( s \). Because the data are only fit to \( v(s) \), we cannot separate \( q(s) \) either from \( M_{\text{in}}(s) \) or from the denominator \( s^2 \). This deduction is based on recent analysis of ambiguities arising in diverse models with lumped (multiplied) factors [30].
Because the three functions $s^2 q(s) M_{in}(s)$ combine in the fitting to $v(s)$, how the force differs from the Newtonian inverse dependence on $s^2$ is obscured. We do not know $M_{in}$ independently. Mass is modelled from the luminosity, but such a determination can be equivocal [18] because luminosity is reduced by the presence of gas and dim white dwarfs and by extinction (e.g., [31, 32]). Thus, the product of two unknowns ($q \times M_{in}$) is being compared to only one measured quantity ($v$).

The right hand side (RHS) of (3) contains the essentials of Milgrom’s proposal, where we have lumped the difference in forces into $M_{in}^*$ and have morphed his acceleration constant ($a_0$) into a dimensionless number ($Q$). We do not imply that either $M_{in}^*$ or $M_{in}$ is the true mass. True mass cannot be ascertained from fitting because any variation of $v$ with $r$ can be mapped into a variation of $M_{in}^*$ with $r$, no matter what the radial force law is. Likewise, obtaining a match in NOMs does not guarantee that these formulations are correctly cast. Transtrum et al. [30] discusses how good fits can be obtained with incorrect physics.

We propose that the factor $Q$, and Milgrom’s $a_0$, effectively accounts for shape. This essential characteristic is missing from NOMs, yet is important to rotational energy and angular momentum [33].

2.2 Newtonian models involving multicomponent fitting

NOMs are based on two equations. The traditional approach equates centripetal to Newtonian forces, while assuming that the mass is distributed among 3 to 4 components [3-5]:

$$v^2(r) = \frac{G \sum M_{in}(r)}{r} = \frac{G}{r} \left[ M_{\text{point}}(r) + M_{\text{sphere}}(r) + M_{\text{disk}}(r) \right] = \frac{G}{r} M_{\text{dynamic}}(r)$$  (4a)

Here, the subscript “point” refers to a central black hole, whereas “sphere” refers to dark matter surrounding the galactic “disk”. Because the galactic bulge, which contributes as a point mass for large $r$ and as a sphere at small $r$, is mathematically equivalent to other components, the effect of a bulge is implicit in our analysis.

Potentials are now typically used to fit RC, based on modifying Poisson’s equation to:
\[-4\pi G \sum_j \rho_j(r) = \nabla^2 \psi \]  

(4b)

[34, see e.g., 5,6]. This approach sums densities, a step forbidden by thermodynamics because intensive variables do not sum. This process superimposes solutions, which is valid only for homogeneous differential equations but not for Poisson’s inhomogeneous equation [35]. Because the potential-density approach gives similar proportions for mass components [cf. 4,5] as does the force-balance approach, which is easier to understand and directly tied to mass, we probe the underlying physics by considering (4a).

Luminosity of a galaxy can be matched by different combinations of the alleged mass components, because the dark matter is entirely non-luminous and the disk is partially non-luminous. Furthermore, these mass components depend differently on \( r \), so summing different proportions of these various contributions always permits a good match to \( \nu(r) \). For example, consider constant density (\( \rho_o \)). The point mass is constant, while the mass of the sphere goes as \( \rho_o r^3 \) and that of the flat disk goes as \( \rho_o r^2 \). When these mass functions are combined with the prefactor \( G/r \), the three terms in (4) roughly go as \( 1/r \), \( r^2 \) and \( r \), respectively. Some combination of these different functions will invariably provide a reasonable fit to galactic \( \nu(r) \) data, which typically first increases linearly with \( r \), gradually becomes constant, and then may decrease with \( r \) (Figs. 1, 2).

Typically, an exponential decline for matter in the disk is assumed: \( \rho = \rho_o \exp(-br) \) where \( b \) is a constant [27]. This decline weakens the disk contribution from the above linear dependence, thereby making the spherical component crucial to providing constant \( \nu \) at large \( r \). This assumption, coupled with the spherical halo volume being \(~13\) times larger than an inscribed disk of height to radius of 1:10 that approximates spiral galaxies ([36]; Fig. 3), gives rise to large estimated amounts of dark matter.

More specifically, fits are obtained with assumed density functions by varying 4 to 9 parameters for 3 to 4 components [4-6]. By considering multiple geometries, each with some specified yet arbitrary dependence of density on \( r \), many “good” fits are possible. Importantly, because only \( \nu(r) \) is being fitted, the only quantity actually being modelled in (4) is \( M_{\text{dynamic}}(r) \). Yet, the RHS of (4) is valid only if the total
mass interior to some radius is spherically distributed (e.g. [33]). Thus, (4) is inappropriate for a disk. Equation (4) describes Keplerian orbits in our Solar System, wherein planetary motions are controlled by the massive Sun, with minor perturbations. Galaxies cannot possibly be represented by this reduced 2-body problem, because their many stars are similar in mass, and most mass resides not near the center but in the outer reaches of these flat objects. Lastly, a different moment of inertia ($I$) than a sphere or point mass governs flat shapes. From comparing (3) and (4), and the above analysis, indicates that the constant acceleration in Milgrom’s original formulation [9], which we transformed into $Q$, is related to a shape factor missing from NOMs.

Incorrectly addressing shape causes overestimation of galactic mass in the NOMs (Fig. 1c), thus creating the postulated large amounts of non-baryons. Possible effects of non-luminous baryons inside (e.g. [17,18,31]) and outside (e.g. [37]) spiral galaxies have been discussed, but the importance of shape was overlooked, as is evident in the RHS of (4), and in the lack of discussion of $I$ for galaxies in Binney and Tremaine’s tome [27].

2.3 Poisson’s orbits or Newton-Maclaurin’s spin?

Recent Mondian analyses of galactic rotation modify Poisson’s equation to:

$$\nabla \left[ \mu \left( \frac{\nabla \psi}{a_0} \right) \nabla \psi \right] = 4\pi G \rho$$  \hspace{1cm} (5)

where $\psi$ is the potential and Milgrom’s function $\mu$ now depends on generalized acceleration ($\nabla \psi$) [38]. In Newtonian analyses, $\mu = 1$, reducing (5) to the familiar equation: $\nabla^2 \psi = 4\pi G \rho$. Most forward models assume a density function and calculate the associated potential, via numerical analysis, iterating until a fit to $v(r)$ is obtained (e.g., [6, 39]). Potentials so obtained from (5) describe the attraction of an exterior (test) mass to the object, as is known [13,29]. This approach cannot constrain the mass (or density) exactly at the radius of interest because the mass of the satellite is irrelevant to its circular orbit [33].

The shape of the Earth was of great importance to navigation ~500 years ago, motivating many prominent mathematicians and physicists to focus on the problem of a spinning, self-gravitating object.
[22]. Newton and Maclaurin showed that forces of rotation and gravitation balance in an oblate spheroid shape [40]; this result is confirmed by the shapes of planets and stars. Regarding galaxies, oblate spheroid geometry is compatible with images (Fig. 3) and with their organized, circular motions. Because gravity is to a good approximation the sole force operating in galaxies, the results of Newton and Maclaurin pertain and greatly restrict the potentials. Last, but not least, it is known from potential theory [29] that all parts of a body attract each other and that the self-potential, and not the ordinary potential, describes the interior workings. Differences between the self-potential and the ordinary potential listed in Table 2 demonstrate defects of the latter as a descriptor of object interiors.

Setting Maclaurin’s contribution aside momentarily, solving (5) with $\mu \equiv 1$ can lead to potentials that are non-Newtonian, since two integrations are performed which involve two constants of integration, but no constraint other than radial velocities is available for galaxies. Modelling galactic disks in cylindrical coordinates $(r, z)$ has led to peculiar logarithmic forms (e.g., table 1 in [41]). For example:

$$\psi_{EB}(r, z) = -\frac{GM}{a} \log \left[ a + \sqrt{b^2 + z^2} + \sqrt{(a + \sqrt{b^2 + z^2})^2 + r^2} \right]$$ (6)

where the constants are restricted to $a + b > 0$ with $b > 0$ [41], and thus $a > 0$. The dependence of velocity on radius for relatively large $r$ is particularly important to RC because this controls the amount of dark matter. Equation (6) does not reduce to the correct limit for orbits at very large $r$, far beyond RC measurements, that associated with a massive central point (Table 2; Fig. 1), and thus cannot be correct.

Whether numerical analyses of Poisson’s equation provide the correct limit is not clear because only fits to $v(r)$ are provided, e.g., [6]). However, even if some assumed density model for a galaxy were correct, the potential obtained from (6) does not include the shape restriction discovered by Maclaurin on an object in equilibrium during spin, and summing densities for different volumes is invalid.

2.4 The disk?

Neither a disk of finite thickness nor a thick ring, with their vertical edges and corners, can be gravitationally stable. This should be obvious from the presence of vertical gravitational force components
that cannot be balanced by radial centrifugal forces. In the oblate shape, the force of spin balances that of Newtonian gravity everywhere (e.g., [40]). Also, the disk has unrealistic corners (Fig. 3).

Early astronomical literature recognized that the oblate spheroid shape described galaxies (e.g., [25, 26, 42]). Cylindrical coordinates were used because the focus was on orbits in the equatorial plane, yielding cumbersome equations. In contrast, Maclaurin’s evaluation of the double integral for the gravitational self-potential, provided a simple dependence on \( s \) only [40].

Pernicious treatment of \( z \) and \( r \) as independent variables in assessing galactic rotation (e.g., [2, 27, 41]) stems from Perek’s [42] claim that density in the \( z \) direction is independent of its variation in \( r \) for an oblate spheroid. This claim is incorrect, as follows. The minor axis \((C)\) is related to the major \((A)\) via the ellipticity \((e)\):

\[
e = \left(1 - \frac{C^2}{A^2}\right)^{1/2}.
\]

The definition for the oblate spheroid surface can then be used to relate \( z \) to cylindrical radius \( r \) for each of the nested homeoids (Fig. 3), which necessarily has the same shape factor of \( e \) and an equatorial radius proportional to \( A \):

\[
z^2 = (1-e^2)(A^2-r^2).
\]

The dependence of density on \( z \) is thus completely specified by the variation of \( \rho \) with \( r \), making the problem of galactic rotation unidimensional. This finding is consistent with radial information on mass being the only information than can be extracted from radial velocities (Sect. 2.1), and with data on the internal layers of spinning planets.

An oblate spheroid is simply a flattened sphere, and this transformation only involves a scaling factor (8). For this reason, and the spherical symmetry of Newton’s law, it is the required shape of spinning galaxies. Perek’s claim provides an additional parameter, which leads to erroneous results for \( v(r) \), even if the orbital approximation were correct.
2.5 Summary

Investigations of galactic rotation using either force balance or Poisson’s equation have utilized potentials that are inconsistent with results of Newton and Maclaurin, whether purposeful, after Milgrom [9], or not. The presently available information on \( v(r) \) and luminosity are insufficient to simultaneously and unambiguously determine mass and a force law in a forward model of orbits. The present paper assumes that Newton’s law holds, and develops a formulation of spin (Table 2) for a stable galaxy. This model is needed to evaluate whether the dark matter component of the mainstream orbital constructs is a mathematical artifact of not addressing shape. Because spin has not heretofore been considered, we provide a first-order approach, the solid-body approximation, after [28].

3. Theory

3.1 Energy is a function of mass

The concept of energy conservation postdates Newton and Maclaurin, who based their work on force balance and momentum conservation. Energy, a scalar, is simpler to deal with than forces, and is directly tied to the desired quantity, mass. This tie is evident in well-known relationships, such as translational kinetic energy = \( \frac{1}{2} mv^2 \), rest energy = \( mc^2 \) (where \( c \) is the speed of light), or the rotational energy of a spinning, rigid body:

\[
\text{R.E.} = \frac{1}{2} I\omega^2,
\]

where \( \omega \) is its angular velocity. Moments of inertia for the homogeneous (subscript h) oblate spheroid about its polar axis and for the sphere are:

\[
I_{\text{obl, h}} = \frac{2}{5} MA^2 = \frac{8\pi \rho_h}{15} A^4 \left(1 - e^2\right)^{\frac{1}{2}}; \quad I_{\text{sphere, h}} = \frac{2}{5} MS^2.
\]

Emden [28] derived the gravitational self-energy of a sphere whose density is described by a polytropic equation of state:

\[
U_{\text{g, self, polytrope, sphere}} = -\frac{3}{5 - n} \frac{GM^2}{S}; \quad U_{\text{g, self, h, sphere}} = -\frac{3}{5} \frac{GM^2}{S}.
\]
where index $n = 0$ corresponds to homogeneous density with higher values ($n$ up to 5) representing progressive mass concentration towards the center. Emden [28] derived (11) for nebulae and rainclouds, where $n$ is related to adiabatic compression of a gas (see e.g., [43]).

Maclaurin derived the gravitational self-energy (i.e., the self-potential) for a rigid oblate spheroid:

$$U_{\text{g, self, oblate}, h} = -\frac{3}{5} \left(1 - e^2\right)^{\frac{3}{2}} \frac{\arcsin(e) GM^2}{e} S = -\frac{3}{5} \frac{GM^2}{S} = -\frac{3}{5} \frac{\arcsin(e) GM^2}{A}. \quad (12)$$

where $S$ is the equivalent body radius, obtained from the volume:

$$V_{\text{obl}} = \frac{4\pi}{3} S^3 = \frac{4\pi}{3} A^2 C = \frac{4\pi}{3} A^3 \left(1 - e^2\right)^{\frac{1}{2}} \quad (13)$$

and we have used (7) to provide the RHS of (12). See Dankova and Rosensteel [40] for a detailed discussion, formulae for three unequal axes, and limiting values for high and low $e$.

3.2 The Virial theorem for galactic spin

Galaxies are spinning, gravitationally bound objects that are restricted in space. Although galaxies clearly evolve with time, as is evident in their varied but gradational forms (i.e., Hubble’s tuning fork), Doppler measurements provide a snapshot in time, which then can only be evaluated in the context of quasi-equilibrium. Because existing analyses of galactic RC err fundamentally (Sect. 2; Table 2), the present paper considers a simple case, that of a spinning, rigid oblate spheroid with different internal density distributions, governed by Newtonian forces. This approach, after [28], provides the outermost velocities, which need to be better understood. Our paper oversimplifies the dynamical coherence of spinning galaxies as rigidity, but this corrects a fundamental flaw in orbital models, which neglect coherence entirely. Coherence is obvious in images and ubiquitous circular rotation.

A bound, conservative system is described by Clausius’ Virial theorem (VT). The VT imposes a constraint in addition to energy and momentum conservation: because the VT is derived from linear momentum balance over the space occupied, only one conservative force or phenomenon pertains to any given analysis [24].
According to the VT, the rotational energy (R.E.) of a spinning galaxy equals negative one-half of its gravitational self-potential energy. The self-potential is relevant \([24,29]\). For any shape, the VT for a spinning object yields:

\[
\omega^2 = -\frac{U_{g,\text{self}}}{I} \quad \text{or} \quad v^2 = -r^2 \frac{U_{g,\text{self}}}{I} \quad . \tag{14}
\]

Note that NOMs \((4)\) are based on Keplerian orbits which are straightforwardly derived from Clausius’ Virial theorem, but require a dominant central mass and concern the ordinary potential \(-G M m / r\). Previous references to the VT concern Keplerian orbits only.

Approximating a galaxy as a spinning, rigid body is reasonable. First and foremost, the observed velocities do increase linearly with \(r\) in the inner zones (Fig. 1). Second, we are pursuing forward models, which are inherently equivocal. The input is density as a function of radius and shape. Shape can be narrowed down to \(C / A \approx 0.1\) for most spirals, but this is not exact. Density is unknown, and while this quantity might be expected to vary smoothly with radius, images of spirals show that this is incorrect in detail. Thus, forward models can only approximate the density, and can only provide approximate results. Third, density and shape trade off, as is evident in \((4)\) and \((14)\). Fourth, our goal is to elucidate the general trends of RC and why constant velocity at large radius is ubiquitous. An exact solution, allowing for differential rotation, requires an inverse model based on homeoids, which can spin independently, and is given in our companion paper.\(^1\) Here we focus on typical or general features of RC.

The gravitational self-potential is obtained by double integration, and can be constructed by successively stripping away concentric shells, and banishing each shell to infinity. For the sphere:

\[
U_{g, \text{self, sphere}} = -\int_0^M G \frac{M_{\text{in}}}{r} \, dm = \int_0^S G \left[ \int_0^r 4\pi f^2 \rho(f) \, df \right] 4\pi s \rho(s) \, ds \tag{15}
\]

Other useful formulae are:

\[
M = \int_0^M \, dm; \quad M_{\text{sphere}} = \int_0^S 4\pi s^2 \rho(s) \, ds \tag{16}
\]

\[
I = \int_0^M s^2 \cos^2 \varphi \, dm; \quad I_{\text{sphere}} = \frac{2}{3} \int_0^S 4\pi s^4 \rho(s) \, ds \tag{17}
\]
where $90^\circ-\varphi$ is the angle from the vertical axis of rotation and equals 0 when $z = 0$.

3.3 Analytical results for a spinning oblate spheroid with varying internal density

We begin by comparing results for the homogeneous sphere and oblate spheroid, using (14) and the results respectively obtained by Emden [28] and Maclaurin (see [40]):

$$v_{s, h}^2 = \frac{3}{2} \frac{GM}{S}; \quad v_{o, h}^2 = \frac{3}{2} \frac{GM \arcsin(e)}{e}$$

From examining (10) and (18), and by considering the definitions of (15)-(17), the Viral result for the oblate spheroid (14) will differ from that of the sphere only by a geometric factor ($\times \arcsin(e)/e$) and by the substitution of $A$ for $S$, given that density in the oblate spheroid integrates to provide the same mass ($M$ for the equivalent radius $S$, see (13)). Essentially, this stems from the coordinate transformation noted in Sect. 2.4. Therefore, we can integrate various spherical radial functions for density to obtain analytical families of outermost velocities for rigid oblate spheroids. Examples are:

$$\rho(r) = br^k; \quad 0 \geq k > 2.5: \quad v^2 = \frac{3}{2} \frac{(5+k)}{5+2k} \frac{GM \arcsin(e)}{e}$$

$$\rho(r) = \rho_0 [1-(r/a)^j]; \quad j > 0: \quad v^2 = \frac{3}{2} \frac{(2j+11)}{2j+5} \frac{GM \arcsin(e)}{e}$$

polytropic oblate; $0 \leq n < 5 \quad v^2 \approx \frac{3}{2} \left(\frac{5}{5-n}\right)^3 \frac{GM \arcsin(e)}{e}$$

For (21), we fit $I$ for polytropic sphere [43] to an approximate relationship of

$$I \approx \frac{2}{5} \left(\frac{5-n}{5}\right)^2 MS^2.$$

For brevity, we denote the above models as power law (19), arc (20), and polytrope (21).

The above examples embody diverse density variations (Fig. 4), with (20) and (21) featuring constant central densities and zero edge densities, while describing different degrees of compression, in accord with images and the bound state being spatial restricted. The polytrope [28] is associated with
adiabatic compression of an ideal gas. Formation of the galaxy from an immense gas cloud that was compressed under its own weight, suggests that the present density distribution might be derived from that of the earlier state. Likewise (20) provides a finite central density and an edge (Fig. 3b), and is thus a reasonable model for a galaxy. The shape of $\rho(r)$ suggests the name “arc” for this construction.

Both the arc and polytrope models provide gradual but complete termination with zero density at some large but finite value of $r$. In contrast, a weak power law provides an abrupt, non-zero density cut-off at the galactic edge, as would a weak exponential function, which may be unrealistic. With a strong power law, edge density is low and therefore reasonable but density would be highly concentrated in the center, much more so than for an exponential function (Fig. 4c) or the low index arc and polytrope models.

The variety of densities in (19)-(21) should suffice to provide a first order explanation of galactic spin. A strong exponential with density $\sim 0$ at the edge is reasonably represented by the $n = 3$ polytrope (Fig. 4a), and so our approach does not exclude this approximation, which is popular in RC studies [27].

4. Forward models

4.1 Dynamical mass of spiral galaxies

Spirals typically have $C/A$ close to 0.1 [36,44], giving $\arcsin(e)/e = 1.48$, rounded to 3/2 for convenience. For reference, the sphere with $e = 0$ has $\arcsin(e)/e = 1$ and the infinite oblate with $e = 1$ has $\arcsin(e)/e = \pi/2 \approx 1.57$. Hence, this geometrical factor varies little from 3/2 for the various flat spiral forms of galaxies, so (19)-(21) provide a total (dynamical) mass of:

$$M_{\text{dynamical}} = \frac{2}{3} \arcsin(e) \frac{v_{\text{outermost}}^2}{G} A f(n, k, j) \leq \frac{4}{9} \frac{v_{\text{outermost}}^2}{G} A$$

where $f$ is a function derived from the density formulation. This factor is unity for constant density, and positive but smaller than unity for all density models (see (19)-(21)).

From (23), the dynamical mass required by the VT for spin is no more than 4/9 that required by “Newtonian” orbital models, i.e., (4). Reduction by $\frac{3}{5}$ results just from considering spin, which is more restricting than orbits. This factor results from combining the 3/5 factor in $U_{\text{g, self}}$ with the 2/5 factor in $I$.
associated with the sphere and spheroids. Another reduction of \( \frac{2}{3} \) results from the high degree of flattening of spinning spiral galaxies, whose shape is constrained by images. Together, these two factors greatly reduce the dynamical mass from that required by orbital models to explain RC (grey bar at top of Fig. 5b; also see Fig. 1). The total mass as deduced from the physics of spin is consistent with models of the baryonic mass inferred from luminosity (Fig. 5). In MOND, total mass is effectively reduced from prior Newtonian models by the combined acceleration constant and the power law.

The third factor \( (f) \) arising from the concentration of mass inwards reduces the total galactic mass from that expected for constant density (Fig. 5a). For arc models, this reduction nears \( \frac{1}{2} \) for low \( j \), whereas high \( j \) provides a dense large core, a diffuse exterior, and requires a larger dynamical mass, approaching that for constant density. The additional reduction can be large for polytropes and strong power laws (Fig. 5). Whereas \( n = 2.5 \) describes \( \text{H}_2 \) gas and \( n = 1.5 \) describes H or He gas [43], both of which are present in spiral galaxies [27], \( n = 3 \), indicating compressions of Sun’s interior, seems unreasonably large. Hence, strong exponentials with negligible density near the edge are also unlikely candidates for the density distributions inside spiral galaxies. Power laws provide increasingly lower dynamic mass with higher inward concentration (Fig. 5). Positive values of \( k \) are not reasonable for galaxies. Large |\( k \)| is not reasonable due to its drastic reduction of total mass, whereas very low |\( k \)| provides undesirable non-zero density near the edge (Fig. 4).

From Fig. 5, power laws with \(-0.4 > k > -2\), polytropes with \( n \) from \(~1\) to \(~2.5\), and arc models with \( j \) from \(~2\) to \(~6\) describe a galaxy with a moderately concentrated interior with low densities at the edge. This subset of models provides a total mass similar to the baryonic masses of the orbital models (Fig. 5ab). Furthermore, we infer galactic mass inside the visible edge assuming a polytrope with \( n = 2 \), representing a mixture of H, \( \text{H}_2 \) and He gas. The results (Table 1) compare well with luminosity. Galaxies actually extend beyond the visible edge and have large (gas) mass outboard, which can be calculated from RC using (21), but no measurement exists for comparison.
4.2 Newtonian physics of the Tully-Fisher relationship

The Tully-Fisher relationship connects the luminosity with maximum velocity: \( L \sim v^\alpha \) where exponent \( \alpha \) is similar to 3 for spirals (e.g., [27]). This relationship is incompatible with the presence of variable but major quantities of dark matter in galaxies.

Equations (19)-(21) and (23) indicate that \( v^2 \sim M/A \). For spheroidal galaxies with similar densities, \( M \sim A^3 \), so roughly \( v \sim A^{1/3} \). With \( L \sim M \), as should be the case without dark matter, then \( v^3 \sim L \). Thus, Newtonian physics with any degree of rotational flattening reproduces the Tully-Fisher relationship.

In contrast, a disk shape has constant height, so \( M \sim A^2 \). Hence, \( L \sim A^3 \) for a baryonic disk-shaped galaxy. Disk moment of inertia goes as \( MA^2 \), and \(-U_{\text{self}} \sim M^2/A \) should reasonably approximate the disk.

From (16), the disk will roughly have \( v^2 \sim M/A \), as do the oblates. This finding is robust, due to the form of the fundamental solution for Poisson’s law and because \( M^2 \) will always appear in the double integral of any shape, after (17). Hence, the disk requires \( v^4 \sim L \), which is inconsistent with the Tully-Fisher relationship for spirals. In short, the Tully-Fisher relationship results from Newtonian physics, and is consistent with the 3-dimensional balance of centrifugal and gravitational forces that occur in the oblate spheroid shape.

4.3 Dependence of the outermost velocity on galaxy size

Equation (23) can be inverted and recast to provide velocity as a function of \( \rho \) and \( A \):

\[
v(A) = \sqrt[3]{\frac{9\pi G C}{f(n, j, k)}} \frac{A}{A} \int_0^A s^2 \rho(s) ds \propto F(A, n, j, k) \quad (24)
\]

where the factor \( C/A \sim 0.1 \) arises from the limit of the integral and we assumed \( \arcsin(e)/e \sim 3/2 \) for simplicity. Again, we are investigating spin of a rigid body.

Analytical solutions for many power law and arc models are shown in Fig. 6, where both the \( v \) and \( A \) axes are normalized. Mass in arc models is effectively some average density times \( A^3 \), yielding a linear dependence of velocity on size (Fig. 6). The polytrope is similarly constrained, although the average density would differ from that of an arc model, and so a linear dependence of \( v \) on \( A \) is also expected in our
normalized plots. Such behavior is due to the abrupt edge becoming less abrupt for the same index $k$ and
same central density as the radius grows (Fig. 4). In contrast, for the power law, the normalized velocity
depends on the normalized radius in another simple power law (Fig. 6).

In comparing our models with measurements, we must recognize that even the extensive
measurements in Fig. 2 have not reach the termination of the object, because some minimum density is
required to collect velocity data on gas. Figure 7 explores limitations inherent to most measurements.
Velocity at the visible edge depends linearly on the visible radius (Fig. 7). Although considerable scatter
exists, no curvature of this trend (dotted line) is evident. All arc and polytrope models, including the
limiting case of constant density, provide a linear dependence of $v$ on $A$ (Fig. 6). From this
correspondence, we deduce that density near the center of spirals changes slowly, and so near their centers,
galaxies indeed spin like rigid tops (Fig. 1).

In contrast to behavior at the visible edge, velocity at the outermost radius of the galaxies in Fig. 2
shows a curved trend (Fig. 7). The only galaxies with data across the entire object are the Milky Way and
neighboring Andromeda, where the most distant material could be influenced by both objects. To
consistently compare these large proximal galaxies with the spirals of Fig. 2a, which have flat RC out to
~2 to 3 times the visible edge, Fig. 7 uses flat spots in the RC of Fig. 2b near 2-3 times the visible radius.
Changing $r$ of the flat spots will make the power law stronger or weaker, but will not eliminate curvature.

Data from various RC surveys show similar dependence of $v$ on the outermost radius studied. The
dependence of $v$ on $A$ is best determined from deBlok et als. [21] study of extended RC of small and large
galaxies. Curvature clearly exists (Fig. 8a). Dwarf galaxies [45] are few in number and occupy a
restricted range in size. Nonetheless, tiny galaxies are consistent with the trend for larger spirals (Fig. 8a).

Maximum velocities of 72 non-interacting, mostly large galaxies surveyed in 2014 by Wiegert and
English [7] clearly depend on the outermost radius in a power law dependence (Fig. 8b). This survey
covers measurements from 1986 to 2011 and includes a few galaxies in Table 1 and some from deBlok et
al. [21]. Eliminating these duplicates does not change the curvature. For most of these spirals, the
maximum velocities are similar to the outermost, allowing direct comparison with our models (Fig. 6).
A curved trend for velocity as a function of size is seen when measurements extend beyond the visible edge (Figs. 7, 8). This behavior is consistent with a power law model having $-1.5 < k < -1$ (Fig. 6). High indices are compatible with the visible parts of galaxies having very steep and linear dependence of velocity on the visible edge, and importantly with the density being low and flat over most of the galaxy (Fig. 4c). The correspondence of our forward model with data is good, given that we are comparing our normalized curves (Fig. 7) to non-normalized curves based on measurements (Figs. 7, 8). To first order, spiral galaxies indeed spin like tops: they are dynamically coherent to the point that rigidity is a reasonable first approximation. One may view the situation as the constituent stars being linked by gravitation into a network that is difficult to distort by the same force that created the network.

Scatter in the data (Fig. 7, 8) is interpreted as density not being a simple function of radius. The averaged function may be smooth, but it need not conform to a simple analytical function. This should be obvious from the RC (Fig. 2), where the velocities have peaks and troughs. Nonetheless, the consistency in the data indicates that average density does not vary greatly amongst the galaxies of different sizes and that to a first approximation galaxies spin as tops with density concentrated towards their centers.

### 4.4 Rotation curves of a spinning, Newtonian galaxy

It is tempting to consider Fig. 6 as an approximation to the interior spin of a galaxy. Although this cannot be correct in detail, Fig. 6 must describe facets of the interior because (24) provides the central and edge limits. Indeed, Fig. 6 recapitulates the essential features of RC (Figs. 1 and 2) and generic models [46]. This rough comparison shows that spiral galaxies spin like a top where mass is concentrated below a certain radius (similar to the visible edge), whereas at larger radii, rigidity diminishes, and orbital speeds lag those of the inner regions. In some rare cases, dynamical coherence is abruptly lost, rather than gradually reduced, as indicated by change in rotation direction or axes (the very rare counter-rotating and polar ring types). Understanding such complexities requires considering spin of nested spheroidal shells.¹

Because density decreases outwards and other objects influence the motions of outermost material, the distant gas and stars necessarily grade into Keplerian orbits. For the Milky Way and Andromeda (Fig.
2b), at distances ~10× the visible edge, $v$ decreases with $r$ and likely involves orbits rather than spin. For distant spirals, RC measurements are limited to within ~2× of their visible edge, and are unlikely to meet conditions appropriate to the Keplerian limit.

5 Discussion

The expectation that spiral galaxies, whose mass mostly resides at large radii, should behave analogously to our Solar System, whose mass almost exclusively resides at its center, is unreasonable. That available NOMs incorporate this error was overlooked. Instead, the validity of Newton’s law in galaxies has been questioned. The MOND formulation originated by Milgrom [9] has successfully reproduced RC of diverse galaxies with masses consistent with luminosity, lending credence to non-Newtonian models, since dark matter in any amount has not been detected [1].

The deadlock is fueled by extreme ambiguities in NOMs and improper use of Poisson’s equation being ignored. From Sections 2.1 and 2.2, it is possible to fit RC with only a dark matter component. In contrast, force calculations of Feng and Gallo [12] and Marr [2] showed that dark matter is not needed to explain RC under NOMs with certain assumed densities. These forward models specifically show that for disks, forces operate from both inside and outside any radius of interest [12], consistent with this shape being inconsistent with gravitational control. Although the requisite oblate spheroids were initially explored, early and subsequent orbital models have incorrectly parameterized density in the vertical direction as being independent of the radial [27, 41, 42], which creates corners (Fig. 3) and is unlike gravitational shapes. This error contributed to widespread exploration of the vertically unstable thick disk.

The present paper explores RC of spiral galaxies from the perspective of these being dynamically coherent, spinning, gravitationally bound states, by combining Clausius’ Viral theorem with Newtonian forces assuming the oblate spheroid shape deduced by Maclaurin. We show that regardless of how density is concentrated inwards, the total mass of spiral galaxies from our spin model is, at most, only 4/9 times the dynamical mass of the NOMs. The mass required by the physics of spin is compatible with the luminosity of a medium comprised of stars, gas, and obscuring dust. Clearly, huge amounts of dark matter
are an artifact of the focus on Keplerian orbits in previous NOMs, and that MOND [9, 10, 38, 39] largely addresses the neglected factors associated with spin and flattening (Table 2). Furthermore, our Newtonian-Maclaurian spin model explains the physics of the Tully-Fisher relationship which is extremely important. Our forward model for galactic spin is a first approximation because we have treated these objects as being rigid. Nonetheless, or results are compatible with existing measurements of velocity sharply increasing with size for small galaxies, but leveling off for large galaxies, and possibly decreasing with size if $v$ is measured at immense distance. Our model also suggests that spin of dense, luminous galactic interiors, within the visible edge, is indeed like that of a rigid body, and leads to the good agreement of calculated mass with luminosity (Table 1). This high degree of dynamic coherence results from the strong forces between massive, proximal stars. The outer regions, with large amounts of gas, do not spin as a rigid body, but neither are the motions consistent with those of orbits, until far beyond the visible edge. Our results for various models of density suggest that the flat trend in $v$ for large galaxies results from the density going as a power law, roughly between $r^{-1}$ and $r^{-3/2}$, which provides low density in the outer reaches of galaxies, but does not overcome the rapid growth of mass with radius. Interestingly, interiors of elliptical galaxies have been modeled with such a power law from the perspective of orbital dynamics. Investigating the detailed behavior of the interior of spirals requires considering behavior of homeoids, which we pursue in inverse models.1

6. Conclusions

The rotational motions of spiral galaxies are highly organized and indicate an object undergoing axial spin. We show that this point of view greatly simplifies the physics, and addresses this organization and many other previously unexplained features: 1) Shapes of spiral galaxies are unlike disks, but rather taper like highly flattened oblate spheroids. 2) Well-known characteristics of RC are consistent with inward concentrations of ordinary matter in a spinning, flattened object; such that the inner regions of spiral galaxies spin like a top, and the outer regions spin as dictated by size and inward densification.
3) Orbital motions are expected at great distance, but while $v$ decreasing with $r$ is observed, the $1/r$ trend is not attained even at distances $10\times$ the visible edge (Fig. 2).

Although we cannot rule out the possibility that non-baryonic matter exists, we have shown that huge amounts of dark matter surrounding spiral galaxies are an artifact of an ill-posed physical problem, involving over-interpretation of ambiguous fits and casting the problem of galactic spin in terms of orbiting particles. Using equations describing orbits overestimates the dynamical mass, because spin redistributes the energy, causing high equatorial velocity but no rotational motion at the poles (Fig. 1c).

Our calculated mass of 14 important galaxies up to the visible edge for polytropic densification with no free parameters (21) agrees well with their measured luminosities (Table 1). Regarding cosmology, we reiterate Disney’s deduction that the latter models are under-constrained [47], and in view of Transtrum et als. [30] findings, should be amenable to retooling to incorporate our results.

Acknowledgement. This research has made use of publically available data from the NASA/IPAC Extragalactic Database (NED) which is operated by the Jet Propulsion Laboratory, California Institute of Technology, under contract with NASA and the CHANG-ES project (Continuum Halos in Nearby Galaxies) for radio emissions.
Table 1. List of galaxies with characteristics from the NED website and references for RC data. Edge denotes the isophote in the visible at 25 B-mag arcsec$^2$. The first 10 galaxies are considered to have desirable characteristics and to represent RC of diverse forms [8]. The last 6 galaxies are important due to proximity or distinct structural characteristics. Mass at the visible edge is calculated for a spinning galaxy with a polytropic index of 2 (inverted equation 21 with $C/A = 1/10$).

<table>
<thead>
<tr>
<th>NGC</th>
<th>Type</th>
<th>Notes</th>
<th>Distance</th>
<th>$L_{\text{vis}}$</th>
<th>Edge</th>
<th>v(edge)</th>
<th>Mass(edge)</th>
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<td></td>
<td></td>
<td></td>
<td>Mpc</td>
<td>$L_{\text{sun,vis}}$</td>
<td>kpc</td>
<td>km s$^{-1}$</td>
<td>Solar masses</td>
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<td>3.28</td>
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<td>68</td>
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<td>2841</td>
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<td>$1.49 \times 10^{10}$</td>
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<td>n.a.</td>
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<td>14.4</td>
<td>356</td>
<td>$4.24 \times 10^{10}$</td>
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</table>

*Seen edge on
†Low surface brightness
‡Lacks a bulge

§Dwarf, DDO 154.

¶Inside view of the Milky Way, for which RC depends on the Sun’s true rotational velocity. RC data from Sofue [4,14].

A Andromeda RC data from Sofue [4]

T Triangulum RC data from Kam et al. [5]

H Sombrero has a huge bulge. RC data from Jardel et al. [13]

BP Representative and extensive RC from Bottema and Pestaña [8]; digitized.

S Subset of RC in [8] that were originally measured by Sofue et al. [48]; downloaded from Sofue’s website.
Table 2. Differences between orbital and spin models

<table>
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<tr>
<th>Characteristic</th>
<th>Orbits</th>
<th>Spin</th>
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<td>circular orbits</td>
<td>axial spin</td>
</tr>
<tr>
<td>galaxy shape</td>
<td>disk</td>
<td>oblate spheroid</td>
</tr>
<tr>
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<td>assumed coplanar</td>
<td>about special axis</td>
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<tr>
<td>size</td>
<td>no limit</td>
<td>$A$</td>
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<tr>
<td>moment of inertia</td>
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<td>$2MA^2/5$ for solid</td>
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<td>$\rho(z)$</td>
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<td>$-\infty$</td>
<td>0</td>
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<tr>
<td>limit large $r$</td>
<td>$r^{-1}$</td>
<td>requires $r &lt; A$</td>
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<td>sphere, const. $\rho$</td>
<td>$GMm/r$</td>
<td>$3/5 GM^2/R$</td>
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<td>velocity</td>
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<td>$r$ is master variable</td>
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<td>at $r \to 0$</td>
<td>$+\infty$</td>
<td>0</td>
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<td>at large $r$</td>
<td>$\sim r^{-5/2}$</td>
<td>flatter than $\sim r$</td>
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<tr>
<td>at $r = A$</td>
<td>$GM/A$</td>
<td>$3/2 GM/A$ (equator)</td>
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<td></td>
<td>0 (poles)</td>
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Notes: $m =$ test particle mass; $M =$ body mass; $A =$ body radius in the equatorial plane. Spin models assume Maclaurin’s equilibrium shape associated with spin of a gravitationally bound object. Velocities are tangential. For the present spin models, velocity as a function of internal $r$ is approximated from the dependence of velocity on $A$. 

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Figures and captions

Fig. 1. Indications that galaxies spin. (a) Comparison of rotation curve types to motions associated with spin and orbits. Thick line = spin of a rigid body. Thin line = suggesting how differential rotation of internal layers affects spin of a solid body. Dashed lines = the three types of RC derived from surveying 79 non-interacting spiral galaxies [7], all of which spin like tops near their centers. Grey curve = orbit of a satellite \((m)\) or thin ring around a point or spherical distribution of mass \((M)\). Schematics suggest the configurations associated with the various curves. For the galaxies, density is low, so the differential spin of spheroidal layers appear as rotating rings. (b) Additional schematics illustrating ambiguities in orbital descriptions. Left, interior spin need not be related to the axis or direction of the orbit. Middle, at any given radius, orbits of any orientation are permitted around a sphere. Right, looking down, orbits can be elliptical; more importantly, the test mass can be huge compared to the nearby masses in the galaxy, as long as it is small compared to the total mass. (c) How spin of a shell differs from particle orbits. Even if the shell and particle have the same mass, \(m\), their moments of inertia differ, as listed. The tangential velocity of the shell at the equator is bigger than that of the particle, but the polar velocity is null, so the total energy can be the same for orbiting and spinning systems, but the velocity distributions differ.

Fig. 2. Extended RC of individual galaxies. Arrows indicates the visible edge from NED. Table 1 lists galaxy characteristics. (a) Galaxies studied by Bottema and Pestaña [8] who considered this set to represent diverse galaxies with data collected under favorable conditions to high radius. (b) Large proximal spirals. The 2013 data on the Milky Way (with error bars) from Sofue [14]. Andromeda data from [4] and the 2015 update for the Milky Way from [4] is an update based on adjustments of the Sun’s absolute velocity. Galaxy centers are separated only by 780 kpc. Note that at these huge distances (~10× the visible edge), where dynamic coherence is unexpected, the velocities begin to decrease with \(r\) similar to orbits.
Fig. 3. Collage of edge-on galaxies and model density contours. Images from NED [44] were acquired at 3.56 microns [49]. Radio (~6 GHz) intensity contours are shown for NGC 891 [50]. UGC 8550 is also known as NGC 5229. RC of these galaxies are used in subsequent figures. Bottom, density element contours in the oblate, for which the $z$ and $r$ dependence of $\rho$ are linked via (8), and in the disk, for which the $z$ and $r$ dependence of $\rho$ are assumed to be independent, after Perek [42]. Both profiles are shown in a vertical plane through the center. Pointy perimeters of galaxies are inconsistent with a disk geometry.

Fig. 4. Densities normalized to the central density as a function of scaled radius normalized to unity at the center and zero at the surface. (a) Emden’s [28] polytropes. Index $n = 0$ (not shown) would correspond to constant density. Medium grey = linearly decreasing density for comparison. Light grey = strongly declining exponential for comparison. (b) Arc model with index $j$. Grey = polytropes for comparison. (c) Power law (black) and exponential (grey) densities, as labeled.

Fig. 5. Total (dynamical) mass at the edge of a spiral galaxy. Grey line along upper axis is the total mass required by orbital models. (a) Models with $C/A$ near 0.1 from (23) for various models as fractions of $v^2A/G$ (black curves) compared to the ratio of baryonic to total mass from recent data and orbital models as a function of galaxy size (grey symbols and lines). Power law from (19). Arc models from (20) where $j > 0$. For this construction, $j = 0$ corresponds to $\rho = 0$, so very low $j$ are not useful. Polytrope models from (21), where $n < 5$. (b) Total mass and baryonic components from orbital models Dot-dashed line = galaxies in Table 1 (top) with the ratios given by Bottema and Pestaña [8]. Milky Way and the average of 14 studies on Andromeda from Sofue [4]. Also shown are Sombrero (Jardel et al. [13]) and Triangulum (Kam et al. [5]).

Fig. 6. Dependence of model velocities on galaxy size. Labels at the top emphasize that these are not RC, but show a size dependence, assuming spiral galaxies spin like tops. For constant density, the velocity at
the edge depends linearly on size. Arc and polytrope models have masses which grow as $A^3$, due to their densities being pinned at the center and edge, providing curves like that of constant density. Power law models have masses growing in a different manner due to their abrupt edge, which becomes less abrupt for larger absolute values of the indices. Because we forced $v = 1$ at $r = 1$, the curves for different $k$ form an arc. These curves are fit with simple power laws, $v_{\text{norm}} = r_{\text{norm}}^h$, where $h$ is listed next to the inset.

**Fig. 7.** Velocity data plotted against two measures of galaxy size, the distance to the visible edge and the outermost radius, for the objects of Table 1. To treat Andromeda and the Milky Way similar to the others (Fig. 2) we took the maximum as $\sim 3 \times$ the visible edge, where their RC were fairly flat.

**Fig. 8.** Velocity data plotted against the outermost radius of various galaxies. The scales are the same as in Fig. 7. (a) Comparison of small and large galaxies. Small diamonds = dwarf spheroidal data from Salucci et al. [45]. Triangles and solid line = “Things” survey by deBlok et al. [21]. (b) Non-interacting galaxies from the survey of Wiegert and English [7]. About 10 of their list had estimated radii which we replaced with data from NED [44].

**References**

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solid object spin without differential rotation

spin with differential rotation

spiral galaxy RC types (Wiegert and English, 2014)

solid sphere: $v_{\text{equator}}^2 = \frac{3}{2} \frac{GM}{A}$

orbits:

- $v^2 = \frac{GM}{r}$
- $I = \frac{1}{2} mr^2$
- any axis

$I_{\text{shell}} = \frac{2}{5} mA^2$

spin: $v_{\text{polar}}^2 = 0$
fig. 2ab

279x361mm (300 x 300 DPI)
fig. 3
fig4abc

279x361mm (300 x 300 DPI)
fig7

279x361mm (300 x 300 DPI)
fig8

279x361mm (300 x 300 DPI)