Superconducting Interfaces
Equivariant Solutions to a System of Nonlinear Wave Equations with
Ginzburg-Landau Type Potential

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Abstract

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In this thesis, we look for solutions to a two-component system of nonlinear wave equations with the properties that one component has an interface and the other is exponentially small except near the interface of the first component. The second component can be identified with a superconducting current confined to an interface. A formal analysis suggests that for suitable initial data, the energy of solutions concentrate about a codimension one timelike surface $\Gamma$ whose dynamics are coupled in a highly nonlinear way to the phase of the superconducting current. We provide a rigorous verification of the predictions these formal arguments make for solutions with an equivariant symmetry in two spatial dimensions subject to a non-degeneracy condition. The contents of this thesis are based on the results presented in [32].
Dedication

To Linda.
Acknowledgements

My thesis could not have been possible without the support of some special people in my life.

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Lastly, I would like to thank my wife and best friend Linda. Her unremitting encouragement made this whole endeavour possible and all the more worthwhile.
1 Introduction

1.1 Synopsis

For this thesis we consider two-component systems of hyperbolic PDEs qualitatively similar to
\[
\begin{cases}
\partial_{tt}\phi - \Delta \phi + \frac{\lambda}{\epsilon^2} (\phi^2 - 1)\phi = -\frac{\beta}{\epsilon^2} |\sigma|^2 \phi \\
\partial_{tt}\sigma - \Delta \sigma + \frac{\lambda}{\epsilon^2} (|\sigma|^2 - 1)\sigma = -\frac{\beta}{\epsilon^2} \phi^2 \sigma,
\end{cases}
\]
where \((\phi, \sigma) : \mathbb{R}^{1+n} \to \mathbb{R} \times \mathbb{C}, 0 < \epsilon \ll 1\) is a small parameter of the model, and \((\lambda_\phi, \lambda_\sigma, \beta)\) are real, non-negative constants. We are interested in solutions to (1.1) with the properties that

1a) \(\phi\) has an interface.

1b) \(\sigma\) is exponentially small except near the interface.

For the first equation of (1.1), if \(\beta = 0\) or if \(\sigma = 0\), then the right hand side of (1.1) vanishes and the system decouples. In this case, \(\phi\) satisfies
\[
\partial_{tt}\phi - \Delta \phi + \frac{\lambda_\phi}{\epsilon^2} (\phi^2 - 1)\phi = 0
\]
and it was shown in [17] that there exists \(\phi\) with an interface solving this equation. We, however, would like to consider regimes where \(\phi\) and \(\sigma\) are coupled (i.e. \(\beta \neq 0\)) and where \((\lambda_\phi, \lambda_\sigma, \beta)\) are chosen so that \((\phi, \sigma)\) have the properties described above, which in particular stipulate that \(\sigma \neq 0\) near the interface of \(\phi\). For these regimes, it follows from the physics literature on superconducting strings, reviewed in section 2 below, that the \(\sigma\)-field can naturally be identified with a superconducting current confined to the interface of \(\phi\). Hence, we call (1.1) the superconducting interface model. A goal of this thesis is to understand the coupling between the current and the interface and, in particular, understand how the current affects the dynamics of the interface.

As discussed in appendix A, a formal asymptotic expansion suggests that in suitable local coordinates \((y^\tau, y^\nu) = (y_0, \ldots, y_n)\) near a codimension one timelike surface \(\Gamma\), with \(y^\tau = (y_0, \ldots, y_{n-1})\) parameterizing \(\Gamma\) and with \(y^\nu = 0\) corresponding to \(\Gamma\), then there should exist a solution to (1.1) satisfying
\[
\begin{cases}
\phi(y^\tau, y^\nu) \approx \phi_0(y^\nu; \zeta(y^\tau)) \\
\sigma(y^\tau, y^\nu) \approx e^{i\theta(y^\tau)} \sigma_0(y^\nu; \zeta(y^\tau)),
\end{cases}
\]
where...
(2a) $\theta$ is a function of $y^\tau$ only.

(2b) $\zeta(y^\tau) := \gamma(\nabla_\tau \theta, \nabla_\tau \theta)$, where $\nabla_\tau$ denotes the tangential gradient along $\Gamma$ and $\gamma_{ij}$ is the induced metric on $\Gamma$ (the ambient metric for this problem is the Minkowski metric - denoted $\eta$).

(2c) For each $\rho \in \mathbb{R}$, $\Phi_0(\cdot; \rho) := (\phi_0, \sigma_0)(\cdot; \rho) : \mathbb{R} \to \mathbb{R}^2$ satisfies the minimization problem.

$$
\mu(\rho) = \inf_{(f, s) \in \mathcal{A}} \int \left\{ \frac{1}{2} |(f', s')|^2 + \frac{\lambda_s}{4} (f^2 - 1)^2 + \frac{\lambda_{\sigma}}{4} (s^2 - 2) s^2 + \frac{\beta}{2} f^2 s^2 + \frac{1}{2} \rho s^2 \right\}
$$

$$
\mathcal{A} := \left\{ (f, s) \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) : \lim_{y^\tau \to \pm \infty} f(y^\tau) = \pm 1, \ f(0) = 0 \right\}.
$$

In particular, the profiles $\phi_0$ and $\sigma_0$ in (1.2) are determined by $\zeta(y^\tau)$.

(2d) $\theta$ and $\Gamma$ satisfy the highly nonlinear, coupled system of PDEs

$$
\Box_\Gamma \theta = -\gamma(\nabla_\tau \log [\mu'(\zeta)], \nabla_\tau \theta) \quad \text{(1.3)}
$$

$$
\vec{H} = 2 \frac{\mu'(\zeta)}{\mu(\zeta)} \mathbb{I}(\nabla_\tau \theta, \nabla_\tau \theta), \quad \text{(1.4)}
$$

where $\vec{H}$ is the mean curvature vector of $\Gamma$ and $\mathbb{I}$ is the second fundamental form of $\Gamma$.

In this thesis, we verify, subject to a non-degeneracy condition, that there does indeed exist a solution to (1.1) satisfying (1.2) when $n = 2$ and when $\Phi$ is an equivariant map.

![Figure 1](image_url)

(a) Nodal Set

(b) Profiles

Figure 1: The formal asymptotic expansion suggests that there exists a solution $\Phi = (\phi, \sigma)$ to (1.1) so that for $\theta$ and $\Gamma$ satisfying (1.3) - (1.4), then at each $p \in \Gamma$ we expect that as we move away from $\Gamma$ in the transverse direction $\phi$ looks like the black curve in (1b) and $\sigma$ looks like $e^{\frac{1}{2} \theta(\varnothing M(p))} \sigma_0$ where $\sigma_0$ looks like the red curve in (1b). Looking at figure 1a, this means that $\sigma$ is exponentially small except near $\Gamma$, $\phi \approx -1$ inside $\Gamma$, $\phi \approx 1$ outside of $\Gamma$, and $\phi$ transitions from $-1$ to $1$ near $\Gamma$. 


It can be shown that if the winding number density $\gamma^{i/j}\partial_i\theta\partial_j\theta$ is sufficiently large, then the $\sigma_0$-field of the approximate solution is 0. It is believed that there are regimes where a solution may initially have a non-zero current (i.e. $\sigma_0(z; \gamma^{i/j}\partial_i\theta\partial_j\theta) \neq 0$), but as the system evolves the solution may lose its current. This type of phenomena is referred to as **current quenching** \[33\] and we show in section \[3.4.2\] below that given suitable initial conditions that the solutions we find undergo current quenching.

To the best of our knowledge, this is the first work to consider solutions to a two-component hyperbolic system with interfaces.

**Mathematical Background**

There is an extensive mathematical literature with results that are of the type we obtain in this thesis. The unifying theme of these types of results is

- For certain PDEs, there exist solutions which have interfaces, point vortices, or vortex filaments whose dynamics are approximately described by solutions to some associated geometric problem.

See \[17\] for a detailed account of these types of results for the scalar elliptic, scalar parabolic, and scalar hyperbolic counterparts of \[1.1\].

The scalar analogue of \[1.1\] is

$$\partial_u u - \Delta u + \frac{\lambda}{\epsilon^2} (u^2 - 1)u = 0,$$  \[1.5\]

where $u : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$. In \[24\], a formal analysis of solutions to \[1.5\] with interfaces is carried out in the case when $n = 3$. In \[27\], the authors further develop the techniques and arguments used in \[24\] to carry out a formal analysis of solutions with interfaces to other more complicated nonlinear scalar wave equations when $n = 2$. In appendix \[A\] we will use some of the ideas introduced in \[24\] and \[27\] to carry out a formal analysis of solutions to \[1.1\] with superconducting interfaces. As a result of this formal analysis, we obtain an effective action which, at least formally, describes to leading order the evolution of the interface and current associated to solutions with a superconducting interface.

In \[9\], the author shows that there exists solutions to \[1.5\] with an interface when $n = 3$ and $\epsilon = 1$. In particular, he looks for solutions of the form

$$u(t, x) = \tanh\left(\frac{x^3}{\sqrt{2}}\right) + w(t, x)$$  \[1.6\]
and shows that for suitable initial data, then there exists a global in time solution to (1.5) of the form (1.6) with \( w(t, x) \) satisfying the bound
\[
|w(t, x)| \lesssim \frac{1}{\sqrt{1 + t}}.
\]

More recently, it has been shown [17] that for any smooth codimension one timelike minimal surface \( \Gamma \), then for suitable initial data, there exists a solution to (1.5) with an interface located near \( \Gamma \), at least up to some time \( T \) independent of the initial data and \( \epsilon \). A similar result was then obtained in [11] for nonlinear wave equations like (1.5), but whose potentials are qualitatively similar to
\[
V'(u) = \frac{\lambda}{\epsilon^2}(u^2 - 1)(2u - \kappa),
\]
where \( \kappa \) is some positive constant of the model. Roughly speaking, the results obtained in [17] and [11] use weighted energy estimates to show that for suitable initial data, there exists an solution to (1.5) satisfying
\[
u \approx q\left(\frac{d_M}{\epsilon}\right),
\]
where \( q = \tanh\left(\frac{1}{\sqrt{2}}\right) \) and \( d_M \) is the Minkowski distance to a codimension one timelike minimal surface \( \Gamma \).

We should also mention the results of [7]. In this paper, the authors study the \( \epsilon \to 0^+ \) limit of solutions to (1.5). However, to obtain their main result, they need to make a technical assumption that is not easily verified.

One can also consider a version of (1.5) for which \( u : \mathbb{R}^{1+n} \to \mathbb{C} \). In this case, the goal is to find and describe solutions to (1.5) that have vortices or vortex filaments. Results describing point vortices and/or vortex filaments in (1.5) and a gauged version of (1.5) have been obtained in [16, 21, 17] and [13, 10], respectively. Of particular interest to us are some the results found in [17]. In this paper, the author shows that for any smooth codimension two timelike minimal surface \( \Gamma \), then for suitable initial data, there exists a solution to (1.5) that has a vortex filament located near \( \Gamma \), at least up to some time \( T \) independent of the initial data and \( \epsilon \). A noteworthy observation is that the dynamical law associated to the Nambu-Goto action [23, 12] describes exactly how the vortex filaments of solutions found in [17] evolve.

Similarly for us, we could consider the case when \( \phi : \mathbb{R}^{1+n} \to \mathbb{C} \). We would then like to find solutions to (1.1) so that \( \phi \) has a vortex filament and \( \sigma \) is exponentially small except near the vortex filament of \( \phi \) and we would like to find a geometric description of the evolution of the
vortex filament. For now, though, we focus our attention on the case when $\phi : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ and look for solutions with an interface.

Two-component systems have been considered in the physics literature as models for interfaces, point vortices, or vortex filaments in various physical systems [20, 15]. However, rigorous mathematical descriptions of solutions to two-component systems of the type we consider are sparse in the math literature. For example, progress on the existence and classification of solutions with interfaces or vortices has been made for various two-component, elliptic systems [3, 5, 6, 4] and (potentially very complicated) ground states of other two-component models subject to physically relevant forcing has been studied [22, 1, 2].

In contrast to [17, 11], who use weighted energy estimates, we linearize (1.1) about an approximate solution obtained using a formal asymptotic expansion and use spectral properties of the linearized operator to show that there exists an exact solution of (1.1) which is close to this approximate solution. We were inspired to take this approach by works such as [29, 30, 13] who use a similar approach. The reason we use this “linearization” approach is that in order to resolve the new complexities introduced by the coupling of the current to the interface of $\phi$, a more detailed description of solutions is required that seems hard to obtain using weighted energy estimates.

1.2 Description of Results

1.2.1 Assumptions

In this thesis, we are not necessarily considering (1.1), but rather systems qualitatively similar to (1.1). Furthermore, we will only be considering the $n = 2$ case and we will only be looking for solutions to these systems that are equivariant maps. That is, for $x \in \mathbb{R}^2$, we will look for solutions to (1.1) of the form

$$\Phi = \begin{pmatrix} \tilde{\phi}(t, |x|) \\ e^{i \frac{d^2}{2} \arg(x)} \tilde{\sigma}(t, |x|) \end{pmatrix}$$

(1.7)

where $(\tilde{\phi}, \tilde{\sigma}) : \mathbb{R}^{1+1} \rightarrow \mathbb{R}^2$ and $d \in 2\pi \mathbb{Z}$ is a fixed constant.

Using assumption (1.7), the systems we are interested in reduces to the following

$$\partial_{rr} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \partial_{rr} \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \frac{1}{r} \partial_r \begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} + \frac{1}{\epsilon^2} \begin{pmatrix} \partial \phi V(\tilde{\phi}, \tilde{\sigma}) \\ \partial \sigma V(\tilde{\phi}, \tilde{\sigma}) \end{pmatrix} + \frac{1}{\epsilon^2 r^2} \begin{pmatrix} 0 \\ \tilde{\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

(1.8)
where \((t, r) \in \mathbb{R} \times \mathbb{R}_+\). Throughout this thesis we assume that \(V : \mathbb{R}^2 \to \mathbb{R}\) and \(\nabla_{\Phi} V := (\partial_{\phi} V, \partial_{\sigma} V) : \mathbb{R}^2 \to \mathbb{R}^2\) satisfy

1. \(V \in C^\infty(\mathbb{R}^2, \mathbb{R})\) and \(V(\phi, \sigma) = V(|\phi|, \sigma) = V(\phi, |\sigma|) = V(|\phi|, |\sigma|)\).

2. \(V\) has local minima at \((1, 0)\) and \((-1, 0)\) and \(V(\pm 1, 0) = 0\), \(V\) has saddle points at \((0, 1)\) and \((0, -1)\), and \(V\) has a local max at \((0, 0)\) with \(\nabla_{\Phi} V \neq (0, 0)\) otherwise. Also, for each fixed \(\phi \in [0, 1]\), we want \(V(\phi, 1) \leq V(\phi, \sigma)\) for all \(\sigma \geq 1\). Similarly, for each fixed \(\sigma \in [0, 1]\), we want \(V(1, \sigma) \leq V(\phi, \sigma)\) for all \(\phi \geq 1\).

3. \(|\text{Hess}_{\Phi} V(\phi, \sigma)| \leq 1 + |(\phi, \sigma)|^2\) and \(\text{Hess}_{\Phi} V(\pm 1, 0) \geq cI\) where \(I\) is the \(2 \times 2\) identity matrix and \(c > 0\).

4. \(V\) satisfies a non-degeneracy condition - see (1.21) below.

5. Solutions to an associated minimization problem determined by \(V\) are unique - see (1.23).

Note that the angular part of the Laplacian gives rise to the \(\frac{d^2}{dr^2}\) term. The initial data of (1.8) that we consider in this thesis will be described in section 1.3 below.

For the rest of the thesis, we will be looking for and studying solutions to (1.8) with properties (1a) and (1b). For notational convenience, we drop the \(\sim\)'s from \(\tilde{\phi}\) and \(\tilde{\sigma}\). We also define

\[
W(\Phi, R) := V(\Phi) + \frac{1}{2\sigma^2} \frac{d^2}{d\sigma^2} \sigma^2.
\] (1.10)

We call \(W\) the \textbf{shifted potential} and we will denote the gradient of the shifted potential as

\[
w(\Phi, r) := \nabla_{\Phi} W(\Phi, r).
\] (1.11)

Two important consequences of assuming that \((\phi, \sigma)\) is equivariant is the following. First, if \(\phi\) has an interface, then the interface is a surface of revolution about the \(t\)-axis. Second, if \(\phi\) has a timelike interface \(\Gamma\), then \(\Gamma\) is parameterized by \((y^0, R(y^0) \cos(\theta), R(y^0) \sin(\theta))\) for some function \(R\). In this case, the interface of \(\tilde{\phi}\) is parameterized by \((y^0, R(y^0))\). These two observations will be used in the formulation of an ansatz to be discussed in the next section.
1.2.2 Formal Analysis

Before moving on, we introduce some notation. Let $\Gamma$ be a smooth codimension one timelike surface parameterized by $(\tau, R(\tau))$ for $0 \leq \tau \leq T$.

**Lemma 1.2.1.** There exists a neighbourhood $N$ of $\Gamma$ on which there exists a differentiable solution to
\[
\begin{cases}
-\partial_t d_M^2 + \partial_r d_M^2 = 1 & \text{on } N \\
d_M = 0 & \text{on } \Gamma.
\end{cases}
\]

Furthermore, there exists $s_M : \mathbb{R}^{1+1} \to \mathbb{R}$ satisfying
\[
-\partial_t d_M \partial_t s_M + \partial_r d_M \partial_r s_M = 0 \quad \text{on } N
\]
\[
(s_M(t, r), R(s_M(t, r))) = (t, r) \quad \text{on } \Gamma
\]
so that
\[
(t, r) = (s_M(t, r), R(s_M(t, r))) + \frac{d_M(t, r)}{\sqrt{1 - R'(s_M(t, r))^2}} (R'(s_M(t, r)), 1).
\]

We call $d_M$ the **Minkowski distance to** $\Gamma$ and $s_M$ the **Minkowski projection to** $\Gamma$.

A proof for lemma 1.2.1 can be found in [18].

Following our discussion in appendix A, we expect that for suitable surfaces $\Gamma$ parameterized by $(y^0, R(y^0))$ and profiles $F_0$, that there exists a solution to (1.8) with properties (1a) and (1b) satisfying
\[
\Phi(t, r) \approx F_0(d_M; R(s_M)).
\]

However, when $\sigma \neq 0$ the coupling between the $\phi$-field and $\sigma$-field introduces new subtleties into the nature of the solutions that necessitates a more detailed description. Thus, we include the leading order correction in our analysis and look for solutions of the form
\[
\Phi(t, r) \approx F_0(\frac{d_M}{\epsilon}; R(s_M)) + \epsilon F_1(\frac{d_M}{\epsilon}; R(s_M), R'(s_M)).
\]

Notice that $F_1$ depends on $R$ and $R'$. In a moment we will see how $F_1$ depends on $R'$. We could use the same notation as we use in appendix A and write $F_0 = F_0(x; \frac{d^2}{\epsilon R^2})$, but we write $F_0 = F_0(x; R)$ for convenience.

Motivated by formal arguments, see section 3.2 below, for fixed $R \in \mathbb{R}_+$ we define $F_0(\cdot; R)$ to
be a solution of the minimization problem

$$\inf_{F \in \mathcal{A}} \int \left\{ \frac{1}{2} |F'(x)|^2 + W(F(x), R) \right\} dx$$

$$\mathcal{A} := \{ F = (f, s) \in H^1_{\text{loc}} : \lim_{x \to \pm\infty} F(x) = (\pm 1, 0), \ f(0) = 0 \}$$

and for fixed \((R, v) \in \mathbb{R}_+ \times (-1, 1)\) we define \(F_1(\cdot; R, v)\) to be a solution of

$$L_1(F_0(x; R), R)F_1 = g(R, v)F_0'(x; R) + 2\frac{x}{\sqrt{1 - v^2}} R^3 \left( \begin{array}{c} 0 \\ s_0(x; R) \end{array} \right)$$

$$\lim_{x \to \pm\infty} F_1(x; R, v) = 0,$$

where \(L_1(F_0, R)\) and \(g(R, v)\) are defined to be

$$L_1(F_0, R) := -\frac{d^2}{dx^2} I_{2 \times 2} + \text{Hess}_\phi W(F_0(\cdot); R)$$

$$g(R, v) := \frac{1}{\sqrt{1 - v^2}} \frac{d^2}{R^3} \frac{\|s_0(\cdot; R)\|^2_2}{\|F_0'(\cdot; R)\|^2_2}. \quad (1.18)$$

Here we see that \(F_1\) also depends additionally on a parameter \(v \in (-1, 1)\). Further, we define \(R(\tau)\) to be a solution to

$$\frac{1}{\sqrt{1 - (R')^2}} \left( \frac{R'}{1 - (R')^2} + \frac{1}{R} \right) = g(R, R')$$

$$R(0) \in (r_0, r_1) \quad \text{and} \quad R'(0) = 0. \quad (1.20)$$

We will define \(r_0\) and \(r_1\) in a moment.

Note that a necessary condition for (1.17) to have a solution is that

$$g(R, v)F_0'(x; R) + 2\frac{x}{\sqrt{1 - v^2}} R^3 \left( \begin{array}{c} 0 \\ s_0(x; R) \end{array} \right) \in \ker(L_1(F_0, R))^\perp,$$

where \(\perp = \perp_{L^2}\) (see below for more details). Also note that since \(F_0\) has an interface, then the \(\phi\)-field of any solution to (1.8) satisfying (1.15) also has an interface and the interface of the \(\phi\)-field is near a codimension one timelike surface parameterized by \(R\).
Suppose $F_0$ satisfies the minimization problem (1.16). Then, it solves
\[ -(F_0)'' + \nabla \Phi W(F_0, R) = 0. \]
Differentiating this with respect to $x$, we see that
\[ L_1(F_0, R)F_0' = 0 \]
and hence $F_0' \in \ker(L_1(F_0, R))$. For this thesis, we only consider potentials $V$ satisfying
\[ \text{If } r_0 < R < r_1, \text{ then } \ker(L_1(F_0; R)) = \text{span}\{F_0'\}. \quad (1.21) \]
This is the non-degeneracy condition we alluded to in (1.9). Later in the thesis we will see that if $R = R(\tau)$ is a solution to (1.20) with $r_0 < R(\tau) < r_1$, then
\[ g(R(\tau), R'(\tau))F_0'(x; R(\tau)) + 2 \frac{x}{\sqrt{1 - (R'(\tau))^2}} R(\tau)^3 \begin{pmatrix} 0 \\ s_0(x; R(\tau)) \end{pmatrix} \in \ker(L_1(F_0, R(\tau)))^\perp. \]
This is only true whenever $R(\tau) \in (r_0, r_1)$ and is one of the reasons why we demand that
\[ R(0) \in (r_0, r_1) \quad \text{and} \quad R'(0) = 0 \quad (1.22) \]
In section 3.3, we will show that for every $R \in \mathbb{R}_+$, there exists a minimizer of (1.16), see proposition 3.3.2. Once we have that $F_0(\cdot; R)$ exists, then we can use the non-degeneracy condition to show that $F_1$ exists, see proposition 3.3.4.

We will also show that there exists $R$ solving (1.20), but in order to show this, we need to know that $F_0(\cdot; R)$ is continuous in $R$. In order to show this, it suffices to know that for each fixed $R$, there exists a unique minimizer of (1.16), see proposition 3.3.3. The final condition we assume $V$ satisfies in this thesis is

Let $r_0$ and $r_1$ be the constants from (1.21). We assume that for each $R \in (r_0, r_1)$, if $F_0(\cdot; R)$ and $\tilde{F}_0(\cdot; R)$ are both solutions of the minimization problem (1.16), then
\[ F_0(\cdot; R) = \tilde{F}_0(\cdot; R). \quad (1.23) \]
Remark: We expect that conditions 4 and 5 of (1.9) (i.e. the conditions specified in (1.21) and (1.23), respectively) to hold for most potentials satisfying conditions 1-3 of (1.9). This, however, is not verified in this thesis.
In section 3.5 we show that for each fixed \((R, v) \in \mathbb{R}_+ \times (-1, 1)\), minimizers of \((1.16)\) and solutions to \((1.17)\) are exponentially small. In section 3.6 we show that \(F_0(y; R)\) is smooth in \(y\) and \(R, F_1(y; R, v)\) is smooth in \(y, R,\) and \(v,\) and that \(R\) is a smooth function. We need to establish these properties in order to prove theorem 1.2.2, the main result of this thesis.

**Remark:** Before moving on to the statement of the main result of this thesis, we remark that it is a simple calculation to show that the results of the formal expansion described in this section agree exactly with the results of the formal expansion described in (2a) - (2d). Something noteworthy to observe is that by looking for solutions to (1.1) that are equivariant and satisfy (1.2), then the geometric problem specified in (2d) greatly simplifies.

### 1.2.3 Main Result

The conclusion of theorem 1.2.2 verifies that, up to the correction term \(F_{1,1}\) to be defined in a moment, there exists a solution to (1.1) satisfying (1.2) when \(n = 2\) and when \((\phi, \sigma)\) is equivariant. In particular, we verify that the evolution of interfaces with a current in (1.1) is described as predicted in (1.4).

In the statement of our main result we use the notation

\[
\|\xi\|^2_{H^1} := \|\xi\|^2_{H^1} + \frac{1}{\epsilon^2} \|\xi\|^2_{L^2}
\]

\[
\|\xi\|^2_{L^2} := \int_0^\infty |\xi|^2.
\]

Also define \(F_{1,1}(\cdot; R, v, a)\) as the solution to

\[
L_1(F_0, R)F_{1,1} = -\frac{a}{\sqrt{1 - v^2}} \partial_x w(F_0, R)
\]

\[
\lim_{x \to \pm \infty} F_{1,1}(x; R, v, a) = 0.
\]

Following the statement of theorem 1.2.2 we will discuss what \(F_{1,1}\) is and what role it plays.

Also, throughout this thesis we will use the following notation

\[
\text{We say } a \lesssim b \text{ if and only if there exists a constant } C > 0 \text{ so that } a \leq Cb.
\]

**Remark:** In this thesis, the implicit constant \(C\) appearing in every estimate is independent of \(\epsilon,\) but is possibly dependent on the constants appearing in (1.9), \(N, R, R', R'',\) and the \(L^2\)-norm of
the partial derivatives of \( F_0 \) and \( F_1 \) (we will show later that these are indeed finite).

The main result of this thesis is the following.

**Theorem 1.2.2.** Let \( V \) be a potential satisfying (1.9) and suppose \( F_0 \) minimizes (1.16) and \( F_1 \) and \( R \) solve (1.17) and (1.20), respectively, with initial data (1.22). Let \( T \) be the maximal time of existence of \( R \), chosen so that \( R(\tau) \in (r_0, r_1) \) for all \( 0 \leq \tau < T \), and for \( \Gamma := \{(\tau, R(\tau)) : 0 \leq \tau < T\} \), let \( N \) be the neighbourhood from lemma 1.2.1. Then, for suitable initial data, there exists a solution \( \Phi \) to (1.8), a constant \( 0 < \bar{T} \leq T \), independent of \( \epsilon \), and a function \( a : [0, \bar{T}] \to \mathbb{R} \) so that for

\[
N_T := N \cap [0, \bar{T}] \times \mathbb{R}_+ \\
\bar{F}_0 = F_0 \left( \frac{d_M - a(s_M)}{\epsilon}; R(s_M) \right) \\
\bar{F}_1 = F_1 \left( \frac{d_M - a(s_M)}{\epsilon}; R(s_M), R'(s_M) \right) + F_{1,1} \left( \frac{d_M - a(s_M)}{\epsilon}; R(s_M), R'(s_M), \frac{a(s_M)}{\epsilon} \right)
\]

we have

\[
\text{For all } (t, r) \in N_T, \ |a(s_M)|, |a'(s_M)| \lesssim \epsilon \tag{1.27}
\]

\[
\|\Phi - \bar{F}_0 - \epsilon \bar{F}_1\|_{L^2_t H^1_r(N_T)} \lesssim \epsilon^2 \tag{1.28}
\]

\[
\|\partial_t \left[ \Phi - \bar{F}_0 - \epsilon \bar{F}_1 \right]\|_{L^2_t L^2_r(N_T)} \lesssim \epsilon^2 \tag{1.29}
\]

\[
\Phi(t, r) = (-1, 0) \text{ for } (t, r) \in [0, \bar{T}] \times [0, R(0) - \bar{T}) \tag{1.30}
\]

\[
\Phi(t, r) = (1, 0) \text{ for } (t, r) \in [0, \bar{T}] \times (R(0) + \bar{T}, \infty), \tag{1.31}
\]

where \( d_M \) is the Minkowski distance to \( \Gamma \) and \( s_M \) is the Minkowski projection to \( \Gamma \), both defined in lemma 1.2.1.

The proof of theorem 1.2.2 may be found following the statement of theorem 4.2.1.

Note that \( F_0 \) has a translation symmetry in its first argument, but \( F_1 \) does not. Thus, when we shift \( F_1 \) by \( \frac{\epsilon}{\epsilon} \) we introduce an error into our approximate solution and the \( F_{1,1} \) term corrects this error.

### 1.3 Initial Data and the Existence of Solutions

In appendix B, we prove that for suitable initial data that is sufficiently regular and which decays to \((1, 0)\) at \( \infty \) fast enough, (1.1) is globally well posed. The proof is completely standard and is done for the reader’s convenience.
Recall, for each $R \in \mathbb{R}_+$, we have picked $F_0(\cdot; R)$ to be a minimizer of (1.16) and we have chosen $R$ to be a solution of (1.20) with initial data (1.22). Let $T$ be the maximal time of existence of $R$, chosen so that $R(\tau) \in (r_0, r_1)$ for all $0 \leq \tau < T$. Define $\Gamma := \{(\tau, R(\tau)) : 0 \leq \tau < T\}$. In section 3.6 we show that $R(\tau)$ is smooth for $0 \leq \tau < T$. Thus, lemma 1.2.1 implies that there exists a neighbourhood $\mathcal{N}$ of $\Gamma$ on which $d_M$, also defined in lemma 1.2.1, is well defined.

Since $d_M$ is continuous, we can find a maximal $0 < \hat{T} \leq T$ so that

$$\mathcal{B} := [0, \frac{1}{2}\hat{T}] \times (R(0) - \hat{T}, R(0) + \hat{T}) \subset \mathcal{N}.$$ 

We will always consider initial data satisfying

$$\Phi(0, r) = \begin{cases} (-1, 0) & \text{for } 0 \leq r < \frac{1}{2}\hat{T} \\ (1, 0) & \text{for } r > \frac{1}{2}\hat{T} \end{cases}$$

$$\partial_t \Phi(0, r) = 0 \quad \text{for } 0 \leq r < \frac{1}{2}\hat{T} \text{ and } r > \frac{1}{2}\hat{T}. \quad (1.32)$$

Since (1.8) is a wave equation, there is a finite speed of propagation of data. For system (1.8) we consider in this thesis, the speed of propagation is always 1. Thus, solutions $\Phi$ of (1.8) satisfying (1.32) will also satisfy

$$\Phi(t, r) = \begin{cases} (-1, 0) & \text{for } 0 \leq r < R(0) - \hat{T} \\ (1, 0) & \text{for } r > R(0) + \hat{T} \end{cases}$$

for all $0 \leq t \leq \frac{1}{2}\hat{T}$. This reduces the analysis to controlling the error between $\Phi$ and the right hand side of (1.15) on the region where $\Phi$ transitions from $(-1, 0)$ to $(1, 0)$ - the region $\mathcal{N}_{\frac{1}{2}\hat{T}}$.

Roughly speaking, we will choose the initial data of $\Phi$ on $(R(0) - \frac{1}{2}\hat{T}, R(0) + \frac{1}{2}\hat{T})$ so that $\Phi$ is sufficiently close, in some sense, to the right hand side of (1.15) at $t = 0$. A precise specification of initial data on this set is not important at the moment and will be discussed in detail later.
\[ B = [0, \frac{1}{2} T] \times (R(0) - \hat{T}, R(0) + \hat{T}) \]

Figure 2: $\mathcal{N}$ is a neighbourhood on which the Minkowski normal coordinates are well defined. Pick the largest rectangle $B = [0, \frac{1}{2} \hat{T}] \times (R(0) - \hat{T}, R(0) + \hat{T})$ that lies within $\mathcal{N}$. We then pick the initial data of (1.8) so that $\Phi = (-1, 0)$ to the left of $B$ and $\Phi = (1, 0)$ to the right of $B$.

2 Physical Motivation: Superconducting Strings

Motivated by [25], Witten introduced a two-component model, closely related to the abelian-Higgs model, to describe finite energy solutions with vortex filaments supporting superconducting currents [34]. We call this model the superconducting string model. It was our initial consideration of this model that lead us to study (1.1) - the superconducting interface model. We will describe what led us to consider (1.1), but in order to do so we will first need to describe the superconducting interface model.

In [34], an effective action for the superconducting string model using formal arguments was derived. The effective action found suggests that

(3a) there should be solutions to this model with a vortex filament supporting a superconducting current

(3b) the vortex filament is near a codimension two timelike surface $\Gamma$, where $\Gamma$ satisfies a geometric equation that is coupled in a highly nonlinear way to the phase of the current and an ambient vector potential representing an external electromagnetic field

Initially, we were interested in rigorously verifying that there exists a solution to the superconducting string model satisfying (3a) and (3b). However, we did not find certain aspects of the coupling between the phase of the current, the local string profile, and the geometric description of the vortex filament coming from these effective dynamics to be completely accurate.
By simplifying to the superconducting interface model we are able to obtain an effective action, different in nature to the effective action proposed for the “neutral current” superconducting string in [34]. In fact, theorem 1.2.2 verifies that for \( n = 2 \), in the equivariant case, and subject to a non-degeneracy condition, that there exists a solution to (1.1) with an interface supporting a current whose dynamics are approximately described by the effective action we find in appendix A. The coupling as described in (3b) is much more complicated in our description compared to what is proposed in [34]. Furthermore, we expect that when the phase of the current is decoupled from the ambient vector potential, then the behaviour of superconducting strings should be qualitatively similar to the behaviour of superconducting interfaces as described in (2a) - (2d).

To illustrate how the superconducting interface model is related to the superconducting string model, we first need to state the superconducting string model. At a high level, the superconducting string model as presented in [34] consists of two weakly coupled complex scalar fields with two independent \( U(1) \) gauge symmetries. More precisely, for \( \phi, \sigma : \mathbb{R}^{1+3} \to \mathbb{C} \) let \( A_\phi, A_\sigma : \mathbb{R}^{1+3} \to \mathbb{R}^4 \) denote their associated gauge fields. Also, define the covariant derivatives associated to the \( \phi \) and \( \sigma \) fields as \( \nabla_\phi = \nabla - iq_\phi A_\phi \) and \( \nabla_\sigma = \nabla - iq_\sigma A_\sigma \), respectively, where \( q_\phi, q_\sigma \in \mathbb{R} \) are the coupling constants between \( (\phi, \sigma) \) and their associated gauge fields. As is standard notation, we define the field strength tensor of the \( A_\phi \)-field as \( F_{\phi,\mu\nu} := \partial_\mu A_{\phi,\nu} - \partial_\nu A_{\phi,\mu} \) and similarly define \( F_{\sigma,\mu\nu} \) as the field strength tensor of the \( A_\sigma \)-field. Finally, for \( (\lambda_\phi, \lambda_\sigma, \beta) \in \mathbb{R}^3_+ \) we define the superconducting string potential as

\[
V_S(\phi, \sigma) = \frac{\lambda_\phi}{4} (|\phi|^2 - 1)^2 + \frac{\lambda_\sigma}{4} (|\sigma|^2 - 2) |\sigma|^2 + \frac{\beta}{2} |\phi|^2 |\sigma|^2 .
\]  

(2.1)

The Lagrangian of the superconducting string model is

\[
\mathcal{L} = \frac{1}{2} \eta^{\alpha\beta} \nabla_{\alpha} \phi \nabla_{\beta} \phi + \frac{1}{2} \eta^{\alpha\beta} \nabla_{\alpha} \sigma \nabla_{\beta} \sigma + \frac{1}{\epsilon^2} V_S(\phi, \sigma) + \frac{\epsilon^2}{4} F_{\phi,\mu\nu} F^{\phi,\mu\nu} + \frac{\epsilon^2}{4} F_{\sigma,\mu\nu} F^{\sigma,\mu\nu} ,
\]  

(2.2)

where \( 0 < \epsilon \ll 1 \) and \( \eta = \text{diag}(-1, 1, 1, 1) \) is the Minkowski metric. As stated earlier, an important feature of this model is that the \( \phi \) and \( \sigma \) fields have two, independent \( U(1) \) gauge symmetries. For appropriately chosen \( \lambda_\phi, \lambda_\sigma, \) and \( \beta \) we can arrange it so that \( \phi \) has a broken gauge symmetry while the gauge symmetry associated to the field \( \sigma \) field remains unbroken. We can identify this unbroken gauge symmetry with electromagnetism. See [33] for an in depth discussion of the physics behind this model.

To obtain (1.1), two changes to the superconducting string model will be made. The first is
to consider $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$. In this case, $\phi$ loses its $U(1)$ gauge symmetry and gains a discrete symmetry. In particular, this allows for $\phi$ to have an interface. The second change we make is to simplify the problem by decoupling the current from the ambient vector potential. To do this, set $q_\sigma = 0$. Applying these changes to (2.2), one obtains the Lagrangian for the superconducting interface model
\[
\mathcal{L} = \frac{1}{2} \eta^{\rho\beta} \partial_\rho \phi \partial_\beta \phi + \frac{1}{2} \eta^{\rho\beta} \sigma \partial_\rho \sigma \partial_\beta \sigma + \frac{1}{\epsilon^2} V_S(\phi, \sigma).
\] (2.3)
In this thesis, we consider the $n = 2$ case.

3 Effective Equations

In this section we state an ansatz and derive an effective action based on this ansatz. We then establish certain properties of this ansatz that will be used to prove our main result.

3.1 Change to Minkowski Normal Coordinates

Suppose $\Gamma$ is codimension 1 timelike surface parameterized by $(y^0, R(y^0))$ in polar coordinates. For $\eta := \text{diag}(-1, 1)$, define a new coordinate system $(y^0, y^1)$ centred about $\Gamma$ as
\[
(t, r) = \psi(y^0, y^1) := (y^0, R(y^0)) + y^1 \nu(y^0),
\] (3.1)
where
\[
\eta \left( \partial \nu(y^0, R(y^0)), \nu(y^0) \right) \quad \text{and} \quad \eta \left( \nu(y^0), \nu(y^0) \right) = 1.
\]
If $\nu(y^0)$ satisfies these conditions, then $-\nu(y^0)$ does too. Thus, we have a choice of $\nu(y^0)$ to make and choose
\[
\nu(y^0) = \frac{1}{\sqrt{1 - R'(y^0)^2}} (R'(y^0), 1).
\]
We call these new coordinates $(y^0, y^1)$ Minkowski normal coordinates. Note that this change of coordinates only makes sense on a neighbourhood of $\Gamma$. We will specify a domain on which this coordinate change makes sense at the end of this section.
Figure 3: Lemma 1.2.1 implies that there exists a neighbourhood $\mathcal{N}$ of $\Gamma$ so that for $(t, r) \in \mathcal{N}$, we can identify $(t, r)$ with a point $p(t, r)$ on $\Gamma := \{(y^0, R(y^0)) : 0 < y^0 \leq T\}$. Results from [18] allow us to identify $p(t, r)$ with $y^0$ and the Minkowski distance of $(t, r)$ to $\Gamma$ with $y^1$.

In this new coordinate system, $\Gamma = \{y^1 = 0\}$. By changing to Minkowski normal coordinates we “straightened $\Gamma$ out”. Also, recall the definition of $s_M$ and $d_M$ from lemma 1.2.1. Results from [18] allow us to identify $y^0$, which we call the tangential coordinate, with $s_M$ and $y^1$, which we call the normal or transverse coordinate, with $d_M$.

As discussed in appendix A and in section 1.2.2, we expect that for suitable profiles $(f_0, s_0)$ and $\Gamma$ parameterized by $(y^0, R(y^0))$, there exists a solution to (1.8) of the form

$$\Phi(y^0, y^1) \approx \begin{pmatrix} f_0(\frac{y^1}{\epsilon}; R(y^0)) \\ s_0(\frac{y^1}{\epsilon}; R(y^0)) \end{pmatrix}. \quad (3.2)$$

Thus, we would like to consider (1.8) in Minkowski normal coordinates. To rewrite (1.8) in Minkowski normal coordinates, consider the action integral associated of (1.8)

$$S(\Phi) := \int \left\{ -\frac{1}{2} |\partial_t \Phi|^2 + \frac{1}{2} |\partial_r \Phi|^2 + \frac{1}{\epsilon^2} W(\Phi, r) \right\} rdt \, dr, \quad (3.3)$$

where $\Phi = (\phi, \sigma)$ and where $W$ is the shifted potential defined in (1.10). Define

$$m := (1 - (R')^2)^{-1/2} \quad \text{and} \quad n := 1 + y^1 m^3 R'' \quad (3.4)$$

A computation shows that

$$\begin{pmatrix} \partial_t \\ \partial_r \end{pmatrix} = \frac{m}{n} \begin{pmatrix} m & -n R' \\ -m R' & n \end{pmatrix} \begin{pmatrix} \partial_{y^0} \\ \partial_{y^1} \end{pmatrix}. \quad (3.5)$$
In Minkowski normal coordinates, \( S \) is

\[
S(\Phi) = \int \left\{ -\frac{m^2}{2n^2} |\partial_{\rho}\Phi|^2 + \frac{1}{2} |\partial_{\eta}\Phi|^2 + \frac{1}{\epsilon^2} W(\Phi, R + y^1 m) \right\} (R + y^1 m) \frac{n}{m} dy^0 dy^1. \tag{3.6}
\]

The equations of motion of (3.6) are thus

\[
\frac{m^2}{n^2} \partial_{\rho,\rho}\Phi + B^\alpha \partial_\alpha \Phi - \partial_{\eta,\eta}\Phi + \frac{1}{\epsilon^2} W(\Phi, R(y^0) + y^1 m(y^0)) = 0, \tag{3.7}
\]

where \( w \) was defined in (1.11) and we have defined

\[
B^0 := \frac{m}{n} \partial_{\rho}(\frac{m}{n}) + \frac{1}{(R + y^1 m) n} \frac{m^2}{R'} \tag{3.8}
\]

\[
B^1 := -\frac{m^3}{n} R'' - \frac{1}{(R + y^1 m)} m. \tag{3.9}
\]

**Domain**

It turns out that for \( R \) satisfying (1.20), then Minkowski normal coordinates are not necessarily defined everywhere. In fact, lemma 1.2.1 only guarantees that this coordinate system is well defined on a neighbourhood \( N \) of \( \Gamma := \{ (y^0, R(y^0)) : 0 \leq y^0 < T \} \), where \( T \) is the maximal time of existence of \( R \). We chose initial conditions, see section 1.3, so that we only need to control the difference between solutions to (3.7) and the right hand side of (3.2) on a neighbourhood of \( \Gamma \). The neighbourhood on which we will control this difference is defined as follows.

Pick the maximal \( y^0_\ast \) and \( y^1_\ast \) so that

**Domain**

(4a) The coordinate transformation \( \psi \) defined in (3.1) is well defined for all \( (y^0, y^1) \in [0, y^0_\ast) \times (-y^1_\ast, y^1_\ast) \).

(4b) For all \( 0 \leq y^0 < y^0_\ast \), then \( R \) solves (3.22) below with \( R(0) \in (r_0, r_1) \) and \( R'(0) = 0, R(y^0) \in (r_0, r_1) \), and \( |R'(y^0)| \leq \frac{1}{2} \).

(4c) For all \( [0, y^0_\ast) \times (-y^1_\ast, y^1_\ast) \), we have that \( n \leq \frac{1}{2} \), where \( n \) was defined in (3.4).

(4d) The estimates found in (4.30) below hold for all \( 0 \leq y^0 < y^0_\ast \).

(4e) If \( \Phi \) is a solution to (1.8) in \( (t, r) \) coordinates with initial data satisfying the conditions outlined in section 1.3 and if \( 0 \leq y^0 < y^0_\ast \), then \( \Phi(\psi(y^0, -y^1_\ast + y)) = (-1, 0) \) and \( \Phi(\psi(y^0, y^1_\ast - y)) = (1, 0) \) for all \( 0 < y < \frac{1}{100} y^1_\ast \).
(4f) By possibly picking $\widehat{T}$ from section 1.3 smaller, we want $\psi\left([0, y^0_0) \times (-y^1, y^1_0) \times (R(0) - \widehat{T}, R(0) + \widehat{T})\right)$, where $\widehat{T}$ was defined in section 1.3.

The reason we pick $R'(0) = 0$ was so that for $y^0_0$ and $y^1_0$ satisfying (4a) - (4f), then the Jacobian associated to the coordinate transformation (3.1) satisfies

$$|J_\psi| \geq c > 0.$$  

That is, we only want to consider neighbourhoods of $\Gamma = \{(y^0_0, R(y^0_0)) : 0 \leq y^0_0 < y^0_0\}$ for which this coordinate transformation is far from becoming singular.

### 3.2 Formal Expansion

Recall that for suitable profiles $(f_0, s_0)$ and surfaces $\Gamma$ parameterized by $(y^0, R(y^0))$, we would like to find solutions to (3.7) of the form

$$\Phi(y^0, y^1) \approx \begin{pmatrix} f_0(y^1; R) \\ s_0(y^1; R) \end{pmatrix}.$$  

(3.10)

As we discussed in section 1.2.2, we will need to consider the leading order correction. Thus, for suitable $(f_0, s_0)$, $(f_1, s_1)$, and surfaces $\Gamma$ we would actually like to look for solutions to (3.7) satisfying

$$\Phi(y^0, y^1) \approx \begin{pmatrix} f_0(y^1; R) \\ s_0(y^1; R) \end{pmatrix} + \epsilon \begin{pmatrix} f_1(y^1; R, R') \\ s_1(y^1; R, R') \end{pmatrix}.$$  

(3.11)

Set $F_0(y^1; R) = (f_0(y^1; R), s_0(y^1; R))$ and $F_1(y^1; R, R') = (f_1(y^1; R, R'), s_1(y^1; R, R'))$. In this section we will see that $F_1$’s additional dependence on $R'$ is natural and to be expected.

Observe that

$$\partial_{y^1}w(\Phi, R) = -2\frac{d^2}{r^3} \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \quad \text{and} \quad \partial_{y^0}w(\Phi, R) = 6\frac{d^2}{r^4} \begin{pmatrix} 0 \\ \sigma \end{pmatrix}.$$  

(3.12)

Plugging the right hand side of (3.11) into (3.6) and grouping terms together by their power in $\epsilon$ we find that the first few terms in the formal expansion are

$\epsilon^{-1}$ term:

$$\int \left( \int \left\{ \frac{1}{2} |\partial_{y^1}F_0|^2 + W(F_0, R) \right\} d(y^1; \epsilon) \right) \frac{R}{m} dy^0.$$  

(3.13)
\( \varepsilon^0 \) term:

\[
\int \left( \int \left\{ \partial_{ij} F_1 \cdot \partial_{ij} F_0 + w(F_0, R) F_1 \right\} d \left( \frac{y^i}{\varepsilon} \right) \right) \frac{R}{m} dy^0
+ \int \left( \int \frac{1}{2} \frac{y^i}{\varepsilon} m \partial_i w(F_0, R) \cdot F_0 d \left( \frac{y^i}{\varepsilon} \right) \right) \frac{R}{m} dy^0
+ \int \left( \int \left\{ \frac{1}{2} \partial_{ij} F_0 \right\}^2 + W(F_0, R) \right) = \frac{y^i}{\varepsilon} d \left( \frac{y^i}{\varepsilon} \right) (1 + m^2 RR'') dy^0
\]

\( \varepsilon^1 \) term:

\[
\int \left( \int \left\{ \frac{1}{2} \partial_{ij} F_1 \right\}^2 + \frac{1}{2} F_1 \cdot \text{Hess}_5 W(F_0, R) F_1 + \frac{y^i}{\varepsilon} m \partial_i w(F_0, R) \cdot F_1 \right) d \left( \frac{y^i}{\varepsilon} \right) \frac{R}{m} dy^0
+ \int \left( \int \left\{ \partial_{ij} F_0 + w(F_0, R) \cdot F_1 \right\} \frac{y^i}{\varepsilon} d \left( \frac{y^i}{\varepsilon} \right) \right) (1 + m^2 RR'') dy^0
+ \int \left( \int \left\{ \frac{1}{4} \frac{y^i}{\varepsilon} m^2 \partial_i w(F_0, R) \cdot F_0 - \frac{m^2}{2} \frac{(R')^2}{R^2} \partial_R F_0 \right\}^2 d \left( \frac{y^i}{\varepsilon} \right) \right) \frac{R}{m} dy^0
+ \int \left( \int \left\{ \frac{1}{2} \partial_{ij} F_0 \right\}^2 + W(F_0, R) \right) \frac{y^i}{\varepsilon} d \left( \frac{y^i}{\varepsilon} \right) m^3 R' dy^0,
\]

where every \( F_0 \) is evaluated at \((\frac{y^i}{\varepsilon}; R)\) and every \( F_1 \) is evaluated at \((\frac{y^i}{\varepsilon}; R, R')\). In particular, for suitable \( F_0 \) and \( F_1 \), then the expressions found in (3.13-3.15) are independent of \( \varepsilon \).

As we discuss in appendix A, it is natural to choose profiles \( F_0 \) so that \( f_0 \) has an interface and so that \( F_0 \) minimizes the energy per unit arc length of the interface. Also, we want \( F_0 \) to be independent of \( \varepsilon \) and we want to be able to take \( \varepsilon \) arbitrarily small. Thus, upon examining the leading order term of the above expansion (3.13), for each \( R \in \mathbb{R}_+ \), we pick \( F_0(\cdot; R) \) satisfying the minimization problem

\[
\inf_{F \in \mathcal{A}} \int_R \left\{ \frac{1}{2} \left| F'(x) \right|^2 + W(F(x), R) \right\} dx
\]

\[
\mathcal{A} := \left\{ F = (f, s) \in H^2_{\text{loc}} : \lim_{x \to \pm \infty} F(x) = (\pm 1, 0), \ f(0) = 0 \right\}.
\]

Without the requirement that \( f(0) = 0 \), then any translation of a minimizer would be another minimizer. This condition kills this degeneracy. Note that if \( F_0 \) satisfies this minimization problem, then it also satisfies

\[
-\partial_{ij} F_0 + w(F_0, R) = 0.
\]

Also, we see that for this choice of \( F_0 \), then the first term of (3.14) will always be 0, regardless
of what we choose $F_1$ to be.

Next, we choose $F_1$. Examining (3.15) we see that it is again natural to pick $F_1$ that minimizes the energy per unit arc length of the interface. That is, we choose $F_1$ satisfying

$$\inf_{F \in \mathcal{B}} \int_{\mathbb{R}} \left\{ \frac{1}{2} F(x) \cdot L_1(F_0, R)F(x) - H(R) \partial_{ij} F_0(x) \cdot F(x) + x m \partial_i w(F_0, R) \cdot F(x) \right\} dx$$

$$\mathcal{B} := \left\{ F \in H^1_{loc} : \lim_{x \to \pm \infty} F(x) = (0, 0), F \in \ker(L_1(F_0, R))^\perp \right\},$$

where

$$L_\epsilon(F_0, R) := -\partial_{xx} + \frac{1}{\epsilon^2} \text{Hess}_\Phi W(F_0(\cdot), R)$$

$$H(R) := \frac{1}{\sqrt{1 - (R')^2}} \left( \frac{R''}{1 - (R')^2} + \frac{1}{R} \right).$$

To see why we look for minimizers to (3.18) orthogonal to $\ker(L_1(F_0, R))$, suppose that $F_1$ minimizes the integral appearing in (3.18). Then, $F_1$ satisfies

$$L_1(F_0, R)F_1 = H(R) \partial_{ij} F_0 - x m \partial_i w(F_0, R)$$

and hence $F_1 \notin \ker(L_1(F_0, R))$. In fact, for $R \in (r_0, r_1)$, then theorem [3.2.1] implies that the integral we want to minimize in (3.16) is strongly convex in directions orthogonal to $\ker(L_1(F_0, R))$ and hence critical points of this integral in $\mathcal{B}$, that is solutions to (3.21) in $\mathcal{B}$, are unique and are solutions to the minimization problem (3.18) [14]. Thus, when $R \in (r_0, r_1)$, finding the solution to the minimization problem (3.18) is equivalent to finding the solution to (3.21) in $\mathcal{B}$.

Remark: The operator $L_\epsilon(F_0, R)$, see (3.19), is the linearization of (3.7) about $F_0$ and $H(R)$, see (3.20), is the mean curvature of the surface of rotation generated by $R$ in $\mathbb{R}^{1+2}$.

A necessary condition for (3.21) to be solvable is that the right hand side of (3.21) must be orthogonal to the kernel of $L_1(F_0, R)$. Whenever $R \in (r_0, r_1)$, the non-degeneracy condition implies that this necessary condition is equivalent to requiring that $R$ be a solution to

$$H(R) \int_{\mathbb{R}} \left| \partial_{ij} F_0(\cdot; R) \right|^2 - \frac{d^2}{R^3} m \int_{\mathbb{R}} s_0(\cdot; R)^2 = 0.$$

which is precisely (1.20). Recall that we picked $y^0_*$ in (4a) - (4f) so that among other conditions, $R$ is a solution to (3.22) with $R(y^0) \in (r_0, r_1)$ for all $y^0 \in [0, y^0_*)$. Most importantly, (3.22) is the geometric relation describing the evolution of the interface of the $\phi$-field of our approximate
solution
\[ \Gamma := \{(y^0, R(y^0)) : 0 \leq y^0 < y^0_*\} . \]

In section 3.3 below, we establish the existence of solutions to the minimization problems (3.16) and (3.18) and the existence of solutions \( R \) to (3.22). In section 4, we verify the predictions of the formal calculations made above. That is, we show that if \( F_0 \) and \( F_1 \) satisfy the minimization problems (3.16) and (3.18), respectively, and if \( R \) satisfies (3.22), then for suitable initial data, there exists a solution to (3.7) satisfying (3.11).

Before moving on, though, we establish the following important estimate regarding the linearized operator \( L_\epsilon(F_0; R) \) that will be used to verify that there exists a solution to (3.7) satisfying (3.11). In the statement of the following theorem we use the notation “\( \lesssim \)” which we introduced in (1.26).

**Theorem 3.2.1 (Spectral Estimate).** Suppose \( F_0 \) satisfies (3.17) and \( R \in (r_0, r_1) \). By assumption 4 of (1.9), the non-degeneracy condition (1.21), \( \ker(L_\epsilon(F_0, R)) = \text{span}\{\partial_\gamma F_0\} \). In particular, this implies that for any \( \xi \in \ker(L_\epsilon(F_0, R)) \) we have

\[ \frac{1}{\epsilon^2} \|\xi\|_{L^2(R)}^2 \lesssim \int_R \xi \cdot L_\epsilon(F_0; R)\xi. \quad (3.23) \]

**Proof of theorem 3.2.1.** For fixed \( R \in (r_0, r_1) \) define,

\[ X = \{\xi \in H^1(\mathbb{R}; \mathbb{R}^2) : \|\xi\|_2 = 1 \text{ and } \langle \xi, \partial_\gamma F_0 \rangle_2 = 0\} \]

\[ I_\epsilon(\xi) := \int_R \xi \cdot L_\epsilon(F_0, R)\xi. \]

We want to show that

\[ m_\epsilon := \inf_{\xi \in X} I(\xi) > 0. \quad (3.24) \]

Clearly, if \( m_\epsilon \geq \frac{1}{\epsilon^2} c \), where \( c \) is from assumption 3 of (1.9), then we are done. Suppose \( m_\epsilon < \frac{1}{\epsilon^2} c \).

If \( m_\epsilon = 0 \) and there exists \( \xi \in X \) at which this infimum is attained, then by the non-degeneracy condition (1.21) we have that \( \xi \propto \partial_\gamma F_0 \). Since \( \xi \in X \), then \( \xi \perp \partial_\gamma F_0 \) which implies that \( \xi = 0 \). This contradicts the fact that \( \|\xi\|_2 = 1 \). Thus, if we show that there exists \( \xi \in X \) at which the infimum of \( I_\epsilon \) is attained, then we are done.
Let \( \xi_n \in X \) be a minimizing sequence. First, observe

\[
\|\xi_n\|_2^2 = I_\epsilon(\xi_n) + \frac{1}{\epsilon^2} \int \xi_n \text{Hess}_\Phi W(F_0, R) \xi_n \\
\leq I_\epsilon(\xi_n) + \frac{1}{\epsilon^2} \|\text{Hess}_\Phi W(F_0, R)\|_\infty \|\xi_n\|_2^2 \\
\leq C,
\]

where we used the fact that \( \|\text{Hess}_\Phi W(F_0, R)\|_\infty < \infty \) and \( \|\xi_n\|_2^2 = 1 \) for all \( n \). Thus, we have that there exists some constant \( C \), possibly depending on \( \epsilon \), so that

\[
\|\xi_n\|_2 = 1 \leq C \quad \text{and} \quad \|\xi'_n\|_2 \leq C \quad \text{and} \quad [\xi_n]_{1/2} \leq \|\xi'_n\|_2 \leq C,
\]

where we used the Sobolev embedding \( H^1 \hookrightarrow C^{0,1/2} \) to obtain the third inequality. Thus, by possibly passing to a subsequence we have that \( \xi_{n_k} \rightarrow \xi \) locally uniformly and weakly in \( H^1 \).

Note, we still have that \( \xi \perp \partial y_1 F_0 \). Suppose \( \|\xi\|_2 = t \in [0, 1] \). We have that

\[
I_\epsilon(\xi) = I_\epsilon(\frac{\xi}{t}) t^2 \geq m_\epsilon t^2.
\]

Also, for \( 0 < \delta \ll 1 \), choose \( \rho \) so that

\[
\begin{align*}
(i) \quad & \|\xi\|_{L^2(\mathbb{R}\setminus B_\rho)} < \delta \\
(ii) \quad & \text{For all } |y^1| > \rho, \text{ then } \text{Hess}_\Phi W(F_0, R) \geq (c - \delta) I.
\end{align*}
\]

For \( n_k \) sufficiently large, we have that

\[
1 = \|\xi_{n_k}\|_2^2 = \|\xi_{n_k}\|_{L^2(B_\rho)}^2 + \|\xi_{n_k}\|_{L^2(\mathbb{R}\setminus B_\rho)}^2 \\
\leq \left( \|\xi_{n_k} - \xi\|_{L^2(B_\rho)} + \|\xi\|_{L^2(B_\rho)} \right)^2 + \|\xi_{n_k}\|_{L^2(\mathbb{R}\setminus B_\rho)}^2 \\
\leq (\delta + t)^2 + \|\xi_{n_k}\|_{L^2(\mathbb{R}\setminus B_\rho)}^2.
\]

Thus,

\[
\|\xi_{n_k}\|_{L^2(\mathbb{R}\setminus B_\rho)}^2 \geq 1 - (\delta + t)^2
\]

(3.27)
Furthermore, we have that

\[ m_\epsilon = \lim_{n_k \to \infty} I_\epsilon(\xi_{n_k}) \]

\[ \geq \lim_{n_k \to \infty} \int \left| \frac{1}{2} \xi'_{n_k} \right|^2 + \lim_{n_k \to \infty} \frac{1}{\epsilon^2} \int \xi_{n_k} \cdot \text{Hess}_\Phi \ t_{F_0, R} \xi_{n_k} \]

\[ \geq \int_{R \setminus B_\rho} \left( \frac{c - \delta}{\epsilon^2} \right) \left( 1 - (\delta + t)^2 \right) - \left\| \text{Hess}_\Phi \ t_{F_0, R} \right\|_\infty \delta \]

Using (3.26) we have that

\[ m_\epsilon \geq I_\epsilon(\xi) + \frac{(c - \delta)}{\epsilon^2} \lim_{n_k \to \infty} \int_{R \setminus B_\rho} \left| \xi_{n_k} \right|^2 - \frac{\left\| \text{Hess}_\Phi \ t_{F_0, R} \right\|_\infty \delta}{\epsilon^2} \int_{R \setminus B_\rho} \left| \xi \right|^2 \]

where we needed to use (3.26) to go from the 4th line to the 5th. This is true for all \( 0 < \delta \ll 1 \) and so we actually have that

\[ m_\epsilon = \lim_{n_k \to \infty} I(\xi_{n_k}) \geq m_\epsilon t^2 + \frac{c}{\epsilon^2} (1 - t^2) \geq m_\epsilon \]

with equality if and only if \( t = 1 \). Thus, we necessarily have that \( t = 1 \).

\[ \square \]

3.3 Existence of \( F_0, F_1, \) and \( R \)

Recall the minimization problem set out in (3.16)

\[ \mu(R) = \inf_{F \in \mathcal{A}} \int \mu(F; R) \]  

\[ \mathcal{A} = \left\{ F = (f, s) \in H^1_{\text{loc}}(\mathbb{R}) : F(\pm \infty) = (\pm 1, 0), \ f(0) = 0 \right\}, \]  

where

\[ \mu(F; R) := \frac{1}{2} |F'|^2 + W(F; R). \]

We will now show that there exists \( F_0 = (f_0, s_0) \in \mathcal{A} \) at which \( \mu(F; R) \) attains its infimum.
Lemma 3.3.1. Set
\[ \tilde{A} = \{(f, s) \in A : f(y^1) = 0 \iff y^1 = 0, |f| \leq 1, 0 \leq s \leq 1, f \text{ is odd, } s \text{ is even}\}. \quad (3.31) \]

Then
\[ \inf_{(f, s) \in \tilde{A}} \int_{\mathbb{R}} \mu(f, s; R) = \inf_{(f, s) \in \tilde{A}} \int_{\mathbb{R}} \mu(f, s; R). \]

Thus, we will look for minimizers to (3.28) in \( \tilde{A} \). The proof of lemma 3.3.1 is given below.

Proposition 3.3.2. There exists \( (f_0, s_0) \in \tilde{A} \) that solves the minimization problem (3.28).

Even though the proof of proposition 3.3.4 is completely standard, we include it for the sake of completeness. The proof will be given at the end of this section.

Next, we establish the existence of solutions \( R \) to (3.22).

Proposition 3.3.3. Suppose \( R(0) \in (r_0, r_1) \) and \( R'(0) = 0 \) and suppose that \( F_0(\cdot; R) \) is a solution to the minimization problem (3.16). Then, there exists \( R : \mathbb{R} \to \mathbb{R} \) and a \( y_0^* \), independent of \( \epsilon \), so that \( R(y_0^*) \) is a solution to (3.22) for all \( 0 \leq y_0^* < y_0^* \) and \( R(y_0^*) \in (r_0, r_1) \) for all \( 0 \leq y_0^* < y_0^* \).

Proof of Proposition 3.3.3. Using assumption 5 of (1.9), we can show that for \( R \in (r_0, r_1) \), then \( F_0(\cdot, R) \) is the unique minimizer of (3.16), see step 1 of the proof of proposition 3.6.1. With this, one is then able to use standard ODE techniques to obtain the local existence of a solution to (3.22), as long as \( R \) remains in the interval \((r_0, r_1)\).

\[ \square \]

Next, we show that there exists a solution \( F_1 \) to (3.21).

Proposition 3.3.4 (Existence of \( F_1 \)). Define
\[ g(R, v) := \frac{1}{\sqrt{1 - v^2}} \frac{d^2}{R^3} \| s_0(\cdot; R) \|_2^2. \quad (3.32) \]

For every fixed \( (R, v) \in (r_0, r_1) \times (-1, 1) \), there exists a unique \( F_1 = F_1(y^1; R, v) \in H^1(\mathbb{R}; \mathbb{R}^2) \) solving
\[ L_1(F_0, R)F_1 = g(R, v) \partial_{y^1} F_0(\cdot; R) + 2 \frac{y^1}{\sqrt{1 - v^2}} \partial_{y^1} w(F_0, R) \]
\[ F_1 \perp_{L^2} \partial_{y^1} F_0(\cdot; R), \]
where \( L_1(F_0, R) \) was defined in (3.19) and \( \partial_r w(F_0, R) \) was defined in (3.12).

Proof of Proposition 3.3.4 Since \( L_1(F_0, R) \) is self-adjoint, then the spectral theorem implies that there exists a family of spectral projection operators \( \{ E_\lambda \}_{\lambda \in \mathbb{R}} \) associated to \( L_1(F_0, R) \) satisfying

\[
L_1(F_0, R) \psi = \int_\mathbb{R} \lambda dE_\lambda \psi \quad \text{for} \ \psi \in D(L_1(F_0, R)).
\]

The non-degeneracy condition (1.21) tells us that the eigenvalue 0 of \( L_1(F_0, R) \) is simple. Recall that \( m_1 \), defined in (3.24), is the smallest, non-zero value in the spectrum of \( L_1(F_0, R) \). Since

\[
g(R, v) \partial_{y^1} F_0(\cdot; R) + \frac{y^1}{\sqrt{1 - v^2}} \partial_r w(F_0, R) \perp \ker(L_1(F_0, R)),
\]

then \( F_1 \in H^1 \) satisfying

\[
F_1 := \int_{\frac{1}{m_1}}^{\infty} \frac{1}{\lambda} dE_\lambda \left( g(R, v) \partial_{y^1} F_0(\cdot; R) + \frac{y^1}{\sqrt{1 - v^2}} \partial_r w(F_0, R) \right) \quad (3.33)
\]

for all \( \psi \in H^1 \) solves (3.21). Moreover, from (3.33) we can see that \( F_1 \perp_{L^2} \partial_{y^1} F_0 \) too.

\[ \square \]

Proof of Lemma 3.3.1 Since \( \widetilde{\mathcal{A}} \subset \mathcal{A} \), then

\[
\inf_{(f, s) \in \widetilde{\mathcal{A}}} \int_\mathbb{R} \mu(f, s; R) \geq \inf_{(f, s) \in \mathcal{A}} \int_\mathbb{R} \mu(f, s; R).
\]

We will show that

\[
\inf_{(f, s) \in \widetilde{\mathcal{A}}} \int_\mathbb{R} \mu(f, s; R) \leq \inf_{(f, s) \in \mathcal{A}} \int_\mathbb{R} \mu(f, s; R).
\]

Define

\[
\widetilde{\mathcal{A}}_1 = \{(f, s) \in \mathcal{A} : |f| \leq 1, 0 \leq s \leq 1\}.
\]

For \((f, s) \in \mathcal{A}\) set

\[
\tilde{f}(y^1) = \max(-1, \min(f(y^1), 1)) \quad \text{and} \quad \tilde{s}(y^1) = \max(0, \min(|s(y^1)|, 1)).
\]

25
Based on condition 2 of \[1.9\], we have that \(\mu(\tilde{f}, \tilde{s}; R) \leq \mu(f, s; R)\) and so

\[
\inf_{(f, s) \in \tilde{A}_1} \int_{\mathbb{R}} \mu(f, s; R) \leq \inf_{(f, s) \in A} \int_{\mathbb{R}} \mu(f, s; R). \tag{3.34}
\]

Next, let \((f, s) \in \tilde{A}_1\). Let \(\lambda_1\) be the smallest zero of \(f\) and let \(\lambda_2\) be the largest zero of \(f\). Note that it is possible that \(\lambda_1\) may the same as \(\lambda_2\). Also, since \(\lim_{y^1 \to \pm\infty} f(y^1) = \pm 1\), then \(\lambda_1 > -\infty\) and \(\lambda_2 < \infty\). We necessarily have that

\[
\int_{-\infty}^{\lambda_1} \mu(f, s; R) \leq M/2 \quad \text{or} \quad \int_{\lambda_2}^{\infty} \mu(f, s; R) \leq M/2. \tag{3.35}
\]

Assume the first inequality of \((3.35)\) holds. By translating, we still have that \((f(-\lambda_1), s(-\lambda_1)) \in \tilde{A}_1\). Thus, we may assume that \(\lambda_1 = 0\). Define \((\tilde{f}, \tilde{s})\) so that \(\tilde{f}\) is an odd function and \(\tilde{s}\) is an even function and so that \((\tilde{f}, \tilde{s}) = (f, s)\) on \((-\infty, 0]\). Then \((\tilde{f}, \tilde{s}) \in \tilde{A}\) and

\[
\int \mu(\tilde{f}, \tilde{s}; R) \leq \int \mu(f, s; R).
\]

If the second inequality of \((3.35)\) holds, then a slight modification to the above argument gives the desired result.

\[\square\]

Proof of proposition 3.3.2. Let \((f_j, s_j) \in \tilde{A}\) be a minimizing sequence. That is, let \((f_j, s_j)\) be a sequence satisfying

\[
\mu(R) = \lim_{j \to \infty} \int_{\mathbb{R}} \mu(f_j, s_j, R). \tag{3.36}
\]

Observe that

\[
\int_{\mathbb{R}} \mu(f_j, s_j, R) \geq \frac{1}{2} \|f_j^\prime\|_2^2 + \frac{1}{2} \|s_j^\prime\|_2^2
\]

and so \(\{f_j^\prime\}\) and \(\{s_j^\prime\}\) are uniformly bounded sequences in \(L^2\).

Since \(\|(f_j^\prime, s_j^\prime)\|_2\) is uniformly bounded, then the Sobolev embedding \(H^1 \hookrightarrow C^{0,1/2}\) implies that \(\{(f_j, s_j)\}_{j=1}^\infty\) is an equicontinuous family. Since \((f_j, s_j) \in \tilde{A}\) for all \(j\), then this family is also uniformly bounded. By a variant of the Arzela-Ascoli theorem, we have that there exists a subsequence \((f_{j_i}, s_{j_i})\) converging locally uniformly to some uniformly continuous function
\((f_0, s_0)\). This implies, in particular, that \((f_{j_l}, s_{j_l})\) converges pointwise to \((f_0, s_0)\). By Fatou’s lemma, we have that
\[
\int_{\mathbb{R}} W(f_0, s_0, R) \leq \liminf_{j_l \to \infty} \int_{\mathbb{R}} W(f_{j_l}, s_{j_l}, R).
\] (3.37)

Since \(\|(f_{j_l}', s_{j_l}')\|_2\) is uniformly bounded, then by the Banach Alaoglu theorem we have that there exists a further subsequence so that \((f_{j_l}', s_{j_l}')\) converges weakly to some \((p, q)\) in \(L^2(\mathbb{R})\). Observe that for \(\phi \in C_0^\infty(\mathbb{R})\) we have
\[
\int_{\mathbb{R}} \phi p = \lim_{j_l \to \infty} \int_{\mathbb{R}} \phi f_{j_l}' = - \lim_{j_l \to \infty} \int_{\mathbb{R}} \phi' f_{j_l} = - \int_{\mathbb{R}} \phi' f_0,
\]
where we used the compact support of \(\phi\) and the fact that \(f_{j_l} \to f_0\) locally uniformly to obtain the last equality. Thus, \(f_0' = p\). Similarly, we have that \(s_0' = q\). Using the identity \(x^2 \geq a^2 + 2a(x - a)\), we have that
\[
\|(f_{j_l}', s_{j_l}')\|_2^2 \geq \|(f_0', s_0')\|_2^2 + 2 \int_{\mathbb{R}} (f_0', s_0') \cdot (f_0' - f_{j_l}', s_0' - s_{j_l}').
\]
Since \((f_{j_l}', s_{j_l}') \rightharpoonup (f_0', s_0')\) in \(L^2(\mathbb{R})\), we have that
\[
\|(f_0', s_0')\|_2 \leq \liminf_{j_l \to \infty} \|(f_{j_l}', s_{j_l}')\|_2.
\] (3.38)

Combining (3.37) and (3.38) completes the proof.

\[\square\]

### 3.4 Properties of \(F_0\) and \(R\)

As we stated in the introduction, we are interested in regimes where \(\sigma \neq 0\). In section 3.4.1, we show that there exists a potential so that if \((f_0, s_0)(\cdot; R)\) is a minimizer of (3.16) with this potential, then \(s_0(\cdot; R) \neq 0\) for some range of \(R\). It can easily be verified that this potential satisfies conditions 1-3 of (1.9). However, we were unable to verify that this potential satisfies conditions 4-5 of (1.9).

In section 3.4.2, we show that our main theorem is consistent with the phenomena of current quenching described in the physics literature when the winding number density is sufficiently large [33]. The results of this section are not used anywhere else in this thesis.
3.4.1 Regimes where $s_0$ is nonzero

As we discussed in the introduction, we would like to find solutions to (1.1) so that the $\phi$-field has an interface and so that the $\sigma$-field is exponentially small except near the interface of $\phi$. Furthermore, we want these solutions to be of the form

$$\begin{pmatrix} \phi \\ \sigma \end{pmatrix} \approx \begin{pmatrix} f_0(y'; e; R) \\ e^{i\beta s_0(y'; e; R)} \end{pmatrix},$$

where $y^r = (y^0, \theta)$ and $y^\nu = y^1$ and $(f_0, s_0)$ solve (3.17).

For this section we choose $V = V_S$, where $V_S$ was defined in (2.1). We will show that if the constants of this potential satisfy

$$0 < \lambda_\sigma < \lambda_\phi \quad \text{and} \quad 0 < \beta < \sqrt{\lambda_\phi \lambda_\sigma} \quad \text{and} \quad \lambda_\phi + \beta < 2\lambda_\sigma,$$

then $s_0 \neq 0$. It is also worth noting that if the constants of $V_S$ satisfy (3.40), then this potential satisfies conditions 1-3 of (1.9). We believe that this potential also satisfies conditions 4-5 as well, but we were unable to verify this.

In proposition 3.5.1 below, we show that $s_0(y^1; R)$ goes to zero exponentially fast in $y^1$. Thus, if $(\phi, \sigma)$ is a solution to (1.1) satisfying (3.39) and if $V$ is a potential for which $s_0 \neq 0$, then we would have found solutions with the properties that we want.

Set

$$E(f, s) = \int_{\mathbb{R}} \left\{ \frac{1}{2} (f')^2 + \frac{1}{2} (s')^2 + W(f, s, R) \right\}.$$

When $s = 0$, then it is known that $E_0(f) := E(f, 0)$ is uniquely minimized when

$$f_{\min} = \tanh(\sqrt{\frac{\lambda_\phi}{2}} x).$$

The goal is to find parameters so that $E(f_{\min}, s) < E(f_{\min}, 0)$ for some non-zero $s$ with $(f_{\min}, s) \in \tilde{A}$.

**Proposition 3.4.1.** Suppose that the constants of $V_S$ satisfy (3.40). For sufficiently large $R$, there exists a minimizer $(f, s) \in \tilde{A}$ of

$$\int_{\mathbb{R}} \mu(f, s; R)$$
with $s \neq 0$.

Proof of proposition [3.4.1] Note that

$$E(f_{\min}, s) = E(f_{\min}, 0) + \int_{\mathbb{R}} \left\{ \frac{1}{2}(s')^2 + \frac{\lambda_{\sigma}}{4} (s^2 - 2) s^2 + \frac{\beta}{2} f^2 s^2 + \frac{d^2}{2R^2} s^2 \right\}, \quad (3.41)$$

If we can show that the second term is negative we would be done. To this end, take

$$s = \frac{1}{\cosh(Bx)},$$

where $B = \sqrt{\frac{\lambda_{\phi}}{2}}$. Thus, plugging $s$ into the second term we have that

$$\int_{\mathbb{R}} \left\{ \frac{1}{2}(s')^2 + \frac{\lambda_{\sigma}}{4} (s^2 - 2) s^2 + \frac{\beta}{2} f^2 s^2 + \frac{d^2}{2R^2} s^2 \right\} = \int_{\mathbb{R}} \left[ B^2 - \frac{\lambda_{\sigma}}{4} + \frac{\beta}{2} \sinh^2(By^1) \right] \frac{1}{\cosh^2(By^1)} + \int_{\mathbb{R}} \left[ \frac{d^2}{2R^2} - \frac{\lambda_{\sigma}}{4} \right] \frac{1}{\cosh^2(By^1)} = \frac{1}{3B} \left\{ \lambda_{\phi} + \beta + 3 \frac{d^2}{R^2} - 2\lambda_{\sigma} \right\}.$$

The third constraint of (3.40) ensures that the second term is indeed negative when $R$ is sufficiently large which allows us to conclude that $E(f_{\min}, s) < E(f_{\min}, 0)$.

\[\square\]

3.4.2 Interface Evolution and Current Quenching

We start this section with two observations about the approximate solution $F_0$ and the surface $R$.

The first observation we make is that when $s_0 \neq 0$, that is when the interface has a current, then the interface moves towards the origin. To see why this is true, recall that the surface $R$ satisfies the geometric relation

$$\frac{1}{\sqrt{1 - (R')^2}} \left( \frac{R''}{1 - (R')^2} + \frac{1}{R} \right) = \frac{1}{\sqrt{1 - (R')^2}} \frac{d^2 \|s_0\|_2^2}{\|s_0\|_2^2}.$$
Rearranging, we have that

\[ R'' = \begin{bmatrix} d^2 \frac{\|s_0\|_2^2}{R^2} - 1 \end{bmatrix} \left( 1 - \frac{(R')^2}{R} \right). \]

Since \( F_0 \) minimizes \( \mu(f, s; R) \), then \( F_0 \) satisfies

\[-F''_0 + \nabla_\phi W(F_0, R) = 0.\]

Multiplying this by \( F'_0 \) and integrating, we see that \( F_0 \) also satisfies

\[ \frac{1}{2} |F'_0|^2 = W(F_0, R) \]

Thus,

\[ R'' = \left[ \frac{1}{2} \frac{d^2}{R^2} \left( \frac{\|s_0\|_2^2}{W(F_0, R)} \right) - 1 \right] \left( 1 - \frac{(R')^2}{R} \right). \tag{3.42} \]

Since \( V(F_0) > 0 \), we have that

\[ W(F_0, R) = V(F_0) + \frac{d^2}{2R^2} s_0^2 > \frac{d^2}{2R^2} s_0^2 \]

and hence

\[ \frac{1}{2} \frac{d^2}{R^2} \left( \frac{\|s_0\|_2^2}{W(F_0, R)} \right) > 1. \]

Thus, as long as \( |R'| < 1 \), we have that \( R'' < 0 \). Since \( R'(0) = 0 \), this implies that \( R \) is moving towards the origin.

Figure 4: Since \( R'(0) = 0 \) and \( R'' < 0 \) whenever \( s_0 \neq 0 \), at least for a short time, then \( R \) is decreasing.
The second observation we make is that for \( R \) sufficiently small and for \( F_0 = (f_0, s_0) \in \mathfrak{A} \) satisfying the minimization problem (3.28), then we necessarily have that \( s_0 = 0 \). To see why this is true, consider the following expansion

\[
W(f_0, s_0; R) = W(f_0, 0; R) + \partial_{\sigma} V(f_0, 0) s_0 + \left[ \int_0^1 \partial_{\sigma r} V(f_0, \lambda s_0) d\lambda + \frac{1}{2} \frac{d^2}{R^2} \right] s_0^2.
\]

Since \( |f_0| \leq 1 \) and \( 0 \leq s_0 \leq 1 \), then for \( R \) sufficiently small the second term is non-negative. Furthermore, condition 1 of (1.9) implies that \( \partial_{\sigma} V(f_0, 0) = 0 \) for all \( f_0 \). Thus, for \( R \) sufficiently small we have that

\[
W(f_0, s_0; R) \geq W(f_0, 0; R)
\]

with equality if and only if \( s_0 = 0 \). That is, \( s_0 = 0 \).

Suppose \( (f_0, s_0) \in \mathfrak{A} \) satisfies the minimization problem (3.28) and suppose we have a potential \( V \) for which there exists a range of \( R \) so that \( s_0(y^1; R) \neq 0 \). By the second observation, we see that even though \( s_0(y^1; R) \neq 0 \) for some \( R \), there exists \( R \) sufficiently small for which \( s_0(y^1; R) = 0 \). Suppose \( R_* \) is the largest value of \( R \) so that \( s_0(y^1; R_*) = 0 \). Note that

\[
\int \mu(f_0, s_0; R) = \int \mu(f_0, s_0; R_*) + \int \frac{1}{2} \left( \frac{d^2}{R_*^2} - \frac{d^2}{R^2} \right) s_0^2.
\]

For \( R < R_* \), the second term on the right hand side is positive and hence

\[
\int \mu(f_0, s_0; R) \geq \inf_{(f, s) \in \mathfrak{A}} \int \mu(f, s; R) = \inf_{(f, 0) \in \mathfrak{A}} \int \mu(f, 0; R_*),
\]

where a necessary condition for equality is that \( s_0 = 0 \). Thus, \( s_0(y^1; R) = 0 \) for all \( R \leq R_* \) and \( s_0(y^1; R) \neq 0 \) for \( R > R_* \). Since we are assuming that there exist some \( R \) for which \( s_0 \neq 0 \), then \( R_* < \infty \).

We will examine the case where \( R_* < R(0) < R_* + \delta \) for some \( 0 < \delta \ll 1 \). By the first observation, \( R \) becomes smaller as the system evolves. Since our solution only makes sense when \( |R'| < 1 \), an interesting question to ask is: does \( R \) become smaller than \( R_* \) before \( |R'| = 1 \)? In other words, can the natural evolution of this system kill or quench a current? In the arguments to follow, we will need that \( (f_0, s_0) \) is continuous in \( R \) for \( R \) close to \( R_* \). We will see in section 3.6 below that this is indeed the case if \( R_* \in (r_0, r_1) \), where \( r_0 < r_1 \) come from the non-degeneracy condition (1.21).
Pick \( r_1 \) possibly smaller so that \( R_* \in (r_0, r_1) \) and

\[
-1 \leq \frac{1}{2} \frac{d^2}{R^2} \left\| s_0 \right\|_2^2 - 1 \leq \frac{1}{2}
\]

for all \( r_0 < R < r_1 \). Define \( I := \left\{ y^0 \mid r_0 < R(y^0) < r_1 \right\} \). For \( y^0 \in I \), we have that

\[
-\frac{1 - (R')^2}{r_0} \leq R'' \leq -\frac{1 - (R')^2}{r_1},
\]

where we used the previous observation to conclude that \( R \) is decreasing which then allows us to obtain the upper bound. It then follows that for \( y^0 \in I \) and for \( R'(0) = 0 \), then

\[
-\tanh\left(\frac{y^0}{r_0}\right) \leq R' \leq -\tanh\left(\frac{y^0}{2r_1}\right). \tag{3.43}
\]

Note that \( |R'(y^0)| < 1 \) for all \( y^0 \in I \). Integrating once more in \( y^0 \), we find that

\[
R(0) - r_0 \log \cosh\left(\frac{y^0}{r_0}\right) \leq R \leq R(0) - 2r_1 \log \cosh\left(\frac{y^0}{2r_1}\right). \tag{3.44}
\]

Choosing \( r_0 < R_* < R(0) = R_* + \delta < r_1 \) and \( 0 < \delta \) sufficiently small, then (3.44) and (3.43) imply that there exists \( y^0 \in I \) so that \( R(y^0) = R_* \).

### 3.5 Asymptotics of \( F_0 \) and \( F_1 \)

**Proposition 3.5.1.** Suppose \((f_0, s_0) \in \tilde{A}\) minimizes

\[
\int_{\mathbb{R}} \mu(f, s; R). \tag{3.45}
\]

Then there exists \( \alpha > 0 \), depending only on the constant \( c \) appearing in condition 3 of (1.9), so that

\[
\begin{cases}
1 - |f_0| & \lesssim e^{-\alpha|y^1|} \\
|f'_0| & \lesssim e^{-\alpha|y^1|}
\end{cases}
\quad \text{and} \quad
\begin{cases}
|s_0| & \lesssim e^{-\alpha|y^1|} \\
|s'_0| & \lesssim e^{-\alpha|y^1|}.
\end{cases} \tag{3.46}
\]

**Proposition 3.5.2.** Suppose \( F_1 \) solves (3.21). Then there exists \( \alpha > 0 \) so that for \( \beta = 0, 1 \) we have

\[
\left| \partial^\beta_y F_1 \right| \lesssim e^{-\alpha|y^1|}. \tag{3.47}
\]

Obtaining these asymptotics is a standard exercise, but we include the proofs since they
are used quite a bit in the proof of theorem 1.2.2. Before moving on to the proofs of these propositions, we make a few key observations.

First, observe that (3.46) implies that \((1 - f_0, s_0) \in L^2([0, \infty))\) and \((-1 - f_0, s_0) \in L^2((-\infty, 0])\).

Second, since \((f_0, s_0)\) minimizes (3.45), then \((f_0, s_0)\) satisfies

\[
- \left( \frac{f_0}{s_0} \right)'' + w(f_0, s_0, R) = 0.
\]

(3.48)

Thus, once we know that \((f_0, s_0)\) is smooth in \(y^1\) (see section 3.6), then this relation tells us that for \(\beta \in \mathbb{Z}_+\), there exists \(\alpha > 0\) so that

\[
\left| \partial_{y^1}^\beta (f_0, s_0) \right| \lesssim e^{-\alpha |y^1|}.
\]

(3.49)

This can be shown using the same arguments used to show that (3.49) holds.

Proof of proposition 3.5.1: Since \((f_0, s_0) \in \hat{A}\), then \(f_0\) is odd and \(s_0\) is even. Thus, it suffices to
show that (3.46) holds for $y^1 \to \infty$. Set

$$F_0 = (f_0, s_0) \quad \text{and} \quad v = \frac{1}{2} |F_0(y^1) - (1, 0)|^2.$$ 

Differentiating twice we have

$$v'(y^1) = \left(F_0(y^1) - (1, 0)\right) \cdot F'_0(y^1) \quad \text{and} \quad v''(y^1) = \left|F'_0(y^1)\right|^2 + \left(F_0(y^1) - (1, 0)\right) \cdot F''_0(y^1).$$

Examining $v''$, we have that for sufficiently large $y^1$

$$v'' \geq (F_0 - (1, 0)) \cdot \nabla \Phi W(F_0, R)$$

$$= (F_0 - (1, 0)) \cdot \left(\text{Hess} \Phi W((1, 0); R)(F_0 - (1, 0)) + O(v)\right)$$

$$\geq 2cv + O(v^{3/2}),$$

where we used condition 3 of (1.9) to obtain the last inequality. Since $v \to 0$ as $y^1 \to \infty$, then for all $0 < \delta \ll c$, we have that

$$v'' \geq 2c(\delta)v$$

for all $y^1$ sufficiently large, where $c(\delta) := c - \delta$.

Since $v > 0$ and $v \to 0$ as $y^1 \to \infty$, then there exists $\hat{y}^1 > 0$ so that $v'(\hat{y}^1) < 0$. We can rewrite the estimate $v'' \geq 2c(\delta)v$ as

$$(v' - \sqrt{2c(\delta)v})' + \sqrt{2c(\delta)(v' - \sqrt{2c(\delta)v})} = 0.$$ 

Set $w = v' - \sqrt{2c(\delta)v}$. Then we have that

$$w' + \sqrt{2c(\delta)w} \geq 0.$$ 

Multiplying through by $e^{\sqrt{2c(\delta)y^1}}$ and integrating from $\hat{y}^1$ to $y^1 > \hat{y}^1$, we find that

$$e^{\sqrt{2c(\delta)y^1}}w(y^1) - e^{\sqrt{2c(\delta)y^1}}w(\hat{y}^1) \geq 0.$$ 

(3.51)

Note that our choice of $\hat{y}^1$ implies that $w(\hat{y}^1) < 0$. Rearranging (3.51), we find that

$$-v' + \sqrt{2c(\delta)v} \leq \left[-e^{\sqrt{2c(\delta)y^1}}w(\hat{y}^1)\right] e^{-\sqrt{2c(\delta)y^1}}.$$ 

34
Multiplying through by $e^{-\sqrt{2(c(\delta))}y^1}$ and integrating from $y^1$ to $\infty$, we find that

$$v \lesssim e^{-\sqrt{2(c(\delta))}y^1}$$

(3.52)

for $y^1$ sufficiently large, giving us the desired asymptotics of $f$ and $s$.

Since $F_0 \in \mathcal{A}$ is a solution of the minimization problem (3.28), then $F_0$ satisfies $F_0'' = \nabla_{\Phi} W(F_0, R)$. Multiplying both sides by $F_0'$ and integrating, we have that

$$\frac{1}{2} |F_0'|^2 = W(F_0, R).$$

Using the identity $g(1) = g(0) + \int_0^1 g'(t)dt$ and the fact that $W((1, 0), R) = 0$, we have

$$|F_0'|^2 \lesssim \|\nabla_{\Phi} W(F_0, R)\|_{L^\infty(y^1, \infty)} |F_0 - \left( \begin{array}{c} 1 \\ 0 \end{array} \right)|.$$  

Since $\nabla_{\Phi} W(F_0, R)$ is bounded for $|x, y| < 1$, then

$$|F_0'|^2 \lesssim e^{-\sqrt{2(c(\delta))}y^1}$$

(3.53)

for $y^1$ sufficiently large. This gives us the desired asymptotics for $f_0'$ and $s_0'$.

□

Proof of proposition 3.5.2: Set

$$v = \frac{1}{2} |F_1|^2.$$  

Then

$$v' = F_1 \cdot F_1' \quad \text{and} \quad v'' = |F_1'|^2 + F_1 \cdot F_1''.$$  

Examining $v''$, we have

$$v'' \geq F_1 \cdot \text{Hess}_{\Phi} W(F_0, R) F_1 + \left[ -H(R) \partial_{y^1} F_0 + y^1 m(y^0) \partial_{y^1} W(F_0, R) \right] \cdot F_1$$  

$$\geq F_1 \cdot \text{Hess}_{\Phi} W \left( \begin{array}{c} 1 \\ 0 \end{array} \right), R) F_1 + F_1 \cdot P(y^1) F_1 + Q(y^1) \cdot F_1,$$

where we have defined

$$P(y^1) := \text{Hess}_{\Phi} W(F_0, R) - \text{Hess}_{\Phi} W((1, 0), R)$$

$$Q(y^1) := -H(R) \partial_{y^1} F_0 + y^1 m(y^0) \partial_{y^1} W(F_0, R).$$
Observe that proposition 3.5.1 implies that
\[ |P(y^1)| \to 0 \quad \text{and} \quad |Q(y^1)| \to 0 \]
exponentially fast as \( |y| \to \infty \). By assumption 3 of (1.9), we have that
\[ v'' \geq 2c v - 2|P(y^1)|v - \sqrt{2} |Q(y^1)| v^{1/2}. \]
Cauchy-Schwarz can then be used to show that
\[ v'' \geq \left[ 2c - 2|P(y^1)| - \frac{\delta^2}{\sqrt{2}} \right] v - \frac{1}{\sqrt{2}\delta^2} |Q(y^1)|^2 \]
for all \( 0 < \delta \). Since \( |P(y^1)|, |Q(y^1)| \to 0 \) as \( |y| \to \infty \), then by choosing \( 0 < \delta \ll 1 \) sufficiently small, we have that
\[ v'' \geq 2c(\delta)v - \frac{1}{\sqrt{2}\delta^2} |Q(y^1)|^2 \]
for all \( |y| \) sufficiently large, where again \( c(\delta) := c - \delta \). With estimates (3.52) and (3.53), we can use arguments from the proof of proposition 3.5.1 to proceed.

\[ \square \]

3.6 Regularity of \( F_0, F_1, \) and \( R \)

For each fixed \( R \), standard ODE theory [8] tells us the solution to (3.17) is smooth in \( y^1 \). However, it does not tell us anything about the differentiability of \( F_0 \) with respect to \( R \). It does not even tell us if \( F_0 \) is continuous in \( R \). In this section, we will show that \( F_0 \) is actually smooth in \( R \).

Once we have that \( F_0 \) is smooth in \( R \), standard ODE theory tells us that \( R(y^0) \) satisfying (3.22) is also smooth as long as \( |R^0| < 1 \). Once we have that \( F_0 \) is smooth in \( (y^1, R) \), we would like to then use standard ODE theory to conclude that \( F_1 \) is smooth in \( (y^1, R, v) \). Unfortunately, things are not so simple. Using these types of arguments, one can only show that for each fixed \( R \), \( F_1 \) is smooth in \( (y^1, v) \). It turns out, though, that once the smoothness of \( F_0 \) in \( (y^1, R) \) has been established, then the arguments from step 3 of the proof of proposition 3.6.1 can be used to show that \( F_1 \) is smooth in \( (y^1, R, v) \).
Proposition 3.6.1. Suppose $F_0 \in \tilde{A}$ is the solution to the minimization problem (3.16), then $F_0$ is smooth in $y^1$ and $R$ for $(y^1, R) \in \mathbb{R} \times (r_0, r_1)$.

Proof of proposition 3.6.1. As we stated before, $F_0$ is smooth in $y^1$. The first step in this proof is to show that $F_0$ is differentiable pointwise with respect to $R$ and that $\partial_R F_0(R)$ is smooth in $y^1$. Once we have established this, we will show that $F_0$ is actually smooth in $y^1$ and $R$.

Step 1: In this step, we will show that $F_0$ is pointwise continuous with respect to $R$. We need to know this in order to show that $F_0$ is differentiable pointwise with respect to $R$.

Define

$$I_R(F) := \int \mu(F; R),$$

(3.54)

where $\mu(F; R)$ was defined in (3.30). Let $F_R \in \tilde{A}$ denote the minimizer of $I_R$. We have that

$$I_R(F_R) = I_R(F_R) + \frac{1}{2} \left( \frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2} \right) \int s_R^2 \geq I_R(F_R) + \frac{1}{2} \left( \frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2} \right) \int s_R^2$$

(3.55)

and similarly

$$I_R(F_R) \geq I_R(F_R) - \left( \frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2} \right) \int s_R^2.$$  

text(3.56)

Putting these together, we have that

$$\left( \frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2} \right) \int s_R^2 \leq I_R(F_R) - I_R(F_R) \leq \left( \frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2} \right) \int s_R^2. \quad (3.57)$$

By (3.55), we have that $I_R(F_R) \leq C$ for all $R \in (r_0, r_1)$. Thus, there exists some constant $C$ so that for all $R \in (r_0, r_1)$, then $\|s_R\|_2^2 \leq C$. This implies that

$$\lim_{\tilde{R} \to R} I_R(F_R) = I_R(F_R). \quad (3.58)$$

It follows that if $R_k$ is any sequence in $(r_0, r_1)$ converging to $R \in (r_0, r_1)$, then $F_{R_k}$ is a minimizing sequence of $I_R$. Using the same arguments found in the proof of proposition 3.3.2, we have that there exists a subsequence of $F_{R_k}$ converging to a minimizer of $I_R$ weakly in $\tilde{H}^1$ and locally uniformly. Since we are assuming that minimizers of $I_R$ are unique, this is the 5th property we are assuming $V$ satisfies (1.23), then in fact we have that if $\tilde{R} \to R$, then $F_{\tilde{R}} \to F_R$ weakly in $\tilde{H}^1$ and locally uniformly.
In fact, we actually have that

$$\|F_R - F_{\hat{R}}\|_{H^1} \to 0$$

(3.59)

as $\hat{R} \to R$. To see this, consider the following

$$\|F_{\hat{R}} - F_{R}\|_{H^1} \leq \|F_{\hat{R}} - F_{R}\|_{H^1(B_{\rho})} + \|F_{\hat{R}} - F_{R}\|_{H^1(\mathbb{R}\setminus B_{\rho})} \leq \|F_{\hat{R}} - F_{R}\|_{H^1(B_{\rho})} + \|F_{\hat{R}} - (1, 0)\|_{H^1(\mathbb{R}\setminus B_{\rho})}$$

Since $F_{\hat{R}}$ and $F_{R}$ decay exponentially fast at the same rate, see (3.5.1), then for all $\rho$ sufficiently large and can use the fact that $F_{\hat{R}} \to F_{R}$ locally uniformly to see that (3.59) holds.

**Step 2:** In this step, we will show that $F_{R}$ is differentiable pointwise in $R$.

First, we will show that for all $\xi \in \ker(L_1(F_{R}; R))^\perp$, there exists a constant $C > 0$ so that

$$I_R(F_{R}) + \frac{m_1}{2} \|\xi\|_2^2 - C \|\xi\|_2^{1/2} \|\xi\|_2^{5/2} \leq I_R(F_{R} + \xi),$$

(3.60)

where $m_1$ was defined in (3.24).

This identity is obtained as follows. We have that

$$I_R(F_{R} + \xi) = I_R(F_{R}) + \int \xi \cdot (F''_{R} + \nabla \varphi W(F_{R}, R)) + \frac{1}{2} \int \xi \cdot L_1(F_{R}; R)\xi$$

$$+ \int \left[ W(F_{R} + \xi, R) - W(F_{R}, R) - \nabla \varphi W(F_{R}, R)\xi - \frac{1}{2} \xi \cdot \text{Hess} \varphi W(F_{R}, R)\xi \right]$$

$$\geq I_R(F_{R}) + \frac{m_1}{2} \|\xi\|_2^2 - C \|\xi\|_3$$

where we used the spectral estimate (3.23), $m_1$ was defined in (3.24), and the fact that $-F''_{R} + \nabla \varphi W(F_{R}, R) = 0$ to obtain the last inequality. Using the Gagliardo-Nirenberg inequality, we have that

$$I_R(F_{R}) + \frac{1}{2} \|\xi\|_2^2 - C \|\xi\|_2^{1/2} \|\xi\|_2^{5/2} \leq I_R(F_{R} + \xi)$$

establishing the identity (3.60).

Suppose $F_{R} \in \tilde{A}$ and $F_{\hat{R}} \in \tilde{A}$, see (3.31) for the definition of $\tilde{A}$, are the unique minimizers
of \(I_R\) and \(I_{\tilde{R}}\), respectively. Then, we have that

\[
I_R(F_\tilde{R}) = I_{\tilde{R}}(F_{\tilde{R}}) + \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right) \int s_R^2 \\
\leq I_R(F_{\tilde{R}}) + \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right) \int s_R^2 \\
= I_R(F_{\tilde{R}}) + \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right) \int (s_\tilde{R}^2 - s_R^2). \tag{3.61}
\]

Setting \(\xi = F_\tilde{R} - F_R\) and using (3.60), we have that

\[
I_R(F_{\tilde{R}}) = I_R(F_R + \xi) \\
\geq I_R(F_R) + \frac{1}{2} \|\xi\|^2 - c \|\xi\|^2_2 \|\xi\|^2_2.
\]

Using (3.59), we see that for all \(\tilde{R}\) sufficiently close to \(R\), then

\[
I_R(F_{\tilde{R}}) \geq I_R(F_R) + C \|\xi\|^2_2, \tag{3.62}
\]

where \(C > 0\) is some constant. Combining (3.61) and (3.62) and using Cauchy-Schwarz, we have that

\[
C \|\xi\|^2_2 \leq \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right) \int (s_\tilde{R}^2 - s_R^2) \\
\leq \frac{d^2(R + \tilde{R})}{R^2 \tilde{R}^2} |R - \tilde{R}| \int |s_R + s_{\tilde{R}}| |s_{\tilde{R}} - s_R| \\
\leq \frac{d^2(R + \tilde{R})}{R^2 \tilde{R}^2} |R - \tilde{R}| \|s_R\|_2 \|s_{\tilde{R}}\|_2 \|s_{\tilde{R}} - s_R\|_2 \\
\leq \frac{d^2(R + \tilde{R})}{R^2 \tilde{R}^2} |R - \tilde{R}| \|s_R\|_2 \|s_{\tilde{R}}\|_2 \|\xi\|_2
\]

for all \(\tilde{R}\) sufficiently close to \(R\). Again, since \(I_R(F_R) < \infty\) for all \(R \in (r_0, r_1)\), then \(\|s_R\|_2 \leq C\) for all \(R \in (r_0, r_1)\). Thus,

\[
\|F_{\tilde{R}} - F_R\|_2 \leq C |\tilde{R} - R| \tag{3.63}
\]

for some constant \(C\).

Set

\[
D_R(\tilde{R}) := \frac{F_{\tilde{R}} - F_R}{\tilde{R} - R}.
\]

Suppose \(F_R = (f_R, s_R)\) and \(F_{\tilde{R}} = (f_{\tilde{R}}, s_{\tilde{R}})\). Since \(F_R, F_{\tilde{R}} \in \tilde{A}\), then \(f_R\) and \(f_{\tilde{R}}\) are odd and \(s_R\) and
The non-degeneracy condition implies that \( \ker(L_1(F_R; R)) = \text{span} \{ F_R' \} \). Since \( f_R \) is odd and \( s_R \) is even, the \( f_R' \) is even and \( s_R' \) is odd. Thus, using parity arguments, we see that \( F_R - F_R \in \ker(L_1(F_R; R))^+ \). In particular, we have that \( D_R(\tilde{R}) \in \ker(L_1(F_R; R))^+ \). From (3.63), we see that \( \| D_R(\tilde{R}) \|_2 \leq C \) for all \( \tilde{R} \) sufficiently close to \( R \). Furthermore, \( D_R(\tilde{R}) \) satisfies the following relation

\[
L_1(F_R; R)D_R(\tilde{R}) = \frac{d^2(\tilde{R} + R)}{R^2 R^2} \left( \begin{array}{c} 0 \\ s_\tilde{R} \end{array} \right) \\
- \int_0^1 \left[ \text{Hess}_\Phi V(F_R + t(F_R - F_R)) - \text{Hess}_\Phi V(F_R) \right] \cdot D_R(\tilde{R}) dt.
\]

With this identity, we obtain the estimate

\[
\| D_R(\tilde{R}) \|_2^2 \leq \left| \langle D_R(\tilde{R}), L_1(F_R; R)D_R(\tilde{R}) \rangle \right| + c \| D_R(\tilde{R}) \|_2^2 \\
\leq \sup_{t \in [0,1]} \| \text{Hess}_\Phi V(F_R + t(F_R - F_R)) - \text{Hess}_\Phi V(F_R) \|_\infty \| D_R(\tilde{R}) \|_2^2 \\
+ \frac{d^2(\tilde{R} + R)}{R^2 R^2} \| s_\tilde{R} \|_2 \| D_R(\tilde{R}) \|_2 + c \| D_R(\tilde{R}) \|_2^2 \\
\leq C,
\]

where the last inequality only holds for \( \tilde{R} \) sufficiently close to \( R \).

To summarize, we have shown that \( \| D_R(\tilde{R}) \|_{H^1} \leq C \) for all \( \tilde{R} \) sufficiently close to \( R \). This bound also implies that \( \left[ D_R(\tilde{R}) \right]_{1/2} \leq C \). Thus, for every sequence \( \tilde{R}_k \to R \), there exists some \( G \in H^1 \) and a subsequence of \( \tilde{R}_k \) so that \( D_R(\tilde{R}_k) \to G \) weakly in \( H^1 \) and locally uniformly. Since each \( D_R(\tilde{R}) \perp \ker(L_1(F_R, R)) \), then \( G \perp \ker(L_1(F_R, R)) \) as well. Furthermore, we have
that for each fix $\phi \in C^\infty_0$, then

$$0 = \left\langle \phi, \frac{d^2(\tilde{R}_k) + R}{\tilde{R}_k^2 R^2} \begin{pmatrix} 0 \\ s_{\tilde{R}_k} \end{pmatrix} \right\rangle$$

$$+ \left\langle \phi, \frac{\frac{d^2(\tilde{R}_k) + R}{\tilde{R}_k^2 R^2} \left( \begin{pmatrix} 0 \\ s_{\tilde{R}_k} \end{pmatrix} \right) \right\rangle \right.$$ 

$$= \left\langle L_1(F, R)\phi, D_R(\tilde{R}_k) \right\rangle - \left\langle \phi, \frac{d^2(\tilde{R}_k) + R}{\tilde{R}_k^2 R^2} \left( \begin{pmatrix} 0 \\ s_{\tilde{R}_k} \end{pmatrix} \right) \right\rangle$$

$$+ \left\langle \int_0^1 \left[ \text{Hess}_\phi V(F + t(F_{R_i} - F_R)) - \text{Hess}_\phi V(F_R) \right] D_R(\tilde{R}_k) dt \phi, D_R(\tilde{R}_k) \right\rangle$$

Taking $k_i \to \infty$ we see that

$$\left\langle L_1(F, R)\phi, G \right\rangle - \left\langle \phi, \frac{d^2(F'}{R^3} \left( \begin{pmatrix} 0 \\ s_R \end{pmatrix} \right) \right\rangle = 0$$

In particular, we have that

$$L_1(F, R)G = 2\frac{d^2}{R^3} \left( \begin{pmatrix} 0 \\ s_R \end{pmatrix} \right). \quad (3.65)$$

Suppose $G_1$ and $G_2$ both solve (3.65), then

$$L_1(F, R)(G_1 - G_2) = 0$$

and hence $G_1 - G_2 \in \text{ker}(L_1(F, R))$. By assumption 4 of (1.9), this implies that $G_1 - G_2 = \alpha F'_R$ for some constant $\alpha$. Recall, however, that $G_1, G_2 \perp \text{ker}(L_1(F, R))$ and so $\alpha = 0$. Thus, $G_1 = G_2$. This means every sequence $\tilde{R}_k$ has a subsequence for which $D_R(\tilde{R}_k) \to G$ weakly in $H^1$ and locally uniformly. In particular, this implies that $D_R(\tilde{R}) \to G$ weakly in $H^1$ and locally uniformly. In particular, the locally uniform convergence implies that $F_0$ is differentiable in $R$ for $R \in (r_0, r_1)$.

Since $G$ satisfies (3.65), then standard ODE theory tells us that $G$ is smooth in $y^1$. This completes step 2.

Step 3: In this step, we will show that $F_0$ is smooth in $y^1$ and $R$. Actually, we will only show
that $\partial_R F_0(R)$ is differentiable pointwise in $R$ and that $\partial_{RR} F_0(R)$ is smooth in $y^1$. This argument can then be applied inductively to show that $F_0$ is smooth in $y^1$ and $R$.

Set $G_R = \partial_R F_R$, where $F_R \in \bar{\mathcal{A}}$ is the unique minimizer of $I_R$. In the previous step we showed that $G_R \in H^1$ for all $R \in (r_0, r_1)$. For

$$h(R) := 2 \frac{d^2}{R^3} \begin{pmatrix} 0 \\ s_R \end{pmatrix}$$

we have that $G_R$ satisfies

$$L_1(F_R, R) G_R = h(R).$$

For $0 < |\lambda| \ll 1$, we have that

$$L_1(F_{R+\lambda}, R + \lambda) \frac{G_{R+\lambda} - G_R}{\lambda} + \frac{L_1(F_{R+\lambda}, R + \lambda) - L_1(F_R, R)}{\lambda} G_R = \frac{h(R + \lambda) - h(R)}{\lambda}. \quad (3.66)$$

This implies that

$$\left| \left\langle \frac{G_{R+\lambda} - G_R}{\lambda}, L_1(F_{R+\lambda}, R + \lambda) \frac{G_{R+\lambda} - G_R}{\lambda} \right\rangle \right|$$

$$= \left| \left\langle \frac{G_{R+\lambda} - G_R}{\lambda}, \frac{h(R + \lambda) - h(R)}{\lambda} \right\rangle - \frac{L_1(F_{R+\lambda}, R + \lambda) - L_1(F_R, R)}{\lambda} G_R \right|$$

$$\leq \left\| \frac{G_{R+\lambda} - G_R}{\lambda} \right\|_2 \left\| \frac{h(R + \lambda) - h(R)}{\lambda} - \frac{L_1(F_{R+\lambda}, R + \lambda) - L_1(F_R, R)}{\lambda} G_R \right\|_2. \quad (3.67)$$

Recall that $F_R = (f_R, s_R)$. Since $f_R$ is odd and $s_R$ is even, then $\partial_R f_R$ is still odd and $\partial_R s_R$ is still even. Since $\ker(L_1(F_{R+\lambda}, R + \lambda)) = \text{span} \{ F_{R+\lambda} \}$, then using parity arguments, we can see that

$$\frac{G_{R+\lambda} - G_R}{\lambda} \perp \ker(L_1(F_{R+\lambda}, R + \lambda)).$$

Hence we apply the spectral estimate (3.23) to (3.67) to obtain the following estimate

$$\left\| \frac{G_{R+\lambda} - G_R}{\lambda} \right\|_2 \leq \left\| \frac{h(R + \lambda) - h(R)}{\lambda} - \frac{L_1(F_{R+\lambda}, R + \lambda) - L_1(F_R, R)}{\lambda} G_R \right\|_2.$$
With this $L^2$ bound, we can also bound the $\dot{H}^1$ norm of $\frac{G_{R+\lambda} - G_R}{\lambda}$ for sufficiently small $\lambda$ as

\[
\left\| \frac{G_{R+\lambda} - G_R}{\lambda} \right\|_{\dot{H}^1}^2 \\
\leq \left| \left( \frac{G_{R+\lambda} - G_R}{\lambda}, L_1(F_{R+\lambda}, R + \lambda) \frac{G_{R+\lambda} - G_R}{\lambda} \right) \right| + c \left\| \frac{G_{R+\lambda} - G_R}{\lambda} \right\|_{L^2}^2 \\
\leq \left\| \frac{G_{R+\lambda} - G_R}{\lambda} \right\|_2 \left( h(R + \lambda) - h(R) \right) - \frac{L_1(F_{R+\lambda}, R + \lambda) - L_1(F_R, R)}{\lambda} G_R \right\|_2 + c \left\| \frac{G_{R+\lambda} - G_R}{\lambda} \right\|_2^2 \\
\leq C,
\]

where we used (3.66) and the $L^2$ bound of $\frac{G_{R+\lambda} - G_R}{\lambda}$ to obtain this estimate.

We then proceed as we did at the end of step 1 to show that $\frac{G_{R+\lambda} - G_R}{\lambda}$ has a limit as $\lambda \to 0$. This limit is the derivative of $G_R$ with respect to $R$, which is the second derivative of $F_R$ with respect to $R$. Furthermore, $\partial_R G_R$ satisfies

\[
L_1(F_R, R) \partial_R G_R = \partial_R h(R) - \partial_R L_1(F_R, R) G_R
\]

and since $\partial_R h(R)$ and $\partial_R L_1(F_R, R) G_R$ are smooth in $y^1$, then standard ODE theory implies that $\partial_R G_R$ is smooth in $y^1$ too. Thus, $\partial_{RR} F_0$ exists and is smooth in $y^1$.

\[
\square
\]

4 Effective Dynamics

For the entirety of this section, choose $F_0(\cdot; R)$ satisfying the minimization problem (3.16), $F_1(\cdot; R, v)$ satisfying the minimization problem (3.18), and $h$ satisfying (3.22) with $R(0) \in (r_0, r_1)$ and $R'(0) = 0$, where $r_0 < r_1$ come from the non-degeneracy condition (1.21). We also pick $y_0^1$ and $y_1^1$ as we did in (4a)–(4f).

4.1 Approximation Using Profiles Coming From the Formal Asymptotics

We will show that if $\Phi$ solving (3.7) starts off “close” to the right hand side of (3.11) on $(-y^1_1, y^1_1)$, then it remains close to the right hand side of (3.11) up to some time independent of $\epsilon$. In this section, we will define what we mean by “close” and introduce some objects that show up in the statement of the main theorem of this thesis, theorem 4.2.1.
Let $\Phi$ be a solution to (3.7) whose boundary data satisfies

$$
\begin{align*}
\Phi(0, y^1) &= (\text{sign} y^1, 0) \\
\partial_r \Phi(0, y^1) &= (0, 0) \\
\Phi(y^0, \pm y^1) &= (\pm 1, 0)
\end{align*}
$$

(4.1) (4.2)

Even though $\Phi$ is only defined on $[0, y^0_0) \times (-y^1_*, y^1_*)$, our choice of $y^0_0$ and $y^1_*$ allows us to naturally extend $\Phi$ to all of $[0, y^0_0) \times \mathbb{R}$ as

$$
\tilde{\Phi}(y^0, y^1) := \begin{cases} 
(\text{sign} y^1, 0) & \text{for } y^1 \notin (-y^1_*, y^1_*) \\
\Phi(y^0, y^1) & \text{for } y^1 \in (-y^1_*, y^1_*)
\end{cases}
$$

(4.3)

Using the coordinate transformation $\psi$, defined in (3.1), we make the following important observation. Suppose $\Phi$ is a solution to (3.7) whose boundary data satisfies (4.1) and (4.2). Recall that $R'(0) = 0$ and hence $\psi(0, y^1) = (0, R(0)) + y^1(0, 1)$. Let $\Phi$ be a solution to (1.8) in $(t, r)$ coordinates with initial data

$$(\Phi, \partial_r \Phi)(0, r) = \left( \Phi \circ \psi^{-1}, \partial_r (\Phi \circ \psi^{-1}) \right)(0, r)$$

for $r \in (R(0) - y^1_*, R(0) + y^1_*)$

$$\Phi(0, r) = (-1, 0)$$

for $r \in [0, R(0) - y^1_*)$

$$\Phi(0, r) = (1, 0)$$

for $r \in (R(0) + y^1_*, \infty)$

$$\partial_r \Phi(0, r) = (0, 0)$$

for $r \notin (R(0) - y^1_*, R(0) + y^1_*)$.

Then, $\Phi \circ \psi$ and $\Phi$ are the same on $[0, y^0_0) \times (-y^1_*, y^1_*)$ and the initial data of $\Phi$ satisfies the conditions discussed in section 1.3. Thus, if we can find a solution $\Phi$ to (3.7) which looks like the right hand side of (3.11) and whose initial data satisfies (4.1) and (4.2), then we can use this observation to construct a solution to (1.8) for all of $r \in \mathbb{R}_+$ so that $\Phi \circ \psi$ looks like the right hand side of (3.11).

Consider $\tilde{\Phi}$ from (4.3). We will find a function $a : [0, y^0_0) \to \mathbb{R}$ so that for each $y^0 \in [0, y^0_0)$, the $L^2_y$ distance between $\tilde{\Phi}(y^0, \cdot)$ and $F_0 \left( \frac{y^1 - a}{\epsilon}; R(y^0) \right)$ is minimized. For each $R \in \mathbb{R}_+$ and $\Psi \in L^2(\mathbb{R}, \mathbb{R}^2)$, define

$$h_\epsilon(\Psi, R, a) := \left\| \Psi - F_0 \left( \frac{y^1 - a}{\epsilon}; R \right) \right\|_{L^2_y}^2$$

(4.4)

$$G_\epsilon(\Psi, R, a) := \partial_\epsilon h_\epsilon(\Psi, R, a) = \frac{2}{\epsilon} \left( \left\| \Psi - F_0 \left( \frac{y^1 - a}{\epsilon}; R \right), \partial_\epsilon F_0 \left( \frac{y^1 - a}{\epsilon}; R \right) \right\|_{L^2_y} \right) .$$

(4.5)
Define
\[ U_{\delta, \epsilon} := \left\{ (\Psi, R) \in L^2(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \mid \inf_{a \in \mathbb{R}} h_\epsilon(\Psi, R, a) < \delta \right\}, \]  
(6.4)
\[ V_{\delta, \epsilon}(a_0) := \left\{ (\Psi, R) \in L^2(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \mid h_\epsilon(\Psi, R, a_0) < \delta \right\}. \]
(6.7)

**Lemma 4.1.1.** There exists a maximal \( \delta > 0 \) and a unique \( C^\infty \) map \( \tilde{a} : U_{\delta, \epsilon} \to \mathbb{R}, C^\infty \) with respect to the \( L^2 \times \mathbb{R} \) topology, so that \( G_\epsilon(\Psi, R, \tilde{a}(\Psi, R)) = 0 \) where both \( \delta \) and \( \tilde{a} \) possibly depend on \( \epsilon \).

**Proof:** Since

1. \( G_{R, \epsilon} \) is \( C^\infty \) because it is linear in \( \Psi \) and because \( F_0 \) is smooth in both \( y^1 \) and \( R \)
2. \( G_\epsilon(F_0(\frac{-a_0}{\epsilon}; R), R, a_0) = 0 \)
3. \[ \partial_{\alpha_l} G_\epsilon(F_0(\frac{\cdot - a_0}{\epsilon}; R), R, a) = \left\langle \partial_{\alpha_l} F_0(\frac{\cdot - a_0}{\epsilon}; R), \partial_\beta F_0(\frac{\cdot - a_0}{\epsilon}; R) \right\rangle_{L^2} > 0, \]
then we can apply the implicit function theorem. That is, there exists \( \delta > 0 \) and a unique \( C^\infty \) map \( a : V_{\delta, \epsilon}(a_0) \to \mathbb{R}, \) both \( \delta \) and \( a \) depending on \( \epsilon \), so that \( G_\epsilon(\Psi, R, a(\Psi, R)) = 0 \) for all \( (\Psi, R) \in V_{\delta, \epsilon}(a_0). \)

Observe that
\[ U_{\delta, \epsilon} = \bigcup_{b \in \mathbb{R}} V_{\delta, \epsilon}(a_0 + b). \]

For each \( (\Psi, R) \in U_{\delta, \epsilon} \) there exists \( b \in \mathbb{R} \) so that for \( \tau_b \Psi := \Psi(\cdot - b), \) then \( (\tau_b \Psi, R) \in V_{\delta, \epsilon}(a_0). \)
Define \( \tilde{a}_b(\Psi, R) := a(\tau_b \Psi, R) - b, \) then \( G_\epsilon(\Psi, R, \tilde{a}_b(\Psi, R)) = 0. \) If \( (\tau_b \Psi, R), (\tau_c \Psi, R) \in V_{\delta, \epsilon}(a_0), \) then by the uniqueness of \( a \) one has that \( \tilde{a}_b(\Psi, R) = \tilde{a}_c(\Psi, R). \) Thus, there exists a unique \( \tilde{a} : U_{\delta, \epsilon} \to \mathbb{R} \) so that \( G_\epsilon(\Psi, R, \tilde{a}(\Psi, R)) = 0. \)

□

**Corollary 4.1.2.** For all \( \epsilon > 0, \) there exists a maximal \( \tilde{\delta} > 0 \) independent of \( \epsilon \) and a unique \( C^\infty \) map \( \tilde{b} : U_{\delta, 1} \to \mathbb{R}, \) independent of \( \epsilon, C^\infty \) with respect to the \( L^2 \times \mathbb{R} \) topology, so that for all \( \Psi \in \tilde{U}_{\delta, \epsilon}, \) then
\[ G_\epsilon(\Psi, R, \frac{1}{\epsilon} \tilde{b}(\Psi, R)) = 0, \]

where \( \Psi_\epsilon(y^1) := \Psi(\epsilon y^1). \)

That is, for arbitrary \( \epsilon > 0, \) the \( \delta \) and \( \tilde{a} \) coming from lemma 4.1.1 are of the form
\[ \delta = \epsilon \tilde{\delta} \quad \text{and} \quad \tilde{a}(\Psi, R) = \frac{1}{\epsilon} \tilde{b}(\Psi, R). \]
Problem (3.16), we define $y_0$ term appearing on the right hand side of (4.9). Upon examining (3.21), we see that $F_1(\Psi, R, \Phi(\Psi, R)) = 0$. If $\Psi \in U_{\delta, \epsilon}$, then $\Psi \in U_{\delta, \epsilon}$ and hence

$$0 = G_1(\Psi, R, \Phi(\Psi, R)) = G_1(\Psi, R, \frac{1}{\epsilon} \Phi(\Psi, R)).$$

Consider $\Phi$ defined in (4.3) and suppose that $(\Phi(0), R(0)) \in U_{\delta, \epsilon}$ with $\delta > 0$ coming from corollary 4.1.2. Then, there exists some maximal $0 < y_0(\epsilon)$ so that $(\Phi(y_0), R(0)) \in U_{\delta, \epsilon}$ for all $0 \leq y < y_0(\epsilon)$. An important step in our proof of theorem 4.2.1 is to establish lower bounds for $y_0(\epsilon)$ that are independent of $\epsilon$. By choosing $a(y) := \tilde{a}(\Phi(y), R(y))$, where $\tilde{a}$ is from lemma 4.1.1, then a natural consequence of this discussion is the following corollary.

**Corollary 4.1.3.** Consider $\Phi$ from (4.3) and suppose that $(\Phi(0), R(0)) \in U_{\delta, \epsilon}$ with $\delta > 0$ from corollary 4.1.2. Then, there exists $y_0(\epsilon) > 0$ and a unique $C^\infty$ function $a(y)$ so that $G_1(\Phi(y), R(y), a(y)) = 0$ for all $0 \leq y < y_0(\epsilon)$.

As part of our assumptions for theorem 4.2.1 we assume that $(\Phi(0), R(0)) \in U_{\delta, \epsilon}$. Let $a(y)$ be the function from the conclusion of corollary 4.1.3. For $F_0$ solving the minimization problem (3.16), we define

$$F_0(y, y_1) = F_0(y_1 - \frac{a(y)}{\epsilon}; R(y)).$$

For $R = R(y_0)$ and $v = R'(y_0)$, then $F_1(y, y_1; R, R')$ from proposition 3.3.4 solves

$$\epsilon L_0(F_0; R) \left[ F_1(y_1 - \frac{a(y)}{\epsilon}; R, R') \right] = H(R) \partial_{y_1} F_0 - \frac{1}{\epsilon} m(y_0)(\frac{y_1 - a(y)}{\epsilon}) \partial_{y_1} w(F_0, R),$$

where $m$ was defined (3.4) and $H(R)$ was defined in (3.20). Note that we have a $\frac{a}{\epsilon} m \partial_{y_1} w(F_0, R)$ term appearing on the right hand side of (4.9). Upon examining (3.21), we see that $F_1$ does not have a translation symmetry in $y_1$ as the inhomogeneity of (3.21) depends explicitly on $y_1$ and it is this lack of a translation symmetry which gives rise to this new term. By translating the leading order correction term appearing in our ansatz, we introduce an error. Thus, we need to introduce a correction term. For each fixed $(R, v, a) \in \mathbb{R}_+ \times (-1, 1) \times \mathbb{R}$, define $F(\cdot; R, v, a)$ to be
the solution of

\[ L_1(F_0, R) F_{1,1} = -\frac{a}{\sqrt{1 - v^2}} m \partial_r w(F_0, R) \]

\[ \lim_{x \to \pm\infty} F_{1,1} = 0. \]

(4.10)

**Proposition 4.1.4.** Suppose \( R \in (r_0, r_1) \). Then there exists \( F_{1,1} \in H^1 \) solving (4.10). Furthermore, \( F_{1,1} \perp L^2 \partial_y F_0 \), \( F_{1,1} \) is smooth in \( y^1, R, v, \) and \( a \), and there exists a constant \( \alpha > 0 \), depending only on \( c \) appearing in condition 3 of (1.9), so that for all \( \sigma, \beta, \kappa, \lambda \in \mathbb{Z}^+ \),

\[ \left| \partial^{\sigma}_y \partial^\beta_R \partial^\kappa_v \partial^\lambda_a F_{1,1} \right| \lesssim e^{-\alpha |y^1|}. \]

The arguments used to prove proposition 4.1.4 are the same arguments used to establish the existence, asymptotics, and regulatory of \( F_1 \). We omit the details of the proof. Now define

\[ \tilde{F}_1(y^0, y^1) = F_1\left(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0), R'(y^0)\right) + F_{1,1}\left(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0), R'(y^0), \frac{a(y^0)}{\epsilon}\right) \]

(4.11)

and observe that for each fixed \( y^0 \), then

\[ \epsilon L_\epsilon(\tilde{F}_0, R) \tilde{F}_1 = H(R) \partial_{y^1} \tilde{F}_0 - \frac{1}{\epsilon} \frac{y^1}{\epsilon} m(y^0) \partial_r w(F_0, R). \]

For our result, we will need to control two quantities. The first is the error and the second is the shift \( a(y^0) \). Define the **error** between \( \tilde{\Phi} \) and our approximation \( \tilde{F}_0 + \epsilon \tilde{F}_1 \) as

\[ \xi := \tilde{\Phi} - \tilde{F}_0 - \epsilon \tilde{F}_1. \]

(4.12)

Also, define the quantity

\[ \Delta(y^0) := \left( 1 + \left| \frac{a(y^0)}{\epsilon} \right| + \left| \frac{a'(y^0)}{\epsilon} \right| \right)^3. \]

(4.13)

An observation that we will make use of later is the following. Since \( 0 = \partial_a h(a(y^0)) \), we can
use (4.5) to get that
\[
0 = \int_{\mathbb{R}} (\xi + \epsilon \tilde{F}_1) \cdot \partial_{\xi} \tilde{F}_0
= \int_{\mathbb{R}} \xi \cdot \partial_{\xi} \tilde{F}_0 + \epsilon \int_{\mathbb{R}} \tilde{F}_1 \cdot \partial_{\xi} \tilde{F}_0
= \int_{\mathbb{R}} \xi \cdot \partial_{\xi} \tilde{F}_0,
\]

where we needed to use the fact that \( \tilde{F}_1 \perp \partial_{\xi} \tilde{F}_0 \), from proposition 3.3.4 and proposition 4.1.4, to go from the second line to the third. Thus, we have that
\[
\int_{\mathbb{R}} \xi \cdot \partial_{\xi} \tilde{F}_0 = 0. \tag{4.14}
\]

As we stated at the beginning of this section, we want to show that if \( \xi \) starts off small, then \( \xi \) remains small on \([0, y^0_0)\). Recall that for \( y^1 \in \mathbb{R} \setminus (-y^1_1, y^1_1) \), then \( \tilde{\Phi}(y^0, y^1) = (\text{sign} y^1, 0) \). For each \( y^0 \in [0, y^0_0) \), we can use the asymptotics derived in section 3.5 to control the size of \( \xi \) outside of \((-y^1_1, y^1_1)\) as
\[
\|\xi\|_{H^1((0,\infty) \times (-y^1_1, y^1_1))} \leq \|(\text{sign} y^1, 0) - \tilde{F}_0\|_{L^2((0,\infty) \times (-y^1_1, y^1_1))} + \epsilon \|\tilde{F}_1\|_{L^2((0,\infty) \times (-y^1_1, y^1_1))}
\leq e^{-\epsilon^{1-a}} \tag{4.15}
\]
for some \( \alpha > 0 \). Thus, if we can control the size of \( a \) and if \( \epsilon \) is taken sufficiently small, then we have that \( \xi(y^0) \) is small outside of \((-y^1_1, y^1_1)\) for each \( y^0 \in [0, y^0_0) \).

Next, we will examine \( \xi \) on \([0, y^0_0) \times (-y^1_1, y^1_1)\). Since \( \tilde{\Phi} = \Phi = \tilde{F}_0 + \epsilon \tilde{F}_1 + \xi \) on this set, then we can plug this into (3.7). Doing so, we find that \( \xi \) solves
\[
\frac{m^2}{n^2} \partial_{y^0, y^0} \xi + B^0 \partial_{y} \xi + L_{\epsilon}(\tilde{F}_0, R) \xi + S_{-1} + S_0 + N = 0, \tag{4.16}
\]

where we used the fact that
\[
-\partial_{y^1} \tilde{F}_0 + \frac{1}{\epsilon} w(\tilde{F}_0, R) = 0
\]
\[
\epsilon L_{\epsilon}(\tilde{F}_0, R) \tilde{F}_1 = H(R) \partial_{y^1} \tilde{F}_0 - \frac{1}{\epsilon} \frac{y^1}{\epsilon} m \partial_y w(\tilde{F}_0, R)
\]
to simplify and for $\tilde{F}_\xi = \epsilon \tilde{F}_1 + \xi$ we defined

$$S_{-1} := [H(R) + B^1] \partial_{\gamma^1} \tilde{F}_0$$

(4.17)

$$S_0 := \frac{m^2}{n^2} \partial_{\gamma^0,\phi}(\tilde{F}_0 + \epsilon \tilde{F}_1) + B^0 \partial_{\gamma^0}(\tilde{F}_0 + \epsilon \tilde{F}_1) + \epsilon B^1 \partial_{\gamma^1} \tilde{F}_1$$

(4.18)

$$N := \frac{1}{\epsilon^2} \left[ w(\tilde{F}_0 + \tilde{F}_\xi, R + y^1m) - w(\tilde{F}_0, R) - \text{Hess}_\Phi W(\tilde{F}_0, R) \tilde{F}_\xi - y^1 m(y^0) \partial_{\gamma^0} w(\tilde{F}_0, R) \right].$$

(4.19)

Recall that $w(\Phi, r) := \nabla \Phi_{W}(\Phi, r)$ and so $N$ is really just the Taylor expansion of $\nabla \Phi_{W}(\tilde{F}_0 + \tilde{F}_\xi, R + y^1m)$ about $(\tilde{F}_\xi, y^1m)$.

Now that we have (4.15), we are then left to control $\xi$ on $[0, y^0_0] \times (-y^1_*, y^1_*)$. We will use the following quantities to control the size of $\xi$ on this set.

**Definition 4.1.5.** For $Q = (Q_1, Q_2) : (0, y^0_0) \times (-y^1_*, y^1_*) \to \mathbb{R}^2$ define the energy density

$$e(Q) = \frac{1}{2} m^2 \left| \partial_{\gamma^0} Q \right|^2 + \frac{1}{2} \left| \partial_{\gamma^1} Q \right|^2 + \frac{1}{\epsilon^2} Q \cdot \text{Hess}_\Phi W(\tilde{F}_0, R) Q.$$

(4.20)

Using the energy density, we define the energy of $Q$ as

$$E(Q)(y^0) = \int_{-y^1_*}^{y^1_*} e(Q)(y^0, y^1) \, dy^1.$$

(4.21)

For convenience we set $E(y^0) := E(\xi)(y^0)$.

Using this new definition, we obtain a very useful corollary to theorem 3.2.1 that we will use to control the error term $\xi$.

**Corollary 4.1.6.** For $\xi$ as defined in (4.12), we have that

$$\frac{1}{\epsilon^2} \int_{-y^1_*}^{y^1_*} |\xi|^2 \lesssim E + \frac{1}{\epsilon^2} e^{-\alpha |\xi|/\epsilon}$$

(4.22)

for some $\alpha > 0$.

**Proof of corollary 4.1.6:** Using theorem 3.2.1 we have that

$$\frac{1}{\epsilon^2} \int \xi^2 \lesssim \int \xi \cdot L_\alpha(F_0, R) \xi.$$

49
Integrating by parts and using (4.15), we have that

\[ \frac{1}{\epsilon^2} \int_{-y_1^*}^{y_1^*} |\xi|^2 \lesssim E + \int_{|y| > y_1^*} \left\{ \frac{1}{2} |\xi'|^2 + \frac{1}{\epsilon^2} \xi \cdot \text{Hess}_\Phi W(F_0, R) \xi + \frac{1}{\epsilon^2} |\xi|^2 \right\}. \]

Using the fact that \[ \|\text{Hess}_\Phi W(F_0, R)\|_{\infty} \lesssim 1 \] and (4.15) establishes (4.22).

\[ \square \]

4.2 Main Theorem

Recall that we chose \( y_0^* \) and \( y_1^* \) satisfying (4a) - (4f). The main theorem we want to prove is the following.

**Theorem 4.2.1.** Let \( V \) be a potential satisfying (1.9) and suppose \( F_0 \) and \( F_1 \) minimize (3.16) and (3.18), respectively. Suppose \( R \) solves (3.22) with \( R(0) \in (r_0, r_1) \) and \( R'(0) = 0 \). Further, suppose \( \Phi(0) \) and \( a(0) \) satisfy

\[ \frac{A(0) \lesssim 1}{A(0) \lesssim 1} \quad \text{and} \quad E(0) \lesssim \epsilon^2 \quad (4.23) \]

\[ \left\{ \begin{array}{l} \Phi(0, y_1^*) = (\text{sign} y_1^*, 0) \\ \partial_\phi \Phi(0, y_1^*) = (0, 0) \end{array} \right\} \quad \text{for } \frac{1}{2} < |y_1^*| < y_1^* \quad (4.24) \]

\[ \Phi(y_0^*, \pm y_1^*) = (\pm 1, 0) \quad \text{for } y_0^* \in [0, y_0^*). \quad (4.25) \]

Then there exists a solution \( \Phi \) to (3.7) on \([0, y_0^*) \times (-y_1^*, y_1^*)\) satisfying the boundary conditions (4.23) - (4.24), and a function \( a : [0, y_0^*) \to \mathbb{R} \) so that

\[ \frac{A(y_0^*) \lesssim 1}{A(y_0^*) \lesssim 1} \quad \text{and} \quad E(y_0^*) \lesssim \epsilon^2 \quad (4.26) \]

for all \( 0 \leq y_0^* < y_0^* \).

Theorem 1.2.2 is then obtained from theorem 4.2.1 as follows.

**Proof of theorem 1.2.2.** Let \( \Phi \) be the solution coming from conclusion of theorem 4.2.1. The bounds

\[ |a|, |a'| \lesssim \epsilon \]
easily follow from the bound $A \leq 1$. First, the bound

$$\int_{-y_1^*}^{y_1^*} |\partial_y \xi|^2 \lesssim \epsilon^2$$

easily follows from the bound $E \lesssim \epsilon^2$. Next, we apply the spectral estimate (4.22) to the bound $E \lesssim \epsilon^2$ to show that

$$\frac{1}{\epsilon^2} \int_{-y_1^*}^{y_1^*} |\xi|^2 \lesssim \epsilon^2. \quad (4.27)$$

Then, since

$$\| \text{Hess}_\Phi W(F_0, R) \|_\infty \lesssim 1$$

we can again use $E \lesssim \epsilon^2$ and (4.27) to show that

$$\int_{-y_1^*}^{y_1^*} |\partial_y \xi|^2 \lesssim \epsilon^2.$$

To then obtain (1.27), (1.28), and (1.29) we do a change of coordinates to $(t, r)$ coordinates, see (3.1). Since the domain we are working on is $[0, y_0^*] \times (-y_1^*, y_1^*)$, see (4a) - (4f), this change of coordinates is well defined. Further, results from [18] allow us to identify $y_0^*$ and $y_1^*$ with $s_M$ and $d_M$ from lemma 1.2.1 respectively, finishing the proof.

The proof of theorem 4.2.1 relies on the following two estimates

Theorem 4.2.2. Suppose that $\Phi$ solves (3.7) satisfying the boundary conditions (4.25) - (4.24). Then, for as long as $a(y_0^*)$ is well defined we have

$$E \lesssim E(0) + \left[ \frac{y_1^*}{(R - y_1^* m)^3} + \frac{(y_1^*)^2}{(R - y_1^* m)^4} \right] E[y_0^*] + \epsilon^{1/2} e^{-\alpha \frac{y_1^*}{R}} A[y_0^*]$$

$$+ \frac{1}{\epsilon} (\epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{y_1^*}{R}})^{1/2} E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{y_1^*}{R}}) E^{1/2} A[y_0^*]$$

$$+ \frac{1}{\epsilon^2} \int_0^{y_0^*} (1 + \epsilon E^{1/2} + e^{-\alpha \frac{y_1^*}{R}})(\epsilon^2 + \epsilon E^{1/2} + e^{-\alpha \frac{y_1^*}{R}})(\epsilon E^{1/2} + e^{-\alpha \frac{y_1^*}{R}}) A. \quad (4.28)$$
Theorem 4.2.3. Suppose that $\Phi$ solves (3.7) satisfying the boundary conditions (4.25) - (4.24). Then, for as long as $a(y^0)$ is well defined we have

\[
\frac{|a''|}{\varepsilon} \leq \frac{|a''|}{\varepsilon} \left[ \varepsilon + \sqrt{\varepsilon} \sqrt{E + e^{-\alpha\varepsilon^{-1/2}}} \right] A + \frac{1}{\varepsilon^{5/2}} \left[ \varepsilon + (E + e^{-\alpha\varepsilon^{-1/2}})^{1/2} e^{1/4}(E^{3/2} + E + e^{-\alpha\varepsilon^{-1/2}}) \right].
\] (4.29)

Proof of Theorem 4.2.1: Recall that we choose $\gamma^0$ and $y^1$, see (4a) - (4f) so that

\[
C(y^0) \frac{y_1^1}{(R(y^0) - y_1^1 m(y^0))^3} \leq \frac{1}{4} \quad \text{and} \quad C(y^0) \frac{(y_1^1)^2}{(R(y^0) - y_1^1 m(y^0))^4} \leq \frac{1}{4},
\] (4.30)

where $C(y^0)$ is a continuous function of $y^0$, independent of $\varepsilon$, coming from the energy inequality (4.28). We will also make use of the following two estimates

\[
|a(y^0)| \leq |a(0)| + \int_0^{y^0} |a'| \quad \text{and} \quad |a'(y^0)| \leq |a'(0)| + \int_0^{y^0} |a''|.
\] (4.31)

The conclusions of our discussion following the definition of $\tilde{\Phi}$ in (4.3) and the existence and uniqueness of solutions to (1.8) in $(t, r)$ coordinates, see appendix B, allow us to conclude that there exists a unique solution $\Phi$ to (3.7) satisfying the boundary conditions (4.23) - (4.24).

Since $E(0) \leq \varepsilon^2$, we can use theorem 3.2.1 to show that

\[
\|\tilde{\Phi}(0) - \tilde{F}_0(0)\|_2 \leq \|\tilde{\Phi}(0) - \tilde{F}_0(0) - e\tilde{F}_1(0)\|_2 + \varepsilon^2 \|\tilde{F}_1(0)\|_2 \leq \varepsilon^2,
\]

where $\tilde{\Phi}$ was defined in (4.3). Thus, $(\tilde{\Phi}(0), R(0)) \in U_{\tilde{a}, \varepsilon}$ and hence we can use corollary 4.1.3 to find $a(y^0)$ for $y^0 \in [0, y^0_\varepsilon(\varepsilon))$.

Next, define

\[
E_M(I) := \max_{y^0 \in I} E(y^0) \quad \text{and} \quad A_M(I) := \max_{y^0 \in I} A(y^0).
\]

Using theorem 4.2.2 we have that

\[
E_M(I) \leq E(0) + \frac{1}{2} E_M(I) + \sqrt{\varepsilon} e^{-\alpha \varepsilon^{-1/2}} e^{\alpha A_M(I)} \Delta_M(I) + \frac{1}{\varepsilon} \sqrt{E_M(I) B_M(I)} + \left(1 + e \sqrt{E_M(I)} + e^{-\alpha \varepsilon^{-1/2}} e^{\alpha A_M(I)} \right) e^2 + e \sqrt{E_M(I) + e^{-\alpha \varepsilon^{-1/2}} e^{\alpha A_M(I)}} \sqrt{E_M(I) A_M(I)}
\]

\[
+ \frac{|I|}{\varepsilon^2} (1 + e \sqrt{E_M(I)} + e^{-\alpha \varepsilon^{-1/2}} e^{\alpha A_M(I)}) e^2 + e \sqrt{E_M(I) + e^{-\alpha \varepsilon^{-1/2}} e^{\alpha A_M(I)}} A_M(I) e^{-\alpha \varepsilon^{-1/2}} e^{\alpha A_M(I)},
\] (4.32)
where \( B_M(I) \) is defined as

\[
B_M(I) := \left[ e + (e \sqrt{E_M(I)} + e^{-\frac{5}{2}y^1} e^{\alpha \Delta_M(I)})^{1/2} E_M(I)^{1/4} \right] \left[ e^{3/2} + e \sqrt{E_M(I)} + e^{-\frac{5}{2}y^1} e^{\alpha \Delta_M(I)} \right] A_M(I).
\]

Using theorem 4.2.3 and (4.31), we also have that

\[
A_M(I)^{1/3} \leq (1 + |I|) A(0)^{1/3} + \frac{1}{e^{5/2}} \frac{|I| + |I|^2}{1 - \left[ e + \sqrt{E_M(I)} + e^{-\frac{5}{2}y^1} e^{\alpha \Delta_M(I)} \right] A_M(I)} B_M(I).
\]

Using (4.23), (4.32), and (4.33), we see that for \( I = [0, \min \{y^0(\epsilon), y^0_*\}] \)

\[
E_M(I) \leq \epsilon^2 \quad \text{and} \quad A_M(I) \leq 1.
\]

If \( \min \{y^0(\epsilon), y^0_*\} = y^0_* \), then we are done. If not, then because

\[
E_M(I) \leq \epsilon^2
\]

we can use corollary 4.1.3 to show that \( a(y^0) \) actually exists beyond \( y^0(\epsilon) \). Applying the above argument over and over again allows us to conclude that \( a(y^0) \) exists and is well defined on \( I = [0, y^0] \) and that

\[
E_M(I) \leq \epsilon^2 \quad \text{and} \quad A_M(I) \leq 1.
\]

\( \Box \)

### 4.3 Proof of Energy Estimate (Theorem 4.2.2)

We will need to estimate \( (\gamma^1)^\rho (\frac{\partial}{\partial y^1})^\rho (\frac{\partial}{\partial R})^\rho (\frac{\partial}{\partial v})^\rho (\frac{\partial}{\partial a})^\rho F_i \), for \( \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\} \), in order to prove this theorem and theorem 4.2.3. A point on notation before continuing. We define the third slot of \( F_1 = F_1(y^1; R, v) \) and \( F_{1,1} = F_{1,1}(y^1; R, v, a) \) as \( v \) and the fourth slot of \( F_{1,1} \) as \( a \).

**Lemma 4.3.1.** For \( \alpha, \beta, \kappa, \gamma, \) and \( \sigma \in \mathbb{N} \cup \{0\} \), we have

\[
\left\| (\gamma^1)^\rho (\frac{\partial}{\partial y^1})^\rho (\frac{\partial}{\partial R})^\rho (\frac{\partial}{\partial v})^\rho (\frac{\partial}{\partial a})^\rho \left[ F_0 \left( \frac{\cdot - a}{\epsilon} ; R \right) \right] \right\|_2 \leq \frac{1}{e^{a - \gamma - \frac{1}{2}} \epsilon} \left( 1 + \frac{|a|}{\epsilon} \right)^\gamma.
\]

\[
\left\| (\gamma^1)^\rho (\frac{\partial}{\partial y^1})^\rho (\frac{\partial}{\partial R})^\rho (\frac{\partial}{\partial v})^\rho (\frac{\partial}{\partial a})^\rho \left[ F_1 \left( \frac{\cdot - a}{\epsilon} ; R, v \right) \right] \right\|_2 \leq \frac{1}{e^{a - \gamma - \frac{1}{2}} \epsilon} \left( 1 + \frac{|a|}{\epsilon} \right)^\gamma.
\]

\[
\left\| (\gamma^1)^\rho (\frac{\partial}{\partial y^1})^\rho (\frac{\partial}{\partial R})^\rho (\frac{\partial}{\partial v})^\rho (\frac{\partial}{\partial a})^\rho \left[ F_{1,1} \left( \frac{\cdot - a}{\epsilon} ; R, v, \frac{a}{\epsilon} \right) \right] \right\|_2 \leq \frac{1}{e^{a - \gamma - 1/2} \epsilon} \left( 1 + \frac{|a|}{\epsilon} \right)^\gamma.
\]
Proof of lemma 4.3.1: For (4.34) we have

\[
\left\| (y^1)^\gamma \left( \frac{\partial}{\partial y^1} \right)^\alpha \left( \frac{\partial}{\partial R} \right)^\beta \left[ F_0 \left( \frac{\cdot - a}{\epsilon} ; R \right) \right] \right\|_2 \leq \frac{1}{\epsilon^\alpha} \left\| (y^1)^\gamma \left( \frac{\partial^{\alpha + \beta} F_0}{\partial (y^1)^\gamma \partial R^\beta} \left( \frac{\cdot - a}{\epsilon} ; R \right) \right) \right\|_2 \\
= \frac{1}{\epsilon^{\alpha - \gamma}} \left\| \left( \frac{y^1 - a + a}{\epsilon} \right)^\gamma \left( \frac{\partial^{\alpha + \beta} F_0}{\partial (y^1)^\gamma \partial R^\beta} \left( \frac{\cdot - a}{\epsilon} ; R \right) \right) \right\|_2 \\
= \frac{1}{\epsilon^{\alpha - \gamma}} \left[ \sum_{j=0}^\gamma \left( \frac{\epsilon^j}{\alpha^j} \right)^{\gamma-j} \left( \frac{\partial^{\alpha + \beta} F_0}{\partial (y^1)^\gamma \partial R^\beta} \left( \frac{\cdot - a}{\epsilon} ; R \right) \right) \right]_2 \\
\leq \frac{1}{\epsilon^{\alpha - \gamma}} \sum_{j=0}^\gamma \left( \frac{\epsilon^j}{\alpha^j} \right)^{\gamma-j} \left( \frac{\partial^{\alpha + \beta} F_0}{\partial (y^1)^\gamma \partial R^\beta} \left( \frac{\cdot - a}{\epsilon} ; R \right) \right) \right\|_2 \\
\leq \frac{1}{\epsilon^{\alpha - \gamma - \frac{1}{2}}} \left( 1 + \frac{|a|}{\epsilon} \right)^\gamma,
\]

where we did a change of variables and used the exponential decay of \( F_0 \) and its derivatives to obtain the last inequality.

We estimate (4.35) and (4.36) in the same way.

We will use lemma 4.3.1 to obtain the more useful estimates

**Corollary 4.3.2.** For \( \alpha, \gamma \in \mathbb{N} \cup \{0\} \) and \( \beta = 0, 1, 2 \), then as long as \( a(y^0) \) exists we have for \( i = 1, 2 \)

\[
\left\| (y^1)^\gamma \left( \frac{\partial}{\partial y^1} \right)^\alpha \left( \frac{\partial}{\partial v^0} \right)^\beta \hat{F}_i \right\|_2 \leq \frac{1}{\epsilon^{\alpha - \gamma - 1/2}} \left( 1 + \delta^{i/2} \left| \frac{a''}{\epsilon} \right| \right) (1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon})^{\gamma + \beta},
\]

where \( \delta^i \) is the Kronecker-delta and the constant in the estimate depends on \( \gamma^0 \), but not \( \epsilon \).

**Proof of corollary 4.3.2:**

\( \beta = 0 \): This immediately follows from lemma 4.3.1.

\( \beta = 1 \): We have that

\[
\frac{\partial}{\partial y^0} \left[ \hat{F}_0 \right] = -\frac{a'}{\epsilon} \left[ \frac{\partial F_1}{\partial y^1} \right] + R \left[ \frac{\partial F_1}{\partial v} \right] + R' \left[ \frac{\partial F_1}{\partial \hat{v}} \right],
\]

\( \frac{\partial}{\partial v^0} \left[ \hat{F}_1 \right] = \frac{\partial F_1}{\partial v^1} + \frac{\partial F_{1,1}}{\partial v^1} - \frac{\partial F_{1,1}}{\partial a} + R \left[ \frac{\partial F_{1,1}}{\partial R} \right] + R' \left[ \frac{\partial F_{1,1}}{\partial v} + \frac{\partial F_{1,1}}{\partial \hat{v}} \right].
\]
where $F_0$, $F_1$, $F_{1,1}$, and all of their partial derivatives are evaluated at $(\frac{\tau - a}{\epsilon}; R)$, $(\frac{\tau - a}{\epsilon}; R, R')$, and $(\frac{\tau - a}{\epsilon}; R, R', \alpha)$, respectively (we suppress the arguments of these quantities for notational convenience). Estimating $(y^1)^\gamma \partial_{\gamma y} \partial_{\gamma y}^\alpha F_0$ first, we have

$$\| (y^1)^\gamma \partial_{\gamma y} \partial_{\gamma y}^\alpha F_0 \|_2 \lesssim \| (y^1)^\gamma y' \partial_{\gamma y}^\alpha \left[ F_0 \left( \frac{-a}{\epsilon} ; R \right) \right] \|_2 + \| (y^1)^\gamma R' \partial_{\gamma y} \partial_{\gamma y}^\alpha \left[ F_0 \left( \frac{-a}{\epsilon} ; R \right) \right] \|_2 \lesssim \frac{1}{\epsilon^{\gamma - 1/2}} \left( 1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon} \right)^{\gamma + 1},$$

where we used (4.34) to obtain the last inequality. Estimating $(y^1)^\gamma \partial_{\gamma y} \partial_{\gamma y}^\alpha F_1$ in the same way, we find that

$$\| (y^1)^\gamma \partial_{\gamma y} \partial_{\gamma y}^\alpha F_1 \|_2 \lesssim \frac{1}{\epsilon^{\gamma - 1/2}} \left( 1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon} \right)^{\gamma + 1}.$$

**$\beta = 2$:** We have that

$$\frac{\partial^2}{\partial (y^0)^2} \left[ \tilde{F}_0 \right] = -\frac{a''}{\epsilon} \partial F_0 + \frac{a'}{\epsilon} \partial^2 F_0 + 2R' \frac{a'}{\epsilon} \partial F_0 + (R')^2 \frac{\partial^2 F_0}{\partial R^2} + R'' \frac{\partial F_0}{\partial R},$$

$$\frac{\partial}{\partial (y^0)^2} \left[ \tilde{F}_1 \right] = -\frac{a''}{\epsilon} \left[ \frac{\partial F_1}{\partial y^1} + \frac{\partial F_{1,1}}{\partial y^1} - \frac{\partial F_1}{\partial a} \right] + \frac{(a'')}{\epsilon} \left[ \frac{\partial^2 F_1}{\partial (y^1)^2} + \frac{\partial F_{1,1}}{\partial (y^1)^2} - 2 \frac{\partial F_{1,1}}{\partial y^1 \partial a} + \frac{\partial F_{1,1}}{\partial a^2} \right]$$

$$- 2R' \frac{a'}{\epsilon} \left[ \frac{\partial^2 F_1}{\partial y^1 \partial R} + \frac{\partial^2 F_{1,1}}{\partial y^1 \partial R} - \frac{\partial^2 F_{1,1}}{\partial R \partial a} \right] - 2R'' \frac{a'}{\epsilon} \left[ \frac{\partial^2 F_1}{\partial (y^1)^2} + \frac{\partial^2 F_{1,1}}{\partial (y^1)^2} - \frac{\partial^2 F_{1,1}}{\partial y^1 \partial a} - \frac{\partial^2 F_{1,1}}{\partial a^2} \right]$$

$$+ R'' \left[ \frac{\partial F_1}{\partial a} + \frac{\partial F_{1,1}}{\partial a} \right] + (R')^2 \left[ \frac{\partial^2 F_1}{\partial R^2} + \frac{\partial^2 F_{1,1}}{\partial R^2} \right] + 2R' \left[ \frac{\partial^2 F_1}{\partial R \partial y} + \frac{\partial^2 F_{1,1}}{\partial R \partial y} \right]$$

$$+ R'' \left[ \frac{\partial F_1}{\partial y} + \frac{\partial F_{1,1}}{\partial y} \right] + (R')^2 \left[ \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_{1,1}}{\partial y^2} \right],$$

where again we have suppressed the arguments of $F_0$, $F_1$, and all of their partial derivatives for notational convenience. We then argue as we did in the $\beta = 1$ case to obtain the estimates

$$\| (y^1)^\gamma \partial_{\gamma y} \partial_{\gamma y}^\alpha \tilde{F}_0 \|_2 \lesssim \frac{1}{\epsilon^{\gamma - 1/2}} (1 + \frac{|a'|}{\epsilon} \epsilon^{\gamma + 2}) \left( 1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon} \right)^{\gamma + 1}.$$
\[
\|y^1 \partial_{\gamma} \partial_{\gamma} F_1\|_2 \lesssim \frac{1}{\epsilon^{2 - \gamma/2}} (1 + \|a''\|)(1 + \|a\| + \|a'\|)\gamma^2.
\]

To begin the energy estimate, we will use the following divergence identity.

**Lemma 4.3.3.**
\[
\partial_{\gamma} \xi \cdot \left[ \frac{m^2}{n^2} \partial_{y_{\gamma}} \partial_{y_{\gamma}} \xi + B^\alpha \partial_{\alpha} \xi + L_\gamma (\bar{F}_0, R) \xi \right] = \text{div}_{y_{\gamma}} \bar{X} + Y, \tag{4.40}
\]
where
\[
\bar{X} = (e(\xi), -\partial_{\gamma} \xi \cdot \partial_{\gamma} \xi)
\]
\[
Y = -\frac{1}{\epsilon^2} \xi \cdot \left[ \partial_{\gamma} \text{Hess}_\xi \bar{W} (\bar{F}_0, R) \xi + B^\alpha \partial_{\gamma} \xi \cdot \partial_{\alpha} \xi - \frac{1}{2} \partial_{y_{\gamma}} (\frac{m^2}{n^2}) \|\partial_{\gamma} \xi\|^2 \right].
\]

We omit the proof of lemma [4.3.3] as the proof is a straightforward computation. Using the divergence identity (4.40) and (4.16), we have
\[
\partial_{\gamma} E = \int \left[ \partial_{\gamma} e \right]_{|y|^1 \leq |y^1|} - \int \left[ \text{div}_{y_{\gamma}} \bar{X} + \partial_{\gamma} \xi \cdot \partial_{\gamma} \xi \right]_{|y^1| \leq |y^1|} + \int Y + \partial_{\gamma} \xi \cdot \partial_{\gamma} \xi \bigg|_{|y^1| \leq |y^1|}.
\]

We then integrate \( \partial_{\gamma} E \) with respect to \( y^0 \) to get
\[
E(y_0) - E(0) = -\int_0^{y^0} \int \left[ \partial_{\gamma} \xi \cdot \left[ S_{-1} + S_0 + N \right] \right]_{|y^1| \leq |y^1|} - \int_0^{y^0} \int Y + \partial_{\gamma} \xi \cdot \partial_{\gamma} \xi \bigg|_{|y^1| \leq |y^1|}. \tag{4.41}
\]

We will use this energy identity in order to establish the estimate \( (4.28) \).

We will break the analysis up to simplify things. We will estimate each term on the right hand side of \( (4.41) \) individually and then add all of the individual estimates back up to obtain the energy estimate.
Lemma 4.3.4.

\[-\int_0^{y_0} \int_{|y| \leq |y_0|} \partial_{y^0} \xi \cdot S_{-1} \leq (\epsilon E^{1/2} + e^{-\alpha^{1/2}/\epsilon}) \sqrt{\epsilon A}|_{y_0} + \int_0^{y_0} (\epsilon E^{1/2} + e^{-\alpha^{1/2}/\epsilon}) \sqrt{\epsilon A} \]  

(4.42)

Proof of lemma 4.3.4: Recall that

\[ S_{-1} = \left[ H(R) + B^1 \right] \partial_{y^1} \tilde{F}_0, \]  

(4.43)

Integrating by parts in \( y^0 \) we have

\[-\int_0^{y_0} \int_{|y| \leq |y_0|} \partial_{y^0} \xi \cdot S_{-1} = -\int_0^{y_0} \xi \cdot S_{-1}|_{y_0} + \int_0^{y_0} \xi \cdot \partial_{y^0} S_{-1}. \]  

(4.44)

For \( j = 0, 1 \) we have

\[ \int_{-y_0^j}^{y_0^j} \xi \cdot \partial_{y^0} S_{-1} \leq \left( \int_{-y_0^j}^{y_0^j} |\xi|^2 \right)^{1/2} \| \partial_{y^0} S_{-1} \|_2 \]  

(4.45)

\[ \leq (\epsilon E^{1/2} + e^{-\alpha^{1/2}/\epsilon}) \| \partial_{y^0} S_{-1} \|_2. \]

We estimate \( \| S_{-1} \|_2 \) first as

\[ \| S_{-1} \|_2 \leq \| (y_1^j) \partial_{y^1} \tilde{F}_0 \|_2 \]  

\[ \leq \sqrt{\epsilon A}, \]  

(4.46)

where we used corollary 4.3.2 to obtain the last inequality. Using corollary 4.3.2 and (3.12), we estimate the second term of (4.44) as

\[ \| \partial_{y^0} S_{-1} \|_2 \]  

\[ \leq \| \partial_{y^0} \left( H(R) + B^1 \right) \|_2 + \| \left( H(R) + B^1 \right) \partial_{y^0} \partial_{y^1} \tilde{F}_0 \|_2 \]  

(4.47)

\[ \leq \sqrt{\epsilon A}. \]

Combining (4.45), (4.46), and (4.47) finishes the proof.

□
Lemma 4.3.5.

\[- \int_0^{\eta} \int_{|y| \leq y^1} \partial_\rho \xi \cdot S_0 \leq (\epsilon E^{1/2} + e^{-\alpha^{1/\gamma}}) \sqrt{\epsilon A} \bigg|_0^{y^0} + \int_0^{y^0} (1 + |d''|/\epsilon)(\epsilon E^{1/2} + e^{-\alpha^{1/\gamma}}) \sqrt{\epsilon A} \]  

(4.48)

Proof of lemma 4.3.5 Using the definition of $S_0$, see (4.18), we see that

\[
\int_0^{y^0} \int_{|y| \leq y^1} \partial_\rho \xi \cdot S_0 = \int_0^{y^0} \int_{|y| \leq y^1} \partial_\rho \xi \cdot \left[ \frac{m^2}{n^2} \partial_{\rho, \rho}(\tilde{F}_0 + \epsilon \tilde{F}_1) + B^0 \partial_\rho (\tilde{F}_0 + \epsilon \tilde{F}_1) + \epsilon B^1 \partial_\gamma \tilde{F}_1 \right].
\]

We would like to integrate by parts in $y^0$ to move the derivative from $\xi$ to $S_0$ and use corollaries 4.1.6 and 4.3.2 to estimate. However, then we would have to estimate $\partial_\rho \rho \rho (\tilde{F}_0 + \epsilon \tilde{F}_1)$ which will give rise to $\partial_\rho \rho \rho a$ terms, which we would rather avoid. So, we need to take special care when estimating these two problematic terms and proceed as we would like to for the other terms.

1. $\partial_{\rho, \rho} \tilde{F}_0$ term: Recall that $\tilde{F}_0 = F_0(\frac{y^1-\alpha(\eta)}{\epsilon}; R(y^0))$ and so

\[
\int_0^{y^0} \int_{y^1}^{y^1} \frac{m^2}{n^2} \partial_\rho \xi \cdot \partial_{\rho, \rho} \tilde{F}_0 = \int_0^{y^0} \int_{y^1}^{y^1} \frac{m^2}{n^2} \partial_\rho \xi \cdot \left[ -\alpha''/\epsilon \partial_\rho F_0 + (\alpha'/\epsilon)^2 \partial_{\rho, \rho} F_0 \right]
\]

\[
+ \int_0^{y^0} \int_{-y^1}^{-y^1} \frac{m^2}{n^2} \partial_\rho \xi \cdot \left[ -2R' \alpha'/\epsilon \partial_\rho \partial_\rho F_0 + R'' \partial_\rho F_0 \right] \quad (4.49)
\]

\[
+ \int_0^{y^0} \int_{-y^1}^{-y^1} \frac{m^2}{n^2} \partial_\rho \xi \cdot \left[ (R')^2 \partial_{\rho, \rho} F_0 \right],
\]

where we use the notation $\partial_\rho^\alpha \partial_\rho^\beta F_0 = \partial_\rho^\alpha \partial_\rho^\beta F_0(\frac{y^1-\alpha}{\epsilon}; R)$ for convenience. Observe that for $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ and $\beta, \gamma \in \mathbb{Z}_+$, we have that

\[
\|(y^1)^\gamma \partial_{\rho}^\alpha \partial_\rho^\beta F_0\|_2 \leq \epsilon^{y+1/2} \left( 1 + \frac{|d|}{\epsilon} \right)^y. \quad (4.50)
\]

First, we will bound the $\alpha''$ term. Recall that $m = (1 - (R')^2)^{-1/2}$ and $n = 1 + y^1 m^3 R''$. 

58
Consider the $\partial_\rho \xi \cdot \partial_{y^1} F_0$ term. We have that

$$
\int_0^{y^0} \int_{|\nu|^\leq 1} \frac{m^2 a''}{\epsilon} \partial_\rho \xi \cdot \partial_{y^1} F_0 = \int_0^{y^0} \int_{|\nu|^\leq 1} \frac{m^2 a''}{\epsilon} \partial_\rho \xi \cdot \partial_{y^1} F_0
$$

$$
+ \int_0^{y^0} \int_{|\nu|^\leq 1} \left[ \frac{m^2}{n^2} - m^2 \right] \frac{a''}{\epsilon} \partial_\rho \xi \cdot \partial_{y^1} F_0.
$$

Also, by differentiating

$$
0 = \int \xi \cdot \partial_{y^1} \tilde{F}_0
$$

with respect to $y^0$, we see that

$$
\int_{|\nu|^\leq 1} |\nu|^\leq y^1 \quad \frac{a'}{\epsilon} \int \xi \cdot \partial_{y^1} F_0 - \int R \cdot \partial_{y^1} F_0.
$$

(4.51)

Using (4.51), we control the $m^2$ term as follows

$$
\left| \int_0^{y^0} \int_{|\nu|^\leq 1} m^2 \frac{a''}{\epsilon} \partial_\rho \xi \cdot \partial_{y^1} F_0 \right| \leq \int_0^{y^0} |m|^2 \left| \frac{a''}{\epsilon} \right| \int_{|\nu|^\leq 1} \left| \partial_\rho \xi \cdot \partial_{y^1} F_0 \right|
$$

$$
\leq \int_0^{y^0} \left| \frac{a'}{\epsilon} \right| \left| \frac{a'}{\epsilon} \right| \int_{|\nu|^\leq 1} \left| \xi \cdot \partial_{y^1} F_0 \right| + \int \left| \xi \cdot \partial_R \partial_{y^1} F_0 \right|
$$

where we used the fact that $|m|^2 \leq 1$ for all $0 \leq y^0 \leq y^*_0$ to obtain the second estimate. We then use the Cauchy-Schwarz inequality, corollary 4.3.1, and (4.22) to conclude that

$$
\left| \int_0^{y^0} \int_{|\nu|^\leq 1} m^2 \frac{a''}{\epsilon} \partial_\rho \xi \cdot \partial_{y^1} F_0 \right| \leq \int_0^{y^0} \left| \frac{a''}{\epsilon} \right| \left( \epsilon E^{1/2} + e^{-a' y^1} \right) \sqrt{\epsilon A}.
$$

To control the $\frac{m^2}{n^2} - m^2$, observe that $\frac{m^2}{n^2} - m^2 = O(y^1)$. We use the Cauchy-Schwarz...
inequality and the definition of the energy (4.21) to see that

\[
\left| \int_0^\rho \int_{|\cdot| \leq \delta} \left[ \frac{m^2}{n^2} \epsilon \partial_0 \xi \cdot \partial_0 F_0 \right] \right| \leq \int_0^\rho \frac{|a''|}{\epsilon} \left\| \partial_0 \xi \right\|_{L^2(-\delta, \rho)} \left\| \partial_0 F_0 \right\|_2
\]

\[
\leq \int_0^\rho \frac{|a''|}{\epsilon} \epsilon^{3/2} E^{1/2} A.
\]

To summarize, we have the following estimate

\[
\int_0^\rho \int_{|\cdot| \leq \delta} \frac{m^2}{n^2} \epsilon \partial_0 \xi \cdot \partial_0 F_0 \leq \int_0^\rho \frac{|a''|}{\epsilon} (\epsilon E^{1/2} + e^{-\omega\rho_{\omega, 0}}) \sqrt{\epsilon A}.
\]

We deal with the rest of the terms of (4.49) by shifting the \( \partial_0 \) off of \( \xi \) and estimate using (4.22) along with corollary 4.3.1. We only estimate the \( \partial_0 y^i F_0 \) term of (4.49). The estimates for the other three terms of (4.49) are then obtained using similar arguments.

Estimating the \( \partial_0 y^i F_0 \) term, we see that

\[
\int_0^\rho \int_{|\cdot| \leq \delta} \frac{m^2}{n^2} \partial_0 \xi \cdot \left[ -\frac{a'}{\epsilon} \partial_0 y^i F_0 \right]
\]

\[
= -\int_{|\cdot| \leq \delta} \frac{m^2}{n^2} \left( \frac{a'}{\epsilon} \right)^2 \partial_0 y^i F_0 \bigg|_0^\rho + \int_0^\rho \int_{|\cdot| \leq \delta} \xi \cdot \partial_0 \frac{m^2}{n^2} \left( \frac{a'}{\epsilon} \right)^2 \partial_0 y^i F_0
\]

\[
\leq \left( \frac{a'}{\epsilon} \right)^2 \left\| \xi \right\|_{L^2(-\delta, \rho)} \left\| \partial_0 y^i F_0 \right\|_2 \bigg|_0^\rho + \int_0^\rho \left\| \xi \right\|_{L^2(-\delta, \rho)} \left( \frac{a'}{\epsilon} \right)^2 \left\| \partial_0 y^i F_0 \right\|_2
\]

\[
+ \int_0^\rho \left\| \xi \right\|_{L^2(-\delta, \rho)} \left[ 2 \frac{|a'|}{\epsilon} \frac{|a''|}{\epsilon} \left\| \partial_0 y^i F_0 \right\|_2 + \frac{|a''|^3}{\epsilon^3} \left\| \partial_0 y^i y^i F_0 \right\|_2 + \frac{|a'|^2}{\epsilon^2} \left\| \partial_0 y^i y^i F_0 \right\|_2 \right]
\]

\[
\leq (\epsilon E^{1/2} + e^{-\omega\rho_{\omega, 0}}) \sqrt{\epsilon A} \bigg|_0^\rho + \int_0^\rho \left( 1 + \frac{|a'|}{\epsilon} \right) (\epsilon E^{1/2} + e^{-\omega\rho_{\omega, 0}}) \sqrt{\epsilon A}.
\]
Putting together the above estimates yields

\[- \int_0^{\gamma^0} \int_{|\nu'| \leq t^0} \frac{m^2}{n^2} \partial_{\nu'} \xi \cdot \partial_{\nu' t} \tilde{F}_0 \leq (\epsilon E^{1/2} + e^{-a \frac{1}{\epsilon^2}}) \sqrt{\epsilon} A \bigg|_{t^0}^{\gamma^0} + \int_0^{\gamma^0} (1 + \frac{|a''|}{\epsilon})(\epsilon E^{1/2} + e^{-a \frac{1}{\epsilon^2}}) \sqrt{\epsilon} A.\]  \hspace{1cm} (4.52)

2. $\partial_{\nu t} \tilde{F}_1$ term: Since $\frac{m^2}{n^2} \leq 1$ on $(0, y^0_t) \times (-y^1_t, y^1_t)$, we have

\[-\epsilon \int_0^{\gamma^0} \int_{|\nu'| \leq t^0} \frac{m^2}{n^2} \partial_{\nu'} \xi \cdot \partial_{\nu' t} \tilde{F}_1 \leq \epsilon \int_0^{\gamma^0} E^{1/2} \|\partial_{\nu t} \tilde{F}_1\|_2 \leq \int_0^{\gamma^0} (1 + \frac{|a''|}{\epsilon}) \epsilon^{3/2} E^{1/2} A.\]  \hspace{1cm} (4.53)

where we used Cauchy-Schwarz and the definition of the energy (4.21) to obtain the first inequality and corollary 4.3.2 to obtain the last inequality.

3. $B^0 \partial_{\nu t} \tilde{F}_0$: Using the boundedness of $B^0$ and $\partial_{\nu t} B^0$ on $[0, y^0_t) \times (-y^1_t, y^1_t)$, we see that

\[- \int_0^{\gamma^0} \int_{|\nu'| \leq t^0} B^0 \partial_{\nu t} \xi \cdot \partial_{\nu' t} \tilde{F}_0 \leq \int_0^{\gamma^0} \int_{|\nu'| \leq t^0} \xi \cdot \partial_{\nu t} \Big[ B^0 \partial_{\nu t} \tilde{F}_0 \Big] \leq (\epsilon E^{1/2} + e^{-a \frac{1}{\epsilon^2}}) \sqrt{\epsilon} A \bigg|_{t^0}^{\gamma^0} + \int_0^{\gamma^0} (1 + \frac{|\partial_{\nu' t} a|}{\epsilon})(\epsilon E^{1/2} + e^{-a \frac{1}{\epsilon^2}}) \sqrt{\epsilon} A.\]  \hspace{1cm} (4.54)

where we used Cauchy-Schwarz, corollary 4.1.6 and corollary 4.3.2 to obtain the last inequality.
4. $B^0 \partial_\rho \tilde{F}_1$ term: Using the boundedness of $B^0$ on $[0, y_0^1) \times (-y_1^*, y_1^*)$ and corollary 4.3.2 we have that

$$-\epsilon \int_0^{y_0^1} \int_{-y_1^*}^{y_1^*} B^0 \partial_\rho \xi \cdot \partial_\rho \tilde{F}_1 \lesssim \int_0^{y_0^1} \epsilon^{3/2} E^{1/2} A.$$  \hspace{1cm} (4.55)

5. $B^1 \partial_\rho \tilde{F}_1$ term: Using the boundedness of $B^1$ and $\partial_\rho B^1$ on $(0, y_0^1) \times (-y_1^*, y_1^*)$ and corollary 4.3.2 we get

$$-\int_0^{y_0^1} \int_{|\rho| \leq y_1^*} \epsilon B^1 \partial_\rho \xi \cdot \partial_\rho \tilde{F}_1$$
$$= - \int_0^{y_0^1} \int_{|\rho| \leq y_1^*} \xi \cdot [\epsilon B^1 \partial_\rho \tilde{F}_1] \bigg|_0^{y_0^1} + \int_0^{y_0^1} \int_{|\rho| \leq y_1^*} \xi \cdot \partial_\rho \left[ \epsilon B^1 \partial_\rho \tilde{F}_1 \right]$$
$$\lesssim (\epsilon E^{1/2} + e^{-a \frac{1-a}{2}}) \sqrt{\epsilon A} \bigg|_0^{y_0^1} + \int_0^{y_0^1} (\epsilon E^{1/2} + e^{-a \frac{1-a}{2}}) \sqrt{\epsilon A},$$  \hspace{1cm} (4.56)

where we used Cauchy-Schwarz, corollary 4.1.6 and corollary 4.3.2 to obtain the last inequality.

Putting together the estimates obtained from steps 1-5 we obtain (4.48).

\[
\begin{align*}
\text{Lemma 4.3.6.} \\
- \int_0^{y_0^1} \int_{-y_1^*}^{y_1^*} \tilde{F}_1 \cdot \partial_\rho \tilde{F}_1 &\leq (1 + \epsilon + (\epsilon E^{1/2} + e^{-a \frac{1-a}{2}}) (\epsilon^{5/2} + \epsilon E^{1/2} + e^{-a \frac{1-a}{2}}) (\epsilon E^{1/2} + e^{-a \frac{1-a}{2}}))A \\
&+ \frac{E^{1/2}}{\epsilon} \left( (\epsilon E^{1/2} + e^{-a \frac{1-a}{2}})^{1/4} E^{1/4} \right) \left( \epsilon^{3/2} + (\epsilon E^{1/2} + e^{-a \frac{1-a}{2}}) \right) A \bigg|_0^{y_0^1} \\
&+ \left[ \frac{y_1^1}{(R - y_1^1 m)^3} + \frac{(y_1^1)^2}{(R - y_1^1 m)^4} \right] \bigg|_0^{y_0^1} \bigg] \bigg|_0^{y_0^1}
\end{align*}
\]
In order to prove theorem (4.2.1), we will need to choose $y^1_1$ sufficiently small so that the factor multiplying $E$ in (4.28) is less than one. This is why we need to include the $\frac{y^1_1}{(R-y^1_1m)^3}$ factors in (4.57).

Proof of lemma 4.3.6: Recall that we defined $N$ as

$$N = -\frac{1}{\epsilon^2} \left[ w(\tilde{F}_0 + \tilde{F}_\xi, R + y^1_1m) - w(\tilde{F}_0, R) - \text{Hess}_\phi(\tilde{F}_0, R)\tilde{F}_\xi - y^1_1m (y^0_0) \partial_y w(\tilde{F}_0, R) \right]$$

and $\tilde{F}_\xi$ as $\tilde{F}_\xi = \epsilon \tilde{F}_1 + \xi$. Using the identity

$$g(1) = g(0) + g'(0) + \frac{1}{2} \int_0^1 (1-t)g''(t)dt$$

we can rewrite $N$ as

$$N = -\frac{1}{\epsilon^2} \int_0^1 (1-t) \frac{d^2}{dt^2} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1_1m)dt. \quad (4.58)$$

Examine now the left hand side of (4.57). First, we integrate by parts to move $\partial_y$ from $\xi$ onto $N$. That is,

$$-\int_0^{y^0_1} \int_{-y^1_1}^{y^1_1} \partial_y \xi \cdot N = -\int_0^{y^0_1} \int_{-y^1_1}^{y^1_1} \xi \cdot \partial_y N \bigg|_{0}^{r \xi} + \int_0^{y^0_1} \int_{-y^1_1}^{y^1_1} \xi \cdot \partial_y N.$$

Using Cauchy-Schwarz, we see that

$$-\int_0^{y^0_1} \int_{-y^1_1}^{y^1_1} \partial_y \xi \cdot N \lesssim \epsilon E^{1/2} ||N||_{L^2(-y^1_1m)} \bigg|_{0}^{y^0_1} + \int_0^{y^0_1} \int_{-y^1_1}^{y^1_1} \xi \cdot \partial_y N. \quad (4.59)$$

To simplify things, we will estimate the two terms appearing on the right hand side of (4.59) separately.

Before continuing, though, recall that $\tilde{F}_\xi$ has two-components: one corresponding to the $\phi$-field and the other corresponding to the $\sigma$-field. We write $\tilde{F}_\xi$ as

$$\tilde{F}_\xi = \left( (\tilde{F}_\xi)_\phi, (\tilde{F}_\xi)_\sigma \right).$$
Using \( \|N\|_{L^2(-y_1^1,y_1^1)} \) Estimate: First, examine the \( \frac{d^2}{dt^2} w(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \) term in \( N \). Expanding things out, we have that

\[
\frac{d^2}{dt^2} w(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) = \frac{d}{dt} \left[ \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \tilde{F}_\epsilon + \partial_t w(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) y^1 m \right]
\]

Using (4.58) and (4.60), we have that

\[
\|N\|_{L^2(-y_1^1,y_1^1)} \leq \left\| \frac{1}{\epsilon^2} \int_0^1 (1-t)(\tilde{F}_\epsilon)_\phi \partial_\phi \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \tilde{F}_\epsilon dt \right\|_{L^2(-y_1^1,y_1^1)}
\]

\[
+ \left\| \frac{1}{\epsilon^2} \int_0^1 (1-t) \epsilon \partial_\tau \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \tilde{F}_\epsilon dt \right\|_{L^2(-y_1^1,y_1^1)}
\]

\[
+ \left\| \frac{1}{\epsilon^2} \int_0^1 (1-t) 2y^1 m \partial_\tau \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \tilde{F}_\epsilon \right\|_{L^2(-y_1^1,y_1^1)}
\]

\[
+ \left\| \frac{1}{\epsilon^2} \int_0^1 (1-t)(y^1 m)^2 \partial_{\tau\tau} w(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \right\|_{L^2(-y_1^1,y_1^1)}
\]

We will estimate each of the four terms appearing on the right hand side of the above separately.

1. \((\tilde{F}_\epsilon)_\phi\) term:

\[
\| \int_0^1 (1-t)(\tilde{F}_\epsilon)_\phi \partial_\phi W(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \tilde{F}_\epsilon dt \|_{L^2(-y_1^1,y_1^1)} \leq \| (\tilde{F}_\epsilon)_\phi \|_{L^\infty(-y_1^1,y_1^1)} \| \tilde{F}_\epsilon \|_{L^2(-y_1^1,y_1^1)}
\]

where we used the boundedness of \( \tilde{F}_0 + t\tilde{F}_\epsilon \) and \( R + ty^1 m \) on \((0,y_0^1) \times (-y_1^1,y_1^1)\) for \( 0 \leq t \leq 1 \) to control the operator norm \( \| \partial_\phi W(\tilde{F}_0 + t\tilde{F}_\epsilon, R + ty^1 m) \| \). Next, we use the Gagliardo-
Nirenberg inequality and corollary 4.1.6 to show that

\[ \|\tilde{F}_\xi\|_{L^\infty(-y^1_0,y^1_1)} \leq \epsilon \|\tilde{F}_1\|_{\infty} + \|\xi\|_{L^\infty(-y^1_0,y^1_1)} \leq \epsilon + \|\tilde{F}_1\|_{L^2(-y^1_0,y^1_1)} \|\tilde{F}_\xi\|_{L^2(-y^1_0,y^1_1)} \leq \epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}})^{1/2} E^{1/4}. \] (4.61)

Furthermore, using corollary 4.1.6 and corollary 4.3.2 we see that

\[ \|\tilde{F}_\xi\|_{L^2(-y^1_0,y^1_1)} \leq \epsilon \|\tilde{F}_1\|_2 + \|\xi\|_{L^2(-y^1_0,y^1_1)} \leq \epsilon^{3/2} (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}}). \]

Thus, we obtain the estimate

\[ \left\| \frac{1}{\epsilon^2} \int_0^1 (1-t)(\tilde{F}_\xi)_\phi \partial_\phi \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1_1 m)\tilde{F}_\xi dt \right\|_{L^2(-y^1_0,y^1_1)} \leq \frac{1}{\epsilon^2} \left( \epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}})^{1/2} E^{1/4} \right) \left( \epsilon^{3/2} (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}}) \right). \]

2. \((\tilde{F}_\xi)_\sigma\) term: Similarly, we have that

\[ \left\| \frac{1}{\epsilon^2} \int_0^1 (1-t)(\tilde{F}_\xi)_\sigma \partial_\sigma \text{Hess}_\sigma W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1_1 m)\tilde{F}_\xi dt \right\|_{L^2(-y^1_0,y^1_1)} \leq \frac{1}{\epsilon^2} \left( \epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}})^{1/2} E^{1/4} \right) \left( \epsilon^{3/2} (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}}) \right). \]

3. \(\partial_\tau \text{Hess}_\phi W\) term: Using the boundedness of \(\partial_\tau \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1_1 m)\) and \(\frac{1}{R-y^1_0 m}\) on \((0,y^1_0) \times (-y^1_0,y^1_1)\), corollary 4.1.6 and corollary 4.3.2, we see that

\[ \left\| \frac{1}{\epsilon^2} \int_0^1 (1-t)2y^1_1 m \partial_\tau \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1_1 m)\tilde{F}_\xi dt \right\|_{L^2(-y^1_0,y^1_1)} \leq \frac{1}{\epsilon^2} \left( \epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}})^{1/2} E^{1/4} \right) \left( \epsilon^{3/2} (\epsilon E^{1/2} + e^{-\alpha \frac{2+\alpha}{\tau}}) \right). \]
We then apply corollary 4.1.6 and find that

\[ \left\| \frac{1}{\epsilon^2} \int_0^1 (1 - t)2y^1 m \partial_r \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \right\|_{L^2(\gamma^1, \gamma^1)} \leq \frac{1}{\epsilon^2} \left( \epsilon^{5/2} A + \epsilon \frac{y^1}{(R - y^1 m)^3} E^{1/2} + e^{-\frac{1}{2} \frac{y^1}{R}} \right). \]

4. \((y^1 m)^2 \partial_{rr} w\) term: Since

\[ \partial_{rr} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) = \begin{pmatrix} 0 \\ \frac{3\epsilon}{(R + ry^1 m)^2} (\tilde{s}_0 + t(\tilde{F}_\xi)_{rr}) \end{pmatrix} \]

then we have

\[ \left\| \frac{1}{\epsilon^2} \int_0^1 (1 - t)(y^1 m)^2 \partial_{rr} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \right\|_{L^2(\gamma^1, \gamma^1)} \leq \frac{1}{\epsilon^2} \left( \left\| (y^1 m)^2 \tilde{s}_0 \right\|_2 + \epsilon \left\| (y^1 m)^2 \tilde{s}_1 \right\|_2 + \epsilon \frac{(y^1 m)^2}{(R - y^1 m)^3} \tilde{\xi} \right)_{L^2(\gamma^1, \gamma^1)} \leq \frac{1}{\epsilon^2} \left( \epsilon^{5/2} A + \epsilon^{7/2} A + \epsilon \frac{(y^1 m)^2}{(R - y^1 m)^3} E^{1/2} + e^{-\frac{1}{2} \frac{y^1}{R}} \right), \]

where again we used corollary 4.3.2 to obtain the last inequality.

To summarize, we have that

\[ \left\| N \right\|_{L^2(\gamma^1, \gamma^1)} \leq \left[ \frac{y^1}{(R - y^1 m)^3} + \frac{(y^1 m)^2}{(R - y^1 m)^4} \right] E^{1/2} \epsilon + \frac{1}{\epsilon^2} \left( \epsilon + (\epsilon E^{1/2} + e^{-\frac{1}{2} \frac{y^1}{R}})^{1/2} E^{1/4} \right) \left( \epsilon^{3/2} + (\epsilon E^{1/2} + e^{-\frac{1}{2} \frac{y^1}{R}}) \right) A. \]

\[ \left| \left\langle \xi, \partial_{\gamma^0} N \right\rangle \right|_{L^2(\gamma^1, \gamma^1)} \]

Estimate: Notice that

\[ \partial_{\gamma^0} N = -\frac{1}{\epsilon^2} \int_0^1 (1 - t) \frac{d^2}{dt^2} \left[ \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \cdot \partial_{\gamma^0}(\tilde{F}_0 + t\tilde{F}_\xi) \right] \]

\[ -\frac{1}{\epsilon^2} \int_0^1 (1 - t) \frac{d^2}{dt^2} \left[ \partial_{rr} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_{\gamma^0}(R + ty^1 m(y^0)) \right]. \]

(4.62)
Consider the first term on the right hand side first. Since

\[
\frac{d^2}{dt^2} \left[ \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi) \right]
\]

\[
= (\tilde{F}_\xi)_\phi \partial_\phi \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi)
\]

\[+ 2(\tilde{F}_\xi)_\phi (\tilde{F}_\xi)_\sigma \partial_\rho \partial_\sigma \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi)
\]

\[+ (\tilde{F}_\xi)_\rho \partial_\rho \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi)
\]

\[+ (y^1 m)^2 \partial_{\sigma \rho} \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi)
\]

\[+ (\tilde{F}_\xi)_\sigma \partial_\sigma \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi)
\]

\[+ (y^1 m) \partial_\tau \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi),
\]

where we needed to use the fact that \(\partial_\phi \partial_\tau \text{Hess} W(F, R) = \partial_\phi \partial_\tau \text{Hess} W(F, R) = 0\) to obtain the above. Using Cauchy-Schwarz, corollary 4.1.6, corollary 4.3.2, the boundedness of \(\text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)\) and all of its derivatives on \(0, y^0 \times (-y^1, y^1)\) for all \(0 \leq t \leq 1\), and (4.61), we have that

\[
\left| \int_0^{y^0} \int_{-y^1}^{y^1} \left( \xi, \frac{d^2}{dt^2} \left[ \text{Hess}_\phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \cdot \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi) \right] \right) \right|_{L^2(-y^1, y^1)}
\]

\[\leq \int_0^{y^0} \int_{-y^1}^{y^1} \left( \left\| \tilde{F}_\xi \right\|_{L^2(-y^1, y^1)} \left\| \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi) \right\|_{L^2(-y^1, y^1)} + \left\| \tilde{F}_\xi \right\|_{L^2(-y^1, y^1)} \left\| \tilde{F}_\xi \right\|_{L^2(-y^1, y^1)} \right) \left\| \xi \right\|_{L^2(-y^1, y^1)}
\]

\[+ \int_0^{y^0} \int_{-y^1}^{y^1} \frac{6d^2(y^1 m)^2}{(R + ty^1 m)^4} \xi \cdot \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi) + \int_0^{y^0} \int_{-y^1}^{y^1} \frac{2d^2(y^1 m)^2}{(R + ty^1 m)^4} \xi \cdot \partial_\rho (\tilde{F}_0 + t\tilde{F}_\xi)
\]

\[\leq \int_0^{y^0} \left( \epsilon + \left( \epsilon E^{1/2} + e^{-\alpha \frac{y^1 m}{y^0}} \right)^{1/2} E^{1/4} (\epsilon E^{1/2} + E^{1/2}) \right)^2 \left( \epsilon E^{1/2} + E^{1/2} \right) \left( \epsilon E^{1/2} + e^{-\alpha \frac{y^1 m}{y^0}} \right)
\]

\[+ \int_0^{y^0} \left( \epsilon + \left( \epsilon E^{1/2} + e^{-\alpha \frac{y^1 m}{y^0}} \right)^{1/2} E^{1/4} (\epsilon E^{1/2} + E^{1/2}) \right)^2 \left( \epsilon E^{1/2} + E^{1/2} \right) \left( \epsilon E^{1/2} + e^{-\alpha \frac{y^1 m}{y^0}} \right)
\]

\[+ \int_0^{y^0} \left( \epsilon E^{1/2} + e^{-\alpha \frac{y^1 m}{y^0}} \right)^{3/2} \left( \epsilon E^{1/2} + E^{1/2} + e^{-\alpha \frac{y^1 m}{y^0}} \right) + \left[ \frac{y^1}{(R - y^1 m)^2} + \frac{(y^1)^2}{(R - y^1 m)^3} \right] \epsilon E^{y^0}_0.
\]
Now consider the second term of (4.62). Since $\partial_tw(F,R) = (0,-2\frac{dt}{E} s)$, we have that

$$
\frac{d^2}{dt^2} \left[ \partial_tw(\tilde{F}_0 + t\tilde{F}, R + ty^1m)(R' + ty^1m') \right] = 2(\tilde{F}_\xi)^2(y^1m)\partial_{\sigma}\partial_{\tau}w(\tilde{F}_0 + t\tilde{F}, R + ty^1m)(R' + ty^1m')
$$

Using Cauchy-Schwarz, corollary 4.1.6, corollary 4.3.2, the boundedness of $\partial_tw(\tilde{F}_0 + t\tilde{F}, R + ty^1m)$ and all of its derivatives on $(0,y^1_0) \times (-y^1_0, y^1_0)$ for all $0 \leq t \leq 1$, and (4.61), we have that

$$
\int_0^{y^0} \left| \int_0^{y^0} \left( \xi, \frac{d^2}{dt^2} \left[ \partial_tw(\tilde{F}_0 + t\tilde{F}, R + ty^1m)(R' + ty^1m') \right] \right) \right|_{L^2(-y^1_0, y^1_0)} \leq \int_0^{y^0} \left[ 1 + \|\tilde{F}_\xi\|_{L^\infty(-y^1_0, y^1_0)} \right] \left[ \|\xi\|_{L^2(-y^1_0, y^1_0)} + \|\tilde{F}_\xi\|_{L^2(-y^1_0, y^1_0)} \right] \left[ \|\tilde{F}_{\xi\xi}\|_{L^2(-y^1_0, y^1_0)} + \|\xi\|_{L^2(-y^1_0, y^1_0)} \right] \left( 1 + \epsilon + (\epsilon E^{1/2} + e^{-\frac{1}{\sqrt{E}}} \epsilon^{1/4} E^{1/4} + \epsilon E^{1/2} + e^{-\frac{1}{\sqrt{E}}} \epsilon^{1/4} E^{1/4} + \epsilon E^{1/2} + e^{-\frac{1}{\sqrt{E}}} \epsilon^{1/4} E^{1/4}) \right).
$$

To summarize, we have that

$$
\int_0^{y^0} \int_{-y^1_0}^{y^1_0} \left| \xi \cdot \partial_{\nu}N \right| \leq \frac{1}{\epsilon^2} \int_0^{y^0} \left( 1 + \epsilon + (\epsilon \sqrt{E} + e^{-1/2} \epsilon^{1/4} \sqrt{E}^{1/4} + \epsilon \sqrt{E} + e^{-1/2} \epsilon^{1/4} \sqrt{E}^{1/4} \right) \left( \epsilon \sqrt{E} + e^{-1/2} \epsilon^{1/4} \sqrt{E}^{1/4} \right) A
$$

$$
+ \left[ \frac{y^1_0}{(R - y^1_0m)^3} + \frac{(y^1_0)^2}{(R - y^1_0m)^4} \right] \left| \right|_{0}^{y^0}.
$$
Putting together the estimates for the two terms on the right hand side of (4.59), we see that

\[- \int_0^{y_0^1} \int_{-y_1^*}^{y_1^*} \partial_{y^0} \xi \cdot N \lesssim \frac{1}{\epsilon^2} \int_0^{y_0^1} \int_0^{y_0^1} \left( 1 + \epsilon + \left( \epsilon E^{1/2} + e^{-a \frac{y_1^*}{1 + \epsilon}} \right) \left( \epsilon^{3/2} + \epsilon E^{1/2} + e^{-a \frac{y_1^*}{1 + \epsilon}} \right) \right) \]

\[+ \frac{E^{1/2}}{\epsilon} \left( \epsilon + \left( \epsilon E^{1/2} + e^{-a \frac{y_1^*}{1 + \epsilon}} \right) \right)^{1/2} \left( \epsilon^{3/2} + \epsilon E^{1/2} + e^{-a \frac{y_1^*}{1 + \epsilon}} \right) \int_0^{y_0^1} \left[ \frac{y_1^1}{(R - y_1^* m)^3} + \frac{(y_1^1)^2}{(R - y_1^* m)^4} \right] \int_0^{y_0^1} \left[ E + \frac{1}{\epsilon^2} e^{-a \frac{y_1^*}{1 + \epsilon}} \right] \]

\[+ \left[ \frac{y_1^1}{(R - y_1^* m)^3} + \frac{(y_1^1)^2}{(R - y_1^* m)^4} \right] \int_0^{y_0^1} \left[ E + \frac{1}{\epsilon^2} e^{-a \frac{y_1^*}{1 + \epsilon}} \right]. \]

\[\square\]

**Lemma 4.3.7.**

\[- \int_0^{y_0^1} \int_{-y_1^*}^{y_1^*} Y \lesssim \int_0^{y_0^1} \left( E + \frac{1}{\epsilon^2} e^{-a \frac{y_1^*}{1 + \epsilon}} \right) \]  \hspace{1cm} (4.63)

**Proof of lemma 4.3.7** Recall the definition of $Y$ from lemma 4.40. Using this, we have that

\[- \int_0^{y_0^1} \int_{-y_1^*}^{y_1^*} Y = \int_0^{y_0^1} \int_{-y_1^*}^{y_1^*} \left\{ \xi \cdot \left[ \partial_{y^0} \text{Hess}_\Phi W(\tilde{F}_0, R) \right] \xi - B^{a} \partial_{y^0} \xi \cdot \partial_{y^0} \xi - \frac{1}{2} \partial_{y^0} \left( \frac{n^2}{n^2} \right) \left| \partial_{y^0} \xi \right|^2 \right\} \]

\[\lesssim \int_0^{y_0^1} \left( E + \frac{1}{\epsilon^2} e^{-a \frac{y_1^*}{1 + \epsilon}} \right), \]

where we used corollary 4.1.6 and the boundedness of the operator $\partial_{y^0} \text{Hess}_\Phi W(\tilde{F}_0, R)$ and the $B^a$’s on $(0, y_0^*) \times (-y_1^*, y_1^*)$ to obtain the estimate.

\[\square\]

**Lemma 4.3.8.**

\[\int_{0}^{y_0^1} \partial_{y_1^*} \xi(y_0^0, \pm y_1^*) \cdot \partial_{y_1^*} \xi(y_0^0, \pm y_1^*) \lesssim \int_{0}^{y_0^1} \frac{1}{\epsilon} e^{-a \frac{y_1^*}{1 + \epsilon}} \]  \hspace{1cm} (4.64)

**Proof of lemma 4.3.8** It follows from our choice of $y_0^0$ and $y_1^*$, see (4a) - (4f) that

\[\xi(y_0^0, \pm y_1^*) = \begin{cases} 
\pm 1 \\
0
\end{cases} - \tilde{F}_0(y_0^0, \pm y_1^*) - \epsilon \tilde{F}_1(y_0^0, \pm y_1^*) \]
for all $0 \leq y^0 \leq y^0_0$. Using (4.38), (4.39), and (3.5.1), we have that

$$\left| \partial_{y^0} \xi(y^0, \pm y^1_0) \right| \lesssim e^{-a y^1_0 - A}.$$  

Similarly, using (3.5.1) we see that

$$\left| \partial_{y^1} \xi(y^0, \pm y^1_0) \right| = \left| -\frac{1}{\epsilon} \partial_{y^0} F_0(y^0, \pm a(y^0) ; R(y^0)) - \partial_{y^1} F_1(y^0, \pm a(y^0) ; R(y^0), R'(y^0)) \right| \lesssim \frac{1}{\epsilon} e^{-a y^1_0 - A}.$$  

Combining the estimates obtained in lemma 4.3.4 to lemma 4.3.8 allows us to conclude the proof of theorem 4.2.2.

\[\square\]

4.4 Proof of Bounded Shift Theorem (Theorem 4.2.3)

To prove this theorem we will use (4.14). Differentiating (4.14) with respect to $y^0$ twice we find that

$$0 = \int_{\mathbb{R}} \left( \partial_{y^0} \partial_{y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^1} \tilde{F}_0 - \partial_{y^1} \xi \cdot \partial_{y^0} \partial_{y^0} \tilde{F}_0 \right)$$

$$= \int_{|y^1| \leq y^1_0} \left( \partial_{y^0} \partial_{y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^1} \tilde{F}_0 - \partial_{y^1} \xi \cdot \partial_{y^0} \partial_{y^0} \tilde{F}_0 \right)$$

$$+ \int_{|y^1| > y^1_0} \left( \partial_{y^0} \partial_{y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^1} \tilde{F}_0 - \partial_{y^1} \xi \cdot \partial_{y^0} \partial_{y^0} \tilde{F}_0 \right).$$  

On the one hand, we can use the results of (3.5) to show that

$$\int_{|y^1| > y^1_0} \left( \partial_{y^0} \partial_{y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^1} \tilde{F}_0 - \partial_{y^1} \xi \cdot \partial_{y^0} \partial_{y^0} \tilde{F}_0 \right) \lesssim (1 + \frac{|a''|}{\epsilon}) A e^{-a y^1_0 - a}.$$  

\[\text{(4.65)}\]

\[\text{(4.66)}\]
On the other hand, we can use equation (4.16) for $\xi$ to show that

$$\int_{|y^1| \leq y^1} \frac{n^2}{m^2} \left[ B^i \partial_0 \xi + L_4(\tilde{F}_0, R)\xi + S_{-1} + S_0 + N \right] \cdot \partial_{y^i} \tilde{F}_0. \quad (4.67)$$

Examining the $S_0$ term on the right hand side of (4.67) more closely, we see that

$$\int_{-y^1}^{y^1} \frac{n^2}{m^2} S_0 \cdot \partial_{y^i} \tilde{F}_0 = \int_{-y^1}^{y^1} \left( \partial_{y^0, \rho} (\tilde{F}_0 + \epsilon \tilde{F}_1) + \frac{n^2}{m^2} \partial_{y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1) + \epsilon \frac{n^2}{m^2} B^i \partial_{y^i} \tilde{F}_0 \right) \cdot \partial_{y^i} \tilde{F}_0. \quad (4.68)$$

Next, examine the term containing $\partial_{y^0, \rho} \tilde{F}_0$. Since $\tilde{F}_0(y^0, y^1) = F_0(\frac{y^1 - \eta(y^0)}{\epsilon}; R(y^0))$, we see that

$$\int_{-y^1}^{y^1} \partial_{y^0, \rho} \tilde{F}_0 \cdot \partial_{y^i} \tilde{F}_0 = -\frac{a'}{\epsilon} \int_{-y^1}^{y^1} \left| \partial_{y^0} F_0 \right|^2 + \frac{1}{\epsilon} \int_{-y^1}^{y^1} \left( \frac{a'}{\epsilon} \right)^2 \partial_{y^1} F_0 \cdot \partial_{y^i} F_0$$

$$+ \frac{1}{\epsilon} \left[ -2R' \left( \frac{a'}{\epsilon} \right)^2 \partial_{y^0} \partial_{y^i} F_0 + R'' \partial_{y^0} \partial_{y^i} F_0 + (R')^2 \partial_{y^0} \partial_{y^i} F_0 \right] \cdot \partial_{y^i} F_0, \quad (4.69)$$

where again we have suppressed the arguments of the $F_0$’s and used the notation

$$\partial_{y^0, \rho} F_0 = \partial_{y^0} \partial_{y^0} F_0(\frac{y^1 - \eta}{\epsilon}; R).$$

Remember, the goal is to obtain a bound for $\frac{a''}{\epsilon}$. To do this, we will use (4.65 - 4.69) to isolate the $\left| \partial_{y^0} F_0 \right|^2$ term appearing in (4.69). We will then estimate each of the remaining terms. We will break the analysis up into three parts and then combine them to obtain (4.29).

1. $\partial_{y^0, \rho} \xi \cdot \partial_{y^i} \tilde{F}_0$ upper bound estimate: Using (4.65) and (4.66), we can isolate the $\partial_{y^0, \rho} \xi \cdot \partial_{y^i} \tilde{F}_0$ term. Estimating the resulting expression, we have that

$$\int_{-y^1}^{y^1} \left| \partial_{y^0, \rho} \xi \cdot \partial_{y^i} \tilde{F}_0 \right| \lesssim \left( 1 + \frac{|a''|}{\epsilon} \right) A e^{-\frac{|a''|}{\epsilon}} + \int_{-y^1}^{y^1} 2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^i} \tilde{F}_0 - \partial_{y^i} \xi \cdot \partial_{y^0, \rho} \tilde{F}_0 \right|$$

$$\lesssim \left( 1 + \frac{|a''|}{\epsilon} \right) A e^{-\frac{|a''|}{\epsilon}} + \frac{1}{\sqrt{\epsilon}} E^{1/2} A + \sqrt{\epsilon} \left( 1 + \frac{|d''|}{\epsilon} \right) E^{1/2} A,$$
where we used Cauchy-Schwarz and corollary 4.3.2 to obtain the last inequality.

2. \( \partial_{\rho,\rho} \xi \cdot \partial_{\gamma} \tilde{F}_0 \) lower bound estimate: Estimating \( \partial_{\rho,\rho} \xi \cdot \partial_{\gamma} \tilde{F}_0 \) using (4.67), we get

\[
\left| \int_{\gamma^1} \partial_{\rho,\rho} \xi \cdot \partial_{\gamma} \tilde{F}_0 \right| \geq \left| \int_{\gamma^1} \frac{n^2}{m^2} S_0 \cdot \partial_{\gamma} \tilde{F}_0 \right| - \left| \int_{\gamma^1} \frac{n^2}{m^2} L_e(\tilde{F}_0, R) \xi \cdot \partial_{\gamma} \tilde{F}_0 \right| - \left| \int_{\gamma^1} \frac{n^2}{m^2} \partial_{\rho,\rho} \xi \cdot \partial_{\gamma} \tilde{F}_0 \right| - \left| \int_{\gamma^1} \frac{n^2}{m^2} S_{-1} \cdot \partial_{\gamma} \tilde{F}_0 \right| - \left| \int_{\gamma^1} \frac{n^2}{m^2} N \cdot \partial_{\gamma} \tilde{F}_0 \right|.
\]

We will estimate the \( B^o \partial_{\rho} \xi, L_e(\tilde{F}_0, R) \xi, S_{-1} \), and \( N \) terms separately.

(a) \( B^o \partial_{\rho} \xi \cdot \partial_{\gamma} \tilde{F}_0 \) term: Recall the definitions for \( B^o \) (3.8), \( B^i \) (3.9), and \( E \) (4.21). Using Cauchy-Schwarz and the boundedness of \( \frac{m^2}{n} B^o \) on \((0, y^0) \times (-y^1, y^1)\), we have

\[
\left| \int_{|y^1| \leq y^1} \frac{n^2}{m^2} B^o \partial_{\rho} \xi \cdot \partial_{\gamma} \tilde{F}_0 \right| \leq E^{1/2} \| \partial_{\gamma} \tilde{F}_0 \|_2 \leq \frac{1}{\sqrt{\epsilon}} E^{1/2} A.
\]

where we used Cauchy-Schwarz and corollary 4.3.2 to obtain the last inequality.

(b) \( L_e(\tilde{F}_0, R) \xi \cdot \partial_{\gamma} \tilde{F}_0 \) term: Recall that Hess_{\gamma} W(\tilde{F}_0, R) is symmetric. By integrating by parts with respect to \( y^1 \) twice, we can move the operator \( L_e(\tilde{F}_0, R) \) onto \( \frac{n^2}{m^2} \partial_{\gamma} \tilde{F}_0 \). Doing so, we see that

\[
\int_{|y^1| \leq y^1} \frac{n^2}{m^2} L_e(\tilde{F}_0, R) \xi \cdot \partial_{\gamma} \tilde{F}_0 = \int_{|y^1| \leq y^1} \xi \cdot L_e(\tilde{F}_0, R) \left( \frac{n^2}{m^2} \partial_{\gamma} \tilde{F}_0 \right) + \left[ \frac{n^2}{m^2} \partial_{\gamma} \xi \cdot \partial_{\gamma} \tilde{F}_0 - \xi \cdot \partial_{\gamma} \left( \frac{n^2}{m^2} \partial_{\gamma} \tilde{F}_0 \right) \right]_{y^1}.
\]

Using the estimate (3.49) we can easily bound the second term as

\[
\left| \left[ \frac{n^2}{m^2} \partial_{\gamma} \xi \cdot \partial_{\gamma} \tilde{F}_0 - \xi \cdot \partial_{\gamma} \left( \frac{n^2}{m^2} \partial_{\gamma} \tilde{F}_0 \right) \right]_{y^1} \right| \leq \frac{1}{\epsilon^2} e^{-a \frac{y^1}{\epsilon}} e^{\alpha A}.
\]
Using the fact that $\frac{n^2}{m^2}$ is bounded on $(0, y_0^0) \times (-y_1^+, y_1^+)$, Cauchy-Schwarz, and corollary 4.3.2 we estimate the first term as follows

\[
\left| \int_{|y| \leq y_1^+} \xi \cdot L_\epsilon(\tilde{F}_0, R) \left( \frac{n^2}{m^2} \partial_{y_1^+} \tilde{F}_0 \right) \right| \leq \int_{|y| \leq y_1^+} \frac{n^2}{m^2} \xi \cdot \partial_{y_1^+} \tilde{F}_0 \leq \int_{|y| \leq y_1^+} \left[ \partial_{y_1^+} \left( \frac{n^2}{m^2} \right) \xi \cdot \partial_{y_1^+} \tilde{F}_0 + \partial_{y_1^+} \left( \frac{n^2}{m^2} \right) \xi \cdot \partial_{y_1^+} \tilde{F}_0 \right] \]

\[
\leq \frac{1}{\sqrt{\epsilon}} E^{1/2} + \frac{1}{\epsilon^{3/2}} e^{-\frac{\alpha y_1^+}{2}} e^{\alpha A},
\]

where we needed to use the fact that $\partial_{y_1^+} \tilde{F}_0 \in \ker(L_\epsilon(\tilde{F}_0, R))$ to kill the $L_\epsilon(\tilde{F}_0, R) \partial_{y_1^+} \tilde{F}_0$ term. Thus, we have that

\[
\left| \int_{|y| \leq y_1^+} \frac{n^2}{m^2} L_\epsilon(\tilde{F}_0, R) \xi \cdot \partial_{y_1^+} \tilde{F}_0 \right| \leq \frac{1}{\sqrt{\epsilon}} E^{1/2} + \frac{1}{\epsilon^{3/2}} e^{-\frac{\alpha y_1^+}{2}} e^{\alpha A}.
\]

(c) $S_{-1} \cdot \partial_{y_1^+} \tilde{F}_0$ term: Recall the definition of $S_{-1}$ (4.17). Using the boundedness of $\frac{n^2}{m^2}$ on $(0, y_0^0) \times (-y_1^+, y_1^+)$, Cauchy-Schwarz, and corollary 4.3.2 we have that

\[
\left| \int_{|y| \leq y_1^+} \frac{n^2}{m^2} S_{-1} \cdot \partial_{y_1^+} \tilde{F}_0 \right| = \left| \int_{|y| \leq y_1^+} \frac{n^2}{m^2} \partial_{y_1^+} \tilde{F}_0 \cdot \left[ B^1 + H(R) \right] \partial_{y_1^+} \tilde{F}_0 \right| \leq \int_{|y| \leq y_1^+} \left| \partial_{y_1^+} \tilde{F}_0 \right|^2 \leq A,
\]

where we used the fact that $B^1 = -H(R) + O(y^1)$ to obtain first inequality.

(d) $N \cdot \partial_{y_1^+} \tilde{F}_0$ term: For this term, we proceed as we did in the proof of lemma 4.3.6 when we estimated the $N \cdot \partial_{y_1} \xi$ term. Again, we will use the identity

\[
g(1) = g(0) + g'(0) + \int_0^1 (1 - t)g''(t)
\]
to rewrite $N$. Thus, we have that

$$\int \frac{n^2}{m^2} N \cdot \partial_{yi} \tilde{F}_0$$

$$= -\frac{1}{e^2} \int_0^1 (1 - t) \int_{-y_i^1}^{y_i^1} \frac{n^2}{m^2} (\tilde{F}_\xi)_{yi} \partial_{yi} \Hess \phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \cdot \partial_{yi} \tilde{F}_0$$

$$- \frac{1}{e^2} \int_0^1 (1 - t) \int_{-y_i^1}^{y_i^1} \frac{n^2}{m^2} (\tilde{F}_\xi)_y \partial_{yi} \Hess \phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \cdot \partial_{yi} \tilde{F}_0$$

$$- \frac{1}{e^2} \int_0^1 (1 - t) \int_{-y_i^1}^{y_i^1} \frac{n^2}{m^2} 2y^1 m \partial_{yi} \Hess \phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \cdot \partial_{yi} \tilde{F}_0$$

$$- \frac{1}{e^2} \int_0^1 (1 - t) \int_{-y_i^1}^{y_i^1} \frac{n^2}{m^2} \partial_{yi} \varpi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \cdot \partial_{yi} \tilde{F}_0.$$

We will estimate each of the four terms appearing on the right hand side of the above individually. That is,

i. Since $\partial_{yi} \Hess \phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)$ is bounded on $(0, y_i^0) \times (-y_i^1, y_i^1)$ for $0 \leq t \leq 1$, then

$$\left| \frac{1}{e^2} \int_0^1 (1 - t) \int_{-y_i^1}^{y_i^1} \frac{n^2}{m^2} (\tilde{F}_\xi)_{yi} \partial_{yi} \Hess \phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \cdot \partial_{yi} \tilde{F}_0 \right|$$

$$\leq \frac{1}{e^2} \int_{-y_i^1}^{y_i^1} |\tilde{F}_\xi|^2 |\partial_{yi} \tilde{F}_0|$$

$$\leq \frac{1}{e^2} \|\tilde{F}_\xi\|_{L^\infty(-y_i^1, y_i^1)} \|\tilde{F}_\xi\|_{L^2(-y_i^1, y_i^1)} \|\partial_{yi} \tilde{F}_0\|_2$$

$$\leq \frac{1}{e^{3/2}} (\epsilon \|\tilde{F}_\xi\|_\infty + \|\xi\|_{L^\infty(-y_i^1, y_i^1)} (\epsilon \|\tilde{F}_\xi\|_2 + \|\xi\|_{L^2(-y_i^1, y_i^1)}))$$

$$\leq \frac{1}{e^{3/2}} (\epsilon + \|\xi\|_{L^\infty(-y_i^1, y_i^1)} (\epsilon^{3/2} + \epsilon^{1/2} + e^{-a \frac{1-a}{2}})), $$

where we needed to use corollaries 4.1.6 and 4.3.2. to obtain the last and second to last inequalities. Next, we estimate the $\|\xi\|_\infty$ term. Using Gagliardo-
Nirenberg, we see that
\[ \| \xi \|_{L^\infty(-y_1^*, y_1^*)} \lesssim \| \xi \|_{L^2(-y_1^*, y_1^*)}^{1/2} \| \partial y \xi \|_{L^2(-y_1^*, y_1^*)}^{1/2} \lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{1-m}{r}})^{1/2} E^{1/4}. \]

Thus, we have that
\[
\left| \frac{1}{\epsilon^2} \int_0^1 (1 - t) \int_{-y_1^*}^{y_1^*} \frac{n^2}{m^2} (\hat{F}_\xi)_\phi \partial \phi \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \cdot \partial y \tilde{F}_0 \right| 
\lesssim \frac{1}{\epsilon^{3/2}} (\epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{1-m}{r}})^{1/2} E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{1-m}{r}}).
\]

ii. Similarly, we have that
\[
\left| \frac{1}{\epsilon^2} \int_0^1 (1 - t) \int_{-y_1^*}^{y_1^*} \frac{n^2}{m^2} (\hat{F}_\xi)_\sigma \partial \sigma \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \cdot \partial y \tilde{F}_0 \right| 
\lesssim \frac{1}{\epsilon^{3/2}} (\epsilon + (\epsilon E^{1/2} + e^{-\alpha \frac{1-m}{r}})^{1/2} E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{1-m}{r}}).
\]

iii. Using Cauchy-Schwarz, the boundedness of \( \frac{n^2}{m^2} m(y_0^0) \partial_t \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \) on \((0, y_1^0) \times (-y_1^*, y_1^*)\) for all \(0 \leq t \leq 1\), corollary 4.1.6 and corollary 4.3.2 we find that
\[
\left| \frac{1}{\epsilon^2} \int_0^1 (1 - t) \int_{-y_1^*}^{y_1^*} \frac{n^2}{m^2} 2y^1 m \partial_t \text{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \tilde{F}_\xi \cdot \partial y \tilde{F}_0 \right| 
\lesssim \frac{1}{\epsilon^{3/2}} (\| e^{\tilde{F}_1} \|_2 + \| \tilde{F}_\xi \|_{L^2(-y_1^*, y_1^*)} \| y^1 \partial y \tilde{F}_0 \|_2 
\lesssim \frac{1}{\epsilon^{3/2}} (\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{1-m}{r}}).
iv. Recall that
\[ \partial_{rr}w(\tilde{F}_0, R) = 6 \frac{d^2}{R^4} \begin{pmatrix} 0 \\ \tilde{s}_0 \end{pmatrix}. \]

Thus, we have that
\[ \left| \frac{1}{\epsilon^2} \int_0^1 (1 - t) \int_{-y^1}^{y^1} \frac{n^2}{m^2} (y^1)^2 \partial_{rr}w(\tilde{F}_0 + ty\tilde{\xi}, R + ty^1m) \cdot \partial_{y^1}\tilde{F}_0 \right| \]
\[ = \left| \frac{1}{\epsilon^2} \int_0^1 (1 - t) \int_{-y^1}^{y^1} (y^1)^2 \frac{d^2n^2}{(R + ty^1m)^2} (\tilde{s}_0 + ty\tilde{s}) \partial_{y^1}\tilde{s}_0 \right| \]
\[ \leq \frac{1}{\epsilon^2} \left\| (y^1)^2 \tilde{s}_0 \partial_{y^1}\tilde{s}_0 \right\|_2 + \frac{1}{\epsilon^2} \left\| \epsilon \tilde{s}_1 \right\|_2 \left\| (y^1)^2 \partial_{y^1}\tilde{s}_0 \right\|_2 + \frac{1}{\epsilon^2} \left\| \xi \right\|_{L^2(-y^1,y^1)} \left\| (y^1)^2 \partial_{y^1}\tilde{s}_0 \right\|_2 \]
\[ \leq 1 + \epsilon + \sqrt{\epsilon E^{1/2}} + \frac{1}{\sqrt{\epsilon}} e^{-\alpha \frac{A - a}{\epsilon} \frac{y}{E}}. \]

where we used Cauchy-Schwarz and the boundedness of \( \frac{n^2}{(R + ty^1m)} \) on \((0, y^1_0) \times (-y^1_0, y^1_0)\) for \(0 \leq t \leq 1\) to obtain the second last inequality and corollaries 4.1.6 and 4.3.2 to obtain the last inequality.

Thus, we have that
\[ \left| \int_{-y^1}^{y^1} \frac{n^2}{m^2} N \cdot \partial_{y^1}\tilde{F}_0 \right| \leq 1 + \epsilon + \sqrt{\epsilon E^{1/2}} + \frac{1}{\sqrt{\epsilon}} e^{-\alpha \frac{A - a}{\epsilon} \frac{y}{E}}. \]

Putting the estimates from steps (a)-(d) together, we have that
\[ \left| \int_{|y^1| \leq y^1} \partial_{y^1} \phi \xi : \partial_{y^1}\tilde{F}_0 \right| \geq \left| \int_{|y^1| \leq y^1} \frac{n^2}{m^2} S_0 \cdot \partial_{y^1}\tilde{F}_0 \right| \]
\[ - \left[ \frac{1}{\epsilon^{3/2}} (\epsilon A + (\epsilon E^{1/2} + e^{-\alpha \frac{A - a}{\epsilon} \frac{y}{E}}) \frac{1}{2} E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha \frac{A - a}{\epsilon} \frac{y}{E}}) \right]. \]
3. $S_0 \cdot \partial_{y^i} \tilde{F}_0$ lower bound estimate: Using (4.68) and Cauchy-Schwarz, we see that

$$
\left| \int_{-y^1}^{y^1} \frac{n^2}{m^2} S_0 \cdot \partial_{y^i} \tilde{F}_0 \right| \geq \left| \int_{-y^1}^{y^1} \partial_{\rho^0 \rho} \tilde{F}_1 \cdot \partial_{y^i} \tilde{F}_0 \right|
- \left[ \left\| \partial_{\rho^0 \rho} \tilde{F}_1 \right\|_2 + \left\| \partial_{\rho^0 \rho} \tilde{F}_0 \right\|_2 + \epsilon \left\| \partial_{y^i} \tilde{F}_0 \right\|_2 \right] \left\| \partial_{y^i} \tilde{F}_0 \right\|_2
\geq \left| \int_{-y^1}^{y^1} \partial_{\rho^0 \rho} \tilde{F}_1 \cdot \partial_{y^i} \tilde{F}_0 \right| - \left[ A + \left| a'' \right| A \right],
$$

where we needed to use the fact that $\frac{n^2}{m^2} B^\rho$ is bounded on $(0, y^0) \times (-y^1, y^1)$ to obtain the first inequality and where we used corollary 4.3.2 and the facts that $R'$ and $R''$ are bounded for $y^0 \in (0, y^0)$ to obtain the last inequality.

4. $\partial_{\rho^0 \rho} \tilde{F}_0 \cdot \partial_{y^i} \tilde{F}_0$ lower bound estimate: This is the last term we need to estimate. Using (4.69), we see that

$$
\left| \int_{-y^1}^{y^1} \partial_{\rho^0 \rho} \tilde{F}_1 \cdot \partial_{y^i} \tilde{F}_0 \right| \geq \frac{1}{\epsilon} \left| a'' \right| \left| \int_{-y^1}^{y^1} \left| \partial_{y^i} F_0 \right|^2 \right| - \frac{1}{\epsilon} \left( \frac{d'}{\epsilon} \right)^2 \left| \int_{-y^1}^{y^1} \partial_{y^i} F_0 \cdot \partial_{y^i} F_0 \right|
- \frac{2R'}{\epsilon} \left| \frac{d'}{\epsilon} \right| \left| \int_{-y^1}^{y^1} \partial_{R} \partial_{y^i} F_0 \cdot \partial_{y^i} F_0 \right| - \frac{R''}{\epsilon} \left| \int_{-y^1}^{y^1} \partial_{R} F_0 \cdot \partial_{y^i} F_0 \right|
- \frac{(R')^2}{\epsilon} \left| \int_{-y^1}^{y^1} \partial_{RR} F_0 \cdot \partial_{y^i} F_0 \right|
\approx \left| \frac{a''}{\epsilon} \right| - A,
$$

where we used the boundedness of $R'$ and $R''$ on $(0, y^0)$, Cauchy-Schwarz, and corollary 4.3.1 to obtain the last inequality.

Combining the estimates obtained in steps 1-4, we obtain the estimate

$$
\frac{|a''|}{\epsilon} \leq \frac{|a''|}{\epsilon} \left[ \epsilon + \sqrt{\epsilon} \sqrt{E} + e^{-a\frac{1}{16}} \right] A
+ \frac{1}{\epsilon^{5/2}} \left( \epsilon + (\epsilon \sqrt{E} + e^{-a\frac{1}{16}})^{1/2} E^{1/4} \right)^2 (\epsilon^{3/2} + \epsilon \sqrt{E} + e^{-a\frac{1}{16}}) A.
$$
Rearranging this inequality, we obtain (4.29)

A  Formal Asymptotics

Let $\eta$ be the Minkowski metric on $\mathbb{R}^{1+n}$ and let $\Gamma \subset (\mathbb{R}^{1+n}, \eta)$ be an $n$-dimensional time-like surface in space-time. Suppose that $\Gamma$ is parameterized by some map $H : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{1+n}$. Define a new coordinate system $(y^\tau, y^\nu) \in \mathbb{R}^n \times \mathbb{R}$, called Minkowski normal coordinates, as

$$(t, x) = \psi(y^\tau, y^\nu) = H(y^\tau) + y^\nu \nu(y^\tau),$$

where $\nu(y^\tau) \perp_{\eta} \partial_{y^\tau} H(y^\tau)$ and $|\nu(y^\tau)|_\eta = 1$. We call $y^\tau \in \mathbb{R}^n$ “tangential coordinates” and $y^\nu \in \mathbb{R}$ the “normal coordinate”. Note that this coordinate system may only be well defined on a neighbourhood $\mathcal{N}$ of $\Gamma$.

Recall that we want to find solutions of (1.1) so that $\phi$ has an interface and so that $\sigma$ is exponentially small except near the interface of $\phi$. A discussion in [34] suggests that there should exist “simple” solutions to (1.8) that like

- $\phi$ looks like a profile in directions transverse to its interface.
- $|\sigma|$ looks like a profile in directions transverse to the interface of $\phi$ and the phase of $\sigma$ only varies in directions tangential to the interface of $\phi$.

Based on this discussion and results from [17], we expect that for suitable $\Gamma$ parameterized by $H, \theta : \Omega \to \mathbb{R}$, and $\Phi_0 := (\phi_0, \sigma_0) : \mathbb{R} \to \mathbb{R}^2$ there exists a solution with these characteristics of the form

$$\Phi(y^\tau, y^\nu) \approx \left( \begin{array}{c} \phi_0(y^\nu) \\ e^{i\theta(y^\nu)}\sigma_0(y^\nu) \end{array} \right)$$

(A.1)

We will now carry out a formal asymptotic analysis to find $\Phi_0$ so that $\phi_0$ has an interface and to find $\Gamma$ and $\theta$ for which we expect (A.1) to hold. To do this, we will expand the action integral associated to (1.1) about the right hand side of (A.1). From this expansion, we obtain an effective action. We will then make a choice for the profile $\Phi_0$ and for this choice of $\Phi_0$, we expect, heuristically, that the correction terms coming from expanding the action about the right hand side of (A.1) will be of lower order when $\Gamma$ and $\theta$ are critical points of the effective action.
The Lagrangian associated to (1.1) in Minkowski normal coordinates is

$$\mathcal{L} := \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma + \frac{1}{\epsilon^2} V(\phi, \sigma), \quad (A.2)$$

where $g_{\alpha\beta} := \eta_{\lambda\omega} \partial_\alpha \psi^\lambda \partial_\beta \psi^\omega$ is the Minkowski metric in normal coordinates and, only for this appendix,

$$V = \frac{\lambda_{\phi}}{4} (\phi^2 - 1)^2 + \frac{\lambda_{\sigma}}{4} (|\sigma|^2 - 2)|\sigma|^2 + \frac{\beta}{2} \phi^2 |\sigma|^2.$$

Note that

$$g_{ij} = \gamma_{ij} + (y^\nu) \eta_{\lambda\omega} \left( \partial_i H^\lambda \partial_j H^{\nu\omega} + \partial_j H^\lambda \partial_i H^{\nu\omega} \right) + (y^\nu)^2 \eta_{\lambda\omega} \partial_i \psi^\lambda \partial_j \psi^\omega$$

where $\gamma_{ij} := \eta_{\alpha\beta} \partial_\alpha \psi^\lambda \partial_\beta \psi^\omega$ denotes the induced metric on the surface $\Gamma$ (Latin indices range over the tangential coordinates and Greek indices will range over both tangential and normal coordinates). For $\xi = (\xi_\phi, \xi_\sigma) : \mathbb{R}^{1+n} \rightarrow \mathbb{R} \times \mathbb{R}$, we plug

$$\phi = \phi_0 \left( \frac{y^\nu}{\epsilon} \right) + \xi_\phi$$
$$\sigma = \epsilon^2 \theta(y^\nu) \left[ \sigma_0 \left( \frac{y^\nu}{\epsilon} \right) + \xi_\sigma \right]$$

into the action integral and find that

$$S(\Phi) = \frac{1}{\epsilon^2} \int \left\{ \frac{1}{2} |\Phi_0(y^\nu)|^2 + V(\Phi_0)(\frac{y^\nu}{\epsilon}) + \frac{1}{2} \xi(y^\nu) \sigma_0^2(\frac{y^\nu}{\epsilon}) \right\} \sqrt{-\gamma(y^\nu)} dy^\nu dy^\nu + S_1(\Phi_0, \xi) \quad (A.3)$$

where the first term of $S$ includes all the terms independent of $\xi$. In this expansion we formally think of $\xi$ as being of lower order than $\Phi_0$. In fact, the main result of this thesis establishes that when $n = 2$ and if $(\phi, \sigma)$ is equivariant, then if $\xi$ starts off small, then it remains small under the evolution of (1.1). For now, though, we have formally derived the following effective action

$$\tilde{S} := \int \left\{ \frac{1}{2} |\Phi'_0(y^\nu)|^2 + V(\Phi_0)(y^\nu) + \frac{1}{2} \xi(y^\nu) \sigma_0(y^\nu)^2 \right\} \sqrt{-\gamma(y^\nu)} dy^\nu dy^\nu. \quad (A.4)$$

It is natural to choose $\Phi_0$ so that in transverse directions to $\Gamma$, $\Phi_0$ is energy minimizing and so that $\phi_0$ has an interface. To this end, suppose for $\rho \in \mathbb{R}$, $F = (f, s)(\cdot; \rho)$ satisfies the
minimization problem

\[ \mu(\rho) := \inf_{(f, s) \in \mathcal{A}} \int \left\{ \frac{1}{2} |(f', s')|^2 + V(f, s) + \frac{1}{2} \rho s^2 \right\} dy \quad (A.5) \]

\[ \mathcal{A} := \left\{ (f, s) \in H^1_{loc} : \lim_{y^v \to \pm \infty} f(y^v) = \pm 1, \ f(0) = 0 \right\}. \quad (A.6) \]

In this case, the boundary conditions imposed results in \( f \) having an interface and the condition that \( f(0) = 0 \) kills the translation symmetry that would otherwise be present in any solutions to this minimization problem. Furthermore, for suitable potentials \( V, s \) is exponentially small except near the interface of \( f \). For each \( \rho \in \mathbb{R} \) it is natural to pick \( \Phi_0(\cdot; \rho) = (f, s)(\cdot; \rho) \). Based on this, we modify our ansatz to be that we expect for suitable \( \Gamma \) and \( \theta \), there should exist a solution to (1.1) satisfying

\[ \Phi \approx \begin{pmatrix} \phi_0\left( y^v; \zeta(y^v) \right) \\ e^{i \theta(y^v)} \sigma_0\left( y^v; \zeta(y^v) \right) \end{pmatrix} \quad (A.7) \]

for \( \Phi_0 = \Phi_0(\cdot; \xi) \) minimizing (A.5). Note: The differentiability of \( \Phi_0 \) in both \( y^v \) and \( \zeta \) is important and needs to be established (we did this in section 3.6 for potentials satisfying (1.9)). For this section we assume that \( \Phi_0 \) is sufficiently regular so that we may carry out the calculations that are to follow.

For this choice of \( \Phi_0 \), the effective action becomes

\[ \tilde{S}(H, \theta) = \int \mu(\xi) \sqrt{-\gamma} dy. \quad (A.8) \]

Heuristically, we expect that when \( \theta \) and \( H \) are critical points of \( \tilde{S} \), then \( \xi \) will be of lower order than the right hand side of (A.1). That is, for \( \theta \) and \( H \) satisfying the nonlinear, coupled system

\[ 0 = \frac{\delta \tilde{S}}{\delta \theta} = -2 \partial_j \left( \mu'(\xi) \sqrt{-\gamma} \gamma^{ij} \partial_i \theta \right) \quad (A.9) \]

\[ 0 = \frac{\delta \tilde{S}}{\delta H} = -\eta_{\alpha \beta} \partial_j \left( \mu(\xi) \sqrt{-\gamma} \gamma^{ij} \partial_i H^\alpha \right) + 2 \eta_{\alpha \beta} \partial_j \left( \mu'(\xi) \sqrt{-\gamma} \gamma^{ij} \gamma^{kl} \partial_k \theta \partial_l \theta H^\beta \right), \quad (A.10) \]

then \( \Phi_0(\cdot; \xi) \) should be a good approximate solution. We expect that the coupled system for \( \theta \) and \( H \) is a hyperbolic system, but this is not completely clear by just looking at it. By expanding
\[ \Box_\Gamma \theta = -\gamma (\nabla z \log [\mu'(\zeta)] \cdot \nabla \theta) \quad (A.11) \]

\[ \vec{H} = 2 \frac{\mu'(\zeta)}{\mu(\zeta)} \Pi(\nabla \theta, \nabla \theta), \quad (A.12) \]

where \( \vec{H} \) is the mean curvature vector of \( \Gamma \) and \( \Pi \) is the second fundamental form of \( \Gamma \). We can simplify this further as follows. For \( \Phi_0 \) minimizing (A.5 - A.6), then \( \Phi_0 \) satisfies

\[ -\Phi''_0 + v(\Phi_0) + \zeta(0, \sigma_0) = 0, \quad (A.13) \]

where

\[ v(\Phi_0) := \begin{pmatrix} \lambda_\phi (\phi_0^2 - 1) \phi_0 + \beta \sigma_0^2 \phi_0 \\ \lambda_\sigma (\sigma_0^2 - 1) \sigma_0 + \beta \phi_0^2 \sigma_0 \end{pmatrix}. \]

Multiplying (A.13) by \( \Phi'_0 \) and integrating we find the equipartition of energy identity

\[ \frac{1}{2} \| \Phi'_0 \|^2 = V(\Phi_0) + \frac{1}{2} \zeta \sigma_0^2. \quad (A.14) \]

Using this identity we find that

\[ \mu(\zeta) = \| \Phi'_0 \|^2. \]

Further, if we differentiate \( \mu(\zeta) \) with respect to \( \zeta \) we find

\[ \mu'(\zeta) = \int \left\{ \Phi'_0 \cdot \partial_\zeta \Phi_0 + v(\Phi_0) \cdot \partial_\zeta \Phi_0 + \zeta \sigma_0 \partial_\zeta \sigma_0 + \frac{1}{2} \sigma_0^2 \right\} \]

\[ = \int \left\{ \left\{ -\Phi''_0 + v(\Phi_0) + \zeta \begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix} \right\} \cdot \partial_\zeta \Phi_0 + \frac{1}{2} \sigma_0^2 \right\} \]

\[ = \int \frac{1}{2} \sigma_0^2, \]

where we used (A.13) to obtain the last reduction. Thus, we obtain a nice geometric relation relating the surface about which our approximate surface is concentrated and the phase of \( \sigma \)-field

\[ \Box_\Gamma \theta = -\gamma (\nabla \theta \log \left[ 2 \| \sigma_0 \|^2 \right] \cdot \nabla \theta) \quad (A.15) \]

\[ \vec{H} = \frac{\| \sigma_0 \|^2}{\| \Phi'_0 \|^2} \Pi(\nabla \theta, \nabla \theta). \quad (A.16) \]
B Well Posedness of the Interface with a Current Model

B.1 Local Well Posedness

To show that the superconducting interface model is LWP we will implement a fixed point argument. This argument is classical, but we include it for the sake of completeness. See [28] for a review of wave equations and [31] for a more complete review of fixed point arguments in PDEs.

We will require the following estimate in order to carry out this argument. That is, for \((w_0, w_1) \in H^1_x \times L^2_x\) and \(w\) satisfying

\[
\begin{aligned}
\Box w &= F(w) \\
\partial_t w(0) &= w_0 \\
\partial_x w(0) &= w_1,
\end{aligned}
\]

then

\[
\|w\|_{C^0_T H^1_x (I \times \mathbb{R}^2)} + \|\partial_t w\|_{C^0_T L^2_x (I \times \mathbb{R}^2)} \lesssim \langle |I| \rangle \left( \|w_0\|_{H^1} + \|w_1\|_{L^2} \right) + \|F(w)\|_{L^1_T L^2}. \tag{B.1}
\]

For our application \(u\) is a 2-component vector (i.e. \(u : \mathbb{R}^{1+2} \to \mathbb{R}^2\)). We are interested in a problem whose initial data satisfy

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix} - u_0 \in H^1_x \quad \text{and} \quad u_1 \in L^2_x
\]

and obeys

\[
\begin{aligned}
\Box u &= \nabla \phi V(u) \\
u(0) &= u_0 \\
\partial_t u(0) &= u_1.
\end{aligned} \tag{B.2}
\]

We want to use (B.1) and our problem doesn’t quite satisfy the hypothesis to directly apply it. So we will need to be careful.
Set $\nu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We can rewrite (B.2) as

$$
\begin{align*}
\Box (u - \nu) &= \nabla \Phi V(\nu + (u - \nu)) \\
u(0) - \nu &= u_0 - \nu \in H^1_x \\
\partial_t u(0) &= u_1 \in L^2_x.
\end{align*}
$$

Define

$$\Psi(u - \nu, \partial_t u) = S(t) \begin{pmatrix} u_0 - \nu \\ u_1 \end{pmatrix} + \int_0^t S(t - s) \begin{pmatrix} 0 \\ \nabla \Phi V(u(s)) \end{pmatrix} ds, \tag{B.3}
$$

where $S(t)$ is the semi-group associated to the linear operator $\Box w = 0$ (Note: $\Psi(u - \nu, \partial_t u)(t, x) \in \mathbb{R}^2 \times \mathbb{R}^2$). We will show that for sufficiently small $T$, $\Psi$ has a unique fixed point in $X = X((0, T) \times \mathbb{R}^2)$ using the Banach Fixed Point Theorem, where

$$X = \{ G : \| G \|_X < \infty \}$$

$$\| G \|_X = \| G_1 \|_{C^0_t H^1_x} + \| G_2 \|_{C^0_t L^2_x}$$

with $G = (G_1, G_2) : \mathbb{R}^{1+2} \to \mathbb{R}^2 \times \mathbb{R}^2$.

1. For $\overline{B_R(0)} \subset X$ and $R$ sufficiently large, we will show that $\Psi : \overline{B_R(0)} \hookrightarrow \overline{B_R(0)}$. Using (B.3), we have that

$$\| \Psi(u - \nu, \partial_t u) \|_X \leq \| S(t) \begin{pmatrix} u_0 - \nu \\ u_1 \end{pmatrix} \|_X + \int_0^t \| S(t - s) \begin{pmatrix} 0 \\ \nabla \Phi V(u(s)) \end{pmatrix} \|_X.$$

(a) Using (B.1), we have that

$$\| S(t) \begin{pmatrix} u_0 - \nu \\ u_1 \end{pmatrix} \|_X \leq \langle T \rangle \left( \| u_0 - \nu \|_{H^1_x} + \| u_1 \|_{L^2_x} \right).$$
(b) Again using (B.1), we have that
\[
\left\| \int_0^t S(t-s) \left( \begin{array}{c} 0 \\ \nabla \Phi V(u)(s) \end{array} \right) \right\|_X \lesssim \int_0^t \left\| S(t-s) \left( \begin{array}{c} 0 \\ \nabla \Phi V(u)(s) \end{array} \right) \right\|_X ds \\
\lesssim \int_0^t \langle t-s \rangle \left\| \nabla \Phi V(u) \right\|_{L^2_x} ds \\
\lesssim \langle T \rangle \left\| \nabla \Phi V(u) \right\|_{L^2_x L^2_t}.
\]

We are left to now estimate \( \left\| \nabla \Phi V(u) \right\|_{L^1_x L^2_t} \). By assumption 4 of (1.9), we have
\[
\left\| \nabla \Phi V(u) \right\|_{L^1_x L^2_t} = \left\| \nabla \Phi V(\nu) + \int_0^1 \frac{d}{ds} \nabla \Phi V(\nu + s(u-\nu)) ds \right\|_{L^1_x L^2_t} \\
= \left\| \nabla \Phi V(\nu) + \int_0^1 \text{Hess}_\Phi V(s(u-\nu) + \nu)(u-\nu) ds \right\|_{L^1_x L^2_t}.
\]

Using now assumptions 2 and 4 of (1.9), we have
\[
\left\| \nabla \Phi V(u) \right\|_{L^1_x L^2_t} \lesssim \int_0^1 \left[ \left\| 1 + |\nu + s(u-\nu)|^2 \right\| \left\| u-\nu \right\|_{L^1_x L^2_t} ds \\
\lesssim \left\| u-\nu \right\|_{L^1_x L^2_t} + \left\| u-\nu \right\|_{L^1_x L^2_t}^3 \\
\lesssim \left\| u-\nu \right\|_{L^1_x L^2_t} + \left\| u-\nu \right\|_{L^1_x L^2_t}^3 \\
\lesssim T \left( \left\| u-\nu \right\|_{C^0_t L^3_x} + \left\| u-\nu \right\|_{C^1_t L^5_x}^3 \right).
\]

Using the Sobolev embedding theorem, we have that
\[
\left\| \nabla \Phi V(u) \right\|_{L^1_x L^2_t} \lesssim T \left( \left\| (u-\nu, \partial_t u) \right\|_X + \left\| (u-\nu, \partial_t u) \right\|_X^3 \right).
\]

Thus, we obtain the estimate
\[
\left\| \Psi(u-\nu, \partial_t u) \right\|_X \lesssim c \langle T \rangle \left( \left\| u_0 - \nu \right\|_{H^1_t} + \left\| u_1 \right\|_{L^2} \right) + c \langle T \rangle T \left[ \left\| (u-\nu, \partial_t u) \right\|_X + \left\| (u-\nu, \partial_t u) \right\|_X^3 \right],
\]

where \( c \) is the constant independent of \( T \) coming from the above estimates. Choosing \( R \)
so that
\[ c \langle T \rangle \left( \| u_0 - v \|_{H^1_0} + \| u_1 \|_{L^2} \right) \leq \frac{R}{2} \]
and taking \( T \) sufficiently small, depending on \( R \), we have that
\[ \| \Psi(u - v, \partial_t u) \|_X \leq R. \]

2. By possibly taking \( T \) smaller, we will next show that the map \( \Psi \) is a contraction mapping. Fix \( u \) and \( \tilde{u} \) with \((u - v, \partial_t u), (\tilde{u} - v, \partial_t \tilde{u}) \in B_R(0) \) and examine
\[ \| \Psi(u - v, \partial_t u) - \Psi(\tilde{u} - v, \partial_t \tilde{u}) \|_X. \]

Using (B.1) again and estimating as we did in step 1, then
\[
\begin{align*}
\langle T \rangle \left\| \nabla_\phi V(u) - \nabla_\phi V(\tilde{u}) \right\|_{L^1_t L^2_x} & \leq \langle T \rangle \left\| \frac{d}{ds} \nabla_\phi V(s(u - v) + (1 - s)(\tilde{u} - v)) \right\|_{L^1_t L^2_x} ds \\
& \leq \langle T \rangle \left\| \text{Hess}_\phi V(s(u - v) + (1 - s)(\tilde{u} - v))(u - \tilde{u}) \right\|_{L^1_t L^2_x} ds \\
& \leq \langle T \rangle \left\| 1 + |s(u - v) + (1 - s)(\tilde{u} - v)|^2 \right\|_{L^1_t L^2_x} \| u - \tilde{u} \|_{L^1_t L^2_x},
\end{align*}
\]
where we used assumption 4 of (1.9) to obtain the last inequality. Using the Holder inequality we thus have that
\[
\begin{align*}
\| \Psi(u - v, \partial_t u) - \Psi(\tilde{u} - v, \partial_t \tilde{u}) \|_X & \leq \langle T \rangle \left\| u - \tilde{u} \right\|_{L^1_t L^2_x} + \| u - \tilde{u} \|_{L^1_t L^2_x} + \| u - v \|_{L^1_t L^2_x} + \| \tilde{u} - v \|_{L^1_t L^2_x} \\
& \leq \langle T \rangle \left[ \left\| u - \tilde{u} \right\|_{C^0_t L^2_x} + \| u - \tilde{u} \|_{L^1_t L^2_x} + \| u - v \|_{C^0_t L^2_x} + \| \tilde{u} - v \|_{C^0_t L^2_x} \right] \\
& \leq \langle T \rangle \left[ \left\| u - \tilde{u} \right\|_{C^0_t L^2_x} + \| u - \tilde{u} \|_{C^0_t L^2_x} + \left\| u - \tilde{u} \right\|_{C^0_t L^2_x} + \| u - \tilde{u} \|_{C^0_t L^2_x} \right] \\
& \leq \langle T \rangle \left[ 1 + \| (u - v, \partial_t u) \|_X + \| (\tilde{u} - v, \partial_t \tilde{u}) \|_X + \| (u - \tilde{u}, \partial_t (u - \tilde{u})) \|_X \right],
\end{align*}
\]
where we used the Sobolev embedding theorem to obtain the last estimate. Thus,

$$
\| \Psi(u - v, \partial_t u) - \Psi(\tilde{u} - v, \partial_t \tilde{u}) \|_X \\
\leq \langle T \rangle T \left[ 1 + \| (u - v, \partial_t u) \|_X^2 + \| (\tilde{u} - v, \partial_t \tilde{u}) \|_X^2 \right] \| (u - \tilde{u}, \partial_t (u - \tilde{u})) \|_X
$$

and so by taking $T$ sufficiently small we then get that $\Psi$ is also a contraction mapping. Thus, $\Psi$ is a contraction mapping.

Thus (B.2) is locally well-posed.

\subsection*{B.2 Global Well Posedness}

We would like to show that a solution to (B.2) exists for all time. To show this, we first iterate the process outlined above. That is, we find a solution from $(0, T_1)$ using the fixed point argument given above. We then repeat this process, but now we take the initial data to be $u(T_1) = u_0$ and $\partial_t u(T_1) = u_1$. This will then give us another time interval $(T_1, T_2)$ for which (B.2) has a solution. We have thus found a solution on $(0, T_2)$. We can keep repeating this argument as long as

$$
\| u(T_i) \|_{H^1} + \| \partial_t u(T_i) \|_{L^2} < \infty.
$$

Let $(0, T_*)$ be the maximal time of existence and suppose $T_* < \infty$. This means that

$$
\lim_{t \to T_*} \left[ \| u(t) \|_{H^1} + \| \partial_t u(t) \|_{L^2} \right] = \infty,
$$

but for all times $0 < t < T_*$, then

$$
\| u(t) \|_{H^1} + \| \partial_t u(t) \|_{L^2} < \infty.
$$

We will use a conservation law to show that $T_*$ can’t be finite. The energy of (B.2) is given by

$$
E(u)(t) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \partial_t u^2 + \frac{1}{2} |\nabla u|^2 + V(u) \right\}
$$
and this is a conserved quantity (i.e. $E(u)(t) = E(u)(0)$). First, we have

$$
\|u\|_{L^2_x} \leq \|u(0)\|_{L^2_x} + \int_0^t \|\partial_t u\|_{L^2_x} \leq \|u(0)\|_{L^2_x} + \int_0^t E(u)^{1/2} \leq \|u(0)\|_{L^2_x} + tE(u)(0)^{1/2}.
$$

Secondly, we have that

$$
\|\partial_t u\|_{L^2_x} + \|\nabla u\|_{L^2_x} \leq E(u)(0)^{1/2}.
$$

Combining these two estimates yields

$$
\|u\|_{H^1_x} + \|\partial_t u\|_{L^2_x} \leq \|u(0)\|_{L^2_x} + (1 + t)E(u)(0)^{1/2}.
$$

Thus, we have that

$$
\lim_{t \to T_*} \left[ \|u\|_{H^1_x} + \|\partial_t u\|_{L^2_x} \right] \leq \|u(0)\|_{L^2_x} + (1 + T_*)E(u)(0)^{1/2} < \infty.
$$

Thus, we can extend the solution to (B.2) contradicting the maximality of the time of existence. Thus, $T_* = \infty$ and so (B.2) is globally well posed.

5 References


