THE EQUIVARIANT $K$-THEORY OF COMMUTING 2-TUPLES IN $SU(2)$

by

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Abstract

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In this thesis, we study the space of commuting $n$-tuples in $G = SU(2), Hom(\mathbb{Z}^n, SU(2))$. We describe this space geometrically via providing an explicit $G$-CW complex structure, an equivariant analog of familiar CW-complexes. For the $n = 2$ case, this geometric description allows us to compute various cohomology theories of this space, in particular the $G$-equivariant $K$-Theory $K^*_G(Hom(\mathbb{Z}^2, SU(2)))$, both as an $R(SU(2))$-module and as an $R(SU(2))$-algebra, where $R(SU(2))$ denotes the representation ring of $SU(2)$.

This space is of particular interest in the study of quasi-Hamiltonian spaces $M$, which come equipped with a moment map that takes values in the Lie group, $\phi : M \to G$. For the specific example $M = G \times G$ with moment map given by the commutator map, $\phi^{-1}(e) = Hom(\mathbb{Z}^2, G)$. Finite dimensional quasi-Hamiltonian spaces have a bijective correspondence with certain infinite dimensional Hamiltonian spaces, and we compute relevant components of this larger picture in addition to $\phi^{-1}(e) = Hom(\mathbb{Z}^2, SU(2))$ for this example.
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# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Overview</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Outline of the Thesis</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>Classical Results and Context</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>Preliminaries</td>
<td>5</td>
</tr>
<tr>
<td>3.1</td>
<td>$G$-CW Complexes</td>
<td>5</td>
</tr>
<tr>
<td>3.2</td>
<td>Equivariant $K$-theory</td>
<td>7</td>
</tr>
<tr>
<td>3.3</td>
<td>Equivariant cohomology</td>
<td>9</td>
</tr>
<tr>
<td>3.4</td>
<td>On $SU(2)$</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>$G$-CW Complex Calculations</td>
<td>11</td>
</tr>
<tr>
<td>4.1</td>
<td>$G$-CW structures for various $G$-spaces</td>
<td>11</td>
</tr>
<tr>
<td>4.2</td>
<td>$G$-CW structure for $Hom(\mathbb{Z}^2, G)$</td>
<td>16</td>
</tr>
<tr>
<td>4.3</td>
<td>$G$-CW structure for $Hom(\mathbb{Z}^n, G)$</td>
<td>17</td>
</tr>
<tr>
<td>5</td>
<td>Equivariant $K$-theory computations</td>
<td>22</td>
</tr>
<tr>
<td>5.1</td>
<td>$K_G^*(G)$</td>
<td>22</td>
</tr>
<tr>
<td>5.2</td>
<td>Module structure of $K_G^*(Hom(\mathbb{Z}^2, G))$</td>
<td>24</td>
</tr>
<tr>
<td>5.2.1</td>
<td>The 1-skeleton of $Hom(\mathbb{Z}^2, G)$</td>
<td>24</td>
</tr>
<tr>
<td>5.2.2</td>
<td>The 2-skeleton of $Hom(\mathbb{Z}^2, G)$</td>
<td>26</td>
</tr>
<tr>
<td>5.3</td>
<td>Other cohomology theories of $Hom(\mathbb{Z}^2, G)$</td>
<td>30</td>
</tr>
<tr>
<td>5.3.1</td>
<td>$K_T^*(Hom(\mathbb{Z}^2, G))$</td>
<td>30</td>
</tr>
<tr>
<td>5.3.2</td>
<td>$H^*(Hom(\mathbb{Z}^2, G); \mathbb{Z})$</td>
<td>32</td>
</tr>
<tr>
<td>5.3.3</td>
<td>$H_T^*(Hom(\mathbb{Z}^2, G); \mathbb{Z})$</td>
<td>33</td>
</tr>
<tr>
<td>5.3.4</td>
<td>$H_T^*(Hom(\mathbb{Z}^2, G); \mathbb{Z})$</td>
<td>36</td>
</tr>
</tbody>
</table>
5.4 Algebra structure for $K^*_G(\text{Hom}(\mathbb{Z}^2, G))$ .................................................... 37
5.5 Module Structure of various other spaces ................................................................. 39
5.6 Future Work ............................................................................................................. 43

6 Quasi-Hamiltonian Systems ...................................................................................... 44
  6.1 Quasi-Hamiltonian Systems .................................................................................. 44
  6.2 Kirwan Surjectivity ............................................................................................... 46
  6.3 Relating to future work ......................................................................................... 47

Bibliography ................................................................................................................. 50
Chapter 1

Introduction

1.1 Overview

In this thesis, our principal objective is the study of the commuting \( n \)-tuples in \( SU(2) \), \( \{(g_1, \cdots, g_n) \in (SU(2))^n \mid g_ig_j = g_jg_i \ \forall i,j \} \). Since a homomorphism is determined by the mappings of the generators, this is the space \( Hom(\mathbb{Z}^n, SU(2)) \), as it will be henceforth denoted in this thesis. This is an \( SU(2) \)-space where \( SU(2) \) acts on \( Hom(\mathbb{Z}^n, SU(2)) \) by conjugation on each component.

A standard methodology to compute homology and cohomology groups associated to topological spaces is to establish a CW-complex structure on the spaces. The appropriate equivariant analog of the familiar CW-complexes is the notion of a \( G \)-CW complex developed independently by [Mat] and [Ill1]. From the skeleta filtration \( X^n \) of a \( G \)-CW space \( X \), we can consider the corresponding long exact sequences on various cohomology theories corresponding to \( X^n \hookrightarrow X^{n+1} \rightarrow X^{n+1}/X^n \), much as we would for CW-complexes.

A fundamental result in equivariant \( K \)-theory is Bott periodicity, which we describe in more detail following [Seg] in section 3.2. Bott periodicity is the fact that \( K^q_G(X) \cong K^{q+2}_G(X) \) for a compact \( G \)-space \( X \). Thus all even \( K^q_G(X) \) are equal to each other, as are all odd \( K^q_G(X) \) equal to each other. One consequence of this is that on equivariant \( K \)-theory the long exact sequence induced by \( X^n \hookrightarrow X^{n+1} \rightarrow X^{n+1}/X^n \) yields a six term cyclical exact sequence. As quotients \( X^{n+1}/X^n \) can be well described, these six term exact sequences provide a methodology to compute the unknown terms in the sequence. Thus by establishing an explicit \( G \)-CW complex for a space, this geo-
metric description gives us considerable power in computing equivariant cohomology theories.

We apply this methodology to a range of $G$-spaces, but our principal computation is to establish a $G$-CW complex structure for $G = SU(2)$ on $\text{Hom}(\mathbb{Z}^n, G)$. From this, in the case of $n = 2$, we go on to use the long exact sequences corresponding to the skeletal filtration to compute $X = \text{Hom}(\mathbb{Z}^2, SU(2))$ on various cohomology theories such as $K^*_G(X)$, ordinary $H^*(X, \mathbb{Z})$, $K^*_T(X)$, $H^*_G(X)$, and $H^*_T(X)$ where $T$ is the maximal torus of diagonal elements in $SU(2)$. The former two, as modules, have previously been computed by [AG] and [BJS], respectively, for $X = \text{Hom}(\mathbb{Z}^2, G)$.

Beyond providing an alternate and pleasingly geometric computation that motivates future work using this $G$-CW approach, our results extend the literature in three ways. Firstly, our geometric $G$-CW decompositions allow us to also establish the multiplicative structure, thus computing $K^*_G(\text{Hom}(\mathbb{Z}^2, G))$ as an $R(G)$-algebra, not just as an $R(G)$-module. Additionally, our methods allow us to compute the additive and multiplicative structure on the other cohomology theories mentioned above. Finally, we equip the commuting $n$-tuples in $SU(2)$ with a $G$-CW structure that generalizes the 2-tuples case, from which the various cohomology theories mentioned above could be computed analogously. However, we lack a closed form solution for the general $n$ case and note that computations quickly become cumbersome. As such, only the $n = 2$ case has been explicitly written out in this thesis.

One motivation for this study of $\text{Hom}(\mathbb{Z}^2, G)$ is the quasi-Hamiltonian systems introduced by [AMM]. Indeed, the commutator map from $G \times G \to G$ provides an example of such a quasi-Hamiltonian system with $\text{Hom}(\mathbb{Z}^2, G)/G$ as the symplectic quotient where $G$ acts regularly on $\text{Hom}(\mathbb{Z}^2, G)$. The space $\text{Hom}(\mathbb{Z}^2, G)/G$ is of importance in symplectic geometry. As described in [Jef], the moduli space $M = \text{Hom}(\pi_1(\Sigma), G)/G$ can by identified via the holonomy map as gauge equivalence classes of flat connections on a trivial principal $G$ bundle over a closed, oriented 2-manifold $\Sigma$ of genus $g$. In the genus 1 case we get $\pi_1(\Sigma) = \mathbb{Z}^2$. See [Jef2] for background. There are a range of other spaces related to the larger pictures of quasi-Hamiltonian systems and their bijective correspondence with particular Hamiltonian $LG$ systems, and for this example with $G = SU(2)$ we additionally compute $K^*_G(G)$ and $K^*_G(G \times G)$ as modules. Finally, we relate them to larger questions in quasi-Hamiltonian systems.
1.2 Outline of the Thesis

This thesis is organized by chapter as follows:

- In chapter 2 we outline various results in the literature relevant to this thesis. In particularly, we discuss known computations on equivariant $K$-theory and cohomology for several relevant spaces and where our results match or extend them.

- In chapter 3 we provide an introduction into the objects and tools that we will use in this thesis. In particular, we define $G$-CW complexes in section 3.1 and discuss a few important facts. In section 3.2 we introduce Equivariant $K$-theory, following [Seg]; in particular, we cite various propositions that will be critical for our computations on equivariant $K$-theory. In section 3.3 we provide a brief overview of equivariant cohomology. Finally, in section 3.4 we discuss various facts about the Lie group $SU(2)$.

Henceforth, all of the $G$-spaces under consideration in this thesis are for $G = SU(2)$.

- In chapter 4 we establish $G$-CW complex structures on all the various spaces we are interested in computing in this thesis. In particular, we study $G$, $G \times G$ and $Hom(\mathbb{Z}^n, G)$. Of critical importance is describing the attaching maps in our $G$-CW structure sufficiently clearly so that we can compute equivariant $K$-theory from them.

- In chapter 5, we compute the equivariant $K$-theory of our range of spaces as $R(G)$-modules. Note that while we have a $G$-CW structure on commuting $n$-tuples, we only go on to compute equivariant $K$-theory in the $n = 2$ case. In section 5.3 we then study other equivariant cohomology theories of $Hom(\mathbb{Z}^2, G)$. Additionally, we make explicit the $R(G)$-algebra structure for $K^*_c(Hom(\mathbb{Z}^2, G))$ in section 5.4. In section 5.5 the module structure of various other spaces is computed. In 5.6, we discuss some potential for future work.

- In chapter 6, we discuss the relationship between $Hom(\mathbb{Z}^2, G)$ and the other spaces we compute here and the quasi-Hamiltonian systems that originally motived this study. We conduct a review of the literature on analogs of Kirwan surjectivity, and conclude the thesis with suggestions for further work along these themes.
Chapter 2

Classical Results and Context

In this chapter we summarize prior computations of various cohomology theories in the literature that overlap with our results and describe where we match the literature.

\( K^*_G(G) \) was first computed in 2000 by [BZ] for simply connected Lie groups \( G \) acting on themselves by conjugation as the algebra of Grothendieck differentials on the representation ring, \( K^*_G(G) \cong G \Omega^*_R(G)/\mathbb{Z} \). We will match, as expected, this result in the case of \( G = SU(2) \) in section 5.1.

The space \( \text{Hom}(\pi, G) \) for \( \pi \) a finitely generated discrete group, and \( G \) a Lie group, was studied by [AC]. In particular, for \( \pi \) a free Abelian group of rank equal to \( n \), cohomology groups were explicitly computed in the \( n = 2 \) and \( n = 3 \) cases for \( G = SU(2) \). [BJS] further study \( \text{Hom}(\mathbb{Z}^n, SU(2)) \), computing the integral cohomology groups for all positive \( n \), in agreement with [AC2], an erratum to the original [AC] in the \( n = 3 \) case. Our results in this thesis match the \( n = 2 \) case on integral cohomology as noted in section 5.4.

[AG] provide a computation for \( K^*_{SU(2)}(\text{Hom}(\mathbb{Z}^2, G)) \) as an \( R(G) \)-module. This result uses the Atiyah-Hirzebruch-Segal Spectral Sequence and Segal’s localization to reduce the problem to computing to studying the fixed point set \( H^*(\text{Hom}(\mathbb{Z}^2, G)^T; R(G)) \). We reproduce this result in section 5.2 through computing \( K^*_{SU(2)}(\text{Hom}(\mathbb{Z}^2, G)) \) via providing a \( G \)-CW structure on \( \text{Hom}(\mathbb{Z}^2, G) \).

Finally, [HJS1] and [HJS2] are twin papers that compute \( K^*_G(\Omega G) \) for \( G = SU(2) \) acting on itself by conjugation as a module and as an algebra.
Chapter 3

Preliminaries

This section contains a brief overview of $G$-CW complexes, equivariant $K$-theory and a few facts on $SU(2)$ that will be used in our computations. It is by no means intended to be exhaustive on these subjects.

3.1 $G$-CW Complexes

An equivariant analog of the familiar CW-Complexes, referred to as $G$-CW complexes in this thesis, was developed independently by [Mat] and [Ill1]. Let $X$ be a $G$-space for a compact Lie group $G$; that is, $X$ is a Hausdorff topological space with a left action of $G$ on $X$. Following [May],

**Definition 1.** For each $k$, let $H_k$ denote a closed subgroup of $G$. A $G$-CW complex $X$ is a union of $G$-spaces $X^n$ where $X^0$ is a disjoint union of orbits $G/H_k$, referred to as 0-cells, and $X^{n+1}$ is determined inductively by attaching $(n+1)$-cells $(G/H_k) \times D^{n+1}$ to $X^n$ via attaching $G$-maps $\sigma_k : (G/H_k) \times S^n \to X^n$.

That is, $X^{n+1}$ is determined by the pushout of the following diagram:

$$
\begin{array}{c}
\bigsqcup_k (G/H_k) \times S^n & \xrightarrow{\sqcup_k \sigma_k} & X^n \\
\downarrow & & \downarrow \\
\bigsqcup_k (G/H_k) \times D^{n+1} & \longrightarrow & X^{n+1}.
\end{array}
$$

(3.1)

A point of the form $(eH_k, t) \in (G/H_k) \times S^n$ is fixed under the $H_k$ action, and so maps via $\sigma_k$ into the fixed point set of $H_k$, denoted $(X^n)^{H_k}$. Thus, an attaching $G$-map $\sigma_k$ is determined by its restriction $S^n \to (X^n)^{H_k}$. 

5
The degree of a cell of the form \((G/H) \times D^n\) cell is \(n\), and such cells are referred to as \(n\)-cells. The dimension of such a cell is \((\text{Dim}(G/H) + n)\). The space \(X^n\), formed by attaching cells of degree up to and including \(n\), is referred to as the \(n\)-skeleton of \(X\). One interesting characteristic is that lower degree cells can sometimes have higher dimensions than higher degree cells. A \(G\)-CW subcomplex \(Y\) of a \(G\)-CW complex \(X\) is a subcollection of cells from the \(G\)-CW complex for \(X\) that is itself a \(G\)-CW complex.

Given two \(G\)-CW complexes \(X\) and \(Y\), one can form a \((G \times G)\)-CW complex on \(X \times Y\) by taking all products of cells

\[
\left(\left(G/H_k \times D^n\right) \times \left(G/K_m \times D_m\right)\right) \approx \left(\left(G \times G\right)/(H \times K)\right) \times D^{n+m}.
\]

If a \(G\)-CW complex for each \((G \times G)/(H \times K)\) can be determined, then every \((G \times G)/(H \times K) \times D^{n+m}\) cell can be written as a \(G\)-cell and thus a \(G\)-CW complex for \(X \times Y\) can be determined.

The following lets us interpret the quotients of skeleta from our \(G\)-CW complex as suspensions. For \(E\) a \(G\)-bundle, let \(T(E)\) denote the Thom Space. In particular, we write \(T((G/H_k) \times D^n)\) to refer to the Thom space of the product \(G\)-bundle with base \(G/H_k\) with the usual action and trivial action on \(D^n\). We will also let \(S(X)\) denote the unreduced suspension, a \(G\)-space where \(g\) acts on an element in \(X \times I\) via \(g(x,t) = (gx,t)\).

**Proposition 1.** The \(n\)-skeleton \(X^n\) of a \(G\)-CW complex \(X\) satisfies

\[
X^n/X^{n-1} \cong\bigvee_k T((G/H_k) \times D^n) \cong\bigvee_k (S^n \wedge ((G/H_k) \cup pt)) \cong\bigvee_k (S^n((G/H_k) \cup pt))
\]

with \(k\) indexing all \(n\)-cells of our \(G\)-CW complex.

**Proof.** The first identification follows from the observation that an attaching map for a cell of the form \(G/H_k \times D^n\) glues \(G/H_k \times S^n\) into the \((n-1)\)-skeleton resulting in the quotient of the product disk \(G\)-bundle by the product sphere \(G\)-bundle, hence \(T((G/H_k) \times D^n)\). The second identification follows from the more general fact that for a vector bundle \(\xi\), the fibrewise direct sum \(T(\xi \oplus \mathbb{R}^n) \cong G S^n \wedge T(\xi)\) applied to the case of \(\xi\) being the zero dimensional \(G\)-bundle over base \(G/H_k\) which has Thom space \(T(\xi) = G/H_k \cup pt\). \(\square\)

We now quote Theorem J on page 368 of [Mat2] which shows that the inclusion
of a $G$-subskelton is a $G$-cofibration. That is, that it satisfies the homotopy extension property.

**Proposition 2** (Homotopy Extension Property). Let $f_0 : X \to Z$ be a $G$-map of $X$ into an arbitrary $G$-space $Z$. Let $g_i : Y \to Z$ be a $G$-homotopy of $g_0 = f_0|_Y$ where $Y$ is a $G$-subcomplex of $X$. Then there is a $G$-homotopy $f_t : X \to Z$ such that $f_t|_Y = g_t$.

The following standard property of cofibrations is also valid in the equivariant case. Standard proofs of the nonequivariant case can be found in [May2] and [Sel]. The proof of the extension from the nonequivariant to the equivariant case for the dual theorem regarding $G$-fibrations is Theorem A.3 of [HS], and the proof for $G$-cofibrations follows analogously.

**Proposition 3.** Let $j : A \to X$ be a $G$-cofibration for a $G$-space $A$ with $A$ closed in the $G$-space $X$. Let $f, g : A \to Y$ be $G$-homotopic. Then the pushouts of $j$ with $f$ and $g$ are $G$-homotopic.

In particular, if $A$ is $G$-contractible, we get that $X \simeq_G X/A$.

### 3.2 Equivariant $K$-theory

We will consider the case of $G$ being a compact Lie group and $X$ a compact $G$-space. The group $K_G(X)$ is defined as the Grothendieck group of isomorphism classes of $G$-equivariant complex vector bundles over $X$. Throughout this section, we follow [Seg]'s approach outlining the basic definitions and propositions.

Consider the map $j : X \to \{pt\}$. We define the reduced equivariant $K$-theory on $X$ as $K_G(X) = \coker(j* : K_G(pt) \to K_G(X))$.

Equivariant $K$-theory generalizes from two trivial cases. Firstly, when $X = \{pt\}$,

$$K_G(pt) = R(G)$$

where $R(G)$ is the representation ring of $G$. Indeed, a $G$-bundle over a point is a representation of $G$. Secondly, when $G$ is trivial, $K_G(X) \cong K(X)$, ordinary $K$-theory. The map $X \to pt$ induces a map $K_G(pt) \cong R(G) \to K_G(X)$ which makes equivariant $K$-theory an $R(G)$-algebra.

Further, there are two extreme cases of the action of $G$ on $X$. Firstly, if $G$ acts trivially on $X$, then any vector bundle can be thought of as an equivariant vector
bundle, giving us a map $K(X) \to K_G(X)$. Together with the ring homomorphism $K_G(pt) \cong R(G) \to K_G(X)$ induced by the map $X \to pt$, we get a ring homomorphism $\mu : R(G) \otimes K(X) \to K_G(X)$.

**Proposition 4.** If $G$ acts trivially on $X$, the map $\mu$ described above is an isomorphism.

Secondly, for $G$ acting freely:

**Proposition 5.** If $G$ acts freely on $X$ then the quotient map $\pi : X \to X/G$ induces an isomorphism

$$\pi^* : K(X/G) \to K_G(X) \quad (3.3)$$

For a right $G$-space $X$, and a left $G$-space $Y$, let $X \times_G Y := (X \times Y)/G$ where $X \times Y$ admits the $G$-action defined by $g \cdot (x, y) = (xg^{-1}, gy)$.

**Proposition 6.** For $H$ a closed subgroup of $G$, and an $H$-space $X$, we can form the compact $G$-space $(G \times_H X)/H := (G \times H X)$. The map $j : H \hookrightarrow G$ induces an isomorphism $j^* : K_G(G \times_H X) \to K_H(X)$. In particular, we have $K_G(G/H) \cong K_H(pt) \cong R(H)$.

Perhaps the most important theorem in equivariant $K$-theory is the Thom isomorphism, presented here in the form of Theorem 3.1 of [Gre]. An alternate reference is Theorem 6.1.4 in [AS].

**Proposition 7 (Thom Isomorphism).** For $E$ a $G$-bundle over a compact $G$-space $X$, there is a natural Thom isomorphism

$$K_G(X) \xrightarrow{\cong} \tilde{K}_G(E)$$

For $q \geq 0$, we can define

$$K^q_G(X) = \tilde{K}_G(S^q \wedge (X \sqcup pt)). \quad (3.4)$$

A consequence of the Thom isomorphism gives us Bott periodicity which is an isomorphism $K^q_G(X) \cong K^{q+2}_G(X)$, as described in Theorem 3.2 of [Gre] or Theorem 2.7.1 of [AS]. We thus only need to consider $K^0_G(X)$ and $K^1_G(X)$ as $K^q_G(X)$ is the same for all even $q$, and the same for all odd $q$. We will therefore refer to equivariant $K$-theory as being $\mathbb{Z}/2$-graded.

Another consequence of Bott periodicity is that the typical long exact sequence becomes a cyclical six term sequence that will be critical for many of our computations:
**Proposition 8** (Exact Sequences). Let $A \hookrightarrow X$ be a $G$-cofibration with $A$ closed. Then the following six term sequence is exact.

\[
\begin{array}{c}
\tilde{K}_0^G(A) \leftarrow \tilde{K}_0^G(X) \leftarrow \tilde{K}_0^G(X/A) \\
\downarrow \hspace{5cm} \uparrow \\
\tilde{K}_1^G(X/A) \longrightarrow \tilde{K}_1^G(X) \longrightarrow \tilde{K}_1^G(A)
\end{array}
\]  

(3.5)

In particular, we will apply these exact sequences in the case where $A$ is a $G$-subskelleton as proposition 2 of section 3.1 asserts that the inclusion of a $G$-subskelleton is a $G$-cofibration.

### 3.3 Equivariant cohomology

While we are largely interested in equivariant $K$-theory, we will also record various results on equivariant cohomology as well. We thus recall some essential facts regarding equivariant cohomology in this section.

Let $X$ be a $G$-space for a topological group $G$. We can construct the homotopy quotient $EG \times_G X := (EG \times X)/G$ where $EG \to BG := EG/G$ is a universal principal $G$-bundle. The space $EG \times X$ is homotopy equivalent to $X$, but has the advantage that $G$ acts freely on it. Further, the homotopy quotient is well defined as the classifying space $BG$ is unique up to homotopy. We then define the equivariant integral cohomology ring as

\[
H^*_G(X; \mathbb{Z}) := H^*(EG \times_G X; \mathbb{Z}).
\]

We will stop making the $\mathbb{Z}$ explicit from now on.

If $G$ acts freely on $X$, then the fibres of $EG \times_G X \to X/G$ are just $EG$ which is $G$-contractible, so $EG \times_G X \cong_G X/G$. This gives that $H^*_G(X) = H^*(X/G)$. For $X$ a point, we get that $EG \times_G \{pt\} = EG/G$ which was defined to be $BG$. Much like equivariant $K$-theory, the map $X \to pt$ thus induces an $H^*_G(pt)$-algebra structure on $H^*_G(X)$.

Now let $G$ be a compact Lie group, and $T \subset G$ be a maximal torus. Finally, let
$W := N(T)/T$ be the corresponding Weyl group. Proposition 1 of [Bri] shows that

$$H^*_G(pt) = H^*_T(pt)^W.$$ 

We will consider our specific case of $G = SU(2)$ and $T = S^1$ in section 5.4.

### 3.4 On $SU(2)$

Let $G = SU(2)$ which consists of all $2 \times 2$ unitary matrices with determinant 1. Topologically, $SU(2)$ is homeomorphic to $S^3$. All maximal tori in $SU(2)$ are conjugate to $T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi] \right\}$, which is homeomorphic to $S^1$, and any two maximal tori intersect at $\pm e$, where $e$ denotes the identity $2 \times 2$ matrix. It is helpful to visualize $e$ and $-e$ as the poles on $S^3$. Two elements in $SU(2)$ commute precisely if they lie on a common maximal torus. Finally, $G/T$ is homeomorphic to $S^2$.

Elements in $G\{\pm e\}$ can be diagonalized as $g = h t h^{-1}$ with $t \in T^+ := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in (0, \pi) \right\}$, where $h$ is uniquely determined up to right multiplication by elements in $T$. Alternatively, such elements can be diagonalized as $g = h t h^{-1}$ with $t \in T^- := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in (\pi, 2\pi) \right\}$, where $h$ is uniquely determined up to right multiplication by elements in $T$. $G$ has a $G$-projection $G \to G/T$ and for $G\{\pm e\}$, denoting $G$ set minus the poles, a projection $G\{\pm e\} \to T^+$. We will use these facts extensively when describing $G$-CW complexes on $SU(2)$ and spaces related to $SU(2)$ in chapter 4.

Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the Weyl group $W := N(T)/T$ is given by $W = \{T, wT\}$ and has a $W$ action on $T$ that takes $t \mapsto t^{-1}$. The element $w$ acts on $G/T$ corresponding to the antipodal map on $S^2$.

For $G = SU(2)$ and $T$ the maximal torus, $R(T) \cong \mathbb{Z}[b, b^{-1}]$, the representation ring of $T$ with $b$ the canonical one dimensional representation of $T$. Also, $R(G) \cong \mathbb{Z}[v]$, the representation ring of $G$ with $v$ the standard two dimensional representation of $G$. The nontrivial action of the Weyl group is given by $b \mapsto b^{-1}$ and the restriction of $v$ to $T$ is given by $b \oplus b^{-1}$. The ring $R(T)$, as a group, is $R(G) \oplus R(G)$. 

Chapter 4

$G$-CW Complex Calculations

Henceforth, let $G = SU(2)$, acting on itself by conjugation. In this chapter we identify $G$-CW structures on several $G$-spaces, in particular for $Hom(\mathbb{Z}^n, G)$. We begin with the more elementary examples to develop our intuition. Then, in sections 4.2 and 4.3, we discuss commuting 2-tuples and $n$-tuples, respectively. These $G$-CW structures will be used to compute the equivariant $K$-theory of these spaces in chapter 5.

A list of the $G$-CW complexes computed in this section is tabulated below, excluding $Hom(\mathbb{Z}^n, G)$:

<table>
<thead>
<tr>
<th>Space</th>
<th>Cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$\sqcup_2 (G/G) \times D^0$, $(G/T) \times D^1$</td>
</tr>
<tr>
<td>$(G/T) \times (G/T)$</td>
<td>$\sqcup_2 (G/T) \times D^0$, $(G/\ast) \times D^1$</td>
</tr>
<tr>
<td>$G \times G$</td>
<td>$\sqcup_4 (G/G) \times D^0$, $\sqcup_4 (G/T) \times D^1$, $\sqcup_2 (G/T) \times D^2$, $(G/\ast) \times D^3$</td>
</tr>
<tr>
<td>$G \lor G$</td>
<td>$\sqcup_3 (G/G) \times D^0$, $\sqcup_2 (G/T) \times D^1$</td>
</tr>
<tr>
<td>$Hom(\mathbb{Z}^2, G)$</td>
<td>$\sqcup_4 (G/G) \times D^0$, $\sqcup_4 (G/T) \times D^1$, $\sqcup_2 G/T \times D^2$</td>
</tr>
<tr>
<td>$((G \times G) \setminus Hom(\mathbb{Z}^2, G))^+$</td>
<td>$G/G \times D^0$, $G/(\mathbb{Z}/2) \times D^3$</td>
</tr>
</tbody>
</table>

4.1 $G$-CW structures for various $G$-spaces

Example 1. $G = SU(2)$

The group $SU(2)$ has two $G$-fixed points under the conjugation action, $\pm e$. We thus let the zero skeleton $X^0$ consists of two 0-cells of the form $(G/G) \times D^0$ corresponding
to the two points $\pm e \in SU(2)$. Topologically, we view this as the poles of $SU(2)$, which is topologically $S^3$.

Recall from page 9 of section 3.4, that elements in $G \{\pm e\}$ can be diagonalized as $g = hth^{-1}$ with $t \in T^+ := \{t \in T \mid 0 < \theta < \pi\}$ where $h$ is uniquely determined up to right multiplication by elements in $T$. As described in section 3.4, $G \{\pm e\}$ thus has projections to $G/T$ and to $T^+$. Thus, we let $X^1$ consist of a 1-cell of the form $(G/T) \times D^1$ attached to $X^0$ via the attaching map that collapses each of the two connected components of $G/T \times S^0$ to the two 0-cells, respectively. Identifying $D^1$ with $T^+$, described earlier, we have a $G$-homeomorphism from $(G/T) \times D^1$ to $G \{\pm e\}$ given by $(gT, t) \mapsto gtg^{-1}$ where $G$ acts on $(G/T) \times D^1$ by left multiplication on the first factor and trivially on the second. Note that we had two choices for this diagonalization, depending on whether we assert that $t \in T^+$ or $T^-$. We chose the former.

Topologically, one can think of this as the construction of $S^3$ by taking $S^2 \times D^1$ and collapsing the ends $S^2 \times \partial D^1$ to two points. Another way to think of this space is that $SU(2)$ is the $G$-space $S(G/T)$ where the $G$ action on the suspension satisfies $h \cdot (gT, t) = (hgT, t)$.

**Example 2.** $G \times G$ (as a $(G \times G)$-CW complex)

For $G \times G$, we begin by considering the collection of cells formed by taking formal products of cells in $G$. This results in four 0-cells $(G/G) \times (G/G) \times D^0 \times D^0$, four 1-cells $(G/T) \times (G/G) \times D^1 \times D^0$ and one 2-cell $(G/T) \times (G/T) \times D^1 \times D^1$.

Our larger goal, however, is to rewrite this as a $G$-CW complex, not a $G \times G$-CW complex. The zero and one degree cells in the resulting $(G \times G)$-CW complex can be immediately rewritten as four 0-cells $(G/G) \times D^0$ and four 1-cells $(G/T) \times D^1$ to form the 1-skeleton of a $G$-CW complex. The only nontrivial $(G \times G)$ cell to rewrite as a $G$-cell, to be done shortly, is the 2-cell.

The 1-skeleton for $G \times G$, denoted $(G \times G)^1$, thus consists of a hollow square where the four vertices represent the four 0-cells consisting of pairs $(\pm e, \pm e)$, and the four segments represent the four 1-cells, the pairs with $\pm e$ in one factor and any elements from $G/T$ in the other.
In the following diagram, representing \((G \times G)^1\), a non vertex point on the square represents a copy of \(G/T\). The space represented then consists of four copies of the suspension of \(G/T\), topologically \(S^3\), labeled A,B,C,D, and glued together as per the diagram. Note that this can be visualized as \(S^2 \times S^1\) that has been ‘pinched’ at the four vertices.

**Diagram 1.** \((G \times G)^1\)

![Diagram 1](image)

Attaching \(G\)-maps for the 1-skeleton are determined as per the diagram, where for a specific 1-cell \((G/T) \times D^1\), \((G/T) \times \partial D^1\) attaches to the two adjacent vertices induced by the \(G\)-map from \(G/T \to *\). For instance, the 1-cell consisting of pairs \\{(e, g), \forall g \in G\}\ is attached to the vertices \((e, e)\) and \((e, -e)\). Each of the four segments in Diagram 1 can be thought of as a copy of \(G\). Note that for each such segment, as in Example 1, we have two choices of diagonalization to use in the homeomorphism from \(G/T \times D^1\) to the \(G \setminus \{\pm e\}\). We choose to diagonalize \(g = hth^{-1}\) with \(t \in T^+\). This choice will become relevant when we aim to attach higher degree cells when discussing the \(G\)-CW complex for \(\text{Hom}(\mathbb{Z}^2, G)\) in section 4.2.

Finally, we have the top 2-cell \((G/T \times G/T) \times D^2\). Here the attaching map \(\sigma : (G/T) \times (G/T) \times \partial D^2 \to (G \times G)^1\) has the \(\partial D^2 = S^1\) factor identified with the boundary square of Diagram 1. At each of the four vertices, the left factors have the collapsing map \(G/T \times G/T \to *\). Along the A and C segments, we have the projection map onto the first factor \(G/T \times G/T \to G/T\). Along the B and D segments, we have the projection map onto the first second \(G/T \times G/T \to G/T\).

Thus far we have managed to rewrite the 0-skeleton and 1-skeleton as \(G\)-CW complexes. However, we still have a top 2-cell \(G/T \times G/T \times D^2\) that is not written as a \(G\)-CW complex. We thus turn to studying the space \(G/T \times G/T\).

**Example 3.** \(G/T \times G/T\)

\(G/T \times G/T\) is a \(G\)-space with the diagonal action, and we aim to describe it as a \(G\)-CW complex.
Let $\tau$ denote the tangent bundle of $G/T$, and let $\epsilon$ denote the trivial complex line bundle over $G/T$. Lemma 3.1 of [HJS2] provides a $G$-equivariant homeomorphism $\Theta : \mathbb{P}(\tau \oplus \epsilon) \to (G/T) \times (G/T)$. Here, the diagonal elements are given by the subspace $\mathbb{P}(\tau) \cong_G G/T$.

We describe a $G$-CW complex on $\mathbb{P}(\tau \oplus \epsilon)$ as follows, noting that the fibre of this bundle is $S^2$. There is a single 0-cell of the form $G/T \times D^0$. These map to the points at infinity in $\mathbb{P}(\tau \oplus \epsilon)$, which, under $\Theta$, become the diagonal elements. Describing the non-diagonal elements are a single 2-cell of the form $G/T \times D^2$ with attaching map $G/T \times \partial D^2 = G/T \times S^1 \to G/T \times D^0$ induced by collapsing $S^1$ to a point. On the right hand factors, this is the construction of the sphere $S^2$ by attaching the boundary of $D^2$, which is homeomorphic to the complex plane, to a point.

**Example 4.** $G \times G$ (now as a $G$-CW Complex)

The 0- and 1-skeletons of $G \times G$, originally written as $(G \times G)$-CW cells, were rewritten as $G$-CW cells in Example 2.

Now, however, we can use the result of Example 3 to rewrite the top 2-cell $(G \times G) \times D^2$ as a $G$-CW complex. As such in addition to the four 0-cells and four 1-cells described previously, we also have one $(G/T \times D^0) \times D^2 \cong_G G/T \times D^2$ 2-cell and one 4-cell $(G/T \times D^2) \times D^2 \cong_G G/T \times D^4$. Here, the attaching map for the 2-cell (recall we thought of these as the diagonal elements) is to attach $G/T \times \partial D^2$. Here the $\partial D^2 = S^1$ wraps around the boundary square of Diagram 1, while the $G/T$ maps by the identity map, except for the corners where it collapses to a point.

For the 4-cell, $\partial D^4 = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$. For the $G/T \times S^1 \times D^2$ component, the middle factor corresponds to the same $\partial D^2 = S^1$ collapsing to a point to form a sphere, as when we considered $(G/T) \times (G/T)$. Thus this component is attached by collapsing to $G/T \times pt \times D^2$ and identifying this with the 2-cell $G/T \times D^2$ thought of as the diagonal elements. For the $G/T \times D^2 \times S^1$ component, we use the the $G$-homeomorphism of Example 3 to identify this with a point in $G/T \times G/T \times S^1$ where the $S^1$ is thought of as the boundary square of Diagram 1. The attaching map onto the four 1-cells of the form $G/T \times D^1$ in the 1-skeleton is thus projection from the first factor along the A and C segments of $S^1$ and projection from the second factor along the B and D segments. Finally, all of $G/T \times G/T$ collapses to a point at the four vertices in $S^1$. 
Example 5. $G \vee G$

For $G \vee G$, with the wedge attaching the North pole of the first copy to the South pole of the second copy, $X^0$ consists of three 0-cells $(G/G) \times D^0$, labeled $S_1, N_1 = S_2, N_2$, and two 1-cells $(G/T) \times D^1$. The attaching maps for one of the two 1-cells, thought of as the bottom copy of $SU(2)$, collapse to a point the $G/T$ factor while for the second factor the $\partial D^1 = S^0$ endpoints are attached to $S_1$ and $N_1 = S_2$. For the second 1-cell, thought of as the top copy of $SU(2)$, the $G/T$ factor likewise collapses to a point while the endpoints $\partial D^1 = S^0$ are attached to $N_1 = S_2$ and $N_2$.

Example 6. Non-commuting 2-tuples

In this example we reinterpret work following the presentation of page 25, Proposition 6.11 of [AG] in our $G$-CW language, and in a way that makes for slightly easier computations. We will give a $G$-CW structure on the one point compactification of the non-commuting 2-tuples.

Following [AG], set $Y := (G \times G) \setminus Hom(\mathbb{Z}^2, G)$. For the commutator map $\phi : G \times G \to G$, let $\phi_1$ denote the restriction of $\phi$ to the noncommuting 2-tuples, $\phi_1 : Y \to G \setminus \{1\}$. As per Proposition 4.7 of [AC], this is a locally trivial $G$-fibre bundle with fibre

$$F := \phi^{-1}(-1) = \{(x_1, x_2) \in G \times G \mid [x_1, x_2] = -1\},$$

and base $su_2$, the Lie algebra of $SU(2)$ with the adjoint representation. As $su_2$ is $SU(2)$-contractible we get a homotopy equivalence $Y \simeq_G F \times su_2 \simeq_G F \times D^3$ by identifying the Lie algebra of $SU(2)$ with the Lie algebra $\mathbb{R}^3$. Following from [AG], $F \cong_G G/Z(G) \cong_G G/(\mathbb{Z}/2)$ where $G$ acts on $G/(\mathbb{Z}/2)$ by left translation.

Now consider the one-point compactification $Y^+$ which identifies $F \times \partial D^3$ with a point. We can thus equip $Y^+$ with a $G$-CW structure via a single point, $G/G \times D^0$ as the 0-skeleton, and then a 3-cell $G/(\mathbb{Z}/2) \times D^3$ identified by the trivial map on both factors $G/(\mathbb{Z}/2) \times \partial D^3 \to G/G \times D^0$. 
4.2 \textbf{G-CW structure for }\textit{Hom}(\mathbb{Z}^2, G)\textbf{ }

We build our intuition first through an explicit description of commuting \(n\)-tuples in the \(n = 2\) case (which is the only case we will go on to compute the equivariant \(K\)-theory of) before describing the general \(n\)-tuple case in the next section. Consider commuting 2-tuples in \(G\),

\[ \text{Hom}(\mathbb{Z}^2, G) = \{(g_1, g_2) \in G \times G \mid g_1g_2g_1^{-1} = g_2\}. \]

Elements in \(G\) commute if they are on the same maximal torus; all maximal tori are conjugate to each other and intersect at the two poles, \(\pm e\). Let the 0-skeleton consist of pairs \((\pm e, \pm e)\), described by four 0-cells \((G/G) \times D^0\). Let the 1-skeleton consist of pairs \((\pm e, g)\) or \((g, \pm e)\) for \(g \in G\), described by four 1-cells \((G/G) \times D^1\). The attaching maps all work precisely as in Diagram 1 and thus \((\text{Hom}(\mathbb{Z}^2, G))^1 = (G \times G)^1\).

For a pair \((g_1, g_2) \in \text{Hom}(\mathbb{Z}^2, G)\backslash(\text{Hom}(\mathbb{Z}^2, G))^1\), recall from section 3.4 that we had two options (up to right multiplication by elements of \(T\)) for how to diagonalize \(g_1\). The action of the Weyl group alternates between these presentations by acting on \(G/T\) by the antipodal map and on the maximal torus \(T\) by taking \(t \to t^{-1}\). We choose to diagonalize as \(g_1 = h_1t_1h_1^{-1}\) for \(h_1 \in G/T\) and \(t_1 \in T^+\). Aiming to determine a cell of the form \((G/T) \times D^2\), now that a choice in \(G/T\) has been fixed by our choice for diagonalizing \(g_1\), for \(g_2\) we must now use \(g_2 = h_1t_2h_1^{-1}\) where \(t_2\) may be in either \(T^-\) or \(T^+\). That is,

\[ \text{Hom}(\mathbb{Z}^2, G)\backslash(\text{Hom}(\mathbb{Z}^2, G))^1 \cong (G/T \times (T\backslash \{\pm e\})^2)/W \cong ((G/T) \times \sqcup D^2)/W. \]

The \(W\) action is needed because otherwise we would be double counting given our two ways to diagonalize; the nontrivial element of the Weyl group alternates between these presentations. Applying \(W\),

\[ \text{Hom}(\mathbb{Z}^2, G)\backslash(\text{Hom}(\mathbb{Z}^2, G))^1 \cong (G/T) \times \sqcup D^2. \]

We thus get two \((G/T) \times D^2\) 2-cells where the first copy refers to pairs where \(t_1, t_2\) are both in \(T^+\) and the second copy refers to pairs where \(t_1 \in T^+\) but \(t_2 \in T^-\). We can think of the two copies as representing pairs that are on the same side of the maximal torus, and on opposite sides, respectively.
To explicitly define our $G$-homeomorphism

$$\sigma : (G/T) \times \sqcup_2 D^2 \to Hom(\mathbb{Z}^2, G) \setminus (Hom(\mathbb{Z}^2, G))^1$$

we first identify $((G/T) \times \sqcup_2 D^2)$ with $((G/T) \times T^+ \times T^+) \sqcup ((G/T) \times T^+ \times T^-)$ where $t \in T^-$ if $t^{-1} \in T^+$. Our maps should be compatible with our choice of writing elements in the 1-skeleton in terms of $T^+$. For the first connected component, elements are already written this way so our $G$-map takes $(gT, t_1, t_2) \to (gt_1 g^{-1}, gt_2 g^{-1})$. For the second connected component, the third factor is written in terms of $T^-$ and so our $G$-map becomes $(gT, t_1, t_2) \to (gt_1 g^{-1}, (w \cdot g)t_2^{-1}(w \cdot g)^{-1})$. As such, the attaching map for the first connected component $(G/T) \times \partial D^2 \to (Hom(\mathbb{Z}^2, G))^1$ is induced by the identity map on $G/T$ and the identification of $\partial D^2$ with the boundary square in Diagram 1. The attaching map for the second connected component is induced by the same map along the boundary square but has the identity map on $G/T$ along the A and C segments of $\partial D^2$ and the antipodal map on $G/T$ along the segments B and D.

Therefore our $G$-CW complex for $Hom(\mathbb{Z}^2, G)$ consists of the same four 0-cells of the form $(G/G) \times D^0$ and four 1-cells of the form $(G/T) \times D^1$ as $(G \times G)^1$, along with two 2-cells $(G/T) \times D^2$, attached as above.

### 4.3 $G$-CW structure for $Hom(\mathbb{Z}^n, G)$

Now consider the generalization to commuting $n$-tuples in $G$,

$$Hom(\mathbb{Z}^n, G) = \{(g_1, \ldots, g_n) \in G^n \mid g_i g_j g_i^{-1} = g_j \forall i, j\} \quad (4.1)$$

which for $G = SU(2)$ can be described as

$$Hom(\mathbb{Z}^n, G) = \{(g_1, \ldots, g_n) \in G^n \mid \text{all } g_i \text{ lie in a common maximal torus}\}. \quad (4.2)$$

We define a filtration on $Hom(\mathbb{Z}^n, G)$ via

$$(Hom(\mathbb{Z}^n, G))^k = \{(g_1, \ldots, g_n) \in Hom(\mathbb{Z}^n, G) \mid g_i \notin \{\pm e\} \text{ for at most } k \text{ of the } g_i\} \quad (4.3)$$

which satisfies

$$(Hom(\mathbb{Z}^n, G))^n = Hom(\mathbb{Z}^n, G). \quad (4.4)$$
We equip each $\left(\text{Hom}(\mathbb{Z}^n, G)\right)^k$ with a $G$-CW structure inductively as follows. For convenience, define

$$F_n(m) := \text{the number of } m\text{-faces of an } n\text{-hypercube} = 2^{n-m}\binom{n}{m}. \quad (4.5)$$

The $0$-skeleton consists of $F_n(0) = 2^n$ points:

$$\left(\text{Hom}(\mathbb{Z}^n, G)\right)^0 = \{(g_1, \ldots, g_n) \in \text{Hom}(\mathbb{Z}^n, G) \mid g_i \in \{\pm e\} \forall i\} \quad (4.6)$$

and thus can be given a $G$-CW structure by considering $2^n$ discrete cells $(G/G) \times D^0$. Now consider

$$\left(\text{Hom}(\mathbb{Z}^n, G)\right)^k \setminus \left(\text{Hom}(\mathbb{Z}^n, G)\right)^{k-1} = \{(g_1, \ldots, g_n) \in \text{Hom}(\mathbb{Z}^n, G) \mid g_i \notin \{\pm e\} \text{ for exactly } k \text{ of the } g_i\}. \quad (4.7)$$

In each $n$-tuple, we have $n - k$ of the coordinates $g_i \in \{\pm e\}$ and each such choice can be thought of as defining a $k$-face where we are generalizing the notion of the $k$-faces of an $n$-hypercube. There are $F_n(k)$ such $k$-faces, and taking interiors in our chosen manner leaves each face disjoint. The points in each face then consist of $k$ nontrivial coordinates, and so for notational convenience we will shorten our $n$-tuple with $k$ nontrivial coordinates to a $k$-tuple. That is,

$$\left(\text{Hom}(\mathbb{Z}^n, G)\right)^k \setminus \left(\text{Hom}(\mathbb{Z}^n, G)\right)^{k-1} \cong \sqcup_{F_n(k)} \{ (g_1, \cdots, g_k) \in \text{Hom}(\mathbb{Z}^k, G) \mid g_i \notin \{\pm e\} \forall i \}. \quad (4.8)$$

Recall that $(g_1, \cdots, g_k) \in \text{Hom}(\mathbb{Z}^k, G)$ precisely when all $g_i$ lie in a common maximal torus. Given an element $(g_1, \cdots, g_k) \in \text{Hom}(\mathbb{Z}^k, G)$, our goal is to diagonalize this as $(g_1, \cdots, g_k) = (h_1 t_1 h_1^{-1}, \cdots, h_k t_k h_k^{-1})$. Recall that for each component $g_i$ we have two choices on how to diagonalize and we choose that $t_i \in T^+$ for each $i$. With $h_1$ determined, then either $h_i = h_1$ or $h_i = w \cdot h_1$ for all $2 \leq i \leq k$. That is, either $g_i = h_1 t_i h_1^{-1}$ or $g_i = (w \cdot h_1)t_i(w \cdot h_1)^{-1}$ for $2 \leq i \leq k$.

Insisting that all $g_i \notin \{\pm e\}$, breaks $\left(\text{Hom}(\mathbb{Z}^n, G)\right)^k \setminus \left(\text{Hom}(\mathbb{Z}^n, G)\right)^{k-1}$ into disjoint sections, characterized by whether each of the 2nd through $k$th nontrivial coordinates are located on the same side of the maximal torus as $g_1$ or on the opposite side. For instance, in the commuting 2-tuples, there are two top cells corresponding to pairs on the same side of the maximal torus, and pairs on the opposite sides of the max-
imal torus. Thus elements in a particular $k$-face of $(\text{Hom}(\mathbb{Z}^n, G))^k \setminus (\text{Hom}(\mathbb{Z}^n, G))^{k-1}$ are uniquely determined by the following three pieces of data: an element of $G/T$, a $k$-tuple $(t_1, \cdots, t_k) \in (T^+)^k$, and the knowledge of which of the latter $k - 1$ coordinates we ought to apply the antipodal map to, the latter having $2^{k-1}$ possibilities giving us what we will term the $2^{k-1}$ leaves for each $k$-face. A particular $(G/T) \times D^k$ cell records the first two pieces of data in the first and second factor, respectively, while the third is brought by the attaching map to be discussed shortly.

Algebraically, we have

\[
(\text{Hom}(\mathbb{Z}^n, G))^k \setminus (\text{Hom}(\mathbb{Z}^n, G))^{k-1} \cong \sqcup_{F_n(k)}(G/T \times \{\pm e\}^k)/W \\
\cong \sqcup_{F_n(k)}((G/T) \times \sqcup_{2k-1}I^k)/W
\]

(4.9)

where again the $W$ action eliminates the double counting. Applying $W$,

\[
(\text{Hom}(\mathbb{Z}^n, G))^k \setminus (\text{Hom}(\mathbb{Z}^n, G))^{k-1} \cong \sqcup_{F_n(k)}((G/T) \times \sqcup_{2k-1}I^k) \\
\cong \sqcup_{F_n(k)2k-1}((G/T) \times I^k) \cong \sqcup_{2n-1}(k)\!(G/T) \times I^k). \tag{4.10}
\]

This can be thought of as having $2^{k-1} k$-cells of the form $(G/T) \times D^k$ corresponding to each of our $F_n(k)$ $k$-faces.

Putting this together, we state the cells in the $G$-CW structure:

\[
0 - \text{skeleton} : \quad 2^n \text{ cells of the form } (G/G) \times D^0 \tag{4.11}
\]

\[
k - \text{skeleton}, \quad 1 \leq k \leq n : \quad 2^{n-1}\!(\begin{array}{c} n \\ k \end{array}) \text{ cells of the form } (G/T) \times D^k \tag{4.12}
\]

While the cells have been listed, we still need to specify the attaching maps. The $1$-cells are attached analogously to the attaching map for the (single) $1$-cell in our $G$-CW complex for $G$. That is, we have a $G$-map from $(G/T) \times \partial D^1 \rightarrow \{\pm e\}$ that identifies $S^0$ with the points $\{\pm e\}$ along with the map $G/T \rightarrow \{\ast\}$. The difference now is we identify the right hand factor in each $(G/T) \times D^1$ with one of the $F_n(1)$ edges in an $n$-hypercube and $(G/T) \times \partial D^1$ is attached to the corresponding vertices.

The analogy with the right hand factors and an $n$-hypercube continue, where we continue to attach the boundaries of the $2^{k-1} G/T \times D^k$ cells corresponding to each of the various $F_n(k)$ $k$-faces in the $n$-hypercube into the $(k - 1)$-skeleton surrounding
each such $k$-face where the map on the right hand factor sends $\partial D^k$ to various $D^{k-1}$ in the way expected for cubes. The trick, however, comes in the first factor and we need to be careful about the map from $G/T \to G/T$ even if the maps on the second factor follow precisely as one expects from the skeleta of an $n$-hypercube.

To define our $G$-homeomorphism

$$
\sigma : (G/T) \times D^k \to a \text{-} k\text{-}face \text{ of } (\text{Hom}(\mathbb{Z}^n, G))^k \backslash (\text{Hom}(\mathbb{Z}^n, G))^{k-1} \tag{4.13}
$$

we send $(gT, t_1, \cdots, t_k)$ to $(g_1, \cdots, g_k)$ where $g_1 = gt_1g^{-1}$ and for $2 \leq i \leq k$ either $g_k = gt_kg^{-1}$ or $(w \cdot g)t_k(w \cdot g)^{-1}$, depending on which cell we are identifying. Each of the $2^{k-1}$ cells for a particular $k$-face correspond with a particular choice of whether or not to apply the antipode in coordinates 2 through $k$, and this data is now recorded in the $G$-map.

We can make the data encapsulated by the $G$-homeomorphism precise by defining an index $\Lambda^k = \mathbb{Z}_2^k$ with a 1 in the $i$th coordinate corresponding to applying the antipode at that coordinate and a 0 means not doing this. Given our above preference, the first coordinate is thus always a 0. A cell corresponding to index $(0, 1, 1)$, for instance, would consist of those 3-tuples without any coordinates being $\pm e$ which had the first element on one side of the maximal torus, and the other two on the other side. Then a cell consists of a $(G/T) \times D^k$ together with a choice of $\lambda \in \Lambda^k$ which determines a $G$-map 

$$
\sigma_\lambda : (G/T) \times D^k \to a \text{-} k\text{-}face \text{ of } (\text{Hom}(\mathbb{Z}^n, G))^k \backslash (\text{Hom}(\mathbb{Z}^n, G))^{k-1} \text{ where}
$$

$$
\sigma_\lambda(gT, t_1, \cdots, t_k) = ((w^{\lambda_1} \cdot g)t_1(w^{\lambda_1} \cdot g)^{-1}, \cdots, (w^{\lambda_k} \cdot g)t_k(w^{\lambda_k} \cdot g)^{-1}). \tag{4.14}
$$

For the attaching $G$-maps from $(G/T) \times \partial D^k \to (\text{Hom}(\mathbb{Z}^n, G))^{k-1}$ we begin with a $k$-cell $(G/T) \times D^k$ identified by choice of index $\lambda \in \Lambda^k$ that has a 0 in the first coordinate. We identify $D^k$ with a solid $k$-cube. Then $\partial D^k$ can be written as a union of $(k-1)$-faces to that $k$-cube. Now consider a particular $(k-1)$-face of $(G/T) \times \partial D^k$ analogously to considering a particular $(k-1)$-face of $\partial D^k$. Choosing a face is equivalent to ignoring a particular coordinate in our index so the face can be thought of as being indexed by an element $\tilde{\lambda} \in \Lambda^{k-1}$ whose first coordinate is not necessarily a 0. On the corresponding $(k-1)$-face in $(\text{Hom}(\mathbb{Z}^n, G))^{k-1}$, we have $2^{k-2}$ $(k-1)$-cells $(G/T) \times D^{k-1}$, each identified with a particular choice of index $\delta \in \Lambda^{k-1}$ that has a 0 in the first coordinate.
We thus have two possibilities for an attaching map:

\[(gT, t_1, \ldots, t_{k-1}) \mapsto (gT, t_1, \cdots, t_{k-1})\]  
\[\text{or}\]  
\[(w \cdot g)T, t_1, \cdots, t_{k-1}).\]

These possibilities correspond to whether the induced index $\tilde{\lambda}$ has a 0 in the first coordinate, in which case there is a cell from the $(k-1)$-skeleton with a corresponding index and we don’t need to apply $w$, or whether it has a 1, in which case we do apply $w$. Notice that there will always be two $k$-cells whose boundary attaches to particular $(k-1)$-cell, given the two-fold choice in the first coordinate.
Chapter 5

Equivariant $K$-theory computations

In this chapter we compute the module structure of the various $G$-CW complexes determined in chapter 4, in particular, in section 5.2, $K^*_G(\text{Hom}(\mathbb{Z}^2, G))$. In sections 5.3 and 5.4 we additionally compute the module structure on other cohomology theories for $\text{Hom}(\mathbb{Z}^2, G)$. In section 5.5 we discuss the algebra structure on various cohomology theories, in particular for $K^*_G(\text{Hom}(\mathbb{Z}^2, G))$. In 5.6 we consider some of the other spaces previously discussed. Finally, we discuss some possible generalizations in section 5.7.

Let $X^n$ denote the $n$-skeleton of the various spaces described in the table at the beginning of chapter 4. The module structure for the equivariant $K$-theory of these spaces computed in this chapter are tabulated below.

<table>
<thead>
<tr>
<th>Space</th>
<th>$\tilde{K}^0_G(X)$</th>
<th>$\tilde{K}^1_G(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>0</td>
<td>$\mathbb{R}(G)$</td>
</tr>
<tr>
<td>$(G/T) \times (G/T)$</td>
<td>$\mathbb{R}(G) \oplus \mathbb{R}(G) \oplus \mathbb{R}(G)$</td>
<td>0</td>
</tr>
<tr>
<td>$G \times G$</td>
<td>$\mathbb{R}(G)$</td>
<td>$\mathbb{R}(G) \oplus \mathbb{R}(G)$</td>
</tr>
<tr>
<td>$\text{Hom}(\mathbb{Z}^2, G)$</td>
<td>$\mathbb{R}(G) \oplus (\mathbb{R}(G) \oplus \mathbb{R}(G))/\langle(v, -2)\rangle$</td>
<td>$\mathbb{R}(G) \oplus \mathbb{R}(G)$</td>
</tr>
</tbody>
</table>

5.1 $\tilde{K}^*_G(G)$

For this example, $X^1 = G$. Recalling our $G$-CW complex for $G$, we can identify $X^0$ and $X^1/X^0$ as follows

$$X^0 \cong_G S^0$$

$$X^1/X^0 \cong_G \text{Thom}(G/T \times D^1) \cong_G S((G/T) \sqcup pt)$$
where \( pt \) is used to denote the point at infinity in the Thom space.

We now compute, making use of Proposition 1 of section 3.1.

\[
\tilde{K}_0^0(X^0) = \tilde{K}_0^1(2 \text{ pts}) = K_G^0(pt) = R(G)
\]

\[
\tilde{K}_1^0(X^0) = \tilde{K}_1^1(2 \text{ pts}) = K_G^1(pt) = 0
\]

\[
\tilde{K}_0^0(X^1/X^0) = \tilde{K}_0^0(\text{Thom}((G/T) \times D^1)) = \tilde{K}_0^0(S((G/T) \cup pt))
\]

\[
= \tilde{K}_1^1((G/T) \cup pt) = K_G^1(G/T) = 0
\]

\[
\tilde{K}_1^1(X^1/X^0) = \tilde{K}_1^1(\text{Thom}((G/T) \times D^1)) = \tilde{K}_1^1(S((G/T) \cup pt))
\]

\[
= \tilde{K}_0^0((G/T) \cup pt) = K_G^0(G/T) = R(T)
\]

As such, the six term exact sequence of Proposition 6 of section 3.2 corresponding to \( X_0 \rightarrow X_1 \rightarrow X_1/X_0 \) is the following.

\[
\begin{array}{ccccccccc}
R(G) & \longrightarrow & \tilde{K}_G^0(X) & \longrightarrow & 0 \\
\downarrow i^* & & \downarrow & & \\
R(T) & \longrightarrow & \tilde{K}_G^1(X) & \longrightarrow & 0
\end{array}
\] (5.1)

We thus aim to study the kernel and cokernel of \( i^* \).

The map of spaces \( i : X^1/X^0 \hookrightarrow S(X^0) \) is given by \( i : S(G/T \cup pt) \rightarrow S(pt \cup pt) \). The map is induced by the map \( G/T \rightarrow \ast \).

To aid in visualizing this, the space \( X^1/X^0 \) consists of \( G \) (visualized as \( S^3 \)) with the north and south poles identified. Alternatively consider the \( G \)-space \( Y \) consisting of \( G \) with a line attaching the north and south poles where all points on this line are fixed under the action of \( G \). As the line is \( G \)-contractible, proposition 3 of section 3.1 gives that \( X^1/X^0 \cong_G Y \). To include into \( S(X^0) \cong_G S^1 \) we attach a cone over \( X^1 \cong_G S(G/T) \). This is thus equivalent to collapsing \( G/T \) to a point.

We thus need to study the map \( i^* : K_G^1(pt) = R(G) \rightarrow K_G^1(G/T) = R(T) \). We saw in section 3.4 that we could write \( R(G) = \mathbb{Z}[v] \) and \( R(T) = \mathbb{Z}[b, b^{-1}] \) where \( v = b + b^{-1} \). The ring \( R(T) \) can be identified as an \( R(G) \)-module as \( R(G) \oplus R(G) \). Indeed, identifying \( b^{-1} = v - b \), we can give \( R(T) \) the \( \{1, b\} \) basis as an \( R(G) \)-module. Hence the collapsing map \( G/T \rightarrow \ast \) induces the map \( R(G) \rightarrow R(G) \oplus R(G) \) that takes \( v \mapsto (v, 0) \). Finally,
\[ \tilde{K}_0^G(X) = \ker(i^*) = 0 \]
\[ \tilde{K}_1^G(X) = \coker(i^*) = R(G) \]

Note that this corresponds with our expected answer from ordinary reduced cohomology with integer coefficients which, as \( G \) is topologically just \( S^3 \), has a single copy of \( \mathbb{Z} \) in degree three (which appears here in degree one due to the \( \mathbb{Z}_2 \) grading in equivariant \( K \)-theory). Indeed, one can write out the corresponding exact sequence for ordinary integral cohomology and obtain the same result. This result also corresponds to [BZ]'s result for \( K^G_q(G) \) in the case of \( G = SU(2) \).

### 5.2 Module structure of \( K^*_G(Hom(\mathbb{Z}^2, G)) \)

From the \( G \)-CW structure described in section 4.2, we will compute the module structure of its \( G \)-equivariant \( K \)-theory in this section.

#### 5.2.1 The 1-skeleton of \( Hom(\mathbb{Z}^2, G) \)

Recall from the \( G \)-CW structured described in section 4.2, that the 0-skeleton, denoted \( X^0 \) consisted of four disjoint points (thought of as cells of the form \( G/G \times D^0 \)) corresponding to the four fixed points \((\pm e, \pm e)\). Our 1-skeleton had four 2-cells of the form \( G/T \times D^1 \) attached according to Diagram 1.

**Diagram 1.** \((Hom(\mathbb{Z}^2, G))^1\)

\[
\begin{array}{cccc}
(e,e) & B & (e,-e) \\
A & & C \\
(-e,e) & D & (-e,-e)
\end{array}
\]

By Proposition 1 of section 3.1, we have \( X^1/X^0 \simeq_G \bigsqcup_4 (S((G/T) \sqcup \infty)) \). We can now compute the \( G \)-equivariant \( K \)-theory of \( X^0 \) and \( X^1/X^0 \).

\[
K_0^G(X^0) = K_0^G(\sqcup_4 pt) = \oplus_4 K_0^G(pt) = \oplus_4 R(G) \tag{5.2}
\]
\[
K_1^G(X^0) = K_1^G(\sqcup_4 pt) = \oplus_4 K_1^G(pt) = 0 \tag{5.3}
\]
\[ \tilde{K}_G^0(X^1/X^0) = \tilde{K}_G^0(\bigvee_4 (S((G/T) \sqcup \{\infty\}))) = \tilde{K}_G^1(\bigvee_4 ((G/T) \sqcup \{\infty\})) \]
\[ = \tilde{K}_G^1((\sqcup_4 G/T) \sqcup \{\infty\})) = K_G^0(\sqcup_4 G/T) = \oplus_4 K_G^1(G/T) = \oplus_4 K_T^0(pt) = 0 \] (5.4)

\[ \tilde{K}_G^1(X^1/X^0) = \tilde{K}_G^1(\bigvee_4 (S((G/T) \sqcup \{\infty\}))) = \tilde{K}_G^0(\bigvee_4 ((G/T) \sqcup \{\infty\})) \]
\[ = \tilde{K}_G^0((\sqcup_4 G/T)) \sqcup \{\infty\}) = K_G^0(\sqcup_4 G/T) = \oplus_4 K_G^0(G/T) \]
\[ = \oplus_4 K_T^0(pt) = \oplus_4 R(T) = \oplus_8 R(G) \] (5.5)

In the above computations, we have made repeated use of two facts. First, we have an isomorphism \( K_H^0(pt) = R(G) \) where in our case we’ve used \( H = T \) and \( H = G \). Second, we have an isomorphism \( K_G^0(A \sqcup B) = K_G^0(A) \oplus K_G^0(B) \).

Let us give explicit generators for the two above spaces with nontrivial equivariant \( K \)-theory. In particular, in equation 5.2 we observed that \( K_G^0(X^0) = K_G^0(\sqcup_4 pt) = \oplus_4 K_G^0(pt) \). Let \( x_i \in K_G^0(X^0) \) denote the image under the disjoint union isomorphism of a generator for \( K_G^0(pt) \), which is isomorphic to \( R(G) \). Here, the \( i \) represents the \( i \)th point in the disjoint union. Likewise, from equation 5.5, we have that \( \tilde{K}_G^1(X^1/X^0) = \tilde{K}_G^0(\sqcup_4 G/T) = \oplus_4 K_T^0(pt) \). Let \( y_i \in \tilde{K}_G^1(X^1/X^0) \) be the image under the disjoint union isomorphism of a generator for \( K_T^0(pt) = R(T) \) where the \( i \) represents the \( i \)th point in the disjoint union.

For the unreduced version of the six term exact sequence described in Proposition 6 of section 3.2 for \( X^0 \hookrightarrow X^1 \to X^1/X^0 \) we get
\[ \begin{array}{c}
\oplus_4 R(G) & \stackrel{i^*}{\longrightarrow} & K_G^0(X^1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\oplus_4 R(T) & \longrightarrow & K_G^1(X^1) & \longrightarrow & 0
\end{array} \] (5.6)

Our goal is thus to study \( i^* \).

We can visualize the 1-skeleton as four copies of \( S(G/T) \) glued together according to Diagram 1. The attaching maps work by having, for each of the four cells \( G/T \times D^1 \) in the 1-skeleton, two versions of the collapsing map \( G/T \to pt \). This is analogous to how, when studying \( SU(2) \) itself, the two connected components of \( G/T \times \partial D^1 \) collapse down to the two endpoints. The difference now is that we are doing the same thing four times and have to keep track of which two points the \( G/T \times \partial D^1 \) components
collapse down to.

We will write the map $i^*: \oplus_4 R(G) \to \oplus R(T)$ as a basis with the domain and codomain written in the bases $x_1, \cdots, x_4$ and $y_1, \cdots, y_4$ described above, respectively. We will denote by $j^*$ the inclusion $j^*: R(G) \to R(T)$ given by $j^*(v) = (v, 0) \in R(G) \oplus R(G)$ induced by the collapsing map $j: G/T \to pt$. The kernel of this map is $0$ and the cokernel is $R(G)$. Putting this all together, we have the map $i^*: \oplus_4 R(G) \to \oplus R(T)$ described by the matrix

$$
\begin{pmatrix}
  j^* & j^* & 0 & 0 \\
  0 & j^* & j^* & 0 \\
  0 & 0 & j^* & j^* \\
  j^* & 0 & 0 & j^*
\end{pmatrix}
$$

(5.7)

Note that this is completely analogous to how we could compute the homology of the edges of a square by giving it at a CW complex with four vertices and four line segments.

Our goal is to compute the kernel and cokernel of this matrix. Because the matrix has rank $3$, the kernel is one copy of $R(G)$. The copy of $R(G)$ coming from only having rank $3$ and the four copies of $R(G)$ coming from each $j^*$ map having cokernel $R(G)$ gives us that

$$K^0_G(X^1) = \ker(i^*) = R(G)$$

(5.8)

$$K^1_G(X^1) = \text{coker}(i^*) = \oplus_5 R(G)$$

(5.9)

Note that our computation matches what we would expect on ordinary cohomology. Indeed, ignoring the $G$ action, $X^1$ is homotopy equivalent to $(\bigsqcup_4 S^2) \setminus S^1$ and thus has five copies of the integers in odd degrees, and one copy in even degrees.

### 5.2.2 The 2-skeleton of $\text{Hom}(\mathbb{Z}^2, G)$

Recall that in addition to the 1-skeleton of the previous section, the $G$-CW complex for $\text{Hom}(\mathbb{Z}^2, G)$, as described in section 4.2, had two 2-cells of the form $G/T \times D^2$. Proposition 1 of section 3.1 thus gives us that.
\[ X^2/X^1 \cong_G \bigvee_2 (S^2(G/T \sqcup pt)) \quad (5.10) \]

Computing \( \tilde{K}_G^*(X^2/X^1) \) gives us

\[
\tilde{K}_G^0(X^2/X^1) = \tilde{K}_G^0 \left( \bigvee_2 ((S^2(G/T) \sqcup pt) \right) = \tilde{K}_G^0((G/T) \sqcup (G/T) \sqcup pt) = 
= K_G^0((G/T) \sqcup (G/T)) = K_G^0(G/T) \oplus K_G^0(G/T) = \bigoplus_2 R(T) \quad (5.11)
\]

\[
\tilde{K}_G^1(X^2/X^1) = \tilde{K}_G^1 \left( \bigvee_2 ((S^2(G/T) \sqcup pt) \right) = \tilde{K}_G^1((G/T) \sqcup (G/T) \sqcup pt) = 
= K_G^1((G/T) \sqcup (G/T)) = K_G^1(G/T) \oplus K_G^1(G/T) = 0 \quad (5.12)
\]

The second to last isomorphism in equation 5.11 comes from the general isomorphism that \( K_G^q(A \sqcup B) = K_G^q(A) \oplus K_G^q(B) \) for \( G \)-spaces \( A \) and \( B \). For \( i \in \{1, 2\} \), let \( y_i \in \tilde{K}_G^0(X^2/X^1) \) denote the image under the disjoint union isomorphism of a generator for \( K_G^0(G/T) \) which is in turn isomorphic to \( R(T) \). Here the \( i \) represents the \( i \)th copy of \( G/T \) in the disjoint union.

Combining with our prior computation of \( X^1 \) which we saw had an isomorphism \( K_G^*(X^1) = \bigoplus_5 R(G) \) we get the six term exact sequence corresponding to \( X^1 \hookrightarrow X^2 \rightarrow X^2/X^1 \):

\[
0 \longrightarrow \tilde{K}_G^0(\text{Hom}(\mathbb{Z}^2, G)) \longrightarrow \bigoplus_4 R(G) \quad (5.13)
\]

Consider the maps \( i : X^2/X^1 \hookrightarrow S(X^1) \) and \( j : S(X^1) \hookrightarrow S(X^1/X^0) \). The composition is the cellular map \( d = j \circ i : X^2/X^1 \rightarrow S(X^1/X^0) \). We thus get

\[
d^* : \tilde{K}_G^1(X^1/X^0) \rightarrow \tilde{K}_G^1(X^1) \rightarrow \tilde{K}_G^0(X^2/X^1) \quad (5.14)
\]

Substituting our prior computations for the quotient spaces we get

\[
d^* : K_G^0(\sqcup_4(G/T)) \rightarrow K_G^0(\sqcup_4(G/T)). \quad (5.15)
\]

Note that the corresponding map being in even degree has all spaces zero.
Our goal is thus to study this map $d^*$. We first turn to the study of a different map that will be used in this consideration.

We denote by $a$ the antipodal map $a : G/T \to G/T$ given by $a(gT) = (w \cdot g)T$ where $w$ is the nontrivial element of the Weyl group. We denote by $a^* : K_0^G(G/T) \to K_0^G(G/T)$ the induced map in even degree. Via the isomorphism $K_0^G(G/T) = K_1^G(pt) = R(T)$, this gives a map that we will denote with the same symbol, $a^* : R(T) \to R(T)$.

Now recall that our 1-skeleton was described by the following diagram.

**Diagram 2.** $(\text{Hom}(\mathbb{Z}^2, G))^1$

![Diagram 2](image)

To this diagram we attach our two top cells. One of the two cells is induced by the identity map $Id : G/T \to G/T$ along all four segments. The other cell is induced by the identity map $Id : G/T \times G/T$ along the A and C segments, but by the antipodal map $a : G/T \to G/T$ on the B and D segments. Recalling from section 5.2.1 that we defined $x_1, \cdots, x_4$ as generators for each factor in $\tilde{K}_0^G(X^1/X^0) = \bigoplus_4 R(T)$ thought of as coming from each of the four $(G/T)$ factors in a disjoint union. Further, we had defined previously in this section generators $y_1$ and $y_2$ for $\tilde{K}_1^G(X^2/X^1) = \bigoplus_2 R(T)$. The map $d^*$ in these bases is then given by the matrix

$$
\begin{pmatrix}
Id^* & Id^* & Id^* & Id^* \\
Id^* & a^* & Id^* & a^*
\end{pmatrix}
$$

which reduces to

$$
\begin{pmatrix}
Id^* & Id^* & Id^* & Id^* \\
0 & a^* - Id^* & 0 & a^* - Id^*
\end{pmatrix}
$$

To compute the kernel and cokernel of this matrix, we need to study the map $a^* - Id^* : R(T) \to R(T)$.

Recall that $R(T) = \mathbb{Z}[b, b^{-1}]$; that is, Laurent polynomials in the variable $b$. To
express this as an $R(G)$-module, we set $v = b + b^{-1}$ and so we can express a generic element as $p + qb$ where $p, q \in \mathbb{Z}[v]$. For instance, in this basis $b^{-1} = v(1) + (-1)b$.

Recall that the antipodal map $a : G/T \to G/T$ was given by $a(gT) = (w \cdot g)T$. As the induced action of $w$ on $K$-theory is to interchange $b$ and $b^{-1}$, so we get the map $a^* : \mathbb{Z}[b, b^{-1}] \to \mathbb{Z}[b, b^{-1}]$ given by $(a^*)(p + qb) = p + qb^{-1}$. Hence, $(a^* - Id)(p + qb) = qv - 2qb$. Hence,

$$\ker((a^* - Id) : R(T) \to R(T)) = R(G)$$  \hspace{1cm} (5.18)

$$coker((a^* - Id) : R(T) \to R(T)) = (R(G) \oplus R(G))/\langle(v, -2)\rangle$$  \hspace{1cm} (5.19)

We now return to the study of $d^*$ in its entirety, where the reduced matrix gives us

$$\ker(d^*) = \oplus_5 R(G)$$  \hspace{1cm} (5.20)

$$coker(d^*) = (R(G) \oplus R(G))/\langle(v, 2b)\rangle$$  \hspace{1cm} (5.21)

Recall that $d^* = i^* \circ j^*$ was a composition and our real goal is to study $i^*$ that appear in our six term exact sequence.

Firstly, note that $j^*$ is surjective. Indeed we had the following short exact sequence coming from our computation of the 1-skeleton:

$$\tilde{K}_1^1(G)(X^1/X^0) \xrightarrow{j^*} \tilde{K}_1^1(G)(X^1) \to 0.$$  \hspace{1cm} (5.22)

Hence,

$$coker(i^*) = coker(d^*) = (R(G) \oplus R(G))/\langle(v, 2b)\rangle$$  \hspace{1cm} (5.23)

Finally, we note that $d^*$ has five copies of $R(G)$ in its kernel, while $j^*$ has three copies of $R(G)$ in its kernel, combined such that two copies of $R(G)$ are left in the kernel of $i^*$.

Putting this all together, we get

$$\tilde{K}_0^0(G(Hom(\mathbb{Z}^2, G))) = coker(i^*) = (R(G) \oplus R(G))/\langle(v, -2)\rangle$$  \hspace{1cm} (5.24)

$$\tilde{K}_1^1(G(Hom(\mathbb{Z}^2, G))) = ker(i^*) = R(G) \oplus R(G).$$  \hspace{1cm} (5.25)

Note that this result matches that of [AG].
5.3 Other cohomology theories of $\text{Hom}(\mathbb{Z}^2, G)$

We repeat the computation, firstly of $T$-equivariant $K$-theory. Secondly we verify that it works out as expected on ordinary integral cohomology, matching the results from [BJS] and [AC]. We then turn to $G$- and $T$-equivariant $K$-theory.

5.3.1 $K^*_T(\text{Hom}(\mathbb{Z}^2, G))$

We can repeat the computation of section 5.2 tersely on $T$-equivariant $K$-theory. Speaking broadly, the interesting parts of our computation are encoded by the various attaching maps between $G/T$ and $G/T$ and we aim to study these maps. In [HJS2] it was shown that, as an $R(T)$-algebra, $K^*_T(G/T)$ is computed to be

$$R(T)[L]/\langle L^2 - (b^{-1} - b)L + 1 \rangle,$$

for $L$ a complex line bundle over $G/T$ described in [HJS2] that, for our purposes, we need merely know is Weyl invariant. Thus on reduced $T$-equivariant $K$-theory, as $R(T)$-modules, we get $K^0_T(G/T) = R(T) \oplus R(T)$ and $K^1_T(G/T) = 0$.

$$\tilde{K}^0_T(X^2/X^1) = \tilde{K}^0_T\left(\bigvee_2 ((S^2(G/T) \sqcup pt))\right) = \tilde{K}^0_T((G/T) \sqcup (G/T) \sqcup pt) = K^0_T((G/T) \sqcup (G/T)) = \bigoplus_2 (R(T) \oplus R(T)) \tag{5.26}$$

$$\tilde{K}^1_T(X^2/X^1) = \tilde{K}^1_T\left(\bigvee_2 ((S^2(G/T) \sqcup pt))\right) = \tilde{K}^1_T((G/T) \sqcup (G/T) \sqcup pt) = K^1_T((G/T) \sqcup (G/T)) = 0 \tag{5.27}$$

The computation for the the 1-skeleton $X^1$ works exactly analogously, only with $R(T)$ everywhere instead of $R(G)$. We won’t repeat the computation explicitly but we nonetheless get in an identical way

$$K^0_T(X^1) = \ker(i^*) = R(T) \tag{5.28}$$

$$K^1_T(X^1) = \coker(i^*) = \bigoplus_2 R(T). \tag{5.29}$$

We thus get the following six term exact sequence on equivariant $K$-theory corre-
sponding to the exact sequence of spaces $X^1 \hookrightarrow X^2 \to X^2/X^1$

$$
\begin{array}{c}
0 \longrightarrow \tilde{K}_T^0(Hom(\mathbb{Z}^2, G)) \longrightarrow \bigoplus_4 R(T) \\
\downarrow \quad \rho \quad \downarrow \\
0 \longrightarrow \tilde{K}_T^1(Hom(\mathbb{Z}^2, G)) \longrightarrow \bigoplus_5 R(T)
\end{array}
\tag{5.30}
$$

To study $i^*$ we again study the map given by the composition

$$
d^*: \tilde{K}_T^1(X^1/X^0) \overset{j^*}{\longrightarrow} \tilde{K}_T^1(X^1) \overset{i^*}{\longrightarrow} \tilde{K}_T^0(X^2/X^1)
\tag{5.31}
$$

where substituting our prior computations for the quotient spaces we get

$$
d^*: K_T^0(\sqcup_4 (G/T)) \to K_T^0(\sqcup_4 (G/T))
\tag{5.32}
$$

The map $d^*$ can be described by the same matrix, only now we have to slow down from sketching what has been, thus far, a completely analogous computation to carefully study the key map $a^* - Id^*$ that appears upon reducing this matrix, much as we did when we worked $G$-equivariantly.

Recalling that we had described $K_G^0(G/T)$ as $R(T)[L]/\langle L^2 - (b^{-1} - b)L + 1 \rangle$, we can write a generic element $p_1 + q_1 b + p_2 L + q_2 bL$ where $p_1, q_1, p_2, q_2$ are polynomials in $v = b + b^{-1}$. The antipodal map acts by taking $b$ to $b^{-1}$ while taking $L$ to itself. Hence $a^*: K_G^0(G/T) \to K_G^0(G/T)$ takes

$$
p_1 + q_1 b + p_2 L + q_2 bL \mapsto p_1 + q_1 b^{-1} + p_2 L + q_2 b^{-1} L
\tag{5.33}
$$

Now we can consider $a^* - Id^*: K_G^0(G/T) \to K_G^0(G/T)$ which is given by

$$
(a^* - Id^*)(p_1 + q_1 b + p_2 L + q_2 bL) = q_1(b - b^{-1}) + q_2(b - b^{-1})L
\tag{5.34}
$$

Now that we understand this key map, we can assemble the rest of the argument as in the $G$-equivariant case to get

$$
\tilde{K}_T^0(Hom(\mathbb{Z}^2, G)) = coker(i^*) = R(T) \oplus (R(T)/\langle b^{-1} - b \rangle)
\tag{5.35}
$$

$$
\tilde{K}_T^1(Hom(\mathbb{Z}^2, G)) = ker(i^*) = R(T) \oplus R(T).
\tag{5.36}
$$

Note that we thus get a torsion term in $T$-equivariant $K$-theory whose corresponding term on $G$-equivariant $K$-theory is not a torsion term. For an alternative way to
view this, we can consider \( (\sum_{i=0}^{\infty} a_i v^i, \sum_{i=0}^{\infty} b_i v^i) \in R(G) \oplus R(G) \) with at most a finite number of nonzero \( a_i, b_i \in \mathbb{Z} \). Under the identification that \( (v, -2) = (0, 0) \), we can thus identify \( (\sum_{i=0}^{\infty} a_i v^i, \sum_{i=0}^{\infty} b_i v^i) = (a_0, \sum_{i=0}^{\infty} (b_i + 2a_{i+1}) v^i) \) so this nontrivial summand in our cokernel can be identified with \( \mathbb{Z} \oplus \mathbb{Z}[v] \). Hence this is not a free \( R(G) \)-module. Yet another representation of a generic element is

\[
\left( \sum_{i=0}^{\infty} a_i v^i, \sum_{i=0}^{\infty} b_i v^i \right) = \left( a_0 + \sum_{i=1}^{\infty} (1/2b_{i-1} + a_i) v^i, 0 \right).
\]

Hence, if we invert the \( 2 \in R(G) \), this becomes a free module. This is analogous to the \( 2 \)-torsion in degree 4 that occurs in the integral cohomology calculation that disappears upon inverting 2.

We note that in this case \( K^*_G(Hom(\mathbb{Z}^2, G)) = (K^*_T(Hom(\mathbb{Z}^2, G)))^W \). Indeed, \( R(T)^W = R(G) \). Further, \( T \)-equivariantly we have a torsion term consisting of \( R(T)/\langle b^{-1} - b \rangle \) which is a polynomial in just the variable \( b \). Hence, upon taking Weyl invariance, only the constant polynomials survive, and so \( (R(T)/\langle b^{-1} - b \rangle)^W = \mathbb{Z} \). This is the same copy of the integers that appears in the \( G \)-equivariant computation, hence the equality after taking Weyl invariants.

### 5.3.2 \( H^*(Hom(\mathbb{Z}^2, G); \mathbb{Z}) \)

We begin on ordinary integral cohomology and verify that it works as expected. Recall that the 1-skeleton consists of \( X^1 = (\bigvee_4 S^3/G) \vee S^1 \). Here we can exploit the fact that nonequivariantly this is homotopy equivalent to \( (\bigvee_4 S^3) \vee S^1 \). The quotient \( X^2/X^1 = \bigvee_2(S^2(G/T) \vee S^2) \) which is topologically \( \bigvee_2(S^4 \vee S^2) \). The long exact sequence on cohomology for \( X^1 \hookrightarrow X^2 \twoheadrightarrow X^2/X^1 \) thus has all zeros in even degree for \( X^1 \) terms and zeros in odd degree for \( X^2/X^1 \) terms. We are left with the following exact sequences of the form:

\[
0 \leftarrow \tilde{H}^{2q}(X^2) \leftarrow \tilde{H}^{2q}(\bigvee_2(S^4 \vee S^2)) \leftarrow \tilde{H}^{2q-1}(\bigvee_4 S^3 \vee S^1) \leftarrow \tilde{H}^{2q-1}(X^2) \leftarrow 0
\]

which by desuspending can be rewritten for \( q \geq 1 \).
and so we aim to compute the kernel and cokernel of $i^*$ as before.

For the rightmost $S^0$ factors in the wedge, this induces on integral cohomology the diagonal map $H^0(pt) = \mathbb{Z} \to H^0(pt) \oplus H^0(pt) = \mathbb{Z} \oplus \mathbb{Z}$. This results in a single factor of $\mathbb{Z}$ coming from the cokernel of this map in degree 2.

For the left-hand factors, first let us consider the antipodal map $a : G/T \to G/T$ thought of now topologically as $a : S^2 \to S^2$. Using the isomorphism $H^2(G/T) = \mathbb{Z}$, this induces the map $a^* : H^2(S^2; \mathbb{Z}) = \mathbb{Z} \to H^2(S^2; \mathbb{Z}) = \mathbb{Z}$ that is just multiplication by $-1$. The map $a^* - Id$ we saw previously is thus multiplication by $-2$. Analogously to section 5.2.2 we denote $x_1, \cdots, x_4$ as generators for each factor in $H^2(\bigvee_4 G/T) = \mathbb{Z}^4$ and denote $y_1, y_2$ as generators for each factor in $H^2(\bigvee_2 S^2) = \mathbb{Z}^2$. We thus get the map from $H^2(\bigvee_2 S^2) \to H^2(\bigvee_4 G/T)$ given by the matrix in this basis:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1
\end{pmatrix}.
$$

As this matrix has kernel $\mathbb{Z}^2$ and cokernel $\mathbb{Z}_2$, putting all of this together gives us

$$
\tilde{H}^q(Hom(\mathbb{Z}^2, G)) = \begin{cases}
\mathbb{Z} & q = 2 \\
\mathbb{Z} \oplus \mathbb{Z} & q = 3 \\
\mathbb{Z}/2 & q = 4 \\
0 & \text{else.}
\end{cases}
$$

Note that this result matches that of [BJS] and [AC].

5.3.3 $H^*_T(Hom(\mathbb{Z}^2, G); \mathbb{Z})$

Now let us consider $H^*_T(Hom(\mathbb{Z}^2, G); \mathbb{Z})$, following from the introduction to equivariant cohomology in section 3.3.

Following the presentation of [HJS2], let $b$ denote the weight 1 one-dimensional
representation of $T$, as before. Let $ar{b} = c^T_1(b) \in H^2_T(pt; \mathbb{Z}) = H^2(BT; \mathbb{Z})$ be the equivariant Chern class of $b$. We thus get $H^*_T(pt; \mathbb{Z}) = \mathbb{Z}[ar{b}]$. The nontrivial element of the Weyl group $W$ acts on $\mathbb{Z}[ar{b}]$ via $w(\bar{b}) = -\bar{b}$.

[HJS2] computed that

$$H^*_T(G/T; \mathbb{Z}) = \mathbb{Z}[\bar{b}] [L]/\langle L^2 - \bar{b}^2 \rangle$$

where $\bar{L}$ is the first Chern class of the bundle $L$ used in section 5.3.1; for our current purposes here it suffices to note that it is a complex line bundle that is Weyl invariant.

In particular, we observe that as a $\mathbb{Z}$ module we get that

$$H^{2q+1}_T(G/T; \mathbb{Z}) = 0$$

and

$$H^{2q}_T(G/T; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

with the factors generated by $\bar{b}^q$ and $\bar{L}\bar{b}^{q-1}$.

Now consider the antipodal map between $G/T$ and $G/T$ which is given by multiplying by the nontrivial element of the Weyl group. This clearly induces the trivial map on odd degrees. Further, $a^*: H^{4q}_G(G/T) = \mathbb{Z} \oplus \mathbb{Z} \rightarrow H^{4q}_G(G/T) = \mathbb{Z} \oplus \mathbb{Z}$ is given by

$$a^*(p\bar{b}^{2q} + q\bar{L}\bar{b}^{2q-1}) = pw \cdot \bar{b}^{2q} + qw \cdot (\bar{L}\bar{b}^{2q-1}) = p\bar{b}^{2q} - q\bar{L}\bar{b}^{2q-1}$$

for $p, q \in \mathbb{Z}$, as the Weyl group acts trivially on $\bar{L}$ and even powers of $\bar{b}$, and by multiplication by -1 on odd powers of $\bar{b}$. Repeating this story in degree $4q + 2$, we get

$$a^*(p\bar{b}^{2q+1} + q\bar{L}\bar{b}^{2q}) = pw \cdot \bar{b}^{2q+1} + qw \cdot (\bar{L}\bar{b}^{2q}) = -p\bar{b}^{2q+1} + q\bar{L}\bar{b}^{2q}.$$ 

Repeating these computations for the map we will really care about, $a^* - Id$, gives us

$$(a^* - Id)(p\bar{b}^{2q} + q\bar{L}\bar{b}^{2q-1}) = -2q\bar{L}\bar{b}^{2q-1}$$

in degree $4q$ and

$$a^*(p\bar{b}^{2q+1} + q\bar{L}\bar{b}^{2q}) = -p\bar{b}^{2q+1}$$
in degree 4q-2. In either even degree case, we thus get a cokernel with a \( \mathbb{Z} \) summand and a \( \mathbb{Z}_2 \) summand, only with the order of the factors differing between the two cases.

Now let us consider \( \text{Hom}(\mathbb{Z}^2, G) \). Following the computation from section 5.2, recall that our 1-skeleton consists of \( X^1 = (\vee_4 SG/T) \vee S^1 \) and our quotient has the form \( X^2/X^1 = \vee_2(S^2(G/T) \vee S^2) \). The long exact sequence on \( T \)-equivariant cohomology for \( X^1 \to X^2 \to X^2/X^1 \) thus has all zeros in even degree for \( X^1 \) terms and zeros in odd degree for \( X^2/X^1 \) terms. We are left with

\[
0 \leftarrow \tilde{H}_T^{2i}(X^2) \leftarrow \tilde{H}_T^{2q}(\bigvee_2(S^2(G/T) \vee S^2)) \overset{i^*}{\leftarrow} \tilde{H}_T^{2q-1}(\bigvee_4 SG/T) \vee S^1 \leftarrow \tilde{H}_T^{2q-1}(X^2) \leftarrow 0
\]

which by desuspending can be rewritten for \( q \geq 1 \) as

\[
0 \leftarrow \tilde{H}_T^{2i}(X^2) \leftarrow \tilde{H}_T^{2q-2}(\bigvee_2((G/T) \vee S^0)) \overset{i^*}{\leftarrow} \tilde{H}_T^{2q-2}(\bigvee_4 G/T) \vee S^0 \leftarrow \tilde{H}_T^{2q-1}(X^2) \leftarrow 0.
\]

We thus need to compute the kernel and cokernel of \( i^* \). For the right hand \( S^0 \) factors in the wedge, we get the diagonal map \( H_T^{2q-2}(pt) = \mathbb{Z} \to H_T^{2q-2}(pt) \oplus H_T^{2q-2}(pt) = (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}). \) This has trivial kernel, but picks up a factor of \( (\mathbb{Z} \oplus \mathbb{Z}) \) coming from the cokernel in all even degrees \( 2q \) for \( q \geq 2 \).

We have previously analyzed the map \( a^*: H_G^{4q}(G/T) = \mathbb{Z} \oplus \mathbb{Z} \to H_G^{4q}(G/T) = \mathbb{Z} \oplus \mathbb{Z}. \) For the left factors in the wedge, for \( q \geq 1 \), and for \( 1 \leq i \leq 4 \), denote by \( x_i \in \tilde{H}_T^{2q-2}((G/T)) \) the image under the isomorphism for a wedge, \( \tilde{H}_T^{2q-2}(\vee_i(G/T)) = \oplus_i \tilde{H}_T^{2q-2}(G/T) \), of a generator for \( \tilde{H}_T^{2q-2}(G/T) \). Here \( i \) denotes the \( i \)th \( G/T \) factor. Likewise \( y_1, y_2 \) be generators coming from the two summands in \( \tilde{H}_T^{2q-2}(G/T) \). The attaching maps work as before, giving us with respect to these bases the matrix

\[
\begin{pmatrix}
Id & Id & Id & Id \\
Id & a^* & Id & a^*
\end{pmatrix}
\]

Recall that we had previously studied the map \( a^* - Id \), which occurs upon reducing the matrix, and observed that the cokernel of this map was \( \mathbb{Z} \oplus \mathbb{Z}_2 \) and the kernel of this map was of two copies of \( \tilde{H}_T^{2q-2}((G/T); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \) in degree \( 2q - 2 \) for \( q \geq 2 \). Putting all of this together gives us
$\tilde{H}_T^0(\text{Hom}(\mathbb{Z}^2, G)) = 0$
$\tilde{H}_T^1(\text{Hom}(\mathbb{Z}^2, G)) = 0$
$\tilde{H}_T^2(\text{Hom}(\mathbb{Z}^2, G)) = \mathbb{Z}$
$\tilde{H}_T^{2q-1}(\text{Hom}(\mathbb{Z}^2, G)) = \oplus_q \mathbb{Z}$ for $q \geq 2$
$\tilde{H}_T^{2q}(\text{Hom}(\mathbb{Z}^2, G)) = (\mathbb{Z}_2 \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z})$ for $q \geq 2$.

5.3.4 $H^*_G(\text{Hom}(\mathbb{Z}^2, G); \mathbb{Z})$

Finally, let us consider the case of $H^*_G(\text{Hom}(\mathbb{Z}^2, G); \mathbb{Z})$. Following [HJS2] again, we denote $\bar{t} := \bar{b}^2$ which lives in degree 4. Then

$$H^*_G(pt; \mathbb{Z}) = H^*(BG; \mathbb{Z}) = (H^*(BT; \mathbb{Z}))^W = \mathbb{Z}[\bar{t}].$$

The arguments of section 3 of [HJS2] while stated rationally all work integrally as well. In particular we have that

$$H^*_G(G/T; \mathbb{Z}) = \mathbb{Z}[\bar{t}][\bar{L}] / (L^2 - \bar{t})$$

where $\bar{L}$ is as before.

In particular, we observe that as a $\mathbb{Z}$ module we get that

$$H^{2q+1}_G(G/T; \mathbb{Z}) = 0$$

and

$$H^{2q}_G(G/T; \mathbb{Z}) = \mathbb{Z}$$

with the factors generated by $\bar{t}^q$ in degrees $4q$ and generated by $\bar{L} \bar{t}^{q-1}$ in degrees $4q-2$.

Since $w(\bar{b}) = -\bar{b}$, $w(\bar{t}) = \bar{t}$. As such, all of $H^*_G(G/T; \mathbb{Z})$ is invariant under the action of $w$. The core map that we will have to study is $a^* - \text{Id} : H^*_G(G/T; \mathbb{Z}) \to H^*_G(G/T; \mathbb{Z})$ is thus just the zero map. Repeating what we did for $H^*_T(\text{Hom}(\mathbb{Z}^2, G))$, we get segments of the long exact sequence corresponding to $X^1 \hookrightarrow X^2 \to X^2/X^1$ that look like

$$0 \hookrightarrow \tilde{H}_G^{2q}(X^2) \hookrightarrow \tilde{H}_G^{2q}(\bigvee_{2} (S^2(G/T) \vee S^2)) \xleftarrow{i^*} \tilde{H}_G^{2q-1}(\bigvee_{4} SG/T) \vee S^1) \hookrightarrow \tilde{H}_G^{2q-1}(X^2) \hookrightarrow 0$$
which by desuspending can be rewritten for $q \geq 1$ as

$$0 \leftarrow \tilde{H}_G^{2q}(X^2) \leftarrow \tilde{H}_G^{2q-2}(\bigvee_2 ((G/T) \vee S^0)) \leftarrow \tilde{H}_G^{2q-2}(\bigvee_4 (G/T) \vee S^0) \leftarrow \tilde{H}_G^{2q-1}(X^2) \leftarrow 0.$$

As before, for the right hand $S^0$ factors in the wedge, we get the diagonal map $H^{2q-2}_G(pt) = \mathbb{Z} \to H^{2q-2}_G(pt) \oplus H^{2q-2}_T(pt) = \mathbb{Z} \oplus \mathbb{Z}$. This has trivial kernel, but picks up a factor of $\mathbb{Z}$ coming from the cokernel in all even degrees $2q$ for $q \geq 1$.

Exactly as for the $T$-equivariant case, we get, for $q \geq 1$, generators $x_1, \ldots, x_4$ coming from $\bigoplus_4 \tilde{H}_G^{2q-2}(G/T)$ and generators $y_1, y_2$ coming from $\bigoplus_2 \tilde{H}_G^{2q-2}(G/T)$. The attaching maps work as before giving us the matrix with respect to these bases

$$
\begin{pmatrix}
Id & Id & Id & Id \\
Id & a^* & Id & a^*
\end{pmatrix},
$$

which reduces to

$$
\begin{pmatrix}
Id & Id & Id & Id \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Because of our earlier analysis that the antipodal map acts via the identity. Since as a $\mathbb{Z}$-module each $\tilde{H}_G^{2q}(G/T)$ is a single factor of $\mathbb{Z}$ in all even degrees, we get three copies of the integers in the kernel, and 1 in cokernel in degree $2q$ for $q \geq 1$.

Putting this all together we get:

$$
\begin{align*}
\tilde{H}_G^0(Hom(\mathbb{Z}^2, G)) &= 0 \\
\tilde{H}_G^1(Hom(\mathbb{Z}^2, G)) &= 0 \\
\tilde{H}_G^2(Hom(\mathbb{Z}^2, G)) &= \mathbb{Z} \\
\tilde{H}_G^{2q-1}(Hom(\mathbb{Z}^2, G)) &= \bigoplus_3 \mathbb{Z} \text{ for } q \geq 2 \\
\tilde{H}_G^{2q}(Hom(\mathbb{Z}^2, G)) &= \bigoplus_2 \mathbb{Z} \text{ for } q \geq 2.
\end{align*}
$$

### 5.4 Algebra structure for $K^*_G(Hom(\mathbb{Z}^2, G))$

Recall our six term exact sequence for the 1-skeleton:
CHAPTER 5. EQUIVARIANT $K$-THEORY COMPUTATIONS

\[
K^0_G(\sqcup_{4\text{pt}}) \leftarrow K^0_G(X^1) \leftarrow 0. \tag{5.37}
\]

Recall from proposition 1 of section 3.1 that the quotient $X^1/X^0$ is a suspension. Hence, both spaces on the left have trivial multiplicative structure and so we get a graded ring homomorphism onto $K^*_G(X^1)$, which thus also has trivial multiplicative structure.

Similarly, we have a six term exact sequence for the 2-skeleton

\[
0 \leftarrow \tilde{K}^0_G(\text{Hom}(\mathbb{Z}^2, G)) \leftarrow \bigoplus_4 R(G) \leftarrow \text{coker} \circ i^* \rightarrow \tilde{K}^1_G(\text{Hom}(\mathbb{Z}^2, G)) \leftarrow \bigoplus_5 R(G) \tag{5.38}
\]

where $\tilde{K}^*_G(\text{Hom}(\mathbb{Z}^2, G))$ is determined by the kernel and cokernel of $i^*$. As computed in section 5.2, this works out, on unreduced, to be

\[
K^0_G(\text{Hom}(\mathbb{Z}^2, G)) = \text{coker}(i^*) = R(G) \oplus (R(G) \oplus R(G)) / \langle (v, -2) \rangle \tag{5.39}
\]

\[
K^1_G(\text{Hom}(\mathbb{Z}^2, G)) = \text{ker}(i^*) = R(G) \oplus R(G). \tag{5.40}
\]

Denoting $K^*_G(X) = K^0_G(X) \oplus K^1_G(X)$ we get

\[
K^*_G(\text{Hom}(\mathbb{Z}^2, G)) = (R(G) \oplus (R(G) \oplus R(G)) / \langle (v, -2) \rangle) \oplus (R(G) \oplus R(G)) \tag{5.41}
\]

Let us denote by $x_1, \cdots, x_5$ five generators for $K^*_G(\text{Hom}(\mathbb{Z}^2, G))$ coming from each of the five $R(G)$ factors. Note that $x_1, x_2, x_3$ come from even degree where $x_4, x_5$ come from odd degree. To determine the $R(G)$-algebra structure we need to know how to multiply these. Observe that, again by Proposition 1 of section 3.1, $X^2/X^1$ is a suspension. Hence, like $X^1$, the space $X^2/X^1$ has trivial multiplicative structure on equivariant $K$-theory. We thus have a surjective graded ring homomorphism $K^0_G(X^2/X^1) \oplus K^1_G(X^1) \twoheadrightarrow K^*_G(\text{Hom}(\mathbb{Z}^2, G))$. The domain has trivial multiplicative structure and hence $K^*_G(\text{Hom}(\mathbb{Z}^2, G))$ also has trivial multiplicative structure.
The only interesting further feature to record comes from the \((R(G) \oplus R(G))/\langle(v, -2)\rangle\) term which imposes the relation \(vx_2 - 2x_3 = 0\) for this choice of generators. Putting this all together, the \(R(G)\)-algebra structure for \(\tilde{K}_G^*(\text{Hom}(\mathbb{Z}^2, G))\) is thus

\[
\tilde{K}_G^*(\text{Hom}(\mathbb{Z}^2, G)) = R(G)[x_1, x_2, x_3, x_4, x_5]/\langle\{x_i x_j = 0 \forall i, j\}, vx_2 - 2x_3\rangle,
\]

(5.42)

where \(x_1, x_2, x_3\) are in even degree and \(x_4, x_5\) are in odd degree.

Likewise, the same argument gives trivial multiplicative structure on the other cohomology theories under consideration.

### 5.5 Module Structure of various other spaces

**Example 1.** \(G/T \times G/T\)

Recall that our \(G\)-CW complex for \((G/T) \times (G/T)\) yields the following spaces

\[
X^0 \cong_G G/T \times D^0
\]

\[
X^2/X^0 \cong_G \text{Thom}(G/T \times D^2) \simeq_G S^2(G/T \sqcup pt)
\]

by Proposition 1 of section 3.1.

Computing on \(\tilde{K}_G^*()\) gives us

\[
\tilde{K}_G^q(X^0) = K_G^q(G/T) = \begin{cases} R(T) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd} \end{cases}
\]

and

\[
\tilde{K}_G^q(X^2/X^0) = \tilde{K}_G^q(S^2((G/T) \sqcup pt)) = \tilde{K}_G^{q+2}((G/T) \sqcup pt)
\]

\[
= K_G^{q+2}(G/T) = \begin{cases} R(G) \oplus R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}
\]

(5.43)
We thus get the following six term exact sequence corresponding to $X^0 \hookrightarrow X^2 \to X^2/X^0$. Note that we do unreduced as $G/T$ does not have a $G$ fixed point.

\[
\begin{array}{cccccc}
R(G) \oplus R(G) & \hookrightarrow & K^0_G(X^2) & \hookrightarrow & R(G) \oplus R(G) . \\
\downarrow & & \downarrow & & \uparrow & \\
0 & \hookrightarrow & K^1_G(X^2) & \hookrightarrow & 0
\end{array}
\]

The sequence splits, and thus we get

\[
\tilde{K}^0_G((G/T \times G)/T) = \oplus_4 R(G) \quad (5.45)
\]

\[
\tilde{K}^1_G((G/T \times G)/T) = 0 \quad (5.46)
\]

Note that computing the analogous exact sequence on ordinary reduced cohomology with integer coefficients for $(G/T) \times (G/T)$, which is topologically $S^2 \times S^2$, gives two copies of $\mathbb{Z}$ in degree 2 and one copy of $\mathbb{Z}$ in degree 4 as expected.

**Example 2. $G \times G$**

Let us first recall Diagram 1 that was used in describing the G-CW complex.

**Diagram 1.** $(G \times G)^1$

\[
\begin{array}{ccc}
\text{(e,e)} & \text{B} & \text{(e,-e)} \\
\text{A} & & \text{C} \\
\text{(-e,e)} & \text{D} & \text{(-e,-e)}
\end{array}
\]

Diagram 1 depicts our 1-skeleton $X^1 \cong_G (G \times G)^1$ which, being the same as the 1-skeleton for $\text{Hom}(\mathbb{Z}^2, G)$, we have already computed in section 5.2. Denoting $G \times G$ by $X$, we note that as per Proposition 1 of Section 3.1

\[
X/X^1 \cong_G \text{Thom}((G/T) \times (G/T) \times D^2) \cong_G S^2((G/T) \times (G/T)) \lor S^2.
\]

We thus get

\[
\tilde{K}^0_G(X^1) = 0
\]

\[
\tilde{K}^1_G(X^1) = \bigoplus_5 R(G)
\]
\[
\tilde{K}^0_G(X/X^1) = \tilde{K}^0_G(S^2(((G/T) \times (G/T)) \cup pt)) = \tilde{K}^0_G((G/T) \times (G/T)) \bigoplus \tilde{K}^0_G(S^0) = \bigoplus_4 R(G)
\]

\[
\tilde{K}^1_G(X/X^1) = \tilde{K}^1_G(S^2(((G/T) \times (G/T)) \cup pt)) = \tilde{K}^1_G((G/T) \times (G/T)) \bigoplus \tilde{K}^1_G(S^0) = 0
\]

We thus get the following six term exact sequence on equivariant \(K\)-theory corresponding to the exact sequence of spaces \(X^1 \hookrightarrow X^2 \twoheadrightarrow X^2/X^1\)

\[
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{K}^0_G(G \times G) & \longrightarrow & \bigoplus_4 R(G) & \\
& & \downarrow i^* & & & \\
0 & \longrightarrow & \tilde{K}^1_G(G \times G) & \longrightarrow & \bigoplus_5 R(G) & \\
\end{array}
\]

(5.47)

To determine \(i^*\), corresponding to the identity map from \(S^1 \rightarrow S^1\) we get an identity between the right-most \(R(G)\) factors. Recall that the attaching map for the 2-cell of the form \((G/T) \times (G/T) \times D^2\) is determined by maps \((G/T) \times (G/T) \rightarrow G/T\) that are projections onto the first factor along the A and C segments of Diagram 1, and projections onto the second factor along the B and D segments of Diagram 1. In \(\tilde{K}^q_G((G/T) \times (G/T))\), there were two copies of \(R(G)\), one for each of the two copies of \(G/T\), and a final copy of \(R(G)\) coming from the \(G/T \times D^2\) cell representing the diagonal elements. If we denote \(x_1, \ldots, x_5\) as generators of the summands in \(\bigoplus_5 R(G)\) and \(y_1, \ldots, y_4\) as generators of the summands in \(\bigoplus_4 R(G)\) then the map \(i^* : \bigoplus_5 R(G) \rightarrow \bigoplus_4 R(G)\) is described in this basis is given by

\[
\begin{pmatrix}
\text{Id} & 0 & \text{Id} & 0 & 0 \\
0 & \text{Id} & 0 & \text{Id} & 0 \\
\text{Id} & \text{Id} & \text{Id} & \text{Id} & 0 \\
0 & 0 & 0 & 0 & \text{Id}
\end{pmatrix}
\]

Thus,

\[
\tilde{K}^0_G(G \times G) = \text{coker}(i^*) = R(G)
\]

\[
\tilde{K}^1_G(G \times G) = \text{ker}(i^*) = R(G) \oplus R(G)
\]

Note that doing the analogous computation on reduced ordinary cohomology gives the same result as that expected when considering the space topologically as \(S^3 \times S^3\);
that is, two copies of $\mathbb{Z}$ in degree 3 and one in degree 6.

**Example 3.** \((G \times G) \backslash \text{Hom}(\mathbb{Z}^2, G)^+\)

Note that $\mathbb{Z}/2$ is the centre of $SU(2)$. Using our $G$-CW complex for \(((G \times G) \backslash \text{Hom}(\mathbb{Z}^2, G))^+\), we compute:

\[
K_q^G(((G \times G) \backslash \text{Hom}(\mathbb{Z}^2, G))^+) = K_q^G(T\text{hom}(G/(\mathbb{Z}/2) \times D^3))
\]

\[
= K_q^G(S^3(G/(\mathbb{Z}/2) \sqcup *)) = K^{q+1}_G(G/(\mathbb{Z}/2) \sqcup *)
\]

Recall that for normal subgroups $H$ of $G$,

\[
K_q^G(G/H) \cong K_q^H(pt) \cong \begin{cases} 
R(H) & \text{if } q \text{ even;} \\
0 & \text{if } q \text{ odd.}
\end{cases}
\]

Thus,

\[
\tilde{K}_q^G(((G \times G) \backslash \text{Hom}(\mathbb{Z}^2, G))^+) \cong \begin{cases} 
0 & \text{if } q \text{ even;} \\
R(\mathbb{Z}/2) & \text{if } q \text{ odd.}
\end{cases}
\]

This computation lets us state a result regarding $i^* : K^q_G(G \times G) \to K^q_G(\text{Hom}(\mathbb{Z}^2, G))$, which is equivalent to Proposition 6.11 of [AG].

**Theorem 1.** Let $i : \text{Hom}(\mathbb{Z}^2, G) \to G^2$ be the inclusion map. Then

\[
i^* : K^1_G(G \times G) \xrightarrow{\cong} K^1_G(\text{Hom}(\mathbb{Z}^2, G))
\]

is an isomorphism and there is a short exact sequence of $R(G)$-modules

\[
0 \to K^0_G(G \times G) \to K^0_G(\text{Hom}(\mathbb{Z}^2, G)) \to R(\mathbb{Z}/2) \to 0
\]

**Proof.** Consider the six term exact sequence corresponding to the pair \((G \times G, \text{Hom}(\mathbb{Z}^2, G))\).

The result follows from the above computation for the noncommuting 2-tuples together with the isomorphism

\[
K^q_G(G \times G, \text{Hom}(\mathbb{Z}^2, G)) = \tilde{K}^q_G(((G \times G) \backslash \text{Hom}(\mathbb{Z}^2, G))^+)
\]

$\square$
5.6 Future Work

There are two immediate generalizations of our present computation of the commuting 2-tuples.

- Commuting \( n \)-tuples in \( SU(2) \). The \( G \)-CW complex is written down in chapter 3, from which \( K_G \) should follow. Heuristically, the important information encoded by our \( G \)-CW complex are various maps between \( G/T \) and \( G/T \); either the identity map or the antipodal map. Hence, analogously to the \( n = 2 \) case, one will get matrices whose components are either the identity map or the antipodal map (or the zero map). A closed form solution may not be possible in the general case, but at least it should be able to be programmed into a computer for arbitrary \( n \). The relevant information is encoded in the cellular maps \( K_G^{q+1}(X^k/X^{k-1}) \to K_G^q(X^{k+1}/X^k) \) which is given by an \( 2^n \binom{n}{k} \times 2^n \binom{n}{k+1} \) matrix (by the results of section 4.3 that works by analogy with the \( k \)-faces of an \( n \)-cube). The entries of this matrix consist of either the identity map or the antipodal map. As the attaching maps are done analogously to how one attaches the canonical CW structure for an \( n \)-cube, a convenient description of the entries of these matrices should be able to be determined. The \( n = 2 \) case we did explicitly is also aided by the fact that we have only one nontrivial step \( X^2/X^1 \hookrightarrow S(X^1) \) to write out an exact sequence for; for higher \( n \) there will need to be multiple steps computing the skeleta in turn, and the earlier skeleta will not be free.

- Commuting \( m \)-tuples (or just 2-tuples) in \( SU(n) \). For \( SU(n) \) the maximal torus is a torus of dimension \( n - 1 \) opposed to only being dimension 1 as it was in our \( SU(2) \) case. This adds some complexity, but nonetheless constructing a \( G \)-CW complex on \( \text{Hom}(\mathbb{Z}^m, SU(n)) \) from which one can compute various cohomology theories seems promising.
Chapter 6

Quasi-Hamiltonian Systems

In this chapter we outline the key ideas regarding quasi-Hamiltonian systems. In section 6.1, we will introduce the concept of quasi-Hamiltonian systems and note that the spaces we have been computing throughout this thesis are motivated, in part, by the spaces in a particular example of a quasi-Hamiltonian system. In section 6.2, we outline the results in the literature about Kirwan surjectivity. In section 6.3, we discuss some potential areas for future work.

6.1 Quasi-Hamiltonian Systems

Alekseev, Malkin, and Meinrenken [AMM] introduced quasi-Hamiltonian $G$-spaces which are certain finite-dimensional $G$-spaces $M$ equipped with a $G$-valued quasi-Hamiltonian moment map $\Phi : M \to G$. Following their presentation, for a compact Lie group $G$, we make a choice of an invariant positive definite inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$. Let $\theta, \bar{\theta} \in \Omega^1(G, \mathfrak{g})$ denote the left- and right-invariant Maurer-Cartan forms, respectively. Let $\chi = \frac{1}{12} \langle [\theta, \bar{\theta}], \xi \rangle$ denote the canonical bi-invariant 3-form on $G$. Finally, let $\nu_\xi$ denote the generating vector field on $M$ for an element $\xi \in \mathfrak{g}$. Then a quasi-Hamiltonian $G$-space is a $G$-manifold $M$ with an invariant 2-form $\omega \in \Omega(M)^G$ and a $G$-map $\phi \in C^\infty(M, G)^G$ such that

1. $d\omega = -\phi^* \chi$
2. $i(\nu_\xi) \omega = \frac{1}{2} \phi^* \langle \theta + \bar{\theta}, \xi \rangle$
3. for each $x \in M$, $\ker \omega_x = \{ \nu_\xi, \xi \in \ker(Ad_{\phi(x)} + 1) \}$

These finite dimensional quasi-Hamiltonian $G$-spaces are in bijective correspondence with certain infinite dimensional Hamiltonian $L_s G$-spaces $\mathcal{M}$ that fit into the
following diagram where $\mathcal{M}$ is a pullback of the bottom right square and the columns are fibrations with fibre the space $\Omega G$. The space $\Phi^{-1}(e)/G$ is referred to as the symplectic quotient $M//G$.

\[
\begin{array}{ccc}
\Omega G & \xrightarrow{=} & \Omega G \\
\Phi^{-1}(0) & \xrightarrow{\Phi} & L\mathfrak{g}^* \\
\Phi^{-1}(e) & \xrightarrow{\Phi} & G
\end{array}
\]

where $Hol$ is the holonomy map to the path space $PG$ composed with the end-point evaluation map.

Consider the following specific example of a quasi-Hamiltonian $G$-space. Let $G = SU(2)$ which acts on itself by conjugation. The space $M = (G \times G)^h$ with the diagonal action and equipped with the $G$-map $\Phi : (G \times G)^h \rightarrow G$ given by taking the commutator map $\Phi(a_1, \ldots, a_h, b_1, \ldots, b_h) = \prod_{i=1}^{h} [a_i, b_i]$ provides an example of a quasi-Hamiltonian $G$-space. The corresponding symplectic quotient $M//L_sG$ is the moduli space of flat connections on the trivial principal $G$-bundle over a compact, connected 2-manifold $\Sigma$ of genus $h$ with boundary $\partial \Sigma = S^1$ (See [AMM]). In particular, we will consider the $h = 1$ case which has kernel $\Phi^{-1}(e) = Hom(\mathbb{Z}^2, G)$. For this specific case we relabel parts of the diagram:

\[
\begin{array}{ccc}
\Omega G & \xrightarrow{=} & \Omega G \\
Hom(\mathbb{Z}^2, G) & \xrightarrow{\Phi} & L\mathfrak{g}^* \\
\Phi^{-1}(e) & \xrightarrow{\Phi} & G \times G
\end{array}
\]

While this chapter is near the end of this thesis, a key motivation for our work is to understand this diagram on equivariant $K$-theory. In so doing, while the principal computation is the equivariant $K$-theory of the symplectic quotient $Hom(\mathbb{Z}^2, G)$, we have also matched [BZ]'s result for $G$, and we remind the reader of the computation in [AG] of $G \times G$ and the work on $\Omega G$ done by [HJS1].
6.2 Kirwan Surjectivity

We collect in this section an overview of the original Kirwan surjectivity and its various analogs.

Kirwan [Kir] showed that for a compact Lie group $G$ and a compact Hamiltonian $G$-space $M$ with proper moment map $\mu : M \to \mathfrak{g}^*$, the function $\|\mu\|^2 : M \to \mathbb{R}$ is a $G$-equivariantly perfect Morse function on $M$. It follows that the inclusion $\mu^{-1}(0) \hookrightarrow M$ induces a surjection on rational $G$-equivariant cohomology

$$H^*_G(M; \mathbb{Q}) \twoheadrightarrow H^*_G(\mu^{-1}(0); \mathbb{Q}). \quad (6.3)$$

When $0$ is a regular value of $\mu$, $\mu^{-1}(0)/G$ is a symplectic orbifold called the symplectic quotient $M//G$. $G$ then acts locally freely on $\mu^{-1}(0)$ and so we have a surjection called the Kirwan map

$$\kappa : H^*_G(M; \mathbb{Q}) \twoheadrightarrow H^*(M//G; \mathbb{Q}). \quad (6.4)$$

Harada and Landweber [HL] extended this result to $G$-equivariant $K$-theory

$$K^*_G(M) \twoheadrightarrow K^*_G(\mu^{-1}(0)). \quad (6.5)$$

If the $G$ action is free, the second space is isomorphic to $K^*(M//G)$.

There is an infinite-dimensional analogue of the above situation which results in analogous surjectivity theorems. The compact group $G$ is replaced with the Banach Lie group $L_sG := \{\gamma : S^1 \to G \mid \gamma \text{ is of Sobolev class } s\}$ for $s > 3/2$. A Hamiltonian $L_sG$-space consists of an infinite-dimensional symplectic Banach manifold $M$ together with an $L_sG$-action and moment map $\tilde{\Phi} : M \to L_s\mathfrak{g}^*$.

Bott, Tolman, and Weitsman [BTW] proved that the inclusion $\tilde{\Phi}^{-1}(0) \hookrightarrow M$, for $0$ a regular value, induces a surjection on rational $G$-equivariant cohomology:

$$H^*_G(M; \mathbb{Q}) \twoheadrightarrow H^*_G(\tilde{\Phi}^{-1}(0); \mathbb{Q}) \cong H^*(M//L_sG; \mathbb{Q}). \quad (6.6)$$

Harada and Selick [HS] proved the $G$-equivariant $K$-theory analog

$$K^*_G(M) \twoheadrightarrow K^*_G(\tilde{\Phi}^{-1}(0)) \cong K^*(M//L_sG), \quad (6.7)$$

where the latter isomorphism holds when $G$ acts freely.
The third setting to be considered is that of the quasi-Hamiltonian $G$-spaces introduced by Alekseev, Malkin, and Meinrenken [AMM] which are certain finite-dimensional $G$-spaces equipped with a $G$-valued quasi-Hamiltonian moment map $\Phi : M \to G$. These finite dimensional quasi-Hamiltonian $G$-spaces are in bijective correspondence with certain infinite dimensional Hamiltonian $L_sG$-spaces $M$ that fit into the following diagram where $M$ is a pullback of the bottom right square and the columns are fibrations with fibre $\Omega G$.

\[
\begin{array}{ccc}
\Omega G & \to & \Omega G \\
\downarrow & & \downarrow \\
\Phi^{-1}(0) & \to & L^*G \\
\downarrow & & \downarrow \\
\Phi^{-1}(e) & \to & G \\
\end{array}
\] (6.8)

Indeed, this correspondence is instrumental in [BTW] and [HS]'s proofs of surjectivity in the Hamiltonian $L_sG$-space situation for rational $G$-equivariant cohomology and $G$-equivariant $K$-theory, respectively.

In the quasi-Hamiltonian case, the inclusion $\Phi^{-1}(e) \hookrightarrow M$ fails to necessarily induce surjections on either rational $G$-equivariant cohomology or $G$-equivariant $K$-theory. Nonetheless, [BTW] proved a similar result in a way that depended on the surjectivity for Hamiltonian $L_sG$-spaces. Loosely, they proved that $H^*_G(\Phi^{-1}(e))$ is generated as a ring by the image of $H^*_G(M) \to H^*_G(\Phi^{-1}(e))$ together with classes that originate in the fibre.

### 6.3 Relating to future work

For $G = SU(2)$, one of our motivations for studying $\text{Hom}(\mathbb{Z}^2, G)$ was to understand this example in the larger context of quasi-Hamiltonian $G$-spaces. We have computed the equivariant $K$-theory of the symplectic quotient, the base $G \times G$, and the fibre $\Omega G$ for our example, but we have not computed this for the infinite dimensional space $M$. We note a few facts regarding $M$. The space is defined as a pullback. If we restrict the base to just commuting pairs in $G \times G$ then we get the map $(G \times G)|_{\text{Hom}(\mathbb{Z}^2, G)} \to G$ via
the commutator map, and the pullback for this restriction is a product as all such pairs
map to e. However, M itself is not a product topologically which is unfortunate as it
prevents us from easily writing down a G-CW complex from our knowledge of the
base and the fibre. Note, however, that it is indistinguishable on rational cohomology
from a product because the commutator map factors through the smash product which
is topologically $S^3 \wedge S^3 \cong S^6 \rightarrow S^3$ and $\pi_6(S^3) = \mathbb{Z}_{12}$.

As such, an immediate future goal is to attempt to resolve this difficulty and be
able to describe M in our example more fully, or perhaps, at least, the low dimensional
skeleta of it. [BTW] and [HS] provide a filtration of this space where it is proven that the
surjectivity of the inclusion $\phi^{-1}(0) \hookrightarrow M$ induces a surjection on rational cohomology
and equivariant K-theory, respectively. This surjection is true for the inclusion into
each space in the filtration, and thus even understanding the first of these filtration
components would be an advantage.

One immediate generalization from our present situation is to study the example
of a quasi-Hamiltonian system where $M = SU(2)^{2n}$ with the moment map being a
product of commutators as before. This has the symplectic quotient the moduli space
of flat connections on a closed 2-manifold $\Sigma$ of genus $n$. We have done the $n = 1$
case. Note that this does not have commuting $n$-tuples in $SU(2)$ as the symplectic quotient.
The $G$-CW complex for $M$ is simple enough and of course $\Omega G$ remains as the fibre, but
work is needed to establish the $G$-CW structure on the symplectic quotient in this case.

While the above has all been specific to $G = SU(2)$ we now consider $G$ a compact
Lie group. An important long term goal is to extend this body of Kirwan surjectivity
results to the quasi-Hamiltonian setting on G-equivariant K-theory. For our specific
element, $\phi^{-1}(e) \hookrightarrow M$ does not induce a surjection on equivariant $K$-theory. However,
we conjecture that a $K$-theoretic analog of [BTW]’s result in rational equivariant coho-
mology for the quasi-Hamiltonian case holds true. Specifically, from [HS] the inclusion
$\Phi^{-1}(0) \hookrightarrow M$ induces a surjection on $G$-equivariant $K$-theory. As $\Phi^{-1}(0) = \Phi^{-1}(e)$,
some set of generators of $K^*_G(M)$ are needed to generate $K^*_G(\Phi^{-1}(e))$. On cohomology,
the result of [BTW] uses cochains and the Leray-Hirsch theorem, unavailable to us on
equivariant $K$-theory. Nonetheless, it is conjectured that an analog can be stated on
$G$-equivariant $K$-theory. Here, one source is the image $\pi^*(K^*_G(M)) \subset K^*_G(M)$. Unlike
the Hamiltonian $G$-space and Hamiltonian $L_GG$-space situations, this source alone is
insufficient to ensure surjectivity. Additionally, it is conjectured that one needs genera-
tors originating in an appropriate sense from the fibre $\Omega G$. 
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