Essays on Information Acquisition, Auction Design, and Persuasion

by

Xin Zhao

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Graduate Department of Economics
University of Toronto

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Abstract

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Chapter 1 studies how the composition and voting rule of a decision-making committee affect the incentives for its members to acquire information. Fixing the voting rule, a more polarized committee acquires more information. If a committee designer can choose the committee members and voting rule to maximize her payoff from the collective decision, she forms a heterogeneous committee adopting a unanimous rule, in which one member moderately biased toward one decision serves as the decisive voter, and all others are extremely opposed to the decisive voter and serve as information providers. The decisive voter is not identical to the designer.

Chapter 2 studies mechanism design by a seller privately informed of the quality of an indivisible object. The privacy of the seller’s information matters for mechanism design: selecting a mechanism that maximizes the seller’s profit when her information is public is not incentive compatible for the seller when her information is private, as a lower-quality seller has an incentive to mimic a higher-quality seller. I show that reserve prices are the least costly device to separate higher-quality sellers from lower-quality ones. In equilibria that maximize the expected profit of every type of the seller among all separating equilibria, the lowest-quality seller adopts her public-information optimal mechanism, and each higher-quality seller adopts a mechanism that differs from her public-information optimal mechanism only in that the reserve prices are higher.

Chapter 3 studies how a privately informed persuader should persuade a group of
listeners if the listeners can imperfectly verify his information at a cost. Should he bring the listeners together and persuade them simultaneously or communicate with them sequentially? The answer depends on the verification costs of the listeners. Simultaneous persuasion outperforms sequential persuasion when it is not very costly for the listeners to verify the persuader’s information. The opposite can be true if verifying the persuader’s information is very costly. In sequential persuasion, it is optimal for the persuader to first approach the listener who is hardest to persuade. For both persuasion modes, in the persuader-optimal equilibria, the persuader pools extreme private information, and “truthfully reveals” moderate private information.
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To Tengteng and my parents.
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Chapter 1

Heterogeneity and Unanimity: Optimal Committees with Information Acquisition
1.1 Introduction

Committees are commonly employed to make important economic, political, and business decisions. In many situations, a decision-making committee is composed by an empowered individual, or committee designer. For instance, a city mayor may appoint an ad hoc committee to deal with a particular issue, and a manager or chair of an organization may form a hiring committee to screen and select job candidates. In these situations, how should the designer choose the membership of the committee, and what voting rule should the committee employ?

In this paper, I study how the composition and voting rule of a decision-making committee affect the incentives for its members to acquire information (and thus the quality of its collective decision), and characterize the optimal committee design when the designer can choose both the members and the voting rule. I find that when the voting rule is fixed, a more polarized committee acquires more information in equilibrium. To maximize her expected payoff from the collective decision, the designer will form a heterogeneous committee that adopts a unanimous rule. The heterogeneous committee consists of two types of members who have opposing interests.

The committee in the model makes a binary decision. All of its members share a common goal: matching their decision with the value of a binary underlying state. Because information about the state is imperfect, two kinds of erroneous decision can be made. The committee members differ in their losses from the two possible errors, so they have heterogeneous preferences over the decisions. The collective decision is made through voting. Prior to voting, each member can unilaterally make continuous efforts to acquire a continuous signal about the state. I focus the analysis on the case where all the information acquired is publicly observable, to exclude issues related to strategic information transmission between the members.

A member’s incentive to acquire information has two components. The first component is the incentive to prevent errors: a member would like to have more information to
reduce the probability of making an erroneous decision. This incentive exists regardless of the composition of the committee. The second component is the incentive to reduce disagreement. This appears only in heterogeneous committees. Consider a member who has an extreme preference for one decision over the other. If he wants to increase the probability that another member—who has an opposite but non-extreme preference—votes for his favorite decision, he will provide that member with more precise information to induce a larger response and thus make that member more likely to switch from his initially preferred decision. The magnitudes of the two incentives above are affected by the incentive to free ride: when other members are present and acquired information is shared within the committee prior to the final decision, a member’s incentive to collect information decreases if other members acquire information, as information is a public good that is costly to obtain.

The two components above, incentive to prevent errors and incentive to reduce disagreement, depend on the preferences of all committee members and the voting rule used by the committee. However, I find that regardless of the voting rule, there is always a monotonic relationship between the preferences of the members and their incentives to acquire information in equilibrium: when the voting rule favors one of the alternatives (in a sense that will be made clear), the members who lean more toward the other alternative have a stronger incentive to acquire information. This monotonicity is primarily due to the incentive to reduce disagreement. If the voting rule favors one alternative, a member who prefers the other alternative would like to make the aggregate information available to the group more informative, to increase the probability that the members leaning toward the alternative favored by the voting rule switch their votes to his preferred decision.

The monotonicity result implies that if we make the committee members more polarized while fixing the voting rule, the members who dislike the decision favored by the voting rule have more incentive to acquire information, and the ones who prefer the de-
cision favored by the voting rule have less incentive to acquire information. I show that the former effect dominates the latter effect, so the aggregate information acquired by the committee increases with the degree of polarization.

The voting rule, though irrelevant for the monotonicity result, is crucial for the direction of the monotonicity, because it determines which decision is favored and to what extent that decision is favored. In practice, unanimity is often required to make collective decisions. In this paper, I show that unanimous rules may not be optimal in incentivizing information acquisition. The intuition is that when we increase the number of votes required to implement one decision, the other decision becomes more favored by the voting rule, and the members leaning toward the other decision have less incentive to acquire information. This may lead the aggregate information acquired to decrease.

If a committee designer can freely choose the committee members and voting rule, how will she design the committee? An optimal committee turns out to have the following features. First, voting is unanimous: one alternative is endogenously chosen as the status quo and unanimity is required to overturn it. Second, the committee consists of two types of members: one member favors the status quo, and all others are extremely opposed to the status quo. Given the unanimity requirement, the single member favoring the status quo is decisive, and the rest of the members are mainly responsible for providing information. The monotonicity result regarding the preferences and incentive to acquire information intuitively explains why, at the optimum, the information collectors should have extreme preferences against the status quo. Third, the decisive member’s preferences are typically not perfectly aligned with the designer’s preferences. This indicates that sometimes a collective decision made by an optimal committee based on acquired information may differ from the one desired by the designer. This is because having a decisive member not aligned with her may incentivize other members to acquire more information and improve the precision of the final decision. After characterizing the optimal committees, I show that it is always optimal for the designer to delegate the
decision to the designed committee, even if she can perfectly observe all the information acquired by the group.

In this paper, I use a Gaussian information structure to model the information acquisition behavior of committee members, in which every committee member has access to a signal normally distributed conditional on the true state. Members choose the precisions of their signals, and the information cost is linear in the precision. In Section 1.5 I extend the model to allow for convex cost and show that for the committee design problem, the main ideas carry over. In Section 1.6 I examine the case in which the information acquired by each member is private.

1.2 Related Literature

This paper is related to the literature on decision-making committees with endogenous information. Unlike this paper, most of this literature assumes that committee members are \textit{ex ante} identical. Li (2001) and Gershkov and Szentes (2009) study \textit{ex ante} efficient decision rules. Li (2001) finds that a distortionary decision rule that is more conservative than the \textit{ex post} efficient one can mitigate the free-riding problem in information acquisition and improve \textit{ex ante} efficiency. Gershkov and Szentes (2009) examine the socially optimal decision rule from a mechanism design perspective. The current paper focuses on standard voting rules, and examines their roles in information acquisition, instead of social efficiency.

Persico (2004) and Gerardi and Yariv (2008a) study homogeneous committee design with endogenous information, taking the size and decision rule of the committee as the choice variables. In both papers, each committee member can purchase a binary signal with fixed precision. Persico (2004) shows that the optimal committee adopts the \textit{ex post} efficient voting rule, which aggregates all acquired information. This voting rule requires a high level of agreement to overturn the \textit{status quo} only if the signals available
to the committee members are very precise. Compared with Persico (2004), Gerardi and Yariv (2008a) consider more general decision rules that allow communication among the voters. They find that the \textit{ex ante} optimal decision rule may be \textit{ex post} inefficient. I allow each member to choose the precision of his signal, and allow the committee to be heterogeneous.

The impact of heterogeneity—either in preferences and in priors—among the members of a decision-making group on the information acquisition behavior of group members has been studied in the literature. Chan et al. (2015) employ a dynamic model to study the impact of \textit{preference heterogeneity} on a collective decision-making process. In their paper, costly information acquisition is modeled as a collective stopping problem—public information keeps arriving over time until the committee collectively decides to stop it. They demonstrate that greater heterogeneity in committee members’ preferences can induce a more stringent stopping rule, and consequently more information acquisition on average. I have a similar result regarding the impact of preference heterogeneity on information acquisition, but my result is due to the increased incentives of some members to reduce disagreement and prevent errors when preference heterogeneity increases.

Che and Kartik (2009), Van den Steen (2010), and Hirsch (2015) study decision-making groups consisting of a decision maker and an information provider. They all show that when the decision maker and an information collector have \textit{heterogeneous priors}, there exists a “persuasion effect” that incentivizes the information provider to acquire more information than in the case of a common prior. This persuasion effect is similar to the \textit{incentive to reduce disagreement} studied in the current paper. A key difference is that the former appears only when there is prior heterogeneity, whereas the latter exists regardless of whether the heterogeneity of committee members is in prior, or preference, or both.

Another strand of literature related to this paper is the one on advisory committees, or informational committees, with information acquisition. Unlike a decision-making
committee, an advisory committee has no decision power, but provides information to a decision maker. Gerardi and Yariv (2008b) find that the optimal advisory committee for a decision maker is composed of identical members whose preferences are opposed to that of the decision maker. Beniers and Swank (2004) study the relationship between the magnitude of the cost of information and optimal committee composition. They find that when the cost of information is low, the optimal committee is homogeneous; when the cost of information is high, the optimal committee is heterogeneous. Cai (2009) shows that more uncertainty in other members’ preferences may raise one member’s incentive to acquire information.

1.3 Model

Suppose that an $n$-member committee is assembled to collectively decide to accept, $A$, or reject, $R$, a proposal. A random variable $\theta \in \{0, 1\}$ captures the quality of the proposal, with $\theta = 0$ meaning that the proposal should be accepted and $\theta = 1$ meaning that the proposal should be rejected. The payoff of each member depends on the collective decision $d \in \{A, R\}$ and $\theta$. Specifically, the payoff function $u_i : \{A, R\} \times \{0, 1\} \to \mathbb{R}$ of member $i = 1, \ldots, n$ is given by

\[
\begin{align*}
    u_i (A, 0) &= 0, & u_i (R, 1) &= 0, \\
    u_i (A, 1) &= -(1 - q_i), & u_i (R, 0) &= -q_i,
\end{align*}
\]

where $q_i \in [0, 1]$ for all $i$. That is, the payoffs of the member from correct decisions (correctly accepting the proposal or correctly rejecting the proposal, i.e., $(d, \theta) \in \{(A, 0), (R, 1)\}$) are normalized to 0, the loss from false acceptance is $1 - q_i$, and the loss from false rejection is $q_i$. Let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ denote the preference profile of the committee. I assume that $\mathbf{q}$ is common knowledge.

Committee members can differ in their preferences. Without loss of generality, I
assume, throughout the paper, that

\[ q_1 \leq q_2 \leq \ldots \leq q_{n-1} \leq q_n. \quad (1.1) \]

This assumption implies that a lower indexed member (weakly) leans more toward rejection than does a higher indexed member, as his loss from false rejection is smaller.

Two points are worth mentioning regarding the members’ preferences. Firstly, whenever there is perfect information about the true state, members have no conflicts over the final decision, as all of them would like to make the correct decision. Secondly, the sum of a member’s losses from the two types of errors is normalized to 1, so the heterogeneity in the members lies only in their relative concerns over the two types of error.

Members have a common prior over the state \( \theta \), with

\[ \Pr (\theta = 1) = 1 - \Pr (\theta = 0) = \gamma, \]

where \( \gamma \in (0, 1) \). Given this prior, if \( q_i < \gamma \), member \( i \) would prefer to reject the proposal, as \( (1 - q_i) \gamma > q_i (1 - \gamma) \). We call this member pro-rejection. Similarly, if \( q_i > \gamma \), we say that the member is pro-acceptance.\footnote{The common prior assumption is standard in the literature on committees. Although assuming heterogeneous priors would complicate the analysis, the main insights would carry over if we reparameterize the model and assume that the expected losses of a member from the two types of error are summed up to be a constant, which is invariant across committee members.}

Prior to making the final decision, each member \( i \) acquires a signal \( s_i \), which has the structure

\[ s_i = \theta + \varepsilon_i, \text{ where } \varepsilon_i \sim N (0, 1/\rho_i), \forall i, \]

where \( \rho_i \) represents the precision of signal \( s_i \). Member \( i \) chooses the value of \( \rho_i \), incurring the cost \( C (\rho_i) \). Besides the costly signals, the committee has access to a free signal \( s_0 \), which has the same structure as the other signals and has fixed precision \( \rho_0 \). I assume that \( s_0, s_1, s_2, \ldots, s_n \) are independent conditional on \( \theta \).
The free public signal $s_0$ is important for the analysis. I need to assume that $s_0$ exists and is sufficiently precise to avoid a non-concavity issue in members’ expected payoffs (see Assumption 1.3).

Furthermore, I impose the following assumption for the rest of the analysis. The public observability of the signals excludes the issue of strategic information transmission among the members and enables us to focus on the role of information acquisition in committee decision-making. The public observability of the precision profile gives rise to the incentive to reduce disagreement, an important component of members’ incentives to acquire information that can appear only in heterogeneous committees. In Section 1.6, I extend the analysis to the case in which the information acquired by each member is private and discuss the conditions under which the main results carry over.

**Assumption 1.1.** The precision profile $(\rho_0, \rho_1, \rho_2, \ldots, \rho_n)$ and realization of $(s_0, s_1, s_2, \ldots, s_n)$ are observable to all members.

Given $(\rho_0, \rho_1, \rho_2, \ldots, \rho_n)$ and $(s_0, s_1, s_2, \ldots, s_n)$, I define the “aggregate signal”,

$$s = \frac{\sum_{i=0}^{n} \rho_is_i}{\rho},$$

where $\rho = \sum_{i=0}^{n} \rho_i$. This signal has distribution $N(\theta, 1/\rho)$ conditional on $\theta$. Let $F_1(s|\rho)$ and $F_0(s|\rho)$ respectively denote the cumulative distribution functions of $s$ conditional on $\theta = 1$ and $\theta = 0$, and let $f_1(s|\rho)$ and $f_0(s|\rho)$ denote the corresponding density functions. Given Assumption 1.1 and the properties of a Gaussian information structure, each member’s preferred decision depends only on $s$. Specifically, member $i = 1, \ldots, n$.

---

2 In reality, decision-making committees are usually provided with information additional to the information acquired by themselves before they make the final decisions. For example, in a jury trial, before the jury determines the verdict, the judge sometimes provides the jury with a summary of the facts. In regular meetings of the Federal Open Market Committee of Federal Reserve, prior to determining the monetary policies for the upcoming periods, the staff of the Federal Reserve present information concerning “business and credit conditions and domestic and international economic and financial developments as will assist the committee in the determination of the open market policies.” See Federal Open Market Committee Rules of Organization, as amended effective January 27, 2015. http://www.federalreserve.gov/monetarypolicy/rules_authorizations.htm

3 When observing $(\rho_0, \rho_1, \rho_2, \ldots, \rho_n)$ and $(s_0, s_1, s_2, \ldots, s_n)$, member $i$ prefers to reject the proposal.
prefers to reject the proposal if and only if \( s \geq s_i (\rho) \), where \( s_i (\rho) \) satisfies

\[
(1 - q_i) \gamma f_1 (s_i (\rho) | \rho) = q_i (1 - \gamma) f_0 (s_i (\rho) | \rho).
\] (1.2)

Thus, \( s_i (\rho) \) is the value of \( s \) at which the expected losses of member \( i \) from the two possible errors are equal, and the inequality \( s \geq s_i (\rho) \) indicates that the loss of member \( i \) from false acceptance is not smaller than that from false rejection. From equation (1.2), we have

\[
s_i (\rho) = \frac{1}{2} + \frac{1}{\rho} \ln \frac{q_i (1 - \gamma)}{(1 - q_i) \gamma}.
\] (1.3)

Since \( \rho > 0 \), it is obvious that \( s_i (\rho) \) is increasing in \( q_i \). This is intuitive, because a member more concerned with false rejection is more reluctant to vote for rejection. Given (1.1), \( s_i (\rho) \) is increasing in \( i \). To simplify the notation, in the rest of the analysis I write \( s_i \) rather than \( s_i (\rho) \).

After the signals are realized, members vote to determine the collective decision. I if and only if

\[
q_i (1 - \gamma) \prod_{i=0}^n f_{0,i} (s_i | \rho_i) \leq (1 - q_i) \gamma \prod_{i=0}^n f_{1,i} (s_i | \rho_i),
\]

where \( f_{1,i} (s_i | \rho_i) \) and \( f_{0,i} (s_i | \rho_i) \) represent the density functions of signal \( s_i \) conditional on \( \theta = 1 \) and \( \theta = 0 \). This inequality is equivalent to

\[
\frac{q_i (1 - \gamma)}{(1 - q_i) \gamma} \leq \prod_{i=0}^n \frac{f_{1,i} (s_i | \rho_i)}{f_{0,i} (s_i | \rho_i)} = \frac{f_1 (s | \rho)}{f_0 (s | \rho)},
\]

where the equality is based a property of the normal distribution that can be proved using simple algebra.

Since \( f_1 (s | \rho) / f_0 (s | \rho) \) is increasing in \( s \), if \( s \geq s_i (\rho) \), which is defined in (1.2), then it is optimal for member \( i \) to choose rejection. If \( s < s_i (\rho) \), then it is optimal for member \( i \) to choose acceptance. Therefore, \( s_i (\rho) \) completely characterizes the \textit{ex post} optimal decision rule of member \( i \), and is the cutoff that minimizes his expected loss, i.e.,

\[
s_i (\rho) = \arg \max_{\hat{s}} - (1 - q_i) \gamma F_1 (\hat{s} | \rho) - q_i (1 - \gamma) [1 - F_0 (\hat{s} | \rho)].
\]

See Li (2001) for more details.
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focus on voting rules that take acceptance as the *status quo* and require \( k \) votes to reject the proposal, with \( 1 \leq k \leq n \). I call \( k \) the voting threshold. The assumption below allows us to focus on a reasonable set of voting strategies.

**Assumption 1.2.** At the voting stage, no member uses a weakly dominated strategy.

Given this assumption, each member \( i \) votes for rejection if and only if \( s \geq s_i \). Thus, when the voting threshold is \( k \), member \( k \) becomes a decisive voter: the proposal is rejected if and only if \( s \geq s_k \). This implies that voting with threshold \( k \) is equivalent to delegating the decision to voter \( k \), in terms of equilibrium outcomes.

The goal of this paper is to study the equilibrium features of information acquisition in heterogeneous committees. Some of the features depend on the properties of the cost function \( C (\rho_i) \). I first illustrate the main results using a linear cost function \( C (\rho_i) = c \rho_i \), \( c > 0 \), then I discuss how the results extend to the case of a convex cost function.

### 1.4 Equilibrium Analysis

I start by analyzing committees that have pre-determined compositions. For such committees, I study how the members’ preferences are related to their incentives to acquire information in equilibrium, and examine the effects of preference heterogeneity and voting rules on information collection. After that, I study the problem of optimal committee design.

Suppose the cost of information for every member \( i \) is \( C (\rho_i) = c \rho_i \), where \( c > 0 \). Let \( L_i (\rho_i, \rho_{-i}, s_k; q_k) \) denote the expected payoff of member \( i \) from the collective decision given the precision profile \( (\rho_i, \rho_{-i}) \) and the \( s_k \) of the threshold voter \( k \), which depends on preference \( q_k \) of the threshold voter \( k \) exogenous to member \( i \)'s problem, so that

\[
L_i (\rho_i, \rho_{-i}, s_k; q_k) = - (1 - q_i) \gamma F_1 (s_k | \rho) - q_i (1 - \gamma) [1 - F_0 (s_k | \rho)], \forall i. \quad (1.4)
\]
I use $V_i(\rho_i, \rho_{-i}, s_k; q_k)$ to denote the total expected payoff of $i$ in the game, so

$$V_i(\rho_i, \rho_{-i}, s_k; q_k) = L_i(\rho_i, \rho_{-i}, s_k; q_k) - C(\rho_i)$$

$$= -(1 - q_i) \gamma F_1(s_k|\rho) - q_i (1 - \gamma) [1 - F_0(s_k|\rho)] - c_{\rho_i}, \forall i. \quad (1.5)$$

It is easy to see that if the decisive voter has an extreme preference, i.e., $q_k = 0$ or 1, then no member acquires information in equilibrium, and the collective decision is always the one preferred by voter $k$. When $q_k$ is sufficiently close to 0 or 1, we have the same problem, as voter $k$ is very unlikely to switch from his ex ante preferred decision; the value of acquired information is small. Thus, to keep the analysis interesting, I impose a range $I$ on the value of $q_k$ in the following assumption for the remaining analysis in this paper.

For a committee with $q_k \in (0, 1)$, $V_i(\rho_i, \rho_{-i}, s_k; q_k)$ of a member $i$ may fail to be quasiconcave in $\rho_i$ given $q_k$, as I argue in Section 1.4.4. However, there exists a value $\rho(q_k, \gamma)$, which is a function of $q_k$ and $\gamma$, such that for $\rho \geq \rho(q_k, \gamma)$, we have $\partial^2 V_i(\rho_i, \rho_{-i}, s_k; q_k) / \partial \rho_i^2 < 0$, for any $i$. According to this property, I impose the second part of the following assumption to ensure that $V_i(\rho_i, \rho_{-i}, s_k; q_k)$ is strictly concave in $\rho_i$, for any $i$ and any $q_k \in I$. With this assumption, the committee members’ efforts to acquire information are strategic substitutes, and the first order conditions of the members’ payoff maximization problems are sufficient for characterizing an equilibrium. Detailed discussion of this issue is relegated to Section 1.4.4.

**Assumption 1.3.** There exists a closed interval $I$ satisfying

$$[\min \{1/2, \gamma\}, \max \{1/2, \gamma\}] \subset I \subset (0, 1)$$

such that (1) $q_k \in I$ and (2) $\rho_0 \geq \max \{\rho(q_k, \gamma) : q_k \in I\}$.

To proceed, I first examine the marginal benefit of information to a member. For any
i, taking the derivative of \( L_i (\rho_i, \rho_{-i}, \xi_k; q_k) \) with respect to \( \rho_i \), we obtain

\[
\frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i} = \frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i} \bigg|_{\xi_k} + \frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \xi_k} \frac{\partial \xi_k}{\partial \rho_i} \tag{1.6}
\]

Incentive to Prevent Errors \hspace{1cm} \text{Incentive to Reduce Disagreement}

in which

\[
\frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i} \bigg|_{\xi_k} = (1 - q_i) \gamma f_1 (\xi_k | \rho) \frac{(1 - \xi_k)}{2\rho} + q_i (1 - \gamma) f_0 (\xi_k | \rho) \frac{\xi_k}{2\rho},
\]

\[
\frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \xi_k} \frac{\partial \xi_k}{\partial \rho_i} = \left[ -(1 - q_i) \gamma f_1 (\xi_k | \rho) + q_i (1 - \gamma) f_0 (\xi_k | \rho) \right] \frac{\partial \xi_k}{\partial \rho_i}.
\]

The first term of (1.6) reflects the incentive of member \( i \) to prevent errors, taking \( \xi_k \) as fixed.\footnote{To derive the expression of the term \( \frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i} \bigg|_{\xi_k} \), we use the fact that \( F_1 (\xi_k | \rho) = \Phi (\sqrt{p} (\xi_k - 1)) \), and \( f_0 (\xi_k | \rho) = \frac{\partial \Phi (\sqrt{p} (\xi_k - 1))}{\partial \xi_k} \). From (1.4), we have

\[
\frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i} \bigg|_{\xi_k} = (1 - q_i) \gamma \frac{\partial \Phi (\sqrt{p} (\xi_k - 1))}{\partial \rho_i} \bigg|_{\xi_k} - q_i (1 - \gamma) \frac{\partial \Phi (\sqrt{p} \xi_k)}{\partial \rho_i} \bigg|_{\xi_k} = (1 - q_i) \gamma \phi (\sqrt{p} (\xi_k - 1)) \frac{1 - \xi_k}{2\sqrt{p}} + q_i (1 - \gamma) \phi (\sqrt{p} \xi_k) \frac{\xi_k}{2\sqrt{p}},
\]

where \( \phi \) denotes the density function of the standard normal distribution. Since \( f_1 (s | \rho) = \frac{\partial F_1 (s | \rho)}{\partial s} = \sqrt{p} \phi (\sqrt{p} (s - 1)) \) and \( f_0 (s | \rho) = \frac{\partial F_0 (s | \rho)}{\partial s} = \sqrt{p} \phi (\sqrt{p} s) \), we obtain the expression of \( \frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i} \bigg|_{\xi_k} \) shown below (1.6).} This incentive is present regardless of the preference heterogeneity of the members. The second term of (1.6), which reflects the incentive of member \( i \) to reduce disagreement, however, appears only in heterogeneous committees.\footnote{To derive the expression of the term \( \frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \xi_k} \bigg|_{\xi_k} \), we still use the facts that \( F_1 (\xi_k | \rho) = \Phi (\sqrt{p} (\xi_k - 1)) \), and \( F_0 (\xi_k | \rho) = \Phi (\sqrt{p} \xi_k) \). From (1.4), we have

\[
\frac{\partial L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \xi_k} \bigg|_{\xi_k} = -(1 - q_i) \gamma \frac{\partial \Phi (\sqrt{p} (\xi_k - 1))}{\partial \xi_k} - q_i (1 - \gamma) \frac{\partial \Phi (\sqrt{p} \xi_k)}{\partial \xi_k} = -(1 - q_i) \gamma \phi (\sqrt{p} (\xi_k - 1)) + q_i (1 - \gamma) \sqrt{p} \phi (\sqrt{p} \xi_k).
\]

Because \( f_1 (\xi_k) = \sqrt{p} \phi (\sqrt{p} (\xi_k - 1)) \) and \( f_0 (\xi_k) = \sqrt{p} \phi (\sqrt{p} \xi_k) \), we obtain the expression of

...}
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is a public good that is costly to obtain, there is a free-rider problem in information acquisition: if other members acquire information, the magnitudes of the two incentives above of member \( i \) decrease and thus his incentive to acquire information decreases.

The incentive to reduce disagreement is important for the analysis of heterogeneous committees. In the expression for this incentive (see (1.6)), the value of \( \frac{\partial L_i}{\partial s_k} \) satisfies

\[
\frac{\partial L_i}{\partial s_k} (\rho_i, \rho_{-i}, s_k; q_k) \begin{cases} < 0, & \text{if } i < k; \\ > 0, & \text{if } i > k. \end{cases}
\]

(1.7)

The first inequality means that a member \( i \) who leans more toward rejection than the threshold voter would like to decrease \( s_k \). This is intuitive, because for a member \( i \) with \( q_i < q_k \), his *ex post* optimal decision rule \( s_i \) is lower than \( s_k \). A similar intuition applies to the second inequality of (1.7). Regarding \( \frac{\partial s_k}{\partial \rho_i} \), we have

\[
\frac{\partial s_k}{\partial \rho_i} = -\frac{1}{\rho^2} \ln q_k \frac{(1 - \gamma)}{(1 - q_k) \gamma}.
\]

(1.8)

This indicates that if the threshold voter \( k \) is pro-rejection (i.e., \( q_k < \gamma \)), then increasing \( \rho \) can raise his cautiousness in voting for rejection (i.e., \( \frac{\partial s_k}{\partial \rho_i} > 0 \)), as he responds more to signals pointing to acceptance. Combining (1.7) and (1.8), I find that when \( q_k < \gamma \), the second term of (1.6) is negative if \( i < k \) and positive if \( i > k \). This is because a member \( i \) who is more (less, respectively) inclined to reject the proposal than the threshold voter would like to acquire less (more, respectively) information to decrease (increase, respectively) \( s_k \), so as to reduce the distance between \( s_k \) and his *ex post* optimal cutoff \( s_i \), i.e., reduce the chance that the threshold voter disagrees with him.

There is a subtle but important difference between my model and the one in Li (2001). In my model, the collective decision is determined by comparing the realization of the aggregate signal with \( s_k \). In Li (2001), the collective decision of a committee is determined also by comparing the realization of the aggregate signal to a cutoff \( s \). But the cutoff \( s \) in Li (2001) is exogenously imposed and does not change with \( \rho \). Thus, the incentive to reduce disagreement does not appear in his model. It is easy to verify that if we drop this incentive from our model, we get the same comparative statics as Li (2001).
Similar results can be obtained for the case where the threshold voter is pro-acceptance (i.e., \( q_k > \gamma \)). If \( q_k = \gamma \), which means that the threshold voter is \textit{ex ante} unbiased toward any decision, then \( \partial s_k / \partial \rho_i = 0 \), and the incentive to reduce disagreement disappears.

Plugging (1.8) into (1.6), we obtain that for each member \( i \),

\[
\frac{\partial L_i(\rho_i, \rho_{-i}, s_k; q_k)}{\partial \rho_i} = (1 - q_i) \gamma f_1(s_k|\rho) \frac{s_k}{2\rho} + q_i (1 - \gamma) f_0(s_k|\rho) \frac{(1 - s_k)}{2\rho}. \tag{1.9}
\]

The first term of this derivative is the marginal benefit of member \( i \) from reducing the probability of false acceptance. More specifically, in this term, \( (1 - q_i) \) is the loss of member \( i \) from false acceptance and \( \gamma f_1(s_k|\rho) \frac{s_k}{2\rho} \) is the marginal decrement in the probability of false acceptance when increasing \( \rho \). Similarly, the second term of (1.9) is the marginal benefit of member \( i \) from reducing the probability of false rejection.

According to (1.5), we have

\[
\frac{\partial V_i(\rho_i, \rho_{-i}, s_k; q_k)}{\partial \rho_i} = (1 - q_i) \gamma f_1(s_k|\rho) \frac{s_k}{2\rho} + q_i (1 - \gamma) f_0(s_k|\rho) \frac{(1 - s_k)}{2\rho} - c. \tag{1.10}
\]

In equilibrium, \( \partial V_i(\rho_i, \rho_{-i}, s_k; q_k) / \partial \rho_i \leq 0 \) for all \( i \), because if \( \partial V_j(\rho_j, \rho_{-j}, s_k; q_k) / \partial \rho_j > 0 \) for some member \( j \), then member \( j \) has incentive to increase \( \rho_j \). It is possible to have \( \partial V_i(\rho_i, \rho_{-i}, s_k; q_k) / \partial \rho_i < 0 \), i.e., \( \rho_i = 0 \), for all \( i \). This happens when \( \rho_0 \) is large and \( c \) is large. I state this result formally in the lemma below.

**Lemma 1.1.** For any equilibrium \( \rho \), we have \( \frac{\partial V_i(\rho_i, \rho_{-i}, s_k; q_k)}{\partial \rho_i} \leq 0 \), with \( \frac{\partial V_i(\rho_i, \rho_{-i}, s_k; q_k)}{\partial \rho_i} = 0 \) if \( \rho_i > 0 \), for any \( i \).

The next lemma shows that the equilibrium value of \( \rho \) is unique, even if there are possibly multiple equilibria. This is mainly because \( \partial V_i(\rho_i, \rho_{-i}, s_k; q_k) / \partial \rho_i \) depends only on \( \rho \) and is decreasing in \( \rho \).

**Lemma 1.2.** Given Assumption 1.3, the equilibrium level of \( \rho \) is unique.

The proof of this result is in Appendix A.


1.4.1 Preferences and Information Acquisition

This subsection examines how members of a heterogeneous committee differ in their incentives to acquire information. From (1.10), we can see that a member’s incentive to acquire information depends on the profile \(q\) and the voting rule, which affects \(q_k\). The analysis below shows that regardless of the voting rule, there always exists a monotonic relationship between the preferences of members and their incentives to acquire information. The voting rule determines the direction of the monotonic relationship through \(q_k\).

I then study the impact of preference heterogeneity on information acquisition.

Using (1.9) and (1.2), we obtain

\[
\frac{\partial L_i (\rho_i; \rho_{-i}; \bar{s}_k; q_k)}{\partial \rho_i} \ - \ \frac{\partial L_j (\rho_j; \rho_{-j}; \bar{s}_k; q_k)}{\partial \rho_j} = \frac{(1 - \gamma) f_0 (s_k | \rho) (q_i - q_j)}{2 \rho (1 - q_k)} \left[ \left( \frac{1}{2} - q_k \right) - \frac{1}{\rho} \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma} \right].
\]

(1.11)

If \(q_i > q_j\), the sign of this difference is determined by the term in the brackets. For convenience, I define this term as

\[
h (q_k, \rho, \gamma) \equiv \left( \frac{1}{2} - q_k \right) - \frac{1}{\rho} \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma}.
\]

(1.12)

The sign of \(h (q_k, \rho, \gamma)\) is the same as the sign of the difference between \((1 - \gamma) f_0 (s_k | \rho) (1 - s_k)\frac{\gamma f_1 (s_k | \rho) q_k}{2 \rho}\), the marginal decrement in the probability of false rejection when increasing \(\rho\), and \((1 - s_k)\frac{\gamma f_1 (s_k | \rho) q_k}{2 \rho}\), the marginal decrement in the probability of false acceptance when increasing \(\rho\), given \(q_k\) and \(\gamma\). If \(h (q_k, \rho, \gamma) > 0\), increasing \(\rho\) marginally reduces the probability of false rejection more than the probability of false acceptance, so a member with a larger \(q\) has more incentive to acquire information than does one with a smaller \(q\). The result is reversed if \(h (q_k, \rho, \gamma) < 0\). The lemma below summarizes how the value of \(h (q_k, \rho, \gamma)\) is related to the members’ incentives to acquire information in equilibrium.

**Lemma 1.3.** In equilibrium,
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1. If \( h(q_k, \rho, \gamma) < 0 \), then \( \frac{\partial L_i(q_k, \rho, \gamma)}{\partial \rho_i} < \frac{\partial L_j(q_k, \rho, \gamma)}{\partial \rho_j} \) if \( q_i > q_j \), which implies that \( \rho_i > 0 \), only if \( q_i \) is equal to \( \min_{1 \leq l \leq n} \{q_l\} \);

2. If \( h(q_k, \rho, \gamma) > 0 \), then \( \frac{\partial L_i(q_k, \rho, \gamma)}{\partial \rho_i} > \frac{\partial L_j(q_k, \rho, \gamma)}{\partial \rho_j} \) if \( q_i > q_j \), which implies that \( \rho_i > 0 \), only if \( q_i \) is equal to \( \max_{1 \leq l \leq n} \{q_l\} \);

3. If \( h(q_k, \rho, \gamma) = 0 \), then \( \frac{\partial L_i(q_k, \rho, \gamma)}{\partial \rho_i} = \frac{\partial L_j(q_k, \rho, \gamma)}{\partial \rho_j} \), \( \forall i, j \).

This lemma can be simply derived using (1.11) and Lemma 1.1, so I omit its proof.

From this lemma, we see that in equilibrium there is a monotonic relationship between the members’ incentives to acquire information and their preferences. Specifically, in the case where \( h(q_k, \rho, \gamma) < 0 \), if \( q_i > q_j \) (i.e., \( i \) leans more toward acceptance than \( j \)), then member \( j \) has stronger incentive to acquire information than does member \( i \), and a member acquires information in equilibrium only if he is among the ones most willing to reject the proposal. In the case where \( h(q_k, \rho, \gamma) > 0 \), the direction of the monotonicity is reversed—if \( q_i > q_j \), then member \( i \) has more incentive to acquire information than does member \( j \), and a member acquires information in equilibrium only if he is among the ones least willing to reject the proposal. When \( h(q_k, \rho, \gamma) = 0 \), all the members have the same incentive to acquire information.

For an established committee, \( \rho \) is endogenous, so \( h(q_k, \rho, \gamma) \) is also endogenous. Thus, without knowing the equilibrium, Lemma 1.3 cannot tell us which type of monotonicity arises at the stage of information acquisition.

At first glance, it seems that which type of monotonicity we will observe in equilibrium depends on the entire preference profile \( q \), given the other primitives. However, the following proposition shows that the monotonicity is determined only by the preference \( q_k \) of the threshold voter. To proceed, let \( \rho(q_k, q_{-k}; \rho_0, c, \gamma) \) denote the equilibrium aggregate precision, given the preference profile \( q \), voting rule \( k \), and primitives \( \rho_0, c, \) and \( \gamma \). In most of the discussion below, I replace \( \rho(q_k, q_{-k}; \rho_0, c, \gamma) \) by \( \rho(q_k, q_{-k}) \) to simplify the notation.
**Definition 1.1.** The preference \( q_k \) of a threshold voter is virtually unbiased if \( h(q_k, \rho(q_k, q_{-k}), \gamma) = 0 \), regardless of the value of \( q_{-k} \). A threshold voter with a virtually unbiased preference is virtually unbiased.

According to Lemma 1.3, if the threshold voter is virtually unbiased, then all the members have the same incentive to acquire information, regardless of their preferences. That means the three incentives of a member—i.e., the incentive to prevent errors, the incentive to free ride, and the incentive to reduce disagreement—that shape his incentive to acquire information achieve a balance, given that threshold voter is virtually unbiased.

The proposition below establishes the existence and uniqueness of a virtually unbiased preference, and shows how it can be used to determine the monotonic relationship between the members’ preferences and their incentives to acquire information.

**Proposition 1.1.** Given Assumption 1.3, there exists a unique virtually unbiased preference \( \bar{q}^* \), which depends only on \( \gamma, \rho_0 \), and \( c \). When \( \gamma = 1/2 \), \( \bar{q}^* = 1/2 \); otherwise, \( \bar{q}^* \in (\min\{1/2, \gamma\}, \max\{1/2, \gamma\}) \).

1. If \( q_k < \bar{q}^* \), then \( h(q_k, \rho(q_k, q_{-k}), \gamma) > 0 \), \( \forall q_{-k} \), so \( \rho_i \geq \rho_j = 0 \), for \( q_i > q_j \).

2. If \( q_k > \bar{q}^* \), then \( h(q_k, \rho(q_k, q_{-k}), \gamma) < 0 \), \( \forall q_{-k} \), so \( 0 = \rho_i \leq \rho_j \), for \( q_i > q_j \).

This proposition indicates that when the prior is biased (i.e., \( \gamma \neq 1/2 \)), the virtually unbiased threshold voter plays a role in adjusting the bias of the prior. For example, if the prior favors the state \( \theta = 1 \) (i.e., \( 1/2 < \gamma \)), then \( \bar{q}^* \) satisfies \( 1/2 < \bar{q}^* \), that is, the virtually unbiased preference favors acceptance.

Regarding information acquisition, this proposition shows that if the threshold voter is virtually biased toward rejection, i.e., \( q_k < \bar{q}^* \), then the members mostly inclined to accept the proposal are the ones collecting information. In the analysis below, for such committees, I assume that member \( n \) is the only information collector, without loss of
If the threshold voter is virtually biased toward acceptance, i.e., \( q_k > q^* \), then the members most leaning toward rejection are the ones acquiring information. Similarly, for a committee of this kind, I assume that only member 1 collects information. I interpret a voting rule with \( q_k < q^* \) as a voting rule favoring rejection, and a voting rule with \( q_k > q^* \) as a voting rule favoring acceptance.

Why does the monotonic relationship between members’ incentives to acquire information and their preferences hinge on the preference of the threshold voter? The reason primarily lies in the incentive to reduce disagreement. Consider the case where \( \gamma = 1/2 \), i.e., the prior is unbiased. From (1.6), I find that when \( q_k < q^* = 1/2 \), the relationship between a member’s incentive to prevent errors and his preference is indeterminate, while the incentive of member \( i \) to reduce disagreement is increasing in \( q_i \), \( \text{ceteris paribus} \). (See the discussion below (1.8) for intuition.) The monotonicity in the members’ incentives to reduce disagreement is consistent with the monotonicity described in the first case of Proposition 1.1. Similar results can be obtained for \( q_k > q^* = 1/2 \). Therefore, the incentive to reduce disagreement explains the fact that the members’ incentives to acquire information are monotonically related to their preferences hinges on \( q_k \).

If \( \gamma \neq 1/2 \), the incentive to reduce disagreement is not the only driving force for the monotonocity result; the incentive to prevent error also comes into play. However, except for cases in which \( q_k \in (\min \{q^*, \gamma\}, \max \{q^*, \gamma\}) \), the incentive to reduce disagreement is still the dominant force determining the direction of the monotonocity.

In the literature on committee decision-making, preference heterogeneity among committee members is often believed to be detrimental, as it tends to block effective information sharing.\(^7\) In this model, I show that in the absence of strategic information transmission, heterogeneity can be a blessing—a more polarized committee can induce

\(^7\)In an established committee, it is possible that multiple members have preferences equal to \( \max_{1 \leq I \leq n} \{q_I\} = q_n \). For such a committee, when \( q_k < q^* \), there are multiple equilibria if \( \rho > \rho_0 \). According to Lemma 1.2, all these equilibria have the same equilibrium \( \rho \); they differ only in the distribution of information acquisition efforts among the members with preference \( \max_{1 \leq I \leq n} \{q_I\} \).

\(^8\)See Coughlan (2000), Li et al. (2001), Austen-Smith and Feddersen (2006), and Meirowitz (2007) for more details.
more information acquisition in equilibrium.

**Proposition 1.2.** Under Assumption 1.3, given the preference $q_k$ of the threshold voter, making a committee more polarized induces, i.e., decreasing $q_i$ if $i < k$ and increasing $q_i$ if $i > k$, without changing order of members’ preferences, (weakly) more information acquisition in equilibrium.

The proof of this proposition is useful to understand some of the results below, so I put it here instead of putting it in an appendix.

**Proof.** First, I consider the case where the threshold voter of a committee is virtually biased toward rejection (i.e., $q_k < \bar{q}^*$). Then member $n$ is the only one collecting information. Thus, in equilibrium, $\rho = \rho_0 + \rho_n$, $\rho_i = 0$, for $i \leq n - 1$. It is possible that $\rho_n = 0$ when $\rho_0$ is large and $c$ is large. If $\rho_n > 0$, then $\partial L_n (\rho_n, \rho_{-n}, s_k; q_k) / \partial \rho_n = c$. From this condition, we obtain

$$\frac{\partial \rho_n}{\partial q_n} = -\frac{(1 - \gamma) f_0(s_k|\rho) h(q_k, \rho, \gamma)}{2\rho} \frac{\partial^2 L_n (\rho_n, \rho_{-n}, s_k; q_k) / \partial \rho_n^2 > 0}. \quad (1.13)$$

Thus, when $k \neq n$, increasing the degree of polarization of the committee (i.e., decreasing $q_i$ if $i < k$, and increasing $q_i$ if $i > k$, without changing $q_k$ and the order of members’ preferences) increases $\rho$, as $q_n$ increases. When $k = n$, increasing the degree of polarization of the committee by decreasing other members’ preferences, while fixing $q_k$, does not affect $\rho$.

For the case where the threshold voter of a committee is virtually biased toward acceptance (i.e., $q_k > \bar{q}^*$), the argument is similar to the case above.

If the threshold voter of a committee is virtually unbiased (i.e., $q_k = \bar{q}^*$), changing the preferences of non-threshold voters does not change the aggregate precision $\rho$. Thus, increasing the polarization of such a committee, while fixing $q_k$, has no effect on information acquisition. \qed
This result regarding the impact of preference heterogeneity on information acquisition does not rely on the assumption of linear information cost. As I show in the next section, it holds also in certain convex cost environments.

1.4.2 Impact of Voting Rules

For an established committee, the adopted voting rule affects the incentives of members to collect information by affecting $q_k$. I study in this subsection the voting rule that maximizes information acquisition. The result is summarized in the following proposition.

**Proposition 1.3.** Given Assumption 1.3 and $q_i \in I$ for all $i$, one of the unanimous rules, i.e., either $k = 1$ or $k = n$, induces most information acquisition in equilibrium.

To illustrate the reason, I focus on heterogeneous committees in which $q_i \neq q_j$, for $i \neq j$. To begin, consider a voting rule $k$ with $q_k < \bar{q}^*$. In this case, member $n$ is the only one collecting information in equilibrium, according to Proposition 1.1. Now if we decrease $k$, i.e., decrease $q_k$, member $n$ is still the only information collector, but his incentive to reduce disagreement increases, and consequently more information is acquired in equilibrium. Thus, the voting rule $k = 1$ outperforms any other rule with $q_k < \bar{q}^*$ in producing information. Similarly, for voting rules with $q_k > \bar{q}^*$, member 1 is the only one collecting information in equilibrium. These rules induce no more information than does the voting rule $k = n$. For a voting rule $k$ with $q_k = \bar{q}^*$, every member has the same incentive to acquire information in equilibrium. In this case, if $k \neq 1$, we can without loss of generality let member $n$ be the information collector, and increase information acquisition by decreasing $k$. The argument for the case where $q_k < \bar{q}^*$ applies. Similarly, if $k \neq n$, the argument for the case where $q_k > \bar{q}^*$ applies. For any voting rule, I have shown that there is a unanimous rule inducing more information.

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9In my model, the voting rule with $k = 1$ is equivalent to the voting rule that takes $C$ as the *status quo* and requires unanimity to accept the proposal.
acquisition. Therefore, the optimal voting rule that induces most equilibrium information must be one of the unanimous rules. However, this does mean that any unanimous rule outperforms all non-unanimous rules. Consider a committee with $q_n \leq \tilde{q}^*$. The argument above implies that the unanimous rule with $k = 1$ induces most information acquisition, but the unanimous rule with $k = n$ induces the least information acquisition, so is outperformed by all non-unanimous rules.

1.4.3 Committee Design

The analysis above focuses on committees with pre-determined compositions. If a committee designer is authorized to design an $n$-member committee, how should she choose the composition and decision rule of the committee? This subsection is devoted to answering this question. The existing literature on committee design restricts attention to homogeneous committees, taking committee size, the communication rule, and the decision rule as the choice variables. Our analysis here extends the analysis to heterogeneous committees.

Let $q$ denote the preference of the designer, who has the same prior as the committee members. If the committee has a threshold voter with preference $q_k$ and produces aggregate information $\rho$, then the expected payoff of the designer from the collective decision is

$$V_0(\rho, q_k) = - (1 - q_0) \gamma F_1(\tilde{s}_k|\rho) - q_0 (1 - \gamma) [1 - F_0(\tilde{s}_k|\rho)].$$

(1.14)

The objective of the designer is to maximize $V_0(\rho, q_k)$, without taking into account the costs incurred by the members in acquiring information.$^{10}$ I maintain the second part of Assumption 1.3 and impose the following assumption for the rest of the analysis in

10The assumption that the designer does not care about the costs of information incurred by the committee members is appropriate for many collective decision-making situations. Consider the examples of trial juries and monetary policy committees. The collective decisions made by these committees affect not only the payoffs of the committee members, but also the payoffs of many other people in the society. If the objective of the designer is to maximize the expected payoff of all the people that will be affected by the decision (i.e., $q_0$ of the designer represents the average preference of all the people), then the efforts of the committee members devoted to collecting information are negligible.
this section. The first part of this assumption states that $q_0$ is bounded away from 0 and 1. The second part of this assumption ensures that an optimal committee acquires a positive amount of information in equilibrium.

**Assumption 1.4. For the designer,**

1. $q_0$ is in the interior of $I$, and she can choose only $q_k \in I$;
2. $\rho (q_k, q_{-k}; \rho_0, c, \gamma) > \rho_0$ if $q_i = q_0$ for all $i$, which means that a homogeneous committee having the same preference as the designer acquires information in equilibrium.

To begin, I analyze the simplest version of the problem: designing a single-member decision-making committee, i.e., I assume $n = 1$. Such a problem can be interpreted as a delegation problem in which the principal chooses the agent to whom the decision is delegated. Unlike the literature on delegation, which usually assumes that the agent has *exogenous* private information (e.g., Dessein 2002; Li and Suen 2004; Marino 2007; Mylovanov 2008), I assume that the agent’s information is *endogenous*. The problem of the designer can be formulated as follows.

$$\max_{q_1} V_0 (\rho, q_1) \quad (1.15)$$

$$s.t. \ q_1 \in I,$$

$$\rho = \arg \max_{\hat{\rho} \geq \rho_0} - (1 - q_1) \gamma F_1 (\xi_1, \hat{\rho}) - q_1 (1 - \gamma) [1 - F_0 (\xi_1, \hat{\rho})] - c (\hat{\rho} - \rho_0).$$

The following proposition illustrates how the choice of the single member is related to the preference of the designer.

**Proposition 1.4.** A designer with preference $q_0 < q^*$ ($q_0 > q^*$, respectively) optimally chooses $q_1 \in (q_0, q^*)$ ($q_1 \in (q^*, q_0)$, respectively). If $q_0 = q^*$, she optimally chooses $q_1 = q_0$.

As shown in the proof of Proposition 1.1, the information acquired by the single committee member is decreasing in the distance between $q_1$ and $q^*$. Thus, a virtually biased
committee designer faces a trade-off between preference alignment and decision precision. When \( q_0 < \bar{q}^* \) (\( q_0 > \bar{q}^* \), respectively), choosing \( q_1 < \bar{q}^* \) (\( q_1 > \bar{q}^* \), respectively) induces less information acquisition compared with having \( q_1 = \bar{q}^* \), but increases the chance that the member chooses her preferred decision given the acquired information. This proposition shows that, as long as the selected member does not rely entirely on the free signal \( s_0 \) to make his decision, the optimal \( q_1 \) always lies between \( q_0 \) and \( \bar{q}^* \). When \( q_0 = \bar{q}^* \), the trade-off for the designer disappears, because choosing \( q_1 = q_0 \) achieves perfect preference alignment and induces the most information acquisition by the member.

How does the designer choose the committee member if she retains the right to make the final decision and the member is responsible only for collecting information? This question is closely related to a question asked in the literature on informational committee design. The next proposition shows that the choice of agent in this case is very different from the previous case.

**Proposition 1.5.** If it is common knowledge that the designer retains the right to make the final decision and can observe the precision and realization of the signal obtained by the committee member, then it is optimal for her to choose \( q_1 = 1 \) (\( q_1 = 0 \), respectively) when \( q_0 < \bar{q}^* \) (\( q_0 > \bar{q}^* \), respectively). If \( q_0 = \bar{q}^* \), she is indifferent between all values of \( q_1 \).

This proposition indicates that a virtually biased designer optimally chooses a committee member whose preference is extremely opposed to hers. Given the previous analysis (see Proposition 1.2), this result is intuitive. The conflict between the designer and the information collector can incentivize the latter to acquire more information, given that the information is fully transmitted between these two parties. This intuition also underlies the analysis of multi-member committee design.

Che and Kartik (2009) study the problems in Propositions 1.4 and 1.5 using a different model with endogenous information. Unlike Proposition 1.4, they find that if the designer delegates the decision to the single committee member, it is always optimal for the
designer to choose a member identical to herself. This is because in their model, when
the member has the decision right, his incentive to acquire information is independent
of his \textit{ex ante} bias over the alternatives. For the problem of choosing an information
collector, they show that it is always optimal for the designer to choose one different
from herself due to the “persuasion effect”. However, in the current model I show in
Proposition 1.5 that it is necessary for the designer to choose an information collector
different from herself only if she is biased. Moreover, unlike Che and Kartik (2009), I am
able to identify the optimal information collector for each type of designer.

Now we study the design of an $n$-member committee, for $n \geq 2$. In this problem, the
choice variables of the designer are $q$, the preference profile of the committee, and $k$, the
voting rule adopted by the committee to aggregate individual votes. The problem of the
designer is

\[
\max_{q_k} V_0(\rho, q_k) \quad (1.16)
\]

s.t. $0 \leq q_1 \leq \ldots \leq q_n \leq 1$, $q_k \in I$, and

\[
\rho = \rho(q_k, q_{-k}; \rho_0, c, \gamma).
\]

It is convenient to think about the problem by first looking at the value of $q_k$ of
the threshold voter. There are three possible relationships between $q_k$ and $q^*$: $q_k < q^*$,
$q_k > q^*$, or $q_k = q^*$. If $q_k < q^*$, then the non-threshold voters’ incentives to acquire
information are increasing in their inclinations to acceptance. Since the designer cares
only about $q_k$ and $\rho$, given that $q_k < q^*$, she prefers that the non-threshold voters who
collect information lean strongly toward acceptance. A similar analysis can be conducted
for the case $q_k > q^*$. Based on this analysis, it is natural to conjecture that an optimal
committee has a two-party structure, that is, only two types of members are in the
committee. The following proposition confirms this conjecture.

\textbf{Proposition 1.6.} \textit{There exists an optimal decision-making committee satisfying one of}
the two following sets of conditions:

1. \( q_n > \max\{q^*, q_0\}, \) and \( q_i = 0, \) for any \( i < n, \) with \( k = n; \)

2. \( q_1 < \min\{q^*, q_0\}, \) and \( q_i = 1, \) for any \( i > 1, \) with \( k = 1. \)

Three features of the optimal committees in the proposition are worth noting: (1) the adopted voting rule is unanimous, namely, requiring either unanimity to reject or unanimity to accept, (2) the non-threshold voters’ preferences are extreme and opposed to that of the threshold voter, (3) the threshold voter is not perfectly aligned with the designer, and is virtually biased. Features (1) and (2) are due to Proposition 1.1 and Proposition 1.2 as I discussed immediately before the current proposition. Some papers on committee decision-making (e.g., Feddersen and Pesendorfer 1998; Gerardi 2000; Persico 2004; Gerardi and Yariv 2007) show that unanimous votings are suboptimal. Feature (1) provides a new rational for adopting unanimous rules in making collective decisions. Feature (3) indicates that the collective choice made by the committee departs from the one preferred by the designer in some cases. This results from a trade-off facing the designer: having the committee always make a decision the same as the one she would make, or having the committee make a more precise decision. This trade-off is similar to that in the single-member committee design. The difference is that the presence of other members changes the direction of the trade-off: the designer would like the threshold voter to be more biased than she is, so as to incentivize other members to collect information.

In fact, there are infinitely many other committee designs that induce the same amount of information and same expected payoff to the designer as the optimal designs I described in the proposition. However, they are not robust to varying the cost of information. In the next section, we will see that when the cost of information becomes

\[ q' = (q'_1, \ldots, q'_n) \] and voting rule \( k \neq n \) achieves the same payoff for the designer as the optimal committee if \( q'_1 = 0 \) and \( q'_k = q_n. \)
convex, other committee designs are no longer optimal.

In the problem (1.16), I implicitly assume that the designer is committed to delegate the final decision to the committee. If the designer can observe the precisions and realizations of all the signals received by the committee, one may wonder whether she should retain the right to make the final decision. The answer is no. This is because retaining the decision right, in terms of expected payoff, is equivalent to having a committee with a threshold voter with preference $q_0$, so the maximum payoff attainable in this case is dominated by an optimal committee described above. I summarize this result in the following corollary.

**Corollary 1.1.** The designer prefers to delegate the decision to the optimally designed committee rather than to retain the decision right, even if she has perfect knowledge of all the signals received by the committee and her perfect knowledge is commonly known.

In many real world situations, the decision rules adopted to make collective decisions are fixed by institutional rules, though the compositions of the committees are flexible.\footnote{Here are two examples. (1) In criminal law jury trials of some jurisdictions, though the juries are temporary, their guilty verdicts require unanimity. (2) When settling disputes between WTO member countries, ad hoc panels appointed by the Dispute Settlement Body to investigate the disputes reach their decisions using the simple majority rule (whenever consensus is impossible).}

In the rest of this subsection, I consider the case where the voting rule is fixed with $k = n$. Now the designer’s problem can be formulated as

\[
\max_{q_0} V_0 (\rho, q_n) \tag{1.17}
\]

\[
\text{s.t. } 0 \leq q_1 \leq \ldots \leq q_n \leq 1, \quad q_n \in I, \quad \text{and} \quad \rho = \rho (q_n, q_{-n}; \rho_0, c, \gamma).
\]

The following proposition describes how an optimal committee design is related to the preference of the designer.

**Proposition 1.7.** When the voting rule is $k = n$,
1. if $q_0 > q^*$, then there is an optimal committee satisfying $q_n > q_0$ and $q_i = 0$ for any $i < n$;

2. if $q_0 < q^*$, then there is an optimal committee that is either homogeneous with $q_1 = \ldots = q_n \in (q_0, q^*)$ or heterogeneous with $q_n > q^*$ and $q_i = 0$ for any $i < n$.

The second part of this proposition indicates that when unanimity is required to change the status quo, an optimal committee can be homogeneous if the designer is virtually biased against the status quo, i.e., $q_0 < q^*$. I briefly sketch the reason here. Among all homogeneous committee, there is an optimal one for the designer satisfying $q_1 = \ldots = q_n \in (q_0, q^*)$. (This is follows from Proposition 1.4 on single-member committee design.) This optimal homogeneous committee cannot be outperformed by any other committee with $q_n \in (q_0, q^*)$, given the constraint $q_1 \leq \ldots \leq q_n$, in maximizing the designer’s payoff, and may outperform any other committee with $q_n > q^*$.

For the case in which $q_0 > q^*$, a homogeneous committee is never optimal. This is because among all homogeneous committees, the optimal one has $q_1 = \ldots = q_n \in (q^*, q_0)$. This committee is inferior to one with the same $q_n$ but $q_i = 0$ for any $i < n$, given Proposition 1.2 on the impact of group polarization.

### 1.4.4 Non-quasiconcavity

At the beginning of this section, I pointed out that $V_i(\rho_i, \rho_{-i}, s_k; q_k)$ may fail to be quasiconcave in $\rho$. This is because when $q_k \neq \gamma$, as $\rho \to 0$,

$$\frac{\partial V_i(\rho_i, \rho_{-i}, s_k; q_k)}{\partial \rho_i} \to -c,$$
which implies that when \( q_k \neq \gamma \), \( V_i (\rho_i; \rho_{-i}; \xi_k; q_k) \) is decreasing in \( \rho_i \) when \( \rho \) is close to 0.\(^{13}\) If for some \( \rho > 0 \), we have \( \partial V_i (\rho_i; \rho_{-i}; \xi_k; q_k) / \partial \rho_i > 0 \), then \( V_i (\rho_i; \rho_{-i}; \xi_k; q_k) \) is not quasiconcave in \( \rho_i \). The non-concavity issue of \( V_i (\rho_i; \rho_{-i}; \xi_k; q_k) \) is not unique to the current model, and is well-known in problems with endogenous information. Radner and Stiglitz (1984) show that it is a fundamental property in some decision problems that the net value of information is not concave. Their result can be naturally extended to our game-theoretic setup. One consequence is that a pure-strategy equilibrium may not exist (Dasgupta and Maskin, 1986). Another consequence is the possible multiplicity of equilibria when \( V_i (\rho_i; \rho_{-i}; \xi_k; q_k) \) is not concave in \( \rho_i \). This may complicate our comparative statics.

The reason for this non-quasiconcavity is intuitive. Information has value only if it can improve the collective decision. When \( \rho \) is small, marginally increasing \( \rho \) has very limited effect on the collective decision determined by the threshold voter, so the marginal benefit \( \partial L_i (\rho_i; \rho_{-i}; \xi_k; q_k) / \partial \rho_i \) is very small, and can be smaller than the marginal cost \( c \) of information. When \( \rho \) becomes large, \( \partial L_i (\rho_i; \rho_{-i}; \xi_k; q_k) / \partial \rho_i \) may exceed \( c \).

To investigate how \( V_i (\rho_i; \rho_{-i}; \xi_k; q_k) \) changes with \( \rho_i \) given \( q_k \), we derive the expression of \( \partial^2 V_i (\rho_i; \rho_{-i}; \xi_k; q_k) / \partial \rho_i^2 \), and find the following lemma.

---

\(^{13}\)I prove this by showing that \( \lim_{\rho \to 0^+} \frac{\partial L_i (\rho_i; \rho_{-i}; \xi_k; q_k)}{\partial \rho_i} \to 0 \) as \( \rho \to 0 \), when \( q_k \neq \gamma \). From (1.9), we have

\[
\lim_{\rho \to 0^+} \frac{\partial L_i (\rho_i; \rho_{-i}; \xi_k; q_k)}{\partial \rho_i} = \lim_{\rho \to 0^+} \frac{(1 - q_i) \gamma e^{-\frac{\xi_k}{\rho} (1 - k)} \left[ \frac{e}{2} + \ln \frac{2k(1 - \gamma)}{(1 - q_k) \gamma} \right] + q_i (1 - \gamma) e^{-\frac{\xi_k}{\rho} (1 - k)} \left[ \frac{e}{2} - \ln \frac{2k(1 - \gamma)}{(1 - q_k) \gamma} \right]}{2 \sqrt{2\pi} \rho^{3/2}}.
\]

Given the expression of \( \xi_k \), the limit above is proportional to the following limit

\[
\lim_{\rho \to 0^+} e^{-\frac{1}{\rho^{3/2}} \left( \ln \frac{2k(1 - \gamma)}{(1 - q_k) \gamma} \right)^2} = \lim_{\rho \to 0^+} \frac{1}{\rho^{3/2} e^{\frac{1}{2} \left( \ln \frac{2k(1 - \gamma)}{(1 - q_k) \gamma} \right)^2}}.
\]

The value of \( \rho^{3/2} e^{\frac{1}{2} \left( \ln \frac{2k(1 - \gamma)}{(1 - q_k) \gamma} \right)^2} \) is increasing without bound when \( \rho \to 0^+ \). Thus, \( \lim_{\rho \to 0^+} \frac{\partial L_i (\rho_i; \rho_{-i}; \xi_k; q_k)}{\partial \rho_i} = 0 \).
Lemma 1.4. \( \partial^2 V_i (\rho_i, \rho_{-i}, s_k; q_k) / \partial \rho_i^2 < 0 \) if \( \rho \geq \rho (q_k, \gamma) \equiv 6 \ln \frac{q_k(1- \gamma)}{(1- q_k) \gamma} \).

Thus, if the equilibrium level of \( \rho \) exceeds \( \rho (q_k, \gamma) \), then the first order conditions of members’ payoff maximization problems are sufficient for characterizing an equilibrium.

To ensure that \( \rho \geq \rho (q_k, \gamma) \) given \( q_k, \gamma \), I assume that \( \rho_0 \geq \rho (q_k, \gamma) \). This assumption guarantees that \( V_i (\rho_i, \rho_{-i}; s_k; q_k) \) is globally concave in \( \rho_i \) given \( q_k \), regardless of \( \sum_{j \neq i} \rho_j \). The precision choices of committee members are strategic substitutes in this case.

In some of the analysis, such as the impact of voting rules and committee design, we need \( q_k \) to be variable. We need \( q_k \) to be bounded away from 0 and 1, because \( \rho (q_k, \gamma) \rightarrow \infty \) if \( q_k \rightarrow 0 \) or 1, the assumption \( \rho_0 \geq \rho (q_k, \gamma) \) will be violated for any finite \( \rho_0 \). Therefore, I impose the assumption that \( q_k \) belongs to a closed interval \( I \subset (0, 1) \). It is worth mentioning that \( I \) can be an arbitrarily large subset of \( (0, 1) \) if \( \rho_0 \) is sufficiently large.

### 1.5 Convex Information Cost

In this section, I study the case where the “production” of information exhibits diminishing returns. Specifically, I assume that \( C(\rho_i) \) satisfies \( C(0) = 0 \), \( C'(0) = 0 \), and \( C''(\rho_i) > 0 \) for \( \rho_i \geq 0 \). Except the changes in the cost function, I maintain all other assumptions made before. The major insights obtained in the case of linear cost can be extended to this environment. The results in the rest of the paper are proved in Appendix B.

#### 1.5.1 Preferences and Information Acquisition

As before, let us first examine the relationship between the preferences of members and their incentives to acquire information. I still use \( V_i (\rho_i, \rho_{-i}; s_k; q_k) \) defined in (1.5) to denote the expected payoff of member \( i \), given the precision profile \( (\rho_i, \rho_{-i}) \) and the \( s_k \).
of the threshold voter $k$. For any member $i$, the first order condition of his maximization problem is

$$
\frac{\partial V_i(\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i} = (1 - q_i) \gamma f_1(\xi_k|\rho) \frac{\xi_k}{2\rho} + q_i (1 - \gamma) f_0(\xi_k|\rho) \frac{1 - \xi_k}{2\rho} - C'(\rho_i) = 0.
$$

(1.18)

Given that $C'(0) = 0$ and $C''(\rho_i) > 0$ for $\rho_i \geq 0$, the equilibrium is interior. That is, all the members exert positive efforts in acquiring information in equilibrium. This is different from the linear cost case.

Since all the members have the same cost function, to compare their acquired information we need to compare only their values of $\partial L_i/\partial \rho_i$. Regarding this comparison, we have a result similar to Proposition 1.1.

**Proposition 1.8.** There exists a unique virtually unbiased preference $\bar{q}^*$, which depends only on $\gamma$, $\rho_0$, $n$, and $C(\cdot)$. When $\gamma = 1/2$, $\bar{q}^* = 1/2$; otherwise, $\bar{q}^* \in (\min \{1/2, \gamma\}, \max \{1/2, \gamma\})$.

1. If $q_k < \bar{q}^*$, then $h(q_k, \rho(q_k, q_{-k}), \gamma) > 0$, $\forall q_{-k}$, so $\rho_i > \rho_j$, for $q_i > q_j$.

2. If $q_k > \bar{q}^*$, then $h(q_k, \rho(q_k, q_{-k}), \gamma) < 0$, $\forall q_{-k}$, so $\rho_i < \rho_j$, for $q_i > q_j$.

Due to the fact that the equilibrium is interior, the monotonic relationship between members’ preferences and their efforts in acquiring information becomes strong: when the threshold voter is virtually biased toward rejection, the members that are more inclined to reject the proposal acquire less information in equilibrium; when the threshold voter is virtually biased toward acceptance, the members that lean more toward acceptance acquire less information in equilibrium. The intuition behind this result is the same as that for Lemma 1.3 and Proposition 1.1.

To understand the impact of preference heterogeneity on information acquisition, we consecutively study two questions: (1) how changing one non-threshold member’s
preference affects the equilibrium; (2) fixing the preference of the threshold member, how preference heterogeneity affects equilibrium information acquisition. In the case of linear information cost, since only one member acquires information in equilibrium, we have in fact combined the analyses of these two questions into one and summarized the results in Proposition 1.2.

Now given that the cost of information is convex, regarding the first question, we have the following proposition.

**Proposition 1.9.** If \( q_k > \bar{q} \), for all \( i \neq k \), increasing \( q_i \) decreases \( \rho_i \) and increases \( \rho_j \), \( j \neq i \), and \( \rho \) decreases. If \( q_k < \bar{q} \), the results are reversed. For the case in which \( q_k = \bar{q} \), changing a non-threshold voter’s preference does not affect the equilibrium.

Note the response of each member \( j, j \neq i \), to the change in \( q_i \). When \( q_i \) changes, \( \rho_j \) moves in the opposite direction to \( \rho_i \). This results from the incentive to free ride. However, this effect is dominated by the impact of a change in \( q_i \) on \( \rho_i \), so \( \rho \), the aggregate precision, changes in the same direction as \( \rho_i \).

Regarding the second question, we no longer have a result like Proposition 1.2. To illustrate this, I consider the following preference profile,

\[
q_i = \bar{q} + \left( i - \frac{n + 1}{2} \right) \delta, \forall i,
\]

where \( \delta \in \left( 0, \min \left\{ \frac{2(1-\bar{q})}{n-1}, \frac{2\bar{q}}{n-1} \right\} \right) \). In this preference profile, \( \delta \) can be interpreted as a measure of preference heterogeneity in the committee, and the average preference of the committee is \( \bar{q} \). Suppose that \( n \) is an odd number and \( k = (n + 1)/2 \), i.e., the voting rule is a simple majority rule and \( q_k = \bar{q} \). The following proposition shows how the impact of preference heterogeneity on information acquisition depends on \( C^m(\cdot) \) in such a committee.

**Proposition 1.10.** Given \( q_k = \bar{q} \), if \( C^m(\cdot) > 0 \), then the equilibrium value of \( \rho \) is decreasing in \( \delta \); if \( C^m(\cdot) < 0 \), then the equilibrium value of \( \rho \) is increasing in \( \delta \)
A more general version of this proposition, which considers more general preference structures and other voting rules, is proved in Appendix B. The purpose of presenting this proposition here is to show that the concavity (convexity) of \( C''(\cdot) \) is crucial for the impact of preference heterogeneity on information acquisition.

The following proposition compares the performances of a heterogeneous committee, which is not restricted to the preference structure in (1.19), and a homogeneous committee.

**Proposition 1.11.** Consider two committees with the same size \( n \) and average preference \( \bar{q} \):

1. a homogeneous committee with \( q_1 = \ldots = q_n = \bar{q} \);
2. a heterogeneous committee with threshold voter \( q_k = \frac{\sum_1^n q_i}{n} = \bar{q} \).

The heterogeneous committee collects more (less, respectively) information in equilibrium if \( C''(\cdot) < 0 \) (\( C''(\cdot) > 0 \), respectively).

### 1.5.2 Impact of Voting Rules

I now analyze the impact of voting rules on information acquisition. To proceed, I define

\[
\bar{q}^e = \sum_{i=1}^n \lambda_i q_i, \quad \text{where} \quad \lambda_i = \frac{1}{\sum_{i=1}^n \frac{1}{C''(\rho_i)}}. \tag{1.20}
\]

This definition indicates that \( \bar{q}^e \) depends on \( \rho_1, \ldots, \rho_n \) through \( \lambda_1, \ldots, \lambda_n \). The voting rule affects \( \rho_1, \ldots, \rho_n \), thus affects \( \bar{q}^e \). Based on the first order conditions in (1.18), we obtain the following lemma. This lemma can be simply proved using equation (1.47) derived from (1.18), so I omit a detailed proof.

**Lemma 1.5.** If \( \bar{q}^e < q_k \), then increasing the stringency of the voting rule increases the equilibrium value of \( \rho \), i.e., \( d\rho/dq_k > 0 \). Otherwise, decreasing the stringency of the voting rule decreases the equilibrium value of \( \rho \), i.e., \( d\rho/dq_k < 0 \).
Since $\rho_i$ of any member $i$ depends on $q_k$ and are positive, $\bar{q}^e$ depends on $q_k$ and belongs to the open interval $(q_1, q_n)$. From the lemma, we know that the relationship between $\bar{q}^e$ and $q_k$ has implication for the voting rule that maximizes the equilibrium $\rho$. The two figures in Figure 1.1 illustrate two types of relationship between $\bar{q}^e$ and $q_k$, given that $q_k \in [q_1, q_n]$ and $q_1, q_n \in (0, 1)$. In these two figures, the number of intersections between the curve of $\bar{q}^e$ and the 45-degree line is important for determining the optimal voting rule. If $\bar{q}^e$ changes with $q_k$ as in the left panel, i.e., intersects with the 45-degree line once, then the optimal voting rule is a unanimous rule. However, if the relationship between $\bar{q}^e$ and $q_k$ is as in the right panel, then a non-unanimous rule might be optimal. In the next proposition, I show that if $C'''(\cdot) \leq 0$, $\bar{q}^e$ changes with $q_k$ as in the left panel.

**Proposition 1.12.** For a heterogeneous committee, the impact of the voting rule on equilibrium value of $\rho$ depends on $C'''(\cdot)$. If $C'''(\cdot) \leq 0$, then the voting rule inducing the most information acquisition is a unanimous rule.
1.5.3 Committee Design

For the problem of committee design by a designer with preference $q_0$, the change in the cost function changes only the constraint on the equilibrium value of $\rho$. As before, we can formulate the designer’s problem as

$$\max_{q,k} V_0(\rho; q_k)$$

s.t. $0 \leq q_1 \leq \ldots \leq q_n \leq 1$, $q_k \in I$, and

$$\sum_{i=1}^{n} C^{i-1} \left( \frac{\partial L_i(\rho_i; \rho_{-i}; \bar{\rho}_k; q_k)}{\partial \rho_i} \right) = \rho - \rho_0.$$ 

The proposition below characterizes the optimal committee.

**Proposition 1.13.** For a committee designer with $q_0$, the optimal decision-making committee satisfies one of the two following conditions:

1. $q_n > \bar{q}^*$, and $q_i = 0$ for any $i < n$, with $k = n$;

2. $q_1 < \bar{q}^*$, and $q_i = 1$ for any $i > 1$, with $k = 1$.

This proposition, unlike Proposition 1.6, does not make a statement regarding the relationship between $q_0$ and the preference of the threshold voter in the optimal committee. This is primarily because the threshold voter in the environment with convex information cost also collects information in equilibrium. Consider a committee with $q_n = q_0 > \bar{q}^*$, and $q_i = 0$ for any $i < n$, with $k = n$. Increasing $q_n$ in the neighborhood of $q_0$ increases the incentive of the non-threshold voters to acquire information, but will decrease the threshold voter’s efforts in collecting information. The aggregate effect of increasing $q_n$ on $\rho$ is thus indeterminate. So, for the optimal committee having the first set of features, we cannot determine the magnitude of $q_n$ relative to $q_0$. 
Now I consider the problem in which the designer is restricted to design a committee adopting the unanimous voting rule \( k = n \). The problem of the designer is

\[
\max_{\mathbf{q}} V_0(\rho; q_n)
\]

\[s.t. \ 0 \leq q_1 \leq \ldots \leq q_n \leq 1, \ q_n \in I, \text{ and}\]

\[
\sum_{i=1}^{n} C^{n-1} \left( \frac{\partial L_i (\rho_i, \rho_{-i}, \mathbf{z}; q_n)}{\partial \rho_i} \right) = \rho - \rho_0.
\]

For this problem, we have a result similar to Proposition 1.7. The only difference is that we cannot always determine the relationship between \( q_0 \) and the preference of the threshold voter.

**Proposition 1.14.** When the voting rule is \( k = n \),

1. if \( q_0 > \bar{q}^* \), then the optimal committee satisfies \( q_n > \bar{q}^* \) and \( q_i = 0 \) for any \( i < n \);
2. if \( q_0 < \bar{q}^* \), then the optimal committee is either homogeneous with \( q_1 = \ldots = q_n \in (q_0, \bar{q}^*) \) or heterogeneous with \( q_n > \bar{q}^* \) and \( q_i = 0 \) for any \( i < n \).

The proof of this proposition is similar to that of Proposition 1.7, so I omit it.

### 1.6 Private Information

In the analysis above, I rule out strategic information transmission by imposing Assumption 1.1, which states that the precision and realization of the signal acquired by each committee member are observable to others. I show in this section that relaxing this assumption, i.e., assuming instead that both the precision and realization of a signal are private, does not overturn my main findings if the signals are (verifiable) hard evidence regarding the true state and the members are allowed to communicate before they vote.

For this private information case, I specify the timing of the game as follows. In stage 1, every committee member chooses the precision of his private signal. In stage 2, the
public signal and the private signals are all realized, and each member observes the public signal and his own private signal. In stage 3, every member decides whether to reveal his signal, including its precision and realization. (The members cannot manipulate or partially reveal their signals.) In stage 4, the members vote to determine the ultimate decision.

All my assumptions except Assumption 1.1 are left unchanged. The results below hold both in the linear cost case and in the convex cost case.

**Proposition 1.15.** In the private information case, for every committee and every voting threshold, there is an equilibrium in which every member acquires the same amount of information as in the public information case and always reveals his information.

This proposition indicates that for an arbitrary committee, the equilibrium outcome in the public information case is also an equilibrium outcome in the private information case. Preference heterogeneity among the committee members does not prevent full information sharing. Consider, for example, a non-threshold member who leans more toward acceptance than the threshold voter *ex ante*. He has an incentive to conceal his information only when doing so can induce the threshold voter to vote for acceptance more often, given that other members reveal their information. If the threshold voter always votes for rejection when this non-threshold voter refuses to reveal his information, then his incentive to conceal the information is gone. The following corollary is a natural consequence of this proposition, combined with Propositions 1.6 and 1.13.

**Corollary 1.2.** In the private information case, there exists a heterogeneous committee whose full-information-aggregation equilibrium outperforms, from the perspective of the committee designer, the full-information-aggregation equilibrium of every homogeneous committee.

Proposition 1.15 and Corollary 1.2 indicate that the information collected by committee members need not be public as long as we assume that (1) the information is verifiable
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and (2) the committee members can communicate before voting. Without assumption (1), communication among the members becomes cheap talk. In this setting, Li et al. (2001) show that for any two-player heterogeneous committee, full information aggregation is impossible; however, the closer the players’ preferences are, the more information they will share. My conjecture is that the incentive to reduce disagreement still plays a role in incentivizing members to collect information, thus some preference heterogeneity in a committee can improve the designer’s payoff.

Without assumption (2), the members vote strategically to determine the collective decision. A comprehensive analysis of the committee design problem in this case is very complicated. However, I show in a two-player setting that some degree of preference heterogeneity is typically desirable for the designer. The analysis is at the end of Appendix B.

1.7 Discussion and Conclusion

In collective decision-making problems, members of the decision-making committees can collect decision-related information before they make their choices. The amount of information acquired by the members is potentially affected by the committee composition, size, and the decision rule. The existing literature examining the impacts of decision-making environments on information acquisition focuses on homogeneous committees. The current paper studies heterogeneous committees, with an emphasis on the roles of preference heterogeneity and voting rules in information collection.

When studying the problem of committee design, I assume that the size of the committee is fixed. If the designer is allowed to choose the committee size, how large is the optimal committee? The answer depends on the information cost function. With linear cost, the designer does not benefit from increasing the size of the committee beyond two members, as only one non-threshold voter collects information. Thus, a two-member
committee is good enough for the designer. With convex cost, the designer’s payoff is increasing in the size of the committee; adding more voters increases the aggregate information collected. Intuitively, a larger committee can distribute the information acquired across more members, lowering the total cost.

One assumption important for the analysis in the model is Assumption 1.3 which precludes non-concavity in the value of information. Setting aside the technical details, this assumption makes the precision choices strategic substitutes. Analyzing the case in which information acquired by the members can be either substitutes or complements is worth exploring.

Another assumption important for the analysis is that the members of a committee simultaneously acquire information. Some real world situations may not fit this assumption well. For example, in recruitment committees, some members may be responsible for reviewing the applications while others are responsible for interviewing candidates. In this situation, information is acquired sequentially by different members in the committee, with the followers being able to observe the findings of the earlier movers. One can imagine that if the earlier signals induce extreme posteriors, then the followers may not have a strong incentive to collect information; if the earlier signals make the followers quite unsure about the qualities of the candidates, then they will have a strong incentive to collect information. It may be interesting to characterize the optimal order in which the members collect information.
Appendix A

Proof of Lemma 1.2

Suppose that the equilibrium value of $\rho$ is not unique. Let $\rho'$ and $\rho''$ denote two of the equilibrium levels, with $\rho_0 \leq \rho' < \rho''$. Since $\rho'' > \rho_0$, we have $\rho''_j > 0$ for some member $j$ in the $\rho''$-equilibrium. For this member in the $\rho''$-equilibrium, we have

$$\frac{\partial V_j (\rho'_j, \rho''_j, s_k; q_k)}{\partial \rho_j} = 0.$$ 

(1.22)

In the $\rho'$-equilibrium, for the same member $j$, we have $\partial V_j (\rho'_j, \rho'_{-j}, s_k; q_k) / \partial \rho_j \leq 0$, according to Lemma 1.1. Since $\partial V_j (\rho_j, \rho_{-j}, s_k; q_k) / \partial \rho_j$ is a function of the aggregate precision $\rho$, we have

$$\frac{\partial^2 V_j (\rho_j, \rho_{-j}, s_k; q_k)}{\partial \rho_j^2} = \frac{\partial^2 V_j (\rho_j, \rho_{-j}, s_k; q_k)}{\partial \rho_j \partial \rho},$$

for all $(\rho_j, \rho_{-j})$ and $q_k$.

Given Assumption 1.3, we have $\partial^2 V_j (\rho_j, \rho_{-j}, s_k; q_k) / \partial \rho_j \partial \rho < 0$ for $\rho > \rho'$. Thus, we have

$$\frac{\partial V_j (\rho'_j, \rho''_j, s_k; q_k)}{\partial \rho_j} < \frac{\partial V_j (\rho'_j, \rho'_{-j}, s_k; q_k)}{\partial \rho_j} \leq 0,$$

due to $\rho' < \rho''$. This contradicts the equilibrium condition (1.22).

Proof of Proposition 1.1

If $\gamma = 1/2$, then based on the definition of $h(q_k, \rho, \gamma)$ in (1.12), for any $\rho$,

$$h(q_k, \rho, \gamma) \begin{cases} > 0, & \text{if } q_k < 1/2; \\ = 0, & \text{if } q_k = 1/2; \\ < 0, & \text{if } q_k > 1/2. \end{cases}$$
Therefore, for any $q_{-k}$, we have

$$h(q_k, \rho(q_k, q_{-k}), \gamma) = \begin{cases} > 0, & \text{if } q_k < 1/2; \\ = 0, & \text{if } q_k = 1/2; \\ < 0, & \text{if } q_k > 1/2, \end{cases}$$

which means that $q_k = 1/2$ is the unique virtually unbiased preference.

The proof for the case $\gamma \neq 1/2$ is more complex. To begin, consider a homogeneous committee in which all members have the same preference $q_k$. Let $\rho(q_k)$ denote the equilibrium aggregate precision of this committee, $\rho(q_k) \geq \rho_0$. It is easy to verify that $\rho(q_k)$ is continuous in $q_k$, using the maximum theorem. Thus, $h(q_k, \rho(q_k), \gamma)$ is continuous in $q_k$. Since

$$h(q_k, \rho(q_k), \gamma) = \begin{cases} > 0, & \text{if } q_k = \min \{1/2, \gamma\}; \\ < 0, & \text{if } q_k = \max \{1/2, \gamma\}, \end{cases}$$

according to the intermediate value theorem, there exists $\bar{q}^* \in (\min \{1/2, \gamma\}, \max \{1/2, \gamma\})$ such that

$$h(\bar{q}^*, \rho(\bar{q}^*), \gamma) = 0. \quad (1.23)$$

Before proceeding to heterogeneous committees, we first show that $\bar{q}^*$ is unique. At $\bar{q}^*$ satisfying (1.23), we have either $\rho(\bar{q}^*) = \rho_0$ or $\rho(\bar{q}^*) > \rho_0$. I consider these two cases separately below.

**Case 1:** $\rho(\bar{q}^*) = \rho_0$.

In this case, for any member $i$ of the committee, we have

$$\frac{\partial L_i(\rho_i, \rho_{-i}, s_k; \bar{q}^*)}{\partial \rho_i} = (1 - \bar{q}^*) \gamma f_1(s_k | \rho_0) \frac{s_k}{2\rho_0} + \bar{q}^* (1 - \gamma) f_0(s_k | \rho_0) \frac{(1 - s_k)}{2\rho_0} \leq c. \quad (1.24)$$
Fixing $\rho_0$, we examine how $\partial L_i / \partial \rho_i$ changes with $q_k$. We obtain

$$
\frac{\partial^2 L_i (\rho_i; \rho_{-i}; \hat{z}_k; q_k)}{\partial \rho_i \partial q_k} \bigg|_{\rho = \rho_0} = \frac{(1 - \gamma) f_0 (s_k | \rho_0)}{2 \rho_0 (1 - q_k)} h (q_k, \rho_0, \gamma)
$$

$$
+ \frac{1}{2} \left[(1 - q_k) \gamma f_1 (s_k | \rho_0) - q_k (1 - \gamma) f_0 (s_k | \rho_0) \right] \left[ \frac{1}{\rho_0} + \frac{s_k (1 - s_k)}{\rho_0 q_k (1 - q_k)} \right]
$$

$$
= \frac{(1 - \gamma) f_0 (s_k | \rho_0)}{2 \rho_0 (1 - q_k)} h (q_k, \rho_0, \gamma).
$$

(1.25)

The second equality is due to the fact that $(1 - q_k) \gamma f_1 (s_k | \rho_0) = q_k (1 - \gamma) f_0 (s_k | \rho_0)$. (See the discussion for equations (1.2) and (1.3).) From (1.25) and the fact that $h (q_k, \rho_0, \gamma)$ is decreasing in $q_k$, we have

$$
\frac{\partial^2 L_i (\rho_i; \rho_{-i}; \hat{z}_k; q_k)}{\partial \rho_i \partial q_k} \bigg|_{\rho = \rho_0} = \begin{cases} 
> 0, & \text{if } q_k < \bar{q}^*; \\
= 0, & \text{if } q_k = \bar{q}^*; \\
< 0, & \text{if } q_k > \bar{q}^*.
\end{cases}
$$

This implies that for $q_k \neq \bar{q}^*$,

$$
\frac{\partial L_i (\rho_i; \rho_{-i}; \hat{z}_k; q_k)}{\partial \rho_i} \bigg|_{\rho = \rho_0} < \frac{\partial L_i (\rho_i; \rho_{-i}; \bar{q}^*)}{\partial \rho_i} \bigg|_{\rho = \rho_0} \leq c,
$$

where the second inequality is due to (1.24). Therefore, $\rho (q_k) = \rho_0$ for any $q_k$, and $h (q_k, \rho (q_k), \gamma)$ is strictly decreasing in $q_k$.

**Case 2:** $\rho (\bar{q}^*) > \rho_0$

In this case, for any member $i$ of the committee

$$
\frac{\partial L_i (\rho_i; \rho_{-i}; \hat{z}_k; \bar{q}^*)}{\partial \rho_i} = (1 - \bar{q}^*) \gamma f_1 (s_k | \rho (\bar{q}^*)) \frac{s_k}{2 \rho (\bar{q}^*)} + \bar{q}^* (1 - \gamma) f_0 (s_k | \rho (\bar{q}^*)) \frac{(1 - s_k)}{2 \rho (\bar{q}^*)}
$$

$$
= c.
$$

(1.26)
This equation implies that \( \rho(q_k) \) is differentiable in the neighborhood of \( \bar{q}^* \). By total differentiation, we can derive the following equation from (1.26) for \( q_k \) in the neighborhood of \( \bar{q}^* \),

\[
(1 - \gamma) \frac{f_0(\tilde{s}_k|\rho(q_k))}{2\rho(q_k)(1 - q_k)} h(q_k, \rho(q_k), \gamma) + \frac{\partial^2 L_i(\rho, \rho_{-i}, \tilde{s}_k; q_k)}{\partial \rho_i^2} \frac{\partial \rho(q_k)}{\partial q_k} = 0. \tag{1.27}
\]

Since \( h(\bar{q}^*, \rho(\bar{q}^*), \gamma) = 0 \), we have \( \frac{\partial \rho(q_k)}{\partial q_k} |_{q_k = \bar{q}^*} = 0 \). Thus,

\[
\frac{\partial h(q_k, \rho(q_k), \gamma)}{\partial q_k} |_{q_k = \bar{q}^*} = - \left[ 1 + \frac{1}{\rho(\bar{q}^*) \bar{q}^* (1 - \bar{q}^*)} \right] + \frac{1}{\rho(\bar{q}^*)^2} \ln \frac{\bar{q}^* (1 - \gamma)}{(1 - \bar{q}^*) \gamma} \frac{\partial \rho(q_k)}{\partial q_k} |_{q_k = \bar{q}^*} = - \left[ 1 + \frac{1}{\rho(\bar{q}^*) \bar{q}^* (1 - \bar{q}^*)} \right] < 0.
\]

According to the discussion for the two cases above, we can see that \( h(q_k, \rho(q_k), \gamma) \) is decreasing at \( q_k = \bar{q}^* \). This implies that \( \bar{q}^* \) satisfying (1.23) is unique.

We now switch to heterogeneous committees. I first prove that \( \bar{q}^* \) is a virtually unbiased preference, i.e., \( h(\bar{q}^*, \rho(\bar{q}^*, q_{-k}), \gamma) = 0 \), for any \( q_{-k} \). Given that \( h(\bar{q}^*, \rho(\bar{q}^*), \gamma) = 0 \), I show that \( h(\bar{q}^*, \rho(\bar{q}^*, q_{-k}), \gamma) = 0 \) holds for any \( q_{-k} \) by showing that \( \rho(q^*, q_{-k}) = \rho(q^*) \) for any \( q_{-k} \). From Lemma 1.3, it is clear that regardless of \( q_{-k} \), when \( q_k = \bar{q}^* \) and \( \rho = \rho(\bar{q}^*) \),

\[
\frac{\partial^2 L_i(\rho, \rho_{-i}, \tilde{s}_k; q_k)}{\partial \rho_i \partial q_i} = 0, \forall i. \tag{1.28}
\]

Since \( \rho(\bar{q}^*) \) is the equilibrium precision for the homogeneous committee with \( q_i = \bar{q}^* \) for any \( i \), (1.28) and Lemma 1.2 imply that \( \rho(\bar{q}^*) \) is also the unique equilibrium precision for any committee with \( q_k = \bar{q}^* \), i.e., \( \rho(\bar{q}^*, q_{-k}) = \rho(\bar{q}^*) \) for any \( q_{-k} \). Therefore, \( \bar{q}^* \) is a virtually unbiased preference.

Now I show that \( \bar{q}^* \) is the unique virtually unbiased preference. I first prove that the equilibrium precision \( \rho(q_k, q_{-k}) \) under \( (q_k, q_{-k}) \) is continuous in \( q_k \). For any given profile
q_{-k}, according to Lemma 1.1, the equilibrium precision \( \rho(q_k, q_{-k}) \) satisfies

\[
\rho(q_k, q_{-k}) = \min \left\{ \rho \geq \rho_0 : \max_{i \in \{1, \ldots, n\}} \frac{\partial L_i}{\partial \rho_i} (\rho_i, \rho_{-i}, \bar{z}_k; q_k) \leq c \right\}.
\]

Since \( \partial L_i (\rho_i; \rho_{-i}, \bar{z}_k; q_k) / \partial \rho_i \) is continuous in \((\rho, q_k)\) for all \(i\), \(\max_{i \in \{1, \ldots, n\}} \partial L_i (\rho_i, \rho_{-i}, \bar{z}_k; q_k) / \partial \rho_i\) is continuous in \((\rho, q_k)\). Thus, \( \rho(q_k, q_{-k}) \) is continuous in \(q_k\), which implies that the function \(h(q_k, \rho(q_k, q_{-k}), \gamma)\) is continuous in \(q_k\) and satisfies

\[
h(q_k, \rho(q_k, q_{-k}), \gamma) \begin{cases} > 0, & \text{if } q_k = \min \{1/2, \gamma\}; \\ < 0, & \text{if } q_k = \max \{1/2, \gamma\}, \end{cases}
\]

and \(h(\bar{q}^*, \rho(\bar{q}^*, q_{-k}), \gamma) = 0\). Now I show that for any \(q_{-k}\), there is no other value of \(q_k\) satisfying \(h(q_k, \rho(q_k, q_{-k}), \gamma) = 0\). I prove this by contradiction. Suppose that there exists \(\hat{q}^* \in (\min \{1/2, \gamma\}, \max \{1/2, \gamma\})\) such that \(h(\hat{q}^*, \rho(\hat{q}^*, q_{-k}), \gamma) = 0\) for some \(q_{-k}\) and \(\hat{q}^*\). This implies that a homogeneous committee with all members having preference \(\hat{q}^*\) satisfies \(h(\hat{q}^*, \rho(\hat{q}^*, q_{-k}), \gamma) = 0\), following our discussion are (1.28). This contradicts the uniqueness of \(\bar{q}^*\) for homogeneous committees. Therefore, \(\bar{q}^*\) is the unique value satisfying \(h(\bar{q}^*, \rho(\bar{q}^*, q_{-k}), \gamma) = 0\), so it is the unique virtually unbiased preference. Combining this uniqueness with (1.29) and the continuity of \(\rho(q_k, q_{-k})\) in \(q_k\), we have for \(q_k < \bar{q}^*\), \(h(q_k, \rho(q_k, q_{-k}), \gamma) > 0\) for any \(q_{-k}\), and for \(q_k > \bar{q}^*\), \(h(q_k, \rho(q_k, q_{-k}), \gamma) < 0\) for any \(q_{-k}\).

**Proof of Proposition 1.3**

I divide my discussion into three cases below.

**Case 1:** Under voting rule \(k\), \(q_k > \bar{q}^*\).

According to Proposition 1.1, member 1 has the most incentive to acquire information,
i.e., $\frac{\partial L_i}{\partial \rho_i} \geq \frac{\partial L_i}{\partial \rho_j}$ for all $i \neq 1$. Fixing $\rho_1$, taking the derivative of $\frac{\partial L_k}{\partial \rho_1}$ w.r.t. $q_k$ yields

$$\frac{\partial^2 L_1}{\partial \rho_1 \partial q_k} \big|_{\rho_1 = 1} = \frac{1}{2} (1 - q_1) \gamma f_1 (s_k | \rho) - q_1 (1 - \gamma) f_0 (s_k | \rho) \left( \frac{1}{\rho} + s_k (1 - s_k) \right) \rho q_k (1 - q_k).$$

(1.30)

In this equation, $\frac{1}{\rho} + s_k (1 - s_k) > 0$, given Assumption 1.3 and the expression of $\rho (q_k, \gamma)$ in (1.37). Thus, $\frac{\partial^2 L_1}{\partial \rho_1 \partial q_k} \big|_{\rho_1 = 1}$ has the same as $[(1 - q_1) \gamma f_1 (s_k | \rho) - q_1 (1 - \gamma) f_0 (s_k | \rho)]$ on the RHS of (1.30). If $q_k > q_1$, $\frac{\partial^2 L_1}{\partial \rho_1 \partial q_k} \big|_{\rho_1} > 0$, i.e., member 1’s incentive to acquire information is increasing in $q_k$. Since $q_k > q^*$, increasing $q_k$ does not change the fact that member 1 has the most incentive to acquire information. Therefore, the aggregate information acquired is increasing in $q_k$, and (weakly) increasing in $k$. If $q_k = q_1$, we have $\frac{\partial^2 L_1}{\partial \rho_1 \partial q_k} \big|_{\rho_1} = 0$, but the continuity of $\frac{\partial^2 L_1}{\partial \rho_1 \partial q_k} \big|_{\rho_1}$ in $q_k$ implies that the argument above still applies. Therefore, the voting rule $k = n$ induces (weakly) more information acquisition than does any rule in this case.

**Case 2:** Under voting rule $k$, $q_k < q^*$.

For this case, we can use a similar argument as above to show that the voting rule $k = 1$ induces (weakly) more information acquisition than does any other rule.

**Case 3:** Under voting rule $k$, $q_k = q^*$.

For this case, all the members have the same incentive to acquire information, i.e., $\frac{\partial L_i}{\partial \rho_i} = \frac{\partial L_j}{\partial \rho_j}$ for $i \neq j$. We can, without loss of generality, let member 1 or member $n$ be the only information collector, and adopt the arguments in the two cases above to show that at least one of the rules, $k = 1$ or $k = n$, outperforms any other rules.

Therefore, based on the analysis in the three cases above, we can conclude that one of the unanimous rule induces most information acquisition in equilibrium. However, this does not mean that both unanimous rules outperform all non-unanimous rules. Consider
a committee with \( q_o \leq \bar{q}^* \). The argument above implies that the unanimous rule with \( k = 1 \) induces most information acquisition, but the unanimous rule with \( k = n \) induces the least information acquisition, and is outperformed by all non-unanimous rules.

**Proof of Proposition 1.4**

To simplify the notation, we use \( V_0 (q_0, q_1) \) to denote the expected payoff of a designer with preference \( q_0 \) from choosing a committee member with preference \( q_1 \), i.e.,

\[
V_0 (q_0, q_1) = -(1 - q_0) \gamma F_1(s_1|\rho) - q_0 (1 - \gamma) [1 - F_0(s_1|\rho)].
\]

Taking the derivative of \( V_0 (q_0, q_1) \) w.r.t. \( q_1 \), we obtain

\[
\frac{\partial V_0 (q_0, q_1)}{\partial q_1} = \left[ -(1 - q_0) \gamma f_1(s_1|\rho) + q_0 (1 - \gamma) f_0(s_1|\rho) \right] \frac{\partial s_1}{\partial q_1}_{|\rho} + \left[ (1 - q_0) \gamma f_1(s_1|\rho) \frac{s_1}{2\rho} + q_0 (1 - \gamma) f_0(s_1|\rho) \frac{(1-s_1)}{2\rho} \right] \frac{\partial \rho}{\partial q_1},
\]

where

\[
\frac{\partial s_1}{\partial q_1}_{|\rho} = \frac{1}{\rho q_1 (1 - q_1)} > 0.
\]

I should point out here that the equilibrium \( \rho \) may not be differentiable w.r.t. \( q_1 \) everywhere. However, the failure of differentiability at some point does not matter for our proof, as we care only about the direction of the change in the equilibrium value of \( \rho \) w.r.t \( q_1 \). From equations (1.25) and (1.27) in the proof of Proposition 1.1, the equilibrium value of \( \rho \) is (weakly) increasing in \( q_1 \) if \( q_1 < \bar{q}^* \) and (weakly) decreasing in \( q_1 \) if \( q_1 > \bar{q}^* \).

I discuss the three cases, \( q_0 < \bar{q}^* \), \( q_0 > \bar{q}^* \), and \( q_0 = \bar{q}^* \), separately below.

**Case 1**: \( q_0 < \bar{q}^* \).

If \( q_1 < q_0 \), then \( V_0 (q_0, q_1) \) is increasing in \( q_1 \), as the first line of (1.31) is positive and the second line is non-negative. Thus, \( q_1 < q_0 \) is not optimal. If \( q_1 > \bar{q}^* \), then \( V_0 (q_0, q_1) \) is
decreasing in \( q_1 \), as the first line of \((1.31)\) is negative and the second line is non-positive.

For \( q_1 \in [q_0, \tilde{q}^*] \), according to Assumption \( 1.4 \), we have \( \rho > 0 \) in equilibrium. At \( q_1 = q_0 \), we have \( \frac{\partial \rho}{\partial q_1} > 0 \), so \( \frac{\partial V_0(q_0, q_1)}{\partial q_1} > 0 \), as the first line of \((1.31)\) is 0 and the second line is positive. At \( q_1 = \tilde{q}^* \), we have \( \frac{\partial \rho}{\partial q_1} = 0 \), so \( \frac{\partial V_0(q_0, q_1)}{\partial q_1} < 0 \), as the first line of \((1.31)\) is negative and the second line is 0. Therefore, the optimal \( q_1 \) must be in \((q_0, \tilde{q}^*)\).

**Case 2:** \( q_0 > \tilde{q}^* \).

Using an argument similar to the case above, we can conclude that the optimal \( q_1 \) must be in \((\tilde{q}^*, q_0)\).

**Case 3:** \( q_0 = \tilde{q}^* \).

In this case, it is obvious that the optimal \( q_1 = q_0 \). If \( q_1 < q_0 = \tilde{q}^* \), then the first line of \((1.31)\) is positive and the second line is non-negative. If \( q_1 > q_0 = \tilde{q}^* \), then the first line of \((1.31)\) is negative and the second line is non-positive. At \( q_1 = q_0 = \tilde{q}^* \), we have \( \frac{\partial V_0(q_0, q_1)}{\partial q_1} = 0 \).

If Assumption \( 1.4 \) does not hold, then the optimal committee member may have \( q_1 = q_0 \), that is, the designer chooses one having the same preferences as herself. I omit a detailed discussion of this case.

**Proof of Proposition 1.5**

If the designer retains the right of making the final decision, the committee design problem becomes

\[
\max_{q_1} V_0 (\rho, q_0) \\
\text{s.t. } 0 \leq q_1 \leq 1, \text{ and} \\
\rho = \arg \max_{\hat{\rho} \geq \rho_0} L_1(\hat{\rho}, \hat{z}_0; q_0) - c(\hat{\rho} - \rho_0).
\]
where
\[ L_1 (\hat{\rho}, s_0; q_0) = -(1 - q_1) \gamma F_1 (s_0 | \hat{\rho}) - q_1 (1 - \gamma) [1 - F_0 (s_0 | \hat{\rho})], \]

which denotes the expected payoff of member 1 from his decision given the aggregate precision \( \hat{\rho} \).

For \( V_0 (\rho, q_0) \), we have
\[ \frac{\partial V_0 (\rho, q_0)}{\partial \rho} = (1 - q_0) \gamma f_1 (s_0 | \rho) \frac{s_0}{2 \rho} + q_0 (1 - \gamma) f_0 (s_0 | \rho) \frac{(1 - s_0)}{2 \rho} > 0, \]

which implies that the payoff of the designer is increasing in \( \rho \). In equilibrium, \( \rho \) satisfies
\[ \frac{\partial L_1 (\rho, s_0; q_0)}{\partial \rho} - c \leq 0. \quad (1.32) \]

Given Assumption 1.4, we have that equilibrium \( \rho > \rho_0 \) at \( q_1 = q_0 \), so condition (1.32) holds with equality at \( q_1 = q_0 \). Thus, for \( q_1 \) in the neighborhood of \( q_0 \), we have
\[ \frac{\partial \rho}{\partial q_1} = - \frac{(1 - \gamma) f_0 (s_0 | \rho)}{2 \rho} \frac{h (q_0, \rho, \gamma)}{\partial^2 L_1 (\rho, s_0; q_0) / \partial \rho^2}. \]

If \( q_0 < \bar{q}^* \), then according to equation (1.13), we have that \( \frac{\partial \rho}{\partial q_1} > 0 \) for \( q_1 \geq q_0 \). The intuition is the same as (1.13). For \( q_1 < q_0 \), our monotonicity result implies that the amount of information acquired in this case is less than that acquired in the case of \( q_1 = q_0 \). Therefore, a member 1 with \( q_1 = 1 \) acquires the most information.

If \( q_0 > \bar{q}^* \), then the argument is reversed, and a member 1 with \( q_1 = 0 \) acquires the most information.

If \( q_0 = \bar{q}^* \), the definition of \( \bar{q}^* \) directly implies that any \( q_1 \in [0, 1] \) induces the same amount of information.

**Proof of Proposition 1.6**

From the designer’s problem (1.16), we can see that the non-threshold voters affect
the payoff of the designer through $\rho$. Since

$$
\frac{\partial V_0(\rho, q_k)}{\partial \rho} = (1 - q_0) \gamma f_1(s_k|\rho) \frac{s_k}{2\rho} + q_0 (1 - \gamma) f_0(s_k|\rho) \frac{(1 - s_k)}{2\rho} > 0,
$$

the designer would like to have the non-threshold voters collect as much information as possible, given $q_k$. The impact of $q_k$ on the designer’s payoff is more complicated. Let $V_0(q_0, q_k, q_{-k}) \equiv V_0(\rho(q_k, q_{-k}; \rho_0, \gamma), q_k)$. Taking the derivative of $V_0(q_0, q_k, q_{-k})$ w.r.t. $q_k$, we obtain

$$
\frac{\partial V_0(q_0, q_k, q_{-k})}{\partial q_k} = [- (1 - q_0) \gamma f_1(s_k|\rho) + q_0 (1 - \gamma) f_0(s_k|\rho)] \frac{\partial s_k}{\partial q_k} \rho
$$

$$
+ \left[ (1 - q_0) \gamma f_1(s_k|\rho) \frac{s_k}{2\rho} + q_0 (1 - \gamma) f_0(s_k|\rho) \frac{(1 - s_k)}{2\rho} \right] \frac{\partial \rho}{\partial q_k}.
$$

**Case 1: $q_0 < \bar{q}$**

Suppose that the designer chooses $q_k < \bar{q}$. For such a committee, the proof of Proposition 1.5 implies that having $q_i = 1, i \neq k$, induces the most information acquisition. Thus, given $q_k < \bar{q}$, the designer always (weakly) prefers to have $q_i = 1, i \neq k$. Given $q_i = 1$ for all $i \neq k$, if $q_k \in [q_0, \bar{q})$, then Assumption 1.4 implies that $\rho > \rho_0$ in equilibrium, thus we have $\partial V_0(q_0, q_k, q_{-k}) / \partial q_k < 0$, as the first line of (1.33) is non-positive and its second line is negative due to

$$
\frac{\partial \rho_j}{\partial q_k} = \frac{1}{2} (1 - \gamma) f_0(s_k|\rho) \frac{1 + s_k (1 - s_k)}{\rho q_k (1 - q_k)} \frac{\partial^2 L_j(\rho_j, \rho_{-j}, s_k; q_k)}{\partial \rho_j^2} < 0,
$$

for any information collector $j \neq k$. The intuition for (1.34) is the same as that for Proposition 1.3. This result indicates that among all committees with $q_k < \bar{q}$, there is an optimal one satisfying $q_k < q_0$ and $q_i = 1, i \neq k$.

Similar to the argument above, among all committees with $q_k > \bar{q}$, there is an optimal one satisfying $q_k > q_0$ and $q_i = 0, i \neq k$. 
If the designer chooses \( q_k = \bar{q}^* \), it is indifferent for her to choose any non-threshold voters, so the homogeneous committee with \( q_i = \bar{q}^* \) for all \( i \), will give her the highest payoff. However, for this homogeneous committee, we have \( \frac{\partial \rho}{\partial q_k} = 0 \) and \( \partial V_0 / \partial q_k < 0 \). This means that this committee is dominated by one with \( q_k < \bar{q}^* \) and \( q_i = 1 \), \( i \neq k \).

Therefore, we conclude that for a designer with \( q_0 < \bar{q}^* \), there is an optimal committee satisfying either \( q_n \in (\bar{q}^*, 1) \) and \( q_i = 0 \), \( i < n \), with \( k = n \), or \( q_1 \in (0, q_0) \) and \( q_i = 1 \), \( i > 1 \), with \( k = 1 \).

**Case 2:** \( q_0 > \bar{q}^* \)

Similar to the argument above, we conclude that there is an optimal committee satisfying either \( q_n \in (q_0, 1) \) and \( q_i = 0 \), \( i < n \), with \( k = n \), or \( q_1 \in (0, \bar{q}^*) \) and \( q_i = 1 \), \( i > 1 \), with \( k = 1 \).

**Case 3:** \( q_0 = \bar{q}^* \)

For this case, I show that a committee with \( q_k = \bar{q}^* \) is not optimal. Given that \( q_k = \bar{q}^* \), the equilibrium \( \rho \) does not change with the preferences of the non-threshold voters. Thus, the designer is indifferent between a committee with \( q_k = \bar{q}^* \), \( q_i = 1 \), \( i \neq k \) and a committee with \( q_k = \bar{q}^* \), \( q_i = 0 \), \( i \neq k \). It is obvious that for the former one, \( \partial V_0 (q_0, q_k, q_{-k}) / \partial q_k < 0 \), so it is outperformed by a committee with \( q_k < \bar{q}^* \), \( q_i = 1 \), \( i \neq k \). For the latter one, \( \partial V_0 (q_0, q_k, q_{-k}) / \partial q_k > 0 \), so it is outperformed by a committee with \( q_k > \bar{q}^* \), \( q_i = 0 \), \( i \neq k \). Thus, a committee with \( q_k = \bar{q}^* \) is not optimal. Adopting an argument similar to the two cases above, we conclude that there is an optimal committee satisfying either \( q_n \in (q_0, 1) \) and \( q_i = 0 \), \( i < n \), with \( k = n \), or \( q_1 \in (0, q_0) \) and \( q_i = 1 \), \( i > 1 \), with \( k = 1 \).

**Proof of Proposition 1.7**

Proof of this proposition can be easily obtained based on the proof above for Proposition 1.6, taking into account the constraint \( q_i \leq q_k \), for all \( i \neq k \).
Proof of Lemma 1.4

For any $i$, we have

\[
\frac{\partial^2 V_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i^2} = \frac{\partial^2 L_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho^2} \tag{1.35}
\]

\[
= -\frac{(1 - \gamma) f_0 (\xi_k | \rho)}{2 (1 - q_k) \rho} \left\{ -\frac{1}{\rho} + (1 - \xi_k) \xi_k \right\} [(1 - q_i) q_k \xi_k + (1 - q_k) q_i (1 - \xi_k)]
\]

\[
+ \frac{(1 - \gamma) f_0 (\xi_k | \rho)}{(1 - q_k) \rho^3} \ln q_k (1 - \gamma) \frac{(1 - q_i) q_i - (1 - q_i) q_k}{(1 - q_k) (1 - q_i) q_k}. \tag{1.36}
\]

In (1.35), we factor out $\frac{(1 - \gamma) f_0 (\xi_k | \rho)}{2 (1 - q_k) \rho}$ and obtain

\[
\frac{\partial^2 V_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i^2}
\]

\[
= \frac{(1 - \gamma) f_0 (\xi_k | \rho)}{2 (1 - q_k) \rho} \left\{ -\frac{1}{\rho} + (1 - \xi_k) \xi_k \right\} [(1 - q_i) q_k \xi_k + (1 - q_k) q_i (1 - \xi_k)]
\]

\[
+ \frac{2}{\rho^2} \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma} [(1 - q_i) q_i - (1 - q_i) q_k].
\]

Thus, $\frac{\partial^2 V_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i^2} / \frac{(1 - \gamma) f_0 (\xi_k | \rho)}{2 (1 - q_k) \rho}$ is equal to the expression in the large brackets. We rewrite the term $\frac{2}{\rho^2} \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma}$ in the large brackets as $\frac{1}{\rho} (2 \xi_k - 1)$. Then, we have

\[
\frac{\partial^2 V_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i^2} / \frac{(1 - \gamma) f_0 (\xi_k | \rho)}{2 (1 - q_k) \rho}
\]

\[
= - (1 - \xi_k) \xi_k [(1 - q_i) q_k \xi_k + (1 - q_k) q_i (1 - \xi_k)]
\]

\[
- \frac{1}{\rho} \{ (1 - q_i) q_k (3 \xi_k - 1) + (1 - q_k) q_i (2 - 3 \xi_k) \}. \tag{1.36}
\]

We can see that if $3 \xi_k - 1 > 0$ and $2 - 3 \xi_k > 0$, i.e., $\xi_k \in (\frac{1}{3}, \frac{2}{3})$, then $\frac{\partial^2 V_i (\rho_i, \rho_{-i}, \xi_k; q_k)}{\partial \rho_i^2} / \frac{(1 - \gamma) f_0 (\xi_k | \rho)}{2 (1 - q_k) \rho} < 0$. Since $\xi_k = \frac{1}{2} + \frac{1}{\rho} \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma}$, we have $\xi_k \in (\frac{1}{3}, \frac{2}{3})$ if and only if $\frac{1}{\rho} \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma} \in (-\frac{1}{6}, \frac{1}{6})$, i.e.,

\[
\left| \frac{1}{\rho} \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma} \right| < \frac{1}{6}.
\]
or equivalently,

\[ \rho > \rho(q_k, \gamma) \equiv 6 \left| \ln \frac{q_k (1 - \gamma)}{(1 - q_k) \gamma} \right|. \tag{1.37} \]

Therefore, given that \( \frac{(1 - \gamma) f_0(q_k | \rho)}{2(1 - q_k) \rho} > 0 \), if \( \rho > \rho(q_k, \gamma) \), then \( \frac{\partial^2 V_i(\rho, \theta_{-i}, z_k; q_k)}{\partial \rho^2} < 0 \).
Appendix B

Proof of Proposition 1.8

The proof of Proposition 1.1 applies to this proposition, with modifications as follows. First of all, since every committee members acquire information in equilibrium and \( \rho(q_k, q_{-k}) \) is differentiable with respect to \( (q_k, q_{-k}) \), we exclude the discussion on the case \( \rho(\bar{q}^*) = \rho_0 \) and the continuity of \( \rho(q_k, q_{-k}) \) in \( q_k \). Secondly, we replace equations (1.26) and (1.27) by the following ones, respectively,

\[
\frac{\partial L_i}{\partial \rho_i} (\rho_i, \rho_{-i}, \bar{x}_k; \bar{q}^*) = (1 - \bar{q}^*) \gamma f_1 (\bar{x}_k | \rho(\bar{q}^*)) - \frac{\bar{x}_k}{2 \rho(\bar{q}^*)} + \bar{q}^* (1 - \gamma) f_0 (\bar{x}_k | \rho(\bar{q}^*)) \frac{(1 - \bar{x}_k)}{2 \rho(\bar{q}^*)} = C'' (\rho_i),
\]

and

\[
\frac{(1 - \gamma) f_0 (\bar{x}_k | \rho(q_k))}{2 \rho(q_k) (1 - q_k)} h(q_k, \rho(q_k); \gamma) + \frac{\partial^2 L_i}{\partial \rho_i^2} (\rho_i, \rho_{-i}, \bar{x}_k; q_k) \frac{\partial \rho(q_k)}{\partial q_k} = C'' (\rho_i) \frac{\partial \rho(q_k)}{\partial q_k}.
\]

Proof of Proposition 1.9

I examine the impact of change in \( q_j, j \neq k \), on information acquisition. From the first order conditions (1.18), we obtain

\[
\sum_{l=1}^{n} \frac{\partial^2 L_i}{\partial \rho_i \partial \rho_l} (\rho_i, \rho_{-i}, \bar{x}_k; q_k) \frac{\partial \rho_l}{\partial q_j} - C'' (\rho_i) \frac{\partial \rho_i}{\partial q_j} = 0, \forall i \neq j,
\]

or equivalently, using the fact that for any \( i \) and \( l \), \( \frac{\partial^2 L_i}{\partial \rho_i \partial \rho_l} (\rho_i, \rho_{-i}, \bar{x}_k; q_k) / \partial \rho_i \partial \rho_l = \frac{\partial^2 L_i}{\partial \rho_i^2} (\rho_i, \rho_{-i}, \bar{x}_k; q_k) / \partial \rho_i^2 \),

\[
\frac{1}{C'' (\rho_i)} \frac{\partial^2 L_i}{\partial \rho_i^2} (\rho_i, \rho_{-i}, \bar{x}_k; q_k) \frac{\partial \rho_i}{\partial q_j} \frac{\partial \rho_i}{\partial q_j} = \frac{\partial \rho_i}{\partial q_j}, \forall i \neq j, \quad (1.38)
\]
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\[
\frac{1}{C''(\rho_j)} \frac{(1 - \gamma) f_0(z_k | \rho)}{2 \rho (1 - q_k)} h(q_k, \rho; \gamma) + \frac{1}{C''(\rho_j)} \frac{\partial^2 L_j (\rho_j, \rho_{-j}, z_k; q_k)}{\partial \rho_j^2} \frac{\partial \rho}{\partial q_j} = \frac{\partial \rho_j}{\partial q_j}. \tag{1.39}
\]

We add up all the equations above together and obtain

\[
\frac{1}{C''(\rho_j)} \frac{(1 - \gamma) f_0(z_k | \rho)}{2 \rho (1 - q_k)} h(q_k, \rho; \gamma) = \left[ 1 - \sum_{l=1}^{n} \left( \frac{1}{C''(\rho_l)} \frac{\partial^2 L_l (\rho_l, \rho_{-l}, z_k; q_k)}{\partial \rho_l^2} \right) \right] \frac{\partial \rho}{\partial q_j}. \tag{1.40}
\]

Since \( \sum_{l=1}^{n} \left( \frac{1}{C''(\rho_l)} \frac{\partial^2 L_l (\rho_l, \rho_{-l}, z_k; q_k)}{\partial \rho_l^2} \right) \) < 0, we conclude that if \( q_k > \bar{q}^* \), \( \frac{\partial \rho}{\partial q_j} < 0 \), and if \( q_k < \bar{q}^* \), \( \frac{\partial \rho}{\partial q_j} > 0 \). From equation (1.39), we have that \( \frac{\partial \rho_i}{\partial q_j} > 0 \) if \( q_k > \bar{q}^* \), and \( \frac{\partial \rho_i}{\partial q_j} < 0 \) if \( q_k < \bar{q}^* \). The sign of \( \frac{\partial \rho_i}{\partial q_j} \) is consistent with that of \( \frac{\partial \rho}{\partial q_j} \), as \( \frac{\partial \rho_i}{\partial q_j} \), \( i \neq j \), always has a sign opposed to the sign of \( \frac{\partial \rho}{\partial q_j} \).

**Proof of Proposition 1.10**

I prove a more general version of Proposition 1.10. I assume that the voting threshold is \( k \), with \( 1 < k < n \), and the preference profile of the committee has the following structure

\[
q_i = \begin{cases} 
q_k + (i - k) \delta, & \text{if } i \leq k, \\
q_k + (i - k) \tilde{\delta}, & \text{if } i > k,
\end{cases}
\]

where \( \tilde{\delta} \) satisfies

\[
\tilde{\delta} = \frac{k (k - 1) \delta}{(n + 1 - k) (n - k)}, \quad \delta > 0.
\]

With such a preference profile, the average preference of the committee is always \( q_k \) regardless of the value of \( \delta \). Proposition 1.10 is only for the special case of this environment that \( n \) is an odd number and \( k = (n + 1) / 2 \).

Now I examine the change of \( \delta \) on the equilibrium efforts of the committee members in acquiring information. From (1.18), we have that for \( i \leq k \),

\[
\frac{(1 - \gamma) f_0(z_k | \rho)}{2 \rho (1 - q_k)} h(q_k, \rho; \gamma) (i - k) + \frac{\partial^2 L_i (\rho_i, \rho_{-i}, z_k; q_k)}{\partial \rho_i^2} \frac{\partial \rho}{\partial \delta} = C''(\rho_i) \frac{\partial \rho_i}{\partial \delta},
\]
and for $i > k$,

$$\frac{(1 - \gamma) f_0 (s_k | \rho)}{2\rho (1 - q_k)} \frac{k (k - 1)}{(n + 1 - k) (n - k)} h (q_k, \rho, \gamma) (i - k) + \frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i^2} \frac{\partial \rho}{\partial \delta} = C'' (\rho_i) \frac{\partial \rho_i}{\partial \delta},$$

Multiplying $\frac{\delta}{C'' (\rho_i)}$ on both sides of the equations, and then sum them up across $i$, we obtain

$$\frac{(1 - \gamma) f_0 (s_k | \rho)}{2\rho (1 - q_k)} h (q_k, \rho, \gamma) \sum_{i=1}^n \lambda_i (q_i - q_k) \sum_{l=1}^n \frac{1}{C'' (\rho_l)} = \delta \left[ 1 - \left( \sum_{i=1}^n \lambda_i \frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i^2} \right) \sum_{l=1}^n \frac{1}{C'' (\rho_l)} \right] \frac{\partial \rho}{\partial \delta},$$

in which

$$\lambda_i = \frac{1 / C'' (\rho_i)}{\sum_{i=1}^n 1 / C'' (\rho_i)} \in (0, 1), \text{ and } \sum_{i=1}^n \lambda_i = 1.$$  

Since $\frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i^2} < 0$, we have

$$\sum_{i=1}^n \lambda_i \frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i^2} < 0.$$

Thus, the sign of $\frac{\partial \rho}{\partial \delta}$ is the same as that of the term

$$\frac{(1 - \gamma) f_0 (s_k | \rho)}{2\rho (1 - q_k)} h (q_k, \rho, \gamma) \left[ \sum_{i=1}^n \lambda_i (q_i - q_k) \right].$$

If $q_k = \bar{q}^*$, then $h (q_k, \rho, \gamma) = 0$, and $\frac{\partial \rho}{\partial \delta} = 0$. So with a virtually unbiased threshold voter, changing the preference polarization among a group will not affect information collection. However, if $q_k \neq \bar{q}^*$, the degree of preference polarization matters. Let us first consider the case where $C''' (\rho_i) > 0$. In this case, if $q_k > \bar{q}^*$, then $h (q_k, \rho, \gamma) < 0$ and $\rho_i$ is decreasing in $i$. This implies that $\lambda_i$ is increasing in $i$, and

$$\sum_{i=1}^n \lambda_i (q_i - q_k) > 0.$$

So

$$\frac{(1 - \gamma) f_0 (s_k | \rho)}{2\rho (1 - q_k)} h (q_k, \rho, \gamma) \left[ \sum_{i=1}^n \lambda_i (q_i - q_k) \right] < 0,$$

and $\frac{\partial \rho}{\partial \delta} < 0$. If $q_k > \bar{q}^*$, then
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$h(q_k, \rho, \gamma) < 0$ and $\rho_i$ is increasing in $i$, which leads to the result that

$$\sum_{i=1}^{n} \lambda_i (q_i - q_k) < 0.$$ 

So $\frac{\partial p}{\partial q} < 0$ as well. Thus, with $C'''(\rho_i) > 0$, increasing preference polarization of a committee decreases the amount of information collected by the group in equilibrium. The result will be reversed for the case in which $C'''(\rho_i) < 0$.

Proof of Proposition 1.11

We compare the equilibrium aggregate precision $\rho = \sum_{i=0}^{n} \rho_i$ of the heterogeneous committee with that of the homogeneous one. From (1.18), we obtain

$$(1 - \bar{q}) \gamma f_1 (s_k | \rho) \frac{s_k}{2\rho} + \bar{q} (1 - \gamma) f_0 (s_k | \rho) \frac{1 - s_k}{2\rho} = \frac{1}{n} \sum_{i=1}^{n} C' (\rho_i), \quad (1.41)$$

where

$$\bar{q} = \frac{\sum_{i=1}^{n} q_i}{n}.$$ 

If $C'$ is concave, i.e., $C''' < 0$, we have

$$\frac{1}{n} \sum_{i=1}^{n} C' (\rho_i) \leq C' \left( \frac{\sum_{i=1}^{n} \rho_i}{n} \right) = C' \left( \frac{\rho - \rho_0}{n} \right),$$

which implies that, combining with (1.41),

$$\quad (1 - \bar{q}) \gamma f_1 (s_k | \rho) \frac{s_k}{2\rho} + \bar{q} (1 - \gamma) f_0 (s_k | \rho) \frac{1 - s_k}{2\rho} \leq C' \left( \frac{\rho - \rho_0}{n} \right). \quad (1.42)$$

For the homogeneous committee with $q_i = \bar{q}$ for all $i$, let $\rho^s$ be the aggregate precision in equilibrium. Then, we have

$$(1 - \bar{q}) \gamma f_1 (s_k | \rho^s) \frac{s_k}{2\rho^s} + \bar{q} (1 - \gamma) f_0 (s_k | \rho^s) \frac{1 - s_k}{2\rho^s} = C' \left( \frac{\rho^s - \rho_0}{n} \right). \quad (1.43)$$
Given the concavity of \( L_i (\rho_i; \rho_{-i}; s_k; q_k) \) and the convexity of \( C (\rho_i) \) in \( \rho_i \), it is clear that

\[
\rho \geq \rho^*,
\]

that is, the heterogeneous committee acquires more information than the homogeneous one that has the same size and average preference. If the marginal cost function \( C' \) is convex, i.e., \( C'' > 0 \), then the above result will be reversed, that is,

\[
\rho \leq \rho^*.
\]

**Proof of Proposition 1.12**

From \((1.18)\), we obtain

\[
\sum_{i=1}^{n} \frac{1}{C'' (\rho_i)} \left[ \frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i \partial q_k} dq_k + \frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i^2} d\rho \right] = d\rho. (1.46)
\]

Total differentiation of \((1.45)\) w.r.t. \( q_k \) and \( \rho \) gives

\[
\sum_{i=1}^{n} \frac{1}{C'' (\rho_i)} \left[ \frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i \partial q_k} dq_k + \frac{\partial^2 L_i (\rho_i; \rho_{-i}; s_k; q_k)}{\partial \rho_i^2} d\rho \right] = d\rho. (1.46)
\]

We rewrite \((1.46)\) as

\[
\frac{1}{2} \left[ (1 - \bar{q}^e) \gamma f_1 (s_k | \rho) - \bar{q}^e (1 - \gamma) f_0 (s_k | \rho) \right] \left[ \frac{1}{\rho q_k (1 - q_k)} \left( \sum_{i=1}^{n} \frac{1}{C'' (\rho_i)} \right) \right] d\rho.
\]
The definition of $q^e$ indicates that it is a continuous function of $q_k$. We use $q^e(q_k)$ to denote the value of $q^e$ given $q_k$. Since for any $i$, $\rho_i > 0$ and $C''(\rho_i) > 0$, there is

$$q_1 < q^e(q_k) < q_n.$$ 

Thus, $q^e$ is a continuous function mapping from $[q_1, q_n]$ to $[q_1, q_n]$. According to Brouwer’s fixed point theorem, there exists $\hat{q}_k \in (q_1, q_n)$, such that

$$q^e(\hat{q}_k) = \hat{q}_k.$$ 

If the fixed point is unique, then Lemma 1.5, which is proved based on (1.47), implies that the optimal voting rule maximizing information is necessarily a unanimous rule. This is because if $\hat{q}_k$ is the unique fixed point, then $q^e(q_k) > q_k$ for $q_k < \hat{q}_k$ and $q^e(q_k) < q_k$ for $q_k > \hat{q}_k$. Thus, according to Lemma 1.5, voting rule $k = 1$ induces more information acquisition than any other rule $k$ with $q_k < \hat{q}_k$, and voting rule $k = n$ induces more information acquisition than any other rule $k$ with $q_k > \hat{q}_k$. Therefore, the optimal voting rule maximizing information is necessarily a rule with $k = 1$ or $k = n$. However, the fixed point of $q^e(q_k)$ may not be unique. We examine the slope of $q^e(q_k)$ at $q_k = \hat{q}_k$ to see if there is a unique fixed point. If the slope is proved to be smaller than 1, then uniqueness is guaranteed. Otherwise, $q^e(q_k)$ may have multiple fixed points. We take derivative of $q^e(q_k)$ w.r.t. $q_k$ and obtain

$$\frac{dq^e(q_k)}{dq_k} = -\left(\sum_{i=1}^{n} \frac{C''(\rho_i)}{C''(\rho_i) + q^e(q_k)} \frac{\partial q_i}{\partial q_k}\right) \left(\sum_{i=1}^{n} \frac{1}{C''(\rho_i)}\right) + \left(\sum_{i=1}^{n} \frac{q_i i}{C''(\rho_i)} \frac{\partial q_i}{\partial q_k}\right) \left(\sum_{i=1}^{n} \frac{1}{C''(\rho_i)}\right)^2$$

$$= -\left(\sum_{i=1}^{n} \frac{C''(\rho_i)}{C''(\rho_i) + q^e(q_k)} \frac{\partial q_i}{\partial q_k}\right) + q^e \left(\sum_{i=1}^{n} \frac{C''(\rho_i)}{C''(\rho_i) + q^e(q_k)} \frac{\partial q_i}{\partial q_k}\right)$$

$$= \frac{\sum_{i=1}^{n} \frac{1}{C''(\rho_i)} \frac{\partial q_i}{\partial q_k}}{\sum_{i=1}^{n} \frac{1}{C''(\rho_i)}}$$

(1.48)
\[= \sum_{i=1}^{n} \left[ \frac{C''(\rho_i)}{C'(\rho_i)} \frac{\partial \rho_i}{\partial q_k} \lambda_i (\bar{q}_i - q_i) \right].\]

Now we look at the sign of \(\frac{dq_i(q_k)}{dq_k}\) at \(q_k = \hat{q}_k\). From equation (1.44), we have

\[
\frac{1}{C''(\rho_i)} \left\{ \frac{1}{2} \left[ (1 - q_i) \gamma f_1 (\bar{s}_k | \rho) - q_i (1 - \gamma) f_0 (\bar{s}_k | \rho) \right] \frac{[\frac{1}{2} + \delta_k (1 - \bar{s}_k)]}{\rho q_k (1 - q_k)} \right\} = \frac{\partial \rho_i}{\partial q_k}.
\]

Equation (1.47) indicates that at \(q_k = \hat{q}_k\), \(\frac{\partial \rho_i}{\partial q_k} = 0\). Thus, we have \(\frac{\partial \rho_i}{\partial q_k} |_{q_k = \hat{q}_k} > 0\) if \(q_i < \hat{q}_k\), and \(\frac{\partial \rho_i}{\partial q_k} |_{q_k = \hat{q}_k} < 0\) if \(q_i > \hat{q}_k\). Therefore, according to (1.48), when \(C''(\cdot) < 0\), there is \(\frac{dq_i(q_k)}{dq_k} |_{q_k = \hat{q}_k} < 0\), and when \(C''(\cdot) > 0\), \(\frac{dq_i(q_k)}{dq_k} |_{q_k = \hat{q}_k} > 0\). Therefore, for the case in which \(C''(\cdot) < 0\), the fixed point of \(\bar{q}_i^*(q_k)\) is unique, and the optimal voting rule which maximizes the amount of information collected by the committee is a unanimous rule. Otherwise, the optimal voting rule may be a non-unanimous rule.

**Proof of Proposition 1.13**

This proof is similar to the one for Proposition 1.6. Still, let \(V_0(q_0, q_k, q_{-k}) \equiv V_0(\rho(q_k, q_{-k}; \rho_0, c, \gamma), q_k)\), where \(\rho(q_k, q_{-k}; \rho_0, c, \gamma)\) is determined by the equality constraint. Taking derivatives of \(V_0(q_0, q_k, q_{-k})\) with respect to \(q_k\) and \(q_i\) for any \(i \neq k\), we obtain

\[
\frac{\partial V_0(q_0, q_k, q_{-k})}{\partial q_k} = \left[ - (1 - q_0) \gamma f_1 (\bar{s}_k | \rho) + q_0 (1 - \gamma) f_0 (\bar{s}_k | \rho) \right] \frac{\partial s_k}{\partial q_k} |_{q_k} \\
+ \left[ (1 - q_0) \gamma f_1 (\bar{s}_k | \rho) \frac{s_k}{2 \rho} + q_0 (1 - \gamma) f_0 (\bar{s}_k | \rho) \frac{(1 - s_k)}{2 \rho} \right] \frac{\partial \rho}{\partial q_k},
\]

\[
\frac{\partial V_0(q_0, q_k, q_{-k})}{\partial q_i} = \left[ (1 - q_0) \gamma f_1 (\bar{s}_k | \rho) \frac{s_k}{2 \rho} + q_0 (1 - \gamma) f_0 (\bar{s}_k | \rho) \frac{(1 - s_k)}{2 \rho} \right] \frac{\partial \rho}{\partial q_i}, \forall i \neq k.
\]

According to equation (1.40) in the proof of Proposition 1.9, we have

\[
\frac{\partial \rho}{\partial q_i} = \frac{(1 - \gamma) f_0 (s_k | \rho)}{2 \rho (1 - q_k)} \frac{h(q_k, \rho, \gamma)}{C''(\rho_i) \left[ 1 - \sum_{l=1}^{n} \left( \frac{1}{C''(\rho_l)} \frac{\partial^2 L_i (\rho, \rho_{-l}, \delta_k | q_k)}{\partial \rho_l^2} \right) \right]}.
\]
To obtain the expression of \( \frac{\partial p}{\partial q_k} \), we employ the F.O.C’s in (1.18), and obtain

\[
\frac{(1 - \gamma) f_0 (s_k | \rho)}{2 \rho (1 - q_k) C'' (\rho_k)} h (q_k, \rho, \gamma) \\
+ \frac{1}{2} \left[ (1 - \bar{q}^* \gamma f_1 (s_k | \rho) - \bar{q}^* (1 - \gamma) f_0 (s_k | \rho) \right] \left[ \frac{1}{\rho} \frac{\partial L_i (\rho_i, \rho_{-i}, s_k; q_k)}{\partial q_i^2} \right] \sum_{l=1}^{n} \frac{1}{C'' (\rho_l)} \left[ \frac{\partial \rho}{\partial q_k} \right].
\]

(1.49)

Since this proof is very similar to that for Proposition 1.6, we elaborate only the argument for the case \( q_0 < \bar{q}^* \). The proof for the cases with \( q_0 > \bar{q}^* \) and \( q_0 = \bar{q}^* \) can be similarly derived.

First, suppose that the designer chooses \( q_k < \bar{q}^* \). For such a committee, we have \( \partial V_0 (q_0, q_k, q_{-k}) / \partial q_i > 0 \), due to \( \frac{\partial \rho}{\partial q_i} > 0 \). This implies that among all committees with \( q_k < \bar{q}^* \), the optimal one satisfies \( q_i = 1, i \neq k \). However, unlike Proposition 1.6 we cannot determine whether the optimal \( q_k \) is smaller than \( q_0 \) or not. If \( q_k \in (q_0, \bar{q}^*) \), the sign of the second line of \( \partial V_0 (q_0, q_k, q_{-k}) / \partial q_k \) is indeterminate, due to the indeterminacy of \( \frac{\partial \rho}{\partial q_k} \), which is because on the LHS of equation (1.49), the first term is negative, as \( \bar{q}^* > q_k \), and the second is positive, as \( q_k < \bar{q}^* \).

Similar to the argument above, we find that all committees with \( q_k > \bar{q}^* \), the optimal one satisfies \( q_i = 0 \), for \( i \neq k \).

If the designer chooses \( q_k = \bar{q}^* \), it is indifferent for her to choose any non-threshold voters, so the homogeneous committee with \( q_i = \bar{q}^* \) for all \( i \), will give her the highest payoff. However, for this homogeneous committee, we have that \( \frac{\partial \rho}{\partial q_k} = 0 \) and \( \partial V_0 / \partial q_k < 0 \). This means that this committee is dominated by one with \( q_k < \bar{q}^* \) and \( q_i = 1, i \neq k \).

Therefore, we conclude that for a designer with \( q_0 < \bar{q}^* \), there is an optimal committee satisfying either \( q_n \in (\bar{q}^*, 1) \) and \( q_i = 0, i < n \), with \( k = n \), or \( q_1 \in (0, \bar{q}^*) \) and \( q_i = 1, i > 1 \), with \( k = 1 \).

To obtain the proof of Proposition 1.14, we only need to impose the constraint that
\( q_i \leq q_k \) for all \( i \neq k \), in the proof above.

**Proof of Proposition 1.15**

To construct an equilibrium of the game in which every committee member fully reveals his private information, specifying the beliefs of the members in cases where some members do not reveal their signals is crucial. The argument in Milgrom and Roberts (1986) does not apply here. In this proof, I specify the beliefs of the members in these cases and their associated voting strategies as follows:

1. If the threshold voter \( k \) (or a non-threshold voter \( i \) with \( q_i = q_k \)) is the only one that does not reveal his information, then all other members believe that the threshold voter collected no information. In the following voting stage, every member votes based on his belief, and no one adopts a weakly dominated voting strategy.

2. If a non-threshold voter \( i \) with \( q_i \neq q_k \) is the only member that does not reveal his information, then all other members upon observing the aggregate signal \( s_{-i} \) with

\[
s_{-i} = \frac{\sum_{j \neq i} \rho_j s_j}{\rho_{-i}},
\]

where \( \rho_{-i} = \sum_{j \neq i} \rho_j \), hold a common posterior belief that the signal \( s_i \) of voter \( i \) has precision \( \rho_i > 0 \) and satisfies

\[
s_i < \frac{(\rho_{-i} + \rho_i) s_m - \rho_{-i} s_{-i}}{\rho_i}, \text{ if } q_i < q_k, \text{ or } s_i > \frac{(\rho_{-i} + \rho_i) s_M - \rho_{-i} s_{-i}}{\rho_i}, \text{ if } q_i > q_k,
\]

where

\[
\begin{align*}
s_m &= \frac{1}{2} + \frac{1}{\rho_{-i} + \rho_i} \ln \frac{q_m}{(1 - q_m)} \gamma, \text{ with } q_m = \min_{1 \leq j \leq n} \{q_j : q_j > 0\}, \text{ and } \\
{s}_M &= \frac{1}{2} + \frac{1}{\rho_{-i} + \rho_i} \ln \frac{q_M}{(1 - q_M)} \gamma, \text{ with } q_M = \max_{1 \leq j \leq n} \{q_j : q_j < 1\}.
\end{align*}
\]

(Since \( q_k \in (0,1) \), \( q_m \) and \( q_M \) always exist in a committee.) Thus, if \( q_i < q_k \), then all...
non-extreme voters believe that the status quo should be chosen; if \( q_i > q_k \), then all non-extreme voters believe that the status quo should be overturned. In the following voting stage, every member votes based on his belief, and no one adopts a weakly dominated voting strategy.

(3) If there are more than one voter concealing their information, then each member, no matter he reveals his information or not, believes that other members not revealing their information collected no information. In the following voting, every member votes based on his belief as in the two cases above.

Now I show that given the beliefs of the members above in the cases where not everyone reveals his information, it is incentive compatible for every member to reveal his information in stage 3. We first examine the incentive of the threshold voter to conceal his information. It is obvious that given that other members always reveal their information, if the threshold voter also reveal his information, the collective decision is always his optimal decision based on the fully aggregated information; there is no room to strictly improve his payoff by concealing his information. Thus, it is incentive compatible for the threshold voter to reveal his signal.

Suppose a non-threshold voter \( i \) with \( q_i \neq q_k \) observes signal \( s_{i0} \) with

\[
s_{i0} = \frac{\rho_i s_i + \rho_0 s_0}{\rho_{i0}},
\]

where \( \rho_{i0} = \rho_i + \rho_0 \) and \( \rho_i \geq 0 \). If he conceals his information, then his expected payoff from the collective decision is

\[
\left\{ \begin{array}{ll}
\frac{-q_i}{\gamma f_1(s_{i0}|\rho_{i0})} & \text{if } q_i < q_k; \\
\frac{-q_i(1-\gamma)}{\gamma f_1(s_{i0}|\rho_{i0})+(1-\gamma)f_0(s_{i0}|\rho_{i0})} & \text{if } q_i > q_k.
\end{array} \right.
\]

(1.50)

If he reveals his signal, then his expected payoff is
\[
\frac{-(1 - q_i) \gamma f_1 (s_{i0} | \rho_{i0})}{\gamma f_1 (s_{i0} | \rho_{i0}) + (1 - \gamma) f_0 (s_{i0} | \rho_{i0})} F_1 \left( \frac{\rho_{sk} - \rho_{io}s_{i0}}{\rho - \rho_{io}} | \rho - \rho_{i0} \right) + \frac{-q_i (1 - \gamma) f_0 (s_{i0} | \rho_{i0})}{\gamma f_1 (s_{i0} | \rho_{i0}) + (1 - \gamma) f_0 (s_{i0} | \rho_{i0})} \left[ 1 - F_0 \left( \frac{\rho_{sk} - \rho_{io}s_{i0}}{\rho - \rho_{i}} | \rho - \rho_{i0} \right) \right],
\]

(1.51)

where \( \rho - \rho_{i0} \) is the aggregate precision of the signals acquired by all other members that \( i \) expects. Suppose \( q_i < q_k \). If \( \rho - \rho_{i0} = 0 \), then the expected payoff is reduced to

\[
\left\{ \begin{array}{ll}
\frac{-(1 - q_i) \gamma f_1 (s_{i0} | \rho_{i0})}{\gamma f_1 (s_{i0} | \rho_{i0}) + (1 - \gamma) f_0 (s_{i0} | \rho_{i0})}, & \text{if } s_{i0} < s_k; \\
\frac{-q_i (1 - \gamma) f_0 (s_{i0} | \rho_{i0})}{\gamma f_1 (s_{i0} | \rho_{i0}) + (1 - \gamma) f_0 (s_{i0} | \rho_{i0})}, & \text{if } s_{i0} \geq s_k.
\end{array} \right.
\]

(1.52)

Since \( q_i < q_k \), we have \( s_i < s_k \). By comparing (1.50) and (1.52), we can see that if \( s_{i0} < s_k \), member \( i \) is indifferent between concealing and revealing his information, while if \( s_{i0} \geq s_k \), member \( i \) gets strictly better off from revealing his information, as in this case \( s_{i0} > s_i \), which implies

\[
\frac{-(1 - q_i) \gamma f_1 (s_{i0} | \rho_{i0})}{\gamma f_1 (s_{i0} | \rho_{i0}) + (1 - \gamma) f_0 (s_{i0} | \rho_{i0})} < \frac{-q_i (1 - \gamma) f_0 (s_{i0} | \rho_{i0})}{\gamma f_1 (s_{i0} | \rho_{i0}) + (1 - \gamma) f_0 (s_{i0} | \rho_{i0})}.
\]

If \( \rho - \rho_{i0} > 0 \), then it is easy to verify that the expected payoff (1.51) reaches its maximum at \( s_k = s_i \) and is decreasing in \( s_k \) for \( s_k > s_i \). The expected payoff of member \( i \) from concealing his information is equal to that from revealing his information with \( s_k = \infty \). Thus, for every finite \( s_k \), which is assumed in this paper, revealing information is optimal for \( i \). The case where \( q_i > q_k \) can be similarly proved. Therefore, it is incentive compatible for every non-threshold voter \( i \) with \( q_i \neq q_k \) to reveal his information.

Since it is incentive compatible for every member to reveal his acquired information in stage 3, the problem facing the members in stage 1 is identical to the problem facing them in stage 1 of the public information case. Thus, every member collects the same amount of information as he does in public information case. This completes the proof.

Committee Design: Voting without Communication
In this supplementary material, I study the problem that a committee designer composes a two-member committee to make a collective decision. Same as in the main model of this paper, the designer chooses the composition and voting rule of the committee. The departure of this problem from the main model is that the information collected by the members, including the precisions and realizations of the signals, is private, and there is no free public signal. The members aggregate their information through voting. Instead of conducting a comprehensive analysis of this problem, I show only that it is typically not optimal for the designer to compose a homogeneous committee whose members are perfectly aligned with her. I consider only the case where the cost of information is convex.

I first analyze arbitrarily composed committees. Let $q_1$ and $q_2$ denote the preferences of the two committee members, 1 and 2, respectively. The voting rule is $k = 2$, i.e., unanimity is required to overturn the status quo. Suppose that the precisions of their private signals are respectively $\rho_1 > 0$ and $\rho_2 > 0$, and each member forms a correct belief about the precision of the other member’s signal. Then, an equilibrium of the voting game can be characterized by a pair of cut-off values $(\bar{s}_1, \bar{s}_2)$, which satisfy

$$
\frac{q_i (1 - \gamma)}{(1 - q_i) \gamma} = \frac{f_1 (\bar{s}_i | \rho_i)}{f_0 (\bar{s}_i | \rho_i)} \left[ 1 - F_1 (\bar{s}_j | \rho_j) \right] \text{, for } i, j = 1, 2, \ i \neq j,
$$

such that member $i$ votes for overturning the status quo if and only if $s_i \geq \bar{s}_i$, $i = 1, 2$.

Because $f_1 (s_i | \rho_i) / f_0 (s_i | \rho_i)$ satisfies the monotone likelihood ratio property (MLRP) and $f_1 (s_i | \rho_i) / [1 - F_1 (s_i | \rho_i)]$ is increasing in $s_i$, there exists a unique pair $(\bar{s}_1, \bar{s}_2)$ satisfying (1.53). (See Li et al. (2001) for detailed discussion.) The equilibrium values of $\bar{s}_1$ and $\bar{s}_2$ depend on $\rho_1$, $\rho_2$, $q_1$, $q_2$, and $\gamma$.

In this model, $\rho_1$ and $\rho_2$ are endogenous. I impose the following assumption to ensure the existence of an equilibrium of this committee game with information acquisition.

In this assumption, $\bar{\rho}_j$ is the maximum precision that member $j$ is willing to choose;
choosing a precision higher than \( \tilde{p}_j \) is so costly that \( j \)'s payoff is lower than that he can obtain without any information. Since \( [1 - F_0(\bar{s}_j|\rho)] / [1 - F_1(\bar{s}_j|\rho)] \) is decreasing in \( \bar{s}_j \) and has maximum equal to 1, the assumption is not empty.

**Assumption 1.5.** Each committee member has a free private signal \( s_0 \) with precision \( \rho_0 > 0 \). The preferences of the members are non-extreme, i.e., \( q_1, q_2 \in (0, 1) \), and satisfy

\[
\rho_0 > 2 \left| \ln \frac{q_i (1 - \gamma) [1 - F_0(\bar{s}_j|\rho)]}{(1 - q_i) \gamma [1 - F_1(\bar{s}_j|\rho)]} \right|
\]

for all \( \rho \in [\rho_0, \tilde{\rho}_j] \), where \( \tilde{\rho}_j = \rho_0 + C^{-1}(2 \min \{q_j (1 - \gamma), (1 - q_j) \gamma\}) \), and all \( \bar{s}_j \in (-\infty, \frac{1}{2} + \frac{1}{\rho} \ln q_j^{(1 - \gamma)/(1 - q_j \gamma)}], i, j = 1, 2 \) and \( i \neq j \).

In the information acquisition stage of the game, if member \( i \) believes that member \( j \neq i \) chooses precision \( \rho'_j \) and cut-off \( \bar{s}'_j \), and he chooses precision \( \rho_i \) and consequently \( \bar{s}_i \) satisfying

\[
\frac{q_i (1 - \gamma)}{(1 - q_i) \gamma} = \frac{f_1(\bar{s}_i|\rho_i) [1 - F_1(\bar{s}'_j|\rho'_j)]}{f_0(\bar{s}_i|\rho_i) [1 - F_0(\bar{s}'_j|\rho'_j)]},
\]

according to (1.53), then the expected payoff of \( i \) is

\[
- (1 - q_i) \gamma \left\{ 1 - [1 - F_1(\bar{s}_i|\rho_i)][1 - F_1(\bar{s}'_j|\rho'_j)] \right\}
\]

\[
- q_i (1 - \gamma) \left[ 1 - F_0(\bar{s}_i|\rho_i) \right] [1 - F_0(\bar{s}'_j|\rho'_j)] - C (\rho_i - \rho_0).
\]

The first order condition of \( i \)'s payoff maximization problem is

\[
- (1 - q_i) \gamma [1 - F_1(\bar{s}'_j|\rho'_j)] f_1(\bar{s}_i|\rho_i) \left( \frac{\bar{s}_i - 1}{2 \rho_i} + \frac{\partial \bar{s}_i}{\partial \rho_i} \right)
\]

\[
+ q_i (1 - \gamma) [1 - F_0(\bar{s}'_j|\rho'_j)] f_0(\bar{s}_i|\rho_i) \left( \frac{\bar{s}_i}{2 \rho_i} + \frac{\partial \bar{s}_i}{\partial \rho_i} \right) - C' (\rho_i - \rho_0) = 0.
\]

According to (1.54), the first order condition can be reduced to

\[
\frac{(1 - q_i) \gamma [1 - F_1(\bar{s}'_j|\rho'_j)] f_1(\bar{s}_i|\rho_i)}{2 \rho_i} - C' (\rho_i - \rho_0) = 0.
\]

\[(1.56)\]
Whether the second order condition of the payoff maximization problem is satisfied depends on the values of $s_j^*$ and $\rho_j^*$. If member $i$ believes that member $j$ is rational, then $\rho_j^* \in [\rho_0, \bar{\rho}_j]$ and $s_j^* \in (-\infty, \frac{1}{2} + \frac{1}{\rho_j^*} \ln \frac{q_j(1-\gamma)}{(1-q_j)\gamma}]$, because member $j$ would believe that $\bar{s}_i \in \mathbb{R}$ and choose $s_j^*$ following his belief on $\bar{s}_i$ according to (1.53). Given these ranges of $s_j^*$ and $\rho_j^*$, the expected payoff (1.55) is concave in $\rho_i$, thus the second order condition is satisfied.

The concavity of the expected payoff also ensures the existence of an equilibrium. Let $(\rho_1^*, \rho_2^*)$ be the equilibrium precision profile and $(\bar{s}_1^*, \bar{s}_2^*)$ the equilibrium cut-offs associated with the precisions, then according to (1.53) and (1.56), they satisfy

\[
\frac{(1-q_i) \gamma [1 - F_i(s_j^*|\rho_j^*)]f_1(s_i^*|\rho_i^*)}{2\rho_i^*} - C'(\rho_i^* - \rho_0) = 0, \quad \text{and} \quad (1.57)
\]

\[
\frac{q_i(1-\gamma)}{(1-q_i) \gamma} = \frac{f_i(s_i^*|\rho_i^*)[1 - F_i(s_j^*|\rho_j^*)]}{f_0(s_i^*|\rho_i^*)[1 - F_0(s_j^*|\rho_j^*)]}, \quad \text{for } i, j = 1, 2, i \neq j. \quad (1.58)
\]

Now I examine whether a designer with preference $q_0$ would like to compose a committee in which $q_1 = q_2 = q_0$. In this analysis, I assume that the value of $q_0$ satisfies Assumption (1.5), so the game played by the homogeneous committee has an equilibrium. Moreover, I focus on the symmetric equilibrium of the game. It is easy to verify the existence of a symmetric equilibrium, using (1.57) by restricting $s_j^* = s_i^*$ and $\rho_j^* = \rho_i^*$. The expected payoff of the designer from composing a committee with preference profile $(q_1, q_2)$ is

\[-(1-q_0) \gamma \{1 - [1 - F_i(\bar{s}_i^*|\rho_i^*)][1 - F_i(s_j^*|\rho_j^*)]\} - q_0(1-\gamma)[1 - F_0(\bar{s}_i^*|\rho_i^*)][1 - F_0(s_j^*|\rho_j^*)],\]

where $(\rho_1^*, \rho_2^*)$ and $(\bar{s}_1^*, \bar{s}_2^*)$ satisfy (1.57) and (1.58). Starting from a committee with $q_1 = q_2 = q_0$, if the designer marginally moves $q_1$ away from $q_0$, the marginal change of her payoff is

\[
\frac{(1-q_0) \gamma f_1(\bar{s}_j^*|\rho_j^*)[1 - F_i(\bar{s}_i^*|\rho_i^*)]}{2\rho_2} \left( \frac{\partial \rho_i^*}{\partial q_1} + \frac{\partial \rho_j^*}{\partial q_1} \right),
\]

which is derived using the symmetry of the equilibrium and (1.58). It is clear that the
designer has incentive to choose \( q_1 \neq q_0 \) if \( \frac{\partial q_1}{\partial q_1} + \frac{\partial q_2}{\partial q_1} \neq 0 \). I prove that this is typically true.

To proceed, we look at how \( \rho_1 \) and \( \rho_2 \) change with \( q_1 \). From the F.O.C. of member 1 in (1.57), we have

\[
2 [C''(\rho_1^* - \rho_0) \rho_1^* + C''(\rho_2^* - \rho_0)] \frac{\partial \rho_1^*}{\partial q_1} = -\gamma [1 - F_1(\bar{s}_2^*|\rho_2^*)] f_1(\bar{s}_1^*|\rho_1^*)
\]

\[
+ (1 - q_1) \gamma \left\{ -f_1(\bar{s}_2^*|\rho_2^*) f_1(\bar{s}_1^*|\rho_1^*) \left[ \frac{(s_2^*-1)}{2\rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} \right] \right\}
\]

Similarly, from the F.O.C. of member 2 in (1.57), we have

\[
2 [C''(\rho_2^* - \rho_0) \rho_2^* + C''(\rho_1^* - \rho_0)] \frac{\partial \rho_2^*}{\partial q_1} =
\]

\[
- f_1(\bar{s}_1^*|\rho_1^*) f_1(\bar{s}_2^*|\rho_2^*) \left[ \frac{(s_1^*-1)}{2\rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial s_1^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial s_1^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial s_1^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} \right]
\]

\[
(1 - q_2) \gamma \left\{ + [1 - F_1(\bar{s}_1^*|\rho_1^*)] f_1(\bar{s}_2^*|\rho_2^*) \left[ \frac{1}{2\rho_2^*} \frac{\partial \rho_2^*}{\partial q_1} - \frac{(s_2^*-1)}{2} \frac{\partial \rho_2^*}{\partial q_1} \right.ight.
\]

\[
- (s_2^*-1) \rho_2^* \left( \frac{\partial s_2^*}{\partial \rho_2^*} \frac{\partial \rho_2^*}{\partial q_1} + \frac{\partial s_2^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial \rho_1^*} \frac{\partial \rho_1^*}{\partial q_1} \right) \right\}
\]

Suppose that \( \frac{\partial s_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial q_1} = 0 \). Using the fact that \( \frac{\partial s_i^*}{\partial \rho_j^*} = \frac{\partial s_j^*}{\partial \rho_i^*} \), \( i, j = 1, 2 \), in the symmetric equilibrium, we add up the two equations above and obtain,

\[
0 = -\gamma [1 - F_1(\bar{s}_2^*|\rho_2^*)] f_1(\bar{s}_1^*|\rho_1^*)
\]

\[
- (1 - q_1) \gamma \left( \frac{\partial s_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial q_1} \right) f_1(\bar{s}_1^*|\rho_1^*) \left\{ f_1(\bar{s}_2^*|\rho_2^*) + [1 - F_1(\bar{s}_2^*|\rho_2^*)] (s_1^*-1) \rho_1^* \right\}. \tag{1.59}
\]

To learn about \( \frac{\partial s_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial q_1} \), we use (1.58) and obtain

\[
(1 - \gamma) [1 - F_0(\bar{s}_2^*|\rho_2^*)] f_0(\bar{s}_1^*|\rho_1^*)
\]

\[
- q_1 (1 - \gamma) \left( \frac{\partial s_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial q_1} \right) f_0(\bar{s}_1^*|\rho_1^*) \left\{ f_0(\bar{s}_2^*|\rho_2^*) + [1 - F_0(\bar{s}_2^*|\rho_2^*)] s_1^* \rho_1^* \right\}
\]
\[
= -\gamma [1 - F_1(s_1^* | \rho_1^*)] f_1 (s_1^* | \rho_1^*) \\
- (1 - q_1) \gamma \left( \frac{\partial s_1^*}{\partial q_1} + \frac{\partial s_2^*}{\partial q_1} \right) f_1 (s_1^* | \rho_1) \{ f_1(s_2^* | \rho_2^*) + [1 - F_1(s_2^* | \rho_2^*)] (s_1^* - 1) \rho_1^* \}.
\]

Equations (1.59) and (1.60) jointly imply that

\[
\frac{(1 - q_1) [1 - F_0(s_2^* | \rho_2^*)]}{q_1 [1 - F_1(s_1^* | \rho_1^*)]} = - \frac{f_0(s_2^* | \rho_2^*) + [1 - F_0(s_2^* | \rho_2^*)] s_1^* \rho_1^*}{f_1(s_2^* | \rho_2^*) + [1 - F_1(s_2^* | \rho_2^*)] (s_1^* - 1) \rho_1^*}.
\] (1.61)

One should note that (1.61) is independent of \(\gamma\). Thus, the relationship between \(\rho_1^* = \rho_2^*\) and \(s_1^* = s_2^*\) in (1.61) does not depend on \(\gamma\). However, the relationship between \(\rho_1^* = \rho_2^*\) and \(s_1^* = s_2^*\) in (1.58) and (1.57) is governed by \(\gamma\). Therefore, (1.61) typically does not hold, which implies that \(\frac{\partial \rho_1^*}{\partial q_1} + \frac{\partial \rho_2^*}{\partial q_1} = 0\) does not hold typically, and the designer has incentive to change \(q_1\).

The proof for the case of \(k = 1\) can be similarly proved. Thus, I omit it in this supplementary material.
Chapter 2

Auction Design by an Informed Seller: The Optimality of Reserve Price Signaling
2.1 Introduction

Most of the literature assumes that a mechanism designer has no relevant private information. This assumption does not fit real-world situations well. For example, a house owner may have private information valuable to potential buyers in evaluating the house; in keyword auctions, auctioneers are usually better informed than buyers about the frequencies that the keywords are searched.

I study a problem of mechanism design in which the mechanism designer is privately informed. A seller chooses a mechanism to sell a single indivisible object to one of multiple (potentially asymmetric) buyers. The players in the mechanism are the buyers. Before choosing the mechanism, the seller receives a private signal regarding the quality of the object. This signal directly affects the seller’s and buyers’ valuations of the object. This mechanism-selection game involves signaling in that the seller’s choice of the mechanism may (at least partially) reveal her private information to the buyers. However, unlike a typical signaling game, in which the action space of a sender is a finite-dimensional set of actions, the action space of the seller in this game is an infinite-dimensional set of mechanisms.

This game captures environments without an independent mediator to run the mechanism. Unmediated trade is common in practice, especially when the gains from trade are modest (as in, for example, the transactions on customer-to-customer online shopping platforms). As Farrell (1983) argues, finding a neutral mediator can be very costly or impossible in many cases.

I show that in general the mechanism-selection game has no equilibrium in which every type of the seller chooses a mechanism that maximizes her profit in the case that her information is public (henceforth, public-information optimal mechanism). The reason is simple: a lower-quality seller wants to pretend to be a higher-quality seller, to extract

\footnote{In the presence of such a mediator, the seller could also be a participant in the mechanism; see the discussion at the end of this Introduction.}
more profit from the buyers. This implies that to separate lower-quality sellers from higher-quality sellers, the mechanisms adopted by higher-quality sellers should be less profitable than those adopted in the public information case.

In the model, there are (at least) three ways to disincentivize lower-quality sellers from mimicking higher-quality ones: (1) increase the reserve prices of the mechanisms adopted by the higher-quality sellers, to decrease the probabilities of selling the object; (2) decrease the expected payments of the buyers to the higher-quality sellers, or have the higher-quality sellers burn money without changing the rule for allocating the object; (3) change the allocation rules of the higher-quality sellers by, for example, giving some buyers unfair advantages when bidding for the object, to reduce the competitiveness of the bidding games and induce lower bids from the buyers.

I characterize the seller-optimal separating equilibria, which maximize the (interim) expected profit of every type of the seller among all possible separating equilibria. This seemingly complex problem has a simple solution: the equilibrium strategies of the seller differ from the public-information optimal mechanisms only in reserve prices. Specifically, the lowest-quality seller uses the same mechanism that she adopts under public information, while all other types of the seller adopt mechanisms having higher reserve prices than their public-information optimal mechanisms. This result, which is the main result in this paper, shows that the first approach mentioned above is the optimal way of separating the seller types.

To understand why the result holds, let us compare the first two approaches. To eliminate the incentive for a lower-quality seller to mimic a higher type, we need to reduce her revenue gain from doing so. Under the second approach, to reduce the revenue gain by a certain amount we need to force a higher-quality seller to give up that amount of revenue. Under the first approach, which decreases the probability of trade, because a higher-quality seller values the object more, her loss from the decreased trading proba-

\[I \text{ discuss pooling equilibria near the end of Section 2.3 after characterizing the seller-optimal separating equilibria.}\]
bility is less than that of a lower-quality seller; therefore, it is less than her loss using the second approach. For any other approach, including the third one, can be proved to be no better than a combination of the first and second approaches, so is less desirable than the first one.

The setup of my model is similar to those of Jullien and Mariotti (2006) and Cai et al. (2007). These two papers study reserve-price-signaling games in which the buyers are symmetric, the auction is a second-price auction, and the privately informed seller has the freedom to set only the reserve price. Both papers characterize the unique separating equilibrium of their games, and find that the lowest-quality seller sets the same reserve price as in the public information case, whereas other types set higher reserve prices compared with the public information case. In my model, I allow the buyers to be asymmetric and endow the seller with the freedom to design every element of the mechanism, rather than the freedom to vary only the reserve prices. My main result shows that even if the seller has the freedom to choose any mechanism, she ends up adopting a mechanism that differs from her public-information optimal mechanism only in the reserve prices. This result justifies the assumption in reserve-price-signaling games that the informed seller is restricted to choose only the reserve prices.

This paper is closely related to the literature on mechanism design by an informed principal. The game of mechanism selection studied in this literature differs from the main game I study in that the principal is an active participant in the mechanism, so an independent mediator is required to run the mechanism. The seminal work of Myerson (1983) lays the foundation for analyzing the informed-principal problem by developing several solution concepts with different strengths. He also introduces safe mechanisms, which are mechanisms that are incentive compatible and individually rational for the principal and the agents regardless of the agents’ beliefs about the principal’s type. This

\[3\] Cai et al. (2007) consider a more general information structure than I do: they allow for the buyers’ signals to be affiliated. Also see Lamy (2010) for a corrigendum to Cai et al. (2007).

\[4\] Myerson (1985) applies the theories in studying bilateral trading problems.
class of mechanisms is used in the proof of the main result of the current paper. The analyses of Maskin and Tirole (1990, 1992) focus on the one-principal/one-agent case. Maskin and Tirole (1992) consider the environment where the principal’s private information directly affects the agent’s utility, which is similar to the environment considered in the current paper. They find that, under certain conditions, the set of Rothschild-Stiglitz-Wilson (RSW) mechanisms, which are safe mechanisms that maximize the interim expected payoff of the principal among all safe mechanisms, coincides with the set of equilibrium strategies of the principal.

Maskin and Tirole (1990) and Mylovanov and Tröger (2012a, 2012b, 2015) focus on the case in which the principal’s information does not directly enter the agents’ utility functions; it affects the payoffs of the agents only through its effects on the principal’s equilibrium behavior. Mylovanov and Tröger (2012a) provide a solution concept for the informed-principal problem for general private value environments, strongly neologism-proof allocation. Maskin and Tirole (1990) show that generically the principal is better off concealing her information from the buyers. Mylovanov and Tröger (2012b, 2015) study the conditions under which the privacy of the principal’s information does not distort the selection of mechanisms.

I prove my main result using the game of mechanism selection studied in the literature on the informed-principal problem. In this auxiliary game, I characterize the set of RSW mechanisms and show that any RSW mechanism corresponds to the strategy of the seller in some seller-optimal separating equilibrium of the main game I study. Fully characterizing the set of RSW mechanisms is independently interesting, given the aforementioned finding of Maskin and Tirole (1992) regarding the relationship between the RSW mechanisms and the equilibrium strategies of the principal in the informed-principal problem.
2.2 Setup

An indivisible object is for sale. The owner of the object designs a selling mechanism through which she allocates the object to one of $n$ potential buyers.

The seller privately observes a signal $s$, which determines her valuation $v_0(s)$ of the object. I assume that $v_0(s)$ is increasing in $s$ and twice continuously differentiable. The valuation $v_i(s, t_i)$ by buyer $i = 1, 2, \ldots, n$ of the object depends on the seller’s signal $s$ and a private signal $t_i$. I assume that $v_i(s, t_i)$ is twice continuously differentiable in both signals and is increasing in $t_i$. It is common knowledge that $s$ and $t_i$, $i = 1, 2, \ldots, n$, are independently drawn from continuous distributions $F_0: [s, \bar{s}] \rightarrow [0, 1]$ with density $f_0: [s, \bar{s}] \rightarrow \mathbb{R}^+$ and $F_i: [t_i, \bar{t}_i] \rightarrow [0, 1]$ with density $f_i: [t_i, \bar{t}_i] \rightarrow \mathbb{R}^+$, respectively. To simplify the notation, I define $t = (t_1, t_2, \ldots, t_n)$ and use $S, T_i, \text{and } T$ to denote $[s, \bar{s}], [t_i, \bar{t}_i], \text{and } \prod_{i=1}^n [t_i, \bar{t}_i]$, respectively.

The seller chooses a selling mechanism after learning her private signal. A mechanism consists of a message space $\Lambda = \prod_{i=1}^n \Lambda_i$, where $\Lambda_i \subset \mathbb{R}$ is the set of possible messages for buyer $i$, an allocation rule, a payment rule, and a money-burning rule. After the buyers observe the mechanism, they decide whether to participate in the mechanism and, if they choose to participate, report a message. If every buyer participates, the mechanism is implemented; otherwise, the seller keeps the object and each buyer gets the payoff 0. In the rest of the analysis, I call this game the mechanism-selection game. I use perfect Bayesian equilibrium as the equilibrium concept.

In this game, the revelation principle allows us to restrict to direct incentive compatible and individually rational mechanisms on the equilibrium path, although not off the equilibrium path. The message space $\Lambda$ of a direct mechanism is just the signal space $T$, with $T_i$ being the set of possible messages for buyer $i$. I use $x : T \rightarrow [0, 1]^n$.

---

5In my mechanism-selection game, a mechanism is selected after the seller has learned her type, so the choice of mechanism may partially or completely reveal her information. By imbedding money-burning rules into the selling mechanisms, I allow the seller to signal by burning money. I could also allow the seller to engage in “good burning”, but as one will see from the following analysis, it is clearly suboptimal. Thus, I omit it in the model.
$p : T \rightarrow \mathbb{R}^n$, and $b : T \rightarrow \mathbb{R}_+$ to denote the allocation rule, payment rule, and money-burning rule of a direct mechanism. For $x$ and $p$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $p(t) = (p_1(t), p_2(t), \ldots, p_n(t))$, where $x_i(t)$ and $p_i(t)$ are respectively buyer $i$’s probability of getting the object and expected payment to the seller under $t$. The value $b(t)$ is the amount of money burned by the seller under $t$. The allocation rule $x$ satisfies the feasibility constraints

$$0 \leq x_i(t), \sum_{i=1}^n x_i(t) \leq 1 \text{ for all } i \text{ and } t. \tag{2.1}$$

I use $x_0(t)$ to denote the probability that the seller keeps the object when the message profile is $t$, i.e., $x_0(t) = 1 - \sum_{i=1}^n x_i(t)$. By abusing the notation a little, I define

$$x_i(t_i) = \int_{T_{-i}} x_i(t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i} \text{ and } p_i(t_i) = \int_{T_{-i}} p_i(t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i},$$

where $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$, $T_{-i} = \prod_{j \neq i}^n T_j$, and $f_{-i}(t_{-i}) = \prod_{j \neq i}^n f_j(t_j)$. I also define $f(t) = \prod_{i=1}^n f_i(t_i)$.

A direct mechanism is incentive feasible if and only if the feasibility condition (2.1) and the incentive compatibility and individual rationality constraints of the buyers are all satisfied. In general, the incentive feasibility of a mechanism depends on the belief of the buyers about the seller’s type. Suppose that a direct mechanism $M$ with allocation rule $x$ and payment rule $p$ is selected. Let $u_i(M, s, t_i^t | t_i)$ denote buyer $i$’s expected payoff under mechanism $M$ from reporting the type $t_i^t$ when his type is $t_i$ and the seller’s type is $s$, given that all other buyers report their types truthfully. Thus,

$$u_i(M, s, t_i^t | t_i) = v_i(s, t_i) x_i(t_i^t) - p_i(t_i^t).$$

To simplify the notation, I replace $u_i(M, s, t_i^t | t_i)$ by $u_i(M, s | t_i)$ in the rest of the analysis. Suppose that the posterior belief of the buyers about the seller’s type upon observing $M$
is \( f_0 (\cdot | M) \). I define

\[
U_i (M, t'_i | t_i) = \int_S u_i (M, s, t'_i | t_i) f_0 (s | M) \, ds,
\]

which is buyer \( i \)'s expected payoff under mechanism \( M \) from reporting the type \( t'_i \) when his signal is \( t_i \). I replace \( U_i (M, t_i | t_i) \) by \( U_i (M | t_i) \) to simplify the notation. The mechanism \( M \) is incentive feasible under \( f_0 (\cdot | M) \) if and only if it satisfies condition (2.1) and every buyer would like to participate in \( M \) and report his type truthfully, given that all other buyers participate and report their types truthfully, i.e., for any \( i \) and any \( t'_i, t_i \in T_i \) and \( t'_i \neq t_i \),

\[
U_i (M | t_i) \geq U_i (M, t'_i | t_i) \quad \text{and} \quad U_i (M | t_i) \geq 0.
\]

### 2.3 Optimal Selling Mechanisms

The sole departure of the model studied in this paper from the one studied in the standard mechanism design literature is that the seller is privately informed about her signal. Does the privacy of the seller’s signal affect the design of the mechanism? I answer this question in this section and discuss why, in general, selecting a public-information optimal mechanism fails to be an equilibrium strategy of the seller in the mechanism-selection game. Then, I present the main result of this paper, which characterizes the seller-optimal separating equilibria.

#### 2.3.1 Public Information Benchmark

When signal \( s \) is public, the problem of the seller is a standard mechanism design problem. The seller chooses a mechanism \( M \) with an allocation rule \( x \) and payment rule \( p \) to maximize her expected payoff. (In this subsection, I drop the money-burning rule from the analysis, as the seller never burns money in an optimal mechanism.) The problem of
the seller with signal $s$ is

$$\max_{x, p} \int_T \left\{ v_0(s) x_0(t) + \sum_{i=1}^n p_i(t) \right\} f(t) \, dt$$

s.t. $u_i(M, s|t_i) \geq u_i(M, s, t_i'|t_i), \forall i, t_i, t_i' \in T_i,$  

$$u_i(M, s|t_i) \geq 0, \forall i, t_i \in T_i,$$  

$$\sum_{i=1}^n x_i(t) \leq 1, x_i(t) \geq 0, \forall i, \forall t \in T.$$  

(2.2)

According to Milgrom and Segal (2002), constraints (2.2) and (2.3) can be replaced by

$$x_i(t_i) \geq x_i(t_i'), \text{ for each } i, \text{ whenever } t_i > t_i',$$  

(2.4)

$$u_i(M, s|t_i) = \int_{t_i}^{t_i'} \frac{\partial v_i(s, \tilde{t}_i)}{\partial t_i} x_i(\tilde{t}_i) \, d\tilde{t}_i + u_i(M, s|\tilde{t}_i), \text{ for all } i \text{ and } t_i$$  

(2.5)

$$u_i(M, s|\tilde{t}_i) \geq 0, \text{ for all } i.$$  

(2.6)

From (2.5), according to the definition of $u_i(M, s|t_i)$, for all $i$, all $s$, and all $t_i$, we have

$$p_i(t_i) = v_i(s, t_i) x_i(t_i) - \int_{\tilde{t}_i}^{t_i} \frac{\partial v_i(s, \tilde{t}_i)}{\partial t_i} x_i(\tilde{t}_i) \, d\tilde{t}_i - u_i(M, s|\tilde{t}_i).$$

Substituting $p_i(t_i)$ into the objective function, and then using integration by parts, we obtain

$$v_0(s) + \int_T \sum_{i=1}^n [J_i(s, t_i) - v_0(s)] x_i(t) f(t) \, dt - \sum_{i=1}^n u_i(M, s|\tilde{t}_i),$$  

(2.7)

where $J_i(s, t_i)$ is the virtual valuation of buyer $i$ under $(s, t_i)$:

$$J_i(s, t_i) = v_i(s, t_i) - \frac{1 - F_i(t_i)}{f_i(t_i)} \frac{\partial v_i(s, t_i)}{\partial t_i}.$$
Thus, an optimal mechanism maximizes (2.7) subject to the feasibility constraint (2.1) and constraints (2.4), (2.5), and (2.6). Throughout this paper, I impose the following regularity assumption, which is satisfied for commonly used functional forms of $v_i$ when the hazard rate $f_i(t_i)/[1 - F_i(t_i)]$ is non-decreasing in $t_i$.

**Assumption 2.1.** $J_i(s,t_i)$ is increasing in $t_i$ and $J_i(s,t_i) > v_0(s)$ for all $i$ and $s \in S$.

The following proposition characterizes the profit-maximizing mechanisms under this assumption. This result is standard in the literature (see Myerson, 1981); therefore, I omit its proof.

**Proposition 2.1.** Under Assumption 2.1 a mechanism $(x,p)$ is optimal for the seller of type $s \in S$ if and only if the allocation rule $x$ satisfies

$$x \in \arg \max_{\hat{x}} \left\{ \int_T \sum_{i=1}^n [J_i(s,t_i) - v_0(s)] \hat{x}_i(t) f(t) dt : \hat{x} \text{ satisfies } (2.1) \right\},$$

and the payment rule $p$ satisfies

$$p_i(t_i) = v_i(s,t_i) x_i(t_i) - \int_{\tilde{t}_i}^{t_i} \frac{\partial v_i(s,\tilde{t}_i)}{\partial \tilde{t}_i} x_i(\tilde{t}_i) d\tilde{t}_i, \text{ for all } i,t_i.$$ 

In any optimal mechanism $M$ for the type-$s$ seller, the expected payoff of the lowest-type buyer $i$ is equal to 0, i.e., $u_i(M,s|\tilde{t}_i) = 0$ for all $i$.

This proposition indicates that in an optimal mechanism for the seller of type $s$, we have $\sum_{i=1}^n x_i(t) = 1$ if $\max_k \{J_k(s,t_k)\}$ is strictly larger than $v_0(s)$, and $x_i(t) > 0$ only if $J_i(s,t_i)$ is larger than $\max_{k \neq i} \{J_k(s,t_k)\}$. This means that in an optimal mechanism the seller keeps the object if $v_0(s)$ is larger than the virtual valuations of all the buyers and allocates the object to a buyer if the buyer’s virtual valuation is the highest.

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6 For example, when $v_i$ has the linear form $v_i(s,t_i) = \alpha s + \beta t_i, \alpha, \beta > 0$ or the multiplicative form $v_i(s,t_i) = u(s) t_i, u(s) > 0$, it satisfies Assumption 2.1 given $f_i(t_i)/[1 - F_i(t_i)]$ is non-decreasing in $t_i$. 

among all the buyers and is greater than $v_0(s)$. An optimal allocation automatically satisfies the monotonicity constraint \(2.4\), given Assumption \(2.1\).

If the buyers are symmetric, i.e., $T_1 = T_i$, $F_1 = F_i$, and $v_1 = v_i$ for all $i > 1$, then the second-price auction with reserve price $v_1(s, r^F(s))$ is optimal for the seller of type $s \in S$, where $r^F : S \rightarrow T_1$ satisfies $\left[ J_1(s, r^F(s)) - v_0(s) \right] \left[ r^F(s) - t_1 \right] = 0$. The superscript $F$ denotes the public information case, which is sometimes called the full information case in the literature on the informed-principal problem.

Let $(x^{F,s}, p^{F,s})$ be an optimal mechanism for the type-$s$ seller and $U_{0}^{F}(s)$ the optimal expected payoff of the type-$s$ seller. We have

$$U_{0}^{F}(s) = v_0(s) + \int_T \sum_{i=1}^{n} \left[ J_i(s, t_i) - v_0(s) \right] x^{F,s}_i(t) f(t) \, dt$$

$$= v_0(s) x^{F,s}_0 + \int_T \sum_{i=1}^{n} J_i(s, t_i) x^{F,s}_i(t) f(t) \, dt,$$

where $x^{F,s}_0 = 1 - \int_T \sum_{i=1}^{n} x^{F,s}_i(t) f(t) \, dt$. Define

$$g(s, x) = v_0(s) x_0 + \int_T \sum_{i=1}^{n} J_i(s, t_i) x_i(t) f(t) \, dt,$$

where $x_0 = 1 - \int_T \sum_{i=1}^{n} x_i(t) f(t) \, dt$. It is easy to verify that $g(s, x)$ is absolutely continuous and differentiable with respect to $s$ for any feasible allocation rule $x$. Let $g_1(s, x)$ be the partial derivative of $g$ with respect to $s$. Because the derivatives are all bounded, there exists a sufficiently large number $d$ such that for all $s$,

$$\sup_x |g_1(s, x)| = \sup_x \left| v'_0(s) x_0 + \int_T \sum_{i=1}^{n} \frac{\partial J_i(s, t_i)}{\partial s} x_i(t) f(t) \, dt \right| \leq d.$$

Therefore, according to Theorem 2 of Milgrom and Segal (2002), we have

$$U_{0}^{F}(s) = \int_{\tilde{s}}^{s} g_1(\tilde{s}, x^{F,\tilde{s}}) \, d\tilde{s} + U_{0}^{F}(\tilde{s}).$$

\(2.9\)
2.3.2 Optimal Mechanism for Privately Informed Seller

In this subsection, I switch to the case where the seller is privately informed about her signal. I first discuss how the profile of public-information optimal mechanisms \(\{(x^{F,s}, p^{F,s}) : s \in S\}\) fails to be an equilibrium strategy of the seller in this private information environment. Then I characterize the strategies of the seller in the seller-optimal separating equilibria, and show the optimality of reserve prices in signaling the type of the seller.

2.3.2.1 Public Information Benchmark is not Implementable

As I show in (2.18) in the next section, if the profile \(\{(x^{F,s}, p^{F,s}) : s \in S\}\) is an equilibrium strategy of the seller, then

\[
U^F_0(s) = \int_{\bar{s}}^{s} v'_0(\bar{s}) x^F_0(\bar{s}) \, d\bar{s} + U^F_0(\bar{s}).
\]  

(2.10)

Comparing equations (2.9) and (2.10), we have

\[
\int_{\bar{s}}^{s} g_1(\bar{s}, x^{F,\bar{s}}) \, d\bar{s} - \int_{\bar{s}}^{s} v'_0(\bar{s}) x^F_0(\bar{s}) \, d\bar{s} = \int_{\bar{s}}^{s} \sum_{i=1}^{n} \frac{\partial J_i(\bar{s}, t_i)}{\partial s} x^F_i(\bar{s}) f(t) \, dt \, d\bar{s}.
\]

(2.11)

Therefore, as long as the difference in (2.11) is not zero, \(\{(x^{F,s}, p^{F,s}) : s \in S\}\) is not an equilibrium strategy of the seller.

If the seller’s signal does not affect the buyers’ valuations of the object, i.e., the model is a private value model, then \(\partial J_i(s, t_i) / \partial s = 0\) for all \((s, t_i)\), and the difference in (2.11) is equal to 0. In this case, the profile \(\{(x^{F,s}, p^{F,s}) : s \in S\}\) is a seller-optimal separating equilibrium strategy, and the privacy of seller’s information is irrelevant.

The rest of the paper focuses on the environments where the difference in (2.11) is positive by imposing the following assumption, which ensures that the surplus that the seller can extract from any buyer is increasing in \(s\)\(^7\)

\(^7\)This assumption is also satisfied for some commonly adopted functional forms for \(v_i\). For example,
Assumption 2.2. $J_i(s, t_i)$ is strictly increasing in $s$ for all $t_i \in T_i$, $i = 1, \ldots, n$.

Under this assumption, I show in the following proposition the reason that the profile $\{(x^{F,s}, p^{F,s}) : s \in S\}$ fails to be an equilibrium strategy\(^8\).

**Proposition 2.2.** Given Assumption 2.2, a lower-type seller has an incentive to mimic a higher-type one under $\{(x^{F,s}, p^{F,s}) : s \in S\}$ if all buyers truthfully report their types.

The reason is simple: by deviating to the public-information optimal mechanism of a higher-type seller, a lower-type seller can extract more surplus from buyers in trade (Assumption 2.2), even though she may suffer from a decrease in the probability of trade, which is of second order compared with the increase in trade surplus. The proof of this proposition is in the appendix.

### 2.3.2.2 Separating through Reserve Prices

Since under the profile $\{(x^{F,s}, p^{F,s}) : s \in S\}$, lower-type sellers have an incentive to mimic higher-type ones, to separate different types of the seller we should reduce the profitability of the mechanisms adopted by the higher-type sellers, so as to deter lower-type sellers from mimicking higher-type ones. The following theorem implies that raising the reserve prices in the public-information optimal mechanisms of higher-type sellers is the least costly way of achieving separation.

**Theorem 2.1.** Under Assumptions 2.1 and 2.2, there exists a seller-optimal separating equilibrium, which maximizes the (interim) expected profit of every type of the seller among all possible separating equilibria. A strategy $\{(x^s, p^s, b^s) : s \in S\}$ is the equilibrium strategy of the seller in a seller-optimal separating equilibrium if and only if the allocation

\[ (1) \text{ the linear form } v_i(s, t_i) = \alpha s + \beta t_i, \alpha, \beta > 0, \text{ and } (2) \text{ the multiplicative form } v_i(s, t_i) = u(s) \cdot t_i, \text{ with } t_i = (1 - F_i(t_i)) / f_i(t_i) > 0 \text{ for any } t_i. \]

\(^8\)In the case where the buyers are symmetric, to make a lower type seller have an incentive to mimic a higher type seller under strategy $\{(x^{F,s}, p^{F,s}) : s \in S\}$, assuming that $v_i(s, t_i)$ is increasing in $s$, instead of Assumption 2.2 is sufficient. However, for the asymmetric case, it is unclear whether Assumption 2.2 can be replaced.
rule $x^s$ satisfies

$$x^s \in \arg \max_{\hat{x}} \left\{ \int_T \sum_{i=1}^n \left[ J_i(s, t_i) - v_0(s) \right] \hat{x}_i(t) f(t) \, dt: \hat{x} \text{ satisfies (2.1)} \text{ and } \hat{x}_0 = x_0^s \right\},$$

(2.12)

where $x_0^s$ is increasing in $s$ and satisfies $x_0^s = x_0^{F,s}$, $x_0^s > x_0^{F,s}$, for all $s > \underline{s}$, and

$$v_0(s) + \int_T \sum_{i=1}^n \left[ J_i(s, t_i) - v_0(s) \right] x_i^s(t) f(t) \, dt = \int_{\underline{s}}^s v_0'(\tilde{s}) x_0^s \tilde{s} \, d\tilde{s} + U_0^F(s), \text{ for every } s,$$

(2.13)

and the payment rule $p^s$ satisfies

$$p_i^s(t_i) = v_i(s, t_i) x_i^s(t_i) - \int_{\underline{t}_i}^{t_i} \frac{\partial v_i(s, \tilde{t}_i)}{\partial \tilde{t}_i} x_i^s(\tilde{t}_i) \, d\tilde{t}_i, \text{ for every } i, s, \text{ and } t_i.$$  

(2.14)

The lowest type of each buyer gets 0 expected payoff, and the seller never burns money, i.e.,

$$u_i(M^s, s|\underline{t}_i) = 0 \text{ and } b^s(t) = 0, \text{ for every } i, s, \text{ and } t.$$

To understand this theorem, let us compare it with Proposition 2.1, which characterizes the public-information optimal mechanisms. In these two results, the payment rules for the seller have the same structure, and the expected payoff of the lowest type of every buyer and the amount of money burned are both equal to 0. The major difference between the mechanisms in these two results lies in the allocation rules. In the current theorem, condition (2.12) indicates that the object is still allocated to a buyer who has the highest virtual valuation among all the buyers when there is trade, i.e., $x_i(t) > 0$ only if $J_i(s, t_i) \geq \max_{k \neq i} \{ J_k(s, t_k) \}$. The probability of trade, however, is lower than it is in the public-information optimal mechanism if the seller has type $s > \underline{s}$, as $x_0^s > x_0^{F,s}$ for all $s > \underline{s}$. Condition (2.13) ensures that the strategy of the seller is incentive compatible.

The decrease in the probability of trade is associated with raising the reserve prices. The following corollary for the case of symmetric buyers illustrates the increase in the reserve
prices.

**Corollary 2.1.** Under Assumptions 2.1 and 2.2, if the buyers are symmetric, there exists a seller-optimal separating equilibrium in which a seller of type $s$ chooses the second-price auction with reserve price $v_1(s, r(s))$, where $r(s)$ is the minimal type of buyers that can get the object with positive probability. The function $r: S \to T_1$ is increasing in $s$ and satisfies $J_1(s, r(s)) - v_0(s) \geq 0$, for all $s$, which holds with equality if and only if $s = s^\ast$.

In the public information case, as pointed out below Proposition 2.1, it is optimal for a seller of type $s$ to choose the second-price auction with reserve price $v_1(s, r^F(s))$, where $r^F(s)$ satisfies $J_1(s, r^F(s)) - v_0(s) = 0$. In the private information case, the reserve price set by a seller of type $s > s^\ast$ becomes $v_1(s, r(s)) > v_1(s, r^F(s))$, as $J_1(s, r(s)) - v_0(s) > 0$, in a seller-optimal separating equilibrium. The increased reserve prices make the strategy of the seller incentive compatible.\footnote{In the asymmetric case, different buyers potentially face different reserve prices under the same mechanism in a seller-optimal separating equilibrium. However, all buyers’ reserve prices correspond to the same virtual valuation. I illustrate how the seller sets the reserve prices when her type is $s > s^\ast$. Define $r_i(s, J)$ as the type of buyer $i$ having virtual valuation $J$, so $J_i(s, r_i(s, J)) = J$. Given Assumption 2.1, $r_i(s, J)$ is increasing in $J$. If the seller keeps the object with probability $x_0^s$, then she chooses $J$ such that $\prod_{i=1}^n F_i(r_i(s, J)) = x_0^s$. The resulting $r_i(s, J)$ is the minimum type of the buyer $i$ that can get the object with positive probability, and the reserve price for buyer $i$ is consequently $v_i(s, r_i(s, J))$, according to the payment rule (2.14). In a public-information optimal mechanism, $J = v_0(s)$, so $x_0^{F,s} = \prod_{i=1}^n F_i(r_i(s, v_0(s)))$. In Theorem 2.1, since $x_0^s > x_0^{F,s}$ for $s > s^\ast$, we have $J > v_0(s)$ and $r_i(s, J) > r_i(s, v_0(s))$. Condition (2.13), which is to ensure the incentive compatibility of the seller’s strategy, determines the value of $J$ chosen by each type of the seller.}

In the equilibria characterized in the theorem, some high types of the seller may choose to sell the object with probability 0. If this is the case, then the equilibria, to be precise, are partial pooling equilibria. However, the types of the seller that pool are out of the market, while the ones in the market still fully separate from each other. Thus, I still treat these equilibria as separating equilibria.

Now I interpret Theorem 2.1. Proposition 2.2 points out that in the private information case, if the seller adopts ${\{(x^{F,s}, p^{F,s}) : s \in S}\}$ as her strategy and the buyers report their types truthfully, then a lower-type seller has an incentive to pretend to be a higher type, because this allows her to sell the object at a relatively higher price, even
though the probability of trade might be reduced. Given that the expected revenue of every type of seller is determined by \( x \) and \( \sum_{i=1}^{n} u_i (M, s|\ell_i) \) (see (2.7)), we can disincentivize lower-type sellers from mimicking higher-type ones by changing the higher-type ones’ allocation rule \( x \) and/or \( \sum_{i=1}^{n} u_i (M, s|\ell_i) \). For example, (1) we increase the reserve prices of the mechanism adopted by a higher-type seller, so as to decrease the probability of selling the object; (2) we change the allocation rule from always allocating the object to the buyer with the highest virtual valuation, while maintaining the monotonicity of the allocation rule (required by (2.4)); (3) we can increase the expected payoffs \( u_i (M, s|\ell_i), i = 1, 2, \ldots, n \), of the lowest types of the buyers, which is equivalent to uniformly decreasing the payment of the buyers, or burning money. The first two approaches involve changes in the allocation rule \( x \) only, and the third approach changes only \( \sum_{i=1}^{n} u_i (M, s|\ell_i) \). Approaches (1) and (3) are straightforward and easy to implement. Approach (2) is open-ended, as it can be done in many different ways.

Theorem 2.1 tells us that from the seller’s perspective, the first approach outperforms all other possible approaches, including the other two mentioned above. This is because the first approach is less costly than any other approach for higher-type sellers to separate themselves from lower-type ones. The reasons are simple. To eliminate the incentive of a lower-type seller to misreport upward, we need to reduce her revenue gain from doing so. To reduce the revenue gain by a certain amount, the third approach forces a higher-type seller to give up the same amount of revenue. The first approach, which increases the probability that the seller keeps the object with no payment, induces less payoff loss to a higher-type seller, because a higher-type seller values the object more than a lower type.\(^{10}\) For any other approach of deterring lower-type sellers from mimicking higher types, I show through the proof of Proposition 2.3 which is an intermediate step in proving Theorem 2.1 that it is no better than some combination of the first and third

\(^{10}\)When the reserve prices are increased, the payments of the buyers are higher in the case where there is only one buyer reporting a type higher than his reserve price. However, compared with the effect of reserve prices on the probability of trade, this effect is of second order.
approaches, so cannot outperform the first approach.

This theorem provides support to the literature on reserve price signaling (Jullien and Mariotti, 2006; Cai et al., 2007). In the models in that literature, the selling mechanisms have fixed formats: the seller who faces symmetric buyers is allowed to change only the reserve prices. The theorem shows that even if the seller has the freedom to vary every element of the mechanism, she would still stick to using the reserve prices to separate different types when the signals of the players are independent.

Theorem 2.1 characterizes only the seller-optimal separating equilibria of the mechanism selection game. It is challenging to perform standard equilibrium refinements, such as the Intuitive Criterion (Cho and Kreps, 1987) and D1 Criterion (Banks and Sobel, 1987), to rule out other non-separating equilibria, due to the complexity of the action space of the seller. It is also challenging to identify the conditions under which the seller-optimal separating equilibria are not dominated by non-separating equilibrium in terms of the seller’s interim expected payoff. However, the message that signaling through reserve prices is optimal is useful beyond separating equilibria. Consider a partial pooling equilibrium in which, for some $s' \in (s, \bar{s})$, the types of the seller in $[s, s']$ pool, while all other types separate by choosing different mechanisms. If among the separated types, there exists a type $s''$ that does not adopt a mechanism derived by raising only the reserve prices of her public-information optimal mechanism, then there exists another partial pooling equilibrium in which the types in $[s, s']$ also pool and all types get higher expected payoffs, and an open interval of types in $(s'', \bar{s}]$ get strictly higher expected payoff.

Assumption 2.2 is crucial for the optimality of reserve price signaling. If instead we assume that $J_i(s, t_i)$ is decreasing in $s$ for all $i$ and $t_i \in T_i$, then Theorem 2.1 no longer

\[ \text{In contrast, it is easy to find sufficient conditions under which the seller-optimal separating equilibria are dominated by other equilibria. For example, if the belief of buyers is sufficiently concentrated around the highest type $\bar{s}$ of the seller, there exists a fully pooling equilibrium dominating the seller-optimal separating equilibria. The intuition can be borrowed from the analysis of Spence (1973) for the case where the proportion of high type is large.} \]
holds. This alternative assumption fits cases where the object for sale is complementary to other assets owned by the seller, and $s$ captures the degree of complementarity. Under the alternative assumption, a higher-type seller has an incentive to pretend to be a lower-type seller under the strategy that each type of the seller adopts her public-information optimal mechanism. To separate the lower-type sellers from the higher-type ones, we should make the mechanisms adopted by the lower types less profitable. Though Theorem 2.1 does not hold in this new environment, the intuition underlying the theorem still applies in characterizing the seller-optimal separating equilibria: the mechanism chosen by each type of the seller differs from this type’s public-information optimal mechanism only in the expected payoffs of the lowest types of the buyers or the amount of money burned. This is because the third approach mentioned above in interpreting Theorem 2.1 which is to uniformly decrease the payment of the buyers or burn money, now becomes optimal.

I prove Theorem 2.1 using another game of mechanism selection in which the seller herself is an active participant in the mechanism. This auxiliary game is the standard game studied in that literature on mechanism design by an informed principal. Since implementing a mechanism in this game requires a neutral mediator, I call this game the mediated mechanism-selection game, or mediated game for simplicity, in the rest of the analysis, to distinguish it from the main mechanism-selection game I study. Introducing this mediated game facilitates the proof of the theorem by making it easier to show that other separating equilibria are dominated by the ones characterized in Theorem 2.1. For the mediated game, there is a class of mechanisms named safe mechanisms, which are incentive feasible regardless of the buyers’ beliefs about the seller’s type. The mechanisms that maximize the interim expected payoff of the principal among all safe mechanisms are called Rothschild-Stiglitz-Wilson (RSW) mechanisms. The equilibrium strategy of the seller in any separating equilibrium of the mechanism-selection game corresponds to a safe mechanism of the mediated game. I prove that a safe mechanism is an RSW mechanism if and only if it satisfies the conditions in Theorem 2.1 and any
RSW mechanism corresponds to the equilibrium strategy of the seller in some separating equilibrium of the mechanism-selection game. Directly proving the theorem involves a full characterization of the set of separating equilibria, which requires one to specify a belief system for each equilibrium. My indirect approach also yields an independently interesting result: a full characterization of the set of RSW mechanisms. RSW mechanisms play an important role in the equilibrium analysis of informed-principal problems where the principals’ information enters agents’ valuation functions (Maskin and Tirole, 1992). Fully characterizing the set of RSW mechanisms is potentially useful for analyzing this class of informed-principal problems.

2.4 Mediated Game and Proof of Theorem 2.1

In this section, I study the mediated game, in which the seller is an active participant in the selected mechanism. The specifics of this game are the same as those of the main mechanism-selection game, except that now a mechanism has an \((n + 1)\)-dimensional message space \(\Lambda = \Lambda_0 \times \prod_{i=1}^{n} \Lambda_i\), where \(\Lambda_0 \subset \mathbb{R}\) is the set of possible messages for the seller and \(\Lambda_i \subset \mathbb{R}\) is the set of possible messages for buyer \(i\), and the implementation of a mechanism requires the participation of the seller. Most of the analysis is still restricted to direct incentive feasible mechanisms, according to the revelation principle. The message space of a direct mechanism is \(S \times T\). I use \(x : S \times T \to [0, 1]^n\), \(p : S \times T \to \mathbb{R}^n\), and \(b : S \times T \to \mathbb{R}_+\) to denote the allocation rule, payment rule, and money-burning rule of a direct mechanism. The allocation rule \(x\) satisfies the feasibility constraint

\[
0 \leq x_i(s, t), \sum_{i=1}^{n} x_i(s, t) \leq 1 \text{ for all } i, s, \text{ and } t. \tag{2.15}
\]
Let \( x_0(s, t) \) denote the probability that the seller keeps the object when the message profile is \((s, t)\), i.e., \( x_0(s, t) = 1 - \sum_{i=1}^{n} x_i(s, t) \). I define

\[
x_i(s, t_i) = \int_{T_{-i}} x_i(s, t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i} \text{ and } p_i(s, t_i) = \int_{T_{-i}} p_i(s, t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i},
\]

and \( x_0(s) = \int_T x_0(s, t) f(t) dt \), and \( b(s) = \int_T b(s, t) f(t) dt \).

### 2.4.1 Incentive Feasible Mechanisms

Since the mechanism is proposed after the seller learns her type, the choice of mechanism may partially or completely reveal the seller’s information. Suppose that a direct mechanism \( M = (x, p, b) \) is proposed by the seller. Let \( \bar{U}_0(M, s' | s) \) denote the type-\( s \) seller’s expected payoff under mechanism \( M \) from reporting her type as \( s' \) given that all buyers report their types truthfully. Thus,

\[
\bar{U}_0(M, s' | s) = \int_T \left\{ v_0(s) x_0(s', t) + \sum_{i=1}^{n} p_i(s', t) - b(s', t) \right\} f(t) dt.
\]

To simplify the notation, I replace \( \bar{U}_0(M, s | s) \) by \( \bar{U}_0(M | s) \) in the following analysis. If \( M \) is incentive feasible, then condition \( (2.15) \) is satisfied by its allocation rule \( x \), and the seller with \( s \in S \) has an incentive to participate and report truthfully, i.e., for any \( s' \in S \) and \( s' \neq s \),

\[
\bar{U}_0(M | s) \geq \bar{U}_0(M, s' | s) \text{ and } \bar{U}_0(M | s) \geq v_0(s).
\]  \( (2.16) \)

Standard techniques show that constraints \( (2.16) \) are equivalent to

\[
x_0(s) \geq x_0(s') \text{ whenever } s > s', \quad (2.17)
\]

\[
\bar{U}_0(M | s) = \int_{\tilde{s}}^{s} v_0'(\tilde{s}) x_0(\tilde{s}) d\tilde{s} + \bar{U}_0(M | \tilde{s}) \text{ for all } s, \text{ and } (2.18)
\]

\[
\bar{U}_0(M | \tilde{s}) \geq v_0(\tilde{s}). \quad (2.19)
\]
Let \( \bar{u}_i(M, s, t_i'|t_i) \) denote the expected payoff of buyer \( i \) under mechanism \( M \) from reporting his type as \( t_i' \), given that his type is \( t_i \), the seller’s type is \( s \), and all other players report their types truthfully. Thus,

\[
\bar{u}_i(M, s, t_i'|t_i) = v_i (s, t_i) x_i (s, t_i') - p_i (s, t_i') .
\]

As before, I use \( \bar{u}_i(M, s|t_i) \) in place of \( \bar{u}_i(M, s, t_i|t_i) \) to simplify the notation. If the posterior of the buyers about the seller’s type is \( f_0 (\cdot | M) \), then the expected payoff \( \bar{U}_i (M, t_i'|t_i) \) of buyer \( i \) under mechanism \( M \) from reporting the type \( t_i' \), given that his type is \( t_i \) and all other players report their types truthfully, is

\[
\bar{U}_i (M, t_i'|t_i) = \int_S \bar{u}_i(M, s, t_i'|t_i) f_0 (s|M) ds.
\]

Again, I replace \( \bar{U}_i (M, t_i'|t_i) \) by \( \bar{U}_i (M|t_i) \) to simplify the notation. The incentive feasibility of \( M \) requires that for any \( i \) and any \( t_i', t_i \in T_i \) and \( t_i' \neq t_i \),

\[
\bar{U}_i(M|t_i) \geq \bar{U}_i (M, t_i'|t_i) \quad \text{and} \quad \bar{U}_i(M|t_i) \geq 0. \tag{2.20}
\]

Reformulating the constraints of the buyers is not necessary for the analysis. However, one should note that if the seller’s signal is public, then (2.20) can be reformulated to a set of constraints analogous to (2.4), (2.5), and (2.6).

I construct a mechanism \( M^F = (x^F, p^F, b^F) \) with

\[
x^F (s, t) = x^{F,s} (t) , p^F (s, t) = p^{F,s} (t) , \text{ and } b^F (s, t) = 0 ,
\]

for all \( (s, t) \in S \times T \). It is easy to see that \( M^F \) satisfies the IC and IR constraints of the buyers in (2.20), regardless of the belief of the buyers about the seller’s type, as \( \{(x^{F,s}, p^{F,s}) : s \in S\} \) satisfies (2.2) and (2.3). Thus, if all other players report truthfully,
it is incentive compatible for a buyer to report his type truthfully. But the IC constraints of the seller in (2.16) are violated, as I have shown in Proposition 2.2.

2.4.2 Inscrutability Principle

In general, different types of seller can select different mechanisms in an equilibrium of this game. The associated signaling problem potentially complicates the analysis. Thanks to the inscrutability principle introduced in Myerson (1983), we can assume without loss of generality that all types of the seller in equilibrium propose the same mechanism; thus the choice of mechanism reveals no private information of the seller on the equilibrium path, and the posterior of the buyers after observing the mechanism is the same as their prior. The inscrutability principle is formally stated in the following lemma. All the proofs in this section are relegated to the appendix.

Lemma 2.1 (Inscrutability Principle, Myerson (1983)). In the mediated game, for any equilibrium, there exists another equilibrium in which the seller’s choice of mechanism is independent of her type and each type of the seller gets the same expected payoff as in the original equilibrium.

This lemma enables us to focus the analysis on cases where all types of the seller propose the same mechanism.

2.4.3 Safe Mechanisms and Proof of Theorem 2.1

In general, whether a proposed mechanism is incentive feasible for the buyers depends on their beliefs about the seller’s type. However, there is a set of mechanisms whose incentive feasibility is independent of the buyers’ beliefs. These are called safe mechanisms, which were first studied by Myerson (1983).

Definition 2.1. A direct mechanism is safe if it is incentive feasible for the seller, and if it is incentive feasible for the buyers conditional on every possible seller’s type.
According to the definition, if mechanism $M$ is safe, then it satisfies constraints (2.17), (2.18), and (2.19) for the seller, and satisfies the following constraints for the buyers: for any $i, s$, and any $t'_i, t_i \in T_i$ and $t'_i \neq t_i$,

$$\bar{u}_i(M, s|t_i) \geq \bar{u}_i(M, s, t'_i|t_i) \text{ and } \bar{u}_i(M, s|t_i) \geq 0. \quad (2.21)$$

The incentive feasibility of a safe mechanism $M$ is independent of $f_0 (\cdot | M)$, because regardless of $f_0 (\cdot | M)$, (2.21) implies (2.20).

Safe mechanisms are interesting because any separating-equilibrium strategy of the seller in the main mechanism-selection game corresponds to a safe mechanism in the mediated game. If among safe mechanisms, we can find one that dominates every other safe mechanism in terms of the interim expected payoff of the seller, then this safe mechanism must weakly dominate the equilibrium strategy of the seller in any seller-optimal separating equilibrium of the mechanism-selection game. In the analysis below, I show that there indeed exists such an optimal safe mechanism.

The next lemma characterizes the set of safe mechanisms. It can be proved using standard techniques based on (2.17), (2.18) for the seller, and (2.21) for the buyers; therefore, I omit its proof. The set of safe mechanisms is non-empty and convex.

**Lemma 2.2.** A direct feasible mechanism $M$ is safe if and only if it satisfies the following conditions,

1. $x_0(s) \geq x_0(s')$ and, for each $i$, $x_i(s, t_i) \geq x_i(s, t'_i)$, whenever $s > s', t_i > t'_i$.
2. $v_0(s) + \int_T \sum_{i=1}^n [J_i(s, t_i) - v_0(s)] x_i(s, t) f(t) \, dt - \sum_{i=1}^n \bar{u}_i(M, s|t_i) - b(s)$
   $$= \int_s^{s'} v'_0(\tilde{s}) \, x_0(\tilde{s}) \, d\tilde{s} + \bar{U}_0(M|\tilde{s}) \forall s. \quad (2.22)$$
3. $p_i(s, t_i) = v_i(s, t_i) x_i(s, t_i) - \int_{t_i}^{t'_i} \frac{\partial v_i(s, \tilde{t}_i)}{\partial \tilde{t}_i} x_i(s, \tilde{t}_i) \, d\tilde{t}_i - \bar{u}_i(M, s|t_i) \forall i, s, t_i.$
4. $\bar{U}_0(M|\tilde{s}) \geq v_0(\tilde{s}), \bar{u}_i(M, s|t_i) \geq 0 \forall i, s.$
In the following proposition, I characterize the safe mechanisms that maximize the expected payoff of every type of the seller among all safe mechanisms. Maskin and Tirole (1992) call these mechanisms Rothschild-Stiglitz-Wilson mechanisms. Formally, a safe mechanism $M^*$ is an RSW mechanism if for any $s$,

$$
\bar{U}_0 (M^* | s) = \max_{M=(x,p,b)} \bar{U}_0 (M | s)
$$

s.t. $M$ satisfies (2.17), (2.18), (2.19), and (2.21)

RSW mechanisms, if they exist, are of particular interest in the mediated game. First, RSW mechanisms determine the minimum equilibrium payoff of each type of the seller in the mediated game. Second, if RSW mechanisms are not dominated by any other mechanism in terms of the interim expected payoff of the seller, then they coincide with the set of equilibrium strategies of the mediated game.\textsuperscript{12}

The following proposition shows that an RSW mechanism can be derived by raising the reserve prices of $M^F$. The lowest-type seller gets the same expected payoff as she obtains under $M^F$, but all other types of the seller are worse off than they are under $M^F$.

**Proposition 2.3.** Under Assumptions 2.1 and 2.2, a safe mechanism $M^*$ is an RSW mechanism if and only if it satisfies the following three conditions.

1. The allocation rule $x^*$ has the properties that $x^*(s, \cdot)$ solves

$$
\max_{\hat{x}} \int_T \sum_{i=1}^n [J_i (s, t_i) - v_0 (s)] \hat{x}_i (t) f (t) dt
$$

s.t. $\hat{x}$ satisfies (2.1) and $\hat{x}_0 = x_0^* (s)$,

and $x_0^* (s) = x_0^F (s)$, $x_0^* (s) > x_0^F (s)$ for all $s > s$.\textsuperscript{12}

\textsuperscript{12}In the terminology of Myerson (1983), an undominated RSW mechanism is a strong solution to the informed-principal game. A strong solution is an expectational equilibrium, a core mechanism, and a neutral optimum, which are all solution concepts to the informed-principal game, but with increasing strengths.
2. The lowest type of every buyer gets 0 expected payoff, and the seller never burns money, i.e.,
\[ \bar{u}_i (M^*, s_{ij}|s) = 0 \text{ and } b^*(s) = 0, \text{ for all } s, i. \]

3. The lowest type of the seller gets her optimal public-information payoff, i.e.,
\[ \bar{U}_0 (M^*|s) = \bar{U}_0 (M^F|s). \]

All RSW mechanisms have essentially the same allocation rule.

An RSW mechanism differs from \( M^F \), which is characterized in Proposition \ref{prop:2.1}, only in reserve prices. Given this proposition, we can prove the existence of an RSW mechanism by proving the existence of an allocation rule \( x \) satisfying the conditions in Lemma \ref{lem:2.2} with \( \sum_{i=1}^n \bar{u}_i (M, s_{ij}|s) = 0 \) and \( \bar{U}_0 (M|s) = U_0 (M^F|s) \), and the first condition in Proposition \ref{prop:2.3}.

**Proposition 2.4.** There exists an RSW mechanism. If \( M^* \) is an RSW mechanism, then \( x^*_0 (s) \) is continuous and strictly increasing in \( s \) as long as \( x^*_0 (s) < 1 \).

RSW mechanisms are naturally connected with the seller-optimal separating equilibria in the mechanism-selection game. As mentioned above, every separating-equilibrium strategy of the seller in the mechanism-selection game corresponds to a safe mechanism, so any RSW mechanism gives the seller weakly higher interim expected payoffs than does any separating equilibrium. Proposition \ref{prop:2.4} readily shows that in any RSW mechanism, different types of the seller in the market (i.e., the types of the seller who trade with positive probabilities) choose different rules of allocating the object. Thus, RSW mechanisms are natural candidates for the strategies of the seller in seller-optimal separating equilibria. The following proposition, combined with Propositions \ref{prop:2.3} and \ref{prop:2.4} completes the proof of Theorem \ref{thm:2.1}.
Proposition 2.5. If the mechanism $M^* = (x^*, p^*, b^*)$ is an RSW mechanism, then 
\{$(x^*(s, \cdot), p^*(s, \cdot), b^*(s, \cdot)) : s \in S$\} is the equilibrium strategy of the seller in a seller-optimal separating equilibrium of the mechanism-selection game, in which the seller of type $s$ chooses $(x^*(s, \cdot), p^*(s, \cdot), b^*(s, \cdot))$, and the belief of the buyers upon observing any off-equilibrium mechanism is that the seller is of type $s$.

To see why the three propositions above imply Theorem 2.1, first consider the “if” part of the theorem. If the strategy \{$(x^*(s, \cdot), p^*(s, \cdot), b^*(s, \cdot)) : s \in S$\} of the seller in the mechanism-selection game satisfies the conditions in Theorem 2.1, then the mechanism $M^*$ with

$$x^*(s, t) = x^*(t), p^*(s, t) = p^*(t), b^*(s, t) = b^*(t),$$

for every $s, t,$

is an RSW mechanism, due to Lemma 2.2 and the “if” part of Proposition 2.3. Then using Proposition 2.5, we can conclude that \{$(x^*(s, \cdot), p^*(s, \cdot), b^*(s, \cdot)) : s \in S$\}—which is exactly \{$(x^*, p^*, b^*) : s \in S$\}—is the equilibrium strategy of the seller in a seller-optimal separating equilibrium of the mechanism-selection game.

Now consider the “only if” part of the theorem. Suppose that the strategy \{$(x^*, p^*, b^*) : s \in S$\} of the seller is the equilibrium strategy in a seller-optimal separating equilibrium of the mechanism-selection game. Then Propositions 2.4 and 2.5 imply that \{$(x^*, p^*, b^*) : s \in S$\} corresponds to an RSW mechanism of the mediated game; otherwise it cannot be a seller-optimal separating-equilibrium strategy in the mechanism-selection game. According to Lemma 2.2 and the “only if” part of Proposition 2.3, we can conclude that the profile \{$(x^*, p^*, b^*) : s \in S$\} satisfies the conditions in Theorem 2.1.

2.5 Concluding Remarks

In this paper, I study the design of a selling mechanism by a privately informed seller. In general, the privacy of the seller’s information affects mechanism design: a lower-quality seller would like to adopt the public-information optimal mechanism of a higher-quality
seller. However, the privacy of the information does not lead to unfamiliar and complicated selling mechanisms in equilibrium. In characterizing the seller-optimal separating equilibria of this game, I find that the equilibrium mechanism chosen by each type of the seller differs from her public-information optimal mechanism only in reserve prices. This characterization unveils an interesting role of reserve prices: reserve prices are the optimal device for separating different types of the principal.

This finding regarding the role of reserve prices may help to simplify the analysis of some informed-principal problems by reducing the infinite-dimensional signaling problems to finite-dimensional (one-dimensional, in the case of symmetric buyers) ones in which only the reserve prices are the signals. Examples of these informed-principal problems include multi-unit auction design by an informed seller, optimal auction design with resale opportunities, and auction design by competing informed sellers.

The analysis of this paper focuses on a special interdependent value environment in which a buyer’s valuation of the object depends only on the seller’s information and his own private information. Allowing a buyer’s valuation to depend on other buyers’ information does not change the results if the dependence is linear. However, whether the results hold in more general interdependent value environments is unknown.
Appendix

Proof of Proposition 2.2

Let \( U_0^F (\hat{s} | s) \) denote the expected payoff of the type- \( s \) seller from choosing the public-information optimal mechanism of the type-\( \hat{s} \) seller, if the buyers report their types truthfully. We have

\[
U_0^F (\hat{s} | s) = \int_T \left\{ v_0(s) x_0^{F,\hat{s}}(t) + \sum_{i=1}^n p_i^{F,\hat{s}}(t) \right\} f(t) \, dt
\]

\[
= U_0^F (\hat{s}) - [v_0(\hat{s}) - v_0(s)] x_0^{F,\hat{s}},
\]

where the second equality is due to the definition of \( U_0^F (\hat{s}) \). To examine the incentive of the type-\( s \) seller to deviate, we take the difference between \( U_0^F (\hat{s} | s) \) and \( U_0^F (s) \), and obtain

\[
U_0^F (\hat{s} | s) - U_0^F (s) = \int_{\hat{s}}^{s} g_1 (\tilde{s}, x^{F,\hat{s}}) \, d\tilde{s} - \int_{s}^{\hat{s}} v_0'(s) x_0^{F,\hat{s}} \, d\tilde{s}
\]

\[
= \int_{s}^{\hat{s}} \left[ v_0'(s) (x_0^{F,\hat{s}} - x_0^{F,\hat{s}}) + \int_T \sum_{i=1}^n \frac{\partial J_i (\tilde{s}, t_i)}{\partial s} x_i^{F,\hat{s}}(t) f(t) \, dt \right] \, d\tilde{s},
\]

where the first equality is due to (2.9) and the second equality is derived according to the definition of \( g_1(s, x) \) in (2.8).

For \( s \in [\underline{s}, \bar{s}] \), let \( \hat{s} = s + \Delta \). When \( \Delta \) is an arbitrarily small positive number, due to the continuity of \( x_0^{F,\hat{s}} \) in \( s \), \( v_0'(s) (x_0^{F,\hat{s}} - x_0^{F,\hat{s}}) \) is arbitrarily small for \( \tilde{s} \in [s, \hat{s}] \).

However, the second term in the bracket above is bounded from zero. Thus, we have \( U_0^F (\hat{s} | s) - U_0^F (s) > 0 \). That is, the type-\( s \) seller gets better off from deviating (locally) upwards.

---

\(^{13}\)The continuity of \( x_0^{F,s} \) can be easily proved. Let \( r_i (s) \) be the “reserve price” set by the seller of type \( s \) for buyer \( i \). That is, if buyer \( i \) has type \( t_i < r_i (s) \), he has no chance to get this object regardless of the types of other buyers. The value of \( r_i (s) \) is determined by \( J_i (s, r_i (s)) - v_0(s) = 0 \). The differentiability of \( J_i \) with respect to its two arguments implies that \( r_i (s) \) is differentiable in \( s \). The definition of \( x_0^{F,s} \) gives that \( x_0^{F,s} = \prod_{i=1}^n F_i (r_i (s)) \). Distributions \( F_i, i = 1, 2, \ldots, n \), are continuous in \( r_i \), thus \( x_0^{F,s} \) is continuous in \( s \).
Proof of Lemma 2.1

I prove the result only for pure-strategy equilibria. The proof for mixed-strategy equilibria can be derived easily based on the discussion below. Consider a partition \( \{ S^l \}_{l \in I} \) of \( S \), where \( I \) is a set of index that can be finite or infinite. The partition \( \{ S^l \}_{l \in I} \) is defined in a way that different types of the seller in the same element \( S^l \) choose the same mechanism \( M^l \) in equilibrium. By choosing mechanism \( M^l \) in equilibrium, the seller signals the buyers that her type belongs to \( S^l \). According to the revelation principle, we can assume that \( M^l \) is a direct incentive feasible mechanism, without loss of generality. That is, \( M^l \) satisfies (2.15), (2.16), and (2.20). In equilibrium, we have

\[
\bar{U}_0(M^l|s) \geq \bar{U}_0(M'|s), \quad \text{for } s \in S^l, l' \neq l. \tag{2.23}
\]

I construct an inscrutable mechanism \( M = (x, p, b) \): for \( s \in S^l, l \in I \)

\[
x(s, t) = x^l(s, t), \quad p(s, t) = p^l(s, t), \quad b(s, t) = b^l(s, t),
\]

where \( x^l(s, t) \), \( p^l(s, t) \), and \( b^l(s, t) \) are the allocation rule, payment rule, and money-burning rule of mechanism \( M^l \), respectively. It is clear that if every type of the seller chooses the mechanism \( M \), then \( M \) satisfies constraint (2.20) of the buyers, because for any \( i, t_i \),

\[
\bar{U}_i(M|t_i) = \int_I \left[ \bar{U}_i(M^l|t_i) \int_{S^l} f_0(s) \, ds \right] \, dl \\
\geq \int_I \left[ \bar{U}_i(M^l, t_i'|t_i) \int_{S^l} f_0(s) \, ds \right] \, dl = \bar{U}_i(M, t_i'|t_i),
\]

where the first and last equalities are from the definition of \( M \), and the inequality is derived using the incentive compatibility of \( M^l \), and

\[
\bar{U}_i(M|t_i) = \int_I \left[ \bar{U}_i(M^l|t_i) \int_{S^l} f_0(s) \, ds \right] \, dl \geq 0,
\]
due to $\bar{U}_i(M^l|t_i) \geq 0$.

The construction of $M$ immediately implies that any type of the seller has no incentive to misreport her signal due to the incentive compatibility of $M^l$, $l \in I$, and (2.23). By truthfully reporting her type, the seller gets the same payoff as in the original equilibrium strategy $\{M^l\}_{l \in I}$.

**Proof of Proposition 2.3**

I show the sufficiency and necessity of these conditions separately below.

(1) **Sufficiency of the three conditions:**

Suppose that $M^*$ is a safe mechanism satisfying all the three conditions, but there exists a safe mechanism $M$ such that for some $s \in S$, $\bar{U}_0(M|s) > \bar{U}_0(M^*|s)$. If this is true, then we have

$$\int_\frac{a}{2}^s \nu'_0(\tilde{s}) x_0(\tilde{s}) d\tilde{s} + \bar{U}_0(M|\tilde{s}) > \int_\frac{a}{2}^s \nu'_0(\tilde{s}) x^*_0(\tilde{s}) d\tilde{s} + \bar{U}_0(M^F|\tilde{s}),$$

according to (2.22). Since $\bar{U}_0(M|s) \leq \bar{U}_0(M^F|s)$, we have

$$\int_\frac{a}{2}^s \nu'_0(\tilde{s}) x_0(\tilde{s}) d\tilde{s} > \int_\frac{a}{2}^s \nu'_0(\tilde{s}) x^*_0(\tilde{s}) d\tilde{s}.$$ 

This inequality holds only if there is a set $S^M \subset [\tilde{s}, s]$ such that $x_0(s) > x^*_0(s)$ for $s \in S^M$. Let $s^{\text{sup}}$ be the supremum of $S^M$. If $s^{\text{sup}} \notin S^M$, then we can find a $s^\varepsilon \in S^M$ that is arbitrarily close to $s^{\text{sup}}$. Define

$$s^M = \begin{cases} 
  s^{\text{sup}}, & \text{if } s^{\text{sup}} \in S^M; \\
  s^\varepsilon, & \text{if } s^{\text{sup}} \notin S^M.
\end{cases}$$

Thus, $s^M \in S^M$ and

$$x_0(s^M) > x^*_0(s^M). \quad (2.24)$$
Then, we have
\[
\int_{\tilde{s}}^{s^M} v'_0(\tilde{s}) x_0(\tilde{s}) \, d\tilde{s} + U_0(M|\tilde{s}) > \int_{\tilde{s}}^{s^M} v'_0(\tilde{s}) x_0^*(\tilde{s}) \, d\tilde{s} + U_0(M^F|\tilde{s}),
\]

because the set \([s^M, s^{sup}]\) is small or empty and \(x_0(s') \leq x_0^*(s')\) for all \(s' \in (s^{sup}, s]\).

According to equation (2.22) and Condition 2 of this proposition, which indicates \(P_{n} = 1\) and \(u_{i}(M, s^M|t_i) = b^M(s^M)\), we obtain
\[
\int_{T} \sum_{i=1}^{n} [J_i(s^M, t_i) - v_0(s^M)] x_i(s^M, t) \, f(t) \, dt - \sum_{i=1}^{n} u_{i}(M, s^M|t_i) - b^M(s^M)
\]
\[
> \int_{T} \sum_{i=1}^{n} [J_i(s^M, t_i) - v_0(s^M)] x_i^*(s^M, t) \, f(t) \, dt.
\]

Since \(\sum_{i=1}^{n} u_{i}(M, s^M|t_i) + b^M(s^M)\) is nonnegative, we have
\[
\int_{T} \sum_{i=1}^{n} [J_i(s^M, t_i) - v_0(s^M)] x_i(s^M, t) \, f(t) \, dt
\]
\[
> \int_{T} \sum_{i=1}^{n} [J_i(s^M, t_i) - v_0(s^M)] x_i^*(s^M, t) \, f(t) \, dt.
\]

This inequality holds only \(x_0(s^M) < x_0^*(s^M)\), because \(M^*\) satisfies Condition 1 of this proposition. This contradicts (2.24). Therefore, \(M^*\) is an RSW mechanism.

\textbf{(2) Necessity of the three conditions:}

Suppose that \(M^*\) is an RSW mechanism. Without loss of generality, I assume that \(b^*(s) = 0\) for all \(s\), because if for some \(s'\), \(b^*(s') > 0\), then we can construct a new mechanism \(M'\) which is the same as \(M^*\) except that at \(s'\),
\[
\sum_{i=1}^{n} \bar{u}_i(M', s'|t_i) = \sum_{i=1}^{n} \bar{u}_i(M^*, s'|t_i) + b^*(s'), \text{ and } b'(s') = 0.
\]

It is obvious that \(M'\) is also an RSW mechanism. Combining this argument with the
proof below that \( \sum_{i=1}^{n} \tilde{u}_i (M^*, s|t_i) = 0 \) for any \( s \), we can prove \( b^* (s) = 0 \) for all \( s \).

Firstly, I show that Condition 3 is satisfied by \( M^* \). To proceed, I construct a mechanism \( \hat{M} \) as follows: \( \hat{x}_0(\tilde{s}) = x^F_0(\tilde{s}) \), \( \hat{x}_0(s) = \sup_{\tilde{s} \in [\tilde{s}, s]} \{ x^F_0(\tilde{s}) \} \) for \( s > \tilde{s} \), and \( \hat{x}(s, \cdot) \) solves

\[
\max_x \int T \sum_{i=1}^{n} [J_i(s, t_i) - v_0(s)] x_i(t) f(t) dt
\]

\( s.t. \) \( x \) satisfies (2.1) and \( x_0 = \hat{x}_0(s) \),

and the payment rule satisfies

\[
\hat{p}_i(s, t) = v_i(s, t_i) \hat{x}_i(s, t) - \int_{t_i}^{t_i} \frac{\partial v_i(s, \tilde{t}_i)}{\partial t_i} \hat{x}_i(s, \tilde{t}_i, t_{-i}) \tilde{t}_i - \tilde{u}_i(\hat{M}, s|t_i), \text{ for any } i, s, \text{ and } t,
\]

where

\[
\tilde{u}_i(\hat{M}, s|t_i) = \frac{1}{n} \{ v_0(s) + \int T \sum_{i=1}^{n} [J_i(s, t_i) - v_0(s)] \hat{x}_i(s, t) f(t) dt \}
\]

\[ - \int_{\tilde{s}}^{s} \int T \sum_{i=1}^{n} [J_i(s, t_i) - v_0(s)] \hat{x}_i(s, t) f(t) dt \]

The following lemma shows that \( \hat{M} \) is safe.

**Lemma 2.3.** Mechanism \( \hat{M} \) is a safe mechanism.

**Proof.** It is clear that \( \hat{M} \) immediately satisfies all the constraints for a safe mechanism, except that \( \sum_{i=1}^{n} \tilde{u}_i(\hat{M}, s|t_i) \geq 0 \). The definition of \( \hat{x} \) implies that for \( s_1 < s_2 \in S \), either there exists a \( s' \in (s_1, s_2] \) such that \( x^F_0(s') = \hat{x}_0(s_2) \) or \( \hat{x}_0(s_1) = \hat{x}_0(s_2) \). In the first case,

\[
\sum_{i=1}^{n} \tilde{u}_i(\hat{M}, s_2|t_i) - \sum_{i=1}^{n} \tilde{u}_i(\hat{M}, s_1|t_i) \geq v_0(s_2) x^F_0(s') + \int T \sum_{i=1}^{n} J_i(s_2, t_i) x^F_i(s', t) f(t) dt - v_0(s_1) \hat{x}_0(s_1)
\]
Thus, no matter which case happens,

\[ -\int_T \sum_{i=1}^n J_i(s_1, t_i) \hat{x}_i(s_1, t) f(t) dt - \int_{s_1}^{s_2} v'_0(\hat{s}) \hat{x}_0(\hat{s}) d\hat{s} \]

\[ \geq v_0(s_2) x^F_0(s') + \int_T \sum_{i=1}^n J_i(s', t_i) x^F_i(s', t) f(t) dt - v_0(s_1) \hat{x}_0(s_1) \]

\[ -\int_T \sum_{i=1}^n J_i(s_1, t_i) \hat{x}_i(s_1, t) f(t) dt - \int_{s_1}^{s_2} v'_0(\hat{s}) \hat{x}_0(\hat{s}) d\hat{s} \]

\[ \geq U_0(M^F|s') - U_0(M^F|s_1) + [v_0(s_2) - v_0(s')] x^F_0(s') - \int_{s_1}^{s_2} v'_0(\hat{s}) \hat{x}_0(\hat{s}) d\hat{s} \]

\[ = \int_{s_1}^{s'} g'_1(\hat{s}, q^\hat{s}) d\hat{s} - \int_{s_1}^{s'} v'_0(\hat{s}) \hat{x}_0(\hat{s}) d\hat{s} \]

\[ = \int_{s_1}^{s'} \left\{ v'_0(\hat{s}) [x^F_0(\hat{s}) - \hat{x}_0(\hat{s})] + \int_T \sum_{i=1}^n J'_{i1}(\hat{s}, t_i) x^F_i(\hat{s}, t) f(t) dt \right\} d\hat{s}. \]

The first inequality uses the definition of \( \hat{x} \) in (2.25) and the fact that \( x^F_0(s') = \hat{x}_0(s_2) \). The second inequality is derived using the monotonicity of \( J_i \) and the non-negativity of \( x^F_i(s', t) \). The third inequality is based on the definition of \( U_0(M^F|s') \) and the optimality of \( x^F \). The first equality uses the result of (2.9) and the fact that \( \hat{x}_0(s) = x^F_0(s') \) for \( s \in [s', s_2] \). The last equality is obtained by substituting the expression of \( g_1(\hat{s}, q^\hat{s}) \) into the first equality. It is clear that if \( s_1 \) and \( s_2 \) are arbitrarily close to each other, then this difference is positive.

In the second case, i.e., \( \hat{x}_0(s_1) = \hat{x}_0(s_2) \), it is obvious that

\[ \sum_{i=1}^n \bar{u}_i(\hat{M}, s_2|t_i) - \sum_{i=1}^n \bar{u}_i(\hat{M}, s_1|t_i) \geq 0. \]

Thus, no matter which case happens, \( \sum_{i=1}^n \bar{u}_i(\hat{M}, s|t_i) \) is non-decreasing in \( s \). Since

\[ \sum_{i=1}^n \bar{u}_i(\hat{M}, s|t_i) = 0, \]

\( \sum_{i=1}^n \bar{u}_i(\hat{M}, s|t_i) \) is never negative.
It is clear that $U_0 (M^F | s) \geq U_0 (M^* | s)$, given that

$$
U_0 (M^* | s) = v_0 (s) + \int_T \sum_{i=1}^n [J_i (s, t_i) - v_0 (s)] x_i^0 (s, t) f (t) dt - \sum_{i=1}^n \bar{u}_i (M^*, s | t_i),
$$

and $\sum_{i=1}^n \bar{u}_i (M^*, s | t_i) \geq 0$,

according to Lemma 2.2. According to the definition of the RSW mechanism, we have

$U_0 (M^* | s) \geq \tilde{U}_0 (M | s) = U_0 (M^F | s)$. Thus, $U_0 (M^* | s) = U_0 (M^F | s)$.

I prove Condition 1 and Condition 2 together in the rest of the proof. Suppose that Condition 1 is not satisfied by $M^*$, then for some $s$, either $x^* (s, \cdot)$ is not a solution of

$$
\max_{x} \int_T \sum_{i=1}^n [J_i (s, t_i) - v_0 (s)] \hat{x}_i (t) f (t) dt \
\text{s.t. } \hat{x} \text{ satisfies (2.1) and } \hat{x}_0 = x_0^* (s),
$$

or $x_0^* (s) \leq x_0^F (s)$ or both. If the first case happens, then we can construct a new RSW mechanism $\tilde{M}^*$ by adjusting the allocation rule of $M^*$ such that $\tilde{x}^* (s, \cdot)$ solves (2.26), so $\tilde{x}_0^* (s) = x_0^* (s)$. The expected payoff to the lowest type of each buyer under $\tilde{M}^*$ is defined in the following way so that the equation (2.22) holds,

$$
\sum_{i=1}^n \tilde{u}_i (\tilde{M}^*, s | t_i) = \sum_{i=1}^n \bar{u}_i (M^*, s | t_i) + \int_T \sum_{i=1}^n J_i (s, t_i) \tilde{x}_i^* (s, t) f (t) dt \\
- \int_T \sum_{i=1}^n J_i (s, t_i) x_i^* (s, t) f (t) dt
$$

According to the definition of $\tilde{x}^*$ and the supposition that $M^*$ is an RSW mechanism, we have

$$
\sum_{i=1}^n \tilde{u}_i (\tilde{M}^*, s | t_i) > \sum_{i=1}^n \bar{u}_i (M^*, s | t_i) \geq 0.
$$

This inequality implies that the failure of the first part of Condition 1 is equivalent to the failure of Conditions 2. Thus, in the rest of the proof, I assume that the $M^*$ satisfies
the first part of Condition 1, and then show that \( x^*_0(s) \geq x^F_0(s) \) and Condition 2 must hold.

Suppose for some \( s_1, x^*_0(s_1) < x^F_0(s_1) \). According to the continuity of \( x^F_0(s) \), there exists \( s_2 < s_1 \) (\( s_2 \) close to \( s_1 \)) such that \( x^*_0(s_1) < x^F_0(s_2) \). I construct a safe mechanism \( M \) with allocation rule

\[
x^M_i(s, t) = \begin{cases} 
  x^*_i(s, t), & \text{for } s < s_2, \\
  x^F_i(s_2, t), & \text{for } s \geq s_2,
\end{cases}
\]

and payment rule

\[
p^M_i(s, t) = v_i(s, t_i) x^M_i(s, t) - \int_{\tilde{t}_i}^{t_i} \frac{\partial v_i(s, \tilde{t}_i)}{\partial \tilde{t}_i} x^M_i(s, \tilde{t}_i, t_{\cdots i}) \, d\tilde{t}_i - \bar{u}_i(M, s | t_i), \quad \text{for any } i, s, \text{ and } t,
\]

where \( \bar{u}_i(M, s | t_i) \) satisfies that if \( s < s_2 \),

\[
\bar{u}_i(M, s | t_i) = \bar{u}_i(M^*, s | t_i);
\]

if \( s \geq s_2 \),

\[
\bar{u}_i(M, s | t_i) = \frac{1}{n} \left\{ v_0(s) x^M_0(s) + \int_T \sum_{i=1}^n J_i(s, t_i) x^M_i(s, t) f(t) \, dt \right\} - \int_{\tilde{s}}^{s} v'_0(\tilde{s}) x^M_0(\tilde{s}) \, d\tilde{s} - \bar{U}_0(M^F | s) \}
\]

I show that \( M \) is a safe mechanism in the following lemma.

**Lemma 2.4.** Mechanism \( M \) is a safe mechanism.

**Proof.** Since \( M^* \) is safe, the construction of \( M \) immediately implies that it satisfies equation (2.22) and the feasibility condition. According to (2.27), \( x^M_i(s, t_i) \) and \( x^M_0(s) \) satisfy the monotonicity conditions. Now I show that \( \bar{u}_i(M, s | t_i) \) is always nonnegative.
For \( s < s_2 \), we have \( \bar{u}_i(M, s|I_i) = \bar{u}_i(M^*, s|I_i) \geq 0 \). For \( s = s_2 \),

\[
\sum_{i=1}^{n} \bar{u}_i(M, s_2|I_i) = v_0(s) x_0^F(s_2) + \int_{T} \sum_{i=1}^{n} J_i(s_2, t_i) x_i^F(s_2, t) f(t) \, dt \\
- \int_{\frac{s}{2}}^{s_2} v'_0(\bar{s}) x_0^*(\bar{s}) \, d\bar{s} - \bar{U}_0(M^F|\bar{s}) \\
\geq v_0(s) x_0^*(s_2) + \int_{T} \sum_{i=1}^{n} J_i(s_2, t_i) x_i^*(s_2, t) f(t) \, dt \\
- \int_{\frac{s}{2}}^{s_2} v'_0(\bar{s}) x_0^*(\bar{s}) \, d\bar{s} - \bar{U}_0(M^F|\bar{s}) \\
= \sum_{i=1}^{n} \bar{u}_i(M^*, s_2|I_i) \geq 0.
\]

The weak inequality is due to the optimality of \( x^F(s_2, \cdot) \) at \( s_2 \).

For \( s > s_2 \), we have

\[
\sum_{i=1}^{n} \bar{u}_i(M, s|I_i) = v_0(s) x_0^M(s) + \int_{T} \sum_{i=1}^{n} J_i(s, t_i) x_i^M(s, t) f(t) \, dt \\
- \int_{\frac{s}{2}}^{s} v'_0(\bar{s}) x_0^M(\bar{s}) \, d\bar{s} - \bar{U}_0(M^F|\bar{s}) \\
= v_0(s) x_0^F(s_2) + \int_{T} \sum_{i=1}^{n} J_i(s_2, t_i) x_i^F(s_2, t) f(t) \, dt \\
- \int_{\frac{s}{2}}^{s_2} v'_0(\bar{s}) x_0^*(\bar{s}) \, d\bar{s} - \int_{\frac{s}{2}}^{s} v'_0(\bar{s}) x_0^F(s_2) \, d\bar{s} - \bar{U}_0(M^F|\bar{s}) \\
= v_0(s_2) x_0^F(s_2) + \int_{T} \sum_{i=1}^{n} J_i(s_2, t_i) x_i^F(s_2, t) f(t) \, dt \\
- \int_{\frac{s}{2}}^{s_2} v'_0(\bar{s}) x_0^*(\bar{s}) \, d\bar{s} - \bar{U}_0(M^F|\bar{s}) \\
> v_0(s_2) x_0^F(s_2) + \int_{T} \sum_{i=1}^{n} J_i(s_2, t_i) x_i^F(s_2, t) f(t) \, dt \\
- \int_{\frac{s}{2}}^{s_2} v'_0(\bar{s}) x_0^*(\bar{s}) \, d\bar{s} - \bar{U}_0(M^F|\bar{s}) \\
= \sum_{i=1}^{n} \bar{u}_i(M, s_2|I_i) \geq 0.
\]

Therefore, \( M \) is a safe mechanism.
In mechanism $M$, the seller of type $s \in [s_2, s_1]$ gets a higher expected payoff than under $M^*$, because for $s \in [s_2, s_1]$,

$$\int_{\tilde{s}}^{s} v'(\tilde{s}) x_0^M(\tilde{s}) d\tilde{s} + U_0(M^{|\tilde{s}}) = \int_{\tilde{s}}^{s_2} v'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + \int_{s_2}^{s} v'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0(M^{|\tilde{s}}) > \int_{\tilde{s}}^{s} v'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0(M^{|\tilde{s}}) = \int_{\tilde{s}}^{s} v'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0(M^{|\tilde{s}}),$$

where the first equality is from the definition of $x^M$, the inequality is based on that $x_0^*(s_1) < x_0^*(s_2)$. This contradicts that $M^*$ is an RSW mechanism.

Let us turn to Condition 2. Suppose that for some $\tilde{s} > \delta$, $\sum_{i=1}^{n} \bar{u}_i(M^*, \tilde{s}|t_i) > 0$. Then we have the following lemma.

**Lemma 2.5.** There exists some $\tilde{\delta} > 0$ such that over the interval $[\tilde{s} - \tilde{\delta}, \tilde{s}]$,

$$\inf_{s \in [\tilde{s} - \tilde{\delta}, \tilde{s}]} \sum_{i=1}^{n} \bar{u}_i(M^*, s|t_i) > 0.$$

**Proof.** Suppose that for some $\tilde{s}$, $\sum_{i=1}^{n} \bar{u}_i(M^*, \tilde{s}|t_i) > 0$, then there must exist $\delta > 0$, such that for all $s \in [\tilde{s} - \delta, \tilde{s}]$, $\sum_{i=1}^{n} \bar{u}_i(M^*, s|t_i) > 0$. Suppose this is not true, i.e., for any $\delta > 0$, there exists $s \in [\tilde{s} - \delta, \tilde{s}]$ such that $\sum_{i=1}^{n} \bar{u}_i(M^*, s|t_i) = 0$, then we can find a sequence $\{s_n\}_{n=1}^{\infty}$ converging to $\tilde{s}$ with $\sum_{i=1}^{n} \bar{u}_i(M^*, s_n|t_i) = 0$ for every $n$. According to equation (2.22), we have

$$v_0(s_n) + \int_{T} \sum_{i=1}^{n} [J_i(s_n, t_i) - v_0(s_n)] x_i^*(s_n, t) f(t) dt = \int_{\tilde{s}}^{s_n} v'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0(M^{|\tilde{s}}).$$

By continuity,

$$\lim_{n \to \infty} v_0(s_n) + \int_{T} \sum_{i=1}^{n} [J_i(s_n, t_i) - v_0(s_n)] x_i^*(s_n, t) f(t) dt$$
\[
= \int_{\hat{s}}^{\bar{s}} v'_0(\bar{s}) x_0^*(\bar{s}) \, d\bar{s} + \bar{U}_0(M^*|\bar{s}). \quad (2.28)
\]

However, since \(x_0^*(s_n) \leq x_0^*(\hat{s})\) and Condition 1 of Proposition 2.3 is satisfied, there is

\[
v_0(s_n) + \int_T \sum_{i=1}^n [J_i(s_n, t_i) - v_0(s_n)] x_i^*(s_n, t) \, f(t) \, dt \\
\geq v_0(s_n) + \int_T \sum_{i=1}^n [J_i(\hat{s}, t_i) - v_0(\hat{s})] x_i^*(\hat{s}, t) \, f(t) \, dt.
\]

The expression on the RHS of the inequality is continuous. Using (2.28), taking limits on both sides of this inequality yields

\[
\int_{\hat{s}}^{\bar{s}} v'_0(\bar{s}) x_0^*(\bar{s}) \, d\bar{s} + \bar{U}_0(M^*|\bar{s}) \geq v_0(\hat{s}) + \int_T \sum_{i=1}^n [J_i(\hat{s}, t_i) - v_0(\hat{s})] x_i^*(\hat{s}, t) \, f(t) \, dt.
\]

This contradicts equation (2.22), given \(\sum_{i=1}^n \bar{u}_i(M^*, \hat{s}|t_i) > 0\). This completes the proof that there exists \(\delta > 0\), such that for all \(s \in [\hat{s} - \delta, \hat{s}]\), \(\sum_{i=1}^n \bar{u}_i(t_i|s, M^*) > 0\).

Now I show that there exists \(\bar{s} \in (0, \delta)\) such that \(\inf_{s \in [\hat{s} - \delta, \hat{s}]} \sum_{i=1}^n \bar{u}_i(M^*, s|t_i) > 0\). I again prove this by contradiction. Suppose this is not the case, then for any \(\bar{s} \in (0, \delta)\),

\[
\inf_{s \in [\hat{s} - \delta, \hat{s}]} \sum_{i=1}^n \bar{u}_i(M^*, s|t_i) = 0.
\]

If so, for decreasing positive sequences \(\{\delta_m\}_{m=1}^{\infty}\) and \(\{\varepsilon_m\}_{m=1}^{\infty}\) with \(\delta_1 < \delta\), \(\lim_{m \to \infty} \delta_m = 0\) and \(\lim_{m \to \infty} \varepsilon_m = 0\), we can construct a sequence \(\{s_m\}_{m=1}^{\infty}\) such that

\[
s_m \in [\hat{s} - \delta_m, \hat{s}] \text{ and } \sum_{i=1}^n \bar{u}_i(M^*, s_m|t_i) < \varepsilon_m.
\]

This implies that

\[
\lim_{m \to \infty} s_m = \hat{s} \text{ and } \lim_{m \to \infty} \sum_{i=1}^n \bar{u}_i(M^*, s_m|t_i) = 0.
\]
According to equation (2.22),

\[
\lim_{m \to \infty} v_0(s_m) + \int_T^n \sum_{i=1}^{m} [J_i(s_m, t_i) - v_0(s_m)] x_s^*(s_m, t) f(t) \, dt \\
= \lim_{m \to \infty} \int_{\hat{s}}^{\hat{s}'} v_0'(\hat{s}) x_s^*(\hat{s}) \, d\hat{s} + \lim_{m \to \infty} \sum_{i=1}^{n} \tilde{u}_i(M^*, s_m|t_i) + \tilde{U}_0(M^*|s) \\
= \int_{\hat{s}}^{\hat{s}'} v_0'(\hat{s}) x_s^*(\hat{s}) \, d\hat{s} + \tilde{U}_0(M^*|s). \\
(2.29)
\]

However, due to that \( x_s^*(s_m) \leq x_s^*(\hat{s}) \) and Condition 1,

\[
v_0(s_m) + \int_T^n \sum_{i=1}^{m} [J_i(s_m, t_i) - v_0(s_m)] x_s^*(s_m, t) f(t) \, dt \\
\geq v_0(s_m) + \int_T^n \sum_{i=1}^{m} [J_i(s_m, t_i) - v_0(s_m)] x_s^*(\hat{s}, t) f(t) \, dt.
\]

Taking limits on both sides, we obtain

\[
\int_{\hat{s}}^{\hat{s}'} v_0'(\hat{s}) x_s^*(\hat{s}) \, d\hat{s} + \tilde{U}_0(M^*|s) \geq v_0(\hat{s}) + \int_T^n \sum_{i=1}^{m} [J_i(\hat{s}, t_i) - v_0(\hat{s})] x_s^*(\hat{s}, t) f(t) \, dt,
\]

according to (2.29). This contradicts equation (2.22), given \( \sum_{i=1}^{n} \tilde{u}_i(M^*, \hat{s}|t_i) > 0 \). Therefore, I have proved that there exists \( \hat{s} \in (0, \delta) \) such that \( \inf_{s \in [\hat{s} - \delta, \hat{s}] \sum_{i=1}^{n} \tilde{u}_i(M^*, s|t_i) > 0} \).

Given this lemma, we can construct a safe mechanism \( M^\varepsilon \) making all types of the seller in the set \([\hat{s} - \delta, \hat{s}]\) strictly better off. Specifically, the allocation rule of \( M^\varepsilon \) is defined as follows: for \( s < \hat{s} - \delta \), \( x^\varepsilon(s, t) = x^*(s, t); \) for \( s \in [\hat{s} - \delta, \hat{s}] \), \( x^\varepsilon(s, \cdot) \) solves

\[
\max_{\hat{x}} \left\{ \int_T^n \sum_{i=1}^{m} [J_i(s, t_i) - v_0(s)] \hat{x}_i(t) f(t) \, dt : \hat{x} \text{ satisfies } (2.1) \text{ and } \hat{x}_0 = x_s^*(s) + \varepsilon \right\};
\]

and for \( s > \hat{s} \), \( x^\varepsilon(s, t) = x^\varepsilon(\hat{s}, t) \). The \( \sum_{i=1}^{n} \tilde{u}_i(M^\varepsilon, s|t_i) \) of \( M^\varepsilon \) is constructed as follows:
if \( s < \hat{s} \), \( \sum_{i=1}^{n} \bar{u}_i (M^\varepsilon, s|t_i) = \sum_{i=1}^{n} \bar{u}_i (M^\ast, s|t_i) \); if \( s \in [\hat{s} - \hat{\delta}, \hat{s}] \),

\[
\sum_{i=1}^{n} \bar{u}_i (M^\varepsilon, s|t_i) = \sum_{i=1}^{n} \bar{u}_i (M^\ast, s|t_i) - \int T \sum_{i=1}^{n} J_i (s, t_i) x_i^\ast (t) f(t) \, dt + \int T \sum_{i=1}^{n} J_i (s, t_i) x_i^\varepsilon (t) f(t) \, dt - \varepsilon v_0 (\hat{s} - \hat{\delta}) ;
\]

if \( s > \hat{s} \),

\[
\sum_{i=1}^{n} \bar{u}_i (M^\varepsilon, s|t_i) = v_0 (s) x_0^\varepsilon (s) + \int T \sum_{i=1}^{n} J_i (s, t_i) x_i^\varepsilon (s, t) f(t) \, dt - \int_2^s v'_0 (\hat{s}) x_0^\varepsilon (\hat{s}) \, d\hat{s} - \bar{U}_0 (M^F|\hat{s})
\]

**Lemma 2.6.** When \( \varepsilon \) is small enough, \( M^\varepsilon \) is a safe mechanism.

**Proof.** Verifying that \( M^\varepsilon \) satisfies equation (2.22), monotonicity condition, feasibility condition is straightforward, so I need only to show that for all \( s \), \( \sum_{i=1}^{n} \bar{u}_i (M^\varepsilon, s|t_i) \geq 0 \).

For \( s < \hat{s} - \hat{\delta} \), \( \sum_{i=1}^{n} \bar{u}_i (M^\varepsilon, s|t_i) = \sum_{i=1}^{n} \bar{u}_i (M^\ast, s|t_i) \geq 0 \). For \( s \in [\hat{s} - \hat{\delta}, \hat{s}] \),

\[
\sum_{i=1}^{n} \bar{u}_i (M^\varepsilon, s|t_i) \geq \inf_{s \in [\hat{s} - \hat{\delta}, \hat{s}]} \sum_{i=1}^{n} \bar{u}_i (M^\ast, s|t_i) - \left\{ \int T \sum_{i=1}^{n} J_i (s, t_i) x_i^\ast (t) f(t) \, dt - \int T \sum_{i=1}^{n} J_i (s, t_i) x_i^\varepsilon (t) f(t) \, dt - \varepsilon v_0 (\hat{s} - \hat{\delta}) \right\},
\]

which is positive when \( \varepsilon \) is small enough. For \( s > \hat{s} \),

\[
\sum_{i=1}^{n} \bar{u}_i (M^\varepsilon, s|t_i) = v_0 (s) x_0^\varepsilon (\hat{s}) + \int T \sum_{i=1}^{n} J_i (s, t_i) x_i^\varepsilon (\hat{s}, t) f(t) \, dt - \int_2^s v'_0 (\hat{s}) x_0^\varepsilon (\hat{s}) \, d\hat{s} - \bar{U}_0 (M^F|\hat{s}) = v_0 (\hat{s}) x_0^\varepsilon (\hat{s}) + \int T \sum_{i=1}^{n} J_i (s, t_i) x_i^\varepsilon (\hat{s}, t) f(t) \, dt - \int_2^s v'_0 (\hat{s}) x_0^\varepsilon (\hat{s}) \, d\hat{s} - \bar{U}_0 (M^F|\hat{s})
\]
$$\geq v_0(\hat{s}) x_0^\varepsilon(\hat{s}) + \int_T \sum_{i=1}^n J_i(\hat{s}, t_i) x_i^\varepsilon(\hat{s}, t) f(t) dt$$

$$- \int_{\hat{s}}^{s} v_0'(\tilde{s}) x_0^\varepsilon(\tilde{s}) d\tilde{s} - \bar{U}_0(M^F|\tilde{s})$$

$$= \int_T \sum_{i=1}^n \bar{u}_i(M^\varepsilon, \hat{s}|t_i) \geq 0,$$

in which the first equality and second equality are from the definition of $x^\varepsilon(s, \cdot)$ for $s > \hat{s}$, the inequality is due to that $J_i(s, t_i)$ is increasing in $s$, the last equality holds because at $s = \hat{s}$, $M^\varepsilon$ satisfies equation (2.22).

Here I show that the seller with $s \in [\hat{s} - \hat{\Delta}, \hat{s}]$ gets better off. For $s \in [\hat{s} - \hat{\Delta}, \hat{s}]$, the difference of the seller’s payoffs under $M^\varepsilon$ and $M^*$ is

$$\int_{\hat{s}}^{s} v_0'(\tilde{s}) x_0^\varepsilon(\tilde{s}) d\tilde{s} - \int_{\hat{s}}^{\hat{s}} v_0'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} = \int_{\hat{s} - \hat{\Delta}}^{s} v_0'(\tilde{s}) \tilde{\varepsilon} d\tilde{s} > 0.$$

This again contradicts that $M^*$ is an RSW mechanism. Thus, Condition 2 must be satisfied for $M^*$.

Given that Condition 2 is satisfied by $M^*$, it is easy to show that $x_0^*(s) > x_0^F(s)$ for $s > \hat{s}$. Suppose that this is not true for $s' > \hat{s}$, then the type-$s'$ seller gets her public-information optimal payoff. For the types of the seller lower than $s'$, their payoffs are weakly lower than their public-information optimal payoffs. Thus, some of these types would have an incentive to report their types being $s'$, due to Proposition 2.2. This contradicts that $M^*$ is safe. This completes the proof.

**Proof of Proposition 2.4**

The conditions in Proposition 2.1 and Lemma 2.2 imply that to prove the existence of an RSW mechanism, the key is to prove the existence of a function $x_0^*: S \rightarrow [0, 1]$
that is greater than \( x_0^F : S \to [0,1] \) for all \( s \in S \), increasing in \( s \), and satisfies

\[
v_0(s) + \max_x \left\{ \int T \sum_{i=1}^n [J_i(s,t_i) - v_0(s)] x_i(t) f(t) \, dt : x \text{ satisfies } (2.1) \text{ and } x_0 = x_0^*(s) \right\}
= \int_s^1 v'_0(\bar{s}) \, x_0^*(\bar{s}) \, d\bar{s} + \tilde{U}_0(M^F|s).
\] (2.30)

Once there exists such a \( x_0^* \), we can easily derive a pair of \( x^* \) and \( p^* \) using the conditions in Proposition 2.1 and Lemma 2.2, thus derive an RSW mechanism. Specifically, given \( x_0^*(s) \), \( x^*(s,\cdot) \) solving the maximization problem in (2.30) allocates the object to a buyer with the highest virtual surplus; if the maximum virtual surplus under profile \( t' \) is higher than that under \( t \), then \( \sum_{i=1}^n x_i^*(s,t') \geq \sum_{i=1}^n x_i^*(s,t) \). Thus, characterizing \( x^*(s,\cdot) \) is equivalent to finding a value \( J \) of virtual valuation such that the seller keeps the object if and only if the maximum virtual valuation under a profile \( t \) is lower than \( J \). The value \( J \) should satisfy

\[
\max_k \{ J_k(s,t_k) : 1 \leq k \leq n \} = J^{\min}(s) \leq J \leq J^{\max}(s) = \max_k \{ J_k(s,t_k) : 1 \leq k \leq n \}.
\] (2.31)

In line with this intuition, I define \( r_i(s,J) \) as the minimum type of buyer \( i \) that has positive probability to get the object given \( s \) and \( J \). Thus, if \( J_i(s,t_i) \leq J \leq J_i(s,\bar{t}_i) \), \( r_i(s,J) \) satisfies \( J_i(s,r_i(s,J)) = J \); if \( J_i(s,\bar{t}_i) < J \), \( r_i(s,J) = \bar{t}_i \). The continuity and monotonicity of \( J_i(s,t_i) \) in \( s \) and \( t_i \) imply that \( r_i(s,J) \) is continuously decreasing in \( s \) and continuously increasing in \( J \). The probability that the seller keeps the object given \( J \) is \( \prod_{i=1}^n F_i(r_i(s,J)) \), because the object is left unsold if and only if the virtual valuations of the buyers are all smaller than \( J \). Condition (2.30) can be rewritten as

\[
v_0(s) + \sum_{i=1}^n \int_{r_i(s,J^*)(s)}^{\bar{t}_i} \int_{T_{i-1}(s,t_i)} [J_i(s,t_i) - v_0(s)] f(t) \, dt = \int_s^1 v'_0(\bar{s}) \, x_0^*(\bar{s}) \, d\bar{s} + \tilde{U}_0(M^F|s),
\] (2.32)

\footnote{According to condition (2.31), \( J_i(s,\bar{t}_i) > J \) will not happen.}
where $J^*(s)$ is defined as

$$\prod_{i=1}^{n} F_i (r_i (s, J^* (s))) = x_0^* (s),$$

and $T_{-i} (s, t_i)$ is defined to be the set

$$\left\{ t_{-i} \in T_{-i} : J_i (s, t_i) \geq \max_k \{ J_k (s, t_k) : k < i \}, J_i (s, t_i) > \max_k \{ J_k (s, t_k) : k > i \} \right\}.$$

For any type profile $(t_i, \tilde{t}_{-i})$ with $\tilde{t}_{-i} \in T_{-i} (s, t_i)$, buyer $i$ is the highest indexed agent with the maximum virtual valuation. Thus, equation (2.32) corresponds to an allocation rule that allocates the object to the highest indexed member among the buyers with the maximum virtual valuation.

Proving the existence of an RSW mechanism is reduced to proving the existence of a function $x_0^* : S \to [0, 1]$ that is increasing in $s$, bounded by $x_0^F$ and 1, i.e., $x_0^F (s) \leq x_0^* (s) \leq 1$ for all $s$, and satisfies equation (2.32) for all $s$. To proceed, I define:

$$D (J, s, x_0) = v_0 (s) + \sum_{i=1}^{n} \int_{r_i (s, J)}^{\tilde{t}_{-i}} \int_{T_{-i} (s, t_i)} \left[ J_i (s, t_i) - v_0 (s) \right] f (t) \, dt$$

$$- \left[ \int_{s}^{\tilde{s}} v_0' (\tilde{s}) x_0 (\tilde{s}) \, d\tilde{s} + \tilde{U}_0 (M^F | s) \right]$$

So $D (J, s, x_0)$ is the difference between the LHS and RHS of equation (2.32), given $x_0$, $s$, and $J$. Then I define function $J^{x_0} : S \to R$ with

$$J^{x_0} (s) = \arg \min_J \left\{ |D (J, s, x_0)| : \max \{ J^{\min} (s), v_0 (s) \} \leq J \leq J^{\max} (s) \right\}.$$

That is, $J^{x_0} (s)$ is the value of $J$ that minimizes $|D (J, s, x_0)|$, given $s$ and function $x_0$. Let $C (S)$ denote the set of bounded continuous functions $h : S \to R$ endowed with sup norm, $\| h \| = \sup_{s \in S} | h (s) |$. Thus, $C (S)$ is a Banach space. I use $X_0 (S)$ to represent the set of continuous functions $x_0 : S \to R$ that are increasing and bounded by $x_0^F$ and 1, so
$X_0(S) \subset C(S)$. I define a mapping $\Gamma : X_0(S) \rightarrow C(S)$, with

$$\Gamma x_0(s) = \prod_{i=1}^{n} F_i(r_i(s, J^x_0(s))).$$

Once I show that $\Gamma$ has a fixed point $x^*_0$, and $x^*_0$ satisfies $D(J^x_0(s), s, x^*_0) = 0$ for all $s$, then the existence of an RSW mechanism is proved.

I use Schauder’s fixed point theorem to prove that $\Gamma$ has a fixed point. To use this theorem, we need $X_0(S)$ to be non-empty, convex, and compact. This is proved in the following lemma. After that, I show that $\Gamma$ is a continuous mapping.

Lemma 2.7. The set $X_0(S)$ is non-empty, convex, and compact.

Proof. Non-emptiness and convexity are obvious. Here I show its compactness. It is obvious that $X_0(S)$ is bounded, as each of its element is bounded. I show that $X_0(S)$ is closed to complete the proof of compactness. Let $\{x^n_0\}_{n=1}^{\infty}$ be a sequence in $X_0(S)$ converging to $x_0$, so

$$\lim_{n \rightarrow \infty} x^n_0(s) = x_0(s), \forall s \in S. \quad (2.33)$$

Since for any $n$, $x^n_0(s)$ belongs to the closed interval $[x^F_0(s), 1]$, we have $x_0(s) \in [x^F_0(s), 1]$. Thus, $x_0$ is bounded by $x^F_0$ and 1. Also, $x_0$ is increasing in $s$. Suppose not, then there must exist $s < s'$ with $x_0(s) > x_0(s')$. Due to (2.33), for any $\varepsilon/2 > 0$, there exists $N$, for any $n > N$,

$$|x^n_0(s) - x_0(s)| < \varepsilon/2,$$

or equivalently,

$$x_0(s) - \varepsilon/2 < x^n_0(s) < x_0(s) + \varepsilon/2,$$

and there exists $N'$, for any $n > N'$,

$$|x^n_0(s') - x_0(s')| < \varepsilon/2,$$
or equivalently,
\[ x_0(s') - \varepsilon / 2 < x_0^n(s') < x_0(s') + \varepsilon / 2. \]

For \( n > \max \{ N, N' \} \) and \( \varepsilon < x_0(s) - x_0(s') \), we have
\[ x_0^n(s) - x_0^n(s') > x_0(s) - x_0(s') - \varepsilon > 0, \]
which contradicts that \( x_0^n \) is an increasing function. Since \( x_0^n \) is continuous for any \( n \), then Uniform Limit Theorem implies that \( x_0 \) is continuous. Therefore, \( x_0 \in X_0(S) \), and \( X_0(S) \) is compact. \( \square \)

The mapping \( \Gamma \) maps \( X_0(S) \) into a subset \( \hat{C}(S) \) of \( C(S) \), which includes continuous functions bounded by \( x_0^F \) and 1. To prove this, I show that for any \( x_0 \in X_0(S) \), \( \Gamma x_0 \) is bounded by \( x_0^F \) and 1 and is continuous. Given that \( \max \{ J^\text{min}(s), v_0(s) \} \leq J \leq J^\text{max}(s) \), we have
\[ x_0^F(s) = \prod_{i=1}^{n} F_i \left( r_i(s, \max \{ J^\text{min}(s), v_0(s) \}) \right) \leq \Gamma x_0(s) \leq \prod_{i=1}^{n} F_i \left( r_i(s, J^\text{max}(s)) \right) = 1, \forall s. \]
Thus, \( \Gamma x_0 \) is bounded by \( x_0^F \) and 1. Since \( D(J, s, x_0) \) is continuous in \( J \) and \( s \), and the interval \( [\max \{ J^\text{min}(s), v_0(s) \}, J^\text{max}(s)] \) is compact and continuous in \( s \), \( J^x_0(s) \) is continuous in \( s \) according to the Theorem of the Maximum. This consequently implies that \( \Gamma x_0(s) = \prod_{i=1}^{n} F_i \left( r_i(s, J(s)) \right) \) is continuous in \( s \).

**Lemma 2.8.** The mapping \( \Gamma \) is a continuous mapping.

**Proof.** Consider a converging sequence \( \{ x_0^n \}_{n=1}^\infty \) with \( \lim_{n \to \infty} x_0^n = x_0 \) in sup norm. That is, for any \( \varepsilon > 0 \), there exists \( N > 0 \), for any \( n > N \),
\[ ||x_0^n - x_0|| < \varepsilon. \]
Since $\|\cdot\|$ is sup norm, we have

$$
\left| \int_{\tilde{s}}^{s} \nu'_0 (\tilde{s}) x_0^n (\tilde{s}) d\tilde{s} - \int_{\tilde{s}}^{s} \nu'_0 (\tilde{s}) x_0 (\tilde{s}) d\tilde{s} \right| = \left| \int_{\tilde{s}}^{s} \nu'_0 (\tilde{s}) [x_0^n (\tilde{s}) - x_0 (\tilde{s})] d\tilde{s} \right| < \varepsilon \left[ v_0 (\tilde{s}) - v_0 (s) \right],
$$

and for any $s \in S$,

$$
|D (J, s, x_0^n) - D (J, s, x_0)| < \varepsilon \left[ v_0 (\tilde{s}) - v_0 (s) \right]. \tag{2.34}
$$

Now I show that $J^{x_0^n} \to J^{x_0}$ in sup norm. Let

$$
A = \{(s, J) : s \in S, \text{ and } J \in \left[ \max \{ J^{\min} (s), v_0 (s) \}, J^{\max} (s) \right] \}.
$$

It is obvious that $A$ is compact. I define a subset $A_\varepsilon$ of $A$ by

$$
A_\varepsilon = \{(s, J) \in A : |J - J^{x_0} (s)| \geq \varepsilon \}. \tag{2.35}
$$

The set $A_\varepsilon$ is compact, and for $\varepsilon$ small enough, it is non-empty. The result is trivial when $A_\varepsilon$ is empty. For any $\varepsilon$, let

$$
\delta_\varepsilon = \min_{(s, J) \in A_\varepsilon} \|D (J, s, x_0) - D (J^{x_0} (s), s, x_0)\|.
$$

The continuities of $D$ in $J$ and $s$ and the continuity of $J^{x_0} (s)$ in $s$ ensure the existence of $\delta_\varepsilon$. According to (2.34), for any $\delta > 0$, there exists $N_\delta$, for any $n > N_\delta$,

$$
|D (J, s, x_0^n) - D (J, s, x_0)| < \delta / 2, \tag{2.37}
$$
so

\[
\| D(\mathcal{J}^x_0(s), s, x_0) - D(\mathcal{J}^{x_0}(s), s, x_0) \| \\
= | D(\mathcal{J}^x_0(s), s, x_0) - D(\mathcal{J}^{x_0}(s), s, x_0) | \\
\leq | D(\mathcal{J}^x_0(s), s, x_0) - D(\mathcal{J}^{x_0}(s), s, x_0^n) | + | D(\mathcal{J}^{x_0}(s), s, x_0^n) - D(\mathcal{J}^{x_0}(s), s, x_0) | \\
< \delta.
\]

The equality and first inequality are based on the definitions of \( \mathcal{J}^x_0(s) \) and \( \mathcal{J}^{x_0}(s) \), the second inequality is using the triangle inequality of absolute values. The last inequality is from (2.37). Thus, from (2.35) and (2.36), for \( n > N_\delta \),

\[
| \mathcal{J}^x_0(s) - \mathcal{J}^{x_0}(s) | < \varepsilon, \forall s \in S.
\]

This is equivalent to that \( \mathcal{J}^x_0 \rightarrow \mathcal{J}^{x_0} \) in sup norm. Thus, \( \lim_{n \to \infty} \Gamma x_0^n = \Gamma x_0 \) in sup norm, because

\[
\| \Gamma x_0^n - \Gamma x_0 \| = \sup_{s \in S} \left| \prod_{i=1}^{n} F_i \left( r_i \left( s, \mathcal{J}^x_0(s) \right) \right) - \prod_{i=1}^{n} F_i \left( r_i \left( s, \mathcal{J}^{x_0}(s) \right) \right) \right|.
\]

Therefore, the mapping \( \Gamma \) is continuous.

The mapping \( \Gamma \) does not map \( X_0(S) \) into itself, as we cannot guarantee that \( \Gamma x_0 \) is increasing in \( s \). Here I define another mapping \( \Psi \) over \( \hat{C}(S) \), with \( \Psi h(s) = \sup_{s \in [\underline{s}, \overline{s}]} h(\hat{s}) \), for \( h \in \hat{C}(S) \). It is obvious that \( \Psi h(s) \) is increasing in \( s \). So \( \Psi \) maps \( \hat{C}(S) \) into \( X_0(S) \), and the compound mapping \( \Psi \circ \Gamma \) maps \( X_0(S) \) into itself.

**Lemma 2.9.** The mapping \( \Psi \) is a continuous mapping.

**Proof.** Consider a converging sequence \( \{ h^n \}_{n=1}^{\infty} \) with \( \lim_{n \to \infty} h^n = h \) in sup norm, then
for any $\varepsilon > 0$, there exists $N_\varepsilon$, for any $n > N_\varepsilon$,

$$\|h^n - h\| < \varepsilon,$$

which is equivalent to

$$|h^n(s) - h(s)| < \varepsilon, \forall s \in S. \quad (2.38)$$

According to the definition of $\Psi$, for $s \in S$

$$|\Psi h^n(s) - \Psi h(s)| = \left| \sup_{\hat{s} \in [s, s]} h^n(\hat{s}) - \sup_{\hat{s} \in [s, s]} h(\hat{s}) \right| = |h^n(s') - h(s'')|,$$

where $s'$ and $s''$ are values in $[s, s]$ that maximize $h^n$ and $h$, respectively. If $h^n(s') - h(s'') > 0$, then

$$|h^n(s') - h(s'')| = h^n(s') - h(s'') \leq h^n(s') - h(s');$$

if $h^n(s') - h(s'') \leq 0$, then

$$|h^n(s') - h(s'')| = h(s'') - h^n(s') \leq h(s'') - h^n(s'').$$

Thus, given (2.38), for $n > N_\varepsilon$, we have

$$|\Psi h^n(s) - \Psi h(s)| < \varepsilon, \forall s \in S.$$

That is,

$$\|\Psi h^n - \Psi h\| < \varepsilon.$$

This completes the proof that $\Psi$ is continuous. \hfill \Box
Given all the results above, we have that the compound mapping $\Psi \circ \Gamma$ is continuous and maps from $X_0 (S)$, which is non-empty, convex, and compact, into itself. According to Schauder’s fixed point theorem, $\Psi \circ \Gamma$ has a fixed point on $X_0 (S)$, that is, there exists a $x_0^* \in X_0 (S)$ satisfying

$$x_0^* = \Psi \circ \Gamma x_0^*.$$ 

It is obvious that $x_0^* (\bar{s}) = x_0^F (\bar{s})$, according to (2.30).

The function $x_0^*$ is also a fixed point of $\Gamma$. To prove this, I first show that if for some $s \in S$, $x_0^* (s) = 1$, then $x_0^* (s') = \Gamma x_0^* (s') = 1$ for $s' \geq s$. If $x_0^* (s) = 1$, then there exists $\bar{s} \leq s$, $\Gamma x_0^* (\bar{s}) = 1$ which implies that

$$D (J^{\max} (\bar{s}), \bar{s}, x_0^*) = v_0 (\bar{s}) - \left[ \int_{\bar{s}}^{\text{max}} v_0' (\bar{s}) x_0^* (\bar{s}) d\bar{s} + \bar{U}_0 (M^F | \bar{s}) \right] \geq 0,$$

so for all $\bar{s}' \in (\bar{s}, \bar{s}]$,

$$D (J^{\max} (\bar{s}'), \bar{s}', x_0^*) = v_0 (\bar{s}') - \left[ \int_{\bar{s}}^{\bar{s}'} v_0' (\bar{s}) x_0^* (\bar{s}) d\bar{s} + \bar{U}_0 (M^F | \bar{s}) \right] = D (J^{\max} (\bar{s}), \bar{s}, x_0^*) + \int_{\bar{s}}^{\bar{s}'} v_0' (\bar{s}) [1 - x^* (\bar{s})] d\bar{s} \geq 0,$$

Hence, $\Gamma x_0^* (\bar{s}') = 1 = x_0^* (\bar{s}')$.

Now I show that if $x_0^* (s) < 1$, $x_0^* (s)$ is strictly increasing in $s$. I prove this by contradiction. Suppose that for some $s' < s''$, $x_0^* (s') = x_0^* (s'') < 1$. Let $\bar{s} = \inf \{ \bar{s} \in S : x_0^* (\bar{s}) = x_0^* (s'') \}$. The continuity and monotonicity of $x_0^*$ guarantee the existence of $\bar{s}$. For $s < \bar{s}$, $x_0^* (s) < x_0^* (\bar{s})$, and for all $s \in [\bar{s}, s'']$, $x_0^* (s) = x_0^* (s'')$. Moreover, $x_0^* (\bar{s}) = \Gamma x_0^* (\bar{s}) \geq x_0^F (\bar{s})$, because $x_0^* (s) < x_0^* (\bar{s})$ for $s < \bar{s}$. It is not possible to have $D (J^{x_0^*} (\bar{s}), \bar{s}, x_0^*) = 0$, because if so, for $s \in (\bar{s}, s'']$,

$$v_0 (s) x_0^* (s) + \int_T \sum_{i=1}^n J_i (s, t_i) x_i^* (s, t) f (t) dt - \int_{\bar{s}}^s v_0' (\bar{s}) x_0^* (\bar{s}) d\bar{s} + \bar{U}_0 (M^F | \bar{s})$$
\[ \geq v_0(s) x^*_0(\tilde{s}) + \int_T \sum_{i=1}^n J_i(s, t_i) x^*_i(\tilde{s}, t) f(t) dt - \int_{\tilde{s}}^s v'_0(\tilde{s}) x^*_0(\tilde{s}) d\tilde{s} + \tilde{U}_0(M^F|\tilde{s}) \]

\[ > v_0(s) x^*_0(\tilde{s}) + \int_T \sum_{i=1}^n J_i(\tilde{s}, t_i) x^*_i(\tilde{s}, t) f(t) dt - \int_{\tilde{s}}^s v'_0(\tilde{s}) x^*_0(\tilde{s}) d\tilde{s} + \tilde{U}_0(M^F|\tilde{s}) \]

\[ = D\left(J^{x^0}_0(\tilde{s}), \tilde{s}, x^0_0\right) + [v_0(s) - v_0(\tilde{s})] x^*_0(\tilde{s}) - \int_{\tilde{s}}^s v'_0(\tilde{s}) x^*_0(\tilde{s}) d\tilde{s} \]

\[ = 0. \quad (2.39) \]

In the first line, \( x^*(s, \cdot) \) denote the allocation rule maximizing the virtual surplus given \( x^*_0(s) \). The first inequality is due to the optimality of \( x^*(s, \cdot) \). The second inequality is because \( J_i(s, t_i) \) is strictly increasing in \( s \). The first equality is based on the definition of \( D\left(J^{x^0}_0(\tilde{s}), \tilde{s}, x^0_0\right) \) and \( x^*_0(\tilde{s}) = \Gamma x^*_0(\tilde{s}) \), and the last equality is resulted from \( D\left(J^{x^0}_0(\tilde{s}), \tilde{s}, x^*_0\right) = 0 \) and \( x^*_0(s) = x^*_0(s') \) for all \( s \in [\tilde{s}, s'']. \) This sequence of inequalities implies that \( \Gamma x^*_0(s) > x^*_0(s) \). This contradicts that \( x^*_0 \) is a fixed point of \( \Psi \circ \Gamma \). Thus, it is only possible to have \( D\left(J^{x^0}_0(\tilde{s}), \tilde{s}, x^0_0\right) < 0 \), which indicates that \( \Gamma x^*_0(\tilde{s}) = x^F_0(\tilde{s}) \). Since I have shown that for \( s < \tilde{s}, x^*_0(s) < x^*_0(\tilde{s}) = \Gamma x^*_0(\tilde{s}) \), and \( x^F_0(s) < x^*_0(s), \Gamma x^*_0(\tilde{s}) = x^F_0(\tilde{s}) \) implies \( x^F_0(\tilde{s}) = \Psi x^F_0(\tilde{s}) \).

I define

\[ D(s, x^*_0) = v_0(s) x^*_0(s) + \int_T \sum_{i=1}^n J_i(s, t_i) x^*_i(s, t) f(t) dt - \int_{\tilde{s}}^s v'_0(\tilde{s}) x^*_0(\tilde{s}) d\tilde{s} + \tilde{U}_0(M^F|\tilde{s}), \]

\[ \tilde{s} = \sup \{ s \in [\tilde{s}, \tilde{s}] : D(s, x^*_0) = 0 \} . \]

The set \( \{ s \in [\tilde{s}, \tilde{s}] : D(s, x^*_0) = 0 \} \) is non-empty, as it includes \( \tilde{s} \), so \( \tilde{s} \) exists. The continuity of \( x^*_0(s) \) implies that \( \tilde{s} < \tilde{s} \) and \( D(\tilde{s}, x^*_0) = 0 \), \( D(s, x^*_0) > 0 \), for \( s \in (\tilde{s}, \tilde{s}) \).

I prove that \( x^*_0(s') = \Psi x^F_0(s') \), for all \( s' \in (\tilde{s}, \tilde{s}) \). Suppose not, i.e., \( x^*_0(s') > \Psi x^F_0(s') \) for some \( s' \in (\tilde{s}, \tilde{s}) \). Given that \( x^*_0(s') = \Psi \circ \Gamma x^*_0(s') \), there exists \( s'' \leq s' \), such that \( x^*_0(s') = \Gamma x^*_0(s'') \), thus \( \Gamma x^*_0(s'') > \Psi x^F_0(s') \geq x^F_0(s'') \), which implies that \( D(s'', x^*_0) = 0 \) and \( s'' \leq \tilde{s} \), and \( x^*_0(s) = x^*_0(s'') \) for all \( s \in (s'', s'] \). The argument in the paragraph
containing [2.39] already shows that this is impossible. Therefore, \( x^*_0(s') = \Psi x^F_0(s') \), for all \( s' \in (\hat{s}, \tilde{s}] \).

The continuities of \( x^*_0(s) \) and \( \Psi x^F_0(s) \) imply that \( x^*_0(\hat{s}) = \Psi x^F_0(\hat{s}) \). Then we obtain

\[
D(\hat{s}, x^*_0) - D(\tilde{s}, x^*_0) = v_0(\hat{s}) x^*_0(\hat{s}) + \int_T \sum_{i=1}^n J_i(\hat{s}, t_i) x^*_i(\hat{s}, t) f(t) \, dt - \int_T \sum_{i=1}^n J_i(\tilde{s}, t_i) x^*_i(\tilde{s}, t) f(t) \, dt - \int_{\hat{s}}^{\tilde{s}} v_0(\tilde{s}) x^*_0(\tilde{s}) \, d\tilde{s} \geq 0,
\]

according to (2.25) and Lemma 2.3. This contradicts the supposition that

\[
D(\hat{s}, x^*_0) = D(J(\hat{s}), \tilde{s}, x^*_0) < 0.
\]

This completes the proof that \( x^*_0(s) \) is strictly increasing in \( s \) if \( x^*_0(s) < 1 \). Hence, \( x^*_0(s) = \Gamma x^*_0(s) \) for \( x^*_0(s) < 1 \). Combining with the result above that \( x^*_0(s) = \Gamma x^*_0(s) \) for \( x^*_0(s) = 1 \), \( x^*_0(s) \) is a fixed point of \( \Gamma \).

In the rest of the proof, I show that for \( x^*_0 \),

\[
D(J^*\hat{0}(s), s, x^*_0) = 0, \forall s \in S.
\]

I still prove this by contradiction. First, suppose that there exists \( s' \), \( D(J^*\hat{0}(s'), s', x^*_0) > 0 \), which implies that \( x^*_0(s') = 1 \). The continuities of \( x^*_0 \) and \( D(J^*\hat{0}(s), s, x^*_0) \) in \( s \) indicate that there exists \( \hat{s} < s' \), such that \( x^*_0(\hat{s}) = 1 \) and \( D(J^*\hat{0}(\hat{s}), \hat{s}, x^*_0) = 0 \). The monotonicity of \( x^*_0 \) gives us

\[
D(J^*\hat{0}(s'), s', x^*_0) = v_0(s') - \left[ \int_{\hat{s}}^{s'} v_0(\tilde{s}) x^*_0(\tilde{s}) \, d\tilde{s} + U_0(M^F|\tilde{s}) \right]
\]
\[
\begin{align*}
&= D \left( J^{x_0^*} (\tilde{s}, \tilde{s}, x_0^*) \right) + \int_{\tilde{s}}^{s'} v_0' (\tilde{s}) [1 - x_0^* (\tilde{s})] \, d\tilde{s} \\
&= 0,
\end{align*}
\]

which violates the supposition that \( D \left( J^{x_0^*} (s'), s', x_0^* \right) > 0 \). Second, suppose that there exists \( s'' \), \( D \left( J^{x_0^*} (s''), s'', x_0^* \right) < 0 \), which implies that \( x_0^* (s'') = x_0^* (s''). \) Still, using the continuity of \( x_0^* \) and \( D \left( J^{x_0^*} (s), s, x_0^* \right) \), we obtain that there exists \( \hat{s} < s'' \), such that \( x_0^* (s) = x_0^* (s) \) for \( s \in [\hat{s}, \tilde{s}] \), \( D \left( J^{x_0^*} (\hat{s}), \hat{s}, x_0^* \right) = 0 \), and

\[
\begin{align*}
D \left( J^{x_0^*} (s''), s'', x_0^* \right) - D \left( J^{x_0^*} (\hat{s}), \hat{s}, x_0^* \right) \\
= v_0 (s'') x_0^* (s'') + \int_T \sum_{i=1}^n J_i (s'', t_i) x_i^* (s'', t) \, f (t) \, dt \\
- v_0 (\hat{s}) x_0^* (\hat{s}) + \int_T \sum_{i=1}^n J_i (\hat{s}, t_i) x_i^* (\hat{s}, t) \, f (t) \, dt \\
- \int_{\hat{s}}^{\tilde{s}} v_0' (\tilde{s}) x_0^* (\tilde{s}) \, d\tilde{s} \\
> 0.
\end{align*}
\]

The inequality is due to the fact that the profile \( \{ (x^{F,s}, p^{F,s}) : s \in S \} \) is not an incentive compatible strategy of the seller. (See equation (2.11)). Therefore, \( D \left( J^{x_0^*} (s), s, x_0^* \right) = 0 \) for all \( s \in S \).

**Proof of Proposition 2.5**

Suppose that the seller with type \( s \) deviates from \( (x^* (s, \cdot), p^* (s, \cdot), b^* (s, \cdot)) \) to an off-equilibrium mechanism \( M \) in the mechanism-selection game. The mechanism \( M \) may not be a direct mechanism. (There is always an equilibrium for the continuation game following mechanism \( M \) under belief \( \Pr (s) = 1 \), in which no buyer participates in the mechanism. To make the analysis non-trivial, I assume that upon observing \( M \), all the buyers choose to participate if \( M \) has an equilibrium played by the buyers under belief \( \Pr (s) = 1 \).) According to the revelation principle, we can find a direct mechanism
\( \hat{M} = (\hat{x}, \hat{p}, \hat{b}) \) that is incentive feasible under belief \( \Pr (s) = 1 \) and gives every player the same interim expected payoff as in the equilibrium of the continuation game following \( M \). In \( \hat{M} \), the domain of \( \hat{x}, \hat{p}, \) and \( \hat{b} \) is \( T \). The expected payoff of the type-\( s \) seller under \( \hat{M} \) can be expressed as follows,

\[
 v_0 (s) + \int_{T} \sum_{i=1}^{n} [J_i (s, t_i) - v_0 (s)] \hat{x}_i (t) f (t) \, dt - \sum_{i=1}^{n} U_i (\hat{M} | t_i) - \hat{b},
\]

where \( \hat{b} = \int_{T} \hat{b} (t) f (t) \, dt \). From this expression of expected payoff, we can see that if \( \hat{M} \) maximizes the type-\( s \) seller’s expected payoff under the most pessimistic belief, then \( \hat{M} \) satisfies the following conditions.

1. Allocation rule:

\[
\begin{cases}
\hat{x}_i (t) > 0 \text{ only if } J_i (s, t_i) \geq \max \{ v_0 (s), \max_{k \neq i} \{ J_k (s, t_k) \} \}, \text{ and} \\
\sum_{i=1}^{n} \hat{x}_i (t) = 1 \text{ if } \max_{k} \{ J_k (s, t_k) \} > v_0 (s), \forall i, t, s.
\end{cases}
\]

2. Envelope condition:

\[
U_i (\hat{M} | t_i) = \int_{t_i}^{t_i} \frac{\partial v_i (s, \tilde{t}_i)}{\partial t_i} \hat{x}_i (\tilde{t}_i) \, d\tilde{t}_i + U_i (\hat{M} | t_i), \forall i, t_i.
\]

3. Expected payoff to the lowest type of a buyer and money burning:

\[
\hat{b} = 0, U_i (\hat{M} | t_i) = 0, \forall i.
\]

It is not immediately clear whether the optimal \( \hat{M} \) gives a lower expected payoff to the type-\( s \) seller than \( (x^* (s, \cdot), p^* (s, \cdot), b^* (s, \cdot)) \), given that the buyers report their types truthfully. I take an indirect approach to prove this. Let \( \hat{M} (s) = (\hat{x} (s, \cdot), \hat{p} (s, \cdot), \hat{b} (s, \cdot)) \) denote the optimal direct incentive feasible mechanism for the type-\( s \) seller under belief
\( \Pr (s) = 1 \). According to Milgrom and Segal (2002), for all \( s \),

\[
v_0(s) + \int_0^T \sum_{i=1}^n [J_i(s,t) - v_0(s)] \tilde{x}_i(s,t) f(t) \, dt = \int_0^s v'_0(\tilde{s}) \tilde{x}_0(\tilde{s}) \, d\tilde{s} + U_0(M^F|\tilde{s}) . \tag{2.40}
\]

The left-hand side of equation (2.40) can be rewritten as below,

\[
v_0(s) + \int_0^T \sum_{i=1}^n [J_i(s,t) - v_0(s)] \tilde{x}_i(s,t) f(t) \, dt
\]

\[
- \int_0^T \sum_{i=1}^n [J_i(s,t_i) - J_i(s,t_i)] \tilde{x}_i(s,t) f(t) \, dt,
\]

in which the term \( \int_0^T \sum_{i=1}^n [J_i(s,t_i) - J_i(s,t_i)] \tilde{x}_i(s,t) f(t) \, dt \) is positive, due to Assumption 2.2. Below I construct a safe mechanism \( \tilde{M} = (\tilde{x}, \tilde{p}, \tilde{b}) \) in the mediated game, which gives any type \( s \) of the seller the same payoff as in \( \tilde{M}(s) \).

1. Allocation rule:

\[
\tilde{x}(s,t) = \hat{x}(s,t) , \forall s,t.
\]

2. Expected payoff to the lowest type of a buyer and money burning:

\[
\tilde{u}_i(\tilde{M}, s|t_i) = \int_0^T [J_i(s,t_i) - J_i(s,t_i)] \tilde{x}_i(s,t) f(t) \, dt , \text{ and} \\
\tilde{b}(s,t) = 0 , \forall i,s,t.
\]

3. Payment rule:

\[
\tilde{p}_i(s,t) = v_i(s,t_i) \tilde{x}_i(s,t) - \int_{t_i}^{t_i} v'_{12} (s,\tilde{t}_i) \tilde{x}_i(s,\tilde{t}_i,t_{-i}) \, d\tilde{t}_i - \tilde{u}_i(\tilde{M}, s|t_i) , \forall i,s,t.
\]

It is easy to check that the mechanism \( \tilde{M} \) is safe. Since no safe mechanism can give any type of the seller a higher expected payoff than does an RSW mechanism, \( \tilde{M}(s) \) gives type \( s \) of the seller a lower payoff than does \( (x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot)) \), given that the buyers report their types truthfully. Thus, any type \( s \in S \) of the seller has no incentive to
deviate from mechanism \((x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot))\) to any other mechanism, given the off-equilibrium being \(\Pr(s) = 1\). This implies that \(\{ (x^*(s,\cdot), p^*(s,\cdot), b^*(s,\cdot)) : s \in S \}\) is the equilibrium strategy of the seller in a separating equilibrium of the mechanism-selection game. This separating equilibrium is seller optimal, because otherwise \((x^*, p^*, b^*)\) fails to be an RSW mechanism in the mediated game, which raises a contradiction.
Chapter 3

How to Persuade a Group:
Simultaneously or Sequentially
3.1 Introduction

In practice, a proposal from an agent, such as a new policy, a business idea, or a military plan, usually requires the support of various parties. Typically, the agent has more information about the proposal than the parties have, and is more willing to get the proposal approved; he persuades the parties to support the proposal. The parties may not rely on the information provided by the agent; they can acquire additional information to evaluate the proposal before deciding whether to support it. Here are some examples of this situation.

(1) An entrepreneur needs the investment of a group of venture capitalists to launch his project. He persuades the venture capitalists to invest, and the potential investors conduct due diligence to verify the quality of the project before making investment decision.

(2) A firm planning to introduce a new series of products must persuade its suppliers to make specific investment in producing the associated intermediate inputs. The suppliers can do market research to evaluate the potential of the new products, and make their production decisions afterwards.

(3) To get his/her proposal approved, a politician needs to persuade several interest groups. The interest groups can acquire information regarding the benefits of this proposal before casting their votes.

The persuasion by the agent is crucial for the outcome, as it affects the incentive of the other parties to acquire information, and consequently their probabilities of supporting the proposal. In persuasion, not only is the information the agent transmits to the other parties important, but also the order of persuading the parties matters. The current paper studies the impact of the order of persuasion on the outcome of persuasion.

I use the example of venture capital to illustrate the model. A persuader needs the investment of two listeners to launch his project. The expected return of the project to
the listeners is privately known to the persuader. The two listeners have different outside options for investment: one has an outside option with a high return and the other has an outside option with a low return. Before the listeners make their investment decisions, the persuader persuades the listeners through “soft evidence” (costless and unsubstantiated messages), and the listeners can imperfectly verify the return to the project with effort after being persuaded: The more effort a listener exerts, the higher the probability she learns the return to the project.

The analysis focuses on comparing two persuasion modes from the persuader’s perspective: simultaneous public persuasion and sequential private persuasion. Simultaneous public persuasion (henceforth, public persuasion) is a mode in which all the listeners are brought together and persuaded simultaneously through a publicly observed message. Sequential private persuasion (henceforth, sequential persuasion) is a mode in which the listeners are persuaded one by one through private messages, and listeners approached later can observe the decisions of the former listeners, but not the messages they received. Admittedly, to have a more general comparison between simultaneous persuasion and sequential persuasion, the persuader should be allowed to communicate with the listeners through both public and private messages. I restrict attention to the two persuasion modes above for the following reasons. First, the goal of this paper is to illustrate how the order of persuasion can potentially affect the outcome of persuasion, not to do a comprehensive comparison between simultaneous persuasion and sequential persuasion. Second, assuming different communication protocols for different orders of persuasion is to capture the impact of the orders of persuasion on the symmetry of information between the listeners before they make their decisions: The listeners are symmetrically informed when making decisions simultaneously and receive different information when making decisions sequentially.

In both persuasion modes, there are equilibria where only the pickier listener verifies the project upon being persuaded and invests in the project when she finds the return
is better than her outside option. The less picky listener always agrees to invest in such an equilibrium, as her verification has no value given the strategy of the pickier listener. The persuasion modes are equivalent in terms of outcome in such equilibria.

For some parameter values, there exist equilibria where both listeners verify the project (henceforth, joint-verification equilibria), with the pickier listener aiming to prevent a project worse than her outside option from being financed and the less picky listener aiming to prevent a project better than her outside option from not being financed. The persuader wants the less picky listener to exert more verification effort and the pickier listener to exert less verification effort. The persuasion mode matters in such equilibria. In sequential persuasion, it is optimal for the persuader to first persuade the pickier listener, because compared with the opposite order, the pickier listener exerts less verification effort and the less picky listener exerts more verification effort.

The verification effort of the pickier listener in joint-verification equilibria is small, for otherwise the less picky listener would prefer to rely on the pickier listener's verification. When the verification costs of the listeners are low, public persuasion can outperform sequential persuasion, because in the latter mode, joint-verification equilibria do not exist: the pickier listener exerts too much verification effort. When the verification costs are high, joint-verification equilibria exist in both persuasion modes. In this case, the free-riding effect in public persuasion can make the less picky listener exert less verification effort than in sequential persuasion, and sequential persuasion consequently outperforms public persuasion.

This paper is closely related to Caillaud and Tirole (2007). Their paper emphasizes the roles of persuasion cascades and selective communication in group persuasion. They find that it may be optimal for the persuader to selectively communicate with some of the listeners, and then use the approval of these listeners to sway the decisions of others. Unlike the current paper, they assume that (1) the persuader is an “information gatekeeper”, who determines whether the listeners can verify the project, and (2) the
listeners can learn the quality of the project perfectly with a fixed cost. In their model, the order of persuasion is important for inducing persuasion cascades, but as long as the same number of listeners verify the project in equilibrium, the probability of launching the project is independent of the order of persuasion. I show that the order of persuasion affects the probability of launching the project when the two assumptions in their paper are relaxed. I also assume that the persuader privately knows the quality of his project.

Most of the literature on persuasion focuses on the case of one speaker and one listener (e.g., Milgrom 1981; Crawford and Sobel 1982; Glazer and Rubinstein 2004). One-speaker/multiple-listener persuasion, which is common in practice, has attracted less attention. Important papers on multiple-listener persuasion include Farrell and Gibbons (1989), Goltsman and Pavlov (2011), Chakraborty and Harbaugh (2010), and Koessler (2008). These papers do not consider the effects of the order of persuasion and assume that the information used by the listeners to make decisions is provided only by the persuader. The first three papers model persuasion as (costless, non-verifiable) cheap talk. Farrell and Gibbons (1989) and Goltsman and Pavlov (2011) study the differences between public communication and private communication. In their models, listeners are independent decision makers, that is, their payoffs are independent of the actions of others. Chakraborty and Harbaugh (2010) is devoted to analyzing the informativeness of persuasion modeled as multidimensional cheap talk. The analysis of Koessler (2008) is similar to the first two papers, except that persuasion is modeled as communication through certifiable messages.

3.2 One-listener Case

In this section, I analyze the game of persuasion with one listener. The analysis of this section facilitates the analysis of the multiple-listener case.
3.2.1 Setup

There are two players, Persuader (he) and Listener (she). Persuader wants to launch a project, which requires the investment of Listener. Persuader is privately informed about the payoff $\omega$ of Listener from financing the project, where $\omega \in \Omega = [0, 1]$. The prior belief of Listener about $\omega$ is density $f : \Omega \to \mathbb{R}$, which is common knowledge and is continuous and has full support on $\Omega$. Listener is a risk-neutral expected payoff maximizer and has an outside option $R$, so she invests in the project only if the expected value of $\omega$ is strictly larger than $R$. I assume that

$$E[\omega] < R < 1,$$  \hspace{1cm} (3.1)

which means that Listener does not invest in the project, without further information on $\omega$.

Before Listener makes her investment decision, Persuader can persuade, and Listener has the opportunity to investigate and acquire information on $\omega$ after being persuaded. Specifically, in the persuasion stage, Persuader firstly costlessly sends a message $s \in S$ to Listener, where $S$ is a finite message space. Then Listener, based on her posterior belief, chooses her investigation intensity $\alpha \in [0, 1]$, which is the probability of revealing the true value of $\omega$. (With probability $1 - \alpha$, Listener learns nothing from her investigation.)

The cost of investigation for Listener is $c(\alpha)$, which satisfies

$$c(0) = 0, \, c'(0) = 0, \, c'(1) \geq 1, \text{ and } c''(\alpha) > 0 \text{ for all } \alpha.$$  \hspace{1cm} (3.2)

The assumption $c'(1) \geq 1$ makes sure that $\alpha < 1$ in equilibrium. Listener makes investment decision $d \in \{0, 1\}$ after the investigation, where 1 and 0 denote invest and not invest, respectively.
3.2.2 Equilibria

Throughout this paper, I use weak perfect Bayesian equilibrium (wPBE) as the equilibrium concept and consider only pure-strategy equilibria. An equilibrium strategy profile consists of the following elements,

1. The persuasion strategy of Persuader: $p : \Omega \rightarrow S$. This strategy specifies the message reported by each type of Persuader. For example, $p(\omega') = s'$ means that Persuader reports $s'$ if his type is $\omega'$.

2. The strategy of Listener, which includes two parts:

   - Investigation strategy, $\alpha : S \rightarrow [0,1]$. This strategy specifies the investigation intensity of Listener for each message received.

   - Investment strategy, $d : \Omega \sqcup \{\phi\} \rightarrow \{0,1\}$, where $\phi$ means that Listener observes nothing from investigation. This strategy specifies the investment decision of Listener under each possible investigation outcome.

For a wPBE, we need to specify a belief system consistent with the equilibrium strategies. Beliefs of the players on the equilibrium path can be easily constructed using Bayes’ rule, but beliefs off the equilibrium path cannot. In an equilibrium of this game, three types of information sets may be reached with probability 0 (i.e., off the equilibrium path), which are that (1) Listener receives a message not supposed to be reported by any $\omega$ under the persuasion strategy, (2) Listener finds that she chose an investigation intensity that she did not plan to choose, and (3) Listener finds, through her investigation, a value of $\omega$ that is not supposed to report the message she received under the persuasion strategy. The information sets of type (3) include only one value of $\omega$, so I assume that Listener assigns probability 1 to this value. For the information sets of type (2), I assume that if nothing is revealed from investigation, Listener maintains her belief about $\omega$ before the investigation; if a value of $\omega$ is revealed from investigation, Listener assigns
probability 1 to that value. For the information sets of type (1), I assume that Listener holds a posterior belief about $\omega$ inducing her to reject the project.

To begin, I find the equilibrium strategy of Listener, constructing it by backward induction. Suppose that the equilibrium persuasion strategy of Persuader is $p$. Let $f_p(\cdot|s)$ denote the posterior belief of Listener about $\omega$ when she observes message $s$, and $F_p(\cdot|s)$ the corresponding CDF. If she chooses $\alpha$ as her investigation intensity, then her expected payoff is

$$ Eu_p(\alpha|s) = \alpha \int_R^1 \omega f_p(\omega|s) \, d\omega + \alpha F_p(R|s) \, R + (1 - \alpha) \max \{ E_p[\omega|s], R \} - c(\alpha), \quad (3.3) $$

where $E_p[\omega|s]$ is the expected value of $\omega$ under posterior $f_p(\cdot|s)$. The first term on the RHS of $(3.3)$ is the payoff of Listener when $\omega$ is revealed to be above $R$ and she invests. The second term is the payoff of Listener when $\omega$ is revealed to be smaller than $R$ and she rejects the project. The third term is the payoff when the value of $\omega$ is not identified and Listener chooses whether to invest (obtains $E_p[\omega|s]$) or not invest (obtains $R$). With the assumptions on $c(\cdot)$, the value of $\alpha$ maximizing $Eu_p(\alpha|s)$ is uniquely defined and is smaller than 1.

Now we look at the persuasion strategy of Persuader. For any persuasion strategy $p$, the law of iterated expectation (LIE) gives

$$ \sum_{s_i \in \{s \in S : p^{-1}(s) \neq \emptyset\}} E_p[\omega|s_i] \Pr(s_i|p) = E[\omega], \quad (3.4) $$

where $p^{-1}(s) = \{\hat{\omega} \in \Omega : p(\hat{\omega}) = s\}$ is the set of $\omega$ reporting $s$ under persuasion strategy $p$, and $\Pr(s_i|p) = \int_{p^{-1}(s_i)} f(\hat{\omega}) \, d\hat{\omega}$ is the probability of observing $s_i$ under strategy $p$.

For any message $s_i$ reported by some type(s) of Persuader under strategy $p$, i.e., $s_i \in \{s \in S : p^{-1}(s) \neq \emptyset\}$, $E_p[\omega|s_i] \leq R$ holds. This result means that after receiving $s_i$ in equilibrium, if Listener has no further information about the project, she rejects it. Intuitively, if there is a message $s_j$ under strategy $p$ satisfying $E_p[\omega|s_j] > R$, no
Persuader will report a message $s_i$ with $E_p[\omega|s_i] \leq R$, i.e., $\Pr(s_i|p) = 0$ for any $s_i$ with $E_p[\omega|s_i] \leq R$, because (1) any $\omega > R$ will investment for sure when reporting $s_j$, and (2) any $\omega \leq R$ will have positive probability of getting investment when reporting $s_j$ (in the case where Listener finds no further information from her investigation), but 0 probability of getting investment when reporting $s_i$. This contradicts $3.4$.

### 3.2.3 Optimal Equilibrium

I am interested in characterizing the optimal equilibrium for Persuader, which maximizes the ex ante expected probability of launching the project. I first show that when characterizing the optimal equilibrium, we can focus on equilibria in which the number of messages that may be reported by Persuader is no more than 2, i.e.,

$$\# \{s \in S : \ p^{-1}(s) \neq \emptyset\} \leq 2.$$

Let $s_1, s_2, \ldots, s_I$ denote the messages inducing positive investigation intensities of Listener in equilibrium. I define $S_+ \equiv \{s_1, s_2, \ldots, s_I\}$, so $S_+$ is the set of messages that induce investigation in equilibrium. In any equilibrium, $S_+$ is non-empty. For $s_i, s_j \in S_+$, $\alpha(s_i) = \alpha(s_j)$ holds, because if $\alpha(s_i) \neq \alpha(s_j)$, then a Persuader with $\omega \in (R, 1]$ always reports the message inducing the highest investigation, given that Listener invests only if she finds the project desirable; it is therefore impossible to have both $\alpha(s_i) > 0$ and $\alpha(s_j) > 0$, i.e., $s_i, s_j \in S_+$.

**Lemma 3.1.** For an equilibrium in which $S_+$ has more than two messages, there is another equilibrium having $\# \{s \in S : \ p^{-1}(s) \neq \emptyset\} \leq 2$, but generating the same expected payoffs for Persuader and Listener.

This lemma indicates that when characterizing the optimal equilibrium, we can focus on equilibria with at most two equilibrium messages. The reason is simple. For different types of Persuader reporting different messages but facing the same response of Listener, grouping them together and having all of them report identically would induce the same response of Listener as before.
Suppose that in an equilibrium, message $s_+$ induces a positive investigation intensity $\alpha(s_+)$. It is clear that all $\omega > R$ report $s_+$, and Listener invests only if she identifies them from investigation after receiving $s_+$; thus, the \textit{ex ante} expected probability of launching the project is $\alpha(s_+)[1 - F(R)]$. The optimal equilibrium has the maximum value of $\alpha(s_+)$. 

The investigation of Listener when receiving $s_+$ is to prevent a project with $\omega > R$ from being rejected. The incentive for Listener to investigate is therefore increasing in the probability that $\omega > R$. This implies that to maximize $\alpha(s_+)$, the probability that Persuader with $\omega \leq R$ reports $s_+$ should be minimized while maintaining $E_p[\omega|s_+] \leq R$. The next proposition shows that pooling the worse projects with the ones having $\omega > R$ achieves the objective. Proofs are in the Appendix.

\textbf{Proposition 3.1.} \textit{Let $s_+$ denote the message inducing a positive investigation intensity of Listener in equilibrium. In the optimal equilibrium, $p^{-1}(s_+) = [0, \bar{\omega}^*] \cup (R, 1]$, where $\bar{\omega}^*$ satisfies}
\begin{equation*}
E[\omega|\omega \in [0, \bar{\omega}^*] \cup (R, 1)] = R.
\end{equation*}
\textit{That is, the set of $\omega$ reporting $s_+$ has the form $[0, \bar{\omega}^*] \cup (R, 1]$.}

\section{3.3 Public Persuasion}

The rest of this paper discusses persuasion involving two heterogeneous listeners. The analysis focuses on comparing two modes of persuasion, simultaneous public persuasion, in which both listeners are brought together and persuaded simultaneously through a publicly observable message, and sequential persuasion, in which the listeners are persuaded one by one through private messages. I examine which mode of persuasion is better from Persuader’s perspective.

For expositional purposes, I start with public persuasion. This persuasion mode is more closely related to the one-listener case.
3.3.1 Setup

There are three players, Persuader, Listener 1, and Listener 2. Launching the project of Persuader requires the investment of both listeners. If both listeners invest, each of them gets payoff $\omega \in \Omega = [0, 1]$. If the project is not launched, Listener $i$ receives $R_i$, $i = 1, 2$.

The value of $\omega$ is private to Persuader as in the one-listener case. I still use the density $f : \Omega \rightarrow \mathbb{R}$ to denote the prior belief of the listeners about $\omega$. It has the same features as in the one-listener case. To make the analysis interesting, I assume that

$$E[\omega] < R_2 < R_1 < 1.$$ 

This assumption ensures that no listener invests without future information on $\omega$, and Listener 1 is pickier than Listener 2 in investing.

The public-persuasion game also begins with a persuasion stage in which Persuader sends a public message $s \in S$ to the listeners costlessly. The persuasion strategy can be expressed as $p : \Omega \rightarrow S$ as in the one-listener case. After receiving the message $s$, the listeners simultaneously make their decisions on investigation. The investigation strategy of Listener $i$ is a function $\alpha_i : S \rightarrow [0, 1]$, $i = 1, 2$. Listener $i$ pays cost $c_i(\alpha_i)$ for intensity $\alpha_i$. The cost function $c_i(\cdot)$ satisfies the conditions in (3.2). The investment decision of a listener is based on her investigation outcome and the strategy of the other listener. I assume that listeners’ investigation outcomes and the investment decisions are unobservable to each other.

3.3.2 Equilibria

In this subsection, I describe several representative equilibria to illustrate how the players typically interact in the public-persuasion game.

Since launching the project of Persuader requires the investment of both listeners, there is a type of trivial equilibria in which both listeners always reject the project.
without investigation. This type of equilibria generates the worst outcome for Persuader.

1. **Unilateral Investigation**

There are equilibria in which Listener 1, the pickier listener, is the only listener bearing the burden of investigating the project. I illustrate such an equilibrium below. The equilibrium strategy of Persuader, $p$, can be represented using Figure 3.1.

If the project has $\omega \in [0, \omega') \cup (R_1, 1]$, Persuader reports $s_1$. Otherwise, he reports $s_2$. The equilibrium strategy of Listener 1 is that upon observing $s_1$, she investigates the project and invests in it only if $\omega$ is identified to be larger than $R_1$; if $s_2$ is observed, she rejects the project without investigation.\footnote{As in the one-listener case, to have Listener 1 reject the project when she obtains no information from investigation, the choice of $\omega'$ should guarantee that $E_p[\omega|s_1] \leq R_1$.} For Listener 2, she invests in the project without investigation when receiving $s_1$ and rejects the project when observing $s_2$.

In such an equilibrium, Persuader essentially faces only one listener, Listener 1. The existence of such equilibria is intuitive. Listener 1 is pickier than Listener 2; if Listener 1 invests only if she finds $\omega$ is above $R_1$, then doing investigation has no value for Listener 2.

2. **Joint Investigation**

In the type of equilibria described above, a project with $\omega \leq R_1$ is never launched, as Listener 1 invests in the project only if $\omega > R_1$. Below I present an equilibrium in which a project with $\omega \in (R_2, R_1]$ has a positive probability of being launched. For such
Figure 3.2: Public Persuasion Strategy under Joint Investigation

an equilibrium to exist, we require $E[\omega|\omega > R_2] > R_1$, which means that the preferences of the listeners are sufficiently aligned.

The equilibrium strategy of Persuader, $p$, can be represented using Figure 3.2. If $\omega \in (\omega'', R_2]$, Persuader reports $s_2$. Otherwise, he reports $s_1$.

The equilibrium strategy of Listener 1 is that when $s_1$ is received, she investigates with intensity $\alpha_1(s_1)$, and rejects the project only if $\omega$ is revealed to be smaller than $R_1$; if $s_2$ is received, she rejects the project without investigation. For Listener 2, if $s_1$ is received, she investigates with intensity $\alpha_2(s_1)$ and invests in the project only if $\omega$ is revealed to be above $R_2$; if $s_2$ is received, she rejects without investigation. Thus, there is joint investigation under message $s_1$, but no investigation under $s_2$.

In this equilibrium, a project with $\omega \in (R_2, R_1]$ has probability $[1 - \alpha_1(s_1)]\alpha_2(s_1)$ of being launched, though it gives Listener 1 an expected return lower than $R_1$. The existence of this equilibrium relies on the condition $E[\omega|\omega > R_2] > R_1$, which allows Listener 1 to rely on Listener 2’s investigation when she learns nothing from her investigation.

The existence of such an equilibrium also requires that $\alpha_1(s_1)$ not be too high, otherwise Listener 2 would be willing to invest in the project when she learns nothing from her investigation, as given a large $\alpha_1(s_1)$, a project receiving Listener 1’s investment is very likely to have $\omega > R_1$. This constraint for $\alpha_1$ imposes additional requirements for the primitives and the choice of $\omega''$. I will discuss these requirements in more detail when characterizing the optimal equilibria.
3.3.3 Equilibrium Selection

In the equilibria introduced above, the communication between Persuader and the listeners is informative and has a simple structure: when persuading the listeners, Persuader pools the extreme types, but truthfully reveals the moderate undesirable types. In the rest of the paper, I call equilibria with informative persuasion persuasive equilibria. The analysis of this paper is focused on persuasive equilibria.

Theoretically, due to the cheap-talk nature of the persuasion in the model, there can be more than two messages in equilibrium that induce various patterns of interaction between the listeners. But in characterizing the optimal equilibrium of public persuasion, the findings below imply that without loss of generality we can focus on the equilibria whose persuasion strategies are as simple as those in the preceding subsection.

To proceed, I first list what patterns of listeners’ interaction can be induced by messages in a persuasive equilibrium.

**Lemma 3.2.** In a persuasive equilibrium, a message on the equilibrium path induces one of the following patterns of interaction between the listeners.

1. At least one of the listeners rejects the project without investigation.

2. Listener 1 conducts investigation and accepts the project only if $\omega$ is identified to be above $R_1$. Listener 2 invests in the project without investigation.

3. Both listeners investigate the project, with Listener 2 accepting the project only if $\omega$ is identified to be above $R_2$ and Listener 1 accepting the project if $\omega$ is not identified to be smaller than $R_1$.

The proof of this lemma is mechanical. When observing a message, a listener has four alternative strategies: (1) to accept the project without investigation, (2) to reject it

---

2When $E[\omega|\omega > R_2] > R_1$ holds, there may exist an equilibrium in which both listeners investigate the project when receiving a message from Persuader and invest in the project unless they find that $\omega$ is lower than their outside options. However, in this equilibrium, there is essentially no informative communication between Persuader and the listeners: all types of Persuader report the same message to the listeners. I omit this non-persuasive equilibrium.
without investigation, (3) to investigate and accept it if $\omega$ is not identified to be smaller than her outside option, and (4) to investigate and reject it if $\omega$ is not identified to be larger than her outside option. Thus, there are $4 \times 4 = 16$ possible strategy profiles for this two-listener game. We can show that only the profiles listed in the lemma are possible in a persuasive equilibrium.

Different messages in an equilibrium can induce the same pattern of interaction between the listeners. As for Lemma 3.1, we can show that pooling all the types of Persuader facing the same pattern of interaction between the listeners and having them report identically will not change the responses of the listeners, thus generating the same expected probability of success for Persuader and the same \textit{ex ante} expected payoffs for the listeners. Combining this argument with Lemma 3.2, we have the following lemma.

**Lemma 3.3.** For an equilibrium with multiple messages inducing the same pattern of interaction between the listeners, we can find another equilibrium in which there are at most three equilibrium messages, with different messages inducing different patterns of interaction between the listeners, and each player has the same \textit{ex ante} expected payoff as in the original equilibrium.

### 3.3.4 Optimal Equilibrium

In this subsection, I characterize the optimal equilibrium of the public-persuasion game. Lemma 3.3 enables us to focus the analysis on equilibria with no more than three equilibrium messages and different messages inducing different interaction patterns.

I discuss separately three types of equilibria: (1) equilibria in which only one message induces investigation and Listener 1 is the only investigator, (2) equilibria in which only one message induces investigation and both listeners investigate, and (3) equilibria in which two messages induce investigation, with one inducing only Listener 1 to investigate and the other inducing both listeners to investigate. I call the first type of equilibria unilateral-investigation equilibria, the second type of equilibria joint-investigation equi-
libria, and the third type of equilibria mixed-investigation equilibria.

In a unilateral-investigation equilibrium, the strategic situations facing Listener 1 and Persuader are essentially the same as those in the one-listener case, so the analysis in the one-listener game applies. I use \( p^{PU} \) to denote the strategy of Persuader and \( p^{-1,PU} \) to denote the inverse of \( p^{PU} \), where \( P \) and \( U \) in the superscript respectively represent “public persuasion” and “unilateral investigation”. Let \( s_+ \) still denote the message inducing investigation. According to Proposition 3.1, we can conclude that in the optimal unilateral-investigation equilibrium, \( p^{-1,PU} (s_+) = [0, \bar{\omega}^{PU}] \cup (R_1, 1] \), where \( \bar{\omega}^{PU} \) satisfies \( E[\omega|\omega \in [0, \bar{\omega}^{PU}] \cup (R_1, 1)] = R_1 \). The equilibrium investigation intensity \( \alpha_{1}^{PU} (s_+) \) of Listener 1 satisfies

\[
\alpha'(\alpha_{1}^{PU} (s_+)) = \frac{\int_{R_1}^{1} (\omega - R_1) f(\omega) d\omega}{1 - F(R_1) + F(\bar{\omega}^{PU})},
\]

according to (3.3). The ex ante expected probability of launching the project is

\[
\alpha_{1}^{PU} (s_+) [1 - F(R_1)].
\]

Now I examine the joint-investigation equilibria. From Lemma 3.2, we know that in the case where both listeners investigate, Listener 2 accepts the project only if \( \omega \) is identified to be above \( R_2 \), while Listener 1 accepts the project unless \( \omega \) is identified to be smaller than \( R_1 \). For such an equilibrium to exist, a necessary condition is \( E[\omega|\omega > R_2] > R_1 \).

Let \( p^{PJ} \), \( \alpha_{1}^{PJ} \), and \( \alpha_{2}^{PJ} \) denote respectively the persuasion strategy of Persuader, the investigation strategy of Listener 1, and the investigation strategy of Listener 2, where \( P \) and \( J \) in the superscript respectively represent “public persuasion” and “joint investigation”. The message \( s_+ \) still represents the one inducing investigation. It is clear that all \( \omega > R_2 \) report \( s_+ \). According to the discussion on joint-investigation equilibria in
Subsection 3.3.2, the equilibrium investigation intensity $\alpha_1^{PJ} (s_+)$ should be small enough so that Listener 2 would like to reject the project when she learns nothing from her investigation. Thus, we require that

$$\alpha_1^{PJ} (s_+) \leq \bar{\alpha}_{1p}^{PJ} (s_+),$$

where $\bar{\alpha}_{1p}^{PJ} (s_+)$ is the investigation intensity of Listener 1 making Listener 2 indifferent between investing and not investing when she learns nothing from her investigation, under persuasion strategy $p^{PJ}$. I prove in the Appendix that given $p^{PJ}$, there is at most one pair $(\alpha_1^{PJ}, \alpha_2^{PJ})$ that satisfies (3.7) forming an equilibrium.

It is obvious that all $\omega > R_2$ report $s_+$, so $p^{-1,PJ} (s_+)$, the set of Persuader types reporting $s_+$ under strategy $p^{PJ}$, can be uniquely represented by $p^{-1,PJ} (s_+ \cap [0, R_2])$, i.e., the set of $\omega$ that is (weakly) smaller than $R_2$ and reports $s_+$. The next proposition shows that we can focus on the cases where $p^{-1,PJ} (s_+ \cap [0, R_2]$ is an interval of the form $[0, \bar{\omega}]$ when characterizing the optimal equilibrium.

**Lemma 3.4.** For any joint-investigation equilibrium, there is a joint-investigation equilibrium that generates the same probability of launching the project and whose $p^{-1,PJ} (s_+)$ has the form $p^{-1,PJ} (s_+) = [0, \bar{\omega}] \cup (R_2, 1]$, with $\bar{\omega} \leq R_2$.

I briefly explain the intuition behind this proposition. In a joint-investigation equilibrium, under message $s_+$, the reason that Listener 2 investigates is to prevent a project with $\omega > R_2$ from being rejected, while the reason that Listener 1 investigates is to reduce the chance that a project with $\omega \in (R_2, R_1]$ is launched. If in two equilibria, the posterior beliefs of the listeners for both $\omega > R_2$ and $\omega \in (R_2, R_1]$ under message $s_+$ are the same, then the investigation intensities of the listeners in these two equilibria are the same. Therefore, as long as the probability measure of $p^{-1,PJ} (s_+ \cap [0, R_2]$ is the same in two equilibria, the two equilibria are equivalent in outcome from Persuader’s perspective. However, if $p^{-1,PJ} (s_+ \cap [0, R_2]$ has the interval form $[0, \bar{\omega}]$, the value of
\(\bar{\alpha}_{1p} (s_+)\) is larger, which makes the constraint (3.7) easier to be satisfied.

It is worth mentioning that given the reasons that Listeners 1 and 2 investigate upon receiving message \(s_+\), \(\alpha_1^{PJ} (s_+)\) is increasing in \(\alpha_2^{PJ} (s_+)\), because increasing \(\alpha_2^{PJ} (s_+)\) increases the probability that a projected \(\omega \in (R_2, R_1]\) receives the investment of Listener 2. But \(\alpha_2^{PJ} (s_+)\) is decreasing in \(\alpha_1^{PJ} (s_+)\), because increasing \(\alpha_1^{PJ} (s_+)\) reduces the probability that a project with \(\omega \in (R_2, R_1]\) is launched.

The optimal joint-investigation equilibrium can be characterized by solving the following maximization problem.

\[
\max_{\omega \leq R_2} \alpha_2^{PJ} (s_+) \left[ 1 - \alpha_1^{PJ} (s_+) \right] \left[ F (R_1) - F (R_2) \right] + \alpha_2^{PJ} (s_+) \left[ 1 - F (R_1) \right] \quad (3.8)
\]

s.t. \(\alpha_1^{PJ} (s_+) \leq \bar{\alpha}_{1p} (s_+)\).

As I discussed above, a joint-investigation equilibrium may not exist, even if \(E [\omega | \omega > R_2] > R_1\). The investigation intensity \(\alpha_1^{PJ} (s_+)\) is decreasing in \(\bar{\omega}\), and \(\bar{\alpha}_{1p} (s_+)\) is increasing in \(\bar{\omega}\). Thus, the difference \(\alpha_1^{PJ} (s_+) - \bar{\alpha}_{1p} (s_+)\) reaches its minimum when \(\bar{\omega} = R_2\). I define

\[
\varphi^{PJ} (c_1, c_2, R_1, R_2, f) \equiv \min_{\bar{\omega} \leq R_2} \alpha_1^{PJ} (s_+) - \bar{\alpha}_{1p} (s_+),
\]

which is a function of the primitives of the model. Therefore, there exists a joint-investigation equilibrium if and only if

\[
\varphi^{PJ} (c_1, c_2, R_1, R_2, f) \leq 0. \quad (3.9)
\]

I say that a marginal cost function \(c_i^\prime\) is larger than another marginal cost function \(c_i^\prime\) if \(c_i^\prime (\alpha) \geq c_i^\prime (\alpha)\) for all \(\alpha \in [0, 1]\). Given \(c_2, R_1, R_2, \) and \(f\), the existence condition (3.9) is satisfied if the marginal cost function of Listener 1, \(c_1^\prime\), is large enough. This is because as \(c_1^\prime\) becomes larger, \(\alpha_1^{PJ} (s_+)\) becomes smaller for any \(\bar{\omega} \leq R_2\). Also, if \(c_1, R_1, R_2, \) and
If condition (3.9) is satisfied with equality, then there is only one joint-investigation equilibrium in which $p^{-1,PJ}(s_+) = \Omega$, and this equilibrium is optimal. If condition (3.9) is a strict inequality, then there is a set $[\bar{\omega}, R_2]$ with $\bar{\omega}$ satisfying $\alpha_1^{PJ}(s_+) - \bar{\alpha}_1^{PJ}(s_+) = 0$, and each $\bar{\omega} \in [\bar{\omega}, R_2]$ corresponds to a joint-investigation equilibrium. Since the objective function in (3.8) is continuous in $\bar{\omega}$, and $[\bar{\omega}, R_2]$ is compact, there is an optimal $\bar{\omega}^{PJ}$. It is indeterminate whether $\bar{\omega}^{PJ}$ is an interior solution or on the boundary of $[\bar{\omega}, R_2]$.

I compare the optimal joint-investigation equilibrium with the optimal unilateral-investigation equilibrium. Below I provide two sufficient conditions under which the former equilibrium outperforms the latter. These two conditions are identical when $c_1$ and $c_2$ are quadratic.

**Proposition 3.2.** When $E[\omega|\omega > R_2] > R_1$ and (3.9) hold, the optimal joint-investigation equilibrium outperforms the optimal unilateral-investigation equilibrium if

$$
\varphi_2 \left( \int_{R_1}^1 (\omega - R_2) f(\omega) d\omega \right) \geq \varphi_1 \left( \frac{\int_{R_1}^1 (\omega - R_1) f(\omega) d\omega}{1 - F(R_1) + F(\bar{\omega}^{PU})} \right),
$$

where $\varphi_i$ is the inverse of $c_i'$, $i = 1, 2$.

Comparing (3.6) and (3.8), we can see that if $\alpha_2^{PJ}(s_+) > \alpha_1^{PU}(s_+)$, then the optimal joint-investigation equilibrium outperforms the optimal unilateral-investigation equilibrium. The RHS of (3.10) is the investigation intensity of Listener 1 in the optimal unilateral-investigation equilibrium, which is solved from (3.5). The LHS of (3.10) is a lower bound of $\alpha_2^{PJ}(s_+)$, given that $\alpha_2^{PJ}(s_+)$ is decreasing in $\alpha_1^{PJ}(s_+) < 1$ and in $\bar{\omega} \leq R_2$. (See the Appendix for more details.)

The sufficient condition provided in the next proposition imposes more restrictions on the cost functions. This proposition basically means that if the marginal investiga-
tion cost of Listener 1 relative to the marginal investigation cost of Listener 2 is high enough everywhere, then the optimal joint-investigation equilibrium in which Listener 2 is important in determining which projects can be launched outperforms the optimal unilateral-investigation equilibrium in which Listener 1 is crucial for determining which projects can be launched.

**Proposition 3.3.** When $E[\omega > R_2] > R_1$ and (3.9) hold, and

\[
\inf_{\alpha \in (0,1]} \frac{c'_1(\alpha)}{c'_2(\alpha)} \geq \frac{\int_{R_1}^1 (\omega - R_1) f(\omega) d\omega}{[1 - F(R_1) + F(\bar{\omega}^PU)] \int_{R_1}^1 (\omega - R_2) f(\omega) d\omega},
\]

then the optimal joint-investigation equilibrium outperforms the optimal unilateral-investigation equilibrium.

When $E[\omega | \omega > R_2] > R_1$ holds, there may exist mixed-investigation equilibria. Let $s_+^U$ and $s_+^J$ denote the message inducing unilateral investigation and the message inducing joint investigation, respectively. It is clear that all $\omega \in (R_2, R_1]$ report $s_+^J$, otherwise they have no chance to be launched. Let $\alpha_1^{PM}$ and $\alpha_2^{PM}$ denote respectively the investigation strategy of Listener 1 and the investigation strategy of Listener 2. In a mixed-investigation equilibrium, we have

\[
\alpha_1^{PM} (s_+^U) = \alpha_2^{PM} (s_+^J) > 0,
\]

and $\alpha_1^{PM} (s_+^J)$ is small enough so that Listener 2 rejects the project when she learns nothing from her investigation under message $s_+^J$. The condition (3.12) holds because the sets of Persuader types reporting $s_+^U$ and $s_+^J$ both include some $\omega > R_1$, otherwise Listener 1 rejects the project without investigation with receive the message; if $\alpha_1^{PM} (s_+^U) < \alpha_2^{PM} (s_+^J)$, no $\omega > R_1$ will report $s_+^U$, and if $\alpha_1^{PM} (s_+^U) > \alpha_2^{PM} (s_+^J)$, no $\omega > R_1$ will report $s_+^J$. It is indeterminate whether the optimal mixed-investigation equi-
librium outperforms the optimal equilibria of the other two types of equilibria. However, this indeterminacy does not affect much our comparison between public persuasion and sequential persuasion.

3.4 Sequential Persuasion

In this section, I discuss sequential persuasion, in which Persuader approaches the two listeners sequentially. Persuader has two schemes to persuade the listeners under this persuasion mode: approaching Listener 1 first and approaching Listener 2 first. I formalize each scheme as a sequential-persuasion game. For expositional purposes, I use $h_{1,2}$ and $h_{2,1}$ respectively to denote the game in which Listener 1 is approached first and the game in which Listener 2 is approached first.

The basic setup of the games is the same as that of the public-persuasion game, except that the timing is changed as follows.

- The privately informed Persuader chooses a message $s_1$, and sends it to the first listener.

- Upon observing $s_1$, the first listener makes her investigation decision and decides whether to invest in the project based on the investigation outcome.

- If the first listener rejects the project, the game ends. If the first listener invests in the project, the game continues; Persuader approaches the second listener and sends her a message $s_2$. After observing $s_2$ and the investment decision of the first listener without knowing $s_1$, the second listener chooses her investigation intensity and decides whether to invest or not when the investigation outcome is revealed.

Let $p$ still denote the persuasion strategy of Persuader, which specifies a pair of
messages sent to the two listeners for each \( \omega \). Thus, \( p \) can be formulated as

\[
p : \Omega \rightarrow S \times S.
\]

For example, \( p(\omega') = (s', s'') \) means that Persuader sends \( s' \) to the first listener and \( s'' \) to the second listener when he observes that the quality of the project is \( \omega' \). The investigation strategy of Listener \( i, i = 1, 2 \), is a mapping

\[
\alpha_i : S \rightarrow [0, 1].
\]

### 3.4.1 Equilibria

In this subsection, I introduce several representative equilibria of the sequential-persuasion games, and use them to illustrate the typical non-trivial interactions of the players in equilibrium.

1. **Unilateral Investigation**

   For each sequential-persuasion game, there exist equilibria in which only one listener investigates. These equilibria are the counterpart of the unilateral-investigation equilibria in the public-persuasion game. I specify below such an equilibrium of the \( (1, 2) \)-game, in which only Listener 1 investigates and Listener 2 invests without investigation whenever she is approached. It is straightforward to construct a similar equilibrium in the \( (2, 1) \)-game.

   Persuader adopts the following strategy \( p \). If the project has quality \( \omega \in [0, \omega'] \cup (R_1, 1] \), then he sends message \( s' \) to Listener 1; otherwise, he sends message \( s'' \). When Listener 1 invests in the project, he sends the same message to Listener 2. In this strategy, the choice of \( \omega' \) satisfies \( E_p[\omega | s'] \leq R_1 \).

   Listener 1 investigates the project when receiving \( s' \) and invests in it when \( \omega \) is identified to be above \( R_1 \), and rejects the project without investigation when receiving
Chapter 3. How to Persuade a Group

1. Sequential Persuasion Strategy under Unilateral Investigation

As in the public-persuasion game, the existence of such an equilibrium relies on the fact that Listener 1 is pickier than Listener 2; doing investigation has no value for Listener 2 if Listener 1 invests only if she finds $\omega$ is above $R_1$. I show in Subsection 3.4.2 that there is no equilibrium in which only Listener 2 investigates.

2. Joint Investigation

When $E[\omega | \omega > R_2] > R_1$ holds, the sequential-persuasion games may have equilibria in which both listeners investigate. These equilibria are the counterpart of the joint-investigation equilibria in the public-persuasion game.

I introduce two such equilibria, one for each sequential-persuasion game. From these two equilibria, we can see how the order of persuasion matters for Persuader. I first introduce an equilibrium of the $(2, 1)$-game. The persuasion strategy of Persuader in this equilibrium can be illustrated using Figure 3.4. If $\omega \in (\omega', R_2]$, Persuader sends message $s''$ to Listener 2. If $\omega \in [0, \omega'] \cup (R_2, 1]$, he sends message $s'$ to Listener 2. Conditional on Listener 2 being persuaded successfully, he sends Listener 1 the same message that he reported to Listener 2.

I describe the listeners’ strategies backwards. That is, I first look at the strategy of Listener 1, who is approached after Listener 2 is successfully persuaded. For Listener 1, when she is approached and observes message $s'$, she investigates the project with an
intensity $\alpha_1(s')$ that satisfies
\[
c'_1(\alpha_1(s')) = \frac{\int_{R_2}^{R_1} (R_1 - \omega) f(\omega) d\omega}{1 - F(R_2)},
\]
and accepts the project if $\omega$ is not identified to be smaller than $R_1$. When she observes a message different from $s'$, she rejects the project without investigation. For Listener 2, when she observes message $s'$, she investigates the project with an intensity $\alpha_2(s')$, which satisfies
\[
c'_2(\alpha_2(s')) = \frac{\int_{R_2}^{1} (\omega - R_2) f(\omega) d\omega - \alpha_1(s') \int_{R_2}^{R_1} (\omega - R_2) f(\omega) d\omega}{1 - F(R_2) + F(\omega')}.
\]
and invests in the project only if $\omega$ is identified to be above $R_2$. When observing a message different from $s'$, she rejects the project without investigation. Equations (3.13) and (3.14) determine the best responses of Listeners 1 and 2 to each other’s strategy.

To have Listener 2 reject the project when $\omega$ is not revealed under message $s'$, $\alpha_1(s')$ should not be too large. This condition is similar to (3.7) in the joint-investigation equilibria of the public-persuasion game.

Now I specify an equilibrium of the $(1, 2)$-game. The persuasion strategy of Persuader in this equilibrium is the same as that in the equilibrium above, except that the order of persuasion is changed. I still use Figure 3.4 to illustrate the persuasion strategy of Persuader. If $\omega \in (\omega', R_2]$, Persuader sends message $s''$ to Listener 1, and if $\omega \in [0, \omega'] \cup$
(R_2, 1], he sends message s’. Conditional on Listener 1 being successfully persuaded, he persuades Listener 2 with the same message.

For Listener 1, when she observes s’, she investigates the project and invests in it if \( \omega \) is not identified to be smaller than \( R_1 \). When observing a message different from \( s’ \), she rejects the project without investigation. For Listener 2, who is approached after Listener 1 invests in the project, when she observes \( s’ \), she investigates the project and invests in it only if \( \omega \) is identified to be larger than \( R_2 \). When observing a message different from \( s’ \), she rejects the project. The values of \( \alpha_1 (s’) \) and \( \alpha_2 (s’) \) are solutions of the system of equations

\[
c_1' (\alpha_1 (s')) = \frac{\alpha_2 (s') \left[ \int_{R_1}^{R_2} (R_1 - \omega) f (\omega) d\omega \right]}{1 - F (R_2) + F (\omega')}
\]

\[
c_2' (\alpha_2 (s')) = \frac{\int_{R_2}^{1} (\omega - R_2) f (\omega) d\omega - \alpha_2 (s') \int_{R_2}^{R_1} (\omega - R_2) f (\omega) d\omega}{\alpha_1 (s') [1 - F (R_1)] + (1 - \alpha_2 (s')) [1 - F (R_2) + F (\omega')]}. \tag{3.15}
\]

As for the equilibrium above, \( \alpha_1 (s') \) should not be too high to ensure that Listener 2 rejects the project if \( \omega \) is not revealed from her investigation.\(^3\)

Comparing the equilibrium investigation intensities in these two equilibria, we can see that changing the persuasion order only can change the incentives of the listeners to investigate, thus changing the probability of launching the project.

### 3.4.2 Optimal Equilibria

To characterize the optimal equilibria of the sequential-persuasion games, we first need to know which equilibria can arise in each game. As in the public-persuasion game, the analysis is focused on persuasive equilibria, in which the communication between Persuader and the listeners is informative.

\(^3\)According to Brouwer’s fixed-point theorem, there exists a solution to this system of equations on \([0, 1] \times [0, 1]\), but the solution may not be unique.
To begin, I look at the strategies of the listeners. Instead of directly characterizing the equilibrium strategies of the listeners, I first find the possible responses of the listeners that Persuader will face on the equilibrium path. I prove in the Appendix that a message reported by Persuader in the first persuasion stage can induce only the following investigation scenarios on the equilibrium path of a persuasive equilibrium.

For the \((1, 2)\)-game:

1. Listener 1 rejects the project without any investigation.

2. Listener 1 investigates the project and invests in the project only if \(\omega\) is identified to be larger than \(R_1\). When Persuader approaches Listener 2 after getting the investment of Listener 1, Listener 2 invests in the project without investigation.

3. Listener 1 investigates the project and invests in the project unless \(\omega\) is identified to be smaller than \(R_1\). Listener 2 investigates the project and invests in the project only if \(\omega\) is identified to be larger than \(R_2\) when Persuader approaches her with certain messages after getting the investment of Listener 1. If Persuader approaches her with other messages, Listener 2 rejects the project without investigation.

For the \((2, 1)\)-game:

1. Listener 2 rejects the project without any investigation.

2. Listener 2 agrees to invest in the project without investigation. Listener 1 investigates the project when Persuader approaches her with certain messages and accepts the project only if \(\omega\) is identified to be larger than \(R_1\). If Persuader approaches Listener 1 with other messages, she rejects the project without investigation.

3. Listener 2 investigates the project and accepts it only if \(\omega\) is identified to be larger than \(R_2\). Listener 1 investigates the project when Persuader persuades her with certain messages, and accepts the project unless \(\omega\) is identified to be smaller than \(R_1\). Listener 1 rejects the project when Persuader persuades her with other messages.
In an equilibrium, there may be multiple messages inducing the same scenario. Based on the same intuition and using the same technique for proving Lemma 3.3, we have the following lemma.

**Lemma 3.5.** For an equilibrium with multiple messages inducing the same response of the listeners, we can find another equilibrium in which different messages induce different responses of the listeners and each player has the same *ex ante* expected payoff as in the original equilibrium.

For either sequential-persuasion game, there is no equilibrium in which some types of Persuader face Scenario 2 and some other types face Scenario 3. The reason is simple. Suppose that in the \((1,2)\)-game, reporting messages \(s'\) and \(s''\) to Listener 1 respectively induce Scenario 2 and Scenario 3. Then after receiving the investment of Listener 1, when approaching Listener 2, a Persuader reporting \(s''\) to Listener 1 has incentive to mimic the types of Persuader that report \(s'\) to Listener 1 and can receive the investment of Listener 1, as Listener 2 will agree to invest without investigation. Given this observation, when characterizing the optimal equilibria, we can restrict attention to three types of equilibria, without loss of generality, which are (1) Scenario 2 is induced by a message in equilibrium, (2) Scenario 3 is induced by a message in equilibrium with positive probability, and (3) only Scenario 1 happens in equilibrium.

I ignore the type (3) equilibria in characterizing the optimal equilibrium of a sequential-persuasion game, because they always induce 0 probability of launching the Persuader’s project.

For type (1) equilibria (henceforth, unilateral-investigation equilibria), since Listener 1 is the only investigator, the strategic situations facing Persuader and Listener 1 are exactly the same at those in the unilateral-investigation equilibria of the public-persuasion game. Thus, the optimal unilateral-investigation equilibrium for each sequential-persuasion game gives the same expected payoff to Persuader as \((3.6)\).

Now I characterize the optimal type (2) equilibrium (henceforth, joint-investigation
equilibria) of each sequential-persuasion game. To begin, I look at the (2, 1)-game. Let \( p^{SJ}_{(2,1)} \) denote the persuasion strategy of Persuader and \( p^{-1, SJ}_{(2,1)} \) denote the inverse of \( p^{SJ}_{(2,1)} \), where superscripts \( S \) and \( J \) represent “sequential persuasion” and “joint investigation“, respectively. Let \( s_{+,i} \) denote the message inducing Listener \( i \) to investigate, \( i = 1, 2 \). It is clear that all \( \omega \in (R_2, 1] \) report \( s_{+,2} \) and \( s_{+,1} \) in each corresponding stage, as otherwise the project will be rejected for sure. The intuition underlying Lemma 3.4 and the technique for proving that lemma carry over to this game. Therefore, we can assume that \( p^{-1, SJ}_{(2,1)} (s_{+,2}) \cap [0, R_2] = [0, \bar{\omega}], \bar{\omega} \leq R_2 \). I use \( \alpha_{(2,1),1} \) and \( \alpha_{(2,1),2} \) to denote the investigation strategies of Listeners 1 and 2, respectively. To have Listener 2 reject the project when \( \omega \) is not identified from investigation under message \( s_{+,2} \), \( \alpha_{(2,1),1} (s_{+,1}) \) should not be too large. Let \( \bar{\alpha}_{1p^{SJ}_{(2,1)}} (s_{+,2}) \) denote the value of \( \alpha_{(2,1),1} (s_{+,1}) \) that makes Listener 2 indifferent between investing and not investing when \( \omega \) is not identified from investigation under message \( s_{+,2} \). The optimal equilibrium can be characterized by solving the problem

\[
\max_{\omega \leq R_2} \alpha_{(2,1),2} (s_{+,2}) \left( 1 - \alpha_{(2,1),1} (s_{+,1}) \right) \left[ F(R_1) - F(R_2) \right] + \alpha_{(2,1),2} (s_{+,2}) \left[ 1 - F(R_1) \right] \\
\text{s.t. } \alpha_{(2,1),1} (s_{+,1}) \leq \bar{\alpha}_{1p^{SJ}_{(2,1)}} (s_{+,2}).
\]

The value of \( \alpha_{(2,1),1} (s_{+,1}) \) is independent of \( \bar{\omega} \), because only \( \omega > R_2 \) can get investment from Listener 2. Thus, the expected probability of launching the project if the equilibrium has the maximum value of \( \alpha_{(2,1),2} (s_{+,2}) \). The value of \( \alpha_{(2,1),2} (s_{+,2}) \) is decreasing in \( \bar{\omega} \). Intuitively, this is because the investigation of Listener 2 is to prevent a project with \( \omega > R_2 \) from being rejected. (See the Appendix for a detailed characterization of the equilibria.) Therefore, the maximization problem above can be reduced to

\[
\min_{\omega \leq R_2} \omega \\
\text{s.t. } \alpha_{(2,1),1} (s_{+,1}) \leq \bar{\alpha}_{1p^{SJ}_{(2,1)}} (s_{+,2}).
\]
The value of $\bar{\alpha}_{1p_{(2,1)}^{SJ}} (s_{+,2})$ is increasing in $\bar{\omega}$, so the optimal $\bar{\omega}_{(2,1)}^{SJ}$ satisfies $\alpha_{(2,1),1} (s_{+,1}) \leq \bar{\alpha}_{1p_{(2,1)}^{SJ}} (s_{+,2})$. There may not exist a solution to this problem. I define

$$
\phi_{(2,1)}^{SJ} (c_1, c_2, R_1, R_2, f) \equiv \min_{\omega \leq R_2} \alpha_{(2,1),1} (s_{+,1}) - \bar{\alpha}_{1p_{(2,1)}^{SJ}} (s_{+,2}),
$$

which is a function of the primitives. The problem (3.16) has a solution if and only if

$$
\phi_{(2,1)}^{SJ} (c_1, c_2, R_1, R_2, f) \leq 0. \tag{3.17}
$$

In the rest of this subsection, I characterize the optimal joint-investigation equilibrium of the $(1, 2)$-game. As before, let $p_{(1,2)}^{SJ}$ denote the persuasion strategy of Persuader, $p_{(1,2)}^{-1, SJ}$ denote the inverse of $p_{(1,2)}^{SJ}$, and $\alpha_{(1,2),1}$ and $\alpha_{(1,2),2}$ denote the investigation strategies of Listeners 1 and 2, respectively. For this type of equilibria, $\alpha_{(1,2),1} (s_{+,1})$ and $\alpha_{(1,2),2} (s_{+,2})$ are strategic complements.

Similar to Lemma 3.4, we have the following lemma.

**Lemma 3.6.** For any joint-investigation equilibrium of the $(1, 2)$-game, there is a joint-investigation equilibrium that generates the same probability of launching the project and whose $p_{(1,2)}^{SJ}$ has the form $\bigcup_{s \in S} p_{(2,1)}^{-1, SJ} (s_{+,2}, s) = [0, \bar{\omega}_1] \cup (R_2, 1]$ and $p_{(1,2)}^{-1, SJ} (s_{+,1}, s_{+,2}) = [0, \bar{\omega}_2] \cup (R_2, 1]$, with $\bar{\omega}_2 \leq \bar{\omega}_1 \leq R_2$.

The optimal equilibrium can be obtained by solving the problem\footnote{In the case that (3.35) has multiple solutions, we solve the problem for each of these solutions satisfying constraint (3.36), and then choose the optimal one among them.}

$$
\max_{\omega_2 \leq \omega_1 \leq R_2} \alpha_{(1,2),2} (s_{+,2}) \left[ 1 - \alpha_{(1,2),1} (s_{+,1}) \right] [F (R_1) - F (R_2)] + \alpha_{(1,2),2} (s_{+,2}) [1 - F (R_1)] \\
\text{s.t. } \alpha_{(1,2),1} (s_{+,1}) \leq \bar{\alpha}_{1p_{(1,2)}^{SJ}} (s_{+,2}). \tag{3.18}
$$

The value of $\alpha_{(1,2),1} (s_{+,1})$ is decreasing in $\bar{\omega}_2$ and $\bar{\omega}_1$, and $\bar{\alpha}_{1p_{(1,2)}^{SJ}} (s_{+,2})$ is increasing in
Proposition 3.4. The optimal joint-investigation equilibrium of the \((1, 2)\)-game is better than the optimal optimal joint-investigation equilibrium of the \((2, 1)\)-game from the perspective of Persuader, if both of them exist.

The proof is simple. When the \((2, 1)\)-game has an optimal joint-investigation equilibrium, i.e., the problem (3.16) has a solution \(\bar{\omega}_{(2, 1)}^{SJ} \leq R_2\), then we can always find a joint-investigation equilibrium of the \((1, 2)\)-game with \(\bar{\omega}_1 = \bar{\omega}_2 = \bar{\omega}_{(2, 1)}^{SJ}\) that outperforms.

The analysis above indicates that if Persuader can choose the order of persuasion before he observes \(\omega\) of his project, it is optimal for him to persuade Listener 1 first, because for each type of equilibria, either unilateral-investigation or joint-investigation, the optimal equilibrium of that type in the \((1, 2)\)-game gives Persuader a higher probability of launching the project than in the \((2, 1)\)-game. Therefore, in the rest of the analysis, I focus on the \((1, 2)\)-game.

For the \((1, 2)\)-game, when comparing the optimal joint-investigation equilibrium and the optimal unilateral-investigation equilibrium, Proposition 3.2 and Proposition 3.3 in the public-persuasion game apply if we replace the condition “When \(E[\omega | \omega > R_2] > \)
R₁ and (3.9) hold” by “When $E[\omega|\omega > R_2] > R_1$ and (3.20) hold”. The proofs are straightforward, so I omit them.

3.5 Public Persuasion versus Sequential Persuasion

Which mode of persuasion, public persuasion or sequential persuasion, outperforms the other from Persuader’s perspective? To answer this question, I compare the equilibria of these two persuasion modes. The following analysis indicates that the result depends on the parameters. That is, neither mode of persuasion is always better than the other. The analysis focuses on discussing how the investigation costs affect the relative performance of these two persuasion modes.

In this model, we have

$$\phi^{PJ}(c_1, c_2, R_1, R_2, f) < \phi^{SJ}(c_1, c_2, R_1, R_2, f),$$  (3.21)

which implies the existence condition for the joint-investigation equilibria is weaker in the public-persuasion game. Based on (3.21), the comparison between these two persuasion modes can be divided into the following cases.

**Case 1:** $E[\omega|\omega > R_2] \leq R_1$

In this case, only unilateral-investigation equilibria exist in the public-persuasion game and in the (1, 2)-game, because $E[\omega|\omega > R_2] > R_1$ is necessary for the existence of other types of equilibria in these games. Previous analysis shows that the optimal unilateral-investigation equilibria of these two games are equivalent from Persuader’s perspective. Therefore, public persuasion and sequential persuasion are indifferent to Persuader.

**Case 2:** $E[\omega|\omega > R_2] > R_1$ and $0 < \phi^{SJ}(c_1, c_2, R_1, R_2, f)$

---

5 Inequality (3.21) can be proved using (3.31) in public persuasion and (3.35) in sequential persuasion.
In this case, public persuasion outperforms sequential persuasion from Persuader’s perspective.

Given inequality (3.21), we can see that there exists case where \( c'_1 \) and/or \( c'_2 \) are large enough that \( \phi^{PJ}(c_1, c_2, R_1, R_2, f) \leq 0 \), but not large enough that \( \phi^{SJ}(c_1, c_2, R_1, R_2, f) \leq 0 \). In these cases, since \( E[\omega|\omega > R_2] > R_1 \) and \( \phi^{PJ}(c_1, c_2, R_1, R_2, f) \leq 0 \), joint-investigation equilibria in the public-persuasion game exist. But since \( 0 < \phi^{SJ}(c_1, c_2, R_1, R_2, f) \), the existence condition (3.20) is violated, so there are only unilateral-investigation equilibria in the \( <1,2> \)-game. If \( c'_1 \) and \( c'_2 \) satisfy the conditions in either Proposition 3.2 or Proposition 3.3 then the optimal joint-investigation equilibrium of the public-persuasion game generates a higher probability of launching the project of Persuader than the optimal unilateral-investigation equilibrium of the \( <1,2> \)-game. Therefore, public persuasion strictly outperforms sequential persuasion.

If the primitives satisfy \( \phi^{PJ}(c_1, c_2, R_1, R_2, f) > 0 \), then in the public-persuasion game, we have unilateral-investigation equilibria and may have mixed-investigation equilibria. Thus, public persuasion still weakly outperforms sequential persuasion.

**Case 3:** \( E[\omega|\omega > R_2] > R_1 \) and \( \phi^{SJ}(c_1, c_2, R_1, R_2, f) \leq 0 \)

In this case, sequential persuasion may outperform public persuasion from Persuader’s perspective.

Given \( E[\omega|\omega > R_2] > R_1 \), this case is true if \( c'_1 \) and/or \( c'_2 \) are large enough that the condition \( \phi^{SJ}(c_1, c_2, R_1, R_2, f) \leq 0 \) holds. Both the public-persuasion game and the \( <1,2> \)-game have joint-investigation equilibria in this case.

In the joint-investigation equilibria of the public-persuasion game, when both listeners investigate, Listener 2’s investigation intensity is decreasing in Listener 1’s investigation intensity. In joint-investigation equilibria of the \( <1,2> \)-game, when both listeners investigate, their investigation intensities are strategic complements.

Based on these relationships, we can see that if the investigation intensities of Listener 1 in optimal joint-investigation equilibria of the two games are close enough and
large, without violating the constraints for Listener 1’s investigation intensities, then the optimal joint-investigation equilibrium of the \((1, 2)\)-game outperforms that of the public-persuasion game. I first illustrate this result using a limiting example, in which \(c_1(\cdot)\) deviates from the assumptions in (3.2). Specifically, with a small fixed cost \(\bar{c}_1\), Listener 1 investigates with a small fixed intensity \(\bar{\alpha}_1\) satisfying the (3.9) and (3.20) constraints. We can show that the investigation intensity of Listener 2 in the optimal joint-investigation equilibrium of the public-persuasion game is smaller than that in the optimal joint-investigation equilibrium of the \((1, 2)\)-game. \(^6\) If Listener 2’s investigation intensity in the optimal joint-investigation equilibrium of the \((1, 2)\)-game is higher than \(\bar{\alpha}_1\), then the optimal joint-investigation equilibrium of the \((1, 2)\)-game outperforms the mixed-investigation equilibria of the public-persuasion game, if they exist, as in the mixed-investigation equilibria the probability of launching the project is

\[
\bar{\alpha}_1 (1 - \bar{\alpha}_1) [F(R_1) - F(R_2)] + \bar{\alpha}_1 [1 - F(R_1)].
\]

In this case, sequential persuasion outperforms public persuasion.

Given the insights provided by the limiting example, if we maintain the assumptions for \(c_1(\cdot)\) in (3.2), but make \(c'_1\) as close as possible to function

\[
h(\alpha) = \begin{cases} 
0, & \text{if } 0 \leq \alpha < \bar{\alpha}_1; \\
1, & \text{if } \alpha = \bar{\alpha}_1,
\end{cases}
\]

then continuity implies the result in the limiting example holds in this case.

### 3.6 Concluding Remarks

This paper studies a one-persuader/multilistener model of persuasion. The major features of the model are that (1) the persuader is privately informed about the quality

\(^6\) This can be proved based on (3.31) and (3.35).
of his project, (2) he communicates with the listeners about the quality of the project quality through “soft evidence”, and (3) the listeners can investigate the project at a cost, and the investigation intensities are variable.

The objective of the paper is to study the impact of the order of persuasion on the persuader’s payoff. Two modes of persuasion are considered in the analysis, public persuasion and sequential persuasion. In comparing these two modes, the paper finds that the optimal persuasion order depends on the investigation costs of the listeners. If the marginal costs of investigation are low, public persuasion tends to outperform sequential persuasion. The opposite can be true if it is very costly for the listeners to investigate the true state and the preferences of the listeners are sufficiently aligned.

Several questions are left open in this paper. First, I assume in the model that the persuader learns the quality of his project after the persuasion mode is chosen. What happens if the persuader chooses the persuasion mode after he learns the quality of his project? Answering this question can help us understand the relationship between the type of the persuader and the choice of persuasion mode. Needless to say, this problem is more complicated than the one I consider in the current paper, as in this problem the choice of persuasion mode becomes a signaling device of the persuader. Second, I focus on examining the impact of the order of persuasion from the persuader’s perspective in the current paper. However, in reality, it is often the case that listeners determine how a persuader should communicate with them. Thus, it is worth exploring which mode of persuasion is optimal from the listeners’ perspective.
Appendix

Proof of Lemma 3.1

Considering an equilibrium with the persuasion strategy $p$ and $\#S_+ \geq 2$. The equilibrium investigation strategy $\alpha$ satisfies

$$c'(\alpha(s_i)) = \int_R^1 (\omega - R) f_p(\omega|s_i) \, d\omega, \quad i = 1, \ldots, I. \quad (3.22)$$

For all $\omega$, 

$$\sum_{i=1}^I f_p(\omega|s_i) \Pr(s_i|p) = f(\omega) \quad (3.23)$$

$$= \frac{f(\omega)}{\Pr(S_+|p)} \Pr(S_+|p)$$

$$= f_p(\omega|S_+) \Pr(S_+|p),$$

where $\Pr(S_+|p) = \sum_{i=1}^I \Pr(s_i|p)$. I now modify the persuasion strategy of Persuader into one in which all types reporting messages in $S_+$ report the message $s_1$ instead, and other types reporting messages inducing no investigation, if any, report a message, say $s_0$. Let $\tilde{p}$ denote this new persuasion strategy. Under $\tilde{p}$, the optimal investigation intensity $\alpha_{\tilde{p}}(s_1)$ of Listener when receiving $s_1$ satisfies

$$c'(\alpha_{\tilde{p}}(s_1)) = \int_R^1 (\omega - R) f_{\tilde{p}}(\omega|s_1) \, d\omega$$

$$= \int_R^1 (\omega - R) f_p(\omega|S_+) \, d\omega$$

$$= \int_R^1 (\omega - R) \sum_{i=1}^I \frac{f_p(\omega|s_i) \Pr(s_i|p)}{\Pr(S_+|p)} \, d\omega$$

$$= \sum_{i=1}^I \left( \int_R^1 (\omega - R) f_p(\omega|s_i) \, d\omega \right) \frac{\Pr(s_i|p)}{\Pr(S_+|p)}$$

$$= c'(\alpha(s_i))$$
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The second equality is due to the definition of $\bar{p}$. The third equality is obtained using equation (3.23). The last equality is based on (3.22) and that $\alpha(s_i) = \alpha(s_j)$ for any $s_i, s_j \in S_+$. These results imply that $\alpha_{\bar{p}}(s_1) = \alpha(s_i)$. If there is any type of Persuader reporting $s_0$ under $\bar{p}$, then this type must have $\omega \leq R$. Hence, it is optimal to have $\alpha_{\bar{p}}(s_0) = 0$.

The persuasion strategy $\bar{p}$, investigation strategy $\bar{\alpha}$ with $\bar{\alpha}(s_1) = \alpha_{\bar{p}}(s_1)$ and $\bar{\alpha}(s) = 0$ for $s \in S \setminus \{s_1\}$, and the sequentially rational investment strategy, which is to accept the project only if $\omega$ is identified to be strictly larger than $R$, constitute an equilibrium that has at most two equilibrium messages. For all $\omega \in p^{-1}(S_+)$, the probability of success under this constructed strategy profile, which is $\bar{\alpha}(s_1)$ for $\omega \in (R, 1]$ and 0 for $\omega \in [0, R]$, is the same as that under the original equilibrium. For all types reporting message $s_0$, their probability of success is still 0. Therefore, from the perspective of Persuader, this modified strategy profile is equivalent to the original equilibrium.

The ex ante expected payoff of Listener under this new strategy is also the same as that under the original equilibrium. First of all, we have

$$Eu_p(\alpha(s_i) | s_i) = Eu_p(\alpha(s_j) | s_j), \text{ for } s_i, s_j \in S_+. \tag{3.24}$$

That is, the expected payoffs of Listener after receiving the messages in $S_+$ are identical in the original equilibrium. For a message inducing no investigation, if there is any, the expected payoff of Listener is $R$. In the new equilibrium, since $\bar{\alpha}(s_1) = \alpha(s_i)$, we have,

$$Eu_{\bar{p}}(\bar{\alpha}(s_1) | s_1) = Eu_p(\alpha(s_i) | s_i), s_i \in S_+. \tag{3.25}$$

If $s_0$ is reported by any type, the expected payoff of Listener is $R$. Therefore, we have

$$Eu_{\bar{p}}(\bar{\alpha}(s_1) | s_1) \Pr(s_1 | \bar{p}) + R \Pr(s_0 | \bar{p}) = \sum_{i=1}^{I} Eu_p(\alpha(s_i) | s_i) \Pr(s_i | p) + R \Pr(S^0 | p),$$
i.e., the ex ante expected payoffs of Listener under these two equilibria are the same.

**Proof of Proposition 3.1**

To simplify the notation, I define \( L \equiv p^{-1}(s_+) \cap [0, R] \), which is the set of \( \omega \) lower than \( R \) and reporting \( s_+ \). I also define \( \Pr(L) \equiv \int_L f(\omega) \, d\omega \), which is the probability measure of set \( L \). From the F.O.C. of Listener’s payoff maximization problem, we have

\[
c'(\alpha(s_+)) = 1 - R - \int_{R}^{1} F_p(\omega|s_+) \, d\omega
\]

\[
= 1 - R - \frac{\int_{R}^{1} F(\omega) \, d\omega - [F(R) - \Pr(L)] (1 - R)}{1 - F(R) + \Pr(L)}.
\]

The value of \( \Pr(L) \) depends on the specification of the equilibrium. Taking the derivative of \( \alpha(s_+) \) with respect to \( \Pr(L) \) yields

\[
\frac{d\alpha(s_+)}{d\Pr(L)} = -\frac{1}{c''(\alpha(s_+))} \frac{\int_{R}^{1} [1 - F(\omega)] \, d\omega}{[1 - F(R) + \Pr(L)]^2} < 0. \tag{3.26}
\]

Thus, to get the maximum of \( \alpha(s_+) \), we need to have \( \Pr(L) \) as small as possible in equilibrium. Given that \( E_p[\omega|s_+] \leq R \), we have

\[
\int_L \frac{\omega f(\omega)}{1 - F(R) + \Pr(L)} \, d\omega + \int_{R}^{1} \frac{\omega f(\omega)}{1 - F(R) + \Pr(L)} \, d\omega \leq R,
\]

which is equivalent to

\[
\int_{R}^{1} (\omega - R) f(\omega) \, d\omega \leq R \Pr(L) - \int_L \omega f(\omega) \, d\omega. \tag{3.27}
\]

I first characterize the optimal equilibrium among the ones where \( p^{-1}(s_+) \) has the form \([0, \bar{\omega}] \cup (R, 1]\), then I show that any other equilibrium cannot outperform it. Suppose in equilibrium \( p^{-1}(s_+) = [0, \bar{\omega}] \cup (R, 1], \bar{\omega} \leq R \). Following the definition of \( L, \Pr(L) = \)
Based on (3.26) and (3.27), the problem of searching for the optimal equilibrium can be reformulated as the minimization problem below:

\[
\min_{\omega < R} \omega f(\omega) d\omega + \int_{R}^{1} \omega f(\omega) d\omega \leq R [1 - F(R) + F(\omega)].
\]

I define function \(g(\omega)\) as

\[
g(\omega) = R [1 - F(R) + F(\omega)] - \left[ \int_{0}^{\omega} \omega f(\omega) d\omega + \int_{R}^{1} \omega f(\omega) d\omega \right].
\]

There exists a unique \(\omega^*\) such that \(g(\omega^*) = 0\). Any \(\omega \in (\omega^*, R]\) satisfies \(g(\omega) > 0\), and for any \(\omega \in [0, \omega^*), g(\omega) < 0\). Hence, \(\omega^*\) is the solution of the minimization problem.

Any other strategy profile whose persuasion strategy \(p\) achieves a smaller \(Pr(L)\) than does \([0, \omega^*] \cup (R, 1]\) cannot satisfy the constraint \(E_p(\omega|s_+) \leq R\), so does not form an equilibrium.

**Proof of Lemma 3.3**

For an equilibrium in which there are multiple equilibrium messages inducing both listeners to reject the project without investigation, it is straightforward to show that there is another equilibrium, in which all the types of Persuader reporting those messages are pooled together, generating the same expected ex ante payoffs to all the players. Thus, I omit a detailed discussion of this case.

1. **Unilateral Investigation**

Suppose that in an equilibrium, there is a set \(S_2 = \{s_1^2, s_2^2, \ldots, s_n^2\}\) of messages, each of which induces that Listener 1 investigates and accepts the project only if \(\omega\) is identified to be above \(R_1\); Listener 2 invests in the project without any investigation. The strategic situation Listener 1 faces under each message in \(S_2\) is exactly the same as that a listener with outside option \(R_1\) faces in the one-listener case. We can construct a strategy profile
in which all the types of Persuader reporting messages in \( S_2 \) report \( s_1^2 \) instead, and the behavior of each listener on receiving \( s_1^2 \) is the same as that on receiving \( s_1^2 \in S_2 \). This new strategy profile is an equilibrium.

The expected payoff of Persuader is unchanged in the new equilibrium, as the project of each \( \omega \) has the same probability of being launched. Similar to that in the proof of Lemma 3.1, we can show that Listener 1’s expected payoff is unchanged. The payoff of Listener 2 under message \( s_i^2 \), \( i \in \{1, 2, \ldots, n_2\} \), in the original equilibrium is

\[
\alpha_1 (s_i^2) \int_{R_1}^1 \omega f_p (\omega|s_i^2) \, d\omega + \left[ 1 - \alpha_1 (s_i^2) (1 - F_p (R_1|s_i^2)) \right] R_2.
\]

In the new equilibrium, let \( \hat{p} \) and \( \hat{\alpha}_1 \) denote the strategies of Persuader and Listener 1, the expected payoff of Listener 2 under \( s_1^2 \) is

\[
\hat{\alpha}_1 (s_1^2) \int_{R_1}^1 \omega f_{\hat{p}} (\omega|s_1^2) \, d\omega + \left[ 1 - \hat{\alpha}_1 (s_1^2) (1 - F_{\hat{p}} (R_1|s_1^2)) \right] R_2
\]

\[
= \sum_{i=1}^{n_2} \left\{ \alpha_1 (s_i^2) \int_{R_1}^1 \omega f_p (\omega|s_i^2) \, d\omega + \left[ 1 - \alpha_1 (s_i^2) (1 - F_p (R_1|s_i^2)) \right] R_2 \right\} \frac{Pr (s_i^2|\hat{p})}{Pr (S_2|\hat{p})}.
\]

Thus, the \textit{ex ante} payoff of Listener 2 is unchanged.

2. Joint Investigation

Suppose that in an equilibrium, there is a set \( S_3 = \{ s_1^3, s_2^3, \ldots, s_{n_3}^3 \} \) of messages, each of which induces both listeners to investigate the project, and Listener 2 rejects the project unless its \( \omega \) is identified to be larger than \( R_2 \), while Listener 1 rejects the project only if \( \omega \) is identified to be smaller than \( R_1 \). For each \( s_i^3 \), there must be \( E [\omega|\omega \in p^{-1} (s_i^3) \cap (R_2, 1)] > R_1 \), otherwise Listener 1 will not accept the project in the case where she learns nothing from investigation.

I first show that for any \( s_i^3 \in S_3 \), \( p^{-1} (s_i^3) \cap (R_2, R_1] \neq \emptyset \) and \( p^{-1} (s_i^3) \cap (R_1, 1] \neq \emptyset \). Suppose that \( p^{-1} (s_i^3) \cap (R_2, R_1] \) is empty, then Listener 1 will not investigate, as the project accepted by Listener 2 must have \( \omega > R_1 \). If \( p^{-1} (s_i^3) \cap (R_1, 1] \) is empty, then
Listener 1 rejects the project directly.

Let $\alpha_1$ and $\alpha_2$ be the investigation strategies of Listener 1 and Listener 2 respectively. Under a message $s_i^3 \in S_3$, $\alpha_1(s_i^3)$ and $\alpha_2(s_i^3)$ are uniquely defined by

$$
c'_1 (\alpha_1(s_i^3)) = \alpha_2(s_i^3) \int_{R_2}^{R_1} (R_1 - \omega) f_p(\omega|s_i^3) d\omega,
$$

$$
c'_2 (\alpha_2(s_i^3)) = \int_{R_2}^{1} (\omega - R_2) f_p(\omega|s_i^3) d\omega - \alpha_1 (s_i^3) \left[ \int_{R_2}^{R_1} (\omega - R_2) f_p(\omega|s_i^3) d\omega \right],
$$

where

$$
\alpha_1 (s_i^3) \leq \bar{\alpha}_1(s_i^3) = \frac{R_2 - E_p(\omega|s_i^3)}{\int_{R_1}^{1} \omega f_p(\omega|s_i^3) d\omega + F_p(R_1|s_i^3) R_2 - E_p(\omega|s_i^3)},
$$

which ensures that Listener 2 rejects the project when she learns nothing from her investigation. The coexistence of the messages in $S_3$ in equilibrium implies that

$$
\alpha_1 (s_i^3) = \alpha_1(s_j^3), \alpha_2 (s_i^3) = \alpha_2(s_j^3), \text{ for any } s_i^3, s_j^3 \in S_3, i \neq j.
$$

Thus, we have

$$
c'_1 (\alpha_1(s_i^3)) = \alpha_2(s_i^3) \sum_{k=1}^{n_3} \left[ \int_{R_2}^{R_1} (R_1 - \omega) f_p(\omega|s_k^3) d\omega \right] \frac{\Pr(s_k^3|p)}{\Pr(S_3|p)}
$$

$$
= \alpha_2(s_i^3) \int_{R_2}^{R_1} (R_1 - \omega) \frac{\sum_{k=1}^{n_3} f_p(\omega|s_k^3) \Pr(s_k^3|p)}{\Pr(S_3|p)} d\omega
$$

$$
c'_2 (\alpha_2(s_i^3)) = \sum_{k=1}^{n_3} \left[ \int_{R_2}^{1} (\omega - R_2) f_p(\omega|s_k^3) d\omega \right] \frac{\Pr(s_k^3|p)}{\Pr(S_3|p)}
$$

$$
- \alpha_1 (s_i^3) \sum_{k=1}^{n_3} \left[ \int_{R_2}^{R_1} (\omega - R_2) f_p(\omega|s_k^3) d\omega \right] \frac{\Pr(s_k^3|p)}{\Pr(S_3|p)}
$$

$$
= \int_{R_2}^{1} (\omega - R_2) \sum_{k=1}^{n_3} \left[ \frac{f_p(\omega|s_k^3) \Pr(s_k^3|p)}{\Pr(S_3|p)} \right] d\omega
$$

$$
- \alpha_1 (s_i^3) \int_{R_2}^{R_1} (\omega - R_2) \sum_{k=1}^{n_3} \left[ \frac{f_p(\omega|s_k^3) \Pr(s_k^3|p)}{\Pr(S_3|p)} \right] d\omega
$$

Now we consider a strategy profile in which everything is the same as in the equi-
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librium above except that all types of Persuader reporting messages in $S_3$ report the message $s_3^3$, and under message $s_3^3$, the strategies of Listener 1 and Listener 2 are the same as those under message $s_i^3$ in the original equilibrium. The analysis below shows that this strategy profile is an equilibrium.

Let $p$, $\alpha_1$, and $\alpha_2$ denote the strategies of Persuader, Listener 1, and Listener 2 respectively. Under message $s_1^3$, $\alpha_1 (s_1^3)$ and $\alpha_2 (s_1^3)$ should satisfy

\[
c_1' (\alpha_1 (s_1^3)) = \alpha_2 (s_1^3) \int_{R_2}^{R_1} (R_1 - \omega) f_p (\omega|s_1^3) \, d\omega,
\]
\[
c_2' (\alpha_2 (s_1^3)) = \int_{R_2}^{1} (\omega - R_2) f_p (\omega|s_1^3) \, d\omega - \alpha_2 (s_1^3) \left[ \int_{R_2}^{R_1} (\omega - R_2) f_p (\omega|s_1^3) \, d\omega \right].
\]

The definitions of $p$ and $f_p (\omega|s_1^3)$ gives us

\[
f_p (\omega|s_1^3) = \sum_{k=1}^{n_3} \left[ \frac{f_p (\omega|s_k^3) Pr (s_k^3|p)}{Pr (S_3|p)} \right], \text{ for any } \omega \in \Omega.
\]

Since the solution to $\text{3.28}$ is unique, there must be $(\alpha_1 (s_1^3), \alpha_2 (s_1^3)) = (\alpha_1 (s_i^3), \alpha_2 (s_i^3))$.

This does not complete the proof. I still need to show that

\[
\alpha_1 (s_1^3) < \tilde{\alpha}_1 p (s_1^3) = \frac{R_2 - E_p [\omega|s_1^3]}{\int_{R_1}^{1} \omega f_p (\omega|s_1^3) \, d\omega + F_p (R_1|s_1^3) R_2 - E_p [\omega|s_1^3]}.
\]

Based on the expressions of $\tilde{\alpha}_1 p (s_i^3)$, we have

\[
\tilde{\alpha}_1 p (s_1^3) = \frac{\sum_{i=1}^{n_3} [R_2 - E_p [\omega|s_i^3]] \frac{Pr (s_i^3|p)}{Pr (S_3|p)}}{\sum_{i=1}^{n_3} \int_{R_1}^{1} \omega f_p (\omega|s_i^3) \, d\omega + F_p (R_1|s_i^3) R_2 - E_p [\omega|s_i^3]] \frac{Pr (s_i^3|p)}{Pr (S_3|p)}}
\]

\[
= \sum_{i=1}^{n_3} \lambda (s_i^3) \tilde{\alpha}_1 p (s_i^3), \text{ where}
\]
\[ \lambda (s_i^3) = \frac{\int_{R_1} \omega f_p (\omega|s_i^3) \, d\omega + F_p (R_1|s_i^3) R_2 - E_p [\omega|s_i^3]}{\sum_{i=1}^{n_3} \int_{R_1} \omega f_p (\omega|s_i^3) \, d\omega + F_p (R_1|s_i^3) R_2 - E_p [\omega|s_i^3]} \Pr (s_i^3|p) \] 

with

\[ \sum_{i=1}^{n_3} \lambda (s_i^3) = 1. \]

Thus, \( \tilde{\alpha}_1 (s_1^3) < \tilde{\alpha}_{1p} (s_1^3) \).

Applying the same proof scheme as before, it is straightforward to show that the ex ante expected payoffs of Persuader and the listeners are unchanged.

**Uniqueness of Investigation Strategies in Joint-investigation Equilibria**

From the expected payoff functions of the listeners, we can derive their best response functions under message \( s^+ \) as follows:

\[ c'_1 (\alpha_1^{PJ} (s^+)) = \alpha_2^{PJ} (s^+) \int_{R_2}^{R_1} (R_1 - \omega) f_{p^R} (\omega|s^+) \, d\omega; \]

\[ c'_2 (\alpha_2^{PJ} (s^+)) = \int_{R_2}^{R_1} (\omega - R_2) f_{p^R} (\omega|s^+) \, d\omega - \alpha_1^{PJ} (s^+) \int_{R_2}^{R_1} (\omega - R_2) f_{p^R} (\omega|s^+) \, d\omega, \]

and

\[ \alpha_1^{PJ} (s^+) \leq \tilde{\alpha}_{1p^{R^R}} (s^+) \equiv \frac{R_2 - E_{p^R} [\omega|s^+]}{\int_{R_1} \omega f_{p^R} (\omega|s^+) \, d\omega + F_{p^R} (R_1|s^+) R_2 - E_{p^R} [\omega|s^+]}. \]

I show that (3.30) has a unique solution. Define \( b_i (\alpha_j), i, j = 1, 2 \), as

\[ c'_1 (b_1 (\alpha_2)) = \alpha_2 \int_{R_2}^{R_1} (R_1 - \omega) f_{p^R} (\omega|s^+) \, d\omega; \]

\[ c'_2 (b_2 (\alpha_1)) = \int_{R_2}^{R_1} (\omega - R_2) f_{p^R} (\omega|s^+) \, d\omega - \alpha_1 \int_{R_2}^{R_1} (\omega - R_2) f_{p^R} (\omega|s^+) \, d\omega, \]

and \( B (\alpha_1, \alpha_2) = (b_1 (\alpha_2), b_2 (\alpha_1)) \). By ignoring the constraint on \( \alpha_1 \), \( B (\alpha_1, \alpha_2) \) is a
continuous mapping from \([0, 1] \times [0, 1]\) to \([0, 1] \times [0, 1]\), so according to Brouwer’s fixed-point theorem, \(B(\alpha_1, \alpha_2)\) has a fixed point \((\alpha_1^*, \alpha_2^*)\) on \([0, 1] \times [0, 1]\). The fixed point is unique, because if there are two different fixed points \((\alpha_1^*, \alpha_2^*)\) and \((\alpha_1^{*'}, \alpha_2^{*'})\) with \(\alpha_1^* > \alpha_1^{*'}\), then the first equation implies \(\alpha_2^* > \alpha_2^{*'}\), but the second equation implies \(\alpha_2^* < \alpha_2^{*'}\), which is a conflict. If \(\alpha_1^* \leq \bar{\alpha}_{1p^J} (s_+)\), then this fixed point is the profile of equilibrium intensities under \(s_1\), i.e., \((\alpha_1^{P^J} (s_+), \alpha_2^{P^J} (s_+)) = (\alpha_1^*, \alpha_2^*)\). Otherwise, there does not exist an equilibrium of this kind.

Proof of Lemma 3.4

To simplify the notation, I define \(L_P^J \equiv p^{-1,P^J} (s_+) \cap [0, R_2]\). The definition of \(f_{p^J} (\omega|s_+)\) allows us to rewrite (3.30) as

\[
\begin{align*}
\ell_1' (\alpha_1^{P^J} (s_+)) &= \alpha_2^{P^J} (s_+) \int_{R_2}^{R_1} \frac{(R_1 - \omega) f(\omega) d\omega}{1 - F(R_2) + \Pr (L_P^J)}, \\
\ell_2' (\alpha_2^{P^J} (s_+)) &= \int_{R_2}^{1} \frac{(\omega - R_2) f(\omega) d\omega}{1 - F(R_2) + \Pr (L_P^J)} - \alpha_1^{P^J} (s_+) \int_{R_2}^{R_1} \frac{(\omega - R_2) f(\omega) d\omega}{1 - F(R_2) + \Pr (L_P^J)},
\end{align*}
\]

and the constraint is reduced to

\[
\alpha_1^{P^J} (s_+) \leq \bar{\alpha}_{1p^J} (s_+) \equiv 1 - \frac{\int_{R_1}^{1} (\omega - R_2) f(\omega) d\omega}{\int_{L_P^J}^{R_2} (R_2 - \omega) f(\omega) d\omega - \int_{R_2}^{R_1} (\omega - R_2) f(\omega) d\omega}. \quad (3.32)
\]

The composition of \(L_P^J\) determines the values of \((\alpha_1^{P^J} (s_+), \alpha_2^{P^J} (s_+))\) and whether (3.32) is satisfied.

Suppose that there is a set \(L_P^J\) making condition (3.32) satisfied, then we can always find an interval \([0, \bar{\omega} (L_P^J)]\) where the cutoff \(\bar{\omega} (L_P^J)\) satisfies

\[
F (\bar{\omega} (L_P^J)) = \Pr (L_P^J) \quad (3.33)
\]
and
\[
\int_{L_{PJ}} (R_2 - \omega) f(\omega) \, d\omega \leq \int_0^{\bar{\omega}(L_{PJ})} (R_2 - \omega) f(\omega) \, d\omega. \tag{3.34}
\]

The definition of $\bar{\omega}(L_{PJ})$ makes the interval $[0, \bar{\omega}(L_{PJ})]$ induce the same values of $\alpha_1^{P_J}(s_+)$ and $\alpha_2^{P_J}(s_+)$ as does $L_{PJ}$, due to (3.33) and (3.31), and also makes condition (3.32) satisfied, according to (3.32) and (3.34). Therefore, without loss of generality, we can focus on the intervals of the form $[0, \bar{\omega}]$ when characterize the optimal $L_{PJ}$. That is, $p^{P_J}(s_+)$ can be restricted to the form $[0, \bar{\omega}] \cup (R, 1]$.

**Proof of Proposition 3.2**

In the optimal joint-investigation equilibrium, the *ex ante* expected probability of launching the project is

\[
\alpha_2^{P_J}(s_+) \left[ 1 - \alpha_1^{P_J}(s_+) \right] \left[ F(R_1) - F(R_2) \right] + \alpha_2^{P_J}(s_+) \left[ 1 - F(R_1) \right].
\]

The expected probability of launching the project in the optimal unilateral-investigation equilibrium is expressed in (3.6). If $\alpha_2^{P_J}(s_+) \geq \alpha_1^{PU}(s_+)$, then the optimal joint-investigation equilibrium outperforms the optimal unilateral-investigation equilibrium. From (3.31), we have

\[
\alpha_2^{P_J}(s_+) \geq \varphi_2 \left( \int_{R_2}^{1} (\omega - R_2) f(\omega) \, d\omega \right) \int_{R_1}^1 (\omega - R_2) f(\omega) \, d\omega
\]

\[
\geq \varphi_2 \left( \int_{R_1}^1 (\omega - R_2) f(\omega) \, d\omega \right).
\]

According to (3.6), there is

\[
\alpha_1^{PU}(s_+) = \varphi_1 \left( \int_{R_1}^{1} (\omega - R_1) f(\omega) \, d\omega \right) \int_{R_1}^1 (\omega - R_1) f(\omega) \, d\omega.
\]
Condition (3.10) implies that $\alpha^P_{2J}(s_+) \geq \alpha^PU_1(s_+)$. Thus, the optimal joint-investigation equilibrium generates a higher probability of launching the project than the optimal unilateral-investigation equilibrium.

**Proof of Proposition 3.3**

From (3.6) and (3.31), we have

$$\frac{c'_1(\alpha^PU_1(s_+))}{c'_2(\alpha^P_{2J}(s_+))} \leq \frac{\int_{R_1}^1 (\omega - R_1) f(\omega) \, d\omega}{[1 - F(R_1) + F(\bar{\omega}^PU)] \int_{R_1}^1 (\omega - R_2) f(\omega) \, d\omega}.$$

Based on condition (3.11), we have

$$\inf_{\alpha \in (0,1)} \frac{c'_1(\alpha)}{c'_2(\alpha^P_{2J}(s_+))} \geq \frac{c'_1(\alpha^PU_1(s_+))}{c'_2(\alpha^P_{2J}(s_+))}.$$

Thus, it must be that $\alpha^P_{2J}(s_+) \geq \alpha^PU_1(s_+)$, which implies that the optimal joint-investigation equilibrium is the optimal equilibrium of the public-persuasion game.

**Possible Equilibrium Investigation Scenarios in Sequential-persuasion Games**

In sequential-persuasion games, there are two communication stages. Different types of Persuader reporting the same message in the first stage may not behave in the same way in the second stage. In the following analysis, I focus on a set of types who send the same message to the first listener. I analyze what responses of the two listeners that this set of persuader types possibly face.

In the investigation stage, each listener chooses among two types of strategies: (1) Investigate with positive intensity, (2) Do not investigate, i.e., investigate with 0 intensity. For expositional purposes, I use $IN$ and $NI$ to denote these two types of strategies, respectively. In the stage of making investment decision, Listener $i$ accepts the project if $\omega$ is identified to be larger than $R_i$, $i = 1, 2$. Thus, each listener has two alternative
investment strategies, which are (1) Reject the project unless $\omega$ is identified to be larger than $R_i$, and (2) Accept the project unless $\omega$ is identified to be smaller than $R_i$. I use $R$ and $A$ to denote these two alternatives, respectively. Therefore, each listener in equilibrium behaves in one of the four possible ways, $IN+R$, $IN+A$, $NI+R$, and $NI+A$. The strategies $NI+R$ and $NI+A$ mean respectively that rejection without investigation and acceptance without investigation. Below I analyze that for a set of persuader types inducing the first listener to behave in one of the four ways, what strategies of the second listener will be confronted in equilibrium by those who get the investment of the first Listener.

To begin, we look at the $(1,2)$-game, and study that if a subset $T$ of $\Omega$ induces one of the four following strategies of Listener 1, what strategies of Listener 2 will be confronted by Persuader types in $T$ surviving from Listener 1’s scrutiny.

1. $IN+R$

If $T$ induces this strategy of Listener 1, then it must be that both $T \cap (R_1, 1]$ and $T \cap [0, R_1]$ are non-empty, otherwise Listener 1 does not investigate. Only the types with $\omega > R_1$ in $T$ can possibly get the investment of Listener 1. If the types in $T \cap (R_1, 1]$ get Listener 1’s investment, what strategies of Listener 2 they will confront? It is not possible that all of them confront $NI+R$, because if so, Listener 1 would not investigate $T$ at all. If the types in $T \cap (R_1, 1]$ are not mixed with elements with $\omega \leq R_2$ when they approach Listener 2, the optimal strategy of Listener 2 is $NI+A$. What if they are mixed with some $\omega \leq R_2$ types? To answer this question, we first examine whether the types with $\omega \leq R_2$ can get the investment of Listener 1. If so, the strategies of Listener 1 confronted by these types must be $IN+A$ and/or $NI+A$. But it is impossible to have any subset of $\Omega$ induce these strategies in equilibrium while $T$ induces $IN+R$, because if $IN+A$ or $NI+A$ can be induced, then all the member of $T$ will deviate, as $IN+A$ and $NI+A$ can give them higher probabilities of success than does $IN+R$. Therefore,
it is impossible that survivors of $T$ are mixed with $\omega \leq R_2$ at the second stage of persuasion. This implies that if Listener 1 exerts $IN + R$, the types of Persuader in $T$ getting Listener 1’s investment will confront $NI + A$ of Listener 2. If some types in $\Omega/T$ can induce a different strategy of Listener 1, this induced strategy can only be $NI + R$.

2. $IN + A$

Similar to the case above, it is impossible that all the types in $T$ getting Listener 1’s investment confront $NI + R$ of Listener 2, because if that is the case, Listener 1 will deviate from investigation. Also, both $T \cap (R_1, 1]$ and $T \cap [0, R_1]$ should have positive measures, because otherwise Listener 1 has no incentive to investigate as well. Based on the equilibria of joint investigation in Subsection 3.4.1, it is straightforward to show that $IN + R$ and $NI + R$ of Listener 2 can be induced by some messages in the second stage of persuasion. Now we examine whether $IN + A$ or $NI + A$ can be induced in the second stage. First of all, it is not possible that $NI + A$ arises in the second stage of persuasion. Suppose that it arises, then one can verify that $IN + A, IN + R$ and $NI + R$ of Listener 2 cannot be induced by any message in the second stage. If $NI + A$ is the only response of Listener 2 in the second stage, then it must be that $T \cap [0, R_2]$ have zero measure, because otherwise it is not optimal for Listener 2 to adopt $NI + A$. Thus, $T \cap [0, R_2]$ must induce a response of Listener 1 different from $IN + A$. This response can only be $NI + A$, because only this response can give then an expected payoff not lower than does $IN + A$ of Listener 1. But if $NI + A$ exists, $T$ should be empty, as $T$ could mimic the report of $T \cap [0, R_2]$ in the first stage and stick to its report in the second stage, and makes its probability of success equal to 0.

If $IN + A$ is induced by some message, then no message in equilibrium induces $IN + R$ or $NI + R$ in the second stage of persuasion, as the first two give lower
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probabilities of success to any type than does $IN + A$. It is worth mentioning that when $IN + A$ is induced in the second stage, every $\omega \leq R_2$ in $T$ or mimicking the report of $T$ can get a positive probability of success. This implies that if $IN + A$ of Listener 2 is induced by some message in the second stage, $IN + A$ is the only equilibrium response that Listener 1 would have to any possible message in stage one, because if another response of Listener 1 could be induced, we can always find that some types of Persuader have incentive to deviate. Therefore, all types of Persuader report the same messages in stages one and two, and induce the listeners to respond with $IN + A$. This is a complete pooling equilibrium outcome involving no persuasive communication. Same in the public-persuasion game, we exclude this pooling equilibrium from the analysis.

3. $NI + A$

If the report of $T$ induces Listener 1 to adopt $NI + A$, then it should be true that in equilibrium no message at the second stage of persuasion inducing Listener 2 to exert $NI + A$, because otherwise any type mimicking the behavior of $T$ at the first stage and reporting the message inducing $NI + A$ at the second stage can yield probability 1 of success. Also, it is clear that $IN + A$ of Listener 2 cannot be induced when $IN + R$ and/or $NI + R$ can be induced by some message, as no $\omega \leq R_2$ will report the messages inducing $IN + R$ and/or $NI + R$. Suppose that $IN + A$ arises at the second stage, then it must be true that some positive measure of $\omega \leq R_2$ survived from the first-stage persuasion. This positive measure of $\omega \leq R_2$ necessarily confronted $NI + A$ of Listener 1, because all other strategies of Listener 1, even if they can be induced, yields lower probability of success. But it is never optimal for Listener 1 to adopt $NI + A$ facing a message including some of $\omega \leq R_2$, given that Listener 2 adopts $IN + A$. We can use similar argument to exclude $IN + R$ of Listener 2. Thus, it is only possible that $NI + R$ of Listener 2 can be induced at the second stage. This implies that $T \subset [0, R_2]$. Given this
strategy of Listener 2 and that $T$ induces $NI + A$ of Listener 1, the only other strategy of Listener 1 that can be induced in the same equilibrium is $NI + R$. In the equilibrium analysis, I will replace the strategy profile composed of $NI + A$ of Listener 1 and $NI + R$ of Listener 2 which is induced by a subset of $[0, R_2]$ by $NI + R$ of Listener 1. This replacement will facilitate the analysis without changing any result.

4. $NI + R$

If $T$ confronts $NI + R$, the game ends. It is possible that in the same equilibrium other strategies of Listener 1 are induced by some other messages. Specifically, $IN + R$ of Listener 1 can be induced when Listener 2 exerts $NI + A$ for any message received. $IN + A$ of Listener 1 can be induced as well. In this case, Listener 2 exerts $IN + R$ or $NI + R$ to messages received. $NI + A$ of Listener 1 can also arise when Listener 2 adopts $NI + R$ in the second stage to any message. But these three cases cannot coexist in an equilibrium.

Below I analyze the $(2,1)$-game. The pattern of the discussion for these cases is the same as above. We examine that for a subset $T$ of $\Omega$ inducing one of the four following strategies of Listener 2, what strategies of Listener 1 will be confronted by the types in $T$ getting the investment of Listener 2.

1. $IN + R$

The set $T$ inducing this strategy should satisfy that $T \cap (R_2, 1]$ and $T \cap [0, R_2]$ both have positive measures, otherwise Listener 2 makes no effort in investigation. In Subsection 3.4.1 I already provide an equilibrium in which in the second stage, $IN + A$ of Listener 1 is induced. If $IN + A$ of Listener 1 is induced by some message, then no message on the equilibrium path will induce $IN + R$, $NI + R$, or $NI + A$ of Listener 1, because (1) the types reporting the message inducing $IN + A$ include some $\omega \leq R_1$ and some $\omega > R_1$, and (2) $IN + A$ gives all types with $\omega \leq R_1$ a
higher probability of success than the first two strategies and gives the types with 
\( \omega \leq R_1 \) a lower probability of success than \( NI + A \). If \( NI + A \) of Listener 1 is not 
on the equilibrium path, it is also impossible to have \( IN + R, NI + R, \) and \( NI + A \) all 
happen with positive probabilities in equilibrium, because \( NI + A \) gives a higher 
probability of success to any type than does \( IN + R \) and \( NI + R \). Now I show that 
\( NI + A \) cannot be on the equilibrium path. Suppose that \( NI + A \) of Listener 1 is 
induced, then it must be that all the ones in \( T \) getting the investment of Listener 2 
have \( \omega > R_1 \). Thus, \( T \cap (R_2, R_1] \) has zero measure, and the types in \( (R_2, R_1] \) induce 
some responses of Listener 2 in the first stage different from \( IN + R \). The response 
of Listener 2 that can possibly be induced in equilibrium by the types in \( (R_2, R_1] \) is 
\( IN + A \), as all other responses give these types 0 probability of success. However, if 
the types in \( (R_2, R_1] \) confront \( IN + A \) of Listener 2, then all the types with \( \omega \leq R_2 \) 
in \( T \) will mimic the reports of \( (R_2, R_1] \), which contradicts to that \( T \cap [0, R_2] \) has 
positive measure. Therefore, \( NI + A \) of Listener 1 cannot arise in equilibrium in 
the second stage. If \( IN + R \) and/or \( NI + R \) of Listener 1 are induced in the second 
stage, then the optimal response of Listener 2 is to do no investigation. Thus, only 
\( IN + A \) of Listener 1 can happen in equilibrium in this case.

2. \( IN + A \)

If this strategy is induced by \( T \), then it must be that \( T \cap (R_2, 1] \) and \( T \cap [0, R_2] \) 
both have positive measures. In the second stage, it is impossible to have \( NI + A \) 
of Listener 1 induced, because all the types in \( (R_2, R_1] \) have a positive probability 
of getting Listener 2’s investment given that \( IN + A \) of Listener 2 can be induced, 
and these types will report a message inducing \( NI + A \) of Listener 1 in the second 
stage, as this gives them probability 1 of success. The behavior of the types in 
\( (R_2, R_1] \) implies that \( NI + A \) is not optimal for Listener 1 in the second stage. It 
is impossible that \( IN + R, NI + R, \) and \( IN + A \) of Listener 1 are all induced with 
positive probabilities in equilibrium, because \( IN + R \) and \( NI + R \) give any types
getting the investment of Listener a lower probability than does \( IN + A \). Also, \( IN + R \) and/or \( NI + R \) cannot be the only response of Listener 1 in the second stage, because if so, it is not optimal for Listener 2 to do any investigation in the first stage. Thus, it is only possible that \( IN + A \) of Listener 1 is induced in the second stage. Given this result, we can find that if \( IN + A \) is a response of Listener 2 in the first stage, the strategies \( IN + R \), \( NI + R \), and \( NI + A \) of Listener 2 cannot be induced by any message in equilibrium, because \( IN + R \) and \( NI + R \) give lower probabilities of success to any type than does \( IN + A \), and \( NI + A \) gives a higher probability of success to any type in \( T \cap (R_2, 1] \) than does \( IN + A \). Therefore, in this case all types of Persuader report the same messages in stages one and two, and induce the listeners to respond with \( IN + A \). This is a complete pooling equilibrium outcome involving no persuasive communication. Same as before, we exclude this pooling equilibrium from the analysis.

3. \( NI + A \)

If the report of \( T \) induces Listener 2 to adopt \( NI + A \), then it should be true that in equilibrium no message in the second stage of persuasion inducing Listener 1 to exert \( NI + A \), because otherwise any type mimicking the behavior of \( T \) in the first stage and reporting the message inducing \( NI + A \) at the second stage can yield probability 1 of success. Also, it is clear that \( IN + A \) of Listener 1 cannot be induced when \( IN + R \) and/or \( NI + R \) can be induced by some message in the second stage, as no one will report the messages inducing \( IN + R \) and/or \( NI + R \) in that case. Suppose that only \( IN + A \) of Listener 1 arises in the second stage, then one should note that \( T \cap [0, R_2] \) has zero measure, because otherwise it is not optimal for Listener 2 to exert \( NI + A \) in the first stage. However, \( NI + A \) of Listener 2 gives the types in \( [0, R_2] \) a higher probability of success than any other strategy of Listener 2. This is a contradiction. Thus, it is impossible that \( IN + A \) of Listener 1 arises at the second stage of persuasion. It is only possible that \( IN + R \) and/or
NI + R of Listener 1 are induced by some messages.

4. NI + R

If T confronts NI + R, the game ends. It is possible that in the same equilibrium other strategies of Listener 2 can be induced by some other messages. Specifically, NI + A of Listener 2 can be induced and Listener 1 responds with IN + R and/or NI + R to any message reported. IN + R of Listener 2 can be induced as well. In this case, Listener 1 responds with IN + A to any message reported. But these two cases cannot both happen in one equilibrium with positive probabilities.

**Proof of Lemma 3.6**

The technique for proving this lemma is similar to that adopted in Lemma 3.4.

For the (1, 2)-game, the investigation strategies of the listeners should satisfy

\[
c'_1(\alpha_{(1,2),1}(s_{+,1})) = \frac{\alpha_{(1,2),2}(s_{+,2}) \left[ \int_{R_2}^{R_1} (R_1 - \omega) f(\omega) \, d\omega \right]}{1 - F(R_2) + \Pr(L_{(1,2),1}^{SJ})},
\]

\[
c'_2(\alpha_{(1,2),2}(s_{+,2})) = \frac{\int_{R_2}^{1} (\omega - R_2) f(\omega) \, d\omega - \alpha_{(1,2),1}(s_{+,1}) \int_{R_2}^{R_1} (\omega - R_2) f(\omega) \, d\omega}{\alpha_{(1,2),1}(s_{+,1})[1 - F(R_1)] + (1 - \alpha_{(1,2),1}(s_{+,1})) \left[ 1 - F(R_2) + \Pr(L_{(1,2),2}^{SJ}) \right]},
\]

where \( L_{(1,2),1}^{SJ} = p^{1, SJ}_{(1,2)}(s_{+,1}) \cap [0, R_2] \) and \( L_{(1,2),2}^{SJ} = p^{1, SJ}_{(1,2)}(s_{+,1}, s_{+,2}) \cap [0, R_2] \), and \( p^{-1, SJ}_{(1,2)}(s_{+,1}, s) \) is the set of \( \omega \) reporting \( s_{+,1} \) to Listener 1 and \( s_{+,2} \) to Listener 2. Also, the following constraint should be satisfied,

\[
\alpha_{(1,2),1}(s_{+,1}) \leq \bar{\alpha}_{1p^{1, SJ}_{(1,2)}}(s_{+,2}) \equiv 1 - \frac{\int_{R_1}^{1} (\omega - R_2) f(\omega) \, d\omega}{\int_{L_{(1,2),2}^{SJ}} (R_2 - \omega) f(\omega) \, d\omega - \int_{R_2}^{R_1} (\omega - R_2) f(\omega) \, d\omega}.
\]
It is clear that \( (R_2, 1) \subset P^{-1, SJ}_{(1,2)}(s_{+1}, s_{+2}). \)

For any \( L^{SJ}_{(1,2),1} \) and \( L^{SJ}_{(1,2),2} \), I choose \( 0 \leq \bar{\omega}(L^{SJ}_{(1,2),2}) \leq \bar{\omega}(L^{SJ}_{(1,2),1}) \leq R_2 \) such that

\[
\Pr(L^{SJ}_{(1,2),1}) = F(\bar{\omega}(L^{SJ}_{(1,2),1})) \quad \text{and} \quad \Pr(L^{SJ}_{(1,2),2}) = F(\bar{\omega}(L^{SJ}_{(1,2),2})).
\]

This pair of \( F(\bar{\omega}(L^{SJ}_{(1,2),1})) \) and \( F(\bar{\omega}(L^{SJ}_{(1,2),2})) \) induce the same values of \( \alpha_{(1,2),1}(s_{+1}) \) and \( \alpha_{(1,2),2}(s_{+2}) \) as do \( \Pr(L^{SJ}_{(1,2),1}) \) and \( \Pr(L^{SJ}_{(1,2),2}) \), so the only thing we need to check is whether the constraint is satisfied. The LHS of the constraint is unchanged after replacing \( \Pr(L^{SJ}_{(1,2),1}) \) and \( \Pr(L^{SJ}_{(1,2),2}) \) with \( F(\bar{\omega}(L^{SJ}_{(1,2),1})) \) and \( F(\bar{\omega}(L^{SJ}_{(1,2),2})) \). The only term on the RHS related to \( L^{SJ}_{(1,2),1} \) and \( L^{SJ}_{(1,2),2} \) is \( \int_{L^{SJ}_{(1,2),2}} (R_2 - \omega) f(\omega) \, d\omega \). It is easy to show that

\[
\int_{L^{SJ}_{(1,2),2}} (R_2 - \omega) f(\omega) \, d\omega - \int_{0}^{\bar{\omega}(L^{SJ}_{(1,2),2})} (R_2 - \omega) f(\omega) \, d\omega \leq 0.
\]

Therefore, the RHS of the second constraint increases when we change from \( L^{SJ}_{(1,2),2} \) to \( [0, \bar{\omega}(L^{SJ}_{(1,2),2})] \). Therefore, the optimal \( L^{SJ}_{(1,2),1} \) and \( L^{SJ}_{(1,2),2} \) can always be expressed in the form of intervals \( [0, \bar{\omega}(L^{SJ}_{(1,2),1})] \) and \( [0, \bar{\omega}(L^{SJ}_{(1,2),2})] \).

**Characterization of the Joint-investigation Equilibria of Sequential-Persuasion Games**

For the \( (2,1) \)-game, the equilibrium investigation strategies of the listeners satisfy

\[
c_1' (\alpha_{(2,1),1} (s_{+1})) = \frac{\int_{R_2}^{R_1} (R_1 - \omega) f(\omega) \, d\omega}{1 - F(R_2)}, \tag{3.37}
\]

\[
c_2' (\alpha_{(2,1),2} (s_{+2})) = \frac{\int_{R_2}^{1} (\omega - R_2) f(\omega) \, d\omega - \alpha_{(2,1),1} (s_{+1}) \int_{R_2}^{R_1} (\omega - R_2) f(\omega) \, d\omega}{1 - F(R_2) + F(\bar{\omega})}, \tag{3.38}
\]

and
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\[ \alpha_{(2,1)}(s_{+,1}) \leq \bar{\alpha}_{1p_{(2,1)}}(s_{+,2}) \equiv 1 - \frac{\int_{R_1}^{1} (\omega - R_2) f(\omega) d\omega}{\left[ \int_{0}^{\omega} (R_2 - \omega) f(\omega) d\omega - \int_{R_1}^{R_2} (\omega - R_2) f(\omega) d\omega \right]} . \]

For the \( (1,2) \)-game, the equilibria are characterized by (3.35) and (3.36).
Bibliography


