Essays on Financial Market Structure

by

David A. Cimon

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Graduate Department of Economics
University of Toronto

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Abstract

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Graduate Department of Economics
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In this thesis, I model three innovations in modern financial markets. First, I study conflict of interest in the relation between brokers and investors. Second, I study Exchange Traded Funds, and their impact on their constituent assets. Finally, I study crowdfunding as a means for entrepreneurs to resolve uncertainty regarding demand for their projects.

Many investors do not access equity markets directly; instead they rely on a broker who receives their order and submits it to a trading venue. Brokers face a conflict of interest when the commissions they receive from investors differ from the costs imposed by different trading venues. Investors want their orders to be filled with the highest probability, while brokers choose venues in order to maximize their own profits. I construct a model of limit order trading in which brokers serve as an agent for investors who wish to access equity markets.

In just over 20 years, exchange traded funds (ETFs) have gone from a new financial innovation to an industry representing over $1.3 trillion CAD in assets under management. With this rapid rise in popularity, questions naturally arise as to whether ETFs affect the markets for the underlying assets from which they are formed. In this chapter I present a static model of informed limit order book trading in which market participants trade in either cash markets or basket securities ETFs.

Since its advent less than 10 years ago, crowdfunding has grown to a multi-billion dollar industry. There has been debate over whether crowdfunding has competed with or complemented traditional financing methods such as banks and venture capital. One feature of crowdfunding, is a shifting of the risk from the project from the traditional venues, to these consumers themselves. This chapter examines the role of crowdfunding in the financing process for entrepreneurs, specifically in regards to the resolution of uncertainty regarding demand for their projects.
To my parents.
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Chapter 1

Broker Routing Decisions in Limit Order Markets

Many investors, both retail clients as well as large institutions, do not access equity markets directly. Instead, they delegate the decision of which venue to trade on to their broker. In principle, the broker’s and client’s interest are aligned, as the broker earns the commission only if the client’s order executes. However, brokers may prefer to route to a venue which maximizes their own profit, rather than the venue which best serves their client. When testifying before the United States Senate Committee on Homeland Security & Governmental Affairs, Robert Battalio raised the concern that brokers may be maximizing their intakes of rebates, paid to them by trading venues in exchange for order flow, rather than obtaining the best execution for their clients. In fact, concerns about order routing by brokers were raised as early as 2000 by the SEC, when they outlined proposed rule changes mandating disclosure of order routing practices.

The primary focus of this chapter is to study the routing decisions that maximize brokers’ profits. Specifically, how these profit-driven routing decisions affect trader welfare and market quality. Trading fees are one distinguishing feature of trading venues which may drive these decisions by brokers. Recently, many venues have switched to “maker-taker” pricing frameworks, where traders are given a rebate if they supply liquidity, offset by a higher fee by those demanding liquidity. There also exist “inverse” or “taker-maker” exchanges, where traders demanding liquidity are provided a rebate, while those who supply pay a higher fee. Investors often pay their brokers a flat commission per trade, while the broker earns any rebates, or


3Orders either supply liquidity, by specifying a price and remaining available to future traders, or demand liquidity, by removing existing orders at the best price available. Liquidity supplying orders are generally referred to as limit orders, while orders demanding liquidity are referred to as market orders.
pays any trading fees charged by the venue. When clients delegate the choice of venue to their brokerm, a conflict of interest can arise from the presence of these trading fees.

To study the effects of broker routing decisions, I construct a two period model of limit order book trading, in which investors leave the routing decision to their broker and pay him a flat commission upon execution. Brokers have the choice of two possible venues for routing orders. The venues trade a single security at fixed price levels and are each modelled based on the market in Foucault, Moinas, and Theissen (2007). Unlike the previous paper, these venues are differentiated by trading fees for making and taking liquidity. I assume that exchanges have the same net fee, defined as the taker fee plus the maker rebate. Thus, one exchange will have a lower taker fee, while the other will have a higher maker rebate.

In the first period, an uninformed investor arrives and maximizes her expected profit by choosing whether or not to submit a limit order to her broker. If the broker receives an order, he routes it to the venue which will maximize his expected profit. This is defined by the difference between the commission he charges the investor, and the trading fees charged by the exchange. In the second period, either an innovation in the security value occurs and an informed trader arrives to remove mispriced limit orders from all markets or a liquidity trader arrives. If the liquidity trader arrives, he submits an order to his broker who again routes it to maximize his own expected profit.

In equilibrium, brokers route marketable orders to exchanges with lower taker fees, increasing the fill rate at these venues and lowering the risk of adverse selection for limit orders posted there. For limit orders, when brokers charge a fixed commission I find that unless the fees levied by the exchanges are very similar, brokers will route to the exchange with a higher maker rebate and lower execution probability for their clients. Intuitively, this follows from the broker’s profit maximization problem. When fees are similar, the broker benefits from the increased execution probability at the exchange with lesser rebates while incurring a low opportunity cost, as the rebate is only slightly higher at the alternative exchange. Conversely, when the fee structures are very different, the exchange with the higher rebate offers the broker sufficient profit upon filling the order to compensate for the lower execution probability.

Decision making by brokers impacts both their clients and trading venues as a whole. Preferential routing of uninformed market orders to exchanges with lower taker fees lowers the adverse selection costs at these venues. Further, due to a lower concentration of informed trading, I find that the expected value of the trade, conditional on execution, improves for limit orders executed at these exchanges. In this case, the broker’s decision on where to route uninformed market orders directly affects the market conditions for their limit order clients. Since informed traders are equally willing to trade at all exchanges, exchanges favoured by brokers for their uninformed market orders have improved fill rates.

I find that a number of factors improve for investors if fill rates, rather than rebates, drive brokers’ limit order routing decisions. Specifically, more investors will choose to submit limit order, each order will face a lower expected adverse selection cost, and order execution will
occur with a higher probability. Intuitively, this also follows from the improvement of market quality from an increase in the number of uninformed market orders. When brokers route their limit order investors based on fill rates, they route to the same exchanges where they route their uninformed market order clients. When market quality is improved, the expected value of a submitted limit order increases, making these attractive to a larger subset of possible investors. In environments where exchanges have very different fee structures, I show that by raising broker commissions, brokers will switch to routing based primarily on fill rates. When commissions rise, a broker’s interests become increasingly aligned with those of his clients, as the profit from rebates becomes relatively smaller and the higher fill rate also brings an increased probability of earning the commission.

I present two extensions on the model which simplify assumptions made in the primary equilibrium. In the first, I endogenize the commission structure and show that when commissions must be incentive compatible for brokers, there is a loss to clients compared to a case where they pay their own trading fees. In the second, I allow for multiple price levels and endogenous market makers. I find that bid-ask spreads at one exchange can be affected by increases or decreases in spreads at other exchanges. In addition, I find that limit order investor welfare increases when maker rebates are low, as market makers are no longer subsidized for providing liquidity.

1.0.1 Existing Literature

Existing work on maker-taker pricing can be divided into three groups. First is the work which focuses on the incentives for investors. Colliard and Foucault (2012) and Brolley and Malinova (2013) study maker-taker pricing regimes and their effects on investors. Colliard and Foucault (2012) construct a model of a frictionless market, and study the breakdown of the total exchange fee between maker and taker fees. They find that only the total fee has an effect on investor outcomes, and that the breakdown between maker and taker fees has no impact on investor decisions or gains from trade. Brolley and Malinova (2013) construct an alternative model, where investors pay only a flat fee to a broker, who then passes the order on to a single exchange. They find that market orders sent to maker-taker exchanges are subsidized by investors who submit limit orders, when these investors pay a flat fee to their brokers. Empirically, several papers have found that exchanges with a maker-taker structure often have a better spread posted. (Malinova & Park, 2014; Anand, McCormick, & Serban, 2013) However, exchanges with either a maker-taker structure or a higher taker fee have also been found to have a higher concentration of informed trading. (Yim & Brzezinski, 2012; Anand et al., 2013)

The second group is a body of work which focusses on the incentives to exchanges. Theoretical work by Pagnotta and Philippon (2011) focusses the competition between exchanges based on speed. They find that this competition may be beneficial insofar as it increases trading speeds but that with endogenous exchange entry welfare may be lowered. In relation to the present chapter, work on endogenous exchange trading fees by Chao, Yao, and Ye (2015)
focusses on the profitability of exchanges when they are constrained by fixed tick sizes. They argue that, when tick sizes are fixed, the use of varied fee structures by exchanges is welfare improving for market participants.

The final group of work focusses on the incentives for brokers who select the trading venue, rather than investors who submit their orders directly. Empirical work by Boehmer, Jennings, and Wei (2007) focusses on the incentives to brokers created by the introduction of execution quality transparency requirements imposed by SEC Rule 11Ac1-5 (now Rule 605). They find that competition for order flow among broker-dealers drives orders to venues with high fill rates but also that many orders do continue to be routed to venues with low execution quality. The closest work to this chapter is that of Battalio, Corwin, and Jennings (2014) who empirically study problem of the broker-client conflict of interest from trading fees. They find that brokers often make routing decisions based on the presence of liquidity rebates, rather than in the best interests of their clients. Further, they find that clients typically face higher adverse selection costs at exchanges with higher liquidity rebates. The present chapter provides theoretical confirmation of these two empirical results. First, I find that exchanges with higher liquidity rebates will endogenously have worse fill rates, and that a higher percentage of filled orders at the exchanges will be from informed traders. Second, I find that if trading fees are sufficiently different, brokers will route primarily based on rebates, rather than based on fill rates for their clients.

Significant regulatory attention has also been paid to trading fees and routing of investor orders by brokers. Since 2001, the SEC has required brokers to make details available regarding their routing practice through Rule 606.4 Further, the SEC requires brokers to disclose to investors, the routing details of their specific orders upon request, as well as statistics related to execution quality. Regulators have also taken an interest in Canada, where the Ontario Securities Commission (OSC) published proposed regulatory changes, which included a pilot study on prohibitions of maker-taker pricing structures, and disclosure of broker routing practices.5

The remainder of the chapter proceeds as follows. In Section 1.1, I describe the model. In Section 1.2, I describe the benchmark equilibrium results of this model with a single price level at the bid and the ask, exogenous market making and brokers who pass fees through to their clients. In Section 1.3, I present a model with a fixed commission. In Section 1.4, I present a model with endogenous commissions. In Section 1.5, I present an extended model with multiple price levels and endogenous market makers. In section 1.6 I present additional discussion and in Section 1.7 I conclude.

1.1 Model

The model borrows components from the fixed-tick model in Foucault et al. (2007), as well as from trading fee models in Colliard and Foucault (2012) and Brolley and Malinova (2013).
1.1.1 Security

There is a single security, which starts \( t = 1 \) with a value of \( v \) and ends \( t = 2 \) with a value of \( V \). With probability \( \delta \) an innovation in the security value occurs, raising or lowering the value of the security with equal probability by amount \( \sigma \), while with probability \( 1 - \delta \) no change occurs.

The security trades on two exchanges at fixed price ticks of size \( \Delta \). The prices on each exchange are identical, with possible prices at the ask given by \( v + x\Delta \), and possible prices at the bid given by \( v - x\Delta \). The different prices in the grid are represented by \( x \), an exogenous variable which can only take the form of integers.\(^6\) There is room for a single order of quantity \( Q_1 = 1 \) (a buy limit order) or \( Q_1 = -1 \) (a sell limit order) at each tick on each exchange. Each exchange charges fees \( M_i \) and \( T_i \) to traders for orders making liquidity (limit orders) and taking liquidity (market orders) respectively. Both exchanges have the same total cost per order: \( M_i + T_i = e \) \( \forall i \). I make three simplifying assumptions on the parameter set.

**Assumption 1.1:** The total cost per order, \( M_i + T_i \), is set to zero. \( (e = 0) \)

**Assumption 1.2:** Exchange 2 charges a higher taker fee. \( (T_1 < T_2) \)

**Assumption 1.3:** The prices and fees are such that, if an innovation occurs, the value of the security falls outside the grid of prices plus fees at the first tick. \( (v < v + \Delta + T_i < v + \sigma \) and \( v < v + \Delta < v + \sigma) \)

Assumption 1.1 implies that the exchanges have zero marginal cost of processing a trade and do not earn any excess profit from traders. Relaxing this assumption creates a spread between the maker and taker fees at all exchanges. Assumption 1.2 simplifies the solution for market orders, and implies that the broker will preferentially route market orders to exchange 1, easing interpretation of results. Assumption 1.3 ensures that that all orders at the closest tick are profitable for the informed trader and removes the case where taker fees are sufficiently high at one market that informed traders will not trade when the security value changes. Further, it ensures that all orders at the closest tick are subject to adverse selection if an innovation occurs.

1.1.2 Market Participants and Timing

The first period is the liquidity supply period, in which agents post limit orders. The second period is the liquidity demand period, in which agents submit market orders.

i. **Limit Order Investors.** A utility-maximizing limit order investor arrives at the market at the beginning of period \( t = 1 \). She is unable to interact directly with the market and must instead post her order through a broker. Investors are risk neutral, do not discount, and gain utility only in relation to the security being traded.

Analogously to Parlour (1998), a limit order investor arrives with a desire to buy or sell with equal probabilities in the form of a quantity signal \( Q_1 \in \{-1, 1\} \). As in Parlour (1998)

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\(^6\)Examples of feasible prices at the ask include \( v + \Delta \) and \( v + 2\Delta \), where \( x = 1 \) and \( x = 2 \) respectively.
and Foucault (1999), she also receives a private value $y \in Y$, where $Y$ is a uniform distribution centred on zero.

She may submit a limit order for quantity $Q_1$, at a given price $x\Delta$, which will remain in the book until the end of $t = 2$. Orders for $Q_1 = 1$ are at the prices $v - x\Delta$ while orders for $Q_1 = -1$ are at $v + x\Delta$. She pays the broker a commission $c$ upon successful execution of an order.

When submitting an order to a broker, she considers three factors. (1) To which exchange the broker will route her limit order; (2) To which exchange the broker will send market orders; (3) Which price levels market makers will post at. Combining these three, she determines $\theta_i(x\Delta)$, the probability that her order will execute at a given price $x\Delta$ if routed to exchange $i$.

A limit order investor who submits a limit buy order at price $-x\Delta$ has expected utility:

$$U_{LO} = \theta_i(-x\Delta) (E[V|Ex_i] + y + x\Delta - c)$$

where $Ex_i$ represents an order being routed to exchange $i$ and executed in $t = 2$.

**ii. Brokers.** Risk neutral, uninformed, profit maximizing brokers exist in order to provide market access for investors. One group of brokers routes limit orders during $t = 1$, while a second routes market orders during $t = 2$. The brokers receive orders from investors, and route them to one of the trading venues.

When routing orders, the brokers must follow three rules: (1) brokers must accept and place all orders; (2) brokers may route limit orders to any venue, at the price specified by the order; and, (3) brokers must give market orders the best price available.

Brokers incur all costs from the exchanges to which they route orders to ($M_i$ for limit orders and $T_i$ for market orders at exchange $i$). In turn, they profit from the difference between these costs and the commission which they charge clients per order executed, $c$.

For orders routed to exchange $i$, the broker’s profits for market order and limit orders are given by:

$$\pi_{MO} = [c - T_i] \quad \pi_{LO} = \theta_i(x\Delta)[c - M_i]$$

Since under assumption 1.2, $T_1 < T_2$, if limit orders are available at the same price at both exchanges, brokers will always preferentially route market orders to exchange 1. However, when prices are not equal, they must route market orders to the exchange with the best price.

**iii. Market Makers.** Uninformed market makers provide liquidity on each exchange. The market makers arrive immediately after the investor’s order is routed in $t = 1$, and are able to place further orders. Market makers do not go through the broker and instead, pay fees directly to the exchange.

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7In this setup, the broker’s only costs are assumed to come from the exchange fees themselves, while in reality brokers incur several types of costs executing client orders. In an alternative specification, the commission paid by clients and the commission received by the broker can be set to separate values to represent order processing costs. In this case, the role of exchange fees becomes much more prominent, as the profit from the commission alone becomes much lower and can explain how a commission which may appear an order of magnitude larger than an exchange fee may not be the driving factor for routing decisions.
This figure illustrates the timing of this model. Investors and market makers provide liquidity in \( t = 1 \), while liquidity demand takes place in \( t = 2 \). With probability \( \delta \) the liquidity demander is an informed trader, while with probability \( 1 - \delta \) he is uninformed.

Market makers view the order, if any, placed by the limit order investor, and immediately have the option to place orders. They do so at any tick which, given the expected behaviour of all other agents, gives a positive expected value. For buy limit orders, this is any price \( xA \) such that:

\[
E[V|Ex_i] \geq -xA + M_i
\]  

iv. Informed Traders. If an innovation occurs, an informed trader arrives at the market and views the current innovation. This trader has direct access to the market, uses market orders, and pays the fees associated with taking liquidity \( (T_i) \). This trader presents an adverse selection risk to limit order traders who have already placed orders, similar to Glosten and Milgrom (1985), Easley and O’Hara (1987), Glosten (1994) and others.

If the innovation is positive, such that \( V = v + \sigma \), the trader immediately submits market orders for the mispriced orders at both markets at price \( v + \Delta \). If the innovation is negative, such that \( V = v - \sigma \), all orders at price \( v - \Delta \) are filled.

v. Liquidity Traders. If no innovation occurs, a liquidity demanding investor arrives with a quantity signal, distributed evenly over \( Q_2 \in \{-2, -1, 1, 2\} \). He immediately submits market orders for the total amount of his desired quantity. This order is routed by the broker, and the trades execute.\(^8\)

### 1.2 Benchmark Equilibrium

The benchmark equilibrium presented in this model contains 3 simplifying assumptions. First, I assume that brokers charge a commission of \( c = 0 \), pass all fees on to their limit order clients and route according to their clients’ preferences.\(^9\) Second, I allow for only one tick at

\(^8\)The inclusion of these traders removes the no-trade equilibrium as detailed in Milgrom and Stokey (1982).

\(^9\)One assumption which remains unchanged, is that brokers continue to route market orders. Market order traders are assumed to be random and they have no incentive to choose one exchange or another. Brokers continue
the ask \((v + \Delta)\), and one at the bid \((v - \Delta)\). Finally, I assume market makers post exogenously at all empty ticks, following the placement of the order from the limit order investor. The first assumption is relaxed in sections 1.3 and 1.4, while the second and third assumptions are relaxed in section 1.5. Formally, these assumptions are:

**Assumption 1.4:** Brokers charge a commission \(c = 0\), pass all fees incurred \((M_i)\) to their limit order clients, and obey all routing instructions from their limit order clients.

**Assumption 1.5:** There is only one price available at the ask \((v + \Delta)\), and one at the bid \((v - \Delta)\).

**Assumption 1.6:** Following the routing of the limit order from the investor, market makers exogenously place orders at all empty ticks.

These equilibrium is analogous to existing models of limit order submission with exchange fees and fixed tick sizes. The brokers are effectively invisible in the limit order submission process, and costlessly carry out their clients’ directions. This will serve as a benchmark for equilibria in which brokers are active in routing their clients’ order flow.

In this model an equilibrium consists a solution to the limit order investor’s utility maximization problem. This consists of a decision as to whether or not submit an order and to which exchange to route this order for every quantity \(Q_1\) and private signal \(y\).

**Theorem 1.1** (Existence of a Threshold Equilibrium). (1) For fixed parameters \(M_1, M_2, \delta, \sigma\), there exists a unique threshold private value \(\overline{y}_1\), such that for all \(y \geq \overline{y}_1\) limit order investors with \(Q_1 = 1\) will choose to submit a limit buy order and route it to exchange 1.

(2) For fixed parameters \(M_1, M_2, \delta, \sigma\), there exists a unique threshold private value \(\overline{y}_2\). If \(\overline{y}_1 \leq \overline{y}_2\) then all limit order investors prefer being routed to exchange 1. If \(\overline{y}_1 > \overline{y}_2\) then \(\overline{y}_2\) is such that all limit order investors with \(\overline{y}_1 > y \geq \overline{y}_2\) and \(Q_1 = 1\) will choose to submit a limit buy order and route it to exchange 2. Otherwise if \(y < \overline{y}_2\), limit order buyers will abstain.

### 1.2.1 Market Order Routing Decisions

Brokers will route market orders in order to maximize:

\[
\pi_{MO} = c - T_i
\]

(1.4)

Under Assumption 1.2, brokers route to exchange 1 first, since \(T_1 < T_2\) and prices are equal at both exchanges. As there is only a single order available at any given tick, brokers split market orders of size \(|Q_2| > 1\) across both venues.

Market order routing has direct consequences for limit orders. All else equal, brokers will route market orders to the exchange with the lower liquidity taking fee. This decision increases execution probabilities for limit orders at this exchange. Further, because informed orders are to route market orders to exchange 1 prior to exchange 2, due to a lower liquidity taking fees. In principle, were liquidity traders given any utility function increasing in wealth, their incentives would be perfectly aligned with their brokers.
always large, both exchanges receive the same absolute quantity of informed orders, but different quantities of uninformed orders. As a result, the expected value for the security, conditional on execution, will be different across the two exchanges. Put differently, the price impact of market orders will differ across the different venues.

**Proposition 1.1** (Market Order Routing and Limit Order Execution Probability). The execution probability for limit orders at exchange 1, the low taker fee exchange, is always higher than on exchange 2 ($\theta_1 > \theta_2$).

Proposition 1.1 results directly from the preferential routing of market orders to the venue with the lower taker fee. Since only large orders will be sent to both markets, limit orders posted at the low taker fee market necessarily have a higher fill rate. The difference in quantity of market orders sent to each exchange provides the driving force behind the principal-agent problem between brokers and their clients in Section 1.3 onwards. When routing limit orders, brokers will always have the choice between one exchange with a higher execution probability for their clients (exchange 1) and an exchange with a lower fee for them (exchange 2).

Arguably, marketable orders from retail traders comprise only a small portion of total order flow. However, retail order flow is especially important in the context of adverse selection risk, as it generally contains less information content. Therefore, any exchange which receives a higher volume of retail orders compared to other orders likely has lower information costs for limit orders posted there.

One issue with Proposition 1.1, is the implication that the total volume of trading will be much higher at exchanges with an inverse fee structure while, in practice, the majority of trading volume remains concentrated at maker-taker markets. Within the simplified model, this stems from the fact that prices are equal at both markets, while in reality this is not the case. One way to resolve this issue is through the introduction of endogenous market making and multiple price levels, as in Section 1.5. Specifically, if market makers are able to post more aggressively at maker-taker exchanges, volume will be higher.

**Proposition 1.2** (Market Order Routing and Expected Security Value). For limit buy orders, the expected value of the security conditional on execution, is higher if it is routed to exchange 1, than if it is routed to exchange 2 ($E[V|Ex_1] > E[V|Ex_2]$). The result is reversed for limit order sells.

Proposition 1.2 results from the probability of informed trading being higher at exchanges with maker fees that are more attractive for the broker. Since informed traders are willing to remove mis-priced orders from all exchanges regardless of fee structure, exchanges with a lower number of retail orders face relatively higher adverse selection costs. This result supports the conclusions of empirical work from Anand et al. (2013) and Battalio et al. (2014), who find that exchanges with a maker-taker structure have a greater concentration of informed order flow.
1.2.2 Limit Order Investor’s Problem

A limit order investor views her quantity signal $Q_1$, her private value $y$, and anticipates the brokers’ market order routing strategies. The limit order investor’s decision includes the fee pass through by the broker, altering her utility function to the following:

$$U_{LO} = \theta_i \left( E[V|Ex_i] + y - (v - \Delta) - M_i \right)$$

(1.5)

Proposition 1.3 (Market Preference with Fee Pass-Through). When exchange fees are passed on by the brokers, if $M_1 - M_2 \leq \frac{1 - \delta}{1 + \delta} \delta$, then $\bar{y}_2 \geq \bar{y}_1$ and all limit order investors prefer being routed to exchange 1. Otherwise, there may exist some investors who prefer to be routed to exchange 2.

The implication of Proposition 1.3 is that if the difference in the expected value of the security between the exchanges is larger than the difference in exchange fees, all investors will have the same preference to be routed to the same exchange.

Alternatively, if $M_1 - M_2 < \frac{1 - \delta}{1 + \delta} \delta$ the difference in fees passed on to the brokers dominates the difference in security value. If this occurs, depending on the complete distribution of the private value $y$ there may be a preference for some or all investors to be routed to exchange 2. These investors would prefer to pay a lower exchange fee and accept a lower execution probability and expected security value.

Of note, the investors who would prefer to be routed to exchange 2 are those with the lowest private value. While the private value $y$ can have several possible interpretations, the simplest interpretation of Proposition 1.3 is that if investors optimally split orders across exchanges, those with the lowest external value of the security accept worse fill rates and worse execution quality.

1.3 Single Commission

In the benchmark model, brokers are assumed to pass fees through to their clients and route according to their clients’ wishes. There are multiple possible manners in which this commission can be modelled, either by a single broker or by multiple brokers.\(^{10}\) The first is for an arbitrary single commission $c > 0$, where brokers are unable to credibly commit to a routing scheme and must route according to an incentive compatibility constraint. In order to study the incentive compatibility problem faced by brokers with fixed commissions, I assume brokers exogenously charge a commission $c$ to their clients. This involves the relaxation of Assumption 1.4 and the introduction of Assumption 1.7.

\(^{10}\)I do not deal with the commission for market orders as market order traders are random in this model. Therefore, the price setting by the broker for market orders would be arbitrary as the market order traders are not endowed with a utility function. Further, execution at all exchanges is equal for market order traders as limit order investors and market makers are both uninformed.
Assumption 1.7: The broker commission for limit orders is set exogenously such that $c > |T_i|, |M_i|$. In this model an equilibrium consists of: (1) A solution to the broker’s profit maximization problem, and; (2) a solution to the limit order investor’s utility maximization problem. The solution to the broker’s problem consists of decisions on where to route limit orders from clients. The solution to the limit order investor’s problem consists of a decision as to whether or not submit an order, for every quantity $Q_1$, private signal $y$, and broker routing strategy.

Theorem 1.2 (Existence of a Threshold Equilibrium II). (1) For fixed parameters $M_2, \delta, \sigma, c$, there exists a unique threshold maker fee $\overline{M}_1$, such that if $M_1 \leq \overline{M}_1$, brokers will optimally route limit orders to exchange 1. Otherwise, brokers will route limit orders to exchange 2. (2) For fixed parameters $M_2, \delta, \sigma, c$, there exists a unique threshold private value $\overline{y}_i$, for each exchange $i$, such that for all $y \geq \overline{y}_i$ limit order investors with $Q_1 = 1$ will choose to submit a limit buy order. Otherwise if $y < \overline{y}_i$, limit order buyers will abstain.

### 1.3.1 Broker’s Problem

As in the base model, brokers route market order first to exchange 1, where the taker fee is lower, and subsequently to exchange 2. Unlike the initial model, brokers now choose the exchange for limit orders based on their own profit maximization decision. When routing limit orders, brokers choose exchange $i$ in order to maximize:

$$\pi_{LO} = \theta_i [c - M_i]$$

(1.6)

Exchange 1 receives more market orders, and therefore a higher execution probability, as it has a lower liquidity taking fee. For limit orders, brokers will weigh the trade-off between a higher execution probability at exchange 1 and a lower maker fee at exchange 2. This trade-off, should it arise, is at the heart of the broker-client conflict of interest. If one exchange has maker fees which are sufficiently low compared to others, brokers may route there even if the execution probability is low.

### 1.3.2 Limit Order Investor’s Problem

A limit order investor views her quantity signal $Q_1$, her private value $y$, the broker price $c$ and anticipates the brokers’ routing strategies. Using these, she decides whether or not to submit a limit order. Given that her order will be routed to exchange $i$, an investor wishing to buy does so if:

$$\theta_i (E[V|Ex_i] + y - (v - \Delta) - c) > 0$$

(1.7)
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This figure illustrates the equilibrium actions and pay-offs for limit order buyers, and their brokers. The equilibrium is determined through backwards induction. Given the expected pay-offs, the broker will choose to route limit orders to exchange $i$. Given the broker’s routing decision, the limit order investor will submit an order if her private value is above $\gamma_i$. Actions and pay-offs for limit order sellers are symmetric.

As described in Theorem 1.2, in equilibrium there exists a value $\gamma_i$, such that any limit order buyer with a sufficiently large private value $y$ will optimally choose to submit a limit order, while those with a private value below this cut-off will abstain.

**Proposition 1.4 (Limit Order Submission Decision).** *Investors require less favourable private valuations to submit an order if their order will be routed to exchange 1, than if it will be routed to exchange 2 ($\gamma_1 < \gamma_2$).*

Proposition 1.4 describes the difference in decision making behaviour of limit order investors based on broker routing. Limit order traders will optimally choose to submit an order if their expected utility, given routing and execution, is positive. As both the probability of execution and expected value given execution are higher if routed to exchange 1, more traders are willing to submit if their order will be routed there. This is contrary to the fee pass-through model where $\gamma_1 \leq \gamma_2$, depending on exchange fees and adverse selection. With fixed commissions, all investors prefer to be routed to exchange 1, rather than exchange 2.

The decrease in trader volume when routing for rebates also suggests a dilemma for brokers. If they were able to commit to routing to exchange 1, despite a lower profit per order, a larger number of clients would choose to submit orders. Depending on the parametrization, this can lead to a larger expected profit for brokers and the preferred routing scheme for their clients. The increase in the number of clients when orders are routed to exchanges with higher fill rates suggests an important role for routing disclosure by brokers, which may ultimately serve as a commitment mechanism to their clients.\(^{11}\)

\(^{11}\)An example is SEC Rule 606, which mandates the partial disclosure of routing information for non-directed orders by brokers. Information related to this is available online at: [http://www.ecfr.gov/cgi-bin/text-idx](http://www.ecfr.gov/cgi-bin/text-idx)
By combining Propositions 1.2 and 1.4, I am able to make a statement on limit order investor welfare.

**Proposition 1.5** (Limit Order Investor Welfare). *If the two exchanges are sufficiently similar* ($M_1 \leq \bar{M}_1$), *more limit orders will be submitted by investors and each investor will be better off in expectation than if the two exchanges are sufficiently different* ($M_1 > \bar{M}_1$). *Therefore, the expected welfare of limit order investors is higher when exchanges have similar fee structures.*

For simplicity, I define welfare in a strictly utilitarian sense, where total investor welfare is simply the sum of utility of each individual investor. However, in Proposition 1.5, not only does total welfare increase, but every individual investor also has higher expected utility. The increase in utility corresponds to a first-order stochastic dominance relationship, in which the distribution of investor utilities with two similar markets, first order stochastic dominates the distribution of utilities under two very different markets.

### 1.3.3 Comparative Statics

An increase in the probability that an information event occurs ($\delta$) reflects an increase in an uninformed trader’s risk of being adversely selected. An increased probability of an information event can represent several possible scenarios, examples include a period of market turmoil, dates where announcements are expected, or securities which have higher inherent levels of risk.

**Proposition 1.6** (Adverse Selection and Limit Order Investors). *(1)* When adverse selection rises, limit order investors receive lower expected values for their trades ($\frac{\partial E[V | E_x]}{\partial \delta} < 0$ for buyers), fewer investors are willing to submit orders ($\frac{\partial y_i}{\partial \delta} > 0$ for buyers), and the total welfare of investors declines. The reverse is true when adverse selection falls.

*(2)* When adverse selection rises, exchanges must be increasingly similar for brokers to route based on fill rates rather than rebates ($\frac{\partial M_1}{\partial \delta} < 0$). The reverse is true when adverse selection falls.

Proposition 1.6 demonstrates the two negative effects of adverse selection on limit order investors. First, as the probability of trading against an informed trader increases, the expected value of the trade is worse for limit order investors. The lower expected value lowers the number of limit order investors willing to submit orders to their brokers.

Second, during times when more information is reaching the markets, such as during periods of announcements, brokers are more likely to route to based on rebates. The increase in the probability of informed trading causes the total probability of execution to converge across all exchanges and incentivizes brokers to concentrate their orders at exchanges with high maker rebates. Specifically, the equilibrium value $M_1$ falls, meaning that the maximum fee for which brokers will route limit orders to exchange 1 is higher. If the actual maker fee at exchange 1
is now above this new value, the broker will alter his routing habits and divert limit orders to exchange 2. This exchange has an even higher adverse selection risk to investors, increasing the impact of the adverse selection increase. Brokers increase their clients’ risk of being adversely selected during periods when adverse selection is highest.

**Proposition 1.7** (Increase in Broker Commission). There exists a threshold $c$, such that for all $c \geq c$ brokers route limit orders to exchange 1. If $c < c$, brokers route limit orders to exchange 2.

Proposition 1.7 reflects one of the basic ideas in principal-agent problems. Given a sufficient incentive, here in terms of a higher commission, the brokers’ interests will be aligned with their clients interest. In conjunction with Proposition 1.2, this implies that, in some cases a higher commission may increase welfare for both brokers and their clients. Though clients generally prefer paying a lower commission, it implies that when the commission is too low, they may suffer from a conflict of interest.

**Proposition 1.8** (Broker Commissions and Welfare). Consider a market in which brokers charge their clients $c < c$. If the following condition holds:

$$
\bar{c} - c \leq [V|Ex_1] - E[V|Ex_2]
$$

an increase in the commission from $c$ to $\bar{c}$, increases the expected profit of the broker and the expected utility of all investors.

Proposition 1.8 describes when brokers and their clients can be made better off by an increase in commissions. The increase in commission causes the broker to begin routing to exchange 1 which lowers the adverse selection costs to investors. If Equation 1.8 is satisfied, the increase in commission is entirely offset by the decrease in adverse selection costs.

There is a second interpretation to the broker commission, which stems from the costs incurred by the broker. In this model, the costs to brokers of processing an order are set to zero. In a situation where the broker is able to reduce his costs, either through more efficient internal behaviour or through lower external costs, this is effectively equal to an increase in the commission he receives per order, without an increase in the commission each client pays. In this case, a decrease in non-trading fee costs to the broker increases his profit from the commission upon execution, and may also cause him to optimally route to the exchange with a higher fill rate.

### 1.4 Competition Between Brokers

The second commission structure I consider in this chapter is that of competition between identical brokers who charge their clients a fixed price and do not pass through fees. One practical element of the brokerage market is that many brokers offer multiple tiers of services
to clients with varying willingness to pay. While some clients may wish to pay a minimum amount for discount brokerage services, the results of this chapter thusfar have shown that some investors may prefer to pay more in order to control their order routing.

1.4.1 Brokers

I model the broker market as a closed market with $N$ existing brokers. Each of these $N$ brokers is able to offer up to two commission prices, $c_1$ for orders being routed to exchange 1, and $c_2$ for orders being routed to exchange 2.

As in the single commission model, these commissions must be incentive compatible, such that an order at $c_1$ has a higher expected profit when routed to exchange 1 than exchange 2. If multiple brokers offer the same price, orders from investors are evenly distributed among them.

**Proposition 1.9 (Competition Between Brokers).** For $N \geq 2$ brokers, there exists an equilibrium such that:

1. $N$ brokers charge $c_1 = c$ for routing services to exchange 1.
2. $n$ brokers charge $c_2 = M_2$ for routing services to exchange 2, where $2 \leq n \leq N$.

Proposition 1.9 represents an equilibrium which is similar in some respects to a Bertrand equilibrium. As in a Bertrand equilibrium, the prices are not driven any lower if additional brokers beyond the first 2 are included. Contrary to a Bertrand equilibrium, the equilibrium in Proposition 1.9 is not guaranteed to be unique, and the lower possible prices are not equal to marginal cost. This second difference arises from the incentive compatibility problem in the brokerage market.

Given that the model is a single instance game, there is no place for investors to learn about brokers. In this competitive environment, the only signal about a broker available to investors is their posted price. Consider a price at exchange 1, equal to the marginal cost ($c_1 = M_1$). Any order routed to exchange 1 earns zero profit, however if the broker shirks his responsibility and routes to exchange 2, he has the potential of earning $M_1 - M_2 > 0$.

In that sense, a competitive price for routing to exchange 1 cannot be less than the incentive compatible commission $c$. Any broker who charges less than this price cannot credibly demonstrate their routing intentions. This problem does not exist for routing to exchange 2 where $c_2 = M_2$ is an incentive compatible outcome, since routing to exchange 1 would earn the broker $M_2 - M_1 < 0$.

**Proposition 1.10 (Loss From Incentive Compatibility).** There exists a fraction of traders who would be willing to trade under a fee pass-through at exchange 1, but under incentive compatible commissions will only trade at exchange 2. This fraction is characterized by the value $L = \frac{1 + \delta}{1 - \delta} (M_1 - M_2)$. These traders are worse off under incentive compatible commissions.

Proposition 1.10 describes the second issue arising from the need for incentive compatibility, that of a decreased volume of clients trading at exchange 1. The difference between the incentive
compatible price $c$ and the fee pass-through value $M_1$ causes a direct decrease in the proportion of investors willing to trade at exchange 1. Exchange 1 has improved execution probability and security value for investors and by going to exchange 2, they are worse off. Since the value $c$ is characterized by both the exchange fees and adverse selection values, so too is this loss of clients. In line with the other results of the model, this loss increases both in the difference between market fees and in the value of adverse selection.

This loss represents the minimum loss incurred by investors if commissions must be incentive compatible. Any other commission structure with $c > c$ which follows the incentive compatibility constraints of the broker will incur a larger loss to investors. It is also notable that this does not represent a simple transfer between brokers and investors, but a group of investors who are strictly worse off through lower executions probabilities and reduced gains from trade.

1.5 Extension: Endogenous Market Making

In the benchmark model, the price grid at each side of the market was of size one. In this extension, the prices on each exchange remain identical. However, I allow for two ticks at the ask $(v + \Delta, v + 2\Delta)$, and two at the bid $(v - \Delta, v - 2\Delta)$ with room for a single order of quantity $Q_1 = 1$ (a buy limit order) or $Q_1 = -1$ (a sell limit order) at each tick on each exchange. Further, I allow market makers to choose whether or not to post at each available price level. This involves the relaxing of Assumptions 1.5 and 1.6, and the introduction of Assumptions 1.8 and 1.9:

**Assumption 1.8:** There are two prices available at the ask $(v + \Delta, v + 2\Delta)$, and two at the bid $(v - \Delta, v - 2\Delta)$.

**Assumption 1.9:** The grid of prices is such that $\Delta + T_i < \sigma < 2\Delta + T_i$ and $\sigma < 2\Delta - M_i$.

Assumption 1.9 ensures that orders placed at the farther tick are not adversely selected against. Informed traders will only pick off orders at the closest tick and market makers will always be willing to post at the farther ticks. Similar to the Section 1.3, I maintain Assumption 1.7, that the commission $c$ is set exogenously by the broker.

1.5.1 Equilibrium

In the extended model an equilibrium consists of: (1) A decision by market makers to post, or not, at every empty tick at each exchange; (2) A solution to the broker’s profit maximization problem, and; (3) A solution to the limit order investor’s utility maximization problem.

**Theorem 1.3** (Existence of a Threshold Equilibrium III). (1) For fixed parameters $M_1, M_2, \delta, \sigma, c$ there exists a unique market making plan at exchange 1 $\hat{M}_1$, such that market makers will choose to post at prices $\pm \Delta$ at exchange 1 if $M_1 \leq \hat{M}_1$. Otherwise if $M_1 > \hat{M}_1$ they will only post at $\pm 2\Delta$ at exchange 1. For fixed parameters $M_1, M_2, \delta, \sigma, c$ and market making plan $\hat{M}_1$, there exists a market making plan at exchange 2 $\hat{M}_2$, such that market makers will choose to post at
prices $\pm \Delta$ at exchange 2 if $M_2 \leq \tilde{M}_2$. Otherwise if $M_2 > \tilde{M}_2$, they will only post at $\pm 2\Delta$ at exchange 2.

(2) For each market making plan and fixed parameters $M_2, \delta, \sigma, c$, there exist unique threshold maker fees $\tilde{M}_1(\Delta), \tilde{M}_1(2\Delta)$ such that if $M_1(x\Delta) \leq \tilde{M}_1(x\Delta)$ brokers will optimally route limit orders at prices $x\Delta$ to exchange 1. Otherwise brokers will route limit orders at price $x\Delta$ to exchange 2.

(3) For each market making plan and fixed parameters $M_2, \delta, \sigma, c$, there exist unique threshold private values $\tilde{y}_i(\Delta), \tilde{y}_i(2\Delta)$, for each exchange $i$, such that investors with $Q_1 = 1$, will submit an order at price $-\Delta$ if $y \geq \tilde{y}_i(\Delta)$ and at price $-2\Delta$ if $\tilde{y}_i(\Delta) > y \geq \tilde{y}_i(2\Delta)$. Otherwise, if $y < \tilde{y}_i(2\Delta)$, limit order buyers will abstain.

i. Market Makers. Given the expected routing of market orders, market makers choose whether or not to post at each empty tick in order to solve their profit maximization problem. Since there are many market makers, they individually choose whether each tick they may post at is profitable. Market maker behaviour depends, in particular, on the difference in fees between the two exchanges. Market makers are aware that brokers will preferentially route market orders to the exchange with lower taker fees, lowering the chance of being adversely selected at these exchanges.

In equilibrium, there are four possible cases for market making, which will be referred to throughout the remainder of the section. Market makers are always willing to post at the far ticks at both exchanges, and therefore the cases are defined by their willingness to post at the narrow ticks. (1) Market makers post at all ticks at both exchanges; (2) Market makers only post at the narrow tick of the high maker rebate exchange; (3) Market makers only post at the narrow tick of the high fill rate exchange; (4) Market makers only post at the far ticks.

These market making cases are defined by fee thresholds $\tilde{M}_1, \tilde{M}_2$. The fee thresholds exist such that, if $M_i < \tilde{M}_i$, market makers post at the narrow ticks at exchange $i$

**Proposition 1.11** (Market Making Behaviour). (1) If $M_1$ increases from $M_1 \leq \tilde{M}_1$ to $M_1' > \tilde{M}_1$: (i) Market makers will no longer post at prices $\pm \Delta$ at exchange 1; (ii) $\tilde{M}_2$ rises and market makers will begin to post at prices $\pm \Delta$ at exchange 2, if they had not already been doing so.

(2) If $M_2$ increases from $M_2 \leq \tilde{M}_2$ to $M_2' > \tilde{M}_2$ market makers will no longer post at prices $\pm \Delta$ at exchange 2.

Proposition 1.11 describes the results shown in Figure 1.3. Exchange 1 receives a larger proportion of uninformed market orders, and the market makers’ decision to post at this exchange affects the execution probability at exchange 2. This is best seen on the transition from Case 3 to Case 2 in Figure 1.3. The rise in the fee at exchange 1 to $M_1 > \tilde{M}_1$ causes the market makers to cease posting orders at the narrow price levels at exchange 1. This increases the number of uninformed orders that reach exchange 2, and decreases $\tilde{M}_2$ such that market makers will optimally post at the narrow price levels.

These results are driven by the principle of order protection for market orders and provide policy insight. Specifically, the change in fee structure at one exchange may influence the
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Figure 1.3: Types of Market Maker Equilibria

This figure represents the possible equilibria for market makers.

Case 1: Market makers are willing to post at the narrowest ticks in both exchanges, $\Delta$. Case 2: Market makers are willing to post at the narrow tick at exchange 2, $\Delta$, but only at the farther $2\Delta$ in exchange 1. Case 3: Market makers are willing to post at the narrow tick at exchange 1, $\Delta$, but only at the farther $2\Delta$ in exchange 2. Case 4: Market makers are only willing to post at the farthest ticks in both exchanges, $2\Delta$.

Spreads at the competing exchanges. This can occur even if the changes in exchanges’ fees do not change the overall ranking of exchanges by magnitude of fees. In the case presented in Figure 1.3, exchange 1 becomes a taker-maker exchange and market makers are no longer willing to post at the best. Order protection moves market orders to exchange 2, because it quotes a narrower spread.

The behaviour of market makers drives the execution probability at both exchanges in this extension. If the trader chooses to submit at the narrowest tick, there is a constant risk that when the security value undergoes an innovation, the order will be picked off. This occurs with probability $\frac{1}{2}\delta$. Given the optimal market order routing strategy, the total execution probability $\theta_i(x\Delta)$ is a function of the ticks that market makers are willing to post at. Table B.1 in Appendix B gives the execution probabilities for limit orders, at each ask price, given market maker behaviour.

**Proposition 1.12** (Volume with Endogenous Market Making). (1) If market makers post more aggressively at the exchange 2, brokers will always route limit orders at the best to exchange 2. (2) If market makers post more aggressively at exchange 2, total volume will be higher at exchange 2.

One prediction of the base model is that volume will be higher at inverse exchanges, which is not seen in reality. In this extension, exchange fees may allow for a better spread, and therefore a higher volume, at maker taker exchanges. Further, if exchange 2 is the only venue where
market makers post at the best, brokers will send any limit orders at the best to this exchange. Since their orders will not be covered by market makers posting at exchange 1, this exchange will have both a higher execution probability for the client, and a better rebate for the broker.

ii. Limit Order Investors. The presence of endogenous market makers complicates the decision of limit order investors, as their presence affects the execution probability of their limit order, seen in Table B.1 in Appendix B.

![Figure 1.4: Limit Order Investor Expected Utility](image)

This figure represents the expected utility of the limit order investor, under varying exchange fees. Investors have higher expected utility values when exchanges are similar and when exchange fees force market makers to post wider spreads.

Figure 1.4 illustrates two separate effects. The first effect occurs when the trading fees are similar at both venues. This region corresponds to where $M_1(\Delta) \leq \tilde{M}_1(\Delta)$ and $M_1(2\Delta) \leq \tilde{M}_1(2\Delta)$. In this region, brokers optimally route orders to the exchange with the higher fill rate, rather than the higher maker rebate (or lower maker fee), and investor welfare is higher. The second effect occurs on the lower border, where maker fees are high. In these regions investors are better off since market makers no longer post at the narrowest ticks at one, or both venues.

**Proposition 1.13 (Limit Order Investor Welfare in the Extended Model).** (1) If $M_1$ increases from $M_1 < \tilde{M}_1$ to $M_1' > \tilde{M}_1$, limit order investor utility increases. (2) If $M_2$ increases from $M_2 \leq \tilde{M}_2$ to $M_2' > \tilde{M}_2$, limit order investor utility increases.

The increase in investor utility described in Proposition 1.13 comes from two sources. First, when $M_1$ increases such that market makers no longer post at the narrow ticks at exchange 1, the proportion of uninformed orders increases at exchange 2. There is an increase in both the expected value of limit orders routed to exchange 2 and the proportion of investors willing to submit an order. Second, when either $M_1$ or $M_2$ increases, such that market makers no
longer post at the narrow ticks of one exchange, liquidity declines at the best. Some investors who were previously unwilling to submit orders will then choose to submit orders at the wider price levels ($\pm 2\Delta$), as these orders now have a meaningful probability of execution, increasing welfare.

Rebates allow market makers to post more aggressively, and in turn give them a competitive advantage against limit order investors, who don’t receive rebates. It is important to note, that narrower spreads are advantageous to many market participants, notably those trading with market orders. Therefore, while limit order investor utility does increase when market makers choose not to compete at tight spreads, the outcomes for other investors whose utility is not addressed by this model may decline.

1.6 Discussion

The model presented in this chapter is a static model where agents are limited in their roles and whose actions stylized. There are several issues within the brokerage market that may warrant further discussion. This section discusses four such issues in an informal manner, namely those of queuing, commission magnitude, changes to exchange fees and the independence of market participants.\(^{12}\)

1.6.1 Queuing

During trading, multiple limit orders at the same price level are allowed to queue at any given exchange venue to await execution. These operate on the basis of price-time priority, where orders at a better price are executed first, and orders at the same price are executed in order of arrival. The model presented in this chapter simplifies the trading process by preventing multiple orders at the same price level from queuing at a single exchange. As only two agents place limit orders, and market makers are not the primary focus of the model, the removal of this limitation ultimately requires larger changes to the model in order to create an interpretative difference.\(^{13}\)

The primary change to the model when queuing occurs is that the order of the queue depends on which exchange one’s order is placed. Given the optimal decision making of brokers with respect to market orders, one can create a form of “unified queue” across exchanges, where limit orders are sorted first in ascending order of the liquidity taking fee at the exchange on which they are placed, and second in terms of time of arrival.

In the context of this chapter, orders placed at exchange 1, can only have their place in the queue decrease over time. As market orders are sent to exchange 1 with priority (due to

\(^{12}\)Special thanks is required for my discussants, Cheng Zhang at TADC 2015, Larry Glosten and Charles Jones at the SFS Cavalcade and the panel at the Doctoral Consortium of ERIC 2015 for highlighting many of the issues discussed in this section.

\(^{13}\)One simple option to create a need for queueing is the arrival of multiple limit order investors, possibly based on a random process, between each liquidity taking market participant.
the lower taker fee), any limit orders queued at exchange 1 are executed prior to market orders being routed to exchange 2 at the same price. On the contrary, limit orders placed at exchange 2 may have their place in the queue increase or decrease. Consider the case where any order is at the top of the book at exchange 2. If no orders are currently placed at exchange 1, then it remains the first limit order in the queue. However, if a limit order is placed at exchange 1, the order at exchange 2 is now the second order in terms of market order priority.

Of course, both these cases imply that all participants routing orders do so based on exchange fees alone. While it may be optimal for participants to remove equally priced orders at exchange 1 first, in reality this is not always the case, and the “unified queue” across exchanges may be much more fluid than modelling would lead one to believe. It is quite possible that certain brokers or other participants with direct access may prefer one exchange to another, and an order than has time priority at exchange 2 may be executed prior to a similarly priced order at exchange 1.

Regardless, both brokers and limit order investors need to take this unified queue into account when making strategic decisions. In the case of brokers, their routing decision for any given limit order should depend on both the current queue at all applicable exchanges as well as any orders which may be routed ahead in the larger “unified queue”. In that sense, when few orders are available at a price, a broker may have a higher incentive to route for rebates, while when many orders are already on the book across several exchanges, they may route in favour of their clients.

1.6.2 Commission Magnitude

A second abstraction within the model is that of the cost to brokers of providing brokerage services to their clients. Initially, the magnitude of the difference between exchange fees (generally under $0.005 per share), and the commission charged by the broker (often between $0.01 and $0.05 per share for retail clients) may lead one to believe that, for reasonable parametrizations, exchange fees are of little consequence.

In practice, a revealed preference argument based on empirical work (such as Battalio et al. (2014)) suggests that these fees must be relevant. This holds both because brokers are shown to take them into account and because exchanges feel obliged to offer them. In addition, an intuitive theoretical answer is that brokers incur marginal costs (such at settlement costs) other than exchange fees. Any cost that drives the fee paid by the client and profit for the broker apart also increases the importance of exchange fees for broker profitability.

In that sense, in any competitive environment with additional costs to the broker, the difference between the commission and exchange fees should be entirely accounted for by the additional costs of providing brokerage services. In the case of competition between brokers (as in section 1.4), the difference needed in exchange fees for brokers to route in their own interests is much lower between the difference in exchange fees and commissions.
1.6.3 Multiple Exchanges and Fee Changes

In a two exchange model, there is a simple pecking order for market orders between the two. In a practical sense, brokers have the choice of dozens of trading venues, many of which may offer the best price. Exchanges face a trade-off when selecting a fee structure, between incentivizing liquidity and their place in the pecking order. Those exchanges with lower maker fees (or high rebates) may be able to routinely offer a better price, as market makers are able to post more aggressively. However, of those exchanges who do offer the best price, market orders will be prioritized to those with lowest taker fee (and highest maker fee). Therefore, simply setting a large liquidity making rebate may not be enough to drive volume to an exchange, if other exchanges are able to offer the same price. In the extreme case, if many exchanges offer a maker-taker fee structure, those with the highest maker rebate may find themselves uncompetitive, as they could find the volume of uninformed orders insufficient to maintain the best price, despite a large rebate.

When an exchange alters its fee structure, it alters its place in the pecking order of exchanges. If it lowers its taker fee below that of other exchanges, it moves its limit orders ahead in the pecking order. If it is able to do this without making its market makers sufficiently uncompetitive, this increases volume and lowers adverse selection costs at this exchange, while lowering volume and increasing adverse selection costs at those exchanges whom now have higher comparable taker fees. This comes at a risk however, as too low a taker fee (and consequently too high a maker fee), risks handicapping market makers to the point where the exchange is no longer able to regularly offer the best and loses considerable volume.

Similarly, if an exchange is considering lowering its maker fee (raising its taker fee), one of two effects can occur. If this does not enable market makers at this exchange to routinely offer a better price, the exchange would expect to see lower volume and higher adverse selections, as other exchanges will now be ahead in the pecking order. If, on the other hand, this allows market makers to post more aggressively, the exchange may see increased volume.

1.6.4 Independence of Trading Venues

In this chapter, the trading venues are both independent, uninformed entities. All exchanges accept both types of orders without prejudice, all market makers are uninformed, and exchanges do not receive a differing number of informed orders. In reality, many exchanges are not so neutral. Execution services, dark pools and other specialized exchanges may not conform to the simple model of adverse selection presented in this chapter, and may have fill rates that are much more dependent on adverse selection than an open exchange.

In the case of execution services, which are often run by large brokerages, the preference of a broker to route to this service may have more to do with the information available to the broker than any objective ranking of the fee structure may allow. If this were not the case, a broker’s decision to route orders may in itself be a function of private information. In this case,
a higher execution probability (and preferred fee structure for the broker) may not indicate a lower probability of adverse selection, and may be harmful to the client.

1.7 Conclusion

The principal-agent relationship between brokers and their clients is one which has the potential to impact both the individuals involved and markets as a whole. While existing theoretical literature has addressed the concept of exchange fees, specifically maker-taker pricing, a gap remains in explaining how these fees and rebates drive broker behaviour and affect their clients. In order to explain these impacts I construct a static model of limit order trading in which brokers route limit orders from their clients to one of two venues.

I show that in an environment with fixed price levels, rebates for making and taking liquidity are able to drive broker routing decisions for both limit and market orders. These routing decisions, in turn, effect both the fill rates and relative probability of informed trading at both exchanges. Fill rates are higher for limit orders placed at exchanges with smaller maker rebates while the probability of facing informed orders is higher at exchanges with larger maker rebates.

I find that when exchanges have similar fee structures, brokers have less incentive to deviate from their clients’ interests, and will optimally route to the exchange with a higher fill rate. In this case, more of their clients will submit orders, each order will have a higher expected value for the client, and investor welfare will be higher. On the contrary, when exchanges have sufficiently different fee structures, routing will be driven by liquidity rebates, and investor welfare will be lower.

The decision to route is also influenced by the broker’s commission, given exogenously in this model. The results show that when commissions are higher, brokers interests become aligned with their clients, as they profit from a higher fill rate.

These results are furthered in an extended model with multiple price levels and endogenous market making. In this environment limit order investors also benefit when both exchanges charge high liquidity making fees. When maker fees are high, market makers are no longer subsidized when making liquidity, and are less willing to provide liquidity at narrow spreads. Consequently, limit order investors, who pay only flat fees to their brokers, face less competition for execution of their orders and have a welfare improvement. I find that changes in the fee structure at one exchange, may influence the spread at a second exchange. This occurs through the shifting of uninformed traders across exchanges. The utility effects to limit order investors from this change depend on the direction of the shift.

This model has a number of implications for policy regarding both trading venues and brokers. First, it implies that the proliferation of trading venues may not necessarily be beneficial for investor welfare, and in fact may be harmful in the case where certain venues face higher adverse selection costs or lower fill rates. Second, while brokers may be restrained in some of their decisions (such as by the Order Protection Rule), it is important to take factors other than
price into account when defining concepts such as best execution. Specifically, it is beneficial to
investors to consider factors such as fill rates when brokers select venues for their clients. Third, the change in the fee structure at any one exchange, can influence the spreads and market conditions at other exchanges, and that these changes should not and do not occur in a vacuum.

It is important to caution against the implication that fewer trading venues will unambiguously increase investor welfare. One of the primary assumptions in the model, is that all exchanges have the same total fee per order and that this fee is set exogenously. In reality, different venues have different spreads between their maker and taker fees, which result from the competitive environment between them.\textsuperscript{14} In the extreme case of a single monopoly exchange, this spread would likely increase, and the additional costs would likely be passed on by brokers to their clients. Further, there are other features of trading venues, which I do not model, that may appeal to either brokers or their clients.\textsuperscript{15}

\textsuperscript{14}As in Chao et al. (2015)

\textsuperscript{15}Examples include venues with decreased latency or high trading speed (Pagnotta & Philippon, 2011), dark venues, venues with block trading or venues which operate on an auction framework.
Chapter 2

Exchange Traded Funds and Their Impact on Volatility

The popularity of Exchange Traded Funds (ETFs) and other exchange traded products in Canada has increased dramatically in the last decade. In 2000, 3 ETFs were listed in Canada, with a total of $6 Billion CAD in assets under management (AUM). By July 2013, 274 Canadian ETFs existed, with a total of $59.7 Billion CAD AUM.\(^1\) Similarly, the global ETF market increased from 92 funds, with $74.3 Billion USD in 2000 to 2,982 funds with $1.348 Trillion USD in AUM in November 2011.\(^2\)

At the same time as this increase in the popularity of ETFs, conventional mutual funds have not grown to the same degree. In 2000, the United States had 8,155 mutual funds with $7.0 Trillion USD in total assets. By 2011 this had changed to 7,591 funds with $11.6 Trillion USD in total assets.\(^3\) Though this still represents substantial growth in volume, it is nowhere near the more than ten-fold increase in worldwide ETF AUM.\(^4\)

From a practical point of view, ETFs differ from conventional mutual funds in one of two ways, depending on whether the mutual fund in question is a open-ended or close-ended fund. The defining difference between ETFs and open-ended mutual funds is that, while the former trades on an exchange with their values changing in continuous time, the latter trades only at their previous end-of-day net asset values. Due to the continuous nature of ETF trading, its value better reflects the current value of the underlying assets than an open-ended fund.

For closed-funds, the difference lies in the similarity of the price of the ETF to its underlying net asset value. Closed funds often trade at a substantial premium or discount while ETFs, especially index-trackers, are typically very close in price to their value. As such, their price more accurately reflects the value of their underlying assets.

Despite key differences, ETFs function very much like mutual funds. The ETF itself holds

\(^1\)See Canadian ETF Association (2011), Canadian ETF Association (2013)  
\(^2\)See Canadian ETF Association (2011)  
\(^3\)See Investment Company Institute (2013)  
\(^4\)These changes conform with the results of Agapova (2011), who finds that ETFs and mutual funds serve as substitutes rather than complements.
pre-defined baskets of underlying assets, such as equities, bonds, commodities or derivatives, and subsequently issues shares whose values correspond to these baskets. These shares then trade on an exchange, much like traditional equities.

With the rapid increase in popularity, a natural question to pose is whether ETFs affect the market for the underlying assets from which they are constructed. Moreover, if ETFs do affect these underlying markets, how do these changes affect both market quality and the welfare of market participants?

Existing literature contains a numbers of results on introducing ETFs and other basket securities. The primary focus of most empirical work is on the effects of introducing ETFs on liquidity. Hegde and McDermott (2004) and Richie and Madura (2007) both show increases in the liquidity of the underlying assets upon the introduction of ETFs. Similarly, De Winne, Gresse, Platten, et al. (2011) show a decrease in spreads following the introduction of ETFs. Hegde and McDermott (2004) explain this decrease in spreads as a response to lower adverse selections costs arising from a decrease in informed trading in the underlying assets, but De Winne et al. (2011) argue that it is the result of a reduction in order processing and order imbalance costs. In contrast, Van Ness, Van Ness, and Warr (2005) find that spreads increase following the introduction of basket securities.

Yu (2005) finds an increase in liquidity following the introduction of ETFs, and offers insight regarding information generation. Yu suggests that the information generated in the ETF market influences the price of underlying assets, and that trades in the ETF market result in a permanent price impact in cash markets.

Further work by Lin and Chiang (2005) finds that following the introduction of ETFs, volatility increases in underlying asset markets. Finally, Da and Shive (2013) argue that following the introduction of ETFs, return covariance in underlying assets increases as a result of arbitrage between markets.

Several papers attempt to model the effects of either ETFs or similar derivatives on their underlying assets. Subrahmanyan (1991) constructs a model in the tradition of Kyle (1985) and finds that uninformed liquidity traders migrate to the index-based baskets of securities. Following this migration, adverse selection will be higher in the cash markets. He also suggests that ETFs will have a predictive value on the price of their component assets but that their introduction has no effect on the variability of these components.

Gorton and Pennacchi (1993) use a similar model, also based on Kyle (1985), to show that, under certain conditions, liquidity traders are better off trading in these composite securities. In a different vein, Fremault (1991) looks at index future securities using a rational expectations framework in which there exists arbitrage between the securities and the cash markets. Her paper focuses primarily on the role of arbitragers in correcting price inconsistencies across markets, and the information transmission generated by this process.

A number of the empirical findings are difficult to explain using the existing theoretical

---

5 A full review of both empirical and theoretical literature on ETFs exists in Charupat, Miu, et al. (2013).
literature. The decrease in adverse selection, as in Hegde and McDermott (2004) and the increase in liquidity, as in Richie and Madura (2007), are difficult to explain using the current literature. Further, when using a rational expectations framework, as in Fremault (1991), microstructure effects such as the bid-ask spread are not modelled.

This chapter departs from the above works on several fronts. Unlike the framework from Kyle (1985), as used in Subrahmanyam (1991) and Gorton and Pennacchi (1993) or a rational expectations model, as in Fremault (1991), I use a model of informed limit order book trading in the style of Glosten (1994) in which uninformed market makers set a menu of prices contingent on a certain quantity being demanded. I focus on two sources of friction between ETF markets and the underlying cash markets, namely the breadth of agents’ information, and asynchronous trading in different markets.

As a first measure, I introduce the possibility that agents may be informed in different capacities. Some agents are informed broadly, in that they do not have information on the realizations of individual assets, but instead on the market as a whole. I assume that these agents trade in markets for basket securities such as ETFs. Other agents have knowledge on the realizations of individual assets and trade in the cash markets for the assets.

The second friction is the asynchronous timing in trades between ETF markets and the underlying asset markets. For the purpose of this work, the trades in the ETF market take place before trading in the underlying asset markets. Market makers then set quotes in the cash markets, incorporating the information in the ETF market as a noisy signal of the value of each individual asset. While in practice, trades in the ETF and underlying asset markets occur continuously, this assumption allows the isolated study of information transmittal from the ETF market to the asset markets.

My analysis focuses on the changes to the properties of underlying asset markets as a result of the introduction of an ETF market. The base case is one in which two fundamentally unrelated assets trade simultaneously on an exchange. These two assets have no covariance by construction and do not affect one another. Prices in both markets are set by risk-neutral, competitive market makers. These prices are then traded upon by either an informed trader or a liquidity trader.

I then consider the same underlying markets following the introduction of an ETF which is composed equally of both underlying assets. The ETF trades before the underlying assets trade, providing a noisy signal on their value.

I show that the information generated in the ETF market has a number of effects on the underlying markets. Compared to the case in which no ETF exists, the ex-ante price variance of the underlying assets weakly increases, while the posterior difference between the execution price and true value of the assets weakly decreases. I show that, as a result of information generated by ETFs, there is excess correlation, either positive or negative, between the realized prices of the assets in the cash market, and the spreads in underlying asset markets may increase or decrease.
The remainder of the chapter proceeds as follows. In Section 2.1, I construct a model of multiple limit order book markets in which multiple underlying assets are bundled into an exchange traded derivative. In Section 2.2, I consider the equilibrium results for the case of perfect signals. In Section 2.3, I derive comparative static results and present the analysis on the perfect signalling case. In Section 2.4, I consider an extension with imperfect signals. In Section 2.5 I present discussion and in Section 2.6 I conclude. All proofs are contained in Appendix C.

2.1 Model

I consider a model in the style of Glosten (1994), with two underlying assets and one ETF. The liquidation value of one unit of each of the underlying assets is random. For notational consistency, the remainder of the chapter uses the following format for probabilities: $P(\alpha)$ denotes the probability of event $\alpha$ occurring. $V = (\alpha, \beta)$ with $Pr = (X, Y)$, denotes value $\alpha$ occurring with probability $X$, and value $\beta$ occurring with probability $Y$. The distribution of assets is as follows:

\begin{align*}
V_1 &= (0, 1) \quad \text{with} \quad Pr = (p, 1 - p) \\
V_2 &= (0, 1) \quad \text{with} \quad Pr = (q, 1 - q)
\end{align*}

The underlying assets have zero covariance by construction. The model generalizes to a case where the assets have some covariance in their underlying values, however, this generalization complicates the equilibrium and provides little additional insight.

The ETF is composed of equal parts of each of the underlying assets. The ETF takes a value equal to the sum of the two underlying values with:

$$V_e = V_1 + V_2$$

Thus, the unconditional probabilities for each of the possible liquidation values of one unit of the ETF are:

$$V_e = (0, 1, 2) \quad \text{with} \quad Pr = (pq, p + q - 2pq, (1 - p)(1 - q))$$

2.1.1 Agents

Liquidity is provided in all three markets by competitive market makers who post a schedule of bids and asks. Liquidity demand comes from a combination of informed traders seeking to maximize their payoffs and liquidity traders who behave randomly.

All strategic agents are risk neutral and utility is derived solely from the difference between an asset’s liquidation value and the price paid (or received) for it. For all agents submitting
market orders, each unit bought or sold is worth:

\[ U = (V - \text{ask})1\{B\} + (\text{bid} - V)1\{S\} \] (2.5)

Where \(1\{B\}\) and \(1\{S\}\) are indicator functions representing market order buys and sells.\(^6\) Any market order buy corresponds to an accompanying limit order sell, and vice versa for a market order sell.

Both time discounting and inventory concerns are absent in the above formulations. Time periods in this model are considered to be sufficiently short that discounting is not relevant. Further, market maker inventories are not considered in order to isolate the information-externalities imposed by the existence of the ETF market.\(^7\)

**Market Makers**

The markets for the ETF and each of the underlying assets are served by market makers. For each market, market makers set two bid prices at which traders can sell to the market maker and two ask prices at which traders can buy. Traders can exchange one unit at each of these prices.\(^8\)

Each market maker is perfectly competitive and earns zero profit in expectation. They set prices equal to the expectation of the asset’s value conditional on all information available at the time. In the ETF market, this information consists of the type of trade being made. The prices set for each unit \(j\) in the ETF market can be written as:

\[
\text{ask}_j^e = E[V_e|Q_e \geq j] \quad \forall j \in \{1, 2\} \\
\text{bid}_j^e = E[V_e|Q_e \leq j] \quad \forall j \in \{-1, -2\}
\] (2.6) (2.7)

where \(Q_e > 0\) is the total number purchased from the market maker and \(Q_e < 0\) is the total units sold. If a trader wishes to purchase \(Q_e = 2\) units, the first unit is purchased at \(\text{ask}_1^e = E[V_e|Q_e \geq 1]\), and the second unit at \(\text{ask}_2^e = E[V_e|Q_e \geq 2]\). In the underlying markets, market makers set bid and ask prices conditional on the results of trading in the ETF market, as they observe the total quantity transacted. The prices in each of the underlying markets can be written as follows:

\[^6\]For the remainder of the chapter in order to remain consistent with existing literature, I state Buy to represent a market order buy and Sell to represent a market order sell.

\[^7\]An extension could consider the effects of market makers using the cash market to hedge their ETF inventories. End of day rebalancing by leveraged ETFs is considered in empirical studies such as Trainor (2010) and Shum, Hejazi, Haryanto, and Rodier (2015).

\[^8\]From this point forward, a trade of two units at the ask (bid) will be referred to as a large buy (sell), while one unit will be referred to as a small buy (sell).
\[
\text{ask}_j^i(Q_e) = E[V_i|Q_i \geq j, Q_e] \quad \forall j \in \{1, 2\} 
\]
\[
\text{bid}_j^i(Q_e) = E[V_i|Q_i \leq j, Q_e] \quad \forall j \in \{-1, -2\}
\]

where \(Q_i > 0\) is the total number purchased from the market maker and \(Q_i < 0\) is the total units sold. Market makers are unable to observe the type of market order trader (informed or liquidity) which they are trading against prior to the trade. Further, I assume that market makers are unable to discern the identity of the trader post-trade.

### Informed Traders

A fraction of traders in each market are informed, profit maximizing traders. The probability that the trader who arrives in the ETF market is informed is \(\mu_e\), while in the underlying markets it is \(\mu_i\). These probabilities are given exogenously.

An informed trader receives a signal \(S_i\) upon arrival in market \(i\). A signal of strength \(\phi_i\), gives the correct value of the asset with probability \(\phi_i\), and gives a random value with probability \(1 - \phi_i\).

After arriving at the selected market, the informed trader observes her signal \((S_i)\) and the price schedule set by the market maker. She then submits market orders for any profitable limit orders on the book.\(^9\) The strategy of an informed trader in each market is represented by:

\[
\sigma_j(S_i) \in \{0, 1\} \\
i \in \{e, 1, 2\} \\
j \in \{-2, -1, 1, 2\}
\]

\(\sigma_j(S_i)\) takes a value of 1 if the informed trader in market \(i\) selects the \(j^{th}\) unit at the signal \(S_i\), otherwise it takes the value 0. Positive values of \(j\) represent transactions at the ask, while negative values represent transactions at the bid. As a trader must select the first unit before selecting the second, \(\sigma_{|2|}(S_i) = 1 \Rightarrow \sigma_{|1|}(S_i) = 1\) (i.e: a large buy implies a small buy, and a large sell implies a small sell, but not vice versa.) I also define \(\sigma_0(S_i) = 1\) as the strategy in which no trading occurs.

### Liquidity Traders

If the trader who arrives at a market is not informed (probability \(1 - \mu_i\)), a liquidity trader arrives and submits a random market order.\(^{10}\) This liquidity trader submits a large buy, small

---

\(^9\)Any orders at the ask (bid), below (above) the true value of the asset.

\(^{10}\)In the absence of these liquidity traders, there would be no profit to informed traders and no incentive to trade, as per Milgrom and Stokey (1982)
buy, large sell or small sell, or doesn’t trade with equal probability. Thus, the ex-ante probability of a liquidity trader arriving and submitting any given order type is $\lambda_i = \frac{1 - \mu_i}{5}$ in each market $i$.

### 2.1.2 Timing and Market Structure

![Figure 2.1: Market Timing](image)

As shown in Figure 2.1, the model proceeds in five stages as follows. The market maker sets a schedule of prices in the ETF market. This market maker has access only to information on the prior probabilities of asset values and trader types. An agent arrives at the ETF market and places market orders. The quantity transacted, $Q_e$, is visible to all agents in the underlying asset markets, and is referred to as the **ETF history**. Following the ETF stage, market makers set a schedule of prices in each of the underlying asset markets using the prior probabilities of asset values and trader types, as well as the history established in the ETF market. Agents place market orders simultaneously in each of the underlying asset markets. Finally, all agents realize the utility value of their trades.

### 2.2 Equilibrium with Perfect Signals

The primary equilibrium in this model is one in which informed traders have access to perfect signals.

**Assumption 2.1**: Assume $\phi_i = 1$ for $i \in \{e, 1, 2\}$.

An equilibrium in this model consists a schedule of prices set by the market makers in both the underlying asset and ETF markets, as well as solutions to the informed traders’ problems.

To solve for equilibrium, I first solve for the market maker decisions in the ETF market conditional on the behaviour of informed traders. Second, I solve for equilibrium in the underlying asset markets, conditional on each possible result from the ETF market as well as the informed traders in these markets.
Theorem 2.1 (Equilibrium with Perfect Signals). (1) For parameters \( p, q \) and \( \mu_e \), there exist unique equilibrium prices in the ETF market \( \text{ask}_1^e, \text{ask}_2^e, \text{bid}_1^e, \text{bid}_2^e \), such that the market maker earns zero profit in expectation;

(2) For parameters \( p, q \) and \( \mu_e \), and equilibrium ETF prices, there exists a unique solution to the informed trader’s decision in the ETF market;

(3) For parameters \( p, q, \mu_e, \mu_1 \) and \( \mu_2 \), and trading history in the ETF market \( Q_e \), there exist unique equilibrium prices in each the underlying market \( i \), \( \text{ask}_1^i(Q_e), \text{ask}_2^i(Q_e), \text{bid}_1^i(Q_e), \text{bid}_2^i(Q_e) \), such that the market makers earn zero profit in expectation;

(4) For parameters \( p, q, \mu_e, \mu_1 \) and \( \mu_2 \), and equilibrium underlying asset prices, there exists a unique solution to the informed trader’s decision in each underlying asset market.

2.2.1 ETF Market Equilibrium

The ETF market maker sets prices such that:

\[
\text{Price}_i^e = E[V_e|Q_e] \]

By Bayes’ Rule it can be shown that:

\[
\text{ask}_1^e = E[V_e|Q_e \geq 1] = \frac{2 \cdot P(Q_e \geq 1|V_e = 2)P(V_e = 2) + P(Q_e \geq 1|V_e = 1)P(V_e = 1)}{\sum_{i=0}^{2} P(Q_e \geq 1|V_e = i)P(V_e = i)} \quad (2.10)
\]

The second ask price and both bid prices are determined in a similar manner. It can be shown that, regardless of trader action:

\[
\text{ask}_2^e, \text{ask}_1^e \leq 2 \quad \text{bid}_2^e, \text{bid}_1^e \geq 0
\]

The informed trader’s actions in the cases of \( S_e = 0 \) and \( S_e = 2 \) can be pinned down immediately. Informed traders receiving a signal of \( S_e = 2 \) always buy, while those receiving \( S_e = 0 \) always sell. Further, those receiving \( S_e = 2 \) never sell and those with \( S_e = 0 \) never buy.

The actions of informed traders who receive \( S_e = 1 \) depend on the parameters of the underlying assets and the amount of noise in the ETF market. The market maker takes this information into account and sets the bids and asks accordingly.

Depending on prior expectations for the individual assets \( (p, q) \), and the percentage of informed traders in the ETF market \( (\mu_e) \), the equilibrium in the ETF market can take one of several forms. These parameters determine whether an informed trader who receives a middle signal \( (S_e = 1) \) buys or sells any units.

Broadly speaking, in cases where both underlying assets have low expected values given
their priors, the market maker in the ETF market sets low prices and an informed trader may profitably buy at a middle signal, while sells would be unprofitable. Conversely, in the event of high priors in both underlying markets, the market maker sets high prices in the ETF market, and buys are no longer profitable on the middle signal, while sells become profitable.

Figure 2.2: Types of ETF Stage Equilibria

These figures represent the equilibrium in the ETF market, depending on the values of the underlying assets.

The figure shows this with two different probabilities of informed traders ($\mu$), with Panel A representing a larger percentage of informed traders.

Area A: $\text{bid}_1^e > 1$, $\text{bid}_2^e > 1$, informed traders sell two units with a signal of $S_e = 1$.

Area B: $\text{bid}_1^e > 1$, informed traders sell one unit with a signal of $S_e = 1$.

Area C: $\text{bid}_1^e$, $\text{bid}_2^e < 1$, $\text{ask}_1^e$, $\text{ask}_2^e > 1$. Informed traders do not trade with $S_e = 1$.

Area D: $\text{ask}_1^e < 1$, informed traders buy one unit with a signal of $S_e = 1$.

Area E: $\text{ask}_1^e$, $\text{ask}_2^e < 1$, informed traders buy two units with a signal of $S_e = 1$.

Figure 2.2 represents the sets of equilibria with two distributions of informed traders. Note that as the probability of encountering an informed trader ($\mu_e$) increases there is a smaller set of underlying asset parametrizations for which traders will trade on $S_e = 1$. This is a result of market makers setting wider spreads when the probability of encountering an informed trader is higher. When spreads are wider, there is a larger set of parametrizations ($p$ and $q$) for which market makers simultaneously set bids below 1 and asks above 1, making all trades on a middle signal unprofitable.

### 2.2.2 Underlying Asset Market Equilibrium

Following the trading in the ETF market, market makers in each of the underlying asset markets set a schedule of prices, such that:
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\[ Price_1^i(Q_e) = E[V_i|Q_i, Q_e] \] (2.12)

As in the ETF stage, each of these prices can be expressed using Bayes' theorem. In the case of the first ask price, it follows that:

\[ \text{ask}_1^i(Q_e) = E[V_i|Q_i \geq 1, Q_e] \] (2.13)

\[ \text{ask}_1^i(Q_e) = \frac{P(Q_i \geq 1|Q_e, V_i = 1)P(Q_e|V_i = 1)P(V_i = 1)}{\sum_{V_i=0}^{1} P(Q_i \geq 1|Q_e, V_i)P(Q_e|V_i)P(V_i)} \] (2.14)

In Equation (2.14), the \( P(V_i) \) terms are defined by the model parameters, and \( P(Q_e|V_i) \) are given from the results of the ETF stage. As an example, consider the case in which a large buy takes place in the ETF market. \( P(Q_e|V_i) \) accounts for the potential movement in the second asset given any potential value of asset 1:

\[ P(Q_e = 2|V_i = 0) = \lambda_e + \mu_e(q\sigma_2(S_e = 0) + (1 - q)\sigma_2(S_e = 1)) \] (2.15)

\[ P(Q_e = 2|V_i = 1) = \lambda_e + \mu_e(q\sigma_2(S_e = 1) + (1 - q)\sigma_2(S_e = 2)) \] (2.16)

Appendix D contains a complete list of these histories for asset 1. Those for asset 2 are symmetric. Note that, if the underlying value of the asset is known, market makers gain no further information from the ETF market and \( P(Q_i|Q_e, V_i) = P(Q_i|V_i) \). \( P(Q_i|V_i) \) are defined by the strategies of the informed traders, who always buy the maximum number of available units with a signal \( S_i = 1 \) and sell the maximum available if they receive a signal of \( S_i = 0 \).

As all the elements of the bids and asks are well defined and in closed form, substitution into Equation (2.14) or its appropriate counterpart yields the equilibrium solution in the second stage market. The bid-ask spreads can then be computed for any given history:

\[ s^i(Q_e) = \text{ask}_1^i(Q_e) - \text{bid}_1^i(Q_e) \] (2.17)

2.3 Effects of ETF Introduction

In this section, I compare prices for the underlying assets in two situations. In the first case, there is no ETF and the market makers in each underlying market observe only the model’s priors. In the second, there exist assets with an ETF, whose equilibrium is described in Theorem 2.1.

**Assumption 2.2:** Assume there exist identical assets to those described in Section 2.1. These assets have no accompanying ETF and are not interchangeable with the assets described in Section 2.1. The value of these assets is exogenous of the value of the assets in Section 2.1.

The primary difference, between the assets described in Theorem 2.1, and those introduced
in this section, is the presence of an ETF history. Market makers are able to use this history, when present, in their pricing decisions, altering the trading prices of the underlying assets.

2.3.1 Trade Price Variance

The first issue I address is the impact of the presence of an ETF on the variance in the traded prices in the underlying markets. It can be shown that, in a market in which no ETF is traded, the ex-ante expectation of the underlying asset’s execution price is equal to its expected underlying value.

\[
E[\text{Price}] = \frac{1}{\sum Q_i P(Q_i)} \sum P(Q_i) * \text{Price}(Q_i) \tag{2.18}
\]

\[
= P(V_i = 1) \tag{2.19}
\]

Similarly, in the presence of an ETF stage, it can be shown that:

\[
E[\text{Price}] = \frac{1}{\sum Q_e \sum Q_i P(Q_e, Q_i)} \sum \sum P(Q_e, Q_i) * \text{Price}(Q_e, Q_i) \tag{2.20}
\]

\[
= P(V_i = 1) \tag{2.21}
\]

Thus, in both cases, the ex-ante expectation of the final trading price is equal to the unconditional expectation of the asset’s value. The presence of the ETF does not distort the expected value of the asset.

The ex-ante variance of an underlying asset in a market for which no ETF exists is defined by:

\[
Var[\text{Price}] = E[(\text{Price} - P(V_i = 1)^2)] \tag{2.22}
\]

\[
= \frac{\sum Q_i P(Q_i) [\text{Price}(Q_i) - P(V_i = 1)]^2}{\sum Q_i P(Q_i)} \tag{2.23}
\]

For underlying assets which have a corresponding ETF, the variance can be defined by:

\[
Var[\text{Price}] = E[(\text{Price} - P(V_i = 1)^2)] \tag{2.24}
\]

\[
= \frac{\sum Q_e \sum Q_i P(Q_e, Q_i) [\text{Price}(Q_e, Q_i) - P(V_i = 1)]^2}{\sum Q_i P(Q_i)} \tag{2.25}
\]

**Proposition 2.1** (Price Variance of Underlying Assets). (1) The expected execution price of each underlying asset equals its expected underlying value in the presence or absence of an ETF.
(2) The variance in the execution price of each underlying asset weakly increases following the introduction of an ETF.

Proposition 2.1 is derived from a comparison of Equations (2.23) and (2.25). The increase in price variance is caused by an increase in the granularity of information from the presence of an ETF. For any one set of prices that would be set in the underlying markets, a further possible set of prices can be set contingent on any trading which occurs in the ETF market. The range of prices increases and the price may move higher or lower given the ETF market trading. In the context of this model, without an ETF the market makers set four prices (for $Q_i \in \{-2,-1,1,2\}$). Following the introduction of the ETF, which itself has one of five possible outcomes ($Q_e \in \{-2,-1,0,1,2\}$), there are a total of twenty possible prices set by the market maker in each underlying market (in sets of four for each given ETF history).

Proposition 2.1 has utility implications for cash markets. As the assets have the same expected execution price with a higher variance following the introduction of ETFs, each asset can be thought of as mean preserving spreads of itself in the absence of an ETF. While I abstract from risk aversion in the model setup, if instead, one considered a world with risk averse traders using standard utility assumptions, agents may be weakly worse off in the presence of an ETF market.

2.3.2 Ex-Post Price Efficiency

The second issue is whether the posterior difference between the execution price and the true value of the cash market assets changes in the presence of an ETF market. I define ex-post price efficiency as the difference between the execution price and the true value of the asset. In the absence of an ETF, this is given by:

\[
\text{Dif}_i = E_{V_i, Q_i} | V - \text{Price}_i(Q_i) | \tag{2.26}
\]

\[
\begin{align*}
\text{Dif}_i &= \frac{1}{\sum_{Q_i} P(Q_i)} P(V_i = 1)(1 - \sum_{Q_i} P(Q_i | V_i = 1) \text{Price}_i(Q_i)) + \\
&\quad \frac{1}{\sum_{Q_i} P(Q_i)} P(V_i = 0) \sum_{Q_i} P(Q_i | V_i = 0) \text{Price}_i(Q_i) \tag{2.27}
\end{align*}
\]

While the difference in the presence of an ETF is:

\[
\begin{align*}
\text{Dif}_i &= E_{V_i, Q_i, Q_e} | V - \text{Price}_i(Q_e, Q_i) | \tag{2.28}
\end{align*}
\]

\[
\begin{align*}
\text{Dif}_i &= \frac{1}{\sum_{Q_i} P(Q_i)} P(V_i = 1)(1 - \sum_{Q_e} \sum_{Q_i} P(Q_e, Q_i | V_i = 1) \text{Price}_i(Q_e, Q_i)) + \\
&\quad \frac{1}{\sum_{Q_i} P(Q_i)} P(V_i = 0) \sum_{Q_e} \sum_{Q_i} P(Q_e, Q_i | V_i = 0) \text{Price}_i(Q_e, Q_i) \tag{2.29}
\end{align*}
\]
Proposition 2.2 (Ex-Post Price Efficiency). The ex-post difference between the trading price and true asset value in the underlying asset markets is weakly lower in the presence of an ETF.

Corollary 2.1 (Relationship between Price Efficiency and Price Variance). The larger the increase in variance through the introduction of an ETF, the closer that ex-post execution prices will be to the true value of the underlying assets.

Proposition 2.2 suggests that market makers are able to set prices closer to the true value of the asset, by incorporating information generated in the derivatives market. Corollary 2.1 describes the trade-off from this improved information. Namely, that the increased price efficiency created by the information content of the ETF market is directly proportional to the increase in price variance. If market makers move their prices more drastically as a result of trading in the ETF market, variance increases, while at the same time moving the final trading prices closer to the value of the asset.

While an increase in variance can be harmful to traders with risk averse preferences, a smaller difference between prices and asset values is one possible indicator of higher market quality. In this sense, the increase in variance is not necessarily a negative outcome, as it is required to incorporate the new information from the ETF market and improve price setting behaviour.

2.3.3 Spread Size

Market makers set spreads such that they earn zero profit in expectation from trading. The quoted spread in the underlying asset markets in the absence of an ETF is defined by:

\[
S_i = \text{ask}_i \text{^1} - \text{bid}_i \text{^1} = \frac{P(Q_i \geq 1, V_i = 1)}{P(Q_i \geq 1)} - \frac{P(Q_i \leq 1, V_i = 1)}{P(Q_i \leq 1)} \tag{2.30}
\]

This can be transformed such that:

\[
S_i = \frac{P(Q_i \geq 1, V_i = 1)P(Q_i \leq -1, V_i = 0)}{P(Q_i \geq 1)P(Q_i \leq -1)} - \frac{P(Q_i \geq 1, V_i = 0)P(Q_i \leq -1, V_i = 1)}{P(Q_i \geq 1)P(Q_i \leq -1)} \tag{2.32}
\]

Similarly, in the case of an ETF history it can be written as:

\[
S_i(Q_e) = \frac{P(Q_i \geq 1, Q_e, V_i = 1)P(Q_i \leq -1, Q_e, V_i = 0)}{P(Q_i \geq 1, Q_e)P(Q_i \leq -1, Q_e)} - \frac{P(Q_i \geq 1, Q_e, V_i = 0)P(Q_i \leq -1, Q_e, V_i = 1)}{P(Q_i \geq 1, Q_e)P(Q_i \leq -1, Q_e)} \tag{2.33}
\]
Comparison of these two equations leads to the following proposition regarding quoted spreads in the underlying cash markets:

**Proposition 2.3** (Spreads in Underlying Asset Markets). In any ETF history where the following equation holds, there will be narrower quoted spreads in the underlying asset market than had the ETF not existed:

\[ P(V_i = 1|Q_e)P(V_i = 0|Q_e) \leq P(V_i = 1)P(V_i = 0) \]  

(2.34)

If the reverse is true, spreads will be wider in the presence of an ETF than had one not existed. The average spread in an underlying market will be narrower in the presence of an ETF iff:

\[
\sum_{Q_e} P(Q_e) \frac{P(V_i = 1, Q_e)P(V_i = 0, Q_e)}{P(Q_i \geq 1, Q_e)P(Q_i \leq -1, Q_e)} \leq \frac{P(V_i = 1)P(V_i = 0)}{P(Q_i \geq 1)P(Q_i \leq -1)}
\]  

(2.35)

In this model, \( P(V_i = 1|Q_e) = P(V_i = 0|Q_e) = 0.5 \) represents the most uncertain possible assessment of asset \( i \) following an ETF history. If the ETF history causes market makers to be more certain about the value of the asset which they are trading in, they will set tighter spreads. If instead, the ETF creates more uncertainty in the value of the underlying assets, market makers will set wider spreads. The more uncertain the value of the asset is following an ETF history, the higher the left-hand side of Equation (2.34) will be. Similarly, the more uncertain the value of an asset is without an ETF history, the higher the right-hand side of Equation (2.34) will be.

As an example of when spreads may become wider, consider the case of two underlying assets, both of which have a very high expected values (ie: \( P(V_i = 1) \) close to 1). If there is a large amount of selling activity in the ETF market (\( Q_e = -2 \)), the expected value of both underlying assets will fall, causing \(|P(V_i = 1|Q_e) - 0.5| \leq |P(V_i = 1) - 0.5|\), and creating additional uncertainty. As market makers are now less certain of the underlying values following the ETF trading, spreads will increase in both underlying markets. In this case, trading against the prior expectation of the asset (\( Q_e = -2 \)) resulted in increased spreads, while trading in the direction of the prior (\( Q_e = 2 \)) would have narrowed them.

Consider a second example in which spreads become narrower. If the underlying asset has an equal probability of taking a high or low value (\( P(V_i = 1) = 0.5 \)), it is initially as uncertain as possible. Any trading in the ETF market that moves the expected value of the asset away from its initial state (such that \(|P(V_i = 1|Q_e) - 0.5| \geq |P(V_i = 1) - 0.5|\)) will resolve some uncertainty, and spreads will be narrower in the presence of the ETF. In this case trading in either direction will result in narrower spreads.

The above condition applies only to a single post-ETF market trades, and do not provide any information on the average change in spreads. In order to assess this situation, the expectation has to be taken over all \( Q_e \). Spreads decrease on average if:
\[ \sum_{Q_e} P(Q_e) \frac{P(V_i = 1, Q_e)P(V_i = 0, Q_e)}{P(Q_i \geq 1, Q_e)P(Q_i \leq -1, Q_e)} \leq \frac{P(V_i = 1)P(V_i = 0)}{P(Q_i \geq 1)P(Q_i \leq -1)} \] (2.36)

This condition is a restatement of Equations (2.32) and (2.33) summed over possible ETF histories. Spreads do not decrease in all cases and the above equation does not hold under all conditions.

The condition for reduced spreads for a given ETF history reduces substantially, as the predicted order flow \( Q_i \) changes less dramatically than the asset value \( V_i \) due to the presence of liquidity traders. For the individual ETF histories, the difference in magnitude of the two shifts is less important, as the equation can be signed without it. For the average spread, the difference in magnitude is important as it must be weighed over each possible ETF history and Equation (2.36) must be used.

Without fully parametrizing the model, Equation (2.36) is not signed definitively, though interpretation is somewhat similar. If on average, trading in the ETF market creates more certainty in asset values (and therefore order flow) then spreads will decrease. Conversely, if ETF trading, on average, causes more uncertainty in asset values and order flow then spreads will increase. This may account, in part, for the ambiguity regarding spreads in the empirical data, as it suggests that they will either increase or decrease depending on the nature of the underlying assets.

### 2.3.4 Trader Profitability

In the context of this model, informed traders always make positive profits since they have an informational advantage and trade only when profitable. Since the market makers earn zero profit in expectation, any profits earned by the informed traders are offset exactly by losses to the liquidity traders. Any informed trader who enters one of the underlying asset markets, earns an expected profit equal to:

\[ \Pi_i = P[V_i = 0][bid_1 + ask_1] + P[V_i = 1][2 - ask_1 - ask_2] \] (2.37)

The total expected profits going to informed traders can then be represented by \( \mu_i \Pi_i \) while liquidity traders absorb aggregate losses equal to \( -\mu_i \Pi_i \).

The next question I address is how the distribution of profits in the underlying asset markets changes with the introduction of the ETF stage. Unlike previous theoretical work on ETFs (such as Subrahmanyam (1991)), I make no statements on the migration of traders between markets, and instead concentrate on the differences, assuming a constant fraction \( \mu_i \) of traders possessing information. Total profits to informed traders in the absence of an ETF can be rewritten as:
\[
\Pi_i = 2 - \frac{P(Q_i \leq -1|V_i = 0)P(V_i = 0)^2}{P(Q_i \leq -1)} - \frac{P(Q_i = -2|V_i = 0)P(V_i = 0)^2}{P(Q_i = -2)} - \frac{P(Q_i \geq 1|V_i = 1)P(V_i = 1)^2}{P(Q_i \geq 1)} - \frac{P(Q_i = 2|V_i = 1)P(V_i = 1)^2}{P(Q_i = 2)} \tag{2.38}
\]

In the presence of an ETF market, this same profit can be written as:

\[
\Pi_i = 2 - \sum_{Q_e} \frac{P(Q_i \leq -1|V_i = 0)P(V_i = 0, Q_e)^2}{P(Q_i \leq -1, Q_e)} - \sum_{Q_e} \frac{P(Q_i = -2|V_i = 0)P(V_i = 0, Q_e)^2}{P(Q_i = -2, Q_e)} - \sum_{Q_e} \frac{P(Q_i \geq 1|V_i = 1)P(V_i = 1, Q_e)^2}{P(Q_i \geq 1, Q_e)} - \sum_{Q_e} \frac{P(Q_i = 2|V_i = 1)P(V_i = 1, Q_e)^2}{P(Q_i = 2, Q_e)} \tag{2.39}
\]

**Proposition 2.4 (Profitability of Information).** The expected profit of an informed trader in an underlying asset market in the presence of an ETF is less than or equal to the expected profit in its absence.

This result contrasts with Subrahmanyam (1991), in which adverse selection costs increase in underlying asset markets, following the migration of uninformed traders into the ETF market. Through the incorporation of ETF market information, market makers are able to set prices more accurately, reducing adverse selection costs and lowering the total expected profit to informed traders. Though not present in this model, this suggests that informed traders may have incentive to leave underlying asset markets, when the market is served by an ETF.

### 2.3.5 Asset Covariance

By assumption, the underlying assets have zero fundamental correlation. However, as the ETF in this model is composed of equal parts of each asset, any movement in the ETF market may be partially indicative of the underlying values in the cash markets. Trading in the ETF market serves as a noisy signal of these underlying values, as it compresses four possible signals \((V_i = \{0, 1\} \text{ in each asset})\) into three \((V_e = \{0, 1, 2\})\).

As a result of this information flow, the correlation in execution prices in the underlying cash markets is altered. The covariance in cash market execution prices before the introduction of ETFs can be defined by:

\[
\text{Cov} = \sum_{Q_1} \sum_{Q_2} P(Q_1, Q_2)[\text{Price}_1(Q_1) - P(V_1 = 1)][\text{Price}_2(Q_2) - P(V_2 = 1)] \tag{2.40}
\]

In the absence of the ETF, and in the case where the underlying assets are uncorrelated in value, this covariance is equal to 0. In the presence of an ETF history, the equation is altered
such that:

\[
\text{Cov} = \sum Q_1 \sum Q_2 \sum Q_e P(Q_1, Q_2, Q_e)(\text{Price}_1(Q_1, Q_e) - P(V_1 = 1))[\text{Price}_2(Q_2, Q_e) - P(V_2 = 1)]
\]

(2.41)

Unlike the case without the ETF, \( P(Q_1, Q_2|Q_e) \neq P(Q_1|Q_e)P(Q_2|Q_e) \) when information from the ETF market is filtered into the underlying cash markets. Therefore, the covariance will not generally be equal to 0.

**Proposition 2.5** (Correlation in Underlying Asset Market Prices). In the presence of an ETF, the underlying assets have non-zero execution price covariance unless:

\[
\sum Q_1 \sum Q_2 P(Q_1|V_1 = 1)P(Q_2|V_2 = 1) = \sum Q_1 \sum Q_2 P(Q_1|V_1 = 1)P(Q_2|V_2 = 1)P(Q_e|Q_1, Q_2) \frac{P(Q_e|V_1 = 1)P(Q_e|V_2 = 1)P(Q_e|Q_1, Q_2)}{P(Q_e|Q_1)P(Q_e|Q_2)}
\]

(2.42)

Though the covariance can coincidentally remain equal to 0 under some sets of parameters, a scenario which provides meaningful interpretation is:

\[
\sum Q_e \frac{P(Q_e|V_1 = 1)P(Q_e|V_2 = 1)P(Q_e|Q_1, Q_2)}{P(Q_e|Q_1)P(Q_e|Q_2)} = 1 \quad \forall \quad Q_1, Q_2
\]

(2.43)

This condition holds when histories occur randomly in the ETF market, independent of the underlying values of the constituent assets. This randomization occurs when the ETF market consists of only liquidity traders and provides no information.

### 2.3.6 Cross-Market Comparative Statics

Prior to the introduction of an ETF, the only elements that affect an individual asset in the model are its prior distribution \( (p \text{ or } q) \) and the percentage of informed traders within its own market \( (\mu_i) \). However, when an ETF market exists and its history is viewed in the underlying markets, information from other markets is necessary for the market makers to set prices.

Specifically, market makers who are setting prices must take into account two factors that are unrelated to their own markets. These two additional factors are (1) the prior distributions of the other asset in the ETF \( (q \text{ or } p) \) and (2) the percentage of informed traders in the ETF market \( (\mu_e) \). Both these factors are accounted for in the \( P(Q_e|V) \) terms in the pricing equations used by market makers.

Changes in the individual prices set by the market makers can be represented in a the form for asset 1:
\[ \frac{\partial \text{ask}_1}{\partial q} = \eta(\text{ask}_1)P(Q_e|V_1=0) \frac{\partial P(Q_e|V_1=1)}{\partial q} \]
\[ -\eta(\text{ask}_1)P(Q_e|V_1=1) \frac{\partial P(Q_e|V_1=0)}{\partial q} \] (2.44)

\[ \frac{\partial \text{ask}_1}{\partial \mu_e} = \eta(\text{ask}_1)P(Q_e|V_1=0) \frac{\partial P(Q_e|V_1=1)}{\partial \mu_e} \]
\[ -\eta(\text{ask}_1)P(Q_e|V_1=1) \frac{\partial P(Q_e|V_1=0)}{\partial \mu_e} \] (2.45)

The term \( \eta(\text{ask}_1) \) represents a weakly positive scaling term. Results for the other prices, as well as prices for asset 2, are symmetric. As prices adjust from these variables originating in other markets, so too must spreads. In the case of the asset 1, it can be shown that:

\[ \frac{\partial S_1(Q_e)}{\partial q} = \frac{\partial S_1(Q_e)}{\partial P(Q_e|V_1=0)} \frac{\partial P(Q_e|V_1=0)}{\partial q} + \frac{\partial S_1(Q_e)}{\partial P(Q_e|V_1=1)} \frac{\partial P(Q_e|V_1=1)}{\partial q} \] (2.47)

Through substitution of known variables and algebraic manipulation it can be shown that:

\[ \frac{\partial S_1(Q_e)}{\partial q} = \gamma \ast \delta \left( P(Q_e|V_1=1) \frac{\partial P(Q_e|V_1=0)}{\partial q} - P(Q_e|V_1=0) \frac{\partial P(Q_e|V_1=1)}{\partial q} \right) \] (2.48)

Where \( \delta = P(Q_e|V_1=1)^2P(V_1=1)^2 - P(Q_e|V_1=0)^2P(V_1=0)^2 \) and \( \gamma \) is a weakly positive scaling term made up of a number of conditional probabilities. By a similar process, it can be shown that:

\[ \frac{\partial S_1(Q_e)}{\partial \mu_e} = \gamma \ast \delta[P(Q_e|V_1=1) \frac{\partial P(Q_e|V_1=0)}{\partial \mu_e} - P(Q_e|V_1=0) \frac{\partial P(Q_e|V_1=1)}{\partial \mu_e}] \] (2.49)

Thus, changes in the prior distributions of uncorrelated assets, or changes in the percentage of informed traders in the ETF market, affect price setting behaviour and spreads in underlying markets.

### 2.4 Extension: Imperfect Signals

The first extension I consider is one in which traders are imperfectly informed. Specifically, I analyse the case where informed traders in the ETF market receive imperfect signals by relaxing Assumption 2.1.

**Assumption 2.3:** Informed traders in the ETF market receive a signal which is correct with
a known probability ($\phi_e$)

**Assumption 2.4:** With probability $1 - \phi_e$ the signal takes a random value which follows the same distribution as the unconditional value of the ETF.

**Theorem 2.2** (Equilibrium with Imperfect Signals). *In a model with imperfect signals, for given parameters $p, q, \mu_1, \mu_2, \mu_3$ and $\phi_e$, there exists a unique equilibrium consistent with the properties in Section 2.3.*

Unlike the base model, where informed traders know the assets values with certainty, the informed traders now have an expectation of the ETF value, dependent on the strength of the signal. Her valuation of the ETF, given the receipt of a signal, is:

$$E[V_e|S_e] = 2P(V_e = 2, S_e) + P(V_e = 1, S_e) \over P(S_e)$$  \hspace{1cm} (2.50)

Decomposition of the expression above gives a valuation which is a weighted average between the signal, and the unconditional expectation of the ETF:

$$E[V_e|S_e] = S_e\phi_e + (1 - \phi_e)(2 - p - q)$$  \hspace{1cm} (2.51)

As in the case with perfect signalling, the informed trader then adopts a strategy $\sigma_j(S_e)$ for every signal $S_e$ and quantity $j$.

### 2.4.1 Equilibrium Results

As in the perfect signalling case, market makers set prices taking into account informed trader behaviour when receiving both correct and incorrect signals. As before, prices are set equal to the expected value of the asset, given the type of trade occurring. The prices set in the ETF market can be shown to be equal to a weighted combination of the unconditional asset value, and the information provided by the signal, weighted by the strength of the signal.

Although signals are noisy, informed traders maintain an information advantage over market makers, and certain aspects of the initial equilibrium hold. Market makers will continue to set prices such that informed traders receiving a signal of $S_e = 2$ will always buy all available units, as they will always be priced below their valuation. Equally, informed traders receiving a signal of $S_e = 0$ will sell all available units, as their valuation will always be lower than prices set by market makers.

As before, the actions of the informed trader receiving a signal of $S_e = 1$ are conditional on the parameters of the model. Conveniently, due to the fact that both the informed trader and market maker adhere to a valuation scheme which is linear in signal strength, the parameter cutoffs are identical to the perfect signalling case.

**Proposition 2.6** (Informativeness of ETF Trading with Imperfect Signals). *The difference between the probability of an ETF history when an underlying asset takes a high value and the
probability of that same history when an underlying asset takes a low value, decreases under imperfect signalling. This is expressed mathematically as:

$$|P(Q_e|V_i = 1) - P(Q_e|V_i = 0)|_{ Imperfect \leq |P(Q_e|V_i = 1) - P(Q_e|V_i = 0)|_{ Perfect \ (2.52)}$$

Intuitively, imperfect signals in the ETF add additional noisiness to any information transmitted between the ETF market and the underlying asset markets. When informed traders act on an incorrect signal, they act as additional noise traders, increasing the randomness of trading in the ETF market. It is then more difficult to predict the value of these assets from the ETF history. When it becomes more difficult to distinguish whether a history occurs because of an underlying asset taking a low or a high value market makers in underlying markets adjust their prices less in response to any given history in the ETF market.

### 2.5 Discussion

As all components of an ETF are traded separately, they may seem to be a redundant instrument (Yu, 2005). However, the analysis in this chapter shows that they have significant implications with respect to both market quality and trader welfare. This significance arises due to the fact that trading in the ETF markets reveals information to cash markets through a noisy signal. This signal allows traders to incorporate new, albeit imperfect, information into their price setting behaviour.

In terms of market quality, I am able to distinguish two opposing effects. The introduction of an ETF market (1) weakly increases the ex-ante variance in the underlying cash markets, and (2) increases the ex-post efficiency of prices. As the two are mathematically linked in my analysis, any increase in the accuracy at prices comes with a direct increase in price variance.

Another ambiguous change in terms of market quality is that of bid-ask spreads in the cash markets. If market makers are more certain about the value of underlying assets following trading in the ETF market, spreads decrease. Conversely, if market makers are less certain about underlying values, spreads increase. This could, in part, explain the ambiguity in the empirical literature regarding spreads.

For statements regarding trader welfare, there are two possibilities. First, traders may migrate from the underlying cash markets into the ETF market. These effects have been studied by Subrahmanyam (1991) and others. Second, following the introduction of the ETF, traders may remain in the underlying asset markets, which is the behaviour I study. In this case, prices are set closer to their underlying values, increasing the variance and decreasing to profits to informed traders. The increased variance is not a welfare loss in the case where traders are risk neutral, and the reduction in profit is merely a transfer from informed traders to liquidity traders, rather than a loss in surplus.
2.6 Conclusion

In this chapter I construct a model of limit order book trading in which participants trade in either cash markets for individual assets, or in a market for a basket security, comprised of equal parts of all underlying assets. Subsequently, I analyse the effects of the introduction of the ETF market on market quality and trader welfare in the underlying cash markets.

I derive equilibrium results in an ETF market with perfect signals, with underlying cash markets that depend on the trading history of the ETF market. In the cash markets, I show that price setting behaviour by market makers is influenced by the order flow history of the ETF market.

I show that following the introduction of a basket security, underlying cash markets have weakly increased ex-ante variance and ex-post price efficiency. I also show that informed traders in the underlying markets make lower expected profits, with liquidity traders suffering fewer expected losses. Finally, I show that previously uncorrelated underlying assets have excess covariance in their execution prices.

The model contains several avenues for further extension. The model I consider is static in the sense that the trading game is played only once. An extension could consider the dynamics of information flow between alternating cash markets and ETF markets over several periods between random innovations in the underlying asset values.

A second extension concerns the optimal weighting of securities within the basket. While, for simplicity I consider only the case of two equally-weighted underlying securities, this uniformity is not the case for most ETFs. Broadly speaking, many index-tracking ETFs are weighted through some measure such as total market capitalization. Other possible weighting schemes that include asset variance, or covariance with some factor could yield varying results with respect to the welfare or market quality measures in this work.
Chapter 3

Crowdfunding and the Transfer of Risk to Consumers

A major problem faced by entrepreneurs is uncertainty regarding demand for their projects. While they may have confidence in their ability to both design and produce a consumption good, this does not necessarily imply that there is sufficient consumer demand for this to be a profitable venture. Moreover, the demand discovery process can be potentially very costly. This can occur in either the case where the entrepreneur produces too much and is left with an excess, possibly worthless inventory, or too little and is unable to meet demand.

Existing models of entrepreneurship often emphasize the roles of banks or venture capitalists in screening entrepreneurs. Typically, this serves the interest in both the entrepreneur and the source of funding, insofar as this may lower probability of default or create a higher profit. However, many entrepreneurs are unable to access these conventional means of funding and must turn to alternatives. In recent years, crowdfunding has appeared as such an alternative means of funding for some projects, and has rapidly increased in popularity. This chapter seeks to explain the role of crowdfunding as a means of obtaining startup funding. Specifically, I examine: under what circumstances projects are unable to obtain conventional funding despite profitability; under what circumstances crowdfunding is preferred by entrepreneurs to conventional funding methods; and, what the market implications are for products produced under crowdfunding versus conventional funding measures.

One possibility for why many entrepreneurs are unable to access funding is that, for many projects, banks or venture capitalists may simply believe it is unprofitable to fund them. This explanation fits with existing models of credit rationing (such as Williamson (1987)) and seems to describe the crowdfunding market well. Crowdfunding ventures are generally much smaller in size than other investment opportunities, averaging only a few thousand dollars in size (Mollick, 2014). This scale may be well below what is profitable for a conventional financing venue, given reasonable costs to the lender, even if the project had a positive expected cash flow.

A second possible explanation is that conventional sources of funding lack the ability to
discern project quality in some types of project. This is especially likely, considering that many projects seeking crowdfunding are creative efforts, which conventional funding sources may have little expertise in screening. This poses problems to both the entrepreneur and the source of funding. The funding source, without ability to screen, would be forced to assume a larger risk, either increasing the necessary returns to the project or removing its profitability altogether. For the entrepreneur, a funder which is unable to screen their project serves a less useful role, increasing the viability of alternative venues.

To attempt to disentangle potential explanations, I construct a theoretical model of entrepreneurship in which entrepreneurs face uncertain demand for their projects and have a cash-in-advance constraint for production. To cover this constraint they must seek funding through either a conventional bank loan or through crowdfunding. If they seek funding through a bank, the firm pays the bank some interest cost and, in exchange, the firm receives funding. Alternatively, the firm may choose to crowdfund the project.

### 3.0.1 The Crowdfunding Process

The crowdfunding process is relatively new compared to many traditional means of financing small firms. The popular service Kickstarter was only founded very recently, in 2009. Crowdfunding can be roughly divided into groups by what the individuals who fund the projects receive. The first of these, which are the focus of this chapter, are “perks” based projects. In these projects, the backers receive some form of good or service in return for providing the project with funding. In essence, these are somewhat similar to pre-ordering a good prior to receiving it. The second group is equity or debt based crowdfunding, which functions similarly to conventional financial instruments, with a broader audience. These backers receive either a debt or equity stake in the project, rather than a consumption good. Finally, some projects are charitable or philanthropic in motivation, where backers receive no real return for their funding.

While the process behind crowdfunding is not uniform across venues, there are many traits that tend to be common among “perks” based campaigns, that this chapter seeks to explain.\(^2\)

First, rewards based commercial crowdfunding efforts typically offer a series of “perks” or “rewards” that gain the entrepreneur cash in advance for production which has not yet occurred. In economics terms, the person seeking funding outlines a schedule of prices, whereby anyone who contributes an amount that corresponds to this schedule receives some predefined benefit. If the funding campaign involves physically producing some consumption good, typically the schedule of benefits corresponds to the backers receiving some quantity of the final production, which will occur at some point in the future. Generally, there are several tiers in the price schedule, where backers who contribute higher amounts receive higher rewards. In this way,

\(^1\)One good example of such a creative project may be the novelty card game Exploding Kittens on Kickstarter, which raised over $8.7 Million USD. This project initially sought funding of $10,000 USD. See [https://www.kickstarter.com/projects/elanlee/exploding-kittens](https://www.kickstarter.com/projects/elanlee/exploding-kittens)

\(^2\)A much more complete explanation of many of the possible iterations of crowdfunding can be found in Tomczak and Brem (2013) and Agrawal, Catalini, and Goldfarb (2013)
the entrepreneur sets a price and is able to see the demand at that price, before producing. This helps remove inventory risk to the entrepreneur, and can capture a larger portion of the consumer surplus through multiple pricing tiers.

Second, most goods based crowdfunding efforts fall into one of two groups depending on whether there is a minimum revenue required before the project receives funding. In an “all-or-nothing” or goal-based funding campaign, the entrepreneur sets a minimum total revenue goal. If the total revenue pledged falls below this goal, the entrepreneur does not receive any of the pledged funds, and is not required to produce. Conversely, in a “flexible” funding campaign, the entrepreneur receives funds and must produce to fill their orders regardless of how much they receive in total.\(^3\)

The distinction between crowdfunding and other methods of funding, is that the entrepreneur knows, prior to production, what the entire revenue for the initial phase of the project will be. If, for demand below a certain volume, the entrepreneur believes the project would to earn a loss, this can typically be prevented altogether by setting an appropriate minimum funding amount. The risk of default due to a lack of consumer demand will then be either much lower or negated altogether. This chapter shows, that for uncertain consumer demand, the current crowdfunding structure can help to capture increased surplus, when projects would have failed in traditional banking markets. However, with other types of uncertainty (such as uncertainty in production costs), the crowdfunding structure is less effective.

### 3.0.2 Existing Literature

The existing literature on entrepreneurship and barriers to entry in credit markets is well developed and broad in scope. While a complete analysis would be well outside the scope of the present project, the literature on credit rationing is especially relevant. Theoretically, the literature on credit rationing is well established, with multiple possible explanations including incentive problems (Stiglitz & Weiss, 1981), monitoring costs (Williamson, 1987) and collateral constraints (Besanko & Thakor, 1987). Empirically, results on credit rationing have varied. Berger and Udell (1992) suggests that credit rationing is not an empirically significant phenomenon. Alternatively, Carpenter and Petersen (2002) argues that specific industries, specifically high-tech, suffer from limited access to credit markets. The latter explanation fits well with the crowdfunding literature, as many projects fall within industries not typically served by conventional banking.

In the present chapter, true credit rationing (in the sense of Stiglitz and Weiss (1981)), where ex-ante identical loan seekers are either afforded or denied credit does not occur. However, in some cases, borrowers who would be denied credit from the banking system, will be able to find funding for their projects through crowdfunding. In this sense, potentially profitable projects which were passed over by the banking system will be able to enter production as a result of

\(^3\)Empirically, the difference between these two methods is studied by Cumming, Leboeuf, and Schwienbacher (2015)
the new technology.

The model presented in this chapter, which differentiates banking and crowdfunding technologies, also parallels existing literature differentiating banking and venture capital\footnote{A further literature reviews on venture capitalists can be found in Kaplan and Stromberg (2001) and Da Rin, Hellmann, and Puri (2011)}\footnote{Interestingly, existing literature on conventional financing finds that loan rates may decrease with geographical dispersion, as in Degryse and Ongena (2005). Further, Degryse and Ongena (2005) finds that adverse selection does not increase significantly over wider distances.}. One branch of this literature focusses on the role of venture capital in screening new entrepreneurs. Ueda (2004) models this explicitly, by giving venture capitalists superior screening abilities at the cost of an equity stake. Another difference is modelled in Winton and Yerramilli (2008), in which venture capitalists and banks affect the entrepreneur’s riskiness in project choices.

Crowdfunding Literature

In the short time since crowdfunding has gained widespread appeal, a large volume of academic work has sought to examine the process. The majority of this work is empirical or descriptive in nature, and provides a basis to inform theoretical developments.

Agrawal, Catalini, and Goldfarb (2011) focus on the geographic dispersion in crowdfunding backers and find that investors in crowdfunding projects are generally very geographically dispersed compared traditional venues\footnote{Interestingly, existing literature on conventional financing finds that loan rates may decrease with geographical dispersion, as in Degryse and Ongena (2005). Further, Degryse and Ongena (2005) finds that adverse selection does not increase significantly over wider distances.}. They find that towards the end of a funding round, investors come from a farther geographical distance, suggesting that local investors (such as the conventional friends and family) may serve to “vet” the legitimacy of the project.

In terms of timing, Kuppuswamy and Bayus (2014) find that the amount raised in crowdsourced projects tends to follow a “bathtub” shaped distribution, with more funding coming at the first and final weeks of the project, with smaller volumes coming during the middle of a project. Mollick (2014) examines the factors which influence the success of a crowdfunding project. Projects with lower goals and shorter funding periods were associated with higher probabilities of success, as well as projects by entrepreneurs with larger social networks (measured by the number of contacts on their respective Facebook accounts).

Closely linked to the present chapter, Xu (2016) studies the choice between crowdfunding and the conventional banking system empirically and finds that projects which benefit most from early feedback, benefit most from crowdfunding over the conventional banking system.

While the main attention of existing crowdfunding literature is empirical, the closest papers to the present work are theoretical works by Rubinton (2011) and Belleflamme, Lambert, and Schwienbacher (2014). In both cases, as in the present chapter, crowdfunding serves as a method for entrepreneurs to cover some advanced cost of production. Rubinton (2011) constructs a game-theoretical model of crowdfunding in which investors may play various roles in the project. Entrepreneurs can tweak the parameters of the project in order to influence investors decisions. Belleflamme et al. (2014), models the decision making of the entrepreneur when they may set separate pricing between the crowdfunding stage (referred to as the pre-ordering stage) and the second retail stage. Investors in the crowdfunding stage receive a “community benefit”
in exchange for paying a higher price than those in the retail stage. If the entrepreneur is unable to raise sufficient capital during the crowdfunding stage, the second stage does not occur and no retail orders are possible.

The present chapter differs from these two papers in a number of ways. These existing papers focus on projects which have crowdfunding as their exclusive method of funding, while this work allows the endogenous selection of crowdfunding as an alternative to conventional fund raising methods. Second, this chapter focuses on crowdfunding investors as solely consumers who have no outside motives other than consumption. While, it is certain that many crowdfunding efforts have a basis in philanthropy or other motives, the existence of crowdfunding without such motives allows it to be applied to a larger group of possible projects.

3.0.3 Main Results

The primary result of this chapter is that crowdfunding can exist simultaneously to existing funding methods and that entrepreneurs will endogenously sort between traditional bank funding and crowdfunding based on the parameters of their project. In equilibrium, entrepreneurs with larger projects will choose the conventional banking system, while those with smaller or riskier projects will choose to crowdfund. Crowdfunding also enables new projects who would not be able to obtain credit from the banking system to obtain funding. This is because crowdfunding serves as a form of real option. Entrepreneurs whose projects which succeed with a low probability or whose cost of default is high are potentially able to crowdfund instead of being denied credit by banks. The crowdfunding technology also differs from existing pre-order technologies, where entrepreneurs are not able to credibly commit to return funds to consumers if insufficient orders are received.

While I show that crowdfunding is well positioned to handle projects with uncertainty regarding demand, I find that this is not the case for uncertainty regarding production. Specifically, whereas crowdfunding technology allows entrepreneurs to isolate profitable states for uncertain demand, they are unable to do so for uncertain production ability, raising the threshold size of feasible projects.

The remainder of the chapter proceeds as follows. In Section 3.1, I describe a model with a single period model. In Section 3.2, I introduce the crowdfunding technology into the model. In Section 3.3, I present an extension where crowdfunding is compared to existing pre-order technologies. In Section 3.4, I present an extension with production shocks. In Section 3.5 I discuss several additional issues related to crowdfunding and in Section 3.6 I conclude.

3.1 Model

3.1.1 Entrepreneurs

Entrepreneurs have exclusive access to a project, which allows them to produce a consumption good. Each good is demanded by consumers with a linear demand curve, denoted:
Entrepreneurs receive a signal $\phi$, of their demand curve, and receive a demand shock. This demand shock can take either high ($\phi + \sigma$), or low ($\phi - \sigma$) values. Demand, conditional on the signal $\phi$, is distributed discretely:

$$f(\Phi | \phi) \sim \{\phi + \sigma, \phi - \sigma\} \text{ with } Pr = \{p, (1 - p)\}$$

(3.2)

Entrepreneurs require cash in advance in order to produce. They must cover a fixed cost $F$ in order to begin production, as well as a marginal cost $c$ for each unit produced. They are able to obtain this cash from one of two sources, either a bank or through a crowdfunding project.

To resolve corner solutions, I assume that entrepreneurs have a preference for projects to exist in as many states possible. They will proceed with projects that earn exactly 0 profit and, if two options are equally profitable, will select one which produces in both demand states, as opposed to one.

### 3.1.2 Consumers

A mass of risk neutral consumers demand the goods from the projects produced by entrepreneurs. Total demand by the consumers is represented by Equation 3.1. Consumers view the demand signal, the costs of the project and the method of funding chosen by the entrepreneur. Individual consumers within the mass are unable to determine the state of the demand shock or coordinate to determine the demand shock.

If consumers pay for the good prior to receiving it, they assess the probability that they will receive it, and discount their valuation of it appropriately. Projects whose costs are higher than their funding received fail, and consumers obtain no final goods. The legal and administrative costs of retrieving a refund are assumed to be high compared to the size of the project and consumers do not expect to recover any of their cash spent. As an additional assumption, entrepreneurs whose projects fail are able to recover no benefit, as they too must pay these costs from the funding received.

Consumers who pay in advance and receive goods with some probability less than 1, are only willing to pay a fraction of the initial price, leading to a demand curve of:

$$Q = \Phi - B \cdot P$$

(3.3)

### 3.1.3 Banks

Banks are able to offer loans to entrepreneurs at an exogenous interest rate $r_B$. Banks view the signal of entrepreneur demand $\phi$ and, if the entrepreneur approaches them, choose whether to supply credit to cover the fixed costs. After paying the fixed cost $F$, the realization $\Phi$ is
known to both the entrepreneur, and the bank. The bank then chooses whether to extend the loan to cover the remaining marginal costs.

The amount of the initial loan is for $F$. If the bank extends credit following the realization of $\Phi$ and the entrepreneur produces at price $P_B(S)$ in state $S$, the total loan is equal to $(\Phi - B \cdot P_B(S)) \cdot c + F$. If the bank opts not to offer the loan, its value is 0.

I make three further assumptions on the banking technology: First, once the entrepreneur has contracted with the bank for the initial value of the loan, she cannot walk away and must continue the project if the bank extends her credit. Second, I assume that the bank is unable to offer a loan where it does not expect to recover its full value with some positive probability. Finally, I assume that banks, like entrepreneurs, have a preference for projects to succeed, and will engage in projects that earn exactly 0 profit.

Given that she wishes to use the banking technology, the entrepreneur’s banking problem is to maximize her expected profits using banking technology. This expected profit in each state is defined by:

$$\pi_B = (\Phi - B P(S)) \cdot (P(S) - c \cdot (1 + r_B)) - F \cdot (1 + r_B) \quad (3.4)$$

Since the banking technology allows for the discovery of $\Phi$ prior to production, entrepreneurs are able to maximize profit separately in each state. The problem is solved in two stages through backward induction. First the entrepreneur solves for the equilibrium prices in each state $P_B(H)$ and $P_B(L)$. Second, the entrepreneur determines whether the loan is profitable for the bank in expectation. In equilibrium, loans will be profitable for all entrepreneurs with demand signals $\phi \geq \phi_B$.

### 3.1.4 Crowdfunding

As an alternative means of acquiring funds, entrepreneurs may choose to use a form of crowdfunding technology. This technology differs from the loans provided by the banks in a number of ways. In order to crowdfund, an entrepreneur announces the price $P_{CF}$, at which they are willing to sell their product. The entrepreneur receives orders for her product, based on her demand curve, which she must then fill. The entrepreneur deals with the consumers, as opposed to a financial intermediary, effectively receiving payment for the goods before they are produced. Unlike the bank, where entrepreneurs pay an interest rate based on the costs incurred, crowdfunders pay a fee based on their total revenue. This fee, $r_{CF}$, is also set exogenously within the model.

Also unlike banking, consumers must pay in advance for crowdfunding. If the entrepreneur does not receive sufficient cash to cover the total number of orders, the project fails. As consumers assume they will be unable to recover their funds, projects which fail with any positive probability are discounted appropriately.

An additional feature of crowdfunding is that an entrepreneur may set a minimum level of
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revenue which they must acquire before receiving any cash. This level is visible to all consumers. If the entrepreneur receives funding below this level, the funds are returned to the consumers and the entrepreneur is not obliged to produce. The entrepreneur sets this quantity $Q_{\text{min}}$ at the same time as she selects the price.

The entrepreneur’s crowdfunding problem is to select a price, and a minimum order quantity $Q_{\text{min}}$ to maximize:

$$\pi_{\text{CF}} = E[(\Phi - BP)(P(1 - r_{\text{CF}}) - c) - F)|Q \geq Q_{\text{min}}]$$ (3.5)

If the entrepreneur sets $Q_{\text{min}}$ such that she is able to fully cover her costs in all states where $Q(P_{\text{CF}}) \geq Q_{\text{min}}$, consumers have no risk of losing their money, and they do not discount.

This minimum funding level differentiates crowdfunding from existing pre-ordering technologies. By allowing producers to credibly return funds in the event of insufficient demand, consumers do not discount their willingness to pay based on demand-side risk. If producers are unable, or unwilling to set a minimum funding level, consumers discount their willingness to pay based on the probability that they do not receive a final product. In that sense, entrepreneurs with higher risk would necessarily charge a lower price to compensate consumers for that risk. This differs substantially from the traditional banking system, since the risk is on the financial institution, rather than the consumer.\footnote{In practice banks charge an interest rate that is generally increasing in risk. A higher interest rate increases variable costs and leads to a higher monopoly price in equilibrium.}

3.2 Benchmark Equilibrium

An equilibrium in this model consists of three decisions by an entrepreneur: (1) a solution to the entrepreneur’s banking problem; (2) a solution to the entrepreneur’s crowdfunding problem; and, (3) a solution to the entrepreneur’s funding selection problem.

In the case of a bank loan, the entrepreneur assesses the probability she will be offered and accept an initial loan, whether the bank will extend credit in each state, and the probability of receiving both a high or low demand signal. She then solves the banking problem by choosing an optimal price given each signal revelation. In the crowdfunding problem, she receives the same information and chooses the minimum funding amount and the price across both states.

**Theorem 3.1** (Existence of an Equilibrium I). (1) For fixed parameters $B, c, F, p, r_B$, there exist unique prices $P_B(H)$ and $P_B(L)$, which solve the entrepreneur’s banking problem in the high and low states respectively. Given these prices, there exists a unique threshold $\phi_B$, such that for all $\phi \geq \phi_B$ banking is profitable; (2) For fixed parameters $B, c, F, p, r_{\text{CF}}$, there exists a price-minimum quantity pair $P_{\text{CF}}, Q_{\text{min}}$ which solves the entrepreneur’s crowdfunding problem. Given this pair, there exists a unique threshold $\phi_{\text{CF}}$, such that for all $\phi \geq \phi_{\text{CF}}$ crowdfunding is profitable;
(3) For fixed parameters $B, c, F, p, r_B, r_{CF}$ and equilibrium values $P_B(H), P_B(L), P_{CF}$ and $Q_{min}$, there exists a threshold $\bar{\phi}$, such that, for all $\phi > \bar{\phi}$, entrepreneurs prefer banking to crowdfunding. For $\phi < \bar{\phi}$, the entrepreneur's choice of funding is parameter dependent.

### 3.2.1 Banking Problem

Maximization of the entrepreneur’s banking profit function yields equilibrium prices for both high and low demand. These prices are:

$$P_B(H) = \frac{\phi + \sigma + (1 + r_B) \cdot B \cdot c}{2 \cdot B} \quad (3.6)$$

$$P_B(L) = \frac{\phi - \sigma + (1 + r_B) \cdot B \cdot c}{2 \cdot B} \quad (3.7)$$

Given these equilibrium prices, the entrepreneur can determine her profitability in each state $S$, ignoring fixed costs:

$$\pi_B(S) = (\Phi - B P_B(S)) \cdot (P_B(S) - c \cdot (1 + r_B)) \quad (3.8)$$

### Equilibrium Bank Actions

The bank makes its loan decision in two stages: (1) The initial loan to the entrepreneur used to finance the fixed costs $F$ and to discover the demand shock $\Phi$; and (2) the extended credit to the entrepreneur to pay her variable costs $c$. The bank solves this problem through backward induction.

The bank extends credit following the discovery of the demand shock if it is able to recover the full amount of its credit extension. That is to say, that it does not consider the fixed cost in its credit extension decision, as it is a sunk cost. The entrepreneur will produce and the bank will recover its extended credit in state $S$ if $\pi_B(S) \geq 0$.

Given the results of the credit extension decision, the bank makes its initial credit decision. To do so, the bank infers the profitability of the loan, $D(S)$ in each state. The total loan is profitable if:

$$pD(H) + (1 - p)D(L) \geq 0 \quad (3.9)$$

There are one of three possibilities: (1) if the bank would not extend credit in state $S$, the profit from the loan is $D(S) = -F$; (2) if the bank would extend credit, but is not able to recover the full cost of the loan in state $S$, the profit from the loan is $D(S) = (\Phi - B P_B(S)) \cdot (P_B(S) - c) - F$, and; (3) if the bank would extend credit and is able to recover the full cost of the loan, the profit from the loan is $D(S) = r_B((\Phi - B P_B(S)) \cdot c + F)$.

**Proposition 3.1 (Equilibrium Loan Decision).** The bank will only finance a project if it is profitable for both the entrepreneur and itself. There exists a threshold $\phi_B$ such that the bank and entrepreneur earn positive profits iff $\phi \geq \phi_B$. If $\phi < \phi_B$, bank loans are unprofitable for
either the bank or the entrepreneur.

Proposition 3.1 describes a form of credit rationing by the bank. Projects of sufficiently low size \( \phi < \phi_B \), are not profitable for the bank because they default in either one (or both) states. This is captured in a second value, \( \phi_B(S) \), which is defined as the value such that the bank will be willing to extend credit in state \( S \) for all projects with \( \phi \geq \phi_B(S) \). The full decision tree for the banking decision is shown in Figure 3.1.

![Figure 3.1: Banking Decision Model](image)

This figure illustrates the equilibrium actions and pay-offs for the banking decision. Entrepreneurs with \( \phi < \phi_B \) will be willing to engage in a project using banking.

Similar to existing literature on credit rationing, such as Williamson (1987), even were the bank able to raise or lower its interest rate arbitrarily, some projects will not be funded. As an example, for projects of sufficiently small size, the interest rate may not cover fixed costs if the project fails in the low demand state. A rise in the interest rate increases the entrepreneur’s marginal costs, reducing the total quantity the entrepreneur produces in equilibrium. An increase in the interest rate then has the effect of reducing entrepreneur profitability, as well as potentially lowering the quantity of the loan. Even were the entrepreneur still willing to accept the loan, profits may decline. Lowering the interest rate on the same loan, while increasing its total value, may still not cover fixed costs.

### 3.2.2 Crowdfunding Problem

The first stage in the entrepreneur’s crowdfunding problem is to set a price, prior to discovering her true demand. If the entrepreneur wishes to crowdfund regardless of her demand state, she sets an optimal price of as in Proposition 3.6.
**Proposition 3.2** (Crowdfunding Optimal Price I). There exists an equilibrium value $\phi_{Q\text{min}}$ such that, for all $\phi \geq \phi_{Q\text{min}}$ the entrepreneur’s optimal price in crowdfunding is:

$$P_{CF} = \frac{(1 - r_{CF})(\phi - (1 - 2p)\sigma) + Bc}{2B(1 - r_{CF})}$$

(3.10)

This price is feasible iff the entrepreneur sets is able to fully cover her costs in both demand states when this price is posted. The equilibrium minimum funding goal in this case, is any $Q_{\text{min}}$ such that:

$$Q_{\text{min}} \leq \phi - \sigma - BP_{CF}$$

(3.11)

In some cases, it may not be most profitable for the entrepreneur to set a price considering both possible demand schedules. Specifically, she may set a minimum funding goal ($Q_{\text{min}}$) such that she will only receive funding in the case of high demand.

**Proposition 3.3** (Crowdfunding Optimal Price II). For all projects with $\phi < \phi_{Q\text{Min}}$ the entrepreneur’s optimal price in crowdfunding is:

$$P_{CF} = \frac{(1 - r_{CF})(\phi + \sigma) + Bc}{2B(1 - r_{CF})}$$

(3.12)

This price is feasible iff the entrepreneur is able to fully cover her costs in the high demand state when this price is posted. The equilibrium minimum funding goal is any $Q_{\text{min}}$ such that:

$$\phi - \sigma - BP_{CF} < Q_{\text{min}} \leq \phi + \sigma - BP_{CF}$$

(3.13)

**Proposition 3.4** (Crowdfunding Decision). There exists a threshold $\phi_{CF}$ such that the entrepreneur earns positive profits with crowdfunding iff $\phi \geq \phi_{CF}$. If $\phi < \phi_{CF}$, crowdfunding is unprofitable for the entrepreneur.

Proposition 3.4 represents the true strength of crowdfunding technology over existing forms of pre-ordering for consumers. For projects whose risk is on the demand-side, crowdfunding allows entrepreneurs to attempt projects if even a single state, no matter how improbable, is profitable. This change substantially alters the entrepreneur’s decision making versus existing pre-order technologies, and can be seen by comparing Figure 3.3, with the decision tree for crowdfunding, shown in Figure 3.2.

### 3.2.3 Funding Selection Problem

The final stage of the entrepreneur’s problem is to select her method of funding. She optimizes her total expected profit given her expected profits in the banking problem ($\pi_{B}$), the crowdfunding problem ($\pi_{CF}$), and whether each of these methods are feasible ($\phi \geq \phi_{B}$ and $\phi \geq \phi_{CF}$).
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Figure 3.2: Crowdfunding Decision Model

This figure illustrates the equilibrium actions and pay-offs for the crowdfunding decision. Entrepreneurs may set $Q_{\text{min}}$ in order to remove unprofitable states from the project. Allowing the project to fail in a demand state by not setting a $Q_{\text{min}}$ is an out of equilibrium decision with the crowdfunding technology. Entrepreneurs with $\phi < \phi_{CF}$ will be willing to engage in a project using the crowdfunding technology.

Proposition 3.5 (Selection Decision I). There exists a threshold $\bar{\phi}$ such that entrepreneur will always select banking, over crowdfunding if $\phi \geq \bar{\phi}$. If $\phi < \bar{\phi}$ the entrepreneur will select either banking or crowdfunding, depending on the parametrization.

Proposition 3.5 finalizes the entrepreneur’s decision making with respect to her project. Given her expected profit in each of the two possible funding methods, she selects a method with which to proceed. For very large projects ($\phi \geq \bar{\phi}$), she always selects banking as an option. For large projects, banking is preferred both because crowdfunding charges fees based on revenue rather than costs, and because crowdfunding forces the entrepreneur into setting a less profitable price in each state. Since revenue increases faster than costs as $\phi$ increases, crowdfunding fees rise faster than banking interest payments. Further, since crowdfunders must set a price before realizing the demand state, their prices are further from the monopoly price in each state.

The reverse is not true for smaller projects ($\phi < \bar{\phi}$). Projects which are profitable with banking have two potential outcomes: (1) the bank extends the loan in both states or; (2) the bank extends the loan in the high demand state only. Projects which are profitable with crowdfunding technology have two possible outcomes: (1) they are profitable in both states; (2) they earn zero profit in the low demand state. These differences create kinks in the entrepreneur’s profit function. Projects below the threshold ($\phi < \bar{\phi}$) may optimally switch from banking to crowdfunding as their size increases due to these kinks, until their reach the final threshold, $\bar{\phi}$. Thus, while projects above a certain size are strictly better under bank funding, smaller projects are not necessarily strictly better under crowdfunding.

In cases where $\phi_{CF} < \phi_B$, crowdfunding is the only feasible method for the entrepreneur, and is strictly preferred in this interval. By substituting the equilibrium values into the profit
function for crowdfunding, threshold value for a project producing only in the high demand state is:

$$\phi_{CF} = \sqrt{4B(1 - r_{CF})F + Bc} \left(1 - r_{CF}\right) - \sigma$$ \hspace{1cm} (3.14)

The value $\phi_B$ can be the result of either the bank or the entrepreneur being unwilling to engage in a loan. Of these, projects which are unprofitable to the bank, but profitable to the entrepreneur are of particular interest. A special case of the threshold is for projects where the bank extends the loan in the high demand state only. For these single state projects, where the bank’s profit function is the limiting factor, the threshold project can be denoted $\phi_B^*$:

$$\phi_B^* = \frac{2F(1 - p(1 + r_B))}{pr_{BC}} + (1 + r_B)Bc - \sigma$$ \hspace{1cm} (3.15)

In the case where $\phi_{CF} < \phi_B^*$, crowdfunding enables projects which were previously unprofitable for the bank. Two differences between Equations 3.14 and 3.15 are the role of fixed costs $F$ and the probability of the high demand state $p$. The relative size of fixed costs affects both crowdfunding and banking. However, the threshold for banking increases linearly in fixed costs while in crowdfunding it increases only by the root of fixed costs. Further, the threshold value for crowdfunding is independent of the likelihood of the high demand state, while the bank’s threshold for offering a loan is decreasing in this probability. These differences suggest that crowdfunding is well positioned to enable projects with either relatively high fixed costs or a low probability of a high demand states.

### 3.3 Extension: Pre-Ordering

In this extension, I present a limit to crowdfunding technology to differentiate it from simply pre-ordering a consumption good. This pre-ordering technology will not allow producers to set a minimum funding level. For projects which are unprofitable in one state, consumers bear the risk of not receiving any consumption good. The banking problem is identical to the one in Section 3.2.

**Assumption 3.1:** The pre-order technology is identical to the crowdfunding technology with the exception that entrepreneurs are unable to set a minimum funding level.

An equilibrium in this model consists of three decisions by an entrepreneur: (1) a solution to the entrepreneur’s banking problem; (2) a solution to the entrepreneur’s pre-order problem; and, (3) a solution to the entrepreneur’s funding selection problem.

In the case of pre-ordering, the entrepreneur solves the problem by setting the optimal price across both high and low demand. Given the expected profits of these two methods, the entrepreneur will choose whether to seek funding, and which method to use.

**Theorem 3.2 (Existence of an Equilibrium II).** (1) For fixed parameters $B, c, F, p, r_B$, there exist unique prices $P_B(H)$ and $P_B(L)$, which solve the entrepreneur’s banking problem in the
high and low states respectively. Given these prices, there exists a unique threshold $\phi_B$, such that for all $\phi \geq \phi_B$ banking is profitable;

(2) For fixed parameters $B, c, F, p, r_{CF}$, there exists a unique price $P_{PO}$ which solves the entrepreneur’s pre-ordering technology problem. Given this price, there exists a unique threshold $\phi_{PO}$, such that for all $\phi \geq \phi_{PO}$ pre-ordering is profitable;

(3) For fixed parameters $B, c, F, p, r_B, r_{CF}$ and equilibrium values $P_B(H), P_B(L)$ and $P_{CF}$, there exists a threshold $\bar{\phi}$, such that, for all $\phi \geq \bar{\phi}$, entrepreneurs prefer banking to pre-ordering. For $\phi < \bar{\phi}$, the entrepreneur’s choice of funding is parameter dependent.

3.3.1 Pre-ordering Problem

The entrepreneur’s pre-ordering problem is to maximize her expected profits using the pre-ordering technology. The entrepreneur faces additional friction with pre-ordering, as they are unable to set a minimum funding amount and must set a single price across both states. Further, if the entrepreneur does not at least break even in each state, consumers discount their willingness to pay.

**Proposition 3.6 (Pre-Ordering Optimal Price I).** As in the crowdfunding problem, for all $\phi \geq \phi_{Q_{min}}$ the entrepreneur’s optimal price in pre-ordering is:

$$P_{PO} = \frac{(1 - r_{CF})(\phi - (1 - 2p)\sigma) + Bc}{2B(1 - r_{CF})}$$

(3.16)

This price is feasible iff the entrepreneur is able to fully cover her costs in both states when this price is posted.

If the entrepreneur is able to feasibly set a price as described in Proposition 3.6, consumers do not discount their willingness to pay, as the project will be completed with certainty. This is the optimal choice for the entrepreneur, as the earns a positive profit in each state and is able to unconditionally maximize her profit, without hitting her constraint.

Since the entrepreneur must set her price in advance when she engages in crowdfunding, she optimally sets a price over both potential outcomes. If the entrepreneur is able to select her optimal price in this manner, her profit is equal to:

$$\pi_{PO} = (p(\phi + \sigma) + (1 - p)(\phi - \sigma) - BP_{PO}) \left(P_{PO} (1 - r_{CF}) - c\right) - F$$

(3.17)

If this price is infeasible ($\phi < \phi_{Q_{min}}$), she must set a different price through one of two options. Either she must lower the price sufficiently, such that she is able to cover her costs in the low demand state, or she must allow the project to fail in that state. If she allows the project to fail in the low demand state, she must discount consumers’ willingness to pay by the probability of the low demand state, $(1 - p)$.

**Proposition 3.7 (Pre-Ordering Optimal Price II).** If $\phi < \phi_{Q_{min}}$, the entrepreneur’s choice of price in Proposition 3.6 infeasible and her optimal price is one of:
\[ P_{PO}(1) = \frac{(\phi - \sigma) + Bc + \sqrt{((1 - r_{CF})(\phi - \sigma) - Bc)^2 - 4BF}}{2B(1 - r_{CF})} \]  

\[ P_{PO}(2) = \frac{p(1 - r_{CF})(\phi + \sigma) + Bc}{2B(1 - r_{CF})} \]  

She selects the value \( P_{PO}(X) \) minimizes the total value of the function:

\[ |P_{PO}(X) - \frac{(1 - r_{CF})(\phi + \sigma) + Bc}{2B(1 - r_{CF})}| \]  

This price is feasible iff the entrepreneur is able to fully cover her costs in only the high demand state when this price is posted.

The two prices in Proposition 3.7 represent the limitations of the pre-ordering technology compared to crowdfunding. In the first case, the entrepreneur sets the price sufficiently low such that she earns exactly 0 profit in the low demand state, while in the latter she discounts consumers’ willingness to pay and allows the project to fail in that same state. Since, in both cases she earns zero profit in the low demand state, the optimal choice between these two is whichever minimizes the distance from the optimal price in the high-demand state only.

This weakness is not present in the crowdfunding equilibrium in Section 3.3, and represents the true difference between crowdfunding as a technology and existing pre-order schemes.

**Proposition 3.8** (Pre-Ordering Decision). There exists a threshold \( \phi_{PO} \) such that the entrepreneur earns positive profits with pre-ordering iff \( \phi \geq \phi_{PO} \). If \( \phi < \phi_{PO} \), the pre-ordering technology is unprofitable for the entrepreneur.

The threshold for crowdfunding, is always less than or equal to the threshold for the pre-ordering technology \( (\phi_{CF} \leq \phi_{PO}) \).

Proposition 3.8 described the set of projects feasible using the pre-ordering technology. It is highly dependant on whether the project is profitable in one (or both) states, and therefore whether consumers discount the chance of receiving their purchase. The decision tree for the pre-ordering problem is shown in Figure 3.3. This result differs from the results of Cumming et al. (2015), in which crowdfunding presents a risk to entrepreneurs rather than consumers. In Cumming et al. (2015), crowdfunding without a minimum funding goal is presented as an option with both a lower expected return and lower risk to the entrepreneur, while the model in this chapter suggests that consumers actually bear a higher risk when entrepreneurs do not set this minimum goal.

### 3.4 Extension: Production Shocks

Thusfar, this chapter has shown that crowdfunding is well positioned to help entrepreneurs mitigate risk on the demand side of new projects. In this section, I present an extension in
Figure 3.3: Pre-ordering Decision Model

This figure illustrates the equilibrium actions and pay-offs for the pre-ordering decision. As shown, consumers discount their willingness to pay for projects which fail with a positive probability. Entrepreneurs with \( \phi < \phi_{PO} \) will be willing to engage in a project using the pre-ordering technology.

which entrepreneurs face not only a demand shock, but also a shock to their production ability.

This production shock is exogenous and uncorrelated with the demand shock. As in the case of the demand shock, the entrepreneur receives a signal \( c \), of her true marginal production cost \( C \). The true production cost can take either high \( (c + \gamma) \), or low \( (c - \gamma) \) values. As with demand, the conditional distribution of costs given the signal is:

\[
f(C|c) \sim \{c + \gamma, c - \gamma\} \quad \text{with } Pr = \{q, (1 - q)\}
\]  

\[ (3.21) \]

The banking problem is similar to the base model. The entrepreneur assesses her optimal pricing in all possible states, and determines which are profitable. The bank assesses whether it would willingly extend credit after the signal are resolved, and decides whether to allow the initial loan if it is profitable.

Crowdfunding Problem and Equilibrium

As in the base model, the entrepreneur chooses a price-minimum quantity pair in order to maximize her expected profit. Her profit maximization function, is characterized by:

\[
\pi_{CFP} = E[(\Phi - BP)(P(1 - r_{CF}) - C) - F)|Q \geq Q_{min}]
\]

\[ (3.22) \]

As an additional complication, unless crowdfunding is profitable in the case of the both high, and low shock to production, entrepreneurs are unable to use the minimum quantity to shield consumers from all risk. As there are 2 exogenous shocks, there are four possible states the entrepreneur faces.\(^7\)

\(^7\)High production cost with high demand \( (Pr = pq) \), high production cost with low demand \( (Pr = (1 - p)q) \), low production cost with high demand \( (Pr = p(1-q)) \), low production cost with low demand \( (Pr = (1-p)(1-q)) \).
Chapter 3. Crowdfunding and the Transfer of Risk to Consumers

As in the base model, there exists a threshold $\phi_{CFP}$, such that all entrepreneurs with $\phi \geq \phi_{CFP}$ are able to profitably engage in crowdfunding. However, unlike the base model, crowdfunding has a much more limited applicability. In the base model, all projects with at least one profitable state are able to crowdfund. However, since entrepreneurs are unable to screen by production cost, they are unable to isolate a single state, but only a pair of states. Crowdfunding is only feasible if the entrepreneurs are able to set a profitable price in the high demand state across both productions states.

**Proposition 3.9** (Crowdfunding Feasibility with Production Uncertainty). Suppose the low production cost state $C = c - \gamma$, is equal to the production cost with production uncertainty. The threshold for crowdfunding feasibility is strictly higher when production is uncertain, than when it is certain $\phi_{CFP} > \phi_{CF}$.

Proposition 3.9 demonstrates the weakness of crowdfunding technology in mitigating uncertainty from the production process. If the entrepreneur were able to isolate states on the production side, as she is able to do on the demand side, she would be able to remove the risk to her customers. As she does not have this ability, for projects which she is unable to make profitable in the high production cost state, she must discount the price.

A simple interpretation of this result, is that if consumers believe that, with some probability, the cost of production will be higher than the entrepreneur initially believes, the entrepreneur must compensate them appropriately. This results in decreased revenue for the entrepreneur, and reduces the total set of projects she is able to feasibly produce. This suggests that an important factor for crowdfunders is the demonstration of production ability to consumers. If they are able to credibly demonstrate their production ability to consumers (and themselves), the crowdfunding technology is able to resolve the demand risk, as before.

### 3.5 Discussion

The model presented in this chapter is meant to demonstrate how crowdfunding is uniquely suited to help entrepreneurs resolve risk on the demand side of potential consumer products. While, as a relatively young technology, there are many possible issues to discuss around crowdfunding, I discuss three which relate directly to this work. First, whether crowdfunding is suitable for a multi-period production problem, second the role of investory risk in the crowdfunding process and finally the role of banks and other traditional financing venues in screening new entrepreneurs.

#### 3.5.1 Multi-Period Production

While crowdfunding is becoming a much more popular means of achieving early stage funding for projects, many large projects move to more traditional methods of funding for subsequent cash. A prime example of this, the Oculus Rift headset, began with a crowdfunding campaign
which totalled almost $2.5 Million from over 9,500 backers in August 2012.\footnote{The original Kickstarter page can be found here: https://www.kickstarter.com/projects/1523379957/oculus-rift-step-into-the-game/description} The company later turned to venture capital for an additional $75 Million in funding and was eventually purchased by Facebook in a deal worth approximately $2 Billion.\footnote{See the Reuters article “Facebook to buy virtual reality goggles maker for $2 billion” at: http://www.reuters.com/article/us-facebook-acquisition-idUSBREA2D1WX20140326, for an example of media coverage on the issue.}

The model presented in this chapter is a single period model of production, where entrepreneurs face risk in the form of a demand shock. The main benefit from crowdfunding technology over conventional banking or existing pre-order schemes relies on the uncertainty surrounding these demand shocks. Once this shock is resolved, crowdfunding may have substantial drawbacks for subsequent rounds of funding.

One primary drawback is that crowdfunding is a time-limited technology. Entrepreneurs generally have days or weeks in which to obtain funding, before the project closes and they must seek alternate arrangements. Further, the cost of crowdfunding can be much higher for entrepreneurs. Whereas the cost of debt financing is based on the costs incurred by the entrepreneur, crowdfunding platforms charge entrepreneurs based on revenue earned. While, in the existing model, this contributes to large projects preferring banking for the first stage, for projects with a realized demand shock, the result would be the same in subsequent stages. Preferable tax treatment for loan payments, and the absence of startup costs in subsequent stages would also lead to entrepreneurs preferring banking over crowdfunding.

### 3.5.2 Inventory Risk

One problem for new entrepreneurs, outside the context of this model, is that of inventory risks. Overproduction can lead to underpricing or unnecessary expenses towards maintaining a large inventory, while underproduction leaves potentially profitable demand unfilled. One benefit of the crowdfunding process, is that entrepreneurs know the complete revenue, and supply needed, for their initial production run. There is no risk of producing either too much, or too little. In the simplest of cases, the entrepreneur merely produces the profit-maximizing quantity, given the distribution of possible demands, and sells them at the price that clears the market. In this case, inventory costs are zero.

The downside to the crowdfunding process is that consumers may have to wait a substantial period of time before receiving their final product. These delays, as studied by Mollick (2014), could have an impact on consumers’ willingness to pay and the probability that an entrepreneur succeeds.
3.5.3 Screening

One function of banks, venture capitalists and other early sources of financing, is screening successful projects. This function has been well documented in the existing literature\(^{10}\) and, while not modelled explicitly in this chapter, screening can serve as an alternate friction in the model.

In a simple modification, a bank may have access to a perfect yet costly screening technology, which enables them to discern the demand shock faced by the entrepreneur prior to extending a loan and paying the fixed costs. In this case, banks may choose to ration credit, not solely based on default risk, but also on whether the expected profit from the loan covers the cost of the screening technology.

As in the base model, crowdfunding partially resolves this screening problem by allowing the producers to remove unprofitable states. The cost, however, is that crowdfunders are unable to set as precise an optimal price due to the lack of pre-screening process. Further, as demonstrated in Section 3.4, crowdfunding technologies are not useful for screening an entrepreneur’s production ability.

3.6 Conclusion

In the span of less than a decade, crowdfunding has gone from a non-existent technology, to a multi-billion dollar industry. This chapter studies the role of crowdfunding, in transferring the risk of entrepreneurial projects from the traditional financing industry to the consumer market.

Existing models of crowdfunding rely on strong assumptions of consumer behaviour, either in the form of deriving social utility from purchasing from new entrepreneurial ventures (Belleflamme et al., 2014), or in desiring to take equity or participation stakes (Rubinton, 2011). The role of this work is to show that crowdfunding platforms are a viable means of fund-raising, even in the presence of an existing financial system, and without additional motives on consumer demand.

I show, using a simple model of demand, that crowdfunding has unique advantages over the existing start-up technologies. Specifically, it allows entrepreneurs to attempt to engage in projects, if only a single state is profitable. This is especially valuable for projects which may have high fixed start-up costs, high variance in demand, or for projects where profitable states are very unlikely.

For projects of a sufficiently large size, the existing banking system remains the optimal technology, both because producers are able to adjust their prices after demand has been resolved, and because crowdfunding platforms typically charge their fees based on revenue, rather than based on costs (banking) or venture capital (profits).

\(^{10}\)See Bester (1985) and Manove, Padilla, and Pagano (2001) for two of many examples
A further weakness of the crowdfunding technology is that, in the case of production cost uncertainty, risk must be passed onto the consumer. In traditional banking, producers are able to set an optimal price, even in the presence of risk. However, when they pass this production risk onto consumers, as is the case with crowdfunding, they must set a price different from the optimal monopoly price. This reduces profitability and leads to a smaller number of feasible projects.
References


Da, Z., & Shive, S. (2013). When the bellwether dances to noise: Evidence from exchange-traded funds. *Available at SSRN 2158361*.


IEEE Conference on Computational Intelligence for Financial Engineering and Economics (CIFEr).

Appendix A

Proofs of Chapter 1

A.1 Primary Equilibrium

Lemma A.1 (Market Order Routing Equilibrium). In equilibrium, brokers will treat each unit of a multiple unit market order as a separate, order of size one. Brokers will route market orders in the following manner.

- For orders of size $|Q_2| = 1$, to whichever exchange has the lowest liquidity taker fee, which will be exchange $i = 1$ since $T_1 \leq T_2$.
- For orders of size $|Q_2| = 2$, to both exchanges.

Lemma A.2 (Equilibrium Limit Order Trading Decisions). If a limit order investor arrives as a buyer, she will submit an order if her expected utility, conditional on routing and execution, is positive. For an order being routed to exchange $i$, this occurs if:

$$\theta_i (E[V|Ex_i] + y - (v - \Delta) - c) > 0 \quad (A.1)$$

Proof of Theorem 1.1

Part 1

Consider a limit order investor wishing to buy the security. This limit order investor will be willing to trade at exchange 1 versus not trading if\(^1\):

$$E[V|Ex_1] + y + \Delta - M_1 \geq 0 \quad (A.2)$$

This will occur if:

$$y \geq \delta \cdot \sigma - \Delta + M_1 \quad (A.3)$$

Consider the same limit order investor. This investor will be willing to trade at exchange 1, versus trading at exchange 2 if:

\(^1\)For proof of expected values and probability, see proofs of Propositions 1.1 and 1.2.
Appendix A. Proofs of Chapter 1

Pr(Ex1)(E[V|Ex1] + y + Δ - M1) ≥ Pr(Ex2)(E[V|Ex2] + y + Δ - M2) \quad (A.4)
\frac{1}{2}(-\delta \cdot \sigma + y + \Delta - M1) \geq \left(\frac{1}{2} \delta + \frac{1}{4}(1 - \delta)\right)(\frac{-\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)} + y + \Delta - M2) \quad (A.5)

This will occur if:
\[ y \geq -\Delta + \frac{2M_1 - (1 + \delta)M_2}{1 - \delta} \quad (A.6) \]

Define \( \bar{y}_1 \), as the threshold over which investors are more willing to trade at exchange 1 over the options of not trading, or trading at exchange 2. This occurs at:
\[ \bar{y}_1 = \max \left\{ \delta \cdot \sigma - \Delta + M_1, -\Delta + \frac{2M_1 - (1 + \delta)M_2}{1 - \delta} \right\} \quad (A.7) \]

Part 2

Given the results of Part 1, no trader would be willing to trade at exchange 2, over exchange 1, if \( y > \bar{y}_1 \). Investors are willing to trade at exchange 2, rather than not trade if:
\[ E[V|Ex2] + y + \Delta - M_2 \geq 0 \quad (A.8) \]

Traders are willing to trade when this holds with equality and thus \( \bar{y}_2 \) can be defined as:
\[ \bar{y}_2 \geq \frac{\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)} - \Delta + M_2 \quad (A.9) \]

Since all traders with \( y > \bar{y}_1 \) are unwilling to trade at exchange 2, if \( \bar{y}_2 \geq \bar{y}_1 \) no traders are willing to trade at exchange 2. If alternatively, \( \bar{y}_2 < \bar{y}_1 \), then there exist some traders with \( \bar{y}_2 \leq y < \bar{y}_1 \) who would prefer to trade at exchange 2, rather than trade at exchange 1, or not trade.

Proof of Proposition 1.1

Since \( T_1 < T_2 \rightarrow M_1 > M_2 \), \( t \) By lemma A.1, brokers will route all small market orders to exchange 1, and all large market orders to both exchanges. Exchange 1 will receive orders any time a market order trader wishes to buy \( \left(\frac{1}{2}(1 - \delta)\right) \), while exchange 2 will receive orders any time a market order trader wishes to buy \( Q_2 = 2 \left(\frac{1}{4}(1 - \delta)\right) \). Both exchanges will receive an equal quantity of informed orders \( \left(\frac{1}{2}\delta\right) \). Thus, the execution probabilities at the two exchanges are:
\[ \theta_1 = \frac{1}{2} \delta + \frac{1}{2}(1 - \delta) \quad (A.10) \]
\[ \theta_2 = \frac{1}{2} \delta + \frac{1}{4}(1 - \delta) \quad (A.11) \]

Since \( 0 < \delta < 1 \), then \( \theta_1 > \theta_2 \).

Proof of Proposition 1.2
Consider a limit order buy. By an application of Bayes’ Rule, the expected value of the security, conditional on execution at exchange $i$ is:

$$E[V|Ex_i] = \frac{Pr(V = \sigma, Ex_i) \cdot \sigma + Pr(V = 0, Ex_i) \cdot 0 + P(V = -\sigma, Ex_i) \cdot -\sigma}{Pr(V = \sigma, Ex_i) + Pr(V = 0, Ex_i) + Pr(V = -\sigma, Ex_i)}$$  \hspace{1cm} (A.12)

- With probability $\frac{1}{2}\delta$, $V = \sigma$. An informed trader arrives and picks off all orders at the ask. No orders at the bid execute, thus $Pr(V = \sigma, Ex_i) = 0$
- With probability $\frac{1}{2}(1 - \delta)$, $V = 0$, a liquidity trader arrives wishing to sell arrives. He always wishes to buy at least one unit, thus by Lemma A.1, $Pr(V = 0, Ex_1) = \frac{1}{2}(1 - \delta)$. With probability $\frac{1}{4}(1 - \delta)$ he wishes to buy two units, and $Pr(V = 0, Ex_2) = \frac{1}{4}(1 - \delta)$.
- With probability $\frac{1}{2}\delta$, $V = -\sigma$. An informed trader arrives and picks off all orders at the bid, thus $Pr(V = -\sigma, Ex_i) = \frac{1}{2}\delta$.

Through substitution of the above probabilities into Equation A.12 and algebraic manipulation:

$$E[V|Ex_1] = -\delta \cdot \sigma$$  \hspace{1cm} (A.13)
$$E[V|Ex_2] = \frac{-\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)}$$  \hspace{1cm} (A.14)

**Proof of Proposition 1.3**

The execution probability at exchange 1 ($\frac{1}{2}$) is higher than that at exchange 2 ($\delta + \frac{1}{4}(1 - \delta)$). If the expected value of the trade, conditional on execution, is higher at exchange 1 for all investors, than the total expected value of the trade is also higher. This is true if the increase in expected value from being routed to exchange 1 is strictly greater than the increase in fees.

$$E[V|Ex_1] - E[V|Ex_2] \geq M_1 - M_2$$  \hspace{1cm} (A.15)

Substitution of the expected value equation and manipulation of this condition leads to the solution:

$$M_1 - M_2 \leq \frac{1 - \delta}{1 + \delta \sigma}$$  \hspace{1cm} (A.16)

**Lemma A.3** (Limit Order Routing Equilibrium). *For limit orders, brokers are able to route, at the price dictated by their client (P), to any exchange. In equilibrium, brokers will route limit orders in the following manner.*
1. In order to maximize profit, brokers will route an order to whichever exchange has the highest expected profit:

\[ \theta_i[c - M_i] \geq \theta_j[c - M_j] \]  

(A.17)

Proof. Investors are unable to contract with brokers and brokers choose routing behaviour following receipt of an order. Suppose a broker were to agree to route to exchange 1 where \( \theta_1 > \theta_2 \) but \( \theta_1[c - M_1] < \theta_2[c - M_2] \):

- If \( \bar{y}_1 < \bar{y}_2 \) it is possible that more orders would be submitted and total expected profit over all possible investors would be higher such that \( Pr(y \geq \bar{y}_1)\theta_1[c - M_1] > Pr(y \geq \bar{y}_2)\theta_2[c - M_2] \).

- However, for any given order \( \theta_1[c - M_1] < \theta_2[c - M_2] \) and the broker has incentive to deviate following the receipt of the order.

- Therefore, the broker’s promise to route to exchange 1 is not credible, unless every order sent there is more profitable.

\[ \square \]

A.2 Single Commission

Proof of Theorem 1.2

The equilibrium is obtained through two steps in backwards induction. (1) The routing of market orders determines the broker’s expected profit for limit orders at exchanges 1 and 2. This determines the threshold \( M_1 \). (2) Given the threshold \( M_1 \), the limit order investor anticipates that her limit order will be routed to exchange \( i \) and determines the expected utility of order submission. This determines the threshold \( \bar{y}_i \).

Part 1

By Lemma A.3, the broker will optimally route to exchange 1 iff:

\[ \theta_1[c - M_1] \geq \theta_2[c - M_2] \]  

(A.18)

By substitution of \( \theta_i \) and algebraic manipulation this gives the condition that:

\[ M_1 \leq c \frac{1 - \delta}{2} + M_2 \frac{1 + \delta}{2} \]  

(A.19)

Denoting \( M_1 = \bar{M}_1 \) when this condition holds with equality, the broker will optimally route any limit order to exchange 1 if \( M_1 \leq \bar{M}_1 \).

Proof of Theorem 1.2 Part 2

Limit order traders take broker routing as given, following Lemmas A.1 and A.3. Following Lemma A.2 and given routing to exchange \( i \), a limit order trader will submit an order iff:
\[ \theta_i (E[V|Ex_i] + y - (v - \Delta) - c) \geq 0 \]  
(A.20)

Through substitution of \( \theta_i \) from Proposition 1.1 and \( E[V|Ex_i] \) from Proposition 1.2 and algebraic manipulation, the conditions for the markets 1 and 2 respectively are:

\[
y \geq c + \delta \cdot \sigma - \Delta \tag{A.21}
\]
\[
y \geq c + \frac{2\delta \cdot \sigma}{1 + \delta} - \Delta \tag{A.22}
\]

Denoting the conditions \( y_1 \) and \( y_2 \) at equality, any limit order trader with \( y \geq y_i \) will satisfy this condition. These values exist for any parameter set following Assumptions 1.1 and 1.2.

**Proof of Proposition 1.4**

See proof of Theorem 1.2, Part 2.

**Proof of Proposition 1.5**

Consider 2 near identical markets, such that \( |T_1 - T_2| \) is small and \( M_1 < \overline{M}_1 \).

- If market 1 raises its maker fee sufficiently such that \( M_1 > \overline{M}_1 \):
  - Brokers will optimally route limit orders to market 2 via Lemma A.3 and Theorem 1.1.
  - Less limit order traders will submit orders since \( \overline{y}_1 < \overline{y}_2 \).
  - Each executed limit order will have a lower expected welfare gain since by Proposition 1.2 \( E[V|Ex_1] > E[V|Ex_1] \).
  - Welfare will decline.

- If market 1 lowers its maker fee such that \( M_1 < M_2 \), market orders will preferentially be routed to exchange 2.
  - The \( M_1 > \overline{M}_1 \) becomes a condition for \( M_2 > \overline{M}_2 \) and the results from above are symmetric.

**Lemma A.4** (Increase in Adverse Selection Risk). *An increase in \( \delta \):*

1. Decreases the expected value of the security for limit order buyers, given execution at all exchanges.
2. Lowers threshold \( \overline{M}_1 \).

**Proof of Lemma A.4**

Given the equilibrium conditions:

\[
E[V|Ex_1] = -\delta \cdot \sigma \tag{A.23}
\]
\[
E[V|Ex_2] = \frac{-\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)} \tag{A.24}
\]
Derivation gives:

\[
\frac{\partial E[V|Ex_i]}{\partial \delta} < 0 \tag{A.25}
\]

Given the equilibrium condition:

\[
\bar{M}_1 = c \frac{1 - \delta}{2} + M_2 \frac{1 + \delta}{2} \tag{A.26}
\]

The fact that \(c \geq M_2\) and derivation shows:

\[
\frac{\partial \bar{M}_1}{\partial \delta} < 0 \tag{A.27}
\]

**Proof of Proposition 1.6**

Follows from Lemma A.4, proof of Theorem 1.2, Proposition 1.2 and \(0 < \delta < 1, \ c > |M_i|\).

**Proof of Proposition 1.7**

As with Theorem 1.2, by Lemma A.3, the broker will optimally route to exchange 1 iff:

\[
\theta_1[c - M_1] \geq \theta_2[c - M_2] \tag{A.28}
\]

By algebraic manipulation this gives the condition that:

\[
c \geq \frac{2M_1 - (1 + \delta)M_2}{1 - \delta} \tag{A.29}
\]

Denoting \(c = \underline{c}\) when this condition holds with equality, the broker will optimally route any limit order to exchange 1 if \(c \leq \underline{c}\).

**Proof of Proposition 1.8**

Follows from a comparison of the conditions in Propositions 1.2 and 1.7, as well as the welfare statement in Proposition 1.5.

### A.3 Endogenous Commission Structure

**Proof of Proposition 1.9**

Consider \(N \geq 2\) brokers, with \(N\) brokers offering \(c_1 = \underline{c}\) and \(n \geq 2\) offering \(c_2 = M_2\).

Brokers offering \(c_1\) earn profit:

\[
\pi_{LO} = \frac{1}{N} Pr(y \geq y_1)(\underline{c} - M_1) > 0 \tag{A.30}
\]

Any broker offering \(c_1 > \underline{c}\) receives no orders and earns zero profit. Any broker offering \(c_1 < \underline{c}\) is not offering a credible price to route to exchange 1 and will receive no orders. Therefore all \(N\) brokers have an incentive to offer \(c_1 = \underline{c}\) and none have an incentive to deviate.

Consider the \(n \geq 2\) brokers offering \(c_2 = M_2\). Each of these brokers is earning profit zero profit as their price is equal to the constant marginal cost.
As long as at least 2 brokers offer \( c_2 = M_2 \), if either of the brokers individually deviate to \( c_2 > M_2 \), they will receive no orders and earn zero profit. If either offers \( c_2 < M_2 \) they will receive orders and earn losses. Therefore, it is not profitable for any of the \( n \) brokers to deviate and an equilibrium price exists at \( c_2 = M_2 \).

**Proof of Proposition 1.10**

Consider a fee pass through system. The limit order investor’s utility function at exchange 1 is:

\[
Pr(Exe_1)[E[V|Exe_1] + y - M_1 + \Delta]
\]  
\[
(A.31)
\]

The marginal trader under this system is that with a \( y \) such that:

\[
y_1 = M_1 - \Delta - E[V|Exe_1]
\]  
\[
(A.32)
\]

Under a competitive fee system, the utility function of this same investor is:

\[
Pr(Exe_1)[E[V|Exe_1] + y - \zeta + \Delta]
\]  
\[
(A.33)
\]

The marginal trader under this system is that with a \( y \) such that:

\[
y_2 = \zeta - \Delta - E[V|Exe_1]
\]  
\[
(A.34)
\]

A group of traders can be defined between \( y_2 \) and \( y_1 \), that are no longer willing to trade under the competitive system, this group is:

\[
y_2 - y_1 = \zeta - M_1
\]  
\[
(A.35)
\]

Substitution of the condition for \( \zeta \) and algebraic manipulation allows the characterization of the mass of traders as \( L = \frac{1+\delta}{1-\delta}(M_1 - M_2) \).

### A.4 Endogenous Market Making

**Proof of Theorem 1.3 Part 1**

Consider market makers at the bid. Given that brokers will first route market orders to exchange 1, it is optimal for the market makers to post a limit order at exchange 1 iff:

\[
E[V|Ex_1] = -\delta \cdot \sigma - \Delta \geq M_1
\]  
\[
(A.36)
\]

Therefore \( \tilde{M}_1 \) is such that the above equation holds with equality.

If \( M_1 \leq \tilde{M}_1 \), a limit order will always be posted there, either by the market maker or a limit order trader. If this case, it is optimal for the market makers to post at exchange 2 iff:


\[ E[V|Ex_2] = \frac{-\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)} - \Delta \geq M_2 \quad (A.37) \]

In this case, \( \tilde{M}_2 \) is such that the above equation holds with equality.

If \( M_1 > \tilde{M}_1 \), a limit order never be posted at exchange 1. In this case, limit order investors' orders will always be routed to exchange 2 since \( M_1 > M_2 \) and \( \theta_1(\Delta) = \theta_2(\Delta) \) if no market maker posts at exchange 1. If this case, it is optimal for the market makers to post at exchange 2 iff:

\[ E[V|Ex_2] = -\delta \cdot \sigma - \Delta \geq M_2 \quad (A.38) \]

In this case, \( \tilde{M}_2 \) is such that the above equation holds with equality.

**Proof of Theorem 1.3 Part 2**

Proof is similar to Theorem 1.2, Part 1.

The probabilities \( \theta_1(\Delta), \theta_j(\Delta), \theta_1(2\Delta), \theta_2(2\Delta) \) are determined by the market maker behaviour established in Theorem 1.3, Section 1.1.

By Lemma A.3, the broker will optimally route to exchange 1 iff:

\[ \theta_1(\Delta)[c - M_1] \geq \theta_2(\Delta)[c - M_2] \quad (A.39) \]

By algebraic manipulation this gives the condition that:

\[ M_1 \leq \frac{\theta_1(\Delta) - \theta_2(\Delta)}{\theta_1(\Delta)} c + \frac{\theta_2(\Delta)}{\theta_1(\Delta)} M_2 \quad (A.40) \]

Denoting \( M_1 = \overline{M}_1(\Delta) \) when this condition holds with equality, the broker will optimally route any limit order to exchange 1 if \( M_1 \leq \overline{M}_1(\Delta) \).

Proof for the price level \( 2\Delta \) follows through identical reasoning.

**Proof of Theorem 1.3 Part 3**

Proof is similar to Theorem 1.2, Part 2.

Limit order traders take broker routing as given, following Lemmas A.1 and A.3. Limit order traders will submit at price level \( \Delta \), knowing it will be routed to exchange \( i \), given:

\[ \theta_i(\Delta) (E[V|Ex_i] + y - (v - \Delta) - c) \geq \theta_j(2\Delta) (v + y - (v - 2\Delta) - c) \quad (A.41) \]

\[ \theta_i(\Delta) (E[V|Ex_i] + y - (v - \Delta) - c) \geq 0 \quad (A.42) \]

Which is to say that the expected value of submitting at \( \Delta \) is higher than that at \( 2\Delta \) and the value of abstaining. Since \( \theta_1(\Delta), \theta_2(\Delta), \theta_1(2\Delta), \theta_2(2\Delta), E[V|Ex_1], E[V|Ex_2] \) and broker routing decisions are already determined by market maker behaviour, the condition \( \overline{\eta}_i(\Delta) \) can be determined by evaluating the two above conditions at equality, and selection whichever is more stringent.
\[ \bar{y}_1(\Delta) = \max \left\{ \frac{\theta_j(2\Delta)(2\Delta - c) - \theta_i(\Delta)(E[V|Ex_i] - (v - \Delta) - c)}{\theta_i(\Delta) - \theta_j(2\Delta)}, -E[V|Ex_i] + (v - \Delta) + c \right\} \] (A.43)

Algebraic manipulation can show that:

\[ \frac{\theta_j(2\Delta)(2\Delta - c) - \theta_i(\Delta)(E[V|Ex_i] - (v - \Delta) - c)}{\theta_i(\Delta) - \theta_j(2\Delta)} \geq -E[V|Ex_i] + (v - \Delta) + c \] (A.44)

and thus:

\[ \bar{y}_1(\Delta) = \frac{\theta_j(2\Delta)(2\Delta - c) - \theta_i(\Delta)(E[V|Ex_i] - (v - \Delta) - c)}{\theta_i(\Delta) - \theta_j(2\Delta)} \] (A.45)

For traders with \( y < \bar{y}_1 \), they will optimally submit at price level 2\( \Delta \) if:

\[ \theta_i(2\Delta) (v + y - (v - 2\Delta) - c) \geq 0 \] (A.46)

The condition \( \bar{y}_i(2\Delta) \) can be determined by evaluating the preceding condition at equality, in order to obtain:

\[ \bar{y}_i(2\Delta) = -2\Delta + c \] (A.47)

**Proof of Proposition 1.11 Part 1**

Consider an increase from \( M_1 \leq \bar{M}_1 \) to \( M'_1 \geq \bar{M}_1 \). This violates the condition \( E[V|Ex_1] = -\delta \cdot \sigma - \Delta \geq M_1 \) and it is no longer optimal for market makers to post.

Since market makers are no longer posting at exchange 1, the expected value at exchange 2 changes such that \( E[V|Ex_2] = -\delta \cdot \sigma \). Since \( M_1 > M_2 \), then \( E[V|Ex_2] = -\delta \cdot \sigma - \Delta \geq M_2 \). It is now optimal for market makers to post at exchange 2.

**Proof of Proposition 1.11 Part 2**

Consider an increase from \( M_2 \leq \bar{M}_2 \) to \( M'_2 \geq \bar{M}_2 \). This violates either the condition \( E[V|Ex_2] = -\delta \cdot \sigma - \Delta \geq M_2 \) or \( E[V|Ex_2] = -\delta \cdot \sigma - \Delta \geq M_2 \), depending on whether market makers are posting at exchange 1. It is no longer optimal for market makers to post at exchange 2.

**Proof of Proposition 1.12**

The case where market makers post more aggressively at exchange 2 represents case 2 in Table B.1

**Part 1**

For case 2 in Table B.1 the probability of execution at the best is equal at both exchanges.

Since \( M_2 < M_1 \), the profitability given execution is higher at exchange 2 for the broker than at exchange 1. Therefore, the total expected profit for orders at the best is higher for the broker at exchange 2.
Part 2
Consider the case where a limit order is not posted in $t = 1$. Expected volume at exchange 2 will be $\frac{1}{2}\delta$ for informed orders and $\frac{1}{4}(1 - \delta)$ for uninformed orders. Expected volume at exchange 1 will be only $\frac{1}{4}(1 - \delta)$ for uninformed orders, as brokers must first send market orders to the best price.
Total volume at exchange 2 is strictly greater than that at exchange 1 ($\frac{1}{2} > \frac{1}{4}(1 - \delta)$)
Consider the case where a limit order is posted in $t = 1$. If it is placed at the best, given the results of Part 1, it will be routed to exchange 2. Expected volume is the same as the case where no limit order is posted.
If it is placed at the second tick, it will be routed to exchange 1 since, in market making case 2:

$$\theta_2(2\Delta) = 0$$  \hfill (A.48)

The market maker will then post at the best, at exchange 2 and volume will be identical to the case where no limit order is posted in $t = 1$. Therefore, regardless of investor action, volume will always be higher at exchange 2 in market making case 2.

**Proof of Proposition 1.13 Part 1** Consider $\bar{M}_1 < M_1 \leq \bar{M}_1$. Brokers are routing limit orders to exchange 2. If the fees at exchange 1 increase to $M'_1 > \bar{M}_1$, brokers will continue to route to exchange 2, however $\theta_2(\Delta)$ increases as there are no longer any limit orders from market makers at exchange 1. The value $E[V|Ex_2]$ also improves for limit order investors, as a higher concentration of uninformed market orders ($\frac{1}{2}(1 - \delta)$ as opposed to $\frac{1}{4}(1 - \delta)$) reach exchange 2. Therefore, all limit order traders posting at exchange 2 have higher utility in expectation.
In addition, $\theta_1(2\Delta)$ increases from 0 to $\frac{1}{4}(1 - \delta)$. Some measure of traders, between $\bar{y}_i(\Delta)$ and $\bar{y}_i(2\Delta)$ will begin submitting orders at the price level $2\Delta$. Each of these traders will also be better off in expectation.

**Proof of Proposition 1.13 Part 2**
Consider $\bar{M}_2 < M_2 \leq \bar{M}_2$, increasing to $M'_2 > \bar{M}_2$. If $M_1 \leq \bar{M}_1$, there is still liquidity available at the best. $\theta_1(\Delta), \theta_2(\Delta), E[V|Ex_1]$ and $E[V|Ex_2]$ remain the same, however $\theta_1(2\Delta)$ increases from 0 to $\frac{1}{4}(1 - \delta)$. Some measure of traders, between $\bar{y}_i(\Delta)$ and $\bar{y}_i(2\Delta)$ will begin submitting orders at the price level $2\Delta$. Each of these traders will also be better off in expectation.
If $M_1 > \bar{M}_1$, there is now no liquidity available at the best. $\theta_1(2\Delta)$ increases from $\frac{1}{4}(1 - \delta)$ to $\frac{1}{2}(1 - \delta)$ and $\theta_2(2\Delta)$ increases from 0 to $\frac{1}{4}(1 - \delta)$. Each trader submitting an order at price level $2\Delta$ is better off.
Appendix B

Additional Material from Chapter 1

<table>
<thead>
<tr>
<th>MM Case</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
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<td>$\theta_1(\Delta)$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{3}(1+\delta)$</td>
<td>$\frac{1}{4}(1+\delta)$</td>
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<tr>
<td>$\theta_2(\Delta)$</td>
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<td>$\frac{1}{3}$</td>
<td>$\frac{1}{4}(1+\delta)$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>$\theta_1(2\Delta)$</td>
<td>0</td>
<td>$\frac{1}{5}(1-\delta)$</td>
<td>$\frac{1}{4}(1-\delta)$</td>
<td>$\frac{1}{5}(1-\delta)$</td>
</tr>
<tr>
<td>$\theta_2(2\Delta)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{5}(1-\delta)$</td>
</tr>
</tbody>
</table>

Table B.1: Equilibrium Execution Probabilities Given Possible Market Maker behaviour

B.1 Simulation Graphical Parameters

All graphs are based on the following parameters:

- $v = 0$
- $\Delta = 1$
- $\delta = 0.5$
- $\sigma = 1.5$
- $b = 0.5$
- $y = 2$
Appendix C

Proofs of Chapter 2

C.1 Primary Equilibrium

Proof of Theorem 2.1

Derivation of First ETF Asking Price

I provide a derivation of the first asking price. The process for subsequent prices follows without substantial proof, as they follow the same logic as Glosten and Milgrom (1985), Easley and O’Hara (1987), Glosten (1994) and others.

Considering the trader strategies and probabilities of liquidity traders on the ask-side of the market, the \( P(Q^e = 2|V) \) and \( P(Q^e \geq 1|V) \) terms can be replaced as follows:

\[
P(Q^e = 2|V^e = 2) = \lambda_e + \mu_e \sigma_2(S^e = 2) \\
= \lambda_e + \mu_e \\
\]

(C.1)

\[
P(Q^e = 2|V^e = 0) = \lambda_e + \mu_e \sigma_2(S^e = 0) \\
= \lambda_e \\
\]

(C.2)

\[
P(Q^e \geq 1|V^e = 2) = 2\lambda_e + \mu_e \sigma_1(S^e = 2) \\
= 2\lambda_e + \mu_e \\
\]

(C.3)

\[
P(Q^e \geq 1|V^e = 0) = 2\lambda_e + \mu_e \sigma_1(S^e = 0) \\
= 2\lambda_e \\
\]

(C.4)

By substituting the strategies for \( S^e = 0 \) and \( S^e = 2 \), as well as the probability of uninformed trading into the first ask price:

\[
ask^e_1 = \frac{2 \times (2\lambda_e + \mu_e)(1 - p)(1 - q) + (2\lambda_e + \mu_e \sigma_1(S^e = 1))(p + q - 2pq)}{(2\lambda_e + \mu_e)(1 - p)(1 - q) + (2\lambda_e + \mu_e \sigma_1(S^e = 1))(p + q - 2pq) + 2\lambda_e pq} \\
\]

(C.5)

Algebraic manipulation reduces this to:
Appendix C. Proofs of Chapter 2

The strategy in the case of \( S_e = 1 \) is defined differently depending on the underlying parametrization of the model, as the informed trader only engages in buys if the price set is less than her signal. This gives:

\[
\begin{align*}
P(Q_e = 2|V_e = 1) &= \lambda_e + \mu_e \sigma_2 (S_e = 1) \\
P(Q_e \geq 1|V_e = 1) &= 2\lambda_e + \mu_e \sigma_1 (S_e = 1)
\end{align*}
\]

with the probabilities on the bid-side (\( Q_e \leq -1 \) and \( Q_e = -2 \)) being symmetric.

In equilibrium \( \sigma_1(S_e = 1) = 1 \) iff \( \text{ask}^e_1 \leq 1 \). In order to derive the equilibrium, assume \( \sigma_1(S_e = 1) = 1 \), then:

\[
\begin{align*}
\text{ask}^e_1 &= \frac{2\lambda_e(2 - p - q) + 2\mu_e(1 - p)(1 - q) + \mu_e \sigma_1(S_e = 1)(p + q - 2pq)}{2\lambda_e + \mu_e(1 - p)(1 - q) + \mu_e \sigma_1(S_e = 1)(p + q - 2pq)} \\
&= \frac{(2 - p - q)(2\lambda_e + \mu_e)}{2\lambda_e + \mu_e(1 - p)}
\end{align*}
\]

The strategy \( \sigma_1(S_e = 1) = 1 \), holds iff \( \text{ask}^e_1 \leq 1 \), then:

\[
\begin{align*}
1 &\geq \frac{(2 - p - q)(2\lambda_e + \mu_e)}{2\lambda_e + \mu_e(1 - p)} \\
0 &\geq \mu_e(1 - p)(1 - q) + 2\lambda_e(1 - p - q)
\end{align*}
\]

Thus, there exists an equilibrium for the first ask price for \( u_e, p, q \) satisfying the above equation, such that \( \sigma_1(S_e = 1) = 1 \) and ask price as defined by Equation (C.10). On the contrary, assume that \( \sigma_1(S_e = 1) = 0 \), then:

\[
\begin{align*}
\text{ask}^e_1 &= \frac{2\lambda_e(2 - p - q) + 2\mu_e(1 - p)(1 - q)}{2\lambda_e + \mu_e(1 - p)(1 - q)} \\
&= \frac{2\lambda_e(2 - p - q) + 2\mu_e(1 - p)(1 - q)}{2\lambda_e + \mu_e(1 - p)(1 - q)}
\end{align*}
\]

The strategy \( \sigma_1(S_e = 1) = 0 \), holds iff \( \text{ask}^e_1 \geq 1 \), then:

\[
\begin{align*}
1 &\leq \frac{2\lambda_e(2 - p - q) + 2\mu_e(1 - p)(1 - q)}{2\lambda_e + \mu_e(1 - p)(1 - q)} \\
0 &\leq \mu_e(1 - p)(1 - q) + 2\lambda_e(1 - p - q)
\end{align*}
\]

Then, for \( \mu_e, p, q \) satisfying the second condition, there exists an equilibrium such that \( \sigma_1(S_e = 1) = 0 \), with an ask price defined by Equation (C.13). Since the conditions for \( \text{ask}^e_1 \geq 1 \) and \( \text{ask}^e_1 < 1 \) are identical, only one can hold and the equilibrium is unique.
Subsequent ETF Prices

For the second ask price, it can be shown that:

$$\text{ask}^e_2 = \frac{\lambda_e (2 - p - q) + 2\mu_e (1 - p)(1 - q) + \mu_e \sigma_2 (S_e = 1)(p + q - 2pq)}{\lambda_e + \mu_e (1 - p)(1 - q) + \mu_e \sigma_2 (S_e = 1)(p + q - 2pq)}$$  \hfill (C.16)

Assuming $\sigma_2 (S_e = 1) = 1$ results in:

$$\text{ask}^e_2 = \frac{(\mu_e + \lambda_e)(2 - p - q)}{\lambda_e + \mu_e (1 - p)}$$  \hfill (C.17)

Which holds as an equilibrium result iff $\text{ask}^e_2 \leq 1$ or:

$$0 \geq \mu_e(1 - p)(1 - q) + \lambda_e(1 - p - q)$$  \hfill (C.18)

Alternatively, an equilibrium with $\sigma_2 (S_e = 1) = 0$ will occur under the following conditions:

$$\text{ask}^e_2 = \frac{\lambda_e (2 - p - q) + 2\mu_e (1 - p)(1 - q)}{\lambda_e + \mu_e (1 - p)(1 - q)}$$  \hfill (C.19)

$$0 \leq \mu_e(1 - p)(1 - q) + \lambda_e(1 - p - q)$$  \hfill (C.20)

The bid prices also have two sets of equilibria depending on the parametrization of the model. For the first bid, there exists an equilibria where $\sigma_{-1} (S_e = 1) = 1$ at:

$$\text{bid}^e_1 = \frac{2\lambda_e(2 - p - q) + \mu_e(p + q - 2pq)}{2\lambda_e + \mu_e(p + q - pq)}$$  \hfill (C.21)

$$0 \leq 2\lambda_e(1 - p - q) - \mu_e pq$$  \hfill (C.22)

The second type of equilibrium occurs with $\sigma_{-1} (S_e = 1) = 0$ at:

$$\text{bid}^e_1 = \frac{2\lambda_e(2 - p - q)}{2\lambda_e + \mu_e pq}$$  \hfill (C.23)

$$0 \geq 2\lambda_e(1 - p - q) - \mu_e pq$$  \hfill (C.24)

For the second bid, there exists an equilibria where $\sigma_{-2} (S_e = 1) = 1$ at:

$$\text{bid}^e_2 = \frac{\lambda_e (2 - p - q) + \mu_e (p + q - 2pq)}{\lambda_e + \mu_e (p + q - pq)}$$  \hfill (C.25)

$$0 \leq \lambda_e(1 - p - q) - \mu_e pq$$  \hfill (C.26)

Alternatively $\sigma_{-2} (S_e = 1) = 0$ at:
Appendix C. Proofs of Chapter 2

\[ \text{bid}_2^e = \frac{\lambda_e(2 - p - q)}{\lambda_e + \mu_e pq} \]  

(C.27)

\[ 0 \geq \lambda_e(1 - p - q) - \mu_e pq \]  

(C.28)

**Proof of Proposition 2.1**

By expanding the variance terms and then simplifying, it can be shown that the variance for the no ETF case is:

\[
\text{Var}[\text{Price}] = \frac{\sum_{Q_i} P(Q_i)P(V_i = 1|Q_i)^2}{\sum_{Q_i} P(Q_i)|Q_i|} 
+ \frac{\sum_{Q_i \epsilon \{-2, 2\}} P(Q_i)P(V_i = 1|Q_i)^2}{\sum_{Q_i} P(Q_i)|Q_i|} 
- \frac{(1 + P(Q_i = 2) + P(Q_i = -2))P(V_i = 1)^2}{\sum_{Q_i} P(Q_i)|Q_i|} 
\]

(C.29)

This relies on the fact that the first set of each prices is transacted by both small orders and larger orders, hence the extra terms for \(Q_i \epsilon \{-2, 2\}\). With the presence of an ETF market, the equation is similar, such that:

\[
\text{Var}[\text{Price}] = \frac{\sum_{Q_e} \sum_{Q_i} P(Q_i, Q_e)P(V_i = 1|Q_i, Q_e)^2}{\sum_{Q_i} P(Q_i)|Q_i|} 
+ \frac{\sum_{Q_e} \sum_{Q_i \epsilon \{-2, 2\}} P(Q_i, Q_e)P(V_i = 1|Q_i, Q_e)^2}{\sum_{Q_i} P(Q_i)|Q_i|} 
- \frac{(1 + P(Q_i = 2) + P(Q_i = -2))P(V_i = 1)^2}{\sum_{Q_i} P(Q_i)|Q_i|} 
\]

(C.30)

To compare the above equations, the denominators, as well as the final two lines of each equation cancel. The first remaining term in each equation, \(\sum_{Q_e} \sum_{Q_i} P(Q_i, Q_e)P(V_i = 1|Q_i, Q_e)^2\) and \(\sum_{Q_i} P(Q_i)P(V_i = 1|Q_i)^2\) can then be compared such that:

\[
\sum_{Q_e} \sum_{Q_i} [P(V_i = 1, Q_e, Q_i)P(V_i = 1|Q_e, Q_i)] \geq \sum_{Q_i} [P(V_i = 1, Q_i)P(V_i = 1|Q_i)] \]  

(C.31)

\[
\sum_{Q_e} \sum_{Q_i} [P(Q_e, Q_i|V_i = 1)^2] \geq \sum_{Q_i} [P(Q_i|V_i = 1)^2] \]  

(C.32)

\[
\sum_{Q_i} [P(Q_i|V_i = 1)^2] \geq \sum_{Q_e} [P(Q_e|V_i = 1)^2] \]  

(C.33)
Where the last step comes from the fact that $P(Q_i|Q_e, V_i = 1) = P(Q_i|V_i = 1)$ in markets with ETFs, since the presence of a history has no effect on either informed market order traders or noise traders if the underlying value is known. By performing a minimization problem on the final equation, it can be shown that:

$$\min \sum_{Q_e} \left[ \frac{P(Q_e|V_i = 1)^2}{P(Q_e|Q_i)} \right] = 1$$

subject to:

$$\sum_{Q_e} P(Q_e|V_i = 1) = 1$$

$$\sum_{Q_e} P(Q_e|Q_i) = 1$$

Which occurs when:

$$P(Q_e|V_i = 1) = P(Q_e|Q_i) \quad \forall \ Q_e \quad \text{(C.34)}$$

This can be shown by solving for the answer to the more general minimization problem:

$$\min \sum_i \alpha_i^2 \beta_i$$

subject to:

$$\sum_i \alpha_i = 1$$

$$\sum_i \beta_i = 1$$

$$\alpha_i \geq 0$$

$$\beta_i > 0$$

Thus the first term is always greater than or equal to its counterpart when comparing the ETF case with the no ETF case. The proof for the second set of terms, $\sum_{Q_e} \sum_{Q_i \in \{-2, 2\}} P(Q_i, Q_e)P(V_i = 1|Q_i, Q_e)^2$ and $\sum_{Q_i \in \{-2, 2\}} P(Q_i)P(V_i = 1|Q_i)^2$ is similar.

Therefore, the variance of the volume-weighted execution price of the underlying assets when an ETF is present is greater than or equal to the same variance without an ETF.

**Proof of Proposition 2.2 and Corollary 2.1**

To prove this proposition, it must be shown that Equation (2.27) is weakly greater than (2.29). This is equivalent to showing that:

$$- \sum_{Q_i} P(Q_i, V_i = 1) Price_i(Q_i) + \sum_{Q_i} P(Q_i, V_i = 0) Price_i(Q_i) \quad \text{(C.35)}$$

Is weakly greater than:
\[-\sum_{Q_e} \sum_{Q_i} P(Q_e, Q_i, V_i = 1) \text{Price}_i(Q_e, Q_i) + \sum_{Q_e} \sum_{Q_i} P(Q_e, Q_i, V_i = 0) \text{Price}_i(Q_e, Q_i) \quad (C.36)\]

As was the case in the proof of Proposition 2.1, expanding Equations (2.27) and (2.29) will result in an extra set of terms for the cases when \(Q_i \in \{-2, 2\}\), as a result of large orders hitting both sets of prices. Ignoring these momentarily, and dealing with the sums over \(\forall Q_i\), the relevant part of Equation (C.36) can be further transformed by substituting in the pricing equations, such that it is equivalent to:

\[
\sum_{Q_e} \sum_{\forall Q_i} P(Q_e, Q_i, V_i = 1) \left[ \frac{P(\text{Price}_i(Q_e, Q_i), Q_i, V_i = 0) - P(\text{Price}_i(Q_e, Q_i), Q_i, V_i = 1)}{P(Q_e, Q_i)} \right]
\]

\[
\sum_{Q_e} \sum_{\forall Q_i} P(Q_e, Q_i, V_i = 1) \left[ \frac{P(\text{Price}_i(Q_e, Q_i), V_i = 0) - P(\text{Price}_i(Q_e, Q_i), V_i = 1)}{P(Q_e, Q_i)} \right]
\]

\[
\sum_{Q_e} \sum_{\forall Q_i} P(Q_e, Q_i, V_i = 1) \left[ \frac{P(\text{Price}_i(Q_e, Q_i), V_i = 0) - P(\text{Price}_i(Q_e, Q_i), V_i = 1)}{P(Q_e, Q_i)} \right]
\]

By further substituting \(P(V_i = 0|Q_e, Q_i) = 1 - P(V_i = 1|Q_e, Q_i)\) it is equivalent to:

\[
\sum_{Q_e} \sum_{\forall Q_i} P(Q_e, Q_i, V_i = 1) [1 - 2P(V_i = 1|Q_e, Q_i)]
\]

The equivalent transformation to Equation (C.35) results in:

\[
\sum_{\forall Q_i} P(Q_i, V_i = 1) [1 - 2P(V_i = 1|Q_i)]
\]

Comparing the sums over \(\forall Q_i\) is then equivalent to comparing:

\[
\sum_{Q_i} P(Q_i, V_i = 1) P(V_i = 1|Q_i) \leq \sum_{Q_e} \sum_{Q_i} P(Q_e, Q_i, V_i = 1) P(V_i = 1|Q_e, Q_i)
\]

Which is the identical proof to the one above with respect to the price variance. The terms with \(Q_i \in \{-2, 2\}\), which were ignored earlier, can be compared in a similar manner, resulting in the same conclusion.

Therefore, it is shown that the posterior difference between the execution price and the true asset value is weakly less in the presence of an ETF than in the case with no ETF.

**Proof of Proposition 2.3**

By factoring \(P(V_i = 1, Q_e)P(V_i = 0, Q_e)\) out of each of the numerator terms in Equation (2.33), as well as by using the fact that \(P(Q_i|Q_e, V_i) = P(Q_i|V_i)\), it can be rewritten as:
\[ S_i(Q_e) = \frac{P(V_i = 1, Q_e)P(V_i = 0, Q_e)P(Q_i \geq 1|V_i = 1)P(Q_i \leq -1|V_i = 0)}{P(Q_i \geq 1, Q_e)P(Q_i \leq -1, Q_e)} - \frac{P(V_i = 1, Q_e)P(V_i = 0, Q_e)P(Q_i \geq 1|V_i = 0)P(Q_i \leq -1|V_i = 1)}{P(Q_i \geq 1, Q_e)P(Q_i \leq -1, Q_e)} \] (C.43)

Note that both portions of the spread contain the common factor \( P(V_i = 1, Q_e)P(V_i = 0, Q_e) \) in the numerator and \( P(Q_i \geq 1, Q_e)P(Q_i \leq -1, Q_e) \) in the denominator. Similarly, modifying Equation (2.32) by factoring out \( P(V_i = 1)P(V_i = 0) \) results in:

\[ S_i = \frac{P(V_i = 1)P(V_i = 0)P(Q_i \geq 1|V_i = 1)P(Q_i \leq -1|V_i = 0)}{P(Q_i \geq 1)P(Q_i \leq -1) - P(V_i = 1)P(V_i = 0)P(Q_i \geq 1|V_i = 0)P(Q_i \leq -1|V_i = 1)} \] (C.44)

In this case, the common factor \( P(V_i = 1)P(V_i = 0) \) occurs in the numerator and \( P(Q_i \geq 1)P(Q_i \leq -1) \) occurs in the denominator.

By removing the common \( P(Q_i|V_i) \) from Equations (C.43) and (C.44), the equations are left with their respective factors as mentioned above. The spreads can then be compared as in Proposition 2.3.

**Proof of Proposition 2.4**

Each term in Equation (2.39) can be factored such that they can be represented by:

\[ P(Q_i|V_i) \sum_{Q_e} \frac{P(Q_e|V_i)^2P(V_i)^2}{P(Q_e|Q_i)P(Q_i)} \] (C.45)

Each of these terms can be compared to their counterparts in Equation (2.38), which are equal to:

\[ P(Q_i|V_i) \frac{P(V_i)^2}{P(Q_i)} \] (C.46)

By solving the minimization problem for each term, it can be shown, as for Propositions 2.1 and 2.2 that:

\[ \min \sum_{Q_e} \left| \frac{P(Q_e|V_i)^2}{P(Q_e|Q_i)} \right| = 1 \] (C.47)

Therefore, the informed trader makes weakly less profit on any possible trade. This also follows intuitively from the fact that the market maker sets more accurate prices on on each unit, as shown in Proposition 2.2.

**Proof of Proposition 2.5**

The covariance in execution prices in underlying assets can be defined as:
\[ \text{Cov} = \sum_{Q_1} \sum_{Q_2} P(Q_1, Q_2) \left[ \text{Price}_1(Q_1) - P(V_1 = 1) \right] \left[ \text{Price}_2(Q_2) - P(V_2 = 1) \right] \]  

(C.48)

Expanding the terms, this is transformed into:

\[ \text{Cov} = \sum_{Q_1} \sum_{Q_2} P(Q_1, Q_2) \left[ P(V_1 = 1)P(V_2 = 1) - P(V_2 = 1)\text{Price}_1(Q_1) - P(V_1 = 1)\text{Price}_2(Q_2) \right] 
+ \sum_{Q_1} \sum_{Q_2} P(Q_1, Q_2) \text{Price}_1(Q_1)\text{Price}_2(Q_2) \]  

(C.49)

By making use of the fact that \( P(Q_1, Q_2) = P(Q_1)P(Q_2) \), and substituting in the pricing equation, it can be shown that the terms in the first line of the equation can be expanded to:

\[ -P(V_1 = 1)P(V_2 = 1)(1 + P(Q_2 = 2) + P(Q_2 = -2))(P(Q_1 = 2|V_1 = 1) + P(Q_1 = -2|V_1 = 1)) 
- P(V_1 = 1)P(V_2 = 1)(1 + P(Q_1 = 2) + P(Q_1 = -2))(P(Q_2 = 2|V_2 = 1) + P(Q_2 = -2|V_2 = 1)) 
P(V_1 = 1)P(V_2 = 1)[(P(Q_1 = 2) + P(Q_1 = -2))(P(Q_2 = 2) + P(Q_2 = -2)) - 1] \]  

(C.50)

As before, it is important to note in this proof that the extra terms with respect to \( P(Q_i = 2) \) and \( P(Q_i = -2) \) result from the fact that buying or selling large orders triggers two sets of prices. Ultimately all the terms cancel out. By further simplifying the equation with the fact that, within the model, \( P(Q_i = 2) + P(Q_i = -2) = P(Q_i = 2|V_i = 1) + P(Q_i = -2|V_i = 1) \), as well as through further algebraic manipulation it can be shown that the first line of Equation (C.49) exactly cancels out the second, giving a covariance equal to 0.

In the case where an ETF is present, the equivalent expansion of Equation (C.49) is:

\[ \text{Cov} = \sum_{Q_1} \sum_{Q_2} \sum_{Q_e} P(Q_1, Q_2, Q_e)P(V_1 = 1)P(V_2 = 1) 
- \sum_{Q_1} \sum_{Q_2} \sum_{Q_e} P(Q_1, Q_2, Q_e)[P(V_2 = 1)\text{Price}_1(Q_1, Q_e) + P(V_1 = 1)\text{Price}_2(Q_2, Q_e)] 
+ \sum_{Q_1} \sum_{Q_2} \sum_{Q_e} P(Q_1, Q_2, Q_e)\text{Price}_1(Q_1, Q_e)\text{Price}_2(Q_2, Q_e) \]  

(C.51)

The first two lines of Equation (C.51) can be expanded to an identical form to Equation (C.50), which is strictly less than 0. By further algebraic manipulation of the final line of Equation (C.51), it can be shown to cancel out Equation (C.50) iff:
Appendix C. Proofs of Chapter 2

\[
\sum_{Q_1} \sum_{Q_2} P(Q_1|V_1 = 1)P(Q_2|V_2 = 1)
\]

Is equal to:

\[
\sum_{Q_1} \sum_{Q_2} P(Q_1|V_1 = 1)P(Q_2|V_2 = 1) \sum_{Q_e} \frac{P(Q_e|V_1 = 1)P(Q_e|V_2 = 1)P(Q_e|Q_1, Q_2)}{P(Q_e|Q_1)P(Q_e|Q_2)}
\]

(C.52)

C.2 Imperfect Signalling

Proof of Theorem 2.2

Part 1: Expected Value for Informed Traders

The expected of an ETF for an informed trader arriving with a signal \( S_e = i \) is equal to:

\[
E[V_e|S_e = i] = \frac{2P(V_e = 2, S_e = i) + P(V_e = 1, S_e = i)}{P(S_e = i)}
\]

(C.54)

The total probability of receiving any given signal \( S_e = i \) is equal to:

\[
P(S_e = i) = P(V_e = i)\phi_e + \sum_{j=0}^{2} P(V_e = j)P(V_e = i)(1 - \phi_e)
\]

\[
= P(V_e = i)
\]

(C.55)

(C.56)

The valuation can then be factored such that:

\[
E[V_e|S_e = i] = \frac{2P(S_e = i|V_e = 2)P(V_e = 2) + P(S_e = i|V_e = 1)P(V_e = 1)}{P(V_e = i)}
\]

(C.57)

Finally, the values for \( P(S_e|V_e) \) can given by the following:

\[
P(S_e = 2|V_e = 2) = \phi_e + (1 - \phi_e)(1 - p)(1 - q)
\]

(C.58)

\[
P(S_e = 2|V_e = 1) = (1 - \phi_e)(1 - p)(1 - q)
\]

(C.59)

\[
P(S_e = 1|V_e = 2) = (1 - \phi_e)(p + q - 2pq)
\]

(C.60)

\[
P(S_e = 1|V_e = 1) = \phi_e + (1 - \phi_e)(p + q - 2pq)
\]

(C.61)

\[
P(S_e = 0|V_e = 2) = (1 - \phi_e)pq
\]

(C.62)

\[
P(S_e = 0|V_e = 1) = (1 - \phi_e)pq
\]

(C.63)

Substitution of the above and algebraic manipulation gives valuations of:
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\[ E[V_e | S_e = 2] = 2\phi_e + (1 - \phi_e)(2 - p - q) \]  \hspace{1cm} (C.64)

\[ E[V_e | S_e = 1] = \phi_e + (1 - \phi_e)(2 - p - q) \]  \hspace{1cm} (C.65)

\[ E[V_e | S_e = 0] = (1 - \phi_e)(2 - p - q) \]  \hspace{1cm} (C.66)

Part 2: Price Derivation

By taking the conditional action probabilities, given in Appendix D, and substituting them into the following formulas:

\[ \text{ask}_1^e = \frac{2 \times P(Q_e \geq 1 | V_e = 2)P(V_e = 2) + P(Q_e \geq 1 | V_e = 1)P(V_e = 1)}{\sum_{i=0}^{2} P(Q_e \geq i | V_e = i)P(V_e = i)} \]  \hspace{1cm} (C.67)

\[ \text{ask}_2^e = \frac{2 \times P(Q_e = 1 | V_e = 2)P(V_e = 2) + P(Q_e = 1 | V_e = 1)P(V_e = 1)}{\sum_{i=0}^{2} P(Q_e = i | V_e = i)P(V_e = i)} \]  \hspace{1cm} (C.68)

\[ \text{bid}_1^e = \frac{2 \times P(Q_e \leq -1 | V_e = 2)P(V_e = 2) + P(Q_e \leq -1 | V_e = 1)P(V_e = 1)}{\sum_{i=0}^{2} P(Q_e \leq -i | V_e = i)P(V_e = i)} \]  \hspace{1cm} (C.69)

\[ \text{bid}_2^e = \frac{2 \times P(Q_e = -2 | V_e = 2)P(V_e = 2) + P(Q_e = -2 | V_e = 1)P(V_e = 1)}{\sum_{i=0}^{2} P(Q_e = -i | V_e = i)P(V_e = i)} \]  \hspace{1cm} (C.70)

Comparison with the formulas for \( E[V_e | S_e = i] \), along with algebraic manipulation shows that:

\[ E[V_e | S_e = 2] \geq \text{ask}_2^e, \text{ask}_1^e \]  \hspace{1cm} (C.71)

\[ E[V_e | S_e = 0] \leq \text{bid}_2^e, \text{bid}_1^e \]  \hspace{1cm} (C.72)

This results in equilibrium strategies of:

\[ \sigma_1(S_e = 2) = \sigma_2(S_e = 2) = \sigma_{-1}(S_e = 0) = \sigma_{-2}(S_e = 0) = 1 \]

\[ \sigma_{-1}(S_e = 2) = \sigma_{-2}(S_e = 2) = \sigma_1(S_e = 0) = \sigma_2(S_e = 0) = 0 \]

As in the case with perfect signals, the equilibrium strategies for a trader receiving a signal of \( S_e = 1 \) are dependant on the parameters within the model. The process for determining the cut-off values is identical as well. For the first ask price, assume \( \sigma_1(S_e = 1) = 1 \). The trader will only purchase this unit if:

\[ \text{ask}_1^e \leq E[V_e | S_e = 1] \]  \hspace{1cm} (C.73)

By substitution of the equilibrium strategies into the ask price and comparison with expected valuation term, this is shown to occur when:
Appendix C. Proofs of Chapter 2

\[ 0 \geq 2\lambda_e (1 - p - q) + \mu_e (1 - p)(1 - q) \quad (C.74) \]

Likewise \( \sigma_1(S_e = 1) = 0 \) will only occur if:

\[ \text{ask}_e^* \geq E[V_e|S_e = 1] \quad (C.75) \]

This condition is true if:

\[ 0 \leq 2\lambda_e (1 - p - q) + \mu_e (1 - p)(1 - q) \quad (C.76) \]

These two conditions are identical to those in the perfect signalling case. Performing the same substitutions for the remaining prices results in cut-off values which are also identical to the imperfect signalling case, and generates all equilibrium values for \( \sigma_i(S_e = 1) = 0 \).

**Proof of Proposition 2.6**

To compute the difference in probability under imperfect signalling, take the Conditional ETF Action Probabilities Under Imperfect Signalling in Appendix D, and for each history create:

\[
P(Q_E|V_1 = 1) = (1 - p)P(Q_E|V_E = 2) + pP(Q_E|V_E = 1) \quad (C.77)
\]

\[
P(Q_E|V_1 = 0) = (1 - p)P(Q_E|V_E = 1) + pP(Q_E|V_E = 0) \quad (C.78)
\]

Comparison of these values with those in the Conditional ETF History section of Appendix D, result in Proposition 2.6.
Appendix D

Closed Form Probabilities from Chapter 2

D.1 Underlying Asset Action Probabilities

\[
\begin{align*}
P(Q_i \geq 1|Q_e, V_i = 0) &= 2\lambda_i \\
P(Q_i \geq 1|Q_e, V_i = 1) &= 2\lambda_i + \mu_i \\
P(Q_i = 2|Q_e, V_i = 0) &= \lambda_i \\
P(Q_i = 2|Q_e, V_i = 1) &= \lambda_i + \mu_i \\
P(Q_i = -2|Q_e, V_i = 0) &= \lambda_i + \mu_i \\
P(Q_i = -2|Q_e, V_i = 1) &= \lambda_i \\
P(Q_i \leq -1|Q_e, V_i = 0) &= 2\lambda_i + \mu_i \\
P(Q_i \leq -1|Q_e, V_i = 1) &= 2\lambda_i 
\end{align*}
\]

D.2 Conditional ETF History Probabilities

\[
\begin{align*}
P(Q_e = 2|V_1 = 0) &= \lambda_e + \mu_e(q\sigma_2(S_e = 0) + (1 - q)\sigma_2(S_e = 1)) \\
&= \lambda_e + \mu_e(1 - q)\sigma_2(S_e = 1) \\
P(Q_e = 2|V_1 = 1) &= \lambda_e + \mu_e(q\sigma_2(S_e = 1) + (1 - q)\sigma_2(S_e = 2)) \\
&= \lambda_e + \mu_e(q\sigma_2(S_e = 1) + (1 - q))
\end{align*}
\]
\[ P(Q_e = 1|V_1 = 0) = \lambda_e + \mu_e(q\sigma_1(S_e = 0)(1 - \sigma_2(S_e = 0)) + \]
\[ (1 - q)\sigma_1(S_e = 1)(1 - \sigma_2(S_e = 1))) \]
\[ = \lambda_e + \mu_e(1 - q)\sigma_1(S_e = 1)(1 - \sigma_2(S_e = 1)) \]

\[ P(Q_e = 1|V_1 = 0) = \lambda_e + \mu_e(q\sigma_1(S_e = 1)(1 - \sigma_2(S_e = 1)) + \]
\[ (1 - q)\sigma_1(S_e = 2)(1 - \sigma_2(S_e = 2))) \]
\[ = \lambda_e + \mu_e(q\sigma_1(S_e = 1)(1 - \sigma_2(S_e = 1)) \]

\[ P(Q_e = 0|V_1 = 0) = \lambda_e + \mu_e(q\sigma_0(S_e = 0) + (1 - q)\sigma_0(S_e = 1)) \]
\[ = \lambda_e + \mu_e(1 - q)\sigma_0(S_e = 1) \]

\[ P(Q_e = 0|V_1 = 1) = \lambda_e + \mu_e(q\sigma_1(S_e = 0) + (1 - q)\sigma_0(S_e = 2)) \]
\[ = \lambda_e + \mu_e q\sigma_0(S_e = 1) \]

\[ P(Q_e = -1|V_1 = 0) = \lambda_e + \mu_e(q\sigma_{-1}(S_e = 0)(1 - \sigma_{-2}(S_e = 0)) + \]
\[ (1 - q)\sigma_{-1}(S_e = 1)(1 - \sigma_{-2}(S_e = 1))) \]
\[ = \lambda_e + \mu_e(1 - q)\sigma_{-1}(S_e = 1)(1 - \sigma_{-2}(S_e = 1)) \]

\[ P(Q_e = -1|V_1 = 1) = \lambda_e + \mu_e(q\sigma_{-1}(S_e = 1)(1 - \sigma_{-2}(S_e = 1)) + \]
\[ (1 - q)\sigma_{-1}(S_e = 2)(1 - \sigma_{-2}(S_e = 2))) \]
\[ = \lambda_e + \mu_e q\sigma_{-1}(S_e = 1)(1 - \sigma_{-2}(S_e = 1)) \]

\[ P(Q_e = -2|V_1 = 0) = \lambda_e + \mu_e(q\sigma_{-2}(S_e = 0) + (1 - q)\sigma_{-2}(S_e = 1)) \]
\[ = \lambda_e + \mu_e(q + (1 - q)\sigma_{-2}(S_e = 1)) \]

\[ P(Q_e = -2|V_1 = 1) = \lambda_e + \mu_e(q\sigma_{-2}(S_e = 1) + (1 - q)\sigma_{-2}(S_e = 2)) \]
\[ = \lambda_e + \mu_e q\sigma_{-2}(S_e = 1) \]

### D.3 ETF Action Probabilities Under Imperfect Signalling

Define \( \Gamma(i) \) as the actions which occurs at quantity \( i \) as a result of being given incorrect signal. \( \Gamma(i) \) is such that:
\[
\Gamma(i) = pq\sigma_i(S = 0) + (p + q - 2pq)\sigma_i(S = 1) + (1 - p)(1 - q)\sigma_i(S = 2) \quad \text{(D.1)}
\]

The conditional probabilities of any given action occurring in the ETF market, given any underlying value are:

\[
P(Q_e = 2|V_e = 2) = \lambda_e + \mu_e(\phi_e\sigma_2(S_e = 2) + (1 - \phi_e)\Gamma(2))
\]

\[
P(Q_e = 2|V_e = 1) = \lambda_e + \mu_e(\phi_e\sigma_2(S_e = 1) + (1 - \phi_e)\Gamma(2))
\]

\[
P(Q_e = 2|V_e = 0) = \lambda_e + \mu_e(\phi_e\sigma_2(S_e = 0) + (1 - \phi_e)\Gamma(2))
\]

\[
P(Q_e \geq 1|V_e = 2) = \lambda_e + \mu_e(\phi_e\sigma_1(S_e = 2) + (1 - \phi_e)\Gamma(1))
\]

\[
P(Q_e \geq 1|V_e = 1) = \lambda_e + \mu_e(\phi_e\sigma_1(S_e = 1) + (1 - \phi_e)\Gamma(1))
\]

\[
P(Q_e \geq 1|V_e = 0) = \lambda_e + \mu_e(\phi_e\sigma_1(S_e = 0) + (1 - \phi_e)\Gamma(1))
\]

\[
P(Q_e \leq -1|V_e = 2) = \lambda_e + \mu_e(\phi_e\sigma_{-1}(S_e = 2) + (1 - \phi_e)\Gamma(-1))
\]

\[
P(Q_e \leq -1|V_e = 1) = \lambda_e + \mu_e(\phi_e\sigma_{-1}(S_e = 1) + (1 - \phi_e)\Gamma(-1))
\]

\[
P(Q_e \leq -1|V_e = 0) = \lambda_e + \mu_e(\phi_e\sigma_{-1}(S_e = 0) + (1 - \phi_e)\Gamma(-1))
\]

\[
P(Q_e = -2|V_e = 2) = \lambda_e + \mu_e(\phi_e\sigma_{-2}(S_e = 2) + (1 - \phi_e)\Gamma(-2))
\]

\[
P(Q_e = -2|V_e = 1) = \lambda_e + \mu_e(\phi_e\sigma_{-2}(S_e = 1) + (1 - \phi_e)\Gamma(-2))
\]

\[
P(Q_e = -2|V_e = 0) = \lambda_e + \mu_e(\phi_e\sigma_{-2}(S_e = 0) + (1 - \phi_e)\Gamma(-2))
\]

By substituting the \(\sigma_i\) terms which hold for any equilibrium (\(S_e = 2\) always buys and \(S_e = 0\) always sells), \(\Gamma(i)\) is equal to:

\[
\Gamma(2) = (p + q - 2pq)\sigma_2(S = 1) + (1 - p)(1 - q)
\]

\[
\Gamma(1) = (p + q - 2pq)\sigma_1(S = 1) + (1 - p)(1 - q)
\]

\[
\Gamma(-1) = pq + (p + q - 2pq)\sigma_{-1}(S = 1)
\]

\[
\Gamma(-2) = pq + (p + q - 2pq)\sigma_{-2}(S = 1)
\]
Appendix E

Proofs of Chapter 3

E.1 Crowdfunding Equilibrium

Proof of Theorem 3.1, Part 1
The entrepreneur seeks to maximize her profits in each state, given that the bank extends her loan. In the high demand state, her profit function is equal to:

\[ \pi_B(H) = (\phi + \sigma - BP)(P - (1 + r_B)c) - (1 + r_B)F \]  

(E.1)

Optimization of this function leads to a unique best price of:

\[ P_B(H) = \frac{\phi + \sigma + (1 + r_B) \cdot B \cdot c}{2 \cdot B} \]  

(E.2)

Likewise, her profit function in the low state and subsequent optimal price choice are:

\[ \pi_B(L) = (\phi - \sigma - BP)(P - (1 + r_B)c) - (1 + r_B)F \]  

(E.3)

\[ P_B(L) = \frac{\phi - \sigma + (1 + r_B) \cdot B \cdot c}{2 \cdot B} \]  

(E.4)

The entrepreneur is willing to engage in banking if she earns a positive expected profit. Since she has limited liability, if she earns positive profit in at least the high-demand state, this condition is satisfied. By inserting the optimal price from the high demand state into the profit function from the high demand state, this gives the condition:

\[ \pi_B(H) = \frac{(\phi + \sigma - (1 + r_B) \cdot B \cdot c)^2}{4B} - (1 + r_B)F \]  

(E.5)

This equation is quadratic in the demand signal \( \phi \). Subject to parameter constraints \((F > 0)\) such that this quadratic equation has two real roots, this function is positive in the second derivative and therefore this project is profitable to the entrepreneur for all projects with \( \phi \geq \phi_{BE} \) where \( \phi_{BE} \) is the greater of the two real roots of this equation. The lesser of the two real roots corresponds to a negative profit per unit-negative quantity pair, which though
Appendix E. Proofs of Chapter 3

mathematically profitable, is ruled out. This threshold value is:

$$\phi_{BE} = -\sigma + (1 + r_B)Bc + \sqrt{4(1 + r_B)BF} \quad (E.6)$$

Unlike the entrepreneur, the bank must consider both states to determine if the project is profitable. Given that the bank offers the initial loan \( F \), there are 3 possible outcomes for the bank once the state is realized: (1) The project will be profitable if credit is extended and the bank recovers the full value of the loan; (2) the project will be profitable excluding fixed costs if credit is extended and the bank will recover some portion of the loan; (3) the project is completely unprofitable even excluding fixed costs and the bank will not extend credit.

Since the entrepreneur will only engage in the project if she earns positive profits, the bank only considers projects where this is true in at least one state, there are three cases to consider for the bank: (1) full loan values are recovered in both states; (2) the full loan value is recovered in the high demand state and a partial loan value is recovered in the low demand state, and; (3) the full loan value is recovered in the high demand state and credit is not extended in the low demand state.

Case 1 occurs when the project is profitable for the entrepreneur in both states, By inserting the optimal price into her profit function in the low demand state, this occurs if \( \phi \geq \phi_{B1} \), where:

$$\phi_{B1} = \sigma + (1 + r_B)Bc + \sqrt{4(1 + r_B)BF} \quad (E.7)$$

In this case the loan is profitable for the bank and entrepreneur in both states. If this condition does not hold, the project is unprofitable in the low-demand state and the bank will only recover some portion of its loan. It is still profitable for the bank to extend credit in the low demand state if the project is profitable excluding fixed costs. By the entrepreneur’s profit function in the low demand state, this is true if:

$$\phi \geq \sigma + (1 + r_B)Bc \quad (E.8)$$

In this case, the bank recovers a partial value of the loan. The total loan is then profitable if the expectation of the full value recovered in the high demand state is greater than the expected loss in the low demand state. Using the profit functions in each state from the entrepreneur and the loan values from the bank, this gives:

$$p r_B \left( \frac{\phi + \sigma - (1 + r_B)Bc}{2} c + F \right) \geq -(1 - p) \left( \frac{\phi - \sigma - (1 + r_B)Bc}{4B} - F \right) \quad (E.9)$$

If this equation has no real roots, or has real roots that are less than \( \sigma + (1 + r_B)Bc \) then all projects in this range are profitable for the bank and \( \phi_{B2} = \sigma + (1 + r_B)Bc \) via Equation E.8.

The vertex of Equation E.9 is equal to \( \sigma + (1 + r_B)Bc - \frac{p r_B c B}{(1 - p)} \) which is strictly less than
Equation E.8. Therefore, if the equation has real roots, than only one can be greater than Equation E.8 and the threshold \( \phi_{B2} \) is equal to this value.

Finally, if the loan is not profitable for the bank to extend in the low demand state, the bank incurs a loss of \( F \) if this state is realized. The total loan is profitable for the bank if:

\[
pr_B \left( \frac{\phi + \sigma - (1 + r_B)Bc + F}{2} \right) \geq (1 - p)F
\]  

(E.10)

The loan is then profitable in Case 3 if \( \phi \geq \phi_{B3} \), where:

\[
\phi_{B3} = \frac{2F}{pr_B c} (1 - p(1 + r_B)) - (\sigma - (1 + r_B)Bc)
\]  

(E.11)

In all three cases, the entrepreneur must be willing to engage in the loan, which requires that \( \phi \geq \phi_{BE} \). To determine \( \phi_B \), the bank first assesses whether a project is viable where the loan is not extended in the low demand state.\(^1\) If there exists a \( \phi \geq \phi_{BE} \) such that:

\[
\phi_{B3} \leq \phi < \sigma + (1 + r_B)Bc
\]  

(E.12)

Then there exists a profitable project where the bank does not extend the loan in the low demand state, which is profitable in the high demand state. If this is true, the minimum feasible project is equal to:

\[
\phi_B = \max\{\phi_{BE}, \phi_{B3}\}
\]  

(E.13)

If there does not exist a \( \phi \geq \phi_{BE} \) such that Equation E.12 holds, then there is no feasible project in which the bank does not extend credit in the low demand state. If this is true, then the only feasible projects are those in which the bank only extends credit in both states. There is a feasible project where the bank extends credit in both states, but only recovers the full value of the loan in the high demand state if there exists a \( \phi \geq \phi_{BE} \) such that:

\[
\phi_{B2} \leq \phi \leq \phi_{B1}
\]  

(E.14)

Given that there is no feasible \( \phi \) by Equation E.16, and given that there is a feasible \( \phi \) under Equation E.14, then the minimum feasible project threshold is equal to:

\[
\phi_B = \max\{\phi_{BE}, \phi_{B2}\}
\]  

(E.15)

For \( \sigma > 0 \), then \( \phi_{BE} < \phi_{B1} \) and the entrepreneur is willing to engage in some projects where \( \phi_{BE} \leq \phi < \phi_{B1} \). Therefore she will willingly engage in projects which are profitable in the high demand state only. If there does not exist a \( \phi \) such that the Equation E.14 holds, then the bank is only willing to engage in projects in which it recovers its full loan value with certainty. Then, since \( \phi_{BE} < \phi_{B1} \), the minimum threshold value for a loan is:

\(^1\)From before, projects are not extended in the low demand state if \( \phi \leq \sigma + (1 + r_B)Bc \).
Since the conditions under which Equations E.12, E.14 and E.16 are valid are both mutually exclusive and collectively exhaustive, then there exists a unique $\phi_B$. Since the profit functions for both the bank and entrepreneur are increasing in $\phi$ above their respective profitability thresholds, then they are both willing to engage in all projects with $\phi \geq \phi_B$.

**Proof of Theorem 3.1, Part 2**

As in Theorem 3.1, Part 1, the entrepreneur wishes to maximize her profit function. She is able to set a minimum quantity $Q_{min}$ such that, if the total quantity demanded for her project is less than $Q_{min}$, funds are returned to consumers and they do not discount their willingness to pay. Her profit function is:

$$\pi_{CF} = E[(\Phi - BP)(P(1 - r_{CF}) - c) - F)|Q \geq Q_{min}] \quad (E.17)$$

She may unconditionally maximize her profit function across both states, which gives the identical solution of:

$$P_{CF} = \frac{(1 - r_{CF})(\phi - (1 - 2p)\sigma) + Bc}{2B(1 - r_{CF})} \quad (E.18)$$

This solution is feasible if the project succeeds in the low demand state, which is to say:

$$\pi_{CF}(L) = (\phi - \sigma - BP_{CF})(P_{CF}(1 - r_{CF}) - c) - F \geq 0 \quad (E.19)$$

As she is permitted to set a minimum order quantity, she may either choose not to do so, or may do so with any value $Q_{min} \leq \phi - \sigma - BP_{CF}$, and the project will proceed in both states. She may use $Q_{min}$ to isolate the high demand state only.\(^2\) If she does so, her profit function is then:

$$\pi_{CF} = p(\phi + \sigma - BP)(P(1 - r_{CF}) - c) - F) \quad (E.20)$$

Derivation of this equation, leads to an optimal price of:

$$P_{CF} = \frac{(1 - r_{CF})(\phi + \sigma) + Bc}{2B(1 - r_{CF})} \quad (E.21)$$

In order to remove the lower state, the entrepreneur can set any $Q_{min}$ such that:

$$\phi + \sigma - BP_{CF} \geq Q_{min} > \phi - \sigma - BP_{CF} \quad (E.22)$$

By re-inserting the price from Equation E.21 into the profitability equation for the high demand state, and solving the subsequent quadratic equation, it can be shown that the project is

\(^2\)While never optimal, for $\sigma > 0$ it is also not possible to isolate the low demand state
profitable for all:
\[ \phi \geq \frac{\sqrt{4(1 - r_{CF})BF + Bc}}{(1 - r_{CF})} - \sigma \]  
(E.23)

Evaluating Equation E.23 with equality gives the value \( \phi_{CF} \) such that crowdfunding is profitable for all \( \phi \geq \phi_{CF} \).

To select between crowdfunding in the high state only or in both states, the entrepreneur inserts the optimal prices from Equations E.18 and E.21 into their respective profit functions, which yields the condition that the entrepreneur prefers crowdfunding in both states when:

\[
\frac{[(1 - r_{CF})(\phi + (2p - 1)\sigma) - Bc]^2}{4B(1 - r_{CF})} - F \geq p \left[ \frac{[(1 - r_{CF})(\phi + \sigma) - Bc]^2}{4B(1 - r_{CF})} - F \right]
\]  
(E.24)

Algebraic manipulation again gives a quadratic which is positive in the second derivative. The entrepreneur then prefers to crowdfund in both states for all \( \phi \) greater than the greater of the two real roots, which can be designated \( \phi_{QMin} \). The entrepreneur then sets a price equal to Equation E.18 for all \( \phi \geq \phi_{QMin} \), and sets a price equal to Equation E.21 for all \( \phi_{QMin} > \phi \geq \phi_{CF} \).

**Proof of Theorem 3.1, Part 3**

Proof is a simplified proof of Theorem 3.2, Part 3, one small modifications. Namely the profit function for pre-ordering is replaced by that of crowdfunding.

Since, for projects where the entrepreneur earns positive profits in both the high and low demand states, these functions are identical, then the proof is otherwise identical in this section.

**Proof of Proposition 3.1**

See Proof of Theorem 3.1, Part 1

**Proof of Proposition 3.2**

See Proof of Theorem 3.1, Part 2

**Proof of Proposition 3.3**

See Proof of Theorem 3.1, Part 2

**Proof of Proposition 3.4**

See Proof of Theorem 3.1, Part 2

**Proof of Proposition 3.5**

See Proof of Theorem 3.1, Part 3

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**E.2 Pre-Ordering Equilibrium**

**Proof of Theorem 3.2, Part 1**

As the banking technology is unchanged with the addition of the crowdfunding technology, this proof is identical to the proof of Theorem 3.1, Part 1.

**Proof of Theorem 3.2, Part 2**
The entrepreneur seeks to maximize her crowdfunding profit function;

\[ \pi_{PO} = p\pi_{PO}(H) + (1 - p)\pi_{PO}(L) \]  

(E.25)

Which is itself, a weighted average of her profit function in each state \( S \):

\[ \pi_{PO}(S) = (\Phi - BP_{CF})(P_{PO}(1 - r_{CF}) - c) - F \]  

(E.26)

If the project does not have sufficient funds in either state to cover the total costs, the project fails. If the project fails in both states, consumers are unwilling to pay any cost. If the project fails in the low-demand state, consumers reduce their willingness to pay by the probability the project fails \((1 - p)\), meaning they are willing to pay a fraction \(pP\) of the initial price \(P\). Their demand curve in the high-demand state then becomes:

\[ Q = \phi + \sigma - \frac{BP}{p} \]  

(E.27)

Since the entrepreneur’s profit function is quadratic in price with a negative second derivative, the unique solution to the entrepreneur’s optimization unconditional optimization problem is:

\[ P_{PO} = \frac{(1 - r_{CF})(\phi - (1 - 2p)\sigma) + Bc}{2B(1 - r_{CF})} \]  

(E.28)

This solution is feasible if the project succeeds in the low demand state, which is to say:

\[ \pi_{PO}(L) = (\phi - \sigma - BP_{PO})(P_{PO}(1 - r_{CF}) - c) - F \geq 0 \]  

(E.29)

Equation E.29 is quadratic in \(\phi\), designate \(\phi_{PO}(1)\) to be the greater of the two real roots, such that it is greater than zero for all \(\phi \geq \phi_{PO}(1)\). If this solution is unfeasible, the entrepreneur has two further options. She may pick a different price that is profitable in both states, or she may pick a price that fails in the low-demand state, and is discounted by a factor \((1 - p)\).

Since the unconditional optimization is quadratic in price, the best price where the project is profitable in both states if Equation E.29 doesn’t hold is the corner solution to \(\pi_{PO}(L) = 0\). This equation is quadratic in \(P\) where the vertex is the optimal solution to a low-state only demand solution. The highest price where profit is 0, then occurs at the greater of the two roots of this equation, which is at:

\[ P_{PO} = \frac{(\phi - \sigma) + Bc + \sqrt{((1 - r_{CF})(\phi - \sigma) - Bc)^2 - 4BF}}{2B(1 - r_{CF})} \]  

(E.30)

The other option for the entrepreneur is to allow the project to fail in the low demand state, and to allow consumers to discount their willingness to pay. In this case, she optimizes her high demand state profit functions, accounting for the reduced price:

\[ \pi_{PO}(H) = (\phi - \sigma - \frac{BP_{PO}}{p})(P_{PO}(1 - r_{CF}) - c) - F \]  

(E.31)
The unconditional optimization to this problem gives an price of:

\[ P_{PO} = \frac{p(1 - r_{CF})(\phi + \sigma) + Bc}{2B(1 - r_{CF})} \quad (E.32) \]

Given that the project produces 0 profit in the low-demand state with either the prices in Equation E.30 or E.32, the optimal choice between those two prices is parameter dependent, given the prices are entered into the profit function for the high-demand state. In cases where these profits are equal, the entrepreneur has a preference for the project not to fail in either state and selections the price in Equation E.30.

Consider the entrepreneur’s profit function with the prices from Equations E.30 and E.32 inserted. Given these are quadratic equations which are increasing in \( \phi \), with a negative intercept then there exists for each a \( \phi \) such that they equal 0 with equality. Designate the greater of the roots in each of these equations to be \( \phi_{PO}(2) \) and \( \phi_{PO}(3) \) respectively. Since profit is increasing in \( \phi \), then any \( \phi \geq \phi_{PO}(X) \) is also profitable for that same pricing strategy.

Therefore the entrepreneur is able to earn positive profits on any project with \( \phi \geq \min\{\phi_{PO}(2), \phi_{PO}(3)\} \), and the unique equilibrium value \( \phi_{PO} \) exists as:

\[ \phi_{PO} = \min\{\phi_{PO}(2), \phi_{PO}(3)\} \quad (E.33) \]

**Proof of Theorem 3.2, Part 3**

Given the results of Theorem 3.1, Parts 1 and 2, the profit function of the entrepreneur is well defined for both banking and pre-ordering. From these same sections, it is shown, that for a \( \phi \) such that projects are profitable in both states (for either banking or pre-ordering) than any \( \phi + \epsilon \), where \( \epsilon > 0 \), is also profitable in both states. Consider such a project, profitable in both high and low demand states, where the profit functions for pre-ordering and banking are:

\[ \pi_{PO} = p(\phi + \sigma - BP_{PO})(P_{PO}(1 - r_{CF}) - c) \]
\[ + (1 - p)(\phi - \sigma - BP_{PO})(P_{PO}(1 - r_{CF}) - c) - F \quad (E.34) \]

\[ \pi_{B} = p(\phi + \sigma - BP_{B}(H))(P_{B}(H) - (1 + r_{B})c) \]
\[ + (1 - p)(\phi - \sigma - BP_{B}(L))(P_{B}(L) - (1 + r_{B})c) - (1 + r_{B})F \quad (E.35) \]

For projects which are profitable in both states, for both technologies, entrepreneurs prefer banking for all projects where:

\[ \pi_{B} - \pi_{PO} \geq 0 \quad (E.36) \]

Inserting the respective optimal prices into Equation E.36, leads to a quadratic with a positive second derivative. If Equation E.36 has two real roots, than for all \( \phi \) greater than the larger
of the two roots. Banking is more profitable. If this root is within the parameter space where banking and pre-ordering are both profitable in both states \( \phi \geq \max\{\phi_{B1}, \phi_{PO(1)}\} \) then there exists a threshold value \( \bar{\phi} \) which is equal to the larger of the two roots, where banking is more profitable for all \( \phi \geq \bar{\phi} \).

If Equation E.36 had one, or no real roots, or these roots were outside the parameter space where banking and pre-ordering are both profitable \( \phi < \max\{\phi_{B1}, \phi_{PO(1)}\} \) than entrepreneurs always prefer banking for projects profitable in both states. The threshold \( \bar{\phi} \), such that entrepreneurs prefer banking for all \( \phi \geq \bar{\phi} \) is then can then be defined as the lowest value in this parameter space \( \bar{\phi} = \max\{\phi_{B1}, \phi_{PO(1)}\} \).

It is not necessary to consider projects which are not profitable in both states for both banking and crowdfunding, since a threshold is guaranteed to exist above. It is important to note, that while banking is preferred for all \( \phi \geq \bar{\phi} \), it is not necessarily true that pre-ordering is preferred for any or all \( \phi < \bar{\phi} \).

**Proof of Proposition 3.6**
See Proof of Theorem 3.2, Part 2

**Proof of Proposition 3.7**
See Proof of Theorem 3.2, Part 2

**Proof of Proposition 3.8**
See Proof of Theorem 3.2, Part 2. In addition, through algebraic manipulation of the threshold \( \phi_{CF} \) in Theorem 3.1 Part 2, it can be shown that this threshold is less than those in the pre-ordering technology \( \phi_{PO(2)}, \phi_{PO(3)} \). Intuitively, this follows from the value that the entrepreneur is not able to set the price \( P_{CF} = \frac{(1-r_{CF})(\phi+\sigma)+Bc}{2B(1-r_{CF})} \) with the pre-ordering technology, which is the optimal price in the high demand state, since she must discount for project failure. She then earns a lower profit and has a lower minimum project threshold.

### E.3 Production Uncertainty

**Proof of Proposition 3.9**
Consider the threshold project in the model without production uncertainty, and designate it to be the low cost state:

\[
\phi_{CF} \geq \sqrt{4(1-r_{CF})BF + B(c-\gamma) - \sigma} \tag{E.37}
\]

If the entrepreneur was able to isolate the low cost state, and the high demand state, this would remain the threshold project. For \( \gamma > 0 \) and \( q > 0 \), a project which earns at least 0 profit in the high cost state would meet the condition:

\[3\]There may exist a similar such threshold less than this value, but, as this threshold is not unique than the proof that it is at most this value is sufficient for its existence.
\[ \phi \geq \frac{\sqrt{4(1 - r_{CF})BF + B(c + \gamma)}}{(1 - r_{CF})} - \sigma \geq \phi_{CF} \]  

(E.38)

For a project which earns less than 0 profit in the high cost state, the entrepreneur must discount her price by value \( q \). The optimal price in this case is not equal to \( P_{CF} \), which is the unique optimal price derived in the no production uncertainty case. Therefore, profit is lower in the low-cost, high demand case and the minimum feasible \( \phi \) is strictly greater than \( \phi_{CF} \). Therefore, in either case, the minimum feasible project size \( \phi_{CFP} \) is strictly greater than the case with no production uncertainty \( \phi_{CF} \).