Quantomorphisms and Quantized Energy Levels for Metaplectic-c Quantization

by

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Abstract

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Metaplectic-c quantization was developed by Robinson and Rawnsley as an alternative to the classical Kostant-Souriau quantization procedure with half-form correction. This thesis extends certain properties of Kostant-Souriau quantization to the metaplectic-c context. We show that the Kostant-Souriau results are replicated or improved upon with metaplectic-c quantization.

We consider two topics: quantomorphisms and quantized energy levels. If a symplectic manifold admits a Kostant-Souriau prequantization circle bundle, then its Poisson algebra is realized as the space of infinitesimal quantomorphisms of that circle bundle. We present a definition for a metaplectic-c quantomorphism, and prove that the space of infinitesimal metaplectic-c quantomorphisms exhibits all of the same properties that are seen in the Kostant-Souriau case.

Next, given a metaplectic-c prequantized symplectic manifold \((M, \omega)\) and a function \(H \in \mathcal{C}^\infty(M)\), we propose a condition under which \(E\), a regular value of \(H\), is a quantized energy level for the system \((M, \omega, H)\). We prove that our definition is dynamically invariant: if two functions on \(M\) share a regular level set, then the quantization condition over that level set is identical for both functions. We calculate the quantized energy levels for the \(n\)-dimensional harmonic oscillator and the hydrogen atom, and obtain the quantum mechanical predictions in both cases. Lastly, we generalize the quantization condition to a level set of a family of Poisson-commuting functions, and show that in the special case of a completely integrable system, it reduces to a Bohr-Sommerfeld condition.
For David.
Acknowledgements

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I would not be a mathematician if it weren’t for my father. I would not have gotten through the past year if it weren’t for my mother. Thank you both.
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Chapter 1

Introduction

In physics, the phase space for a classical mechanical system contains position and momentum coordinates for the component particles. The physically observable quantities for the system are functions on phase space. The quantum mechanical version of such a system consists of a Hilbert space of quantum states, and an identification of observables with self-adjoint operators.

Geometric quantization is a branch of symplectic geometry that generalizes the concept of quantization to abstract symplectic manifolds. Given a symplectic manifold $(M,\omega)$ and a function $H \in C^\infty(M)$, the corresponding Hamiltonian vector field $\xi_H$ on $M$ has integral curves that satisfy Hamilton’s equations in local canonical coordinates. Thus we can view $(M,\omega)$ as a classical phase space, and $H$ as a Hamiltonian energy function. To quantize $(M,\omega)$, we require a procedure for constructing a suitable Hilbert space of quantum states, and a Lie algebra isomorphism from $C^\infty(M)$, or a subalgebra of it, to operators on those states.

The original formulation of geometric quantization is due to Kostant [14] and Souriau [22]. Their quantization procedure requires that $(M,\omega)$ admit a prequantization circle bundle and a metaplectic structure. In Chapter 2, after we review the necessary elements of symplectic geometry and principal bundles, we present the Kostant-Souriau quantization procedure with half-form correction.

The metaplectic-c group is a circle extension of the symplectic group. Metaplectic-c quantization, which was developed by Hess [13] and Robinson and Rawnsley [17], with earlier work by Czyz [5]. It is a variant of Kostant-Souriau quantization in which the prequantization circle bundle and metaplectic structure are replaced by a single object called a metaplectic-c prequan-
Chapter 1. Introduction

Hess, Robinson and Rawnsley proved that metaplectic-c quantization can be applied to all systems that admit Kostant-Souriau quantizations, and to some where the Kostant-Souriau process fails. In the final section of Chapter 2, we give a detailed description of a metaplectic-c prequantization and its properties.

The objective of this document is to explore some of the ways in which metaplectic-c quantization replicates or improves upon known results for Kostant-Souriau quantization. We begin in Chapter 3 by examining the concept of a quantomorphism: that is, an isomorphism of prequantization bundles that preserves all of their structures. In the context of a Kostant-Souriau prequantization circle bundle for \((M, \omega)\), it is known that the Lie algebra of infinitesimal quantomorphisms is isomorphic to \(C^\infty(M)\). We formulate a definition of a metaplectic-c quantomorphism, and prove that the space of infinitesimal metaplectic-c quantomorphisms is again isomorphic to \(C^\infty(M)\).

The remainder of the document focuses on the concept of a quantized energy level. In quantum mechanics, the energy spectrum for a spatially confined particle is discrete: only certain energy levels are permitted. Various interpretations of this phenomenon can be found in the literature for different forms of geometric quantization. Given \((M, \omega)\) and a function \(H \in C^\infty(M)\), we propose a new definition for a quantized energy level \(E\) of \(H\) in the case where \((M, \omega)\) admits a metaplectic-c prequantization. Our definition is evaluated over a regular level set of \(H\), and it does not require either symplectic reduction or a choice of polarization.

We present our definition in Chapter 4, and show that its properties compare favourably with others that have been studied. Our main result, Theorem 4.3.5, states that if two functions \(H_1, H_2 \in C^\infty(M)\) are such that \(H_1^{-1}(E_1) = H_2^{-1}(E_2)\) for regular values \(E_1, E_2\), then \(E_1\) is a quantized energy level for \((M, \omega, H_1)\) if and only if \(E_2\) is a quantized energy level for \((M, \omega, H_2)\). That is, our condition is a geometric property of the level set, and does not depend on the dynamics of a specific choice of \(H\). As such, we refer to Theorem 4.3.5 as the dynamical invariance theorem. Also in Chapter 4, we demonstrate a computational technique for lifting a local change of coordinates on \(M\) to the level of metaplectic-c prequantizations, and use this technique to evaluate the quantized energy levels for the \(n\)-dimensional harmonic oscillator.

In Chapter 5, we calculate the quantized energy levels of the hydrogen atom, using the physical model that is identical to the Kepler problem. This calculation is more technically de-
manding than that for the harmonic oscillator. We use the Ligon-Schaaf regularization map to transport the problem to $TS^3$, and apply a further transformation to relate the negative quantized energy levels of the hydrogen atom to the positive quantized energies of a free particle on $S^3$. The latter step relies on the dynamical invariance property. We show that the metaplectic-c quantized energy levels agree with the physical prediction from quantum mechanics.

Lastly, in Chapter 6, we show that our quantized energy condition generalizes from one function on $M$ to a family of Poisson-commuting functions. After proving a generalized version of the dynamical invariance theorem, we consider the special case of a completely integrable system, where the size of the Poisson-commuting family is maximal. In that case, the quantized energy condition simplifies to a Bohr-Sommerfeld condition. Thus our quantized energy condition provides a framework that encompasses both the quantized energy levels of a single function $H$ and the Bohr-Sommerfeld leaves of the real polarization generated by a completely integrable system.

Significant portions of this document have already been published or posted on the arXiv. Chapter 3 appears in [23], Chapter 4 appears in [24], and Chapter 5 appears in [25]. Portions of the abstract, Chapter 2 and this introduction are amalgams of material from all three papers. Minor truncations and edits have been performed for the sake of internal consistency and to eliminate redundancy. Additional background material has been added to Chapter 2, and the content of Chapter 6 does not yet appear elsewhere.
Chapter 2

Background

Our starting point is a symplectic manifold $(M^{2n}, \omega)$: that is, a $2n$-dimensional manifold equipped with a closed, nondegenerate two-form. We think of this object as a classical mechanical phase space. Classical observables are elements of $C^\infty(M)$, the smooth real-valued functions on $M$. One objective of geometric quantization is to build a Hermitian line bundle over $M$ and a Lie algebra homomorphism from $C^\infty(M)$ to operators on sections of the bundle, thereby reproducing the transition from classical to quantum mechanics in which observables become operators on a Hilbert space of quantum states. In Section 2.1, we review the basic facts that we will require concerning symplectic manifolds, circle bundles and complex line bundles, and connections.

In Section 2.2, we describe the Kostant-Souriau quantization procedure. We begin with the prequantization stage, in which $(M, \omega)$ is required to admit a prequantization line bundle $(L, \nabla)$. Then we introduce a metaplectic structure and a choice of polarization $F$, and sketch how to construct the complex line bundle of half-forms $\wedge^{1/2} F$. We conclude with the quantization stage, in which the functions in $C^\infty(M)$ whose flows preserve the polarization are mapped to operators on polarized sections of $L \otimes \wedge^{1/2} F$.

Finally, in Section 2.3, we summarize the construction of a metaplectic-c prequantization, as developed by Hess [13] and Robinson and Rawnsley [17]. A metaplectic-c prequantization is an object that performs the functions of a prequantization line bundle and a metaplectic structure simultaneously. As we will see in later chapters, certain features of Kostant-Souriau quantization with half-form correction can be reproduced by a metaplectic-c prequantization.
alone, without requiring a choice of polarization. Since our work focuses on the prequantization stage of Robinson and Rawnsley’s procedure, we omit a detailed discussion of the quantization stage. Some elements of it are presented in Section 3.3.3 in the context of metaplectic-c quantomorphisms.

Some global remarks concerning notation: for any vector field \( \xi \), the Lie derivative with respect to \( \xi \) is written \( L_\xi \). The space of smooth vector fields on a manifold \( P \) is denoted by \( \mathcal{X}(P) \). Given a smooth map \( F : P \to M \) and a vector field \( \xi \in \mathcal{X}(P) \), we write \( F_*\xi \) for the pushforward of \( \xi \) only if the result is a well-defined vector field on \( M \). If \( P \) is a bundle over \( M \), \( \Gamma(P) \) denotes the space of smooth sections of \( P \), where the base is always taken to be the symplectic manifold \( M \). Planck’s constant will only appear in the form \( \hbar \).

## 2.1 Symplectic Manifolds and Bundles

### 2.1.1 Hamiltonian vector fields and the Poisson algebra

Let \((M, \omega)\) be a symplectic manifold. Given \( f \in C^\infty(M) \), define its Hamiltonian vector field \( \xi_f \in \mathcal{X}(M) \) by

\[
\xi_f \cdot \omega = df.
\]

Define the Poisson bracket on \( C^\infty(M) \) by

\[
\{f, g\} = \xi_f g = -\omega(\xi_f, \xi_g), \quad \forall f, g \in C^\infty(M).
\]

A standard calculation establishes that for all \( f, g \in C^\infty(M) \),

\[
[\xi_f, \xi_g] = \xi_{\{f, g\}}.
\]

### 2.1.2 Circle bundles and connection one-forms

Let \( Y \xrightarrow{p} M \) be a right principal \( U(1) \) bundle over \( M \). For any \( \theta \in \mathfrak{u}(1) \), the Lie algebra of \( U(1) \), let \( \partial_\theta \) be the vector field on \( Y \) given by

\[
\partial_\theta(y) = \frac{d}{dt} \bigg|_{t=0} y \cdot \exp(t\theta), \quad \forall y \in Y.
\]
The vector field $\partial_\theta$ is called the vector field generated by the infinitesimal action of $\theta \in \mathfrak{u}(1)$.

A $\mathfrak{u}(1)$-valued one-form $\gamma$ on $Y$ is called a connection one-form if $\gamma$ is invariant under the right principal action, and for all $\theta \in \mathfrak{u}(1)$, $\gamma(\partial_\theta) = \theta$. If $Y$ is equipped with a connection one-form $\gamma$, then there is a two-form $\varpi$ on $M$, called the curvature of $\gamma$, such that $d\gamma = p^*\varpi$.

For any $\xi \in \mathcal{X}(M)$, let $\tilde{\xi}$ be the lift of $\xi$ to $Y$ that is horizontal with respect to $\gamma$. That is, $p_*\tilde{\xi} = \xi$ and $\gamma(\tilde{\xi}) = 0$. For any $\theta \in \mathfrak{u}(1)$, note that $p_*\partial_\theta = 0$, which implies that $p_*[\tilde{\xi}, \partial_\theta] = 0$ and $\gamma([\tilde{\xi}, \partial_\theta]) = -(p^*\varpi)(\tilde{\xi}, \partial_\theta) = 0$. Therefore $[\tilde{\xi}, \partial_\theta] = 0$ for all $\theta$.

### 2.1.3 Complex line bundles and connections

Now let $L \to M$ be a complex line bundle over $M$. A connection $\nabla$ on $L$ allows us to take the derivative of a section $s$ of $L$ in the direction of a vector field $\xi$ on $M$. It can be viewed as a map from sections of $L$ to $L$-valued one-forms on $M$ in the sense that, if $s$ is a section of $L$ and $\xi$ is a vector field on $M$, then at every point $m \in M$, $\nabla s$ acts on the vector $\xi(m)$ to yield a value in $L_m$, which we interpret as the derivative of $s$ in the direction of that vector. Formally, a connection on $L$ is a map

$$\nabla : \Gamma(L) \to \Gamma(L) \otimes \Omega^1(M)$$

such that, for all $r, s \in \Gamma(L)$, all $\xi \in \mathcal{X}(M)$, and all $f \in C^\infty(M)$,

\[
\nabla_\xi (r + s) = \nabla_\xi r + \nabla_\xi s,
\]

\[
\nabla_\xi (fs) = f \nabla_\xi s + (\xi f)s.
\]

If $(Y, \gamma) \xrightarrow{p} M$ is a circle bundle with connection one-form over $M$, then it has an associated Hermitian line bundle with connection $(L, \nabla)$ over $M$. The line bundle $L$ is given by $L = Y \times_{U(1)} \mathbb{C}$, with Hermitian structure induced from the usual Hermitian inner product on $\mathbb{C}$. We write an element of $L$ as an equivalence class $[y, z]$ with $y \in Y$ and $z \in \mathbb{C}$. The connection $\nabla$ on $L$ is constructed from the connection one-form $\gamma$ through the following process.

Given any $s \in \Gamma(L)$, define the map $\tilde{s} : Y \to \mathbb{C}$ so that $[y, \tilde{s}(y)] = s(p(y))$ for all $y \in Y$. Then $\tilde{s}$ has the equivariance property

$$\tilde{s}(y \cdot \lambda) = \lambda^{-1}\tilde{s}(y), \quad \forall \ y \in Y, \ \lambda \in U(1).$$
Conversely, any map $\tilde{s} : Y \to \mathbb{C}$ with the above equivariance property can be used to construct a section $s$ of $L$ by setting $s(m) = [y, \tilde{s}(y)]$ for all $m \in M$ and any $y \in Y$ such that $p(y) = m$.

Let $\xi \in \mathcal{X}(M)$ be given, and let $\tilde{\xi}$ be its horizontal lift to $Y$. If $\tilde{s} : Y \to \mathbb{C}$ is an equivariant map, then so is $\tilde{\xi}\tilde{s}$. This follows from the fact that $[\tilde{\xi}, \partial_\theta] = 0$ for all $\theta \in u(1)$. Define the connection $\nabla$ on $L$ so that for any $\xi \in \mathcal{X}(M)$ and $s \in \Gamma(L)$, $\nabla_\xi s$ is the section of $L$ that satisfies

\[ \nabla_\xi s = \tilde{\xi}\tilde{s}. \]

2.1.4 Holonomy

Let $(Y, \gamma)$ be a circle bundle with connection one-form over $M$. Let $u : \mathbb{R} \to M$ be a path in $M$ such that $u(t + 1) = u(t)$ for all $t$. Given a starting point $y_0 \in Y_{u(0)}$, there is a unique lift of $u(t)$ to a path $\tilde{u}(t)$ in $Y$ such that $\tilde{u}(0) = y_0$ and every tangent vector $\tilde{u}(t)$ satisfies $\gamma(\dot{\tilde{u}}(t)) = 0$. Such a lift is called horizontal with respect to $\gamma$. Since $\tilde{u}(t)$ is a lift of $u(t)$, $\tilde{u}(1) \in Y_{u(0)}$. The resulting map $y_0 \mapsto \tilde{u}(1)$ is automorphism of the fiber $Y_{u(0)}$, called the holonomy$^1$ of $\gamma$ over $u(t)$. If the horizontal lift is itself a closed loop, then $\gamma$ is said to have trivial holonomy over $u(t)$.

There is an equivalent formulation of holonomy on the corresponding line bundle with connection $(L, \nabla)$. Let $\mathcal{C} \subset M$ be the image of $u$ in $M$, and let $\tilde{u}(t)$ be a lift of $u(t)$ to $L$. The image of $\tilde{u}(t)$ can be thought of as a section $s$ of $L$, defined only over $\mathcal{C}$. The lift $\tilde{u}(t)$ is called horizontal if $\nabla_{\dot{u}(t)} s = 0$ for all $t$. Given a starting point $\tilde{u}(0)$ in the fiber $L_{u(0)}$, the condition of being horizontal uniquely determines the rest of the lift. As before, $\tilde{u}(1) \in L_{u(0)}$, so we obtain a linear map $\tilde{u}(0) \mapsto \tilde{u}(1)$ from the fiber $L_{u(0)}$ to itself, called the holonomy of $\nabla$ over $u(t)$.

2.2 Kostant-Souriau Quantization and the Half-Form Correction

Geometric quantization in its original form was developed in the 1960’s by Kostant [14] and Souriau [22]. The half-form correction was added by Blattner, Kostant and Sternberg [2]. The following overview is based on the detailed treatments that can be found in [11, 20, 26].

---

$^1$This material is standard, but our main source is the exposition in [12].
2.2.1 Prequantization

Let \((M, \omega)\) be a symplectic manifold. In the prequantization stage, we require \((M, \omega)\) to admit a prequantization circle bundle.

**Definition 2.2.1.** A **prequantization circle bundle** for \((M, \omega)\) is a right principal \(U(1)\) bundle \(Y \xrightarrow{p} M\), together with a connection one-form \(\gamma\) on \(Y\) satisfying \(d\gamma = \frac{1}{\hbar} p^* \omega\).

If \((M, \omega)\) admits a prequantization circle bundle, then the associated Hermitian line bundle with connection \((L, \nabla)\) is called a **prequantization line bundle**. The manifold \((M, \omega)\) admits a prequantization circle bundle (equivalently, a prequantization line bundle) if and only if the cohomology class \(\left[ \frac{1}{2\pi \hbar} \omega \right] \in H^2(M, \mathbb{R})\) is integral.

Assume that \((M, \omega)\) admits a prequantization circle bundle \((Y, \gamma)\), with corresponding prequantization line bundle \((L, \nabla)\). One of the goals of the prequantization process is to produce a representation \(r : C^\infty(M) \to \text{End} \Gamma(L)\). To be consistent with quantum mechanics in the case of a physically realizable system, the map \(r\) is required to satisfy the following axioms, which are based on an analysis by Dirac [6] on the relationship between classical and quantum mechanical observables.

1. \(r(1)\) is the identity map on \(\Gamma(L)\),
2. for all \(f, g \in C^\infty(M)\), \([r(f), r(g)] = i\hbar r(\{f, g\})\) (up to sign convention).

In the context of Kostant-Souriau prequantization, a suitable map is given by

\[
r(f) = i\hbar \nabla_{\xi_f} + f, \quad \forall f \in C^\infty(M).
\]

The fact that this map satisfies condition (2) can be verified by direct computation; as we will show in Section 3.2.3, it can also be viewed as a consequence of the Lie algebra isomorphism between \(C^\infty(M)\) and the space of infinitesimal quantomorphisms of \((Y, \gamma)\).

2.2.2 Quantization and the half-form correction

The phase space for a classical mechanical system consists of all possible combinations of position and momentum coordinates. However, the wave functions for the corresponding quantum
system depend only on the position coordinates (or, more generally, on a complete set of
commuting observables). This observation motivates the introduction of a structure called a
polarization.

Assume that \((M, \omega)\) admits a prequantization line bundle \((L, \nabla)\). A \textit{polarization} \(^2 \text{F}\) is an
involutive Lagrangian subbundle of the complexified tangent bundle \(TM^\mathbb{C}\) such that \(\dim(F_m \cap
\overline{F}_m)\) is constant over all \(m \in M\). Let a choice of \(F\) be fixed. The \textit{canonical bundle} for \(F\),
denoted by \(K^F\), is the top exterior power of the annihilator of \(F\) in \((T^*M)^\mathbb{C}\). The \textit{half-form bundle}, \(\Lambda^{1/2}F\), is a choice of square root of \(K^F\), when such a square root exists. A \textit{half-form} is
a section of \(\Lambda^{1/2}F\) over \(M\).

To guarantee that the half-form bundle \(\Lambda^{1/2}F\) exists, we require that \((M, \omega)\) admit a meta-
plectic structure, which is a lifting of the structure group for \((M, \omega)\) from the symplectic
group to its unique connected double cover, the metaplectic group. The manifold \((M, \omega)\) ad-
mits such a lifting if and only if the first Chern class \(c_1(TM) \in H^2(M, \mathbb{Z})\) is even: that is,
\[\frac{1}{2}c_1(TM) \in H^2(M, \mathbb{Z}).\] When this condition is satisfied, the equivalence classes of metaplectic
structures for \((M, \omega)\) are in one-to-one correspondence with \(H^1(M, \mathbb{Z}_2)\).

A metaplectic structure on \((M, \omega)\) induces a metalinear structure on the polarization \(F\), from
which the half-form bundle \(\Lambda^{1/2}F\) can be constructed. From \(K^F\), the half-form bundle inherits
a partial connection \(\nabla_\zeta\), defined for all \(\zeta \in \mathcal{X}(M)\) such that \(\zeta\) is parallel to the polarization
\(F\). It also inherits a partial Lie derivative \(L_\xi\), defined for all \(\xi \in \mathcal{X}(M)\) such that the flow of \(\xi\)
preserves \(F\).

Let \(C^\infty_F(M)\) denote the subalgebra of \(C^\infty(M)\) consisting of functions whose Hamiltonian
flows preserve the polarization \(F\). The Kostant-Souriau representation with half-form correction, \(r_F : C^\infty_F(M) \rightarrow \text{End}(L \otimes \Lambda^{1/2}F)\), is defined by
\[r_F(f) = r(f) \otimes I + I \otimes L_\xi f, \quad \forall f \in C^\infty_F(M).\]

This map is a Lie algebra homomorphism.

A section \(s \otimes \nu \in \Gamma(L \otimes \Lambda^{1/2}F)\) is called \textit{polarized} if \(\nabla_\zeta s = 0\) and \(\nabla_\zeta \nu = 0\) for all \(\zeta \in \mathcal{X}(M)\)
such that \(\zeta\) is parallel to \(F\). Let \(\Gamma_P(L \otimes \Lambda^{1/2}F)\) denote the space of polarized sections. The

\(^2\text{Other technical constraints, such as positivity, may be imposed on }F, \text{ but since we will not be performing any explicit computations with polarizations, we will not address these details.}\)
operators $r_F(f)$ restrict to operators on $\Gamma_F(L \otimes \wedge^{1/2} F)$. The polarized sections with compact support form the pre-Hilbert space of quantum states, where the Hermitian inner product arises from the Hermitian structure on $L$, and a pairing from half-forms to densities. This completes the construction of the Lie algebra representation and the Hilbert space as suggested by quantum mechanics.

2.3 Metaplectic-c Quantization

The Kostant-Souriau quantization procedure imposes two conditions on $(M, \omega)$: namely, that it admit a prequantization circle bundle and a metaplectic structure. Each of these requirements comes with a cohomological condition to be satisfied. The motivation behind metaplectic-c quantization is to replace the prequantization circle bundle and metaplectic structure with a single bundle that can play both roles simultaneously.

The material in this section is a summary of results originally presented by Hess [13] and Robinson and Rawnsley [17]. More detail, including proofs of the properties that we state, can be found in [13, 17]. A similar structure, called a spin-c prequantization, was studied in [3, 8, 9, 10].

2.3.1 Definitions on vector spaces

Let $V$ be an $n$-dimensional complex vector space with Hermitian inner product $\langle \cdot, \cdot \rangle$. If we view $V$ as a $2n$-dimensional real vector space, then the action of the scalar $i \in \mathbb{C}$ becomes the real automorphism $J : V \rightarrow V$. Define the real bilinear form $\Omega$ on $V$ by

$$\Omega(v, w) = \text{Im} \langle v, w \rangle, \quad \forall v, w \in V.$$ 

Then $(V, \Omega)$ is a $2n$-dimensional symplectic vector space. The Hermitian and symplectic structures are compatible in the sense that

$$\langle v, w \rangle = \Omega(Jv, w) + i\Omega(v, w), \quad \forall v, w \in V.$$ 

The symplectic group $\text{Sp}(V)$ is the group of real automorphisms $g : V \rightarrow V$ such that
\[ \Omega(gv, gw) = \Omega(v, w) \] for all \( v, w \in V \). Since the fundamental group for \( \text{Sp}(V) \) is \( \mathbb{Z} \), \( \text{Sp}(V) \) has a unique connected double cover called the *metaplectic group*, denoted by \( \text{Mp}(V) \). Let \( \text{Mp}(V) \rightarrow \text{Sp}(V) \) denote the covering map. The *metaplectic-c group* \( \text{Mp}^c(V) \) is defined to be

\[ \text{Mp}^c(V) = \text{Mp}(V) \times_{\mathbb{Z}_2} U(1), \]

where \( \mathbb{Z}_2 \subset U(1) \) is the usual subgroup \( \{1, -1\} \), and where \( \mathbb{Z}_2 \subset \text{Mp}(V) \) consists of the two preimages of \( I \in \text{Sp}(V) \) under the covering map \( \sigma \).

By construction, \( \text{Mp}^c(V) \) contains \( U(1) \) and \( \text{Mp}(V) \) as subgroups. The inclusion of each subgroup into \( \text{Mp}^c(V) \) yields a short exact sequence and a group homomorphism on \( \text{Mp}^c(V) \). One such sequence is

\[ 1 \rightarrow U(1) \rightarrow \text{Mp}^c(V) \xrightarrow{\sigma} \text{Sp}(V) \rightarrow 1, \]

where the group homomorphism \( \sigma \) is called the *projection map*, and its restriction to \( \text{Mp}(V) \subset \text{Mp}^c(V) \) is the double covering map. The other is

\[ 1 \rightarrow \text{Mp}(V) \rightarrow \text{Mp}^c(V) \xrightarrow{\eta} U(1) \rightarrow 1, \]

where the group homomorphism \( \eta \) is called the *determinant map*, and it has the property that for any \( \lambda \in U(1) \subset \text{Mp}^c(V) \), \( \eta(\lambda) = \lambda^2 \). At the level of Lie algebras, the map \( \sigma_* \oplus \frac{1}{2} \eta_* \) yields the identification

\[ \mathfrak{mp}^c(V) = \mathfrak{sp}(V) \oplus \mathfrak{u}(1). \]

Given \( g \in \text{Sp}(V) \), let

\[ C_g = \frac{1}{2}(g - JgJ). \]

By construction, \( C_g \) is an \( \mathbb{R} \)-linear map that commutes with \( J \), so it is a complex linear transformation of \( V \). The determinant of \( C_g \) as a complex transformation is written \( \text{Det}_C C_g \). It can be shown that \( C_g \) is always invertible, so \( \text{Det}_C C_g \) is a nonzero complex number.

A useful embedding of \( \text{Mp}^c(V) \) into \( \text{Sp}(V) \times \mathbb{C} \setminus \{0\} \) can be defined as follows. An element \( a \in \text{Mp}^c(V) \) is mapped to the pair \( (g, \mu) \in \text{Sp}(V) \times \mathbb{C} \setminus \{0\} \), where \( g \in \text{Sp}(V) \) satisfies \( \sigma(a) = g \), and \( \mu \in \mathbb{C} \setminus \{0\} \) satisfies \( \eta(a) = \mu^2 \text{Det}_C C_g \). The latter condition defines \( \mu \) up to a sign, and
we adopt the convention that the parameters of \( I \in \text{Mp}^c(V) \) are \((I, 1)\). This choice uniquely determines the sign of \( \mu \) for all \( a \in \text{Mp}^c(V) \). Following [17], we refer to \((g, \mu)\) as the *parameters* of \( a \).

We note two properties of this parametrization, both of which will be useful in later chapters. First, if \( a \in \text{Mp}(V) \subset \text{Mp}^c(V) \), then \( \eta(a) = 1 \), and so the parameters of \( a \) are \((g, \mu)\) where \( \sigma(a) = g \) and \( \mu^2 \text{Det}_C C_g = 1 \). Second, if \( \lambda \in U(1) \subset \text{Mp}^c(V) \), then the parameters of \( \lambda \) are \((I, \lambda)\), and for any other element \( a \in \text{Mp}^c(V) \) with parameters \((g, \mu)\), the parameters of the product \( a \lambda \) are \((g, \mu \lambda)\).

### 2.3.2 Definitions over a symplectic manifold

Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold, and assume that a \(2n\)-dimensional model symplectic vector space \((V, \Omega)\) with compatible complex structure \(J\) has been fixed. We think of the symplectic frame bundle for \((M, \omega)\) as being modeled on \(\text{Sp}(V)\), in the following sense.

**Definition 2.3.1.** The **symplectic frame bundle** \(\text{Sp}(M, \omega) \xrightarrow{\rho} M\) is a right principal \(\text{Sp}(V)\) bundle over \(M\) given by

\[
\text{Sp}(M, \omega)_m = \{ b : V \to T_m M : b \text{ is a symplectic linear isomorphism} \}, \quad \forall m \in M.
\]

The group \(\text{Sp}(V)\) acts on the fibers by precomposition.

In the previous section, we defined a metaplectic structure for \((M, \omega)\) to be a lifting of the structure group for \((M, \omega)\) from \(\text{Sp}(V)\) to \(\text{Mp}(V)\). Analogously, a metaplectic-c structure for \((M, \omega)\) is a lifting of the structure group to \(\text{Mp}^c(V)\).

**Definition 2.3.2.** A **metaplectic-c structure** for \((M, \omega)\) consists of a right principal \(\text{Mp}^c(V)\) bundle \(P \xrightarrow{\Pi} M\) and a map \(P \xrightarrow{\Sigma} \text{Sp}(M, \omega)\) such that

\[
\Sigma(q \cdot a) = \Sigma(q) \cdot \sigma(a), \quad \forall q \in P, \forall a \in \text{Mp}^c(V),
\]
and such that the following diagram commutes.

$$
\begin{array}{ccc}
P & \xrightarrow{\Sigma} & \text{Sp}(M, \omega) \\
\Pi \downarrow & & \downarrow \rho \\
(M, \omega) & & 
\end{array}
$$

To complete the construction of a metaplectic-c prequantization, we equip the metaplectic-c structure with a $u(1)$-valued one-form whose properties are similar to that of the connection one-form on a prequantization circle bundle.

**Definition 2.3.3.** A **metaplectic-c prequantization** of $(M, \omega)$ is a metaplectic-c structure $(P, \Sigma)$ for $(M, \omega)$, together with a $u(1)$-valued one-form $\gamma$ on $P$ such that:

1. $\gamma$ is invariant under the principal $Mp^c(V)$ action;
2. For any $\alpha \in mp^c(V)$, if $\partial_\alpha$ is the vector field on $P$ generated by the infinitesimal action of $\alpha$, then $\gamma(\partial_\alpha) = \frac{1}{2} \eta_\ast \alpha$;
3. $d\gamma = \frac{1}{i\hbar} \Pi_\ast \omega$.

Note that if $(P, \Sigma, \gamma)$ is a metaplectic-c prequantization for $(M, \omega)$, then $(P, \gamma)$ is a circle bundle with connection one-form over $\text{Sp}(M, \omega)$. The circle that acts on the fibers of $P$ is the center $U(1) \subset Mp^c(V)$. We will view $P$ as a bundle over $M$ or a bundle over $\text{Sp}(M, \omega)$ as the circumstance demands.

For any $\alpha \in mp^c(V)$, we can write $\alpha = \kappa \oplus \tau$ under the identification of $mp^c(V)$ with $\mathfrak{sp}(V) \oplus u(1)$. Then $\gamma(\partial_\alpha) = \tau$, and naturality of the exponential map and equivariance of the map $\Sigma$ with respect to $\sigma$ ensure that $\Sigma_\ast \partial_\alpha = \partial_\kappa$, where $\partial_\kappa$ is the vector field on $\text{Sp}(M, \omega)$ generated by the infinitesimal action of $\kappa \in \mathfrak{sp}(V)$. In fact, $\partial_\alpha$ is uniquely specified by the two conditions $\gamma(\partial_\alpha) = \tau$ and $\Sigma_\ast \partial_\alpha = \partial_\kappa$, an observation that will be useful in Chapter 3.

There is a cohomology constraint that must be satisfied in order for $(M, \omega)$ to admit a metaplectic-c prequantization, and that condition is very closely related to the ones we saw for the prequantization circle bundle and metaplectic structure. Specifically, $(M, \omega)$ is metaplectic-c prequantizable if and only if the cohomology class $\left[ \frac{1}{2\pi i} \omega \right] + \frac{1}{2} c_1(TM) \in H^2(M, \mathbb{R})$ is integral.
Thus, if \((M, \omega)\) admits a prequantization circle bundle and a metaplectic structure, then it is metaplectic-c prequantizable, but the converse is not true.

Two metaplectic-c prequantizations for \((M, \omega)\) are considered equivalent if there is a diffeomorphism between them that preserves the one-forms and commutes with the respective maps to \(\text{Sp}(M, \omega)\). If \((M, \omega)\) is metaplectic-c prequantizable, then the set of equivalence classes of metaplectic-c prequantizations for \((M, \omega)\) is in one-to-one correspondence with the locally constant cohomology group \(H^1(M, U(1))\), which also parametrizes the circle bundles with flat connection over \((M, \omega)\). An observation that will be used repeatedly is that if \(M\) is simply connected, then the metaplectic-c prequantization of \((M, \omega)\) is unique up to isomorphism.
Chapter 3

Metaplectic-c Quantomorphisms

3.1 Introduction

A prequantization circle bundle for a symplectic manifold \((M, \omega)\) consists of a circle bundle \(Y \to M\) and a connection one-form \(\gamma\) on \(Y\) such that \(d\gamma = \frac{1}{i\hbar} \omega\). Souriau [22] defined a quantomorphism between two prequantization circle bundles \((Y_1, \gamma_1) \to (M_1, \omega_1)\) and \((Y_2, \gamma_2) \to (M_2, \omega_2)\) to be a diffeomorphism \(K : Y_1 \to Y_2\) such that \(K^* \gamma_2 = \gamma_1\). This condition implies that \(K\) is equivariant with respect to the principal circle actions. Souriau then defined the infinitesimal quantomorphisms of a prequantization circle bundle \((Y, \gamma)\) to be the vector fields on \(Y\) whose flows are quantomorphisms. Kostant [14] proved that the space of infinitesimal quantomorphisms, which we denote \(Q(Y, \gamma)\), is isomorphic to the Poisson algebra \(C^\infty(M)\). In Section 3.2, we present an explicit construction of the isomorphism from \(Q(Y, \gamma)\) to \(C^\infty(M)\).

The objective of the rest of the chapter is to adapt the concept of an infinitesimal quantomorphism to the case where \((M, \omega)\) admits a metaplectic-c prequantization \((P, \Sigma, \gamma)\). In Section 3.3, we define a metaplectic-c quantomorphism, which is a diffeomorphism of metaplectic-c prequantizations that preserves all of their structures. Our definition is based on Souriau’s, but includes a condition that is unique to the metaplectic-c context. We then use the metaplectic-c quantomorphisms to define \(Q(P, \Sigma, \gamma)\), the space of infinitesimal metaplectic-c quantomorphisms. We show that every property that was proved for \(Q(Y, \gamma)\) has a parallel for \(Q(P, \Sigma, \gamma)\). In particular, \(Q(P, \Sigma, \gamma)\) is isomorphic to the Poisson algebra \(C^\infty(M)\). The construction in Section 3.2 is used as a model for the proofs in Section 3.3. We indicate when the calculations are analogous,
and when the metaplectic-c case requires additional steps.

3.2 Kostant-Souriau Quantomorphisms

In this section, we construct a Lie algebra isomorphism from $C^\infty(M)$ to the space of infinitesimal quantomorphisms. As we have already noted, the fact that these algebras are isomorphic was originally stated by Kostant [14] in the context of line bundles with connection. His proof can be reconstructed from several propositions across Sections 2 – 4 of [14]. Kostant’s isomorphism is also stated by Śniatycki [20], but much of the proof is left as an exercise. We are not aware of a source in the literature for a self-contained proof that uses the language of principal bundles, and this is one of our reasons for performing an explicit construction here.

The other goal of this section is to motivate the analogous constructions for a metaplectic-c prequantization, which will be the subject of Section 3.3. Each result that we present for Kostant-Souriau prequantization will have a parallel in the metaplectic-c case. When the proofs are identical, we will simply refer back to the work shown here, thereby allowing Section 3.3 to focus on those features that are unique to metaplectic-c structures.

3.2.1 Infinitesimal quantomorphisms of a prequantization circle bundle

**Definition 3.2.1.** Let $(Y_1, \gamma_1) \xrightarrow{p_1} (M_1, \omega_1)$ and $(Y_2, \gamma_2) \xrightarrow{p_2} (M_2, \omega_2)$ be prequantization circle bundles for two symplectic manifolds. A diffeomorphism $K : Y_1 \to Y_2$ is called a quantomorphism if $K^* \gamma_2 = \gamma_1$.

Let $K : Y_1 \to Y_2$ be a quantomorphism. Notice that for any $\theta \in u(1)$, the vector field $\partial_\theta$ on $Y_1$ is completely specified by the conditions $\gamma_1(\partial_\theta) = \theta$ and $d\gamma_1(\partial_\theta) = 0$, and the same is true on $Y_2$. Since $K^* \gamma_2 = \gamma_1$, we see that $K^* \partial_\theta = \partial_\theta$ for all $\theta$, and so $K$ is equivariant with respect to the principal circle actions.

**Definition 3.2.2.** Let $(Y, \gamma) \xrightarrow{p} (M, \omega)$ be a prequantization circle bundle. An infinitesimal quantomorphism of $(Y, \gamma)$ is a vector field $\zeta \in \mathcal{X}(Y)$ whose flow $\phi^t$ on $Y$ is a quantomorphism from its domain to its range for each $t$. The space of infinitesimal quantomorphisms of $(Y, \gamma)$ is denoted by $\mathcal{Q}(Y, \gamma)$. 
Let $\zeta \in \mathcal{X}(Y)$ have flow $\phi^t$. The connection form $\gamma$ is preserved by $\phi^t$ if and only if $L_\zeta \gamma = 0$. Therefore the space of infinitesimal quantomorphisms of $(Y, \gamma)$ is

$$Q(Y, \gamma) = \{ \zeta \in \mathcal{X}(Y) \mid L_\zeta \gamma = 0 \}.$$ 

If $K : Y_1 \to Y_2$ is a quantomorphism, then it induces a diffeomorphism (in fact, a symplectomorphism) $K' : M_1 \to M_2$ such that the following diagram commutes.

$$\begin{array}{ccc}
Y_1 & \xrightarrow{K} & Y_2 \\
\downarrow_{p_1} & & \downarrow_{p_2} \\
M_1 & \xrightarrow{K'} & M_2
\end{array}$$

This implies that for any $\zeta \in Q(Y, \gamma)$ with flow $\phi^t$, there is a flow $\phi'^t$ on $M$ that satisfies $p \circ \phi^t = \phi'^t \circ p$. If $\zeta'$ is the vector field on $M$ with flow $\phi'^t$, then $p_* \zeta = \zeta'$. In other words, elements of $Q(Y, \gamma)$ descend via $p_*$ to well-defined vector fields on $M$.

### 3.2.2 The Lie algebra isomorphism

Let $(Y, \gamma) \xrightarrow{p} (M, \omega)$ be a prequantization circle bundle. We will now present an explicit construction of a Lie algebra isomorphism from $\mathcal{C}^\infty(M)$ to $Q(Y, \gamma)$. Recall that the vector field $\partial_{2\pi i}$ on $Y$ satisfies $\gamma(\partial_{2\pi i}) = 2\pi i \in u(1)$ and $p_* \partial_{2\pi i} = 0$.

**Lemma 3.2.3.** For all $f, g \in \mathcal{C}^\infty(M)$,

$$[\tilde{\xi}_f, \tilde{\xi}_g] = \tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi \hbar} p^* \{f, g\} \partial_{2\pi i}.$$  

**Proof.** It suffices to show that

$$p_* [\tilde{\xi}_f, \tilde{\xi}_g] = p_* \left( \tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi \hbar} p^* \{f, g\} \partial_{2\pi i} \right) \quad \text{and} \quad \gamma([\tilde{\xi}_f, \tilde{\xi}_g]) = \gamma \left( \tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi \hbar} p^* \{f, g\} \partial_{2\pi i} \right).$$

First, we have

$$p_* \left( \tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi \hbar} p^* \{f, g\} \partial_{2\pi i} \right) = \xi_{\{f, g\}} = [\xi_f, \xi_g].$$
Since $p_\ast \tilde{\xi}_f = \xi_f$ and $p_\ast \tilde{\xi}_g = \xi_g$, it follows that $p_\ast [\tilde{\xi}_f, \tilde{\xi}_g] = [\xi_f, \xi_g]$. Thus the first equation is verified.

Next, note that
\[
\gamma \left( \tilde{\xi}_{\{f,g\}} - \frac{1}{2\pi \hbar} p^* \{f,g\} \partial_{2\pi i} \right) = \frac{1}{i\hbar} p^* \{f,g\},
\]
and
\[
\gamma([\tilde{\xi}_f, \tilde{\xi}_g]) = -\frac{1}{i\hbar} (p^* \omega)(\tilde{\xi}_f, \tilde{\xi}_g) = \frac{1}{i\hbar} p^* \{f,g\}.
\]
Therefore the second equation is also verified.

\[\Box\]

**Lemma 3.2.4.** The map $E : C^\infty(M) \rightarrow \mathcal{X}(Y)$ given by
\[
E(f) = \tilde{\xi}_f + \frac{1}{2\pi \hbar} p^* f \partial_{2\pi i}, \quad \forall f \in C^\infty(M)
\]
is a Lie algebra homomorphism.

**Proof.** Let $f, g \in C^\infty(M)$ be arbitrary. We need to show that
\[
[\tilde{\xi}_{\{f,g\}} + \frac{1}{2\pi \hbar} p^* \{f,g\} \partial_{2\pi i}] = [\tilde{\xi}_f + \frac{1}{2\pi \hbar} p^* f \partial_{2\pi i}, \tilde{\xi}_g + \frac{1}{2\pi \hbar} p^* g \partial_{2\pi i}].
\]
Using Lemma 3.2.3, the left-hand side becomes
\[
[\tilde{\xi}_f, \tilde{\xi}_g] + 2 \frac{1}{2\pi \hbar} p^* \{f,g\} \partial_{2\pi i}.
\]
Expanding the right-hand side yields
\[
[\tilde{\xi}_f, \tilde{\xi}_g] + \left[ \tilde{\xi}_f, \frac{1}{2\pi \hbar} p^* g \partial_{2\pi i} \right] + \left[ \frac{1}{2\pi \hbar} p^* f \partial_{2\pi i}, \tilde{\xi}_g \right] + \left[ \frac{1}{2\pi \hbar} p^* f \partial_{2\pi i}, \tilde{\xi}_g \right] + \frac{1}{2\pi \hbar} p^* g \partial_{2\pi i}.
\]
The fourth term vanishes because $\partial_\theta(p^* f) = \partial_\theta(p^* g) = 0$ for any $\theta \in u(1)$. To evaluate the third term, recall that that $[\partial_\theta, \tilde{\xi}] = 0$ for any $\theta \in u(1)$ and $\xi \in \mathcal{X}(M)$. Therefore $[\partial_{2\pi i}, \tilde{\xi}_g] = 0$, so this term reduces to
\[
-\frac{1}{2\pi \hbar} (\tilde{\xi}_g p^* f) \partial_{2\pi i} = \frac{1}{2\pi \hbar} p^* \{f,g\} \partial_{2\pi i}.
\]
By the same argument, the second term also reduces to
\[
\frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i}.
\]
Combining these results, we find that the right-hand side of the desired equation is
\[
[\tilde{\xi}_f, \tilde{\xi}_g] + 2 \frac{1}{i\hbar} p^* \{f, g\} \partial_{2\pi i},
\]
which equals the left-hand side.

Lemma 3.2.5. For all \( f \in C^\infty(M) \), \( E(f) \in Q(Y, \gamma) \).

Proof. We need to show that \( L_{E(f)} \gamma = 0 \). We calculate
\[
L_{E(f)} \gamma = E(f) \cdot d\gamma + d(E(f)) \cdot \gamma = \frac{1}{i\hbar} p^* (\xi_f \cdot \omega) - \frac{1}{i\hbar} p^* df = 0.
\]

So far, we have shown that \( E : C^\infty(M) \to Q(Y, \gamma) \) is a Lie algebra homomorphism. We will now construct a map \( F : Q(Y, \gamma) \to C^\infty(M) \), and show that \( E \) and \( F \) are inverses. This will complete the proof that \( C^\infty(M) \) and \( Q(Y, \gamma) \) are isomorphic.

Let \( \zeta \in Q(Y, \gamma) \) be arbitrary. Then \( L_{\zeta} \gamma = \zeta \cdot d\gamma + d(\gamma(\zeta)) = 0 \). This implies that \( \partial_{\theta} \cdot (\zeta \cdot d\gamma + d(\gamma(\zeta))) = 0 \) for any \( \theta \in u(1) \). Since \( d\gamma(\zeta, \partial_{\theta}) = \frac{1}{i\hbar} (p^* \omega)(\zeta, \partial_{\theta}) = 0 \), it follows that \( \partial_{\theta} \cdot d(\gamma(\zeta)) = L_{\partial_{\theta} \gamma}(\zeta) = 0 \). We can therefore define the map \( F : Q(Y, \gamma) \to C^\infty(M) \) so that
\[
- \frac{1}{i\hbar} p^* F(\zeta) = \gamma(\zeta), \quad \forall \zeta \in Q(Y, \gamma).
\]

Theorem 3.2.6. The map \( E : C^\infty(M) \to Q(Y, \gamma) \) is a Lie algebra isomorphism with inverse \( F \).

Proof. Let \( f \in C^\infty(M) \) and \( \zeta \in Q(Y, \gamma) \) be arbitrary. We will show that \( F(E(f)) = f \) and \( E(F(\zeta)) = \zeta \). Using the definitions of \( E \) and \( F \), we have
\[
- \frac{1}{i\hbar} p^* F(E(f)) = \gamma(E(f)) = \gamma (\tilde{\xi}_f + \frac{1}{2\pi\hbar} p^* f \partial_{2\pi i}) = - \frac{1}{i\hbar} p^* f.
\]
This implies that $F(E(f)) = f$.

To show that $E(F(\zeta)) = \zeta$, it suffices to show that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $p_\star E(F(\zeta)) = p_\star \zeta$. By definition,

$$E(F(\zeta)) = \tilde{\xi}_{F(\zeta)} + \frac{1}{2\pi i} p_\star F(\zeta) \partial_{2\pi i} = \tilde{\xi}_{F(\zeta)} + \frac{1}{2\pi i} \gamma(\zeta) \partial_{2\pi i}.$$ 

It is immediate that $\gamma(E(F(\zeta))) = \gamma(\zeta)$, and that $p_\star E(F(\zeta)) = \xi_{F(\zeta)}$. Observe that

$$\zeta \lrcorner p_\star \omega = i\hbar \zeta \lrcorner d\gamma = -i\hbar d(\gamma(\zeta)) = p_\star (dF(\zeta)),$$

having used $L_\zeta \gamma = 0$. Therefore $(p_\star \zeta) \lrcorner \omega = dF(\zeta)$, which implies that $p_\star \zeta = \xi_{F(\zeta)}$. Thus $p_\star E(F(\zeta)) = p_\star \zeta$. This concludes the proof that $E(F(\zeta)) = \zeta$.

Since $E$ and $F$ are inverses, and we know from Lemma 3.2.5 that $E : C^\infty(M) \to \mathcal{Q}(Y, \gamma)$ is a Lie algebra homomorphism, it follows that $E$ and $F$ are the desired Lie algebra isomorphisms.

The primary goal of Section 3.3 is to duplicate the above construction for the infinitesimal quantomorphisms of a metaplectic-c prequantization. However, before moving on to the metaplectic-c case, we will show how the map $E$ can be used to represent the elements of $C^\infty(M)$ as operators on the space of sections of the prequantization line bundle for $(M, \omega)$. This result will also have an analogue in the metaplectic-c case, which we will discuss in Section 3.3.3.

### 3.2.3 An operator representation of $C^\infty(M)$

Let $(L, \nabla)$ be the complex line bundle with connection associated to $(Y, \gamma)$. Recall the association between a section $s$ of $L$ and an equivariant function $\tilde{s} : Y \to \mathbb{C}$ from Section 2.1.2. We note the following properties.

- For any $f \in C^\infty(M)$ and $s \in \Gamma(L)$, the equivariant function corresponding to the section $fs$ is $\tilde{fs} = p_\star f \tilde{s}$.

- The vector field $\partial_{2\pi i}$ is generated by the infinitesimal action of $2\pi i \in u(1)$ on $Y$. Thus,
for all $y \in Y$,

$$(\partial_{2\pi i} \tilde{s})(y) = \left. \frac{d}{dt} \right|_{t=0} \tilde{s}(y \cdot e^{2\pi it}) = -2\pi i \tilde{s}(y).$$

Further recall that the Kostant-Souriau representation map $r : C^\infty(M) \to \End \Gamma(L)$ is defined by

$$r(f)s = (i\hbar \nabla_{\xi_f} + f) s, \ \forall f \in C^\infty(M), \ s \in \Gamma(L).$$

Using the preceding observations, we see that

$$\tilde{r}(f)s = \left( i\hbar \tilde{\xi}_f + p^* f \right) s = \left( i\hbar \tilde{\xi}_f - \frac{1}{2\pi i} p^* f \partial_{2\pi i} \right) s = i\hbar E(f)s.$$

Since we proved in Lemma 3.2.4 that $E(\{f,g\}) = [E(f), E(g)]$ for all $f, g \in C^\infty(M)$, the following is immediate.

**Theorem 3.2.7.** The map $r : C^\infty(M) \to \End \Gamma(L)$ satisfies

$$[r(f), r(g)] = i\hbar r(\{f, g\}), \ \forall f, g \in C^\infty(M).$$

Thus the same map that provides the isomorphism from $C^\infty(M)$ to $\mathcal{Q}(Y, \gamma)$ also yields the usual Kostant-Souriau representation of $C^\infty(M)$ as a space of operators on $\Gamma(L)$. We will see a similar result in the case of metaplectic-c prequantization.

### 3.3 Metaplectic-c Quantomorphisms

Having reviewed the properties of infinitesimal quantomorphisms in Kostant-Souriau prequantization, we will now explore their parallels in metaplectic-c prequantization. In Section 3.3.1, we develop our definition for a metaplectic-c quantomorphism, and use it to define an infinitesimal metaplectic-c quantomorphism. The remainder of the chapter is dedicated to proving the metaplectic-c analogues of the results presented in Section 3.2.

Suppose $(M, \omega)$ admits a metaplectic-c prequantization $(P, \Sigma, \gamma)$. The space of infinitesimal quantomorphisms of $(P, \Sigma, \gamma)$ consists of those vector fields on $P$ whose flows preserve all of the structures on $(P, \Sigma, \gamma)$. Note that one of these structures is the map $P \xrightarrow{\Sigma} \operatorname{Sp}(M, \omega)$, which
does not have a direct analogue in the Kostant-Souriau case. We will show how to incorporate this additional piece of information in the next section.

### 3.3.1 Infinitesimal metaplectic-c quantomorphisms

As in Section 3.2.1, we begin by developing the idea of a quantomorphism between metaplectic-c prequantizations. Let \((P_1, \Sigma_1, \gamma_1) \xrightarrow{\rho_1} \text{Sp}(M_1, \omega_1)\) and \((P_2, \Sigma_2, \gamma_2) \xrightarrow{\rho_2} \text{Sp}(M_2, \omega_2)\) be metaplectic-c prequantizations for two symplectic manifolds, and let \(\Pi_j = \rho_j \circ \Sigma_j\) for \(j = 1, 2\). Let \(K : P_1 \to P_2\) be a diffeomorphism. We will determine the conditions that \(K\) must satisfy in order for it to preserve all of the structures of the metaplectic-c prequantizations.

First, by analogy with the Kostant-Souriau definition, assume that \(K^* \gamma_2 = \gamma_1\).

Fix \(m \in M_1\), and consider the fiber \(P_{1m}\). For any \(q \in P_{1m}\), notice that

\[
T_q P_{1m} = \{ \xi \in T_q P_1 | \Pi_1^* \xi = 0 \} = \ker d\gamma_1 q.
\]

The same property holds for a fiber of \(P_2\) over a point in \(M_2\). By assumption, \(K_*\) is an isomorphism from \(\ker d\gamma_1 q\) to \(\ker d\gamma_2 K(q)\) for all \(q \in P_1\). Therefore \(\Pi_2\) is constant on \(K(P_{1m})\).

Moreover, since \(K\) is a diffeomorphism, we can make the analogous argument with \(K^{-1}\), and conclude that \(K(P_{1m})\) is in fact a fiber of \(P_2\) over \(M_2\), and every fiber of \(P_2\) is the image of a fiber of \(P_1\). Thus \(K\) induces a diffeomorphism \(K'' : M_1 \to M_2\) such that the following diagram commutes.

\[
\begin{array}{ccc}
P_1 & \xrightarrow{K} & P_2 \\
\Pi_1 & \downarrow & \Pi_2 \\
M_1 & \xrightarrow{K''} & M_2 \\
\end{array}
\]

**Lemma 3.3.1.** The map \(K'' : M_1 \to M_2\) is a symplectomorphism.

**Proof.** It suffices to show that \(K''^* \omega_2 = \omega_1\). Using the properties of \(K\), \(\gamma_1\) and \(\gamma_2\), we calculate

\[
\Pi_1^*(K''^* \omega_2) = (K'' \circ \Pi_1)^* \omega_2 = (\Pi_2 \circ K)^* \omega_2 = K^*(i\hbar d\gamma_2) = i\hbar d\gamma_1 = \Pi_1^* \omega_1.
\]

Therefore \(K''^* \omega_2 = \omega_1\), as required. \(\square\)
Assume that a model symplectic vector space \((V, \Omega)\) has been fixed. Recall from Definition 2.3.1 that an element \(b \in \text{Sp}(M_1, \omega_1)_m\) is a map \(b : V \to T_mM_1\) such that \(b^*\omega_{1m} = \Omega\). Since \(K''\) is a symplectomorphism, the composition \(K'' \circ b : V \to T_{K''(m)}M_2\) satisfies \((K'' \circ b)^*\omega_{2K''(m)} = \Omega\), which implies that \(K'' \circ b \in \text{Sp}(M_2, \omega_2)_{K''(m)}\). Let \(\tilde{K}'' : \text{Sp}(M_1, \omega_1) \to \text{Sp}(M_2, \omega_2)\) be the lift of \(K''\) given by

\[
\tilde{K}''(b) = K'' \circ b, \quad \forall b \in \text{Sp}(M_1, \omega_1).
\]

Then \(\tilde{K}''\) is a diffeomorphism, and it is equivariant with respect to the principal \(\text{Sp}(V)\) actions.

Thus, if we assume that \(K^*\gamma_2 = \gamma_1\), we obtain the diffeomorphisms \(K'' : M_1 \to M_2\) and \(\tilde{K}'' : \text{Sp}(M_1, \omega_1) \to \text{Sp}(M_2, \omega_2)\), where both \(K\) and \(\tilde{K}''\) are lifts of \(K''\). However, \(K\) is not necessarily a lift of \(\tilde{K}''\). Indeed, there might not be any map \(K' : \text{Sp}(M_1, \omega_1) \to \text{Sp}(M_2, \omega_2)\) of which \(K\) is a lift. A map \(K\) for which there is no corresponding \(K'\) is constructed in Section 3.4, Example 3.4.1. In Section 3.2.1, we showed that a diffeomorphism of prequantization circle bundles that preserves the connection forms must be equivariant with respect to the principal circle actions. By contrast, Example 3.4.1 demonstrates that it is possible for \(K\) to preserve the prequantization one-forms without being equivariant with respect to the principal \(\text{Mp}^c(V)\) actions.

Suppose we make the additional assumption that \(K(q \cdot a) = K(q) \cdot a\) for all \(q \in P_1\) and \(a \in \text{Mp}^c(V)\). Then \(K\) induces a diffeomorphism \(K' : \text{Sp}(M_1, \omega_1) \to \text{Sp}(M_2, \omega_2)\) that satisfies \(K' \circ \Sigma_1 = \Sigma_2 \circ K\). Combining this with the map \(K'' : M_1 \to M_2\) yields the following commutative diagram.

\[
\begin{array}{ccc}
P_1 & \xrightarrow{K} & P_2 \\
\downarrow{\Sigma_1} & & \downarrow{\Sigma_2} \\
\text{Sp}(M_1, \omega_1) & \xrightarrow{K'} & \text{Sp}(M_2, \omega_2) \\
\downarrow{\rho_2} & & \downarrow{\rho_2} \\
M_1 & \xrightarrow{K''} & M_2
\end{array}
\]

We now have two maps, \(K'\) and \(\tilde{K}''\), which are diffeomorphisms from \(\text{Sp}(M_1, \omega_1)\) to \(\text{Sp}(M_2, \omega_2)\). By construction, \(\rho_2 \circ K' = \rho_2 \circ \tilde{K}''\), and both \(K'\) and \(\tilde{K}''\) are equivariant with respect to the principal \(\text{Sp}(V)\) actions. However, it is still possible for \(K'\) and \(\tilde{K}''\) to be different. A map \(K\) for which \(K' \neq \tilde{K}''\) is given in Example 3.4.2.
As will be shown in Section 3.3.2, this potential discrepancy between $K'$ and $\tilde{K}''$ must be prevented in order to construct the desired isomorphism between $C^\infty(M)$ and the infinitesimal quantomorphisms. We therefore propose the following definition.

**Definition 3.3.2.** The diffeomorphism $K : P_1 \to P_2$ is a **metaplectic-c quantomorphism** if

1. $K^* \gamma_2 = \gamma_1$,
2. the induced diffeomorphism $K'' : M_1 \to M_2$ satisfies $\tilde{K}'' \circ \Sigma_1 = \Sigma_2 \circ K$.

Let $K : P_1 \to P_2$ be a metaplectic-c quantomorphism. Given our concept of a quantomorphism as a diffeomorphism that preserves all of the structures of a metaplectic-c prequantization, we would expect that $K$ is equivariant with respect to the $\text{Mp}^c(V)$ actions. Equivariance is a consequence of the definition, as we now show.

Let $\alpha \in \mathfrak{mp}^c(V)$ be arbitrary, and write $\alpha = \kappa \oplus \tau$ under the identification of $\mathfrak{mp}^c(V)$ with $\mathfrak{sp}(V) \oplus \mathfrak{u}(1)$. The vector field $\partial_\alpha$ on $P_1$ is completely specified by the conditions $\gamma_1(\partial_\alpha) = \tau$ and $\Sigma_1 \ast \partial_\alpha = \partial_\kappa$, and the same is true on $P_2$. Notice that

$$\gamma_2(K \ast \partial_\alpha) = \gamma_1(\partial_\alpha) = \tau,$$

and

$$\Sigma_2 \ast K \ast \partial_\alpha = \tilde{K}'' \ast \Sigma_1 \ast \partial_\alpha = \tilde{K}'' \ast \partial_\kappa = \partial_\kappa,$$

where the final equality follows from the fact that $\tilde{K}''$ is equivariant with respect to $\text{Sp}(V)$. Thus $K \ast \partial_\alpha = \partial_\alpha$ for all $\alpha \in \mathfrak{mp}^c(V)$, which implies that $K$ is equivariant with respect to $\text{Mp}^c(V)$, as desired.

Now consider a single metaplectic-c prequantized space $(P, \Sigma, \gamma) \xrightarrow{\Sigma} \text{Sp}(M, \omega) \xrightarrow{\rho} (M, \omega)$ with $\Pi = \rho \circ \Sigma$.

**Definition 3.3.3.** A vector field $\zeta \in \mathcal{X}(P)$ is an **infinitesimal metaplectic-c quantomorphism** if its flow $\phi^t$ is a metaplectic-c quantomorphism from its domain to its range for each $t$. 
Let $\zeta \in \mathcal{X}(P)$ have flow $\phi^t$. Property (1) of a quantomorphism holds for $\phi^t$ if and only if $L_\zeta \gamma = 0$. If we assume that $\phi^t$ satisfies property (1), then there is a flow $\phi^{\prime\prime}$ on $M$ such that $\Pi \circ \phi^t = \phi^{\prime\prime} \circ \Pi$. The vector field that it generates on $M$ is $\Pi_\ast \zeta$.

Lemma 3.3.1 shows that $\phi^{\prime\prime}$ is a family of symplectomorphisms. Therefore we can lift $\phi^{\prime\prime}$ to a flow on $\text{Sp}(M, \omega)$, denoted by $\tilde{\phi}^{\prime\prime}$, where $\tilde{\phi}^{\prime\prime}(b) = (\phi^{\prime\prime})_\ast \circ b$ for all $b \in \text{Sp}(M, \omega)$. Let the vector field on $\text{Sp}(M, \omega)$ that has flow $\tilde{\phi}^{\prime\prime}$ be $\Pi_\ast \zeta$. Then property (2) of a quantomorphism holds for $\phi^t$ if and only if $\Sigma_\ast \zeta$ is a well-defined vector field on $\text{Sp}(M, \omega)$ and $\Sigma_\ast \zeta = \Pi_\ast \zeta$.

We conclude that the space of infinitesimal metaplectic-c quantomorphisms of $(P, \Sigma, \gamma)$ is

$$
Q(P, \Sigma, \gamma) = \{ \zeta \in \mathcal{X}(P) \mid L_\zeta \gamma = 0 \text{ and } \Sigma_\ast \zeta = \Pi_\ast \zeta \},
$$

where it is understood that the condition $\Sigma_\ast \zeta = \Pi_\ast \zeta$ can only be satisfied if $\Sigma_\ast \zeta$ is well defined.

In the next section, we will construct a Lie algebra isomorphism from $C^\infty(M)$ to $Q(P, \Sigma, \gamma)$.

### 3.3.2 The Lie algebra isomorphism

We begin with a procedure, given by Robinson and Rawnsley in Section 7 of [17], for lifting a Hamiltonian vector field on $M$ to $\text{Sp}(M, \omega)$ and then to $P$. These steps will be used in constructing the isomorphism $E : C^\infty(M) \to Q(P, \Sigma, \gamma)$.

Fix $f \in C^\infty(M)$, and let its Hamiltonian vector field $\xi_f$ have flow $\varphi^t$ on $M$. We know that $\varphi^t_\ast$ preserves $\omega$ because $L_{\xi_f} \omega = 0$. Let $\tilde{\varphi}^t$ be the lift of $\varphi^t$ to $\text{Sp}(M, \omega)$ given by

$$
\tilde{\varphi}^t(b) = \varphi^t_\ast \circ b, \quad \forall b \in \text{Sp}(M, \omega),
$$

and let the vector field on $\text{Sp}(M, \omega)$ with flow $\tilde{\varphi}^t$ be $\tilde{\xi}_f$. We have $\rho_\kappa \tilde{\xi}_f = \xi_f$ by construction. Also, $\tilde{\varphi}^t$ commutes with the right principal $\text{Sp}(V)$ action on $\text{Sp}(M, \omega)$, so $[\tilde{\xi}_f, \partial_\kappa] = 0$ for all $\kappa \in \mathfrak{sp}(V)$. Now let $\tilde{\xi}_f$ be the lift of $\tilde{\xi}_f$ to $P$ that is horizontal with respect to $\gamma$. Then $\Sigma_\ast \tilde{\xi}_f = \tilde{\xi}_f$ and $\gamma(\tilde{\xi}_f) = 0$. A summary of the key properties of $\xi_f$, $\tilde{\xi}_f$ and $\tilde{\xi}_f$ is below.
\[(P, \gamma) \quad \tilde{\xi}_f \quad \gamma(\tilde{\xi}_f) = 0, \quad \Sigma_* \tilde{\xi}_f = \tilde{\xi}_f, \quad \Pi_* \tilde{\xi}_f = \xi_f\]

\[\Sigma \quad \tilde{\xi}_f \quad [\tilde{\xi}_f, \partial_\kappa] = 0 \quad \forall \kappa \in \mathfrak{sp}(V), \quad \rho_* \tilde{\xi}_f = \xi_f\]

In Section 3.2.2, we made use of the vector field \(\partial_{2\pi i}\) on \(Y\). The corresponding object in this context is the vector field \(\partial_{2\pi i}\) on \(P\), where \(2\pi i \in u(1) \subset \mathfrak{mp}^c(V)\). This vector field satisfies \(\gamma(\partial_{2\pi i}) = 2\pi i\) and \(\Sigma(\partial_{2\pi i}) = 0\).

**Lemma 3.3.4.** For all \(f, g \in C^\infty(M)\),

\[ [\hat{\xi}_f, \hat{\xi}_g] = \hat{\xi}_{\{f, g\}} - \frac{1}{2\pi \hbar} \Pi^* \{f, g\} \partial_{2\pi i}. \]

**Proof.** It suffices to show that

\[\Sigma_* [\hat{\xi}_f, \hat{\xi}_g] = \Sigma_* \left( \hat{\xi}_{\{f, g\}} - \frac{1}{2\pi \hbar} \Pi^* \{f, g\} \partial_{2\pi i} \right) \quad \text{and} \]

\[\gamma([\hat{\xi}_f, \hat{\xi}_g]) = \gamma \left( \hat{\xi}_{\{f, g\}} - \frac{1}{2\pi \hbar} \Pi^* \{f, g\} \partial_{2\pi i} \right). \]

A calculation establishes that \(\tilde{\xi}_{\{f, g\}} = [\tilde{\xi}_f, \tilde{\xi}_g]\). The rest of the proof proceeds identically to that of Lemma 3.2.3. \(\square\)

**Lemma 3.3.5.** The map \(E : C^\infty(M) \to \mathcal{X}(P)\) given by

\[E(f) = \hat{\xi}_f + \frac{1}{2\pi \hbar} \Pi^* f \partial_{2\pi i}, \quad \forall f \in C^\infty(M)\]

is a Lie algebra homomorphism.

**Proof.** Precisely analogous to Lemma 3.2.4. \(\square\)

**Lemma 3.3.6.** For all \(f \in C^\infty(M)\), \(E(f) \in \mathcal{Q}(P, \Sigma, \gamma)\).

**Proof.** We need to show that \(L_{E(f)} \gamma = 0\) and \(\Sigma_* E(f) = \Pi_* \tilde{E}(f)\). The verification that \(L_{E(f)} \gamma = 0\) is the same as that in Lemma 3.2.5. Note that \(\Sigma_* E(f) = \tilde{\xi}_f\) and \(\Pi_* E(f) = \xi_f\), so \(\Pi_* \tilde{E}(f) = \tilde{\xi}_f = \Sigma_* E(f)\). Thus the necessary conditions are satisfied, and \(E(f) \in \mathcal{Q}(P, \Sigma, \gamma)\). \(\square\)
As before, we will construct an inverse for $E$, and conclude that $E$ is a Lie algebra isomorphism. Let $\zeta \in \mathcal{Q}(P, \Sigma, \gamma)$ and $\alpha \in \mathfrak{mp}^c(V)$ be arbitrary. Using an identical argument to the one that precedes Theorem 3.2.6, the fact that $\partial_{\alpha \wedge} (L_\zeta \gamma) = 0$ implies that $L_{\partial_{\alpha \wedge}} \gamma(\zeta) = 0$. Therefore we can define $F : \mathcal{Q}(P, \Sigma, \gamma) \to C^\infty(M)$ so that

$$-rac{1}{i\hbar} \Pi^* F(\zeta) = \gamma(\zeta), \quad \forall \zeta \in \mathcal{Q}(P, \Sigma, \gamma).$$

**Theorem 3.3.7.** The map $E : C^\infty(M) \to \mathcal{Q}(P, \Sigma, \gamma)$ is a Lie algebra isomorphism with inverse $F$.

**Proof.** Let $f \in C^\infty(M)$ and $\zeta \in \mathcal{Q}(P, \Sigma, \gamma)$ be arbitrary. From the definitions of $E$ and $F$, $F(E(f))$ satisfies

$$-rac{1}{i\hbar} \Pi^* F(E(f)) = \gamma(E(f)) = \gamma \left( \hat{\xi}_f + \frac{1}{2\pi \hbar} \Pi^* f \partial_{2\pi i} \right) = -\frac{1}{i\hbar} \Pi^* f.$$

Thus $F(E(f)) = f$.

Next, we claim that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $\Sigma_* E(F(\zeta)) = \Sigma_* \zeta$. Observe that

$$E(F(\zeta)) = \hat{\xi}_{F(\zeta)} + \frac{1}{2\pi \hbar} \Pi^* F(\zeta) \partial_{2\pi i} = \tilde{\xi}_{F(\zeta)} + \frac{1}{2\pi i} \gamma(\zeta) \partial_{2\pi i}.$$

It is immediate that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $\Sigma_* E(F(\zeta)) = \tilde{\xi}_{F(\zeta)}$. From the definition of $\mathcal{Q}(P, \Sigma, \gamma)$, we know that $\Sigma_* \zeta = \tilde{\Pi}_* \zeta$. It remains to show that $\Pi_* \zeta = \xi_{F(\zeta)}$. We calculate

$$\zeta \wedge \Pi^* \omega = \zeta \wedge ihd\gamma = -ihd(\gamma(\zeta)) = \Pi^* dF(\zeta).$$

This demonstrates that $(\Pi_* \zeta) \wedge \omega = dF(\zeta)$, which implies that $\Pi_* \zeta = \xi_{F(\zeta)}$ as needed. Thus we have shown that $E(F(\zeta)) = \zeta$, and this completes the proof that $E$ and $F$ are inverses.

If the definition of $\mathcal{Q}(P, \Sigma, \gamma)$ did not include the condition that $\Sigma_* \zeta = \tilde{\Pi}_* \zeta$, this proof would fail in the final step. We would be able to show that $\Sigma_* E(F(\zeta)) = \tilde{\xi}_{F(\zeta)} = \tilde{\Pi}_* \zeta$, but this vector field would not necessarily equal $\Sigma_* \zeta$, and so $F$ would not be the inverse of $E$. This explains why property (2) of a metaplectic-c quantomorphism is necessary in order to obtain a subalgebra of $\mathcal{X}(P)$ that is isomorphic to $C^\infty(M)$. 
3.3.3 An operator representation of $C^\infty(M)$

In [17], Robinson and Rawnsley construct an infinite-dimensional Hilbert space $\mathcal{E}'(V)$ of holomorphic functions on $V \cong \mathbb{C}^n$, on which the group $Mp^c(V)$ acts via the metaplectic representation. They then define the bundle of symplectic spinors for the prequantized system $(P, \Sigma, \gamma) \xrightarrow{\Pi} (M, \omega)$ to be $E'(P) = P \times_{Mp^c(V)} \mathcal{E}'(V)$.

We omit the details of the metaplectic representation here; the only fact we need is that the subgroup $U(1) \subset Mp^c(V)$ acts on elements of $E'(V)$ by scalar multiplication. We write an element of $E'(P)$ as an equivalence class $[q, \psi]$ with $q \in P$ and $\psi \in \mathcal{E}'(V)$.

Section 7 of [17] contains the following construction. Let $s \in \Gamma(E'(P))$ be given, and define the map $\tilde{s} : P \to \mathcal{E}'(V)$ so that $[q, \tilde{s}(q)] = s(\Pi(q))$ for all $q \in P$. This map $\tilde{s}$ satisfies the equivariance condition

$$\tilde{s}(q \cdot a) = a^{-1}\tilde{s}(q), \quad \forall q \in P, a \in Mp^c(V),$$

where the action on the right-hand side is that of the metaplectic representation. Conversely, if $\tilde{s} : P \to \mathcal{E}'(V)$ is any map with the equivariance property above, it can be used to define a section $s \in \Gamma(E'(P))$ by setting $s(m) = [q, \tilde{s}(q)]$ for each $m \in M$ and any $q \in P$ such that $\Pi(q) = m$.

Let $f \in C^\infty(M)$ be arbitrary, and recall the lifting $\xi_f \to \tilde{\xi}_f \to \hat{\xi}_f$ of $\xi_f$ to $P$. A standard calculation establishes that $[\hat{\xi}_f, \partial_\alpha] = 0$ for all $\alpha \in \mathfrak{mp}^c(V)$. Thus, if $\tilde{s} : P \to \mathcal{E}'(V)$ is an equivariant map, then so is $\hat{\xi}_f \tilde{s}$. Define the map $D : C^\infty(M) \to \text{End}\Gamma(\mathcal{E}'(P))$ such that for all $f \in C^\infty(M)$ and $s \in \Gamma(\mathcal{E}'(P))$, $D_{fs}$ is the section of $\mathcal{E}'(P)$ that satisfies

$$\hat{D}_{fs} = \hat{\xi}_f \tilde{s}.$$

Further, define $\delta : C^\infty(M) \to \text{End}\Gamma(\mathcal{E}'(P))$ by

$$\delta_{fs} = D_{fs} + \frac{1}{i\hbar}fs, \quad \forall f \in C^\infty(M), s \in \Gamma(\mathcal{E}'(P)).$$
Theorem 7.8 of [17] states that $\delta$ is a Lie algebra homomorphism.

We see that the construction of $D$ precisely parallels the construction of the connection $\nabla$ on the prequantization line bundle $L$ associated to a prequantization circle bundle $(Y, \gamma)$. As in Section 3.2.3, we make two observations.

- For any $s \in \Gamma(E'(P))$ and $f \in C^\infty(M)$, $\tilde{f} s = \Pi^* f \tilde{s}$.
- For any equivariant map $\tilde{s} : P \to E'(V)$, $\partial_{2\pi i} \tilde{s} = -2\pi i \tilde{s}$.

Therefore

$$\tilde{\delta} f s = \left( \tilde{\xi} f + \frac{1}{2\pi \hbar} \Pi^* f \partial_{2\pi i} \right) \tilde{s} = E(f) \tilde{s}.$$  

The fact that $\delta$ is a Lie algebra homomorphism then follows immediately from Lemma 3.3.5. This construction would apply equally well to any associated bundle where the subgroup $U(1) \subset \text{Mp}^c(V)$ acts on the fiber by scalar multiplication.

### 3.4 Two Diffeomorphisms That Are Not Quantomorphisms

A metaplectic-c quantomorphism $K$ between metaplectic-c prequantizations $(P_1, \Sigma_1, \gamma_1) \xrightarrow{\Sigma_1} \text{Sp}(M_1, \omega_1)$ and $(P_2, \Sigma_2, \gamma_2) \xrightarrow{\Sigma_2} \text{Sp}(M_2, \omega_2)$ is a diffeomorphism $K : P_1 \to P_2$ such that

1. $K^* \gamma_2 = \gamma_1$,
2. the induced diffeomorphism $K'' : M_1 \to M_2$ satisfies $\tilde{K}'' \circ \Sigma_1 = \Sigma_2 \circ K$,

where $\tilde{K}'' : \text{Sp}(M_1, \omega_1) \to \text{Sp}(M_2, \omega_2)$ is the lift of $K''$ given by

$$\tilde{K}''(b) = K''_* \circ b, \quad \forall b \in \text{Sp}(M_1, \omega_1).$$

We claimed that condition (1) is insufficient to guarantee that $K$ is the lift of some map $K' : \text{Sp}(M_1, \omega_1) \to \text{Sp}(M_2, \omega_2)$. In particular, a diffeomorphism $K$ that only satisfies condition (1) might not be equivariant with respect to the principal $\text{Mp}^c(V)$ actions. We further claimed that $K$ might be equivariant and satisfy condition (1), yet fail to satisfy condition (2). We will now construct examples to support these claims.
Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ with Cartesian coordinates $(p,q)$ and polar coordinates $(r,\theta)$. Equip $M$ with the symplectic form $\omega = dp \wedge dq = rdr \wedge d\theta$, and observe that the one-form $\beta = \frac{1}{2} r^2 d\theta$ satisfies $d\beta = \omega$. Let $V = \mathbb{R}^2$ with basis $\{\hat{x}, \hat{y}\}$ and symplectic form $\Omega = \hat{x}^* \wedge \hat{y}^*$, and consider the global trivialization of the tangent bundle $TM$ such that for all $m \in M$, $T_m M$ is identified with $V$ by mapping $\hat{x} \to \partial / \partial p \bigg|_m$ and $\hat{y} \to \partial / \partial q \bigg|_m$. Identify $\text{Sp}(M,\omega)$ with $M \times \text{Sp}(V)$ using this trivialization.

Let $P = M \times \text{Mp}^c(V)$, and define the map $\Sigma : P \to \text{Sp}(M,\omega)$ by $\Sigma(m,a) = (m,\sigma(a))$ for all $m \in M$ and $a \in \text{Mp}^c(V)$. Let $\vartheta_0$ be the trivial connection on the product bundle $M \times \text{Mp}^c(V)$, and let $\gamma = \frac{1}{i\hbar} \beta + \frac{1}{2} \eta^* \vartheta_0$. Then $(P,\Sigma,\gamma)$ is a metaplectic-c prequantization of $(M,\omega)$. In both of the examples below, we will give a diffeomorphism $K : P \to P$.

To facilitate the construction in Example 3.4.1, we introduce a more explicit representation for elements of the metaplectic-c group. By definition, $\text{Mp}^c(V) = \text{Mp}(V) \times \mathbb{Z}_2 U(1)$. The restriction of the projection map $\text{Mp}^c(V) \xrightarrow{\sigma} \text{Sp}(V)$ to $\text{Mp}(V)$ yields the double covering $\text{Mp}(V) \xrightarrow{\sigma} \text{Sp}(V)$. Write an element of $\text{Mp}^c(V)$ as an equivalence class $[h,e^{2\pi it}]$ with $h \in \text{Mp}(V)$ and $t \in \mathbb{R}$. In terms of this parametrization, the projection map is given by

$$
\sigma[h,e^{2\pi it}] = \sigma(h),
$$

and the determinant map $\text{Mp}^c(V) \xrightarrow{\eta} U(1)$ is given by

$$
\eta[h,e^{2\pi it}] = (e^{2\pi it})^2.
$$

**Example 3.4.1.** We will define a diffeomorphism $K : P \to P$ that preserves $\gamma$, but that does not descend through $\Sigma$ to a well-defined map on $\text{Sp}(M,\omega)$.

Let $\mu : \mathbb{R} \to \text{Mp}(V)$ be any smooth nonconstant path such that $\mu(t+1) = \mu(t)$ for all $t \in \mathbb{R}$. Note that the composition $\sigma \circ \mu : \mathbb{R} \to \text{Sp}(V)$ is also nonconstant. Now define $F : \text{Mp}(V) \times U(1) \to \text{Mp}(V) \times U(1)$ by

$$
F(h,e^{2\pi it}) = (h\mu(2t),e^{2\pi it}), \quad \forall h \in \text{Mp}(V), \quad t \in \mathbb{R}.
$$

This map is a diffeomorphism of $\text{Mp}(V) \times U(1)$, and it descends to a diffeomorphism of $\text{Mp}^c(V)$,
which we also denote $F$. For any $[h, e^{2\pi it}] \in \text{Mp}^c(V)$, observe that

$$\eta(F[h, e^{2\pi it}]) = \eta[h \mu(2t), e^{2\pi it}] = (e^{2\pi it})^2 = \eta[h, e^{2\pi it}].$$

This implies that for any $\alpha \in \text{mp}^c(V)$,

$$\frac{1}{2} \eta_* F_* \alpha = \frac{1}{2} \eta_* \alpha.$$

Define the diffeomorphism $K : P \to P$ by $K(m, a) = (T_\lambda(m), a)$ for all $m \in M$ and $a \in \text{Mp}^c(V)$. Since $K$ is the identity on $M$, it preserves $\beta$. From the property of $F$ shown above, $K$ also preserves $\frac{1}{2} \eta_* \vartheta_0$, and thus it preserves $\gamma$. Fix $(m, g) \in \text{Sp}(M, \omega)$, and let $h \in \text{Mp}(V)$ be such that $\sigma(h) = g$. Then the fiber of $P$ over $(m, g)$ is $P_{(m, g)} = \{(m, [h, e^{2\pi it}]) | t \in \mathbb{R}\}$. Notice that

$$\Sigma \circ K(m, [h, e^{2\pi it}]) = \Sigma(m, [h \mu(2t), e^{2\pi it}]) = (m, g \sigma(\mu(2t))),$$

which is not constant with respect to $t$. Thus $K(P_{(m, g)})$ is not contained within a single fiber of $P$ over $\text{Sp}(M, \omega)$, which shows that there is no map $K' : \text{Sp}(M, \omega) \to \text{Sp}(M, \omega)$ such that $K' \circ \Sigma = \Sigma \circ K$.

If $K : P \to P$ is equivariant with respect to $\text{Mp}^c(V)$, then it induces a diffeomorphism $K' : \text{Sp}(M, \omega) \to \text{Sp}(M, \omega)$ that satisfies $K' \circ \Sigma = \Sigma \circ K$. This map and $\tilde{K}''$ are both lifts of $K'' : M \to M$, but they might not be the same map.

**Example 3.4.2.** We will define a diffeomorphism $K : P \to P$ that preserves $\gamma$ and is equivariant with respect to $\text{Mp}^c(V)$, but where $K' \neq \tilde{K}''$.

Let $T_\lambda : M \to M$ be the map that rotates $M$ about the origin by the angle $\lambda$, where $\lambda$ is not an integer multiple of $2\pi$. Define $K : P \to P$ by

$$K(m, a) = (T_\lambda(m), a), \quad \forall m \in M, \ a \in \text{Mp}^c(V).$$

Then $K^* \gamma = \gamma$, and $K(q \cdot a) = K(q) \cdot a$ for all $q \in P$ and $a \in \text{Mp}^c(V)$. The map $K' : \text{Sp}(M, \omega) \to \text{Sp}(M, \omega)$ is given by

$$K'(m, g) = (T_\lambda(m), g), \quad \forall m \in M, \ g \in \text{Sp}(V),$$
and the map $K'' : M \to M$ is simply $T_\lambda$. If we let $T_\lambda$ also denote the automorphism of $V$ given by rotation about the origin by $\lambda$, then under our chosen identification of $TM$ with $M \times V$, we have

$$K''_\lambda(m, v) = (T_\lambda(m), T_\lambda(v)), \quad \forall m \in M, \; v \in V.$$ 

Therefore $\tilde{K}'' : \text{Sp}(M, \omega) \to \text{Sp}(M, \omega)$ is given by

$$\tilde{K}''(m, g) = (T_\lambda(m), T_\lambda \circ g), \quad \forall m \in M, \; g \in \text{Sp}(V).$$ 

Hence $K' \neq \tilde{K}''$. 

$\square$
Chapter 4

Quantized Energy Levels and Dynamical Invariance

4.1 Introduction

It is well known from physics that the quantum mechanical versions of systems such as the harmonic oscillator and the hydrogen atom are restricted to discrete energy levels. More generally, suppose \((M,\omega)\) is a quantizable symplectic manifold and \(H : M \to \mathbb{R}\) is a function on \(M\). If we think of \(H\) as a Hamiltonian energy function, then we can ask what it means for a regular value \(E\) of \(H\) to be a quantized energy level for the system \((M,\omega,H)\).

One possible answer to this question involves the construction of the symplectic reduction. The orbits of the Hamiltonian vector field \(\xi_H\) partition the level set \(H^{-1}(E)\). If \(M_E\), the space of orbits, is a manifold, then \(\omega\) induces a symplectic form \(\omega_E\) on \(M_E\), and \((M_E,\omega_E)\) is the symplectic reduction of \((M,\omega)\) at \(E\). The values of \(E\) for which this new symplectic manifold is quantizable can be taken to be the quantized energy levels of the system. This definition has been applied to the hydrogen atom in the context of Kostant-Souriau quantization [18], and to the \(n\)-dimensional harmonic oscillator (\(n \geq 2\)) in the context of metaplectic-c quantization [17]. In both cases, the physically predicted energy spectrum is obtained. However, technical complications arise in cases where the space of orbits is not a manifold. Also, if the level set is one-dimensional, as it is for the one-dimensional harmonic oscillator, then the space of orbits is
zero-dimensional and the quantization condition is always satisfied trivially. It would be more convenient to have a quantized energy condition that is evaluated over the original manifold, rather than the quotient.

In this chapter, we propose a definition for a quantized energy level of \((M, \omega, H)\) in the case where \((M, \omega)\) admits a metaplectic-c prequantization. If \(E\) is a quantized energy level of \((M, \omega, H)\), and if one additional condition is satisfied, which we describe in Section 4.2.2, then the symplectic reduction \((M_E, \omega_E)\) is metaplectic-c quantizable, provided that it is a manifold. However, the quantized energy condition is evaluated on a bundle over the level set \(H^{-1}(E)\), and does not require the existence of the symplectic reduction.

This scenario was first studied by Robinson [16], and our work is strongly motivated by his results. However, we choose a more robust condition than the one Robinson considered. As we will show, if \(H_1\) and \(H_2\) are two functions on \((M, \omega)\) with regular values \(E_1\) and \(E_2\), respectively, such that \(H_1^{-1}(E_1) = H_2^{-1}(E_2)\), then under our definition, \(E_1\) is a quantized energy level of \((M, \omega, H_1)\) if and only if \(E_2\) is a quantized energy level of \((M, \omega, H_2)\). In other words, the quantization condition only depends on the geometry of the level set, and not on the dynamics of a particular Hamiltonian. We refer to this theorem as the dynamical invariance property. It is not true of the definition considered in [16].

Section 4.2 summarizes Robinson’s results concerning symplectic reduction, and concludes with our proposed definition for a quantized energy level of a metaplectic-c quantized system. The proof of the dynamical invariance of our definition is given in Section 4.3. In Section 4.4, we consider the example of the harmonic oscillator. This example serves two purposes. First, we demonstrate a computational technique in which a local change of coordinates on \(M\) is lifted to the metaplectic-c prequantization. Second, we give an example of a function on \(M\) that has exactly one level set in common with the harmonic oscillator, and show that our quantization condition on that level set is identical for both functions, while Robinson’s is not.

Finally, in Section 4.5, we adapt the quantized energy definition to apply to a Kostant-Souriau quantizable symplectic manifold. We show that the Kostant-Souriau and metaplectic-c definitions have almost identical properties, but the metaplectic-c version is the only one that yields the correct quantized energy levels for the harmonic oscillator.
4.2 Quantized Energy Levels

Given a symplectic manifold \((M, \omega)\) and a smooth function \(H : M \to \mathbb{R}\), the symplectic reduction of \((M, \omega)\) at the regular value \(E\) of \(H\) is constructed by taking the quotient of the level set \(H^{-1}(E)\) by the flow of the Hamiltonian vector field \(\xi_H\). Suppose \((M, \omega)\) admits a metaplectic-c prequantization \((P, \Sigma, \gamma)\). As discussed in Section 3.3.2, there is a natural lift of \(\xi_H\) to a vector field \(\tilde{\xi}_H\) on \(\text{Sp}(M, \omega)\); \(\tilde{\xi}_H\) can then be lifted to a vector field \(\hat{\xi}_H\) on \(P\) that is horizontal with respect to \(\gamma\). Using some of the results that were given in [17], Robinson [16] constructed a certain associated bundle \(P_S \to H^{-1}(E)\), on which \(\hat{\xi}_H\) induces a vector field, and he examined the conditions under which the quotient of \(P_S\) by the induced flow of \(\hat{\xi}_H\) yields a metaplectic-c prequantization for the symplectic reduction of \((M, \omega)\) at \(E\).

In this section, we review the key results from [16, 17], then state our alternative definition for a quantized energy level of the system \((M, \omega, H)\). Our definition is evaluated on the bundle \(P_S\), and ensures that the symplectic reduction acquires a metaplectic-c prequantization under the same conditions as Robinson’s definition. Further, as we will show in Section 4.3, our definition depends only on the geometry of the level set \(H^{-1}(E)\), and not on the specific choice of \(H\). This is an improvement over the definition from [16].

4.2.1 Vector subspaces and quotients

Assume that the model vector space \(V\) has been chosen as in Section 2.3.1. Let \(W \subset V\) be a real subspace of codimension 1. Define the symplectic orthogonal of \(W\) to be

\[
W^\perp = \{ v \in V : \Omega(v, W) = 0 \}.
\]

Then \(W^\perp\) is a one-dimensional subspace of \(W\). The form \(\Omega\) induces a symplectic form on the quotient space \(W/W^\perp\). This new symplectic vector space has symplectic group \(\text{Sp}(W/W^\perp)\) and metaplectic-c group \(\text{Mp}^c(W/W^\perp)\). The complex structure \(J\) on \(V\) induces a complex structure on \(W/W^\perp\) in such a way that

\[
W/W^\perp \cong (W^\perp \oplus JW^\perp)^\perp, \tag{4.2.1}
\]
where the isomorphism respects both the symplectic structures and the complex structures [17]. We write an element of $W/W^\perp$ as an equivalence class $[w]$ for some $w \in W$.

Let $\text{Sp}(V; W)$ be the subgroup of $\text{Sp}(V)$ consisting of those symplectic automorphisms that preserve $W$:

$$\text{Sp}(V; W) = \{ g \in \text{Sp}(V) : gW = W \}.$$ 

Then elements of $\text{Sp}(V; W)$ also preserve $W^\perp$, and there is a group homomorphism

$$\text{Sp}(V; W) \xrightarrow{\nu} \text{Sp}(W/W^\perp),$$

defined by

$$(\nu g)[w] = [gw], \ \forall g \in \text{Sp}(V; W), \ \forall w \in W.$$ 

Let $\text{Mp}^c(V; W)$ be the preimage of $\text{Sp}(V; W)$ under the projection map $\text{Mp}^c(V) \xrightarrow{\sigma} \text{Sp}(V)$. Robinson and Rawnsley [17] constructed a group homomorphism $\text{Mp}^c(V; W) \xrightarrow{\hat{\nu}} \text{Mp}^c(W/W^\perp)$ such that the diagram below commutes.

$$\begin{array}{ccc}
\text{Mp}^c(V) & \supset & \text{Mp}^c(V; W) \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
\text{Sp}(V) & \supset & \text{Sp}(V; W) \\
\downarrow{\nu} & & \downarrow{\nu} \\
& \text{Sp}(W/W^\perp) & \\
\end{array}$$

The map $\hat{\nu}$ has the following property. Given $a \in \text{Mp}^c(V; W)$, let $\chi(a) = \text{Det}_\mathbb{R}(\sigma(a)|W^\perp)$. Then $\chi$ is a real-valued character on $\text{Mp}^c(V; W)$, and it satisfies

$$\eta \circ \hat{\nu}(a) = \eta(a) \text{sign } \chi(a), \ \forall a \in \text{Mp}^c(V; W).$$

Since $\text{sign } \chi(a)$ is identically 1 on a neighborhood of $I$, we see that $\eta_\ast \circ \hat{\nu}_\ast = \eta_\ast$ as maps between Lie algebras.

**4.2.2 Quotients and reduction over a manifold**

Let $(M, \omega)$ be a symplectic manifold that admits a metaplectic-c prequantization $(P, \Sigma, \gamma)$. The definitions from Section 2.3.2 yield the three-level structure

$$(P; \gamma) \xrightarrow{\Sigma} \text{Sp}(M, \omega) \xrightarrow{\rho} (M, \omega), \ \rho \circ \Sigma = \Pi.$$
Let \( H : M \to \mathbb{R} \) be a smooth function on \( M \). We define the Hamiltonian vector field \( \xi_H \) using the convention that
\[
\xi_H \cdot \omega = dH.
\]
Let the flow of \( \xi_H \) be \( \phi^t \), and note that \( \phi^t \) is a symplectomorphism from its domain to its range for each \( t \). The lift of \( \phi^t \) to a flow \( \tilde{\phi}^t \) on \( \text{Sp}(M, \omega) \) is given by
\[
\tilde{\phi}^t(b) = \phi^t_b \circ b, \quad \forall b \in \text{Sp}(M, \omega),
\]
where we interpret elements of \( \text{Sp}(M, \omega) \) as symplectic isomorphisms as discussed in Section 2.3.2. Let \( \tilde{\xi}_H \) be the vector field on \( \text{Sp}(M, \omega) \) whose flow is \( \tilde{\phi}^t \). Define the vector field \( \hat{\xi}_H \) on \( P \) to be the lift of \( \tilde{\xi}_H \) that is horizontal with respect to the one-form \( \gamma \), and let the flow of \( \hat{\phi}^t \) be \( \hat{\phi}^t \).

We now restrict our attention to a particular level set. Let \( E \in \mathbb{R} \) be a regular value of \( H \), and let \( S = H^{-1}(E) \). The null foliation \( TS^\perp \) is defined fiberwise over \( S \) by
\[
T_sS^\perp = \{ \zeta \in T_sM : \omega(\zeta, T_sS) = 0 \}, \quad \forall s \in S.
\]
Then \( T_sS^\perp \) is a one-dimensional subspace of the tangent space \( T_sS \), and \( T_sS^\perp = \text{span} \{ \xi_H(s) \} \).

The constructions that follow are taken from [16]. As in Section 4.2.1, fix a subspace \( W \subset V \) of codimension 1. By definition, an element \( b \in \text{Sp}(M, \omega)_m \) is a symplectic linear isomorphism \( b : V \to T_mM \). Let \( \text{Sp}(M, \omega; S) \) be the subset of \( \text{Sp}(M, \omega)|_S \) given by
\[
\text{Sp}(M, \omega; S)_s = \{ b \in \text{Sp}(M, \omega)_s : bW = T_sS \}, \quad \forall s \in S.
\]
Then \( \text{Sp}(M, \omega; S) \) is a principal \( \text{Sp}(V; W) \) bundle over \( S \). Note that any \( b \in \text{Sp}(M, \omega; S)_s \) maps \( W^\perp \) to \( T_sS^\perp \).

Viewing \( P \) as a circle bundle over \( \text{Sp}(M, \omega) \), let \( P^S \) be the result of restricting \( P \) to \( \text{Sp}(M, \omega; S) \). Then \( P^S \) is a principal \( \text{Mp}(V; W) \) bundle over \( S \). Let \( \gamma^S \) be the pullback of the one-form \( \gamma \) to \( P^S \). After all of these restrictions, we have constructed the new three-level...
structure

\((P^S, \gamma^S) \to \text{Sp}(M, \omega; S) \to S.\)

For each \(s \in S, T_s S/T_s S^\perp\) is a symplectic vector space with symplectic structure induced by \(\omega_s\). Let \(\text{Sp}(TS/TS^\perp)\) be the symplectic frame bundle for \(TS/TS^\perp\), modeled on \(W/W^\perp:\)

\[\text{Sp}(TS/TS^\perp)_s = \left\{ b' : W/W^\perp \to T_s S/T_s S^\perp : b' \text{ is a symplectic linear isomorphism} \right\}, \quad \forall s \in S.\]

Recall the group homomorphism \(\nu : \text{Sp}(V; W) \to \text{Sp}(W/W^\perp)\) from Section 4.2.1. The bundle associated to \(\text{Sp}(M, \omega; S)\) by the map \(\nu\) can be naturally identified with \(\text{Sp}(TS/TS^\perp)\).

Next, let \(P_S \to S\) be the bundle associated to \(P^S \to S\) by the group homomorphism \(\hat{\nu} : \text{Mp}^c(V; W) \to \text{Mp}^c(W/W^\perp)\). Then \(P_S\) is a principal \(\text{Mp}^c(W/W^\perp)\) bundle over \(S\), and a circle bundle over \(\text{Sp}(TS/TS^\perp)\). The one-form \(\gamma^S\) induces a connection one-form \(\gamma_S\) on \(P_S\). This completes the construction of another three-level structure,

\((P_S, \gamma_S) \to \text{Sp}(TS/TS^\perp) \to S.\)

Now let us consider the actions of \(\phi^t, \widetilde{\phi}^t\) and \(\hat{\phi}^t\) on all of these bundles. First, it is clear that \(\phi^t\) preserves \(S\), since \(\xi_H(s) \in T_s S\) at each \(s \in S\). Therefore \(\widetilde{\phi}^t\) restricts to a flow on \(\text{Sp}(M, \omega; S)\), and so \(\hat{\phi}^t\) restricts to a flow on \(P^S\). One can verify that \(\widetilde{\phi}^t\) and \(\hat{\phi}^t\) are equivariant with respect to the principal \(\text{Sp}(V)\) and \(\text{Mp}^c(V)\) actions, respectively, which implies that \(\widetilde{\phi}^t\) induces a flow on the associated bundle \(\text{Sp}(TS/TS^\perp)\), and \(\hat{\phi}^t\) induces a flow on \(P_S\). We let \(\widetilde{\phi}^t\) and \(\hat{\phi}^t\) also denote the respective flows induced on \(\text{Sp}(TS/TS^\perp)\) and \(P_S\).

Suppose the null foliation is fibrating, so that the quotient of \(S\) by the flow of \(\xi_H\) is a manifold. Let the quotient manifold be \(M_E\), and let \(\omega_E\) be the symplectic form on \(M_E\) induced by \(\omega\). Further suppose that the quotient of \(\text{Sp}(TS/TS^\perp)\) by \(\widetilde{\phi}^t\) is well defined, in which case it is naturally isomorphic to the symplectic frame bundle \(\text{Sp}(M_E, \omega_E)\). This is the additional condition that was referenced in the introduction. As discussed in [16], to ensure that the quotient of \(\text{Sp}(TS/TS^\perp)\) by \(\widetilde{\phi}^t\) is well defined, it is sufficient to require that if \(s \in S\) is fixed by \(\phi^t\) for some \(t\), then \(\phi^t_s\) is the identity on \(T_s S\). The culmination of all of these steps is shown below.
Chapter 4. Quantized Energy Levels and Dynamical Invariance

The obvious way to complete the picture is to factor \((P_S, \gamma_S)\) by \(\hat{\phi}^t\). Robinson [16] addressed the question of when this yields a metaplectic-c prequantization for \((M_E, \omega_E)\) with the following theorem.

**Theorem 4.2.1.** (Robinson, 1990) Assume that the symplectic reduction \((M_E, \omega_E)\) is a manifold, and that its symplectic frame bundle \(\text{Sp}(M_E, \omega_E)\) can be identified with the quotient of \(\text{Sp}(TS/TS^\perp)\) by the induced flow \(\tilde{\phi}^t\). If \(\gamma_S\) has trivial holonomy over all of the closed orbits of \(\tilde{\phi}^t\) on \(\text{Sp}(TS/TS^\perp)\), then the quotient of \((P_S, \gamma_S)\) by \(\hat{\phi}^t\) is a metaplectic-c prequantization for \((M_E, \omega_E)\).

Robinson then observed that the holonomy condition in Theorem 4.2.1 is satisfied if \(\gamma_S\) has trivial holonomy over all closed orbits of \(\tilde{\phi}^t\) on \(\text{Sp}(M, \omega; S)\). The quantization condition that was explored in the remainder of [16] is this latter holonomy condition: the regular value \(E\) is quantized if \(\gamma_S\) has trivial holonomy over the closed orbits of \(\tilde{\phi}^t\) on \(\text{Sp}(M, \omega; S)\). However, we argue that a more robust condition arises when we evaluate the holonomy over orbits in \(\text{Sp}(TS/TS^\perp)\). As such, we propose the following definition.

**Definition 4.2.2.** When \(\gamma_S\) has trivial holonomy over all closed orbits of \(\tilde{\phi}^t\) on \(\text{Sp}(TS/TS^\perp)\), we say that \(E\) is a **quantized energy level** of the system \((M, \omega, H)\).

In those cases where \(E\) is a quantized energy level and \((M_E, \omega_E)\) exists, Theorem 4.2.1 gives a sufficient condition for \((M_E, \omega_E)\) to admit a metaplectic-c prequantization. However, Definition 4.2.2 can also be applied to systems where the symplectic reduction does not exist. For the remainder of the chapter, we do not assume that the symplectic reduction exists.

The next section contains the proof of the dynamical invariance property. We will show that if \(S\) is a regular level set of two functions \(H_1\) and \(H_2\) on \(M\), then \(S\) corresponds to a quantized energy level for \((M, \omega, H_1)\) if and only if it does so for \((M, \omega, H_2)\).
4.3 Dynamical Invariance

We continue to use all of the definitions established in Section 4.2. To prove that the quantized energy condition is dynamically invariant, we proceed in two stages. In Section 4.3.1, we prove Theorem 4.3.2, which states that if $E$ is a quantized energy level for the system $(M, \omega, H)$, then for any diffeomorphism $f : \mathbb{R} \to \mathbb{R}$, the value $f(E)$ is a quantized energy level for the system $(M, \omega, f \circ H)$. While this is a less general statement, it allows us to establish some useful preliminary observations. Then, in Section 4.3.2, we prove Theorem 4.3.5, which states that if the functions $H_1, H_2 : M \to \mathbb{R}$ have regular values $E_1$ and $E_2$, respectively, such that $H_1^{-1}(E_1) = H_2^{-1}(E_2)$, then $E_1$ is a quantized energy level for $(M, \omega, H_1)$ if and only if $E_2$ is a quantized energy level for $(M, \omega, H_2)$.

4.3.1 Invariance under a diffeomorphism

Let $H : M \to \mathbb{R}$ be given, and fix a regular value $E$ of $H$. Let $f : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism, and let $S = H^{-1}(E) = (f \circ H)^{-1}(f(E))$.

Denote the Hamiltonian vector field for $H$ by $\xi_H$, and let it have flow $\phi^t$ on $M$. Then $\xi_H$ lifts to $\tilde{\xi}_H$ on $\text{Sp}(M, \omega)$ and to $\tilde{\xi}_H$ on $P$, with flows $\tilde{\phi}^t$ and $\tilde{\rho}^t$, respectively, as in Section 4.2.2. Similarly, denote the Hamiltonian vector field for $f \circ H$ by $\xi_{f \circ H}$, and let it have flow $\rho^t$ on $M$. By the same process, we obtain the lifts $\tilde{\xi}_{f \circ H}$ on $\text{Sp}(M, \omega)$ and $\tilde{\xi}_{f \circ H}$ on $P$, with flows $\tilde{\rho}^t$ and $\tilde{\rho}^t$, respectively.

Observe that for any $m \in M$,

$$d(f \circ H)|_m = \frac{df(H)}{dH}|_{H(m)} \frac{dH}{dH}|_m,$$

which implies that

$$\xi_{f \circ H}(m) = \frac{df(H)}{dH}|_{H(m)} \xi_H(m).$$

Since $S$ is a level set of $H$, $\frac{df(H)}{dH}|_{H(m)}$ is constant over $S$. Let $c \in \mathbb{R}$ be that constant value on $S$. Then $\xi_{f \circ H} = c \xi_H$ everywhere on $S$, which implies that $\rho^t = \phi^{ct}$ everywhere on $S$.

The symplectic vector bundle $\text{Sp}(TS/TS^\perp)$ is naturally identified with the bundle associated to $\text{Sp}(M, \omega; S)$ by the group homomorphism $\nu : \text{Sp}(V; W) \to \text{Sp}(W/W^\perp)$. We write an element
of $W/W^\perp$ as an equivalence class $[w]$ for some $w \in W$, and we write an element of $T_s S/T_s S^\perp$ as an equivalence class $[\zeta]$ for some $\zeta \in T_s S$. As discussed in Section 4.2.2, the lifted flow $\tilde{\phi}^t$ on $\text{Sp}(M, \omega)$ maps $\text{Sp}(M, \omega; S)$ to itself and induces a flow $\tilde{\phi}^t$ on $\text{Sp}(TS/TS^\perp)$. More explicitly, for any $b' \in \text{Sp}(TS/TS^\perp)$, we define $\tilde{\phi}^t(b')$ by choosing $b \in \text{Sp}(M, \omega; S)$ such that $[bw] = b'[w]$ for all $w \in W$, and setting

$$\tilde{\phi}^t(b'[w]) = [\tilde{\phi}^t(b)w], \quad \forall w \in W.$$ 

These remarks apply equally well to the flows $\tilde{\rho}^t$ on $\text{Sp}(M, \omega)$ and $\text{Sp}(TS/TS^\perp)$.

**Lemma 4.3.1.** $\tilde{\rho}^t = \tilde{\phi}^c t$ on $\text{Sp}(TS/TS^\perp)$.

**Proof.** Fix $s \in S$ and $b' \in \text{Sp}(TS/TS^\perp)_s$. Let $b \in \text{Sp}(M, \omega; S)_s$ be such that for all $w \in W$, $[bw] = b'[w]$. Since $\rho^t = \phi^c t$ on $S$, we have $\rho^t_s|TS = \phi^c t|TS$. Therefore, for any $w \in W$,

$$\tilde{\rho}^t(b)w = \rho^t_s|_s(bw) = \phi^c t|_s(bw) = \tilde{\phi}^c t(b)w,$$

where we used the fact that $bw \in T_s S$. Using the definitions of $\tilde{\phi}^t$ and $\tilde{\rho}^t$ on $\text{Sp}(TS/TS^\perp)$, it follows that

$$\tilde{\rho}^t(b'[w]) = [\tilde{\rho}^t(b)w] = [\tilde{\phi}^c t(b)w] = \tilde{\phi}^c t(b'[w]).$$

Thus $\tilde{\rho}^t = \tilde{\phi}^c t$ on $\text{Sp}(TS/TS^\perp)$.

Lemma 4.3.1 implies that $\tilde{\xi}_{f \circ H} = c\tilde{\zeta}_H$ on $\text{Sp}(TS/TS^\perp)$. Since $\tilde{\xi}_{f \circ H}$ is just a constant multiple of $\tilde{\zeta}_H$, it is clear that the flows of both vector fields have the same orbits. Thus, if $\gamma_S$ has trivial holonomy over all closed orbits of $\tilde{\phi}^t$ on $\text{Sp}(TS/TS^\perp)$, then it also has trivial holonomy over all closed orbits of $\tilde{\rho}^t$. The following is now immediate.

**Theorem 4.3.2.** If $E$ is a quantized energy level for $(M, \omega, H)$ and $f : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, then $f(E)$ is a quantized energy level for $(M, \omega, f \circ H)$.

### 4.3.2 Regular level set of multiple functions

Let $H_1, H_2 : M \to \mathbb{R}$ be two smooth functions, and assume that there are $E_1, E_2 \in \mathbb{R}$ such that $E_j$ is a regular value of $H_j$ for $j = 1, 2$, and $H_1^{-1}(E_1) = H_2^{-1}(E_2)$. Let this shared level set be
S. Let the Hamiltonian vector fields $\xi_{H_1}$ and $\xi_{H_2}$ on $M$ have flows $\phi^t$ and $\rho^t$, respectively. The induced flows $\bar{\phi}^t$ and $\bar{\rho}^t$ on $\text{Sp}(TS/TS^\perp)$ are defined as in Section 4.3.1.

The key observation in this scenario is that $TS$ and $TS^\perp$ do not depend on $H_1$ or $H_2$. For all $s \in S$, we have

$$T_s S^\perp = \text{span} \{ \xi_{H_1}(s) \} = \text{span} \{ \xi_{H_2}(s) \},$$

which implies that there is a map $c : S \to \mathbb{R}$ such that $\xi_{H_2}(s) = c(s)\xi_{H_1}(s)$ for all $s \in S$. The map $c$ need not be constant on $S$, and $\xi_{H_1}$ and $\xi_{H_2}$ need not be parallel away from $S$.

In Lemma 4.3.1, we were able to establish a relationship between $\bar{\rho}^t$ and $\bar{\phi}^t$. In the more general case, we will not be able to relate $\bar{\rho}^t$ and $\bar{\phi}^t$ directly, but must instead work in terms of their time derivatives.

For any $s \in S$, $\rho^t|_s$ and $\phi^t|_s$ preserve $TS$. Therefore, for all $\zeta \in T_sS$, the tangent vectors $\frac{d}{dt} \big|_{t=0} (\rho^t|_s \zeta)$ and $\frac{d}{dt} \big|_{t=0} (\phi^t|_s \zeta)$ are elements of $T_\zeta TS$. Note that the vector space $T_sS$ can be naturally identified with its tangent space $T_\zeta T_sS$, which is a subspace of $T_\zeta TS$. Thus $\xi_{H_1}(s) \in T_sS$ can also be viewed as an element of $T_\zeta TS$. This identification is used in the statement of the following result.

**Lemma 4.3.3.** For all $s \in S$ and all $\zeta \in T_sS$,

$$\frac{d}{dt} \bigg|_{t=0} (\rho^t|_s \zeta) = c(s) \frac{d}{dt} \bigg|_{t=0} (\phi^t|_s \zeta) + (\zeta c)\xi_{H_1}(s).$$

**Proof.** We ignore $M$, and treat $S$ as a $(2n-1)$-dimensional manifold. The vector fields $\xi_{H_1}$ and $\xi_{H_2}$ on $S$ have flows $\phi^t$ and $\rho^t$, respectively, and satisfy $\xi_{H_2}(s) = c(s)\xi_{H_1}(s)$.

Fix $s \in S$, and let $U$ be a coordinate neighborhood for $s$ with coordinates $X = (X_1, \ldots, X_{2n-1})$. We write $\phi^t = (\phi^t_1, \ldots, \phi^t_{2n-1})$ and $\rho^t = (\rho^t_1, \ldots, \rho^t_{2n-1})$ with respect to the local coordinates. Then $\phi^t|_s$ becomes a $(2n-1) \times (2n-1)$ matrix with the $(j,k)$th entry given by

$$\left(\phi^t|_s\right)_{jk} = \frac{\partial \phi^t_j(X)}{\partial X_k} \bigg|_{X=s},$$

and similarly for $\rho^t|_s$. Also, since $\xi_{H_2} = c\xi_{H_1}$, we have

$$\frac{d}{dt} \bigg|_{t=0} \rho^t_j = c \frac{d}{dt} \bigg|_{t=0} \phi^t_j, \quad j = 1, \ldots, 2n-1,$$
Let \( \zeta \in T_s S \) be arbitrary. Then
\[
\zeta = \sum_{k=1}^{2n-1} a_k \frac{\partial}{\partial X_k} \bigg|_{X=s}
\]
for some coefficients \( a_1, \ldots, a_{2n-1} \in \mathbb{R} \), and
\[
(\phi^t_s|s\zeta)_j = \sum_{k=1}^{2n-1} a_k \frac{\partial \phi^t_s(X)}{\partial X_k} \bigg|_{X=s} = \zeta \phi^t_j.
\]
The same process establishes that \( (\rho^t_s|s\zeta)_j = \zeta \rho^t_j \). When we take the time derivative of the latter equation, we find
\[
\frac{d}{dt} \bigg|_{t=0} (\rho^t_s|s\zeta)_j = \zeta \left( \frac{d}{dt} \bigg|_{t=0} \rho^t_j \right) = \zeta \left( \frac{d}{dt} \bigg|_{t=0} \phi^t_j \right) = \zeta \left( \frac{d}{dt} \bigg|_{t=0} (\phi^t_s|s\zeta)_j \right).
\]
Now we apply the Leibniz rule and recall that \( \frac{d}{dt} \bigg|_{t=0} \phi^t_j(s) = \xi_{H_1}(s)_j \). The result is
\[
\frac{d}{dt} \bigg|_{t=0} (\rho^t_s|s\zeta)_j = c(s) \frac{d}{dt} \bigg|_{t=0} (\phi^t_s|s\zeta)_j + (\zeta c) \xi_{H_1}(s)_j.
\]
(4.3.1)

The coordinates on \( U \) determine coordinates for \( TS|U \), which in turn provide a basis for \( T_\zeta TS \). In terms of that basis,
\[
\frac{d}{dt} \bigg|_{t=0} (\phi^t_s|s\zeta) = \left( \xi_{H_1}(s)_1, \ldots, \xi_{H_1}(s)_{2n-1}, \frac{d}{dt} \bigg|_{t=0} (\phi^t_s|s\zeta)_1, \ldots, \frac{d}{dt} \bigg|_{t=0} (\phi^t_s|s\zeta)_{2n-1} \right),
\]
\[
\frac{d}{dt} \bigg|_{t=0} (\rho^t_s|s\zeta)_i = \left( \xi_{H_2}(s)_1, \ldots, \xi_{H_2}(s)_{2n-1}, \frac{d}{dt} \bigg|_{t=0} (\rho^t_s|s\zeta)_1, \ldots, \frac{d}{dt} \bigg|_{t=0} (\rho^t_s|s\zeta)_{2n-1} \right),
\]
and
\[
\xi_{H_1}(s) = (0, \ldots, 0, \xi_{H_1}(s)_1, \ldots, \xi_{H_1}(s)_{2n-1})
\]
as an element of \( T_\zeta TS \). Substituting Equation (4.3.1) into the expression for \( \frac{d}{dt} \bigg|_{t=0} (\rho^t_s|s\zeta) \) and comparing the result with the expression for \( \frac{d}{dt} \bigg|_{t=0} (\phi^t_s|s\zeta) \) shows that
\[
\frac{d}{dt} \bigg|_{t=0} (\rho^t_s|s\zeta) = c(s) \frac{d}{dt} \bigg|_{t=0} (\phi^t_s|s\zeta) + (\zeta c) \xi_{H_1}(s),
\]
as desired.
Lemma 4.3.4. For all \( s \in S \) and all \( b' \in \text{Sp}(TS/TS^\perp)_s \),

\[
\tilde{\xi}_{H_2}(b') = c(s)\tilde{\xi}_{H_1}(b').
\]

Proof. Within the model vector space \( V \), fix a symplectic basis \((\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_n)\). Let

\[
W = \text{span}\{\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_2, \ldots, \tilde{y}_n\},
\]

so that

\[
W^\perp = \text{span}\{\tilde{x}_1\} \quad \text{and} \quad W/W^\perp = \text{span}\{[\tilde{x}_2], \ldots, [\tilde{x}_n], [\tilde{y}_2], \ldots, [\tilde{y}_n]\}.
\]

For convenience, let \((\tilde{z}_1, \ldots, \tilde{z}_{2n}) = (\tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_2, \ldots, \tilde{y}_n, \tilde{y}_1)\). Then

\[
W = \text{span}\{\tilde{z}_1, \ldots, \tilde{z}_{2n-1}\}, \quad W^\perp = \text{span}\{\tilde{z}_1\}, \quad W/W^\perp = \text{span}\{[\tilde{z}_2], \ldots, [\tilde{z}_{2n-1}]\}.
\]

For this proof, we will not view an element of the symplectic frame bundle as a map, but as a basis for a tangent space. Explicitly, over \( m \in M \), we identify \( b \in \text{Sp}(M, \omega)_m \) with the ordered 2n-tuple \((\zeta_1, \ldots, \zeta_{2n}) \in (T_mM)^{2n}\), where \( b\tilde{\omega}_j = \zeta_j \) for \( j = 1, \ldots, 2n \). Similarly, over \( s \in S \), we identify \( b' \in \text{Sp}(TS/TS^\perp)_s \) with the ordered \((2n - 2)\)-tuple \(([\zeta_2], \ldots, [\zeta_{2n}] \in (T_sS/T_sS^\perp)^{2n-2}\), where \( b'\tilde{\omega}_j = [\zeta_j] \) for \( j = 2, \ldots, 2n - 1 \).

Let \( s \in S \) and \( b \in \text{Sp}(M, \omega; S)_s \) be given. Let \( b\tilde{\omega}_j = \zeta_j \in T_sM \) for \( j = 1, \ldots, 2n \). Then \( \zeta_j \in T_sS \) for \( j = 1, \ldots, 2n - 1 \) and \( \zeta_1 \in T_sS^\perp \). The flow \( \tilde{\varphi}^t \) on \( \text{Sp}(M, \omega; S) \), evaluated at \( b \), is given by

\[
\tilde{\varphi}^t(b) = \rho^t_s|_m \circ b = (\rho^t_s|_m\zeta_1, \ldots, \rho^t_s|_m\zeta_{2n}),
\]

and the analogous expression holds for \( \varphi^t \).

When we descend to \( \text{Sp}(TS/TS^\perp) \), the image of \( b \) is the element \( b' = ([\zeta_2], \ldots, [\zeta_{2n-1}]) \in (T_sS/T_sS^\perp)^{2n-2} \). The induced flow \( \tilde{\rho}^t \) on \( \text{Sp}(TS/TS^\perp) \), evaluated at \( b' \), is

\[
\tilde{\rho}^t(b') = ([\rho^t_s|_s\zeta_2], \ldots, [\rho^t_s|_s\zeta_{2n-1}]).
\]

As \( t \) varies, this expression describes a curve in \((TS/TS^\perp)^{2n-2}\), through \( b' \). The tangent vector
to this curve at \( b' \) is
\[
\tilde{\xi}_{H_2}(b') = \left. \frac{d}{dt} \right|_{t=0} \tilde{\rho}^t(b') = \left( \left. \frac{d}{dt} \right|_{t=0} \rho^t_{s|\zeta_2}, \ldots, \left. \frac{d}{dt} \right|_{t=0} \rho^t_{s|\zeta_{2n-1}} \right)
\in T_{[\zeta]}(TS/TS^\perp) \times \ldots \times T_{[\zeta_{2n-1}]}(TS/TS^\perp).
\]

The pushforward of the quotient map \( TS \to TS/TS^\perp \), based at \( \zeta_j \), is a linear surjection \( T_{\zeta_j}TS \to T_{[\zeta_j]}(TS/TS^\perp) \) whose kernel is \( T_{\zeta_j}T_sS^\perp \). If we identify \( T_{\zeta_j}T_sS^\perp \) with \( T_sS^\perp \), then we get a natural isomorphism between \( T_{[\zeta_j]}(TS/TS^\perp) \) and \( T_{\zeta_j}TS/T_sS^\perp \). Applying these isomorphisms to the components of \( \tilde{\xi}_{H_2}(b') \) yields
\[
\tilde{\xi}_{H_2}(b') = \left( \left. \frac{d}{dt} \right|_{t=0} \phi^t_{s|\zeta_2}, \ldots, \left. \frac{d}{dt} \right|_{t=0} \phi^t_{s|\zeta_{2n-1}} \right)
\in T_{\zeta_j}TS/T_sS^\perp \times \ldots \times T_{\zeta_{2n-1}}TS/T_sS^\perp.
\]

Since \( \zeta_j \in T_sS \) for \( j = 2, \ldots, 2n - 1 \), we can use Lemma 4.3.3:
\[
\left. \frac{d}{dt} \right|_{t=0} (\rho^t_{s|\zeta_j}) = c(s) \left. \frac{d}{dt} \right|_{t=0} (\phi^t_{s|\zeta_j}) + (\zeta_j c) \xi_{H_1}(s),
\]
where each term in the above equation is viewed as an element of \( T_{\zeta_j}TS \). Upon descending to \( T_{\zeta_j}TS/T_sS^\perp \), the multiple of \( \xi_{H_1}(s) \) vanishes and we find that
\[
\left. \frac{d}{dt} \right|_{t=0} (\rho^t_{s|\zeta_j}) = c(s) \left. \frac{d}{dt} \right|_{t=0} (\phi^t_{s|\zeta_j}).
\]

Therefore
\[
\tilde{\xi}_{H_2}(b') = c(s) \left( \left. \frac{d}{dt} \right|_{t=0} (\phi^t_{s|\zeta_2}), \ldots, c(s) \left. \frac{d}{dt} \right|_{t=0} (\phi^t_{s|\zeta_{2n-1}}) \right) = c(s) \tilde{\xi}_{H_1}(b').
\]

This completes the proof. \( \square \)

We can now prove the result that was the objective of this section.

**Theorem 4.3.5.** If \( H_1, H_2 : M \to \mathbb{R} \) are smooth functions such that \( H_1^{-1}(E_1) = H_2^{-1}(E_2) \) for
regular values \( E_j \) of \( H_j, j = 1, 2 \), then \( E_1 \) is a quantized energy level for \( (M, \omega, H_1) \) if and only if \( E_2 \) is a quantized energy level for \( (M, \omega, H_2) \).

**Proof.** Using Lemma 4.3.4, we see that \( \tilde{\xi}_{H_1} \) and \( \tilde{\xi}_{H_2} \) are parallel on \( \text{Sp}(TS/TS^\perp) \), although the multiplicative factor that relates them is not necessarily constant. This implies that \( \tilde{\phi}^t \) and \( \tilde{\rho}^t \) have identical orbits in \( \text{Sp}(TS/TS^\perp) \), and if \( \gamma_S \) has trivial holonomy over the closed orbits of \( \tilde{\phi}^t \), then the same must also be true for \( \tilde{\rho}^t \). Thus, if \( E_1 \) is a quantized energy level for \( (M, \omega, H_1) \), then \( E_2 \) is a quantized energy level for \( (M, \omega, H_2) \), and vice versa. \( \square \)

### 4.4 The Harmonic Oscillator

In this section, we apply our quantized energy condition to the \( n \)-dimensional harmonic oscillator. The metaplectic-c quantization of this system has already been examined in [16] and [17], and it comes as no surprise that we obtain the same quantized energy levels that are derived in those treatments using prequantization data:

\[
E_N = h \left( N + \frac{n}{2} \right),
\]

where \( N \) is an integer such that \( E_N \) is positive\(^1\). Our first objective in studying this example is to present a useful computational technique: we will locally change coordinates from Cartesian to symplectic polar on the base manifold, then show how to lift this change to the symplectic frame bundle and prequantization bundle. Once symplectic polar coordinates have been established, we will turn to our second objective, which is to construct an explicit example that illustrates the principal of dynamical invariance.

#### 4.4.1 Initial choices

Let \( (\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n) \) be a basis for the model vector space \( V \) such that \( \Omega = \sum_{j=1}^{n} \hat{x}_j^* \wedge \hat{y}_j^* \). All elements of \( V \) will be written as ordered \( 2n \)-tuples with respect to this basis. Assume that \( V \) is identified with \( \mathbb{C}^n \) by mapping the point \( (a_1, \ldots, a_n, b_1, \ldots, b_n) \in V \) to the point

\(^1\)Both [16] and [17] derive the more familiar condition \( N \geq 0 \) by proceeding from prequantization to quantization and introducing a polarization. Since our quantized energy condition uses only the metaplectic-c prequantization, we do not obtain this constraint on the starting point for \( N \).
\((b_1 + ia_1, \ldots, b_n + ia_n) \in \mathbb{C}^n\). Then the complex structure \(J\) on \(V\) is given in matrix form by
\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]
where \(I\) is the \(n \times n\) identity matrix.

A note concerning notation: given expressions \(a_j\) and \(b_j\), \(j = 1, \ldots, n\), we let \((a_j)_{1 \leq j \leq n}\) represent the \(n \times n\) diagonal matrix \(\text{diag}(a_1, \ldots, a_n)\), and we let \(\begin{pmatrix} a_j \\
 b_j \end{pmatrix}_{1 \leq j \leq n}\) represent the \(2n \times 1\) column vector \((a_1, \ldots, a_n, b_1, \ldots, b_n)^T\).

Let \(M = \mathbb{R}^{2n}\), with Cartesian coordinates \((p_1, \ldots, p_n, q_1, \ldots, q_n)\) and symplectic form \(\omega = \sum_{j=1}^{n} dp_j \wedge dq_j\). The energy function for the harmonic oscillator is
\[
H = \frac{1}{2} \sum_{j=1}^{n} (p_j^2 + q_j^2),
\]
which has Hamiltonian vector field
\[
\xi_H = \sum_{j=1}^{n} (q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j}).
\]

Let this vector field have flow \(\phi^t\) on \(M\).

There is a global trivialization of \(TM\) given by identifying \(\tilde{x}_j \mapsto \frac{\partial}{\partial p_j}\big|_m, \tilde{y}_j \mapsto \frac{\partial}{\partial q_j}\big|_m\) at every point \(m \in M\), which induces a global trivialization of \(\text{Sp}(M, \omega)\). Let \(P\) be the trivial bundle \(M \times \text{Mp}^c(V)\), with bundle projection map \(P \xrightarrow{\Pi} M\). Define the map \(P \xrightarrow{\Sigma} \text{Sp}(M, \omega)\) by
\[
\Sigma(m, a) = (m, \sigma(a)), \quad \forall m \in M, \quad \forall a \in \text{Mp}^c(V),
\]
where the ordered pair on the right-hand side is written with respect to the global trivialization stated above. Then \((P, \Sigma)\) is a metaplectic-c structure for \((M, \omega)\).

On \(M\), define the one-form \(\beta\) by
\[
\beta = \frac{1}{2} \sum_{j=1}^{n} (p_j dq_j - q_j dp_j),
\]
so that \(d\beta = \omega\). Let \(\theta_0\) be the trivial connection on the product bundle \(M \times \text{Mp}^c(V)\), and define the one-form \(\gamma\) on \(P\) by
\[
\gamma = \frac{1}{i\hbar} \Pi^* \beta + \frac{1}{2} \eta_\ast \theta_0.
\]

Then \((P, \Sigma, \gamma)\) is a metaplectic-c prequantization for \((M, \omega)\). In fact, it is the unique metaplectic-
c prequantization up to isomorphism, since $M$ is contractible.

Fix $E > 0$, a regular value of $H$, and let $S = H^{-1}(E)$. Let $m_0 \in S$ be given. Observe that the entire system $(P, \Sigma, \gamma) \to (M, \omega, H)$ is invariant under following transformations: (1) rotation of the $p_j q_j$-plane about the origin by any angle, for any $j$, and (2) simultaneous rotations of the $p_j p_k$- and $q_j q_k$-planes about the origin by the same angle, for any $j \neq k$. Thus, after suitable rotations, we can assume without loss of generality that $m_0 = (p_{01}, \ldots, p_{0n}, 0, \ldots, 0)$ in Cartesian coordinates, where $p_{0j} \neq 0$ for all $j$. It is easily established that

$$\phi^t(m_0) = \begin{pmatrix} (\cos t)_{1 \leq j \leq n} & (\sin t)_{1 \leq j \leq n} \\ (-\sin t)_{1 \leq j \leq n} & (\cos t)_{1 \leq j \leq n} \end{pmatrix} \begin{pmatrix} p_{0j} \\ 0 \end{pmatrix}_{1 \leq j \leq n}.$$

This expression describes a periodic orbit with period $2\pi$. Let $C$ be the orbit of $\xi_H$ through $m_0$: $C = \{ \phi^t(m_0) : t \in \mathbb{R} \}$.

### 4.4.2 From Cartesian to symplectic polar coordinates

**On the Manifold**

By symplectic polar coordinates, we mean the local coordinates $(s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$ given by

$$s_j = \frac{1}{2} \left( p_j^2 + q_j^2 \right), \quad \theta_j = \tan^{-1} \left( \frac{q_j}{p_j} \right), \quad j = 1, \ldots, n, \quad (4.4.1)$$

whenever these expressions are defined. The polar angles $\theta_j$ are all defined modulo $2\pi$. For later reference, the inverse coordinate transformations are

$$p_j = \sqrt{2s_j} \cos \theta_j, \quad q_j = \sqrt{2s_j} \sin \theta_j, \quad j = 1, \ldots, n. \quad (4.4.2)$$

Let

$$U = \{(p_1, \ldots, p_n, q_1, \ldots, q_n) \in M : p_j^2 + q_j^2 > 0, \quad j = 1, \ldots, n \},$$

so that symplectic polar coordinates and the corresponding vector fields $\left\{ \frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_n}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n} \right\}$ are defined everywhere on $U$. Observe that the orbit $C$ is contained in $U$. We will construct a local trivialization for $\text{Sp}(M, \omega)$ over $U$, and a local trivialization for $P$ over $C$. 
When we convert to symplectic polar coordinates on \( U \), we find
\[
\omega = \sum_{j=1}^{n} ds_j \wedge d\theta_j,
\]
which implies that for all \( m \in U \), \( \left\{ \frac{\partial}{\partial s_1} \bigg|_m, \ldots, \frac{\partial}{\partial s_n} \bigg|_m, \frac{\partial}{\partial \theta_1} \bigg|_m, \ldots, \frac{\partial}{\partial \theta_n} \bigg|_m \right\} \) is a symplectic basis for \( T_m M \). Further,
\[
\beta = \sum_{j=1}^{n} s_j d\theta_j, \quad H = \sum_{j=1}^{n} s_j, \quad \xi_H = -\sum_{j=1}^{n} \frac{\partial}{\partial \theta_j},
\]
and \( m_0 = (s_{01}, \ldots, s_{0n}, 0, \ldots, 0) \), where \( s_{0j} = \frac{1}{2p^2_{0j}}, \ j = 1, \ldots, n \). The orbit \( \mathcal{C} \) has the form
\[
\mathcal{C} = \{(s_{01}, \ldots, s_{0n}, \tau, \ldots, \tau) : \tau \in \mathbb{R}/2\pi\mathbb{Z}\}.
\]

The goal is to lift this change of coordinates to the symplectic frame bundle, then to the metaplectic-\( c \) prequantization. To facilitate this process, we introduce the following notation. Let \( \Phi_c : U \to \mathbb{R}^{2n} \) be the Cartesian coordinate map, and let \( U_c = \Phi_c(U) \). Similarly, let \( \Phi_p : U \to \mathbb{R}^n \times (\mathbb{R}/2\pi\mathbb{Z})^n \) be the symplectic polar coordinate map, and let \( U_p = \Phi_p(U) \). Let \( F \) denote the transition map \( \Phi_p \circ \Phi_c^{-1} \). Then we have the following commutative diagram.

\[
\begin{array}{ccc}
U & \xrightarrow{\Phi_c} & U_c \subset \mathbb{R}^{2n} \\
\downarrow{\Phi_p} & & \downarrow{\Phi_p} \\
U_p \subset \mathbb{R}^n \times (\mathbb{R}/2\pi\mathbb{Z})^n & \xrightarrow{F} & U_p \subset \mathbb{R}^n \times (\mathbb{R}/2\pi\mathbb{Z})^n \\
\end{array}
\]

This will serve as a model for the changes of coordinates on \( \text{Sp}(M, \omega) \) and \( P \).

**On the Symplectic Frame Bundle**

An element of the fiber \( \text{Sp}(M, \omega)_m \) is a symplectic isomorphism from \( V \) to \( T_m M \). Over \( U \), we define the sections \( b_c \) and \( b_p \) of \( \text{Sp}(M, \omega) \) as follows. At each \( m \in U \), they are given by

\[
b_c(m) : V \to T_m M \text{ such that } \left( \begin{array}{c} \hat{x}_j \\ \hat{y}_j \end{array} \right)_{1 \leq j \leq n} \mapsto \left( \begin{array}{c} \frac{\partial}{\partial \hat{x}_j} \bigg|_m \\ \frac{\partial}{\partial \hat{y}_j} \bigg|_m \end{array} \right)_{1 \leq j \leq n},
\]

\[
b_p(m) : V \to T_m M \text{ such that } \left( \begin{array}{c} \hat{x}_j \\ \hat{y}_j \end{array} \right)_{1 \leq j \leq n} \mapsto \left( \begin{array}{c} \frac{\partial}{\partial \hat{x}_j} \bigg|_m \\ \frac{\partial}{\partial \hat{y}_j} \bigg|_m \end{array} \right)_{1 \leq j \leq n}.
\]
Using these sections, we define two maps $\tilde{\Phi}_c, \tilde{\Phi}_p$ on $\text{Sp}(M, \omega)$ over $U$:

$$
\tilde{\Phi}_c : \text{Sp}(M, \omega)|_U \to U_c \times \text{Sp}(V)
$$

is defined by

$$
\tilde{\Phi}_c(b_c(m) \cdot g) = (\Phi_c(m), g), \quad \forall m \in U, \quad \forall g \in \text{Sp}(V),
$$

and

$$
\tilde{\Phi}_p : \text{Sp}(M, \omega)|_U \to U_p \times \text{Sp}(V)
$$

is defined by

$$
\tilde{\Phi}_p(b_p(m) \cdot g) = (\Phi_p(m), g), \quad \forall m \in U, \quad \forall g \in \text{Sp}(V).
$$

In other words, the section $b_c$ determines the Cartesian trivialization of $\text{Sp}(M, \omega)$ over $U$, and the section $b_p$ determines the symplectic polar trivialization. Lifting the change of coordinates to $\text{Sp}(M, \omega)|_U$ is equivalent to finding a map $\tilde{F} : U_c \times \text{Sp}(V) \to U_p \times \text{Sp}(V)$ that is a lift of $F : U_c \to U_p$ and such that the following commutes.

If such a map $\tilde{F}$ exists, then it must satisfy

$$
\tilde{F} \circ \tilde{\Phi}_c(b_c) = \tilde{\Phi}_p(b_c),
$$

and indeed, once this condition is satisfied, then the value of $\tilde{F}$ everywhere else will be determined by the $\text{Sp}(V)$ group action and the fact that $\tilde{F}$ is a lift of $F$. Let us then evaluate $\tilde{\Phi}_p(b_c)$.

For all $m \in U$, let

$$
G(m) = \begin{pmatrix}
\left( \frac{\partial p_j}{\partial s_j} \right)_m & 1 \leq j \leq n \\
\left( \frac{\partial q_j}{\partial s_j} \right)_m & 1 \leq j \leq n \\
\left( \frac{\partial p_j}{\partial \theta_j} \right)_m & 1 \leq j \leq n \\
\left( \frac{\partial q_j}{\partial \theta_j} \right)_m & 1 \leq j \leq n
\end{pmatrix}.
$$
so that
\[
G(m) \begin{pmatrix}
\left. \frac{\partial}{\partial p_j} \right|_m \\
\left. \frac{\partial}{\partial q_j} \right|_m
\end{pmatrix}_{1 \leq j \leq n} = \begin{pmatrix}
\left. \frac{\partial}{\partial s_j} \right|_m \\
\left. \frac{\partial}{\partial \theta_j} \right|_m
\end{pmatrix}_{1 \leq j \leq n}.
\]

Then \(G(m)\) is a symplectic matrix, and can be viewed as an element of \(\mathrm{Sp}(V)\). From the definitions of \(b_c(m)\) and the group action, we see that
\[
b_c(m) : G(m) \begin{pmatrix}
\tilde{x}_j \\
\tilde{y}_j
\end{pmatrix}_{1 \leq j \leq n} \mapsto G(m) \begin{pmatrix}
\left. \frac{\partial}{\partial p_j} \right|_m \\
\left. \frac{\partial}{\partial q_j} \right|_m
\end{pmatrix}_{1 \leq j \leq n} = \begin{pmatrix}
\left. \frac{\partial}{\partial s_j} \right|_m \\
\left. \frac{\partial}{\partial \theta_j} \right|_m
\end{pmatrix}_{1 \leq j \leq n}.
\]

Thus
\[
\Phi_p(b_c(m)) = (\Phi_p(m), G(m)),
\]
and more generally,
\[
\Phi_p(b_c(m) \cdot g) = (\Phi_p(m), G(m)g), \quad \forall g \in \mathrm{Sp}(V).
\]

Hence we define the map \(F\) by
\[
F(\Phi_c(m), g) = (\Phi_p(m), G(m)g), \quad \forall m \in U, \forall g \in \mathrm{Sp}(V).
\]

Observe, using Equation (4.4.2), that the entries of \(G(m)\) are singly defined with respect to the angles \(\theta_j\), so \(G(m)\) is singly defined as \(m\) traverses the curve \(C\), or any other closed path through \(U\). This is to be expected, since the symplectic polar coordinates are singly defined everywhere on \(U\). However, we will encounter different behavior when we lift the change of variables to the metaplectic-c prequantization.

**On the Metaplectic-c Prequantization**

When we lift the change of variables to \(P\), we restrict our attention further from \(U\) to \(C\). Let \(C_c = \Phi_c(C)\) and let \(C_p = \Phi_p(C)\). To emphasize that all of our calculations take place over the closed curve \(C\), we let \(m(\tau) = (s_{01}, \ldots, s_{0n}, \tau, \ldots, \tau) \in C\) for all \(\tau \in \mathbb{R}/2\pi\mathbb{Z}\), and we abbreviate \(G(m(\tau))\) by \(G(\tau)\). Then \(G(\tau)\) is a closed loop through \(\mathrm{Sp}(V)\) with period \(2\pi\).
Recall that $P = M \times \text{Mp}^{c}(V)$, and that $\Sigma : P \to \text{Sp}(M, \omega)$ is given by $\Sigma(m, a) = (m, \sigma(a))$ for all $(m, a) \in P$, where the right-hand side is written with respect to the Cartesian trivialization. In other words, $P$ was constructed to be consistent with Cartesian coordinates. To formalize this property, we let the map $\hat{\Phi}^c: P|_{C} \to C^c \times \text{Mp}^c(V)$ be defined by

$$\hat{\Phi}^c(m(\tau), a) = (\Phi^c(m(\tau)), a), \ \forall (m(\tau), a) \in P|_{C}.$$ 

Then $\hat{\Phi}^c$ is compatible with $\tilde{\Phi}^c$ in the sense that the following diagram commutes.

$$\begin{array}{ccc}
P|_{C} & \xrightarrow{\Sigma} & \text{Sp}(M, \omega)|_{C} \\
| \downarrow \hat{\Phi}^c \downarrow \hat{\Phi}^c | & & | \downarrow \hat{\Phi}^c \downarrow \hat{\Phi}^c | \\
C^c \times \text{Mp}^c(V) & \xrightarrow{\sigma} & C^c \times \text{Sp}(V)
\end{array}$$

In the bottom line, $\sigma$ maps $C^c \times \text{Mp}^c(V)$ to $C^c \times \text{Sp}(V)$ by acting on the second component.

Our goal is to construct a map $\hat{\Phi}^p: P|_{C} \to C^p \times \text{Mp}^p(V)$ that is compatible with $\tilde{\Phi}^p$ in the same sense. To do so, we will find a map $\bar{F}$ such that the diagram below commutes, then set $\hat{\Phi}^p = \bar{F} \circ \hat{\Phi}^c$.

$$\begin{array}{ccc}
P|_{C} & \xrightarrow{\Sigma} & \text{Sp}(M, \omega)|_{C} \\
| \downarrow \hat{\Phi}^c \downarrow \hat{\Phi}^c | & & | \downarrow \hat{\Phi}^c \downarrow \hat{\Phi}^c | \\
C^c \times \text{Mp}^c(V) & \xrightarrow{\bar{F}} & C^p \times \text{Mp}^p(V) \\
| \downarrow \sigma \downarrow \hat{\Phi}^c | & & | \downarrow \hat{\Phi}^p \downarrow \hat{\Phi}^c | \\
C^c \times \text{Sp}(V) & \xrightarrow{\bar{F}} & C^p \times \text{Sp}(V)
\end{array}$$

Since $\bar{F}$ must be a lift of $\bar{F}$ and therefore of $F$, we assume that $\bar{F}$ takes the form

$$\bar{F}(\Phi^c(m(\tau)), a) = (\Phi^p(m(\tau)), \tilde{G}(\tau)a), \ \forall m(\tau) \in C, \ \forall a \in \text{Mp}^c(V),$$
where $\hat{G}(\tau) \in \text{Mp}^c(V)$. From the condition that
\[
\sigma \circ \hat{F} \circ \hat{\Phi} = \hat{F} \circ \sigma \circ \hat{\Phi},
\]

it follows that we must have $\sigma(\hat{G}(\tau)) = G(\tau)$ for all $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. That is, $\hat{G}(\tau)$ must be a lift of $G(\tau)$ to $\text{Mp}^c(V)$. More specifically, in order for $\hat{F}$ to be singly defined, $\hat{G}(\tau)$ must be a closed loop in $\text{Mp}^c(V)$ with period $2\pi$.

There is another consideration: the effect of this change of coordinates on the one-form $\gamma$. Recall that $\gamma$ is defined on $P$ by
\[
\gamma = \frac{1}{i\hbar} \Pi^* \beta + \frac{1}{2} \eta_* \vartheta_0,
\]
where $\vartheta_0$ is the trivial connection on the product bundle $M \times \text{Mp}^c(V)$. When we change to symplectic polar coordinates, we would like $\gamma$ to retain the same form, where $\beta$ can be written in polar coordinates as in Equation (4.4.3), and where $\vartheta_0$ now represents the trivial connection on the product bundle $\mathcal{C}_p \times \text{Mp}^c(V)$. Since $\ker(\eta) = \text{Mp}(V)$, we can accomplish this by requiring that the path $\hat{G}(\tau)$ lie within $\text{Mp}(V)$.

Recall the parametrization of $\text{Mp}^c(V)$ that was described in Section 2.3.1. If $\hat{G}(\tau)$ is a lift of $G(\tau)$ to $\text{Mp}(V)$, then the parameters of $\hat{G}(\tau)$ have the form $(G(\tau), \mu(\tau))$ where $\mu(\tau)^2 \text{Det}_C C_{G(\tau)} = 1$, and where
\[
C_{G(\tau)} = \frac{1}{2} (G(\tau) - JG(\tau)J).
\]

To determine $\mu(\tau)$, we must examine the matrix $G(\tau) \in \text{Sp}(V)$ explicitly. Using Equation (4.4.2), we find
\[
G(\tau) = \begin{pmatrix}
\left( \frac{1}{\sqrt{2s_0}} \cos \tau \right)_{1 \leq j \leq n} & \left( \frac{1}{\sqrt{2s_0}} \sin \tau \right)_{1 \leq j \leq n} \\
\left( -\sqrt{2s_0} \sin \tau \right)_{1 \leq j \leq n} & \left( \sqrt{2s_0} \cos \tau \right)_{1 \leq j \leq n}
\end{pmatrix}.
\]

Now we calculate $C_{G(\tau)}$ using the complex structure noted in Section 4.4.1, then convert to an
$n \times n$ complex matrix. The result is the diagonal matrix

$$C_{G(\tau)} = \frac{1}{2} \left( \left( \sqrt{2s_{0j}} + \frac{1}{\sqrt{2s_{0j}}} \right) e^{i\tau} \right)_{1 \leq j \leq n},$$

which has complex determinant

$$\text{Det}_C C_{G(\tau)} = \prod_{j=1}^{n} \frac{1}{2} \left( \sqrt{2s_{0j}} + \frac{1}{\sqrt{2s_{0j}}} \right) e^{i\tau} = Ke^{in\tau},$$

where $K$ is a positive real value that is constant over $C$. Thus, a lift of $G(\tau)$ to $\text{Mp}(V)$ is parametrized by

$$\hat{G}(\tau) \mapsto \left( G(\tau), \frac{1}{\sqrt{K}} e^{-in\tau/2} \right). \quad (4.4.4)$$

Notice that $(G(\tau + 2\pi), \mu(\tau + 2\pi)) = (G(\tau), \mu(\tau)e^{-in\pi})$. When $n$ is even, we get a closed loop through $\text{Mp}(V)$ with period $2\pi$, which is the optimal outcome. When $n$ is odd, however, traversing $C$ once multiplies $\mu(\tau)$ by $-1$. This shows that we cannot always lift $G(\tau)$ to a closed path through $\text{Mp}(V)$. Nevertheless, we choose the path $\hat{G}(\tau)$ as defined in Equation (4.4.4) because it preserves the form of $\gamma$. If we let $\varepsilon_n \in \text{Mp}^c(V)$ be the element whose parameters are $(I, e^{-in\pi})$, then $\hat{G}(\tau + 2\pi) = \varepsilon_n\hat{G}(\tau)$.

To prevent $\hat{F}$ from being multi-valued, let $\hat{\mathcal{C}} = \{m(\tau) \in \mathcal{C} : \tau \in (0, 2\pi)\}$, and let $\hat{\mathcal{C}}_{c}$ and $\hat{\mathcal{C}}_{p}$ be the images of $\hat{\mathcal{C}}$ in Cartesian and symplectic polar coordinates, respectively. Then let $\hat{F} : \hat{\mathcal{C}}_{c} \times \text{Mp}^c(V) \to \hat{\mathcal{C}}_{p} \times \text{Mp}^c(V)$ be given by

$$\hat{F}(\Phi_{c}(m(\tau)), a) = (\Phi_{p}(m(\tau)), \hat{G}(\tau)a), \quad \forall (m(\tau), a) \in P|_{\hat{\mathcal{C}}},$$

and identify $P|_{\hat{\mathcal{C}}}$ with $\hat{\mathcal{C}}_{p} \times \text{Mp}^c(V)$ under the map $\hat{\Phi}_{p} = \hat{F} \circ \hat{\Phi}_{c}$. This gives us symplectic polar coordinates for $P$ over $\hat{\mathcal{C}}$. On $\hat{\mathcal{C}}_{p} \times \text{Mp}^c(V)$, $\gamma$ takes the form

$$\gamma = \frac{1}{i\hbar} \sum_{j=1}^{n} s_{0j} d\theta_j + \frac{1}{2} \eta_0 \vartheta_0,$$

where $\vartheta_0$ is the trivial connection on the product bundle, and where the values $s_{0j}$ are constant over $\mathcal{C}_p$. We will manually adjust for the fact that closing the loop multiplies $\hat{G}(\tau)$ by $\varepsilon_n$: if
there is a lift of $\mathcal{C}$ to $P|\mathcal{C}$ whose image in $\dot{\mathcal{C}}_p \times \text{Mp}^c(V)$ has the form $(m(\tau), a(\tau))$ for some path $a(\tau)$ in $\text{Mp}^c(V)$, then this lifted path is closed over $\mathcal{C}$ if and only if $a(\tau + 2\pi) = \varepsilon_n a(\tau)$.

### 4.4.3 Quantized energy levels of the harmonic oscillator

Having established the change of coordinates, we will now drop the explicit use of $\Phi_p$, $F$ and their lifts. Elements of $U$ will be written with respect to the symplectic polar coordinates $(s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$, which for convenience we abbreviate by $X_k$, $k = 1, \ldots, 2n$. Using the identifications $\hat{x}_j \mapsto \frac{\partial}{\partial s_j}|_m, \hat{y}_j \mapsto \frac{\partial}{\partial \theta_j}|_m$ for all $m \in U$, $j = 1, \ldots, n$, we write

$$\text{Sp}(M, \omega)|_U = U \times \text{Sp}(V) \quad \text{and} \quad P|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times \text{Mp}^c(V).$$

At the level of tangent spaces, we have

$$T_{(m, I)} \text{Sp}(M, \omega) = T_m M \times \mathfrak{sp}(V), \quad \forall m \in U,$$

and

$$T_{(m, I)} P = T_m M \times \mathfrak{mp}^c(V) = T_m M \times (\mathfrak{sp}(V) \oplus \mathfrak{u}(1)), \quad \forall m \in \dot{\mathcal{C}}.$$

On $U$, $\xi_H = -\sum_{j=1}^n \frac{\partial}{\partial \theta_j}$. Therefore $T_m S^\perp = \text{span} \left\{ \sum_{j=1}^n \frac{\partial}{\partial \theta_j} |_m \right\}$ at each $m \in U$. Within the model vector space $V$, let

$$W^\perp = \text{span} \{ \hat{y}_1 + \ldots + \hat{y}_n \}.$$

Then

$$W = \text{span} \{ \hat{x}_1 - \hat{x}_2, \ldots, \hat{x}_{n-1} - \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n \},$$

and

$$W/W^\perp = \text{span} \{ [\hat{x}_1 - \hat{x}_2], \ldots, [\hat{x}_{n-1} - \hat{x}_n], [\hat{y}_1 - \hat{y}_2], \ldots, [\hat{y}_{n-1} - \hat{y}_n] \}.$$

The local trivialization of $\text{Sp}(M, \omega)|_U$ induces the local trivializations $\text{Sp}(M, \omega; S)|_{U \cap S} = U \cap S \times \text{Sp}(V; W)$ and $\text{Sp}(T S/T S^\perp)|_{U \cap S} = U \cap S \times \text{Sp}(W/W^\perp)$.

By definition, $P_S$ is the bundle associated to $P^S$ by the group homomorphism $\hat{\nu} : \text{Mp}^c(V) \rightarrow$
M^\text{op}(W/W^\perp). Over \dot{\mathcal{C}}, we have the local trivializations \( P|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times M^\text{op}(V) \), \( P^S|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times M^\text{op}(V; W) \) and \( P^S|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times M^\text{op}(W/W^\perp) \), where the associated bundle map \( \tilde{\nu} : P^S|_{\dot{\mathcal{C}}} \to P^S|_{\dot{\mathcal{C}}} \) acts by

\[ \tilde{\nu}(m, a) = (m, \tilde{\nu}(a)), \quad \forall (m, a) \in P^S|_{\dot{\mathcal{C}}}. \]

Using properties of \( \tilde{\nu} \) noted in Section 4.2.1, we have \( \tilde{\nu}(\varepsilon_n) = \varepsilon_n \in M^\text{op}(W/W^\perp) \) for all \( n \).

Therefore, within \( P^S \), the same factor of \( \varepsilon_n \) is required to close the loop from \( \dot{\mathcal{C}} \) to \( \mathcal{C} \). Further, since \( \eta_\ast \circ \tilde{\nu}_\ast = \eta_\ast \), the one-form \( \gamma_S \) induced on \( P^S \) does not change form under the map \( \tilde{\nu} \):

\[ \gamma_S|_{\dot{\mathcal{C}}} = \frac{1}{i\hbar} \sum_{j=1}^{n} s_{0j} d\theta_j + \frac{1}{2} \eta_\ast \vartheta_0, \tag{4.4.5} \]

where \( \vartheta_0 \) is now the trivial connection on the product bundle \( \dot{\mathcal{C}} \times M^\text{op}(W/W^\perp) \). Lastly, the relevant tangent spaces are

\[ T_{(m,I)} \text{Sp}(TS/TS^\perp) = T_m M \times \text{sp}(W/W^\perp), \quad \forall m \in U, \]

and

\[ T_{(m,I)}P^S = T_m M \times (\text{sp}(W/W^\perp) \oplus \text{u}(1)), \quad \forall m \in \dot{\mathcal{C}}. \]

We chose the initial point \( m_0 = (s_0, \ldots, s_n, 0, \ldots, 0) \), and that \( \mathcal{C} \) is the orbit of \( \xi_H \) through \( m_0 \). We will show that the orbit of \( \tilde{\xi}_H \) through \( (m_0, I) \) is closed in \( \text{Sp}(TS/TS^\perp) \). Note that we already know that the orbits of \( \tilde{\xi}_H \) are closed because \( H \) generates a circle action on \( M \). However, the calculation serves as a useful model for the example that appears in Section 4.4.4. We first need to lift \( \phi^t \) to the flow \( \tilde{\phi}^t \) on \( \text{Sp}(M, \omega) \). By definition, for any \( m \in U \), \( \tilde{\phi}^t(m, I) = (\phi^t(m), \phi^t_\ast|m) \). This implies that

\[ \tilde{\xi}_H(m, I) = \frac{d}{dt} \bigg|_{t=0} \tilde{\phi}^t(m, I) = \left( \xi_H(m), \frac{d}{dt} \bigg|_{t=0} \phi^t_\ast|m \right). \]

We write \( \phi^t = (\phi^t_1, \ldots, \phi^t_{2n}) \) with respect to the symplectic polar coordinates. Then \( \phi^t_\ast|m \) is a \( 2n \times 2n \) matrix, which we interpret as an element of \( \text{Sp}(V) \), and its components are given by
\[(\phi^t_{\ast})_{jk} = \left. \frac{\partial \phi^t_j}{\partial X_k} \right|_m \]. Noting that \( \left. \frac{d}{dt} \phi^t_j \right|_{t=0} = (\xi_H)_j \), we compute

\[
\left( \left. \frac{d}{dt} \phi^t_{\ast} \right|_m \right)_{jk} = \left. \frac{\partial \phi^t_{\ast}}{\partial X_k} \right|_m (\xi_H)_j.
\] (4.4.6)

But \( \xi_H = -\sum_{j=1}^n \frac{\partial}{\partial y_j} \) on \( U \), which has constant components, so \( \left. \frac{d}{dt} \phi^t_{\ast} \right|_{t=0} \) is identically 0. Thus

\[
\tilde{\xi}_H(m, I) = (\xi_H(m), 0), \quad \forall m \in U.
\]

In particular, since the \( \mathfrak{sp}(V) \) component of \( \tilde{\xi}_H \) is constant over the orbit \( \mathcal{C} \), we can find the flow for \( \tilde{\xi}_H \) on \( \text{Sp}(M, \omega) \) through \((m_0, I)\) by exponentiating: it is simply

\[
\tilde{\phi}^t(m_0, I) = (\phi^t(m_0), I).
\]

This is clearly a closed orbit with period \( 2\pi \). The induced flow on the bundle \( \text{Sp}(TS/TS^\perp) \) also takes the form

\[
\tilde{\phi}^t(m_0, I) = (\phi^t(m_0), I).
\]

Now we restrict our attention to the curve \( \dot{\mathcal{C}} \), and lift \( \tilde{\xi}_H \) to \( P_S \), horizontally with respect to \( \gamma_S \), where \( \gamma_S \) is given in Equation (4.4.5). Using the fact that \( \sum_{j=1}^n s_{0j} = E \), we find that for any \( m \in \dot{\mathcal{C}} \), the horizontal lift of \( \tilde{\xi}_H \) to \( T_{(m, I)}P_S \) is

\[
\tilde{\xi}_H(m, I) = \left( \xi_H(m), 0 \oplus \frac{E}{i\hbar} \right).
\]

The \( \mathfrak{mp}^c(V) \) component is constant over \( \dot{\mathcal{C}} \), so the flow is again calculated by exponentiating:

\[
\tilde{\phi}^t(m_0, I) = \left( \phi^t(m_0), \exp \left( 0 \oplus \frac{Et}{i\hbar} \right) \right) = (\phi^t(m_0), e^{Et/i\hbar}),
\]

where \( e^{Et/i\hbar} \in U(1) \subset \text{Mp}^c(W/W^\perp) \).

The quantized energy levels are those values of \( E \) for which this orbit closes. However, we must remember that one circuit about \( \mathcal{C} \) introduces an extra factor of \( \varepsilon_n \). Referring to the properties of the parametrization stated in Section 2.3.1, we see that \( \tilde{\phi}^0(m_0, I) = \tilde{\phi}^{2\pi}(m_0, I) \) if and only if \( e^{2\pi E/i\hbar} = e^{-in\pi} \), or in other words, when \( \frac{2\pi E}{\hbar} + in\pi = -2\pi iN \) for some \( N \in \mathbb{Z} \).
Upon rearranging and recalling that $E$ is positive, we find that the quantized energy levels take the form
\[ E = \hbar \left( N + \frac{n^2}{2} \right), \quad N \in \mathbb{Z}, \quad N > -\frac{n}{2}. \]

Thus the quantization condition correctly reproduces the expected $\frac{n^2}{2}$-shift in the energy levels of the harmonic oscillator.

### 4.4.4 Example of dynamical invariance

Fix $n = 2$. It is convenient to use the definitions $s_j = \frac{1}{2}(p_j^2 + q_j^2)$ and $\frac{\partial}{\partial \theta_j} = p_j \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial p_j}$ everywhere on $M$, for $j = 1, 2$. We will point out when a statement only holds on the neighborhood $U$.

Let $k \in \mathbb{R}$ be a positive constant, and consider the two functions $H_1, H_2 : M \to \mathbb{R}$ given by
\[ H_1 = s_1 + s_2 - k, \quad H_2 = (s_1 + s_2 - k)(s_1 + 2s_2 + 1). \]

The function $H_1$ is just the energy function for the harmonic oscillator, shifted by $k$: its quantized energy levels are
\[ E_N = \hbar N - k, \quad N \in \mathbb{Z}, \quad N > \frac{k}{\hbar}. \]

Notice that $H_1^{-1}(0) = H_2^{-1}(0)$. Let this shared level set be $S$. The energy $E = 0$ is a quantized energy for the system $(M, \omega, H_1)$ if and only if $\frac{k}{\hbar} \in \mathbb{N}$.

The two Hamiltonian vector fields are
\[ \xi_{H_1} = -\frac{\partial}{\partial \theta_1} - \frac{\partial}{\partial \theta_2}, \]
\[ \xi_{H_2} = (s_1 + 2s_2 + 1)\xi_{H_1} - H_1 \left( \frac{\partial}{\partial \theta_1} + 2 \frac{\partial}{\partial \theta_2} \right) \]
\[ = -(2s_1 + 3s_2 - k + 1) \frac{\partial}{\partial \theta_1} - (3s_1 + 4s_2 - 2k + 1) \frac{\partial}{\partial \theta_2}. \]

Since $H_1 = 0$ on $S$, we see that $\xi_{H_2} = (s_1 + 2s_2 + 1)\xi_{H_1}$ everywhere on $S$. Thus the vector fields are parallel on $S$, as expected, and so they share the same orbits in $S$. However, $\xi_{H_1}$ and $\xi_{H_2}$ are not parallel away from $S$. Let $\phi^t$ be the flow of $\xi_{H_1}$, and let $\rho^t$ be the flow of $\xi_{H_2}$. 
Consider the initial point \( m_0 \in S \cap U \), where \( m_0 = (s_{01}, s_{02}, 0, 0) \) with \( s_{01} + s_{02} = k \) and \( s_{01}, s_{02} \neq 0 \). The orbit of both \( \phi^t \) and \( \rho^t \) through \( m_0 \) is

\[
\mathcal{C} = \{(s_{01}, s_{02}, \tau, \tau) : \tau \in \mathbb{R}/2\pi\mathbb{Z}\}.
\]

From Section 4.4.3, we know that

\[
\tilde{\xi}_{H_1}(m, I) = (\xi_{H_1}(m), 0), \quad \forall m \in \mathcal{C},
\]

and therefore

\[
\tilde{\phi}^t(m_0, I) = (\phi^t(m_0), I).
\]

By the same calculation, \( \tilde{\xi}_{H_2}(m, I) = (\xi_{H_2}(m), \frac{d}{dt}\big|_{t=0} \rho^t_m) \) for \( m \in \mathcal{C} \). Applying Equation (4.4.6) to the components of \( \xi_{H_2} \) yields

\[
\frac{d}{dt}\bigg|_{t=0} \rho^t_m = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-2 & -3 & 0 & 0 \\
-3 & -4 & 0 & 0
\end{pmatrix},
\]

which we interpret as an element of \( \mathfrak{sp}(V) \). Let this matrix be denoted by \( \kappa \). Then

\[
\tilde{\xi}_{H_2}(m, I) = (\xi_{H_2}(m), \kappa),
\]

and since the Lie algebra component is constant over \( \mathcal{C} \), we obtain the flow on \( \text{Sp}(M, \omega) \) through \( (m_0, I) \) by exponentiating:

\[
\tilde{\rho}^t(m_0, I) = (\rho^t(m_0), \exp(t\kappa)).
\]
A calculation establishes that

\[
\exp(t\kappa) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2t & -3t & 1 & 0 \\
-3t & -4t & 0 & 1
\end{pmatrix},
\]

which is clearly not periodic. Thus \(\tilde{\rho}^t\) has no closed orbits on \(\text{Sp}(M,\omega)\) over \(S \cap U\).

Now let us transfer to \(\text{Sp}(TS/TS^\perp)\). If we apply the definitions and identifications laid out at the beginning of Section 4.4.3, now setting \(n = 2\), then we find \(W^\perp = \text{span}\{\widehat{y}_1 + \widehat{y}_2\}\), \(W/W^\perp = \text{span}\{[\widehat{x}_1 - \widehat{x}_2], [\widehat{y}_1 - \widehat{y}_2]\}\), and the identification of \(\text{Sp}(M,\omega)|_C\) with \(C \times \text{Sp}(V)\) induces an identification of \(\text{Sp}(TS/TS^\perp)|_C\) with \(C \times \text{Sp}(W/W^\perp)\). Notice that \(\exp(t\kappa)(\widehat{x}_1 - \widehat{x}_2) = \widehat{x}_1 - \widehat{x}_2 + t(\widehat{y}_1 + \widehat{y}_2)\), \(\exp(t\kappa)(\widehat{y}_1 - \widehat{y}_2) = \widehat{y}_1 - \widehat{y}_2\).

Therefore the path through \(\text{Sp}(W/W^\perp)\) induced by \(\exp(t\kappa)\) is

\[
\nu(\exp(t\kappa)) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

Thus, on \(\text{Sp}(TS/TS^\perp)|_C\),

\[
\tilde{\rho}^t(m_0, I) = (\rho^t(m_0), I),
\]

which coincides with \(\tilde{\phi}^t(m_0, I)\).

The above calculation omits certain cases: namely, if the starting point \(m_0\) has \(s_{01} = 0\) or \(s_{02} = 0\). We can no longer eliminate such cases by performing a rotation, because \(H_2\) is not symmetric with respect to \(s_1\) and \(s_2\). If \(m_0 \notin U\), then we have to modify our approach. For example, if \(s_{01} = 0\), then we retain Cartesian coordinates for the \(p_1q_1\)-plane and convert to symplectic polar on the \(p_2q_2\)-plane. The calculation is more complicated, but the result is similar: over \(C\), we find that \(\tilde{\xi}_{H_2} = (\xi_{H_2}, \kappa)\) for some constant value of \(\kappa \in \text{sp}(V)\). The path \(\tilde{\rho}^t(m_0, I) = (\rho^t(m_0), \exp(t\kappa))\) does not close in \(\text{Sp}(M,\omega)\), but the induced path in \(\text{Sp}(TS/TS^\perp)\)
coincides with $\tilde{\phi}_t^i(m_0, I)$. The same pattern holds if we take $s_{02} = 0$.

Thus, if the quantized energy condition were stated in terms of the holonomy of $\gamma^S$ over closed orbits in $\text{Sp}(M, \omega; S)$, as it was in [16], then the value $E = 0$ would satisfy the condition vacuously for the system $(M, \omega, H_2)$, regardless of the value of $k$. It is only by descending to $\text{Sp}(TS/TS^\perp)$ that we recover the quantization condition $\frac{k}{\hbar} \in \mathbb{N}$. Hence our definition of a quantized energy level is dynamically invariant, while that in [16] is not.

### 4.5 Comparison with Kostant-Souriau Quantization

#### 4.5.1 The quantized energy condition and its properties

The quantized energy condition in Definition 4.2.2 can be easily adapted to Kostant-Souriau prequantization. Indeed, the Kostant-Souriau case is simpler, since it does not involve the symplectic frame bundle.

Assume that $(M, \omega)$ admits a prequantization circle bundle $(Y, \gamma)$, and let $H : M \to \mathbb{R}$ be a smooth function. We now mimic the constructions in Section 4.2.2, using $Y$ in place of $P$. Denote the Hamiltonian vector field corresponding to $H$ by $\xi_H$ as before, and let $\tilde{\xi}_H$ be the lift of $\xi_H$ to $Y$ that is horizontal with respect to $\gamma$. Fix $E$, a regular value of $H$, and let $S = H^{-1}(E) \subset M$. Let $Y^S$ be the restriction of $Y$ to $S$, and let $\gamma^S$ be the pullback of $\gamma$ to $Y^S$.

\[(Y, \gamma) \xrightarrow{\text{incl.}} (Y^S, \gamma^S) \quad \text{incl.} \quad \downarrow \quad \text{incl.} \quad \downarrow \quad S\]

The construction stops here, and we give the definition of a quantized energy level using $(Y^S, \gamma^S)$.

**Definition 4.5.1.** If the connection one-form $\gamma^S$ has trivial holonomy over all closed orbits of the Hamiltonian vector field $\xi_H$ on $S$, then $E$ is a **Kostant-Souriau (KS) quantized energy level** for the system $(M, \omega, H)$.

This definition is dynamically invariant, and the proof is much simpler than that in Section 4.3.2.
Theorem 4.5.2. If \( H_1, H_2 : M \to \mathbb{R} \) are smooth functions such that \( H_1^{-1}(E_1) = H_2^{-1}(E_2) \) for regular values \( E_j \) of \( H_j \), \( j = 1, 2 \), then \( E_1 \) is a KS quantized energy level for \((M, \omega, H_1)\) if and only if \( E_2 \) is a KS quantized energy level for \((M, \omega, H_2)\).

**Proof.** We argued in Section 4.3.2 that \( \xi_{H_1} \) and \( \xi_{H_2} \) are parallel on \( S \), which implies that they have the same orbits. Therefore \( \gamma^S \) has trivial holonomy over the orbits of one if and only if it has trivial holonomy over the orbits of the other. The KS version of the dynamical invariance theorem is immediate. \( \square \)

In Section 4.2.2, we examined the case in which the symplectic reduction of \((M, \omega)\) at \( E \) is a manifold. Theorem 4.2.1, due to Robinson [16], gives the conditions under which the quantization condition is sufficient to imply that the symplectic reduction admits a metaplectic-c prequantization. A similar theorem can be given in the context of prequantization circle bundles.

First, we state a general result, which was used in [16] to prove Theorem 4.2.1. Let \( S \) be an arbitrary manifold, and suppose that \((Z, \delta)\) is a principal circle bundle with connection one-form over \( S \). Let the curvature of \( \delta \) be \( \varpi \). Suppose that \( F \) is a foliation of \( S \) whose leaf space \( S_F \) is a smooth manifold. Denote the leaf projection map by \( S \xrightarrow{\pi} S_F \). If \( \delta \) has trivial holonomy over all of the leaves of \( F \), then \( Z \) can be factored to produce a well-defined circle bundle \( Z_F \to S_F \). Further, \( \delta \) descends to a connection one-form \( \delta_F \) on \( Z_F \), and the curvature \( \varpi_F \) of \( \delta_F \) satisfies \( \pi^* \varpi = \varpi \).

Now apply this result to the circle bundle \((Y^S, \gamma^S) \to S\), where the foliation is given by the orbits of the vector field \( \xi_H \) on the level set \( S \), and the symplectic reduction \((M_E, \omega_E)\) is its leaf space.

**Theorem 4.5.3.** Suppose that the symplectic reduction \((M_E, \omega_E)\) for \((M, \omega)\) at \( E \) is a manifold. If \( \gamma^S \) has trivial holonomy over all closed orbits of \( \xi_H \) on \( S \), then the quotient of \((Y^S, \gamma^S)\) by the orbits of \( \tilde{\xi}_H \) is a prequantization circle bundle for \((M_E, \omega_E)\).

Thus the Kostant-Souriau version of the quantized energy condition is sufficient to ensure that the symplectic reduction admits a prequantization circle bundle, whenever the symplectic reduction is a manifold. This is a slight improvement over the metaplectic-c result, since it does not depend on a quotient of the symplectic frame bundle being well defined.
4.5.2 Lack of half-shift in the harmonic oscillator

In this section, we will determine the KS quantized energy levels of the \( n \)-dimensional harmonic oscillator. The calculation will be significantly simpler than that in Section 4.4, but it will yield the wrong answer.

Let \( M = \mathbb{R}^{2n} \), with Cartesian coordinates \((p_1, \ldots, p_n, q_1, \ldots, q_n)\) and symplectic form \( \omega = \sum_{j=1}^n dp_j \wedge dq_j \). The energy function and corresponding Hamiltonian vector field for the harmonic oscillator are

\[
H = \frac{1}{2} \sum_{j=1}^n (p_j^2 + q_j^2), \quad \xi_H = \sum_{j=1}^n \left( q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).
\]

Let the flow of \( \xi_H \) on \( M \) be \( \phi^t \). We know from Section 4.4.1 that all of the orbits of \( \xi_H \) are circles, and that \( \phi^{t+2\pi}(m) = \phi^t(m) \) for all \( m \in M \).

Let \( Y = M \times U(1) \), with projection map \( Y \xrightarrow{\Pi} M \). We define

\[
\beta = \frac{1}{2} \sum_{j=1}^n (p_j dq_j - q_j dp_j)
\]

on \( M \), and let \( \gamma = \frac{1}{i\hbar} \Pi^* \beta + \vartheta_0 \) on \( Y \), where \( \vartheta_0 \) is the trivial connection on the product bundle \( M \times U(1) \). Then \( (Y, \gamma) \) is a prequantization circle bundle for \( (M, \omega) \), and it is unique up to isomorphism.

Let \( E > 0 \) be arbitrary, and let \( S = H^{-1}(E) \). At any point \( s \in S \), we calculate that \( \xi_{H+} \beta = -E \). Therefore the lifted vector field \( \tilde{\xi}_H \) at the point \((s, I) \in Y\) is

\[
\tilde{\xi}_H(s, I) = \left( \xi_H(s), \frac{E}{i\hbar} \right),
\]

where we identify the tangent space \( T_{(s,I)}Y \) with \( T_s M \times \mathfrak{u}(1) \). Note that the \( \mathfrak{u}(1) \) component is constant over all of \( S \), so in particular it is constant over an orbit. Let the flow of \( \tilde{\xi}_H \) be \( \tilde{\phi}^t \). By exponentiation, we find that

\[
\tilde{\phi}^t(s, I) = \left( \phi^t(s), e^{Et/i\hbar} \right).
\]
From this, it follows that the holonomy of $\gamma^S$ over the orbits in $S$ is trivial if and only if $E = Nh$ for some $N \in \mathbb{N}$. This is not consistent with the quantum mechanical prediction of $E = \hbar \left(N + \frac{n}{2}\right)$ when the dimension $n$ is odd.

This shortcoming in the Kostant-Souriau prequantization of the harmonic oscillator is well known, and the standard solution is to proceed from prequantization to quantization while introducing the half-form correction, as described in Section 2.2.2. In this context, the quantized energy levels for the system $(M, \omega, H)$ can be taken to be the eigenvalues of the operator corresponding to the energy function $H$. When the quantization recipe is applied to the harmonic oscillator, the presence of the half-form bundle adds the $\frac{n}{2}$ shift to the energy eigenvalues. However, this correction comes at the cost of introducing a choice of polarization, and the quantized energy definition can no longer be evaluated over a single level set of $H$.

By comparison, when we use the metaplectic-c formulation of a quantized energy level, we find that the correct harmonic oscillator energies are encoded in the geometry of the level sets of the energy function. The result is dynamically invariant, independent of polarization, and consistent with physical prediction. This example illustrates the benefits of metaplectic-c quantization and our quantized energy definition.
Chapter 5

The Hydrogen Atom

5.1 Introduction

The quantized energy levels of the hydrogen atom have been calculated for various physical models, using various flavors of geometric quantization. Notable examples include:

- Simms [18], who used the observation that the space of orbits corresponding to a fixed negative energy is isomorphic to $S^2 \times S^2$, and determined those energies for which the reduced manifold admits a prequantization circle bundle;

- Sniatycki [21], who looked at the 2-dimensional relativistic Kepler problem and computed a Bohr-Sommerfeld condition for the completely integrable system given by considering the energy and angular momentum functions simultaneously;

- Duval, Elhadad and Tuynman [7], who took the phase space to include the spins of the electron and proton, then chose a polarization and determined the Kostant-Souriau quantized operator corresponding to the energy function with fine and hyperfine interaction terms.

These examples exist at one of two possible extremes. On one hand, the quantized energy condition can be evaluated over the symplectic reduction of a particular level set of the energy function, as in [18]. This definition only looks at one energy level at a time, but it requires constructing the symplectic reduction and establishing that the result is a smooth manifold.
On the other hand, the quantized energy levels can be determined from properties of the quantized system as a whole, as in [21] or [7]. These approaches are characterized by requiring a polarization, or the equivalent information – the Hamiltonian vector fields corresponding to the Poisson-commuting functions in a completely integrable system generate a real polarization – and the calculation is not restricted to a single level set of the energy function in question.

Our proposed definition for a quantized energy level acts as a middle ground between these two extremes. The objective of this chapter is to apply the metaplectic-c quantized energy condition to the hydrogen atom, using the physical model that is equivalent to the Kepler problem. We will show that the quantized energy levels are in agreement with the quantum mechanical prediction.

In Section 5.2, we set up our model of the hydrogen atom and construct a metaplectic-c prequantization for its phase space. Section 5.3 presents the Ligon-Schaaf regularization map, which is a symplectomorphism from the negative-energy domain of the hydrogen atom to an open submanifold of $TS^3$. We show how to relate the energy function for the hydrogen atom to that of a free particle on $S^3$, a process that makes use of the dynamical invariance property of our definition. Finally, in Section 5.4, we determine the quantized energy levels of a free particle on $S^3$, and use these to determine the quantized energy levels for the hydrogen atom.

We will apply the definitions and constructions that were given in Section 4.2. In addition, we will make repeated use of the notation and conventions that are described below.

### 5.1.1 Choices for subsequent calculations

In the sections that follow, we will need model symplectic vector spaces of several different dimensions. Let us fix some standardized choices.

For any $n \in \mathbb{N}$, let $(V_n, \Omega_n)$ be a $2n$-dimensional symplectic vector space. Let $(\widehat{v}_1, \ldots, \widehat{v}_n, \widehat{w}_1, \ldots, \widehat{w}_n)$ be a symplectic basis for $V_n$, and write all elements of $V_n$ as ordered $2n$-tuples with respect to this basis. The symplectic form can be written in terms of the dual basis as

$$\Omega_n = \sum_{j=1}^{n} \widehat{v}_j^* \wedge \widehat{w}_j^*.$$
Assume that each real vector \((a_1, \ldots, a_n, b_1, \ldots, b_n) \in V_n\) is identified with the complex vector \((b_1 + ia_1, \ldots, b_n + ia_n) \in \mathbb{C}^n\). The resulting complex structure \(J\) on \(V_n\) is written in matrix form as \(J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\), where \(I\) is the \(n \times n\) identity matrix.

When we require a subspace of \(V_n\) of codimension 1, we choose \(W_n = \text{span}\{\hat{v}_1, \ldots, \hat{v}_n, \hat{w}_1, \ldots, \hat{w}_{n-1}\}\).

Then \(W_n^\perp = \text{span}\{\hat{v}_n\}\) and \(W_n/W_n^\perp = \text{span}\{[\hat{v}_1], \ldots, [\hat{v}_{n-1}], [\hat{w}_1], \ldots, [\hat{w}_{n-1}]\}\).

Using Equation (4.2.1), it is immediate that \(W_n/W_n^\perp\) is isomorphic to \(V_{n-1}\) as a symplectic vector space and a complex vector space. The commutative diagram from Section 4.2.1 containing the group homomorphisms \(\nu\) and \(\hat{\nu}\) can be rewritten as follows.

\[
\begin{array}{ccc}
\text{Mp}^c(V_n) & \overset{\hat{\nu}}{\longrightarrow} & \text{Mp}^c(V_{n-1}) \\
\downarrow & & \downarrow \\
\text{Sp}(V_n) & \overset{\nu}{\longrightarrow} & \text{Sp}(V_{n-1})
\end{array}
\]

### 5.2 Quantizing the Hydrogen Atom

#### 5.2.1 Setup

Let \(\hat{\mathbb{R}}^3\) represent \(\mathbb{R}^3\) with the origin removed, and let \(M = T\hat{\mathbb{R}}^3 = \hat{\mathbb{R}}^3 \times \mathbb{R}^3\). We use Cartesian coordinates \(q = (q_1, q_2, q_3)\) on \(\hat{\mathbb{R}}^3\) and \(p = (p_1, p_2, p_3)\) on \(\mathbb{R}^3\). Equip \(M\) with the symplectic form \(\omega = \sum_{j=1}^3 dq_j \wedge dp_j\).

We consider the model of the hydrogen atom that is equivalent to the Kepler problem. Assume that a proton is fixed at the origin in \(\mathbb{R}^3\), and that an electron of mass \(m_e\) interacts with it via the electrostatic force, which obeys an inverse-square law with constant of proportionality \(k\). Then \(M\) is the phase space for the motion of the electron, where \(q\) and \(p\) represent its position.
and momentum, respectively. The Hamiltonian energy function is

\[ H = \frac{1}{2m_e} |p|^2 - \frac{k}{|q|}, \]

and the corresponding Hamiltonian vector field is

\[ \xi_H = \sum_{j=1}^{3} \left( \frac{1}{m_e} p_j \frac{\partial}{\partial q_j} - \frac{k}{|q|^3} q_j \frac{\partial}{\partial p_j} \right). \]

Let \( \xi_H \) have flow \( \phi^t \) on \( M \).

The solutions to the Kepler problem are well known [1, 4]. The angular momentum vector \( L = q \times p \) is a constant of the motion, as is the eccentricity vector \( e = \frac{1}{m_e k} p \times L - \frac{q}{|q|} \). In position space, the orbits corresponding to a given energy \( E \in \mathbb{R} \) are conic sections with eccentricity

\[ |e| = \sqrt{1 + \frac{2E|L|^2}{m_e k^2}}, \quad (5.2.1) \]

having the origin as a focus.

In particular, suppose \( E < 0 \). Then the value of \( |L| \) lies in the interval \( \left[ 0, \sqrt{-\frac{m_e k^2}{2E}} \right] \). If \( |L| > 0 \), then the orbit is an ellipse, with \( |L| = \sqrt{-\frac{m_e k^2}{2E}} \) being the special case of a circle. All elliptical orbits with energy \( E \) have period \( \frac{2\pi}{\Lambda} \), where \( \Lambda = \sqrt{-\frac{8E^3}{m_e k^2}} \). If \( |L| = 0 \), however, then \( e = 1 \) and the orbit is a line segment. Physically, this represents the case where the electron begins from rest and collapses on a straight-line trajectory into the proton. Such motion is not periodic, which implies that the level set \( H^{-1}(E) \) contains orbits of \( \xi_H \) that are not closed. Further, the collapse occurs in finite time, meaning that the vector field \( \xi_H \) is not complete.

The objective of this chapter is to determine the quantized energy levels for \( (M, \omega, H) \). In the next section, we construct a metaplectic-c prequantization for \( (M, \omega) \), and formulate our approach for performing the quantized energy calculation.
5.2.2 Metaplectic-c prequantization of \((M, \omega)\)

We choose the model symplectic vector space \(V_3\), as described in Section 5.1.1. The tangent bundle \(TM\) can be identified with \(M \times V_3\) with respect to the global trivialization

\[
\hat{v}_j \mapsto \frac{\partial}{\partial q_j} |_m, \quad \hat{w}_j \mapsto \frac{\partial}{\partial p_j} |_m, \quad \forall m \in M, \ j = 1, 2, 3.
\]

This yields an identification of the symplectic frame bundle \(\text{Sp}(M, \omega)\) with \(M \times \text{Sp}(V_3)\).

Let \(P = M \times \text{Mp}^c(V_3)\), with bundle projection map \(P \xrightarrow{\Pi} M\). Define the map \(\Sigma : P \to \text{Sp}(M, \omega)\) by

\[
\Sigma(m, a) = (m, \sigma(a)), \quad \forall m \in M, \forall a \in \text{Mp}^c(V_3),
\]

where the right-hand side is written with respect to the global trivialization above. Let

\[
\beta = \sum_{j=1}^{3} (q_j dp_j + d(q_j p_j)) \tag{5.2.2}
\]

on \(M\), so that \(d\beta = \omega\). The reason for this choice of \(\beta\) will be made clear in Section 5.3.1. Let \(\gamma\) be the \(u(1)\)-valued one-form on \(P\) given by

\[
\gamma = \frac{1}{i\hbar} \Pi^* \beta + \frac{1}{2} \eta_0 \vartheta_0,
\]

where \(\vartheta_0\) is the trivial connection on the product bundle. Then \((P, \Sigma, \gamma)\) is the metaplectic-c prequantization for \((M, \omega)\) (unique up to isomorphism).

By definition, the quantized energy levels of \((M, \omega, H)\) are those regular values \(E\) of \(H\) such that the holonomy of \(\gamma_S\) is trivial over all closed orbits of \(\tilde{\xi}_H\) on \(\text{Sp}(TS/TS^\perp)\), where \(S = H^{-1}(E)\). If \(E \geq 0\), then the quantization condition can be evaluated immediately. From Equation (5.2.1) for the eccentricity of the orbit, we see that if \(E = 0\), then the orbits are parabolas in position space, and if \(E > 0\), then they are hyperbolas. In these cases, \(\xi_H\) has no closed orbits in \(S\), which implies that \(\tilde{\xi}_H\) cannot have any closed orbits in \(\text{Sp}(TS/TS^\perp)\). Therefore the holonomy condition is satisfied vacuously, and all nonnegative energy levels are quantized energy levels. This is consistent with the physical prediction from quantum mechanics: a particle that is not spatially confined has a continuous energy spectrum.
It remains to consider the orbits corresponding to negative energy. Let

\[ N = \{ m \in M : H(m) < 0 \} . \]

Then \( N \) is an open, simply connected submanifold of \( M \). The symplectic form on \( M \) restricts to one on \( N \), and the metaplectic-c prequantization for \( M \) restricts to one for \( N \). We use the same symbols to denote the restricted objects: \((N, \omega)\) is a symplectic manifold, and \((P, \Sigma, \gamma)\) is its metaplectic-c prequantization. Since \( N \) is simply connected, every metaplectic-c prequantization for \((N, \omega)\) is isomorphic to \((P, \Sigma, \gamma)\).

Let \( E < 0 \) be fixed, and let \( S = H^{-1}(E) \subset N \). Through the process described in Section 4.2.2, we obtain the three-level structures

\[ (P^S, \gamma^S) \to \text{Sp}(N, \omega; S) \to S \]

and

\[ (P_S, \gamma_S) \to \text{Sp}(TS/TS^\perp) \to S. \]

Lift \( \xi_H \) to \( \tilde{\xi}_H \) on \( \text{Sp}(N, \omega; S) \). It induces a vector field on \( \text{Sp}(TS/TS^\perp) \), which we also denote by \( \tilde{\xi}_H \).

To evaluate the quantization condition using these bundles, we would have to determine the closed orbits of \( \tilde{\xi}_H \) on \( \text{Sp}(TS/TS^\perp) \), then lift \( \xi_H \) to \( \tilde{\xi}_H \) on \( P_S \), horizontally with respect to \( \gamma_S \), and ensure that every lift of a closed orbit in \( \text{Sp}(TS/TS^\perp) \) is closed in \( P_S \). However, this procedure is computationally prohibitive in all but the special case of the circular orbit. Instead, we will use Ligon-Schaaf regularization to transform the quantized energy calculation on \((N, \omega)\) into one on an open submanifold of \( TS^3 \). This is the subject of Section 5.3.

## 5.3 Ligon-Schaaf Regularization

Let \( T^+S^3 \) represent the result of removing the zero section from \( TS^3 \). In Section 5.2.1, we noted that the vector field \( \xi_H \) is not complete. The Ligon-Schaaf map is a symplectomorphism from \((N, \omega)\) to an open submanifold of \( T^+S^3 \), having the property that \( \xi_H \) is mapped to a vector
field that is complete on $T^+ S^3$. This map was presented in [15], and an in-depth discussion of it can be found in [4].

In this section, we will state the Ligon-Schaaf map and list its relevant properties. Then we will construct a metaplectic-c prequantization for $T S^3$, and show how to lift the Ligon-Schaaf map to the level of symplectic frame bundles, and of metaplectic-c prequantizations. Using the lifted maps and the dynamical invariance property, we will be able to relate the quantized energy levels of the hydrogen atom to those of a free particle on $S^3$.

Our conventions and notation largely follow those in [4]. We state results without proof; much more detail can be found in [4].

5.3.1 $T S^3$ and the Ligon-Schaaf map

Consider $T \mathbb{R}^4 = \mathbb{R}^4 \times \mathbb{R}^4$ with Cartesian coordinates $(x, y)$, where $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. Let $T \mathbb{R}^4$ have symplectic form

$$\omega_4 = \sum_{j=1}^{4} dx_j \wedge dy_j,$$

and let

$$\beta_4 = -\sum_{j=1}^{4} y_j dx_j$$

(5.3.1)
on $T \mathbb{R}^4$, so that $d\beta_4 = \omega_4$. The submanifold $T S^3$ is given by

$$T S^3 = \{(x, y) \in T \mathbb{R}^4 : |x|^2 = 1, x \cdot y = 0\} \subset T \mathbb{R}^4,$$

where we take the usual Euclidean inner product on $\mathbb{R}^4$.

A calculation shows that the restriction of $\omega_4$ to $T S^3$ yields a symplectic form on $T S^3$. Let $(T S^3, \omega_3)$ be the resulting symplectic manifold. Let $\beta_3$ be the restriction of $\beta_4$ to $T S^3$, so that $d\beta_3 = \omega_3$.

Let $T^+ S^3$ represent $T S^3$ with the zero section removed:

$$T^+ S^3 = \{(x, y) \in T S^3 : |y| > 0\}.$$
Define the map \( D : T^+ S^3 \to \mathbb{R} \) by
\[
D(x, y) = -\frac{m_e k^2}{2|y|^2}, \quad \forall (x, y) \in T^+ S^3.
\]

As shown in [4], the Hamiltonian vector field for \( D \) on \( T^+ S^3 \subset T\mathbb{R}^4 \) is
\[
\xi_D = \sum_{j=1}^4 \left( \frac{m_e k^2}{|y|^4} y_j \frac{\partial}{\partial x_j} - \frac{m_e k^2}{|y|^2} x_j \frac{\partial}{\partial y_j} \right),
\]
and its flow is
\[
\psi^t_D(x, y) = \begin{pmatrix}
\cos \frac{m_e k^2 t}{|y|} & \frac{1}{|y|} \sin \frac{m_e k^2 t}{|y|} \\
-|y| \sin \frac{m_e k^2 t}{|y|} & \cos \frac{m_e k^2 t}{|y|}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}.
\]

The map \( D \) is called the Delaunay Hamiltonian, and \( \xi_D \) is the Delaunay vector field. Since the orbits of \( \xi_D \) must preserve \( |y| \), it is clear from the form of \( \psi^t_D \) that \( \xi_D \) is complete on \( T^+ S^3 \), and every orbit is closed.

Now let \( S^3_n \) represent \( S^3 \) with the north pole \((0,0,0,1)\) removed. The Ligon-Schaaf map \( LS : N \to T^+ S^3_n \) is given by
\[
LS(q, p) = (A \sin \varphi + B \cos \varphi, -\nu A \cos \varphi + \nu B \sin \varphi),
\]
where
\[
\nu = \sqrt{-\frac{m_e k^2}{2H(q, p)}}, \quad \varphi = \frac{1}{\nu} (q \cdot p),
\]
\[
A = \left( \frac{q}{|q|} - \frac{1}{m_e k} (q \cdot p)p, \frac{1}{\nu} (q \cdot p) \right), \quad B = \left( \frac{1}{\nu} |q|p, \frac{1}{m_e k} |p|^2|q| - 1 \right).
\]

This map has the following properties:

(1) \( LS \) is a diffeomorphism between \( N \) and \( T^+ S^3_n \);

(2) \( LS^* \beta_3 = \sum_{j=1}^3 (q_j dp_j + d(q_j p_j)) = \beta \) (recall Equation (5.2.2));

(3) \( H = D \circ LS \).

Assertions (1) and (2) imply that \( LS \) is a symplectomorphism; from that and (3), it follows that \( LS_* \xi_H = \xi_D \).
5.3.2 Metaplectic-c prequantization for $TS^3$

We begin by constructing a metaplectic-c prequantization for $(T\mathbb{R}^4, \omega_4)$, and let it induce a metaplectic-c prequantization for $(TS^3, \omega_3)$. For $(T\mathbb{R}^4, \omega_4)$, we proceed in precisely the same way as we constructed $(P, \Sigma, \gamma)$ for $(M, \omega)$ in Section 5.2.2. Choose the model vector space $V_4$, and identify the symplectic frame bundle $\text{Sp}(T\mathbb{R}^4, \omega_4)$ with $T\mathbb{R}^4 \times \text{Sp}(V_4)$ using the global trivialization for the tangent bundle given by

$$\begin{align*}
\hat{v}_j &\mapsto \frac{\partial}{\partial x_j}(x, y), \\
\hat{w}_j &\mapsto \frac{\partial}{\partial y_j}(x, y), \\
\forall (x, y) &\in T\mathbb{R}^4, 
\end{align*}$$

Let $Q_4 = T\mathbb{R}^4 \times \text{Mp}(V_4)$ with bundle projection map $Q_4 \xrightarrow{\Pi_4} T\mathbb{R}^4$. Define the map $\Gamma_4 : Q_4 \to \text{Sp}(T\mathbb{R}^4, \omega_4)$ by

$$\Gamma_4(x, y, a) = (x, y, \sigma(a)), \; \forall (x, y) \in T\mathbb{R}^4, \forall a \in \text{Mp}(V_4).$$

Lastly, define the $u(1)$-valued one-form $\delta_4$ on $Q_4$ by

$$\delta_4 = \frac{1}{i\hbar} \Pi_4^* \beta_4 + \frac{1}{2} \eta_0 \vartheta_0,$$

where $\vartheta_0$ is the trivial connection on $Q_4$, and where $\beta_4$ was defined in Equation (5.3.1). Then $(Q_4, \Gamma_4, \delta_4)$ is the unique metaplectic-c prequantization for $(T\mathbb{R}^4, \omega_4)$ up to isomorphism.

In order to construct the metaplectic-c prequantization of $(TS^3, \omega_3)$ induced by $(Q_4, \Gamma_4, \delta_4)$, we proceed by the process of symplectic reduction. Let $R : T\mathbb{R}^4 \to \mathbb{R}$ be given by

$$R(x, y) = \frac{1}{2} |x|^2, \; \forall (x, y) \in T\mathbb{R}^4.$$

The Hamiltonian vector field for $R$ on $T\mathbb{R}^4$ is

$$\xi_R = -\sum_{j=1}^4 x_j \frac{\partial}{\partial y_j},$$

and its flow on $T\mathbb{R}^4$ is

$$\psi_R^t(x, y) = (x, y - tx), \; \forall (x, y) \in T\mathbb{R}^4.$$
It is straightforward to show that \( TS^3 \) can be identified with the space of orbits of \( \xi_R \) on the level set \( A = R^{-1}(\frac{1}{2}) \). The orbit projection map \( p : A \to TS^3 \) is

\[
p(x, y) = (x, y - (x \cdot y)x), \quad \forall (x, y) \in A.
\]

Let \( i : A \to T\mathbb{R}^4 \) be the inclusion map. Recall that \( \omega_3 \) is the symplectic form on \( TS^3 \) obtained by restricting \( \omega_4 \). A calculation shows that \( p^*\omega_3 = i^*\omega_4 \). Therefore \( (TS^3, \omega_3) \) is identified with the symplectic reduction of \( (T\mathbb{R}^4, \omega_4) \) at the level set corresponding to \( R = \frac{1}{2} \).

Choose the model subspace \( W_4 \) as described in Section 5.1.1, and model \( T_z A \subset T_z T\mathbb{R}^4 \) on \( W_4 \subset V_4 \) for all \( z \in A \). Following the procedure described in Section 4.2.2, we construct the three-level structure

\[
(Q^A_4, \delta^A_4) \xrightarrow{\Gamma^A_4} \text{Sp}(T\mathbb{R}^4, \omega_4; A) \to A
\]

by restriction, then construct the three-level structure

\[
(Q_{4A}, \delta_{4A}) \xrightarrow{\Gamma_{4A}} \text{Sp}(TA/TA^\perp) \to A
\]

by taking associated bundles. Then \( \text{Sp}(TA/TA^\perp) \) is a principal \( \text{Sp}(V_3) \) bundle over \( A, \) and \( Q_{4A} \) is a principal \( \text{Mp}^c(V_3) \) bundle over \( A. \)

Let the symplectic frame bundle \( \text{Sp}(TS^3, \omega_3) \) be modeled on \( V_3, \) so that \( \text{Sp}(TS^3, \omega_3) \) is a principal \( \text{Sp}(V_3) \) bundle over \( TS^3. \) Then \( \text{Sp}(TA/TA^\perp) \) and \( \text{Sp}(TS^3, \omega_3) \) have isomorphic fibers. If we were to complete the symplectic reduction process, we would identify \( \text{Sp}(TS^3, \omega_3) \) with a quotient of \( \text{Sp}(TA/TA^\perp). \) However, since \( TS^3 \subset A, \) it is more convenient to identify \( \text{Sp}(TS^3, \omega_3) \) with a subbundle of \( \text{Sp}(TA/TA^\perp). \)

For each \( z \in A, p_*|_z \) appears in the short exact sequence

\[
0 \to T_z A^\perp \to T_z A \xrightarrow{p_*|_z} T_{p(z)} TS^3 \to 0.
\]

Note that the projection map \( p \) is the identity on \( TS^3 \subset A. \) It follows that for all \( z \in TS^3, \) \( p_*|_z : T_z A/T_z A^\perp \to T_z TS^3 \) is a symplectic isomorphism. Therefore \( \text{Sp}(TS^3, \omega_3) \) and \( \text{Sp}(TA/TA^\perp)|_{TS^3} \) are isomorphic as principal \( \text{Sp}(V_3) \) bundles over \( TS^3. \) Using that isomor-
phism, we view $\text{Sp}(TS^3, \omega_3)$ as a subset of $\text{Sp}(TA/T^A)$. Let $(Q, \Gamma, \delta)$ be the result of restricting $(Q_{A4}, \Gamma_{A4}, \delta_{A4})$ to $\text{Sp}(TS^3, \omega_3) \subset \text{Sp}(TA/T^A)$. Then $(Q, \Gamma, \delta)$ is the metaplectic-c prequantization for $(TS^3, \omega_3)$ (unique up to isomorphism).

We will also use the notation $(Q, \Gamma, \delta)$ to denote the metaplectic-c prequantizations for $T^+S^3$ and $T^+S^3_n$ obtained by restriction.

### 5.3.3 Lifting the Ligon-Schaaf map

Recall that the Ligon-Schaaf map $LS : N \to T^+S^3_n$ is a symplectomorphism. Define the map

$$\tilde{LS} : \text{Sp}(N, \omega) \to \text{Sp}(T^+S^3_n, \omega_3)$$

by

$$\tilde{LS}(b) = LS \circ b, \quad \forall b \in \text{Sp}(N, \omega),$$

and observe that this is an isomorphism of principal $\text{Sp}(V_3)$ bundles.

In Section 5.2.2, we used the global trivialization

$$\hat{v}_j \mapsto \frac{\partial}{\partial q_j}igg|_m, \quad \hat{w}_j \mapsto \frac{\partial}{\partial p_j}igg|_m, \quad \forall m \in N, \quad j = 1, 2, 3,$$

to identify $\text{Sp}(N, \omega)$ with $N \times \text{Sp}(V_3)$. From the map $\tilde{LS}$, we see that $\text{Sp}(T^+S^3_n, \omega_3)$ can be identified with $T^+S^3_n \times \text{Sp}(V_3)$ with respect to the global trivialization

$$\hat{v}_j \mapsto LS_\ast \frac{\partial}{\partial q_j}igg|_{LS(m)}, \quad \hat{w}_j \mapsto LS_\ast \frac{\partial}{\partial p_j}igg|_{LS(m)}, \quad \forall LS(m) \in T^+S^3_n, \quad j = 1, 2, 3.$$

In terms of these trivializations for $\text{Sp}(N, \omega)$ and $\text{Sp}(T^+S^3_n, \omega_3)$, $\tilde{LS}$ is given simply by

$$\tilde{LS}(m, g) = (LS(m), g), \quad \forall m \in N, \quad \forall g \in \text{Sp}(V_3). \quad (5.3.2)$$

As we have seen before, once we have a global trivialization for the symplectic frame bundle, it is straightforward to construct a metaplectic-c prequantization. Let $Q' = T^+S^3_n \times \text{Mp}^c(V_3)$ with bundle projection map $Q' \xrightarrow{\Pi'} T^+S^3_n$. Define the map $\Gamma' : Q' \to \text{Sp}(T^+S^3_n, \omega_3)$ by

$$\Gamma'(z, a) = (z, \sigma(a)), \quad \forall z \in T^+S^3_n, \quad \forall a \in \text{Mp}^c(V_3).$$
Let
\[ \delta' = \frac{1}{i\hbar} \Pi^* \beta_3 + \frac{1}{2} \eta_* \vartheta_0, \]
where \( \beta_3 \) is the restriction to \( T^+ S_n^3 \) of the one-form defined in Equation (5.3.1), and where \( \vartheta_0 \) is the trivial connection on \( Q' \). Then \( (Q', \Gamma', \delta') \) is a metaplectic-c prequantization for \( (T^+ S_n^3, \omega_3) \).

Recall that \( P = N \times \text{Mp}^c(V_3) \). Define \( \hat{L}S : P \to Q' \) by
\[ \hat{L}S(m, a) = (LS(m), a). \]
This is clearly an isomorphism of principal \( \text{Mp}^c(V_3) \) bundles. Since \( LS^* \beta_3 = \beta \), we have \( LS^* \delta' = \gamma \). Lastly, it follows from Equation (5.3.2) that \( \Gamma' \circ \hat{L}S = \tilde{L}S \circ \Sigma \). Therefore \( \hat{L}S \) is an isomorphism of metaplectic-c prequantizations.

All of these observations combine to yield the following commutative diagram.

\[
\begin{array}{ccc}
(P, \gamma) & \xrightarrow{\hat{L}S} & (Q', \delta') \\
\downarrow{\Sigma} & & \downarrow{\Gamma'} \\
\text{Sp}(N, \omega) & \xrightarrow{\tilde{L}S} & \text{Sp}(T^+ S_n^3, \omega_3) \\
\downarrow{LS} & & \downarrow{LS} \\
(N, \omega) & \xrightarrow{LS} & (T^+ S_n^3, \omega_3)
\end{array}
\]

Each of the maps \( LS, \tilde{L}S, \) and \( \hat{L}S \) is an isomorphism. From these isomorphisms and the fact that \( D \circ LS = H \), it follows that \( E < 0 \) is a quantized energy level of \( (N, \omega, H) \) if and only if it is a quantized energy level of \( (T^+ S_n^3, \omega_3, D) \).

Recall that we constructed the metaplectic-c prequantization \( (Q, \Gamma, \delta) \) for \( (T^+ S_n^3, \omega_3) \) in the previous section. Since \( T^+ S_n^3 \) is simply connected, \( (Q', \Gamma', \delta') \) must be isomorphic to \( (Q, \Gamma, \delta) \). Therefore we can calculate the quantized energy levels for the system \( (T^+ S_n^3, \omega_3, D) \) using \( (Q, \Gamma, \delta) \).

Moreover, we claim that \( E < 0 \) is a quantized energy level of \( (N, \omega, H) \) if and only if it is a quantized energy level of \( (T^+ S^3, \omega_3, D) \). When we place the north pole back into \( S^3 \), we acquire more closed orbits: namely, those with \( x \)-components that pass through \((0, 0, 0, 1)\). However, due to the rotational symmetry of the system \( (Q, \Gamma, \delta) \to (T^+ S^3, \omega_3, D) \), an orbit that
passes through \((0, 0, 0, 1)\) can always be transformed into one that does not, without altering the holonomy condition. Thus the quantized energy levels of \((T^+S_n^3, \omega_3, D)\) and \((T^+S^3, \omega_3, D)\) are identical.

### 5.3.4 Rescaling the Delaunay Hamiltonian

So far, we have shown that the negative quantized energy levels of the hydrogen atom are the same as the quantized energy levels of the Delaunay Hamiltonian on \(T^+S^3\). In this section, we will make one final transformation that relates these energies to the quantized energy levels of a free particle on \(S^3\).

Let \(K : TS^3 \rightarrow \mathbb{R}\) be given by

\[
K(x, y) = \frac{1}{2}|y|^2, \quad \forall (x, y) \in TS^3.
\]

When this function is interpreted as a Hamiltonian energy function, it has a kinetic energy term but no potential energy term: that is, it describes a free particle on \(S^3\). It is shown in [4] that the corresponding Hamiltonian vector field on \(TS^3\) is

\[
\xi_K = \sum_{j=1}^{4} \left(y_j \frac{\partial}{\partial x_j} - |y|^2 x_j \frac{\partial}{\partial y_j}\right),
\]

and the flow of this vector field is

\[
\psi^t_K(x, y) = \begin{pmatrix} \cos |y|t & \frac{1}{|y|} \sin |y|t \\ -|y| \sin |y|t & \cos |y|t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \forall (x, y) \in TS^3 \text{ such that } |y| > 0. \tag{5.3.3}
\]

If \(|y| = 0\), then the particle is stationary, and the flow is simply \(\psi^t_K(x, 0) = (x, 0)\). Clearly \(K = 0\) on the zero section of \(TS^3\), and \(K > 0\) on \(T^+S^3\).

The range of the Delaunay Hamiltonian \(D\) is \(\mathbb{R}_{<0}\). Let \(f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{<0}\) be given by

\[
f(z) = -\frac{m_e k^2}{4z}, \quad \forall z \in \mathbb{R}_{>0}. \tag{5.3.4}
\]

Note that \(f\) is a diffeomorphism, and \(D = f \circ K\) on \(T^+S^3\). Using the dynamical invariance
property of quantized energy levels, we see that $E$ is a quantized energy level of $(T^+S^3, \omega_3, D)$ if and only if $f^{-1}(E)$ is a positive quantized energy level of $(TS^3, \omega_3, K)$. It remains to find the positive quantized energy levels for a free particle on $S^3$.

5.4 Quantization of a Free Particle on $S^3$

5.4.1 Orbits of $\xi_K$ and local coordinates for $TS^3$

Let $E > 0$ be arbitrary, and let $S = K^{-1}(E) \subset TS^3$. From the flow of $\xi_K$ in Equation (5.3.3), it is apparent that all orbits of $\xi_K$ in $S$ are closed, with period $\frac{2\pi}{\sqrt{2E}}$. Let $z_0 = (x_0, y_0) \in S$ be an arbitrary initial point, and let $C \subset S$ be the the orbit of $\xi_K$ through $z_0$:

$$C = \{ \psi_t^k(z_0) : t \in \mathbb{R} \} \subset TS^3,$$

We can use the rotational symmetry of the system $(Q, \Gamma, \delta) \to (TS^3, \omega_3, K)$ to make some simplifying assumptions about $C$. If we view $x_0$ and $y_0$ as two perpendicular vectors in $\mathbb{R}^4$, then there is some rotation about the origin in $\mathbb{R}^4$ that carries them both to the $x_3x_4$-plane. By performing this rotation in $x$-space and $y$-space, we can assume without loss of generality that $x_0$ and $y_0$ take the form $x_0 = (0, 0, x_{30}, x_{40})$ and $y_0 = (0, 0, y_{30}, y_{40})$, where $|x|^2 = x_{30}^2 + x_{40}^2 = 1$, $x \cdot y = x_{30}y_{30} + x_{40}y_{40} = 0$, and $|y|^2 = y_{30}^2 + y_{40}^2 = 2E$. The orbit $C$ then lies in $x_3x_4y_3y_4$-space.

Thus far, we have treated $TS^3$ as a submanifold of $T\mathbb{R}^4$, using the coordinates $(x, y)$ on $T\mathbb{R}^4$ to describe points in $TS^3$. Now we make a local change of coordinates on $T\mathbb{R}^4$ that will yield symplectic coordinates for $TS^3$ on a neighborhood that contains $C$. Specifically, we introduce 4-dimensional spherical coordinates and their conjugate momenta. Let $U \subset T\mathbb{R}^4$ be the open set

$$U = \{ (x, y) \in T\mathbb{R}^4 : x_3^2 + x_4^2 > 0 \}.$$
On $U$, let the new spatial coordinates be $(a, b, c, r)$, where
\[ a = \arctan \frac{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}{x_1}, \]
\[ b = \arctan \frac{\sqrt{x_3^2 + x_4^2}}{x_2}, \]
\[ c = \arctan \frac{x_4}{x_3}, \]
\[ r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}. \]

The angles $a$ and $b$ are defined modulo $\pi$, and the angle $c$ is defined modulo $2\pi$. Let $\rho_1 = \sqrt{x_1^2 + x_3^2 + x_4^2}$ and $\rho_2 = \sqrt{x_3^2 + x_4^2}$. The conjugate momenta corresponding to the spherical coordinates are
\[ p_a = \frac{x_1}{\rho_1} (x_2y_2 + x_3y_3 + x_4y_4) - \rho_1 y_1, \]
\[ p_b = \frac{x_2}{\rho_2} (x_3y_3 + x_4y_4) - \rho_2 y_2, \]
\[ p_c = x_3y_4 - x_4y_3, \]
\[ p_r = \frac{1}{r} (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4). \]

Later, we will need the inverse transformations, which are
\[ x_1 = r \cos a, \]
\[ x_2 = r \sin a \cos b, \]
\[ x_3 = r \sin a \sin b \cos c, \]
\[ x_4 = r \sin a \sin b \sin c, \]
\[ (5.4.1) \]
\[ y_1 = p_r \cos a - \frac{p_a}{r} \sin a, \]
\[ y_2 = p_r \sin a \cos b + \frac{p_a}{r} \cos a \cos b - \frac{p_b}{r \sin a} \sin b, \]
\[ y_3 = p_r \sin a \sin b \cos c + \frac{p_a}{r} \cos a \sin b \cos c + \frac{p_b}{r \sin a} \cos b \cos c - \frac{p_c}{r \sin a \sin b} \sin c, \]
\[ y_4 = p_r \sin a \sin b \sin c + \frac{p_a}{r} \cos a \sin b \sin c + \frac{p_b}{r \sin a} \cos b \sin c + \frac{p_c}{r \sin a \sin b} \cos c. \]

(5.4.2)

For convenience, we let $a_j, j = 1, \ldots, 4$, range over $a, b, c, r$. 
On $U$, one can verify that

$$\beta_4 = -\sum_{j=1}^{4} y_j dx_j = -\sum_{j=1}^{4} p_{a_j} da_j,$$

and so

$$\omega_4 = \sum_{j=1}^{4} dx_j \wedge dy_j = \sum_{j=1}^{4} da_j \wedge dp_{a_j}.$$  \hspace{1cm} (5.4.3)

The submanifold $TS^3$ is characterized by the constant values $r = 1$ and $p_r = 0$, which implies

that the restrictions to $TS^3 \cap U$ of $\beta_4$ and $\omega_4$ are

$$\beta_3 = -\sum_{j=1}^{3} p_{a_j} da_j, \quad \omega_3 = \sum_{j=1}^{3} da_j \wedge dp_{a_j}.$$ 

Thus $(a, b, c, p_a, p_b, p_c)$ are symplectic coordinates for $TS^3 \cap U$. On this neighborhood, the map $K$ takes the form

$$K = \frac{1}{2} \left( p_a^2 + \frac{p_b^2}{\sin^2 a} + \frac{p_c^2}{\sin^2 a \sin^2 b} \right).$$

Several times now, once we had a set of symplectic coordinates such as $(a, b, c, r, p_a, p_b, p_c, p_r)$, we used the trivialization of the symplectic frame bundle given by the coordinate vector fields to construct a metaplectic-c prequantization. We could apply this same procedure to $U$; however, since $U$ is not simply connected, it is not necessarily the case that the metaplectic-c prequantization so constructed would be isomorphic to the result of restricting $(Q_4, \Gamma_4, \delta_4)$ to $U$. Instead, we must show how the local change of variables from Cartesian to spherical coordinates can be lifted from $T\mathbb{R}^4$ to $Q_4$. This is the subject of the next section.

### 5.4.2 Change of variables over $C$

Recall that the integral curve of $\xi_K$ through the initial point $z_0 = (x_0, y_0)$ is

$$\psi^t_K(x_0, y_0) = \begin{pmatrix} \cos \sqrt{2E}t & \frac{1}{\sqrt{2E}} \sin \sqrt{2E}t \\ -\sqrt{2E} \sin \sqrt{2E}t & \cos \sqrt{2E}t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

and its image $C$ is a closed curve lying in $x_3 x_4 y_3 y_4$-space. Since $C \subset U$, the points in $C$ can be rewritten in spherical coordinates. Upon converting, we find that any point in $C$ satisfies
$p_a = p_b = 0$ and $a = b = \frac{\pi}{2}$. Further, $p_c$ is a constant value over $C$ satisfying $p_c^2 = 2\mathcal{E}$. Since $C \subset TS^3$, we also have $r = 1$ and $p_r = 0$. Therefore, in spherical coordinates, the orbit takes the form

$$C = \left\{ \left( \frac{\pi}{2}, \frac{\pi}{2}, c, 1, 0, 0, p_c, 0 \right) : c \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$  

Let $z(c)$ represent the point $\left( \frac{\pi}{2}, \frac{\pi}{2}, c, 1, 0, 0, p_c \right) \in C$.

The change of coordinates from Cartesian to spherical must now be lifted to the symplectic frame bundle and the metaplectic-c prequantization for $(T\mathbb{R}^4, \omega_4)$. The change of coordinates on $Sp(T\mathbb{R}^4, \omega_4)$ will take place over $U$, and that on $Q_4$ will take place over $C$. We follow the same steps that we applied to the harmonic oscillator in Section 4.4.2.

On the neighborhood $U$, we have two different coordinate maps: the Cartesian map $\Phi_c : U \to \mathbb{R}^8$, and the spherical map $\Phi_s : U \to \mathbb{R}^4 \times (\mathbb{R}/\pi\mathbb{Z})^2 \times \mathbb{R}^2/2\pi\mathbb{Z} \times \mathbb{R}$. The change of variables on $U$ is simply the transition map $F = \Phi_s \circ \Phi_c^{-1}$. Let $\Phi_c(U) = U_c$ and $\Phi_s(U) = U_s$. Then each of the following maps is a diffeomorphism.

Let $b_c$ be the section of $Sp(T\mathbb{R}^4, \omega_4)$ over $U$ given by

$$b_c(z) : V_4 \to T_z T\mathbb{R}^4 \text{ such that } \hat{v}_j \mapsto \left. \frac{\partial}{\partial x_j} \right|_z, \hat{w}_j \mapsto \left. \frac{\partial}{\partial y_j} \right|_z, \forall z \in U, j = 1, \ldots, 4.$$  

That is, $b_c$ is the section that defines the trivialization of $Sp(T\mathbb{R}^4, \omega_4)|_U$ with respect to Cartesian coordinates. Let the map $\bar{\Phi}_c : Sp(T\mathbb{R}^4, \omega_4)|_U \to U_c \times Sp(V_4)$ be given by

$$\bar{\Phi}_c(b_c(z) \cdot g) = (\Phi_c(z), g), \forall z \in U, \forall g \in Sp(V_4).$$  

Similarly, let $b_s$ be the section of $Sp(T\mathbb{R}^4, \omega_4)|_U$ given by

$$b_s(z) : V_4 \to T_z T\mathbb{R}^4 \text{ such that } \hat{v}_j \mapsto \left. \frac{\partial}{\partial a_j} \right|_z, \hat{w}_j \mapsto \left. \frac{\partial}{\partial p_{aj}} \right|_z, \forall z \in U, j = 1, \ldots, 4.$$  

and define the map $\tilde{\Phi}_s : \text{Sp}(T\mathbb{R}^4, \omega)|_U \to U_s \times \text{Sp}(V_4)$ by

$$
\tilde{\Phi}_s(b_s(z) \cdot g) = (\Phi_s(z), g), \quad \forall z \in U, \forall g \in \text{Sp}(V_4).
$$

To perform the change of coordinates on the level of the symplectic frame bundle, we must lift $F$ to a map $\tilde{F} : U_c \times \text{Sp}(V_4) \to U_s \times \text{Sp}(V_4)$ in such a way that the following diagram commutes.

In Equations (5.4.1) and (5.4.2), we gave explicit formulas for $x_j$ and $y_j$ in terms of $a_k$ and $p_{ak}$. At each $z \in U$, let $G(z)$ be the $8 \times 8$ matrix consisting of the partial derivatives of the Cartesian coordinates with respect to the spherical ones:

$$
G(z) = \begin{pmatrix}
\frac{\partial x_k}{\partial a_j} & \frac{\partial y_k}{\partial a_j} \\
\frac{\partial x_k}{\partial p_{aj}} & \frac{\partial y_k}{\partial p_{aj}}
\end{pmatrix}_{1 \leq j,k \leq 4}.
$$

Then

$$
G(z) \begin{pmatrix}
\frac{\partial x_k}{\partial x_l} \\
\frac{\partial y_k}{\partial y_l}
\end{pmatrix}_{1 \leq k \leq 4} = \begin{pmatrix}
\frac{\partial x_j}{\partial a_j} \\
\frac{\partial y_j}{\partial a_j}
\end{pmatrix}_{1 \leq j \leq 4}, \quad \forall z \in U.
$$

Since the Cartesian and spherical coordinate vectors are both symplectic bases for $T_z T\mathbb{R}^4$, $G(z)$ is a symplectic matrix and can be treated as an element of $\text{Sp}(V_4)$. By an identical argument to that in Section 4.4.2, the desired map $\tilde{F}$ is given by

$$
\tilde{F}(\Phi_c(z), g) = (\Phi_s(z), G(z)g), \quad \forall z \in U, \forall g \in \text{Sp}(V_4).
$$

The final lift to the metaplectic-c prequantization will take place over $\mathcal{C}$. Let $C_c$ and $C_s$ be the images of $\mathcal{C}$ under $\Phi_c$ and $\Phi_s$, respectively. Recall that points in $\mathcal{C}$ are denoted by $z(c)$ with $c \in \mathbb{R}/2\pi\mathbb{Z}$. We write $G(c)$ for $G(z(c))$. The components of $G(c)$ are single-valued with respect to $c$, so $G(c)$ is a closed path through $\text{Sp}(V_4)$.

It is clear from the construction of $Q_4$ how to define a local trivialization with respect to the
Cartesian coordinates. Let \( \Phi_c : Q_4|c \to C_c \times \text{Mp}^c(V_4) \) be given by \( \Phi_c(z(c), a) = (\Phi_c(z(c)), a) \) for all \( z(c) \in C \) and \( a \in \text{Mp}^c(V_4) \). Then \( \Phi_c \) and \( \tilde{\Phi}_c \) make the following diagram commute.

We require a local trivialization of \( Q_4 \) with respect to the spherical coordinates that has the analogous relationship to \( \tilde{\Phi}_s \). To find the appropriate map \( \hat{\Phi}_s : Q_4|c \to C_s \times \text{Mp}^c(V_4) \), we will construct a map \( \hat{F} \) that satisfies the diagram shown below, then define \( \hat{\Phi}_s = \hat{F} \circ \hat{\Phi}_c \).

As before, if \( \hat{F} \) exists, then it has the form

\[
\hat{F}(\Phi_c(z(c)), a) = (\Phi_s(z(c)), \tilde{G}(c)a), \quad \forall z(c) \in C, \forall a \in \text{Mp}^c(V_4),
\]

where \( \tilde{G}(c) \in \text{Mp}^c(V_4) \) and \( \sigma(\tilde{G}(c)) = G(c) \) for all \( c \). To preserve the form of \( \delta_4 \), we lift the path \( G(c) \) to a path in \( \text{Mp}(V_4) \), then check whether \( \tilde{G}(c) \) is single-valued with respect to \( c \).

If \( \tilde{G}(c) \in \text{Mp}(V_4) \), then its parameters (recall Section 2.3.1) take the form \( (G(c), \mu(c)) \), where \( \mu(c)^2 \text{Det}_C G(c) = 1 \) and \( G(c) = \frac{1}{2}(G(c) - JG(c)J) \). To determine \( \mu(c) \), we must calculate \( G(c) \). Using the expressions in Equations (5.4.1) and (5.4.2), we compute the partial derivatives that form the matrix \( G \), and evaluate at the point \( z(c) = (\frac{\pi}{2}, \frac{\pi}{2}, c, 1, 0, 0, p_c, 0) \in C_s \). Using the
abbreviations $S(c) = \sin c$ and $C(c) = \cos c$, the result is

$$G(c) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -S(c) & C(c) & 0 & 0 & -p_c C(c) & -p_c S(c) \\ 0 & 0 & C(c) & S(c) & 0 & 0 & p_c S(c) & -p_c C(c) \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -S(c) & C(c) \\ 0 & 0 & 0 & 0 & 0 & 0 & C(c) & S(c) \end{pmatrix}.$$  

Next, we evaluate $C_G(c)$, using the matrix form for $J$ noted in Section 5.1.1, and convert it to a $4 \times 4$ complex matrix. We find that

$$C_G(c) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -S(c) - \frac{i}{2} p_c C(c) & C(c) - \frac{i}{2} p_c S(c) \\ 0 & 0 & C(c) + \frac{i}{2} p_c S(c) & S(c) - \frac{i}{2} p_c C(c) \end{pmatrix},$$

which has complex determinant

$$\text{Det}_C C_G(c) = -1 - \frac{1}{4} p_c^2.$$  

This value is real and constant over $C$. Thus we can define $\hat{G}(c)$ to be the element of $\text{Mp}(V_4) \subset \text{Mp}^c(V_4)$ with parameters $G(c), i \left(1 + \frac{1}{4}p_c^2\right)^{-1/2}$, for all $c \in \mathbb{R}/2\pi \mathbb{Z}$, whereupon $\hat{G}(c)$ is the desired closed path through $\text{Mp}(V_4)$.

Having determined $\hat{G}(c)$, we now define the map $\hat{F} : \mathcal{C}_c \times \text{Mp}^c(V_4) \to \mathcal{C}_s \times \text{Mp}^c(V_4)$ by

$$\hat{F}(\Phi_c(z(c)), a) = (\Phi_s(z(c)), \hat{G}(c)a) \quad \forall z(c) \in \mathcal{C}, \forall a \in \text{Mp}^c(V_4).$$

The local trivialization of $Q_4|_{\mathcal{C}}$ that is compatible with spherical coordinates comes about by setting $\hat{\Phi}_s = \hat{F} \circ \hat{\Phi}_c$. The one-form $\delta_4$ on $Q_4$ induces a one-form $\delta_{4s}$ on $\mathcal{C}_s \times \text{Mp}^c(V_4)$ that takes
the form
\[ \delta_4^s = \frac{1}{\iota \hbar} \Pi^s \beta_4 + \frac{1}{2} \eta_s \vartheta_0, \]

where \( \beta_4 \) is written in spherical coordinates as in Equation (5.4.3), and where \( \vartheta_0 \) is now the trivial connection on \( C_s \times \text{Mp}^c(V_4) \).

### 5.4.3 Restrictions to \( C \subset T^3 \)

From now on, we no longer write the maps \( \Phi_s, \tilde{\Phi}_s \) and \( \hat{\Phi}_s \) explicitly, but write elements of \( T^4, \text{Sp}(T^4, \omega_4)|_U \) and \( Q_4|_C \) with respect to spherical coordinates and the local spherical trivializations. Further, we treat \( T^3 \) as a six-dimensional manifold with local coordinates \((a, b, c, p_a, p_b, p_c)\) on \( T^3 \cap U \), and with symplectic form \( \omega_3 = \sum_{j=1}^{3} da_j \wedge dp_{a_j} \).

Recall from Section 5.3.2 that we defined the map \( R(x, y) = \frac{1}{2}|x|^2 \) on \( T^4 \), and set \( A = R^{-1}(\frac{1}{2}) \). We argued that the symplectic frame bundle \( \text{Sp}(T^3, \omega_3) \) is isomorphic to \( \text{Sp}(TA/TA^\perp)|_{T^3} \), and that the metaplectic-c prequantization \((Q, \Gamma, \delta)\) for \((T^3, \omega_3)\) is obtained by restricting \((Q_4A, \Gamma_4A, \delta_4A)\) to \( \text{Sp}(TA/TA^\perp)|_{T^3} \).

In spherical coordinates, we have
\[ R(z) = \frac{1}{2}r^2, \quad \forall z \in T^4 \cap U, \]

which has Hamiltonian vector field
\[ \xi_R = r \frac{\partial}{\partial p_r}. \]

It is immediate that at each point \( z \in A \cap U \),

\[ T_z A = \text{span}\left\{ \frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial p_a}, \frac{\partial}{\partial p_b}, \frac{\partial}{\partial p_c}, \frac{\partial}{\partial p_r} \right\} \text{ and } T_z A^\perp = \text{span}\left\{ \frac{\partial}{\partial p_r} \right\}, \]

which implies that
\[ T_z A / T_z A^\perp = \text{span}\left\{ \left[ \frac{\partial}{\partial a} \right], \left[ \frac{\partial}{\partial b} \right], \left[ \frac{\partial}{\partial c} \right], \left[ \frac{\partial}{\partial p_a} \right], \left[ \frac{\partial}{\partial p_b} \right], \left[ \frac{\partial}{\partial p_c} \right], \left[ \frac{\partial}{\partial p_r} \right] \right\}. \]

We see that over \( A \cap U \), the local trivialization of \( \text{Sp}(T^4, \omega_4)|_U \) with respect to spherical
coordinates induces a local trivialization

$$\text{Sp}(T^R 4, \omega_4; A)|_{A\cap U} = A \cap U \times \text{Sp}(V_4; W_4).$$

The group homomorphism $\nu : \text{Sp}(V_4; W_4) \to \text{Sp}(V_3)$ then induces a local trivialization

$$\text{Sp}(TA/TA^\perp)|_{A\cap U} = A \cap U \times \text{Sp}(V_3).$$

Restricting further to $TS^3 \cap U \subset A \cap U$, we find that

$$\text{Sp}(TS^3, \omega_3)|_{TS^3 \cap U} = TS^3 \cap U \times \text{Sp}(V_3),$$

with the local trivialization given by

$$\hat{v}_j \mapsto \frac{\partial}{\partial a_j} \bigg|_z, \quad \hat{w}_j \mapsto \frac{\partial}{\partial p_{a_j}} \bigg|_z, \quad \forall z \in TS^3 \cap U, \; j = 1, 2, 3.$$

On the level of metaplectic-c bundles, we can write

$$Q_4^A|_C = C \times \text{Mp}^c(V_4; W_4)$$

by restricting $Q_4|_C$ to $\text{Sp}(T^R 4, \omega; A)|_{C}$. Then the group homomorphism $\hat{\nu} : \text{Mp}^c(V_4; W_4) \to \text{Mp}^c(V_3)$ induces the local trivialization

$$Q_4A|_C = Q|_C = C \times \text{Mp}^c(V_3).$$

The prequantization one-form on $Q$ is

$$\delta = \frac{1}{i\hbar} \Pi^* \beta_3 + \frac{1}{2} \eta_* \vartheta_0,$$

where $\vartheta_0$ is the trivial connection on $C \times \text{Mp}^c(V_3)$, and where

$$\beta_3 = - \sum_{j=1}^{3} p_j da_j = -p_v dc$$
over $\mathcal{C}$.

### 5.4.4 Quantized energy levels for $(T S^3, \omega_3, K)$

In the local coordinates $(a, b, c, p_a, p_b, p_c)$, the map $K : T S^3 \to \mathbb{R}$ is given by

$$K = \frac{1}{2} \left( p_a^2 + \frac{p_b^2}{\sin^2 a} + \frac{p_c^2}{\sin^2 a \sin^2 b} \right),$$

and its Hamiltonian vector field is

$$\xi_K = p_a \frac{\partial}{\partial a} + \frac{p_b}{\sin^2 a} \frac{\partial}{\partial b} + \frac{p_c}{\sin^2 a \sin^2 b} \frac{\partial}{\partial c} + \left( \frac{p_b^2}{\sin^2 a} \cos a + \frac{p_c^2}{\sin^2 a \sin^2 b} \right) \frac{\partial}{\partial p_a} + \frac{p_c^2 \cos b}{\sin^2 a \sin^2 b} \frac{\partial}{\partial p_b}.$$

Recall that we fixed $\mathcal{E} > 0$ and defined $\mathcal{S} = K^{-1}(\mathcal{E})$, and that $z_0$ is a point in $\mathcal{S}$. The closed curve $\mathcal{C}$ is the orbit of $\xi_K$ through $z_0$. We will show that the orbit of $\tilde{\xi}_K$ through $(z_0, I) \in \text{Sp}(T S^3 / T S^3 \perp)$ is closed, and then we will determine the values of $\mathcal{E}$ for which the orbit of $\hat{\xi}_K$ through $(z_0, I) \in Q_S$ is also closed.

The lift of the flow $\psi^t_K$ to $\tilde{\psi}^t_K$ on $\text{Sp}(T S^3, \omega_3)$ is given by

$$\tilde{\psi}^t_K(z, I) = (z, \psi^t_K|_z), \text{ } \forall z \in T S^3 \cap U,$$

which implies that the lifted vector field $\tilde{\xi}_K$ is

$$\tilde{\xi}_K(z, I) = \left( \xi_K(z), \left. \frac{d}{dt} \right|_{t=0} \psi^t_K|_z \right), \text{ } \forall z \in T S^3 \cap U.$$

As a $6 \times 6$ matrix, $\left. \frac{d}{dt} \right|_{t=0} \psi^t_K|_z$ can be interpreted as an element of the Lie algebra $\mathfrak{sp}(V_3)$, and its components are

$$\left( \left. \frac{d}{dt} \right|_{t=0} \psi^t_K|_z \right)_{jk} = \left. \frac{\partial(\xi_K)_j}{\partial X_k} \right|_{X=z} \bigg|_{X=z},$$

where $X_k$ ranges over $(a, b, c, p_a, p_b, p_c)$. In particular, when we evaluate this matrix of partial
derivatives at $z(c) \in C$, we obtain

$$\left. \frac{d}{dt} \right|_{t=0} \psi^t_K |_{z(c)} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-p_c^2 & 0 & 0 & 0 & 0 \\
0 & -p_c^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}. $$

This value is constant over $C$; we denote it by $\kappa$. Thus

$$\tilde{\xi}_K(z(c), I) = (\xi_K(z(c)), \kappa), \ \forall z(c) \in C,$$

which implies that the flow of $\tilde{\xi}_K$ through $(z_0, I) \in \text{Sp}(TS^3, \omega_3)$ is

$$\tilde{\psi}^t_K(z_0, I) = (\psi^t_K(z_0), \exp(t\kappa)).$$

Let $\lambda = \sqrt{-p_c^2} = \sqrt{2E}$. A calculation shows that

$$\exp(t\kappa) = \begin{pmatrix}
\cos \lambda t & 0 & 0 & \frac{1}{\lambda} \sin \lambda t & 0 & 0 \\
0 & \cos \lambda t & 0 & 0 & \frac{1}{\lambda} \sin \lambda t & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\lambda \sin \lambda t & 0 & 0 & \cos \lambda t & 0 & 0 \\
0 & -\lambda \sin \lambda t & 0 & 0 & \cos \lambda t & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}. $$

Thus the orbit through $(z_0, I)$ is closed, with period $\frac{2\pi}{\lambda}$.

Over $C$, $\xi_K$ reduces to

$$\xi_K = p_c \frac{\partial}{\partial c}. $$
Therefore, for all \( z(c) \in \mathcal{C} \), we have
\[
T_{z(c)}S^\perp = \text{span} \left\{ \frac{\partial}{\partial c} \right\}_{z(c)}, \quad T_{z(c)}S = \text{span} \left\{ \frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial p_a}, \frac{\partial}{\partial p_b} \right\}_{z(c)},
\]
and
\[
T_{z(c)}S/T_{z(c)}S^\perp = \text{span} \left\{ \left[ \frac{\partial}{\partial a} \right], \left[ \frac{\partial}{\partial b} \right], \left[ \frac{\partial}{\partial p_a} \right], \left[ \frac{\partial}{\partial p_b} \right] \right\}_{z(c)}.
\]
We identify \( \text{Sp}(TS/TS^\perp)_{\mathcal{C}} \) with \( \mathcal{C} \times \text{Sp}(V_2) \) in the obvious way.

Upon descending to \( \text{Sp}(TS/TS^\perp)_{\mathcal{C}} \), the induced flow takes the form
\[
\overline{\psi}_K^t(z_0, I) = (\psi_K^t(z_0), \nu(\exp(t\kappa))),
\]
where \( \nu \) is the group homomorphism \( \text{Sp}(V_3; W_3) \to \text{Sp}(V_2) \). We calculate that
\[
\nu(\exp(t\kappa)) = \begin{pmatrix}
\cos \lambda t & 0 & \frac{1}{\lambda} \sin \lambda t & 0 \\
0 & \cos \lambda t & 0 & \frac{1}{\lambda} \sin \lambda t \\
-\lambda \sin \lambda t & 0 & \cos \lambda t & 0 \\
0 & -\lambda \sin \lambda t & 0 & \cos \lambda t
\end{pmatrix}.
\]
The corresponding vector field on \( \text{Sp}(TS/TS^\perp)_{\mathcal{C}} \) is
\[
\overline{\xi}_K(z(c), I) = (\xi_K(z(c)), \overline{\kappa}), \quad \forall z(c) \in \mathcal{C},
\]
where
\[
\overline{\kappa} = \frac{d}{dt} \bigg|_{t=0} \nu(\exp(t\kappa)) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\lambda^2 & 0 & 0 & 0 \\
0 & -\lambda^2 & 0 & 0
\end{pmatrix} \in \text{sp}(V_2).
\]
The local trivializations lift to the level of metaplectic-c bundles, so that \( Q_S|_{\mathcal{C}} = \mathcal{C} \times \text{Mp}^c(V_2) \).

The induced one-form \( \delta_S \) on \( Q_S \) takes the form
\[
\delta_S = -\frac{1}{i\hbar} p_c dc + \frac{1}{2} \eta_* \phi_0
\]
over $\mathcal{C}$. Therefore the horizontal lift of $\tilde{\xi}_K$ to $Q_S|_C$ is

$$\tilde{\xi}_K(z(c), I) = \left( \xi_K(z(c)), \bar{\kappa} \oplus \frac{\lambda^2}{i\hbar} \right), \quad \forall z(c) \in \mathcal{C}.$$ 

Since the $mp^c(V_2)$ component is constant, the orbit of $\tilde{\xi}_K$ through $(z_0, I) \in Q_S$ is

$$\tilde{\psi}_K(z_0, I) = \left( \psi_K(z_0), \exp \left( t\bar{\kappa} \oplus \frac{\lambda^2 t}{i\hbar} \right) \right).$$

We can write the $Mp^c(V_2)$ component as $\exp(t\kappa \oplus 0)e^{\lambda t^2/i\hbar}$, where $\exp(t\bar{\kappa} \oplus 0) \in Mp(V_2) \subset Mp^c(V_2)$ and $e^{\lambda t^2/i\hbar} \in U(1) \subset Mp^c(V_2)$.

The parameters of $\exp(t\bar{\kappa} \oplus 0)$ have the form $(\exp(t\bar{\kappa}, \mu(t)))$ where

$$C_{\exp(t\bar{\kappa})} = \begin{pmatrix} \cos \lambda t + \frac{i}{2} \left( \lambda + \frac{1}{\lambda} \right) \sin \lambda t & 0 \\ 0 & \cos \lambda t + \frac{i}{2} \left( \lambda + \frac{1}{\lambda} \right) \sin \lambda t \end{pmatrix},$$

which has determinant

$$\text{Det}_C C_{\exp(t\bar{\kappa})} = \left( \cos \lambda t + \frac{i}{2} \left( \lambda + \frac{1}{\lambda} \right) \sin \lambda t \right)^2.$$

This value circles the origin twice as $t$ ranges from 0 to $2\pi/\lambda$. Therefore the parameters of $\exp(t\bar{\kappa} \oplus 0)$ are

$$\left( \exp(t\bar{\kappa}), \left( \cos \lambda t + \frac{i}{2} \left( \lambda + \frac{1}{\lambda} \right) \sin \lambda t \right)^{-1} \right),$$

which describes a closed orbit in $Mp(V_2)$ with period $2\pi/\lambda$.

The orbit in $Q_S$ is closed if and only if the $U(1)$ term is also a closed orbit with period $2\pi/\lambda$. We require $e^{\lambda^2 t^2/i\hbar} = 1$ when $t = 2\pi/\lambda$, or equivalently, $\frac{\lambda^2 2\pi}{i\hbar} = -2\pi in$ for some $n \in \mathbb{Z}$. Rearranging and recalling that $\lambda^2 = p^2_c = 2E$ results in $E = \frac{1}{2} n^2 \hbar^2$ for some $n \in \mathbb{Z}$.

We now apply the argument made in Section 5.3.4. We are only concerned with the strictly positive quantized energies of $K$, because those correspond to quantized energies of the Delaunay Hamiltonian $D$ on $T^+ S^3$. Thus we can ignore the energy level corresponding to $n = 0$, and we
can also dismiss $n < 0$ as redundant. This leaves us with

$$\mathcal{E}_n = \frac{1}{2} n^2 \hbar^2, \quad n \in \mathbb{N}.$$  

The positive value $\mathcal{E}_n$ is a quantized energy level of $K$ if and only if $f(\mathcal{E}_n)$ is a quantized energy level of $D$, where the diffeomorphism $f$ is given in Equation (5.3.4), and the quantized energy levels of $D$ are precisely the negative quantized energy levels of the hydrogen atom. At last, we find that the negative quantized energy levels of the hydrogen atom are

$$E_n = f(\mathcal{E}_n) = -\frac{m_e k^2}{2n^2 \hbar^2}, \quad n \in \mathbb{N},$$

which is exactly the quantum mechanical prediction for our model of the hydrogen atom.
Chapter 6

Generalization of the Quantized Energy Condition

The definition of a quantized energy level that was presented in Chapter 4 has a natural generalization from a regular value of a single function to that of a family of \( k \) Poisson-commuting functions. In this chapter, we present the generalized definition, and prove the corresponding generalized version of the dynamical invariance theorem. When the family of functions is of maximal size, we show that the quantization condition reduces to a Bohr-Sommerfeld condition.

A comment concerning notation: in this chapter, superscripts \( j, l \) range over elements of the Poisson-commuting family, while subscripts \( a, c \) range over components with respect to local coordinates.

6.1 The Quantization Condition

Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold, and let \( H^1, \ldots, H^k \) be a family of Poisson-commuting functions, where \( 1 \leq k \leq n \). Then \( H = (H^1, \ldots, H^k) \) is a map from \( M \) to \( \mathbb{R}^k \). Let \( E = (E^1, \ldots, E^k) \) be a regular value of \( H \), and let \( S = H^{-1}(E) \).

Let the Hamiltonian vector field for the function \( H^j \) be denoted by \( \xi^j \), \( 1 \leq j \leq k \). Note that because \( E \) is a regular value of \( H \), the Hamiltonian vector fields \( \xi^1, \ldots, \xi^k \) are linearly independent everywhere on \( S \). Further, for any \( s \in S \), \( T_s S \perp = \text{span} \{ \xi^1(s), \ldots, \xi^k(s) \} \subset T_s S \), which shows that \( T_s S \) is a coisotropic subspace of \( T_s M \).
Let \((V, \Omega)\) be a \(2n\)-dimensional model symplectic vector space, and let \(W \subset V\) be a coisotropic subspace of codimension \(k\). Then \(W/W^\perp\) acquires a symplectic structure from \(\Omega\). We now perform an identical construction to that given in Section 4.2.1. Let \(\text{Sp}(V; W)\) be the subgroup of \(\text{Sp}(V)\) that preserves \(W\), and let \(\text{Mp}^c(V; W)\) be the preimage of \(\text{Sp}(V; W)\) in \(\text{Mp}^c(V)\). Robinson and Rawnsley [17] showed that there is a lift of the natural group homomorphism \(\text{Sp}(V; W) \to \text{Sp}(W/W^\perp)\) to the level of metaplectic-c groups.

\[
\begin{array}{c}
\text{Mp}^c(V) \supset \text{Mp}^c(V; W) \\
\downarrow \sigma \\
\text{Sp}(V) \supset \text{Sp}(V; W) \\
\downarrow \nu \quad \downarrow \sigma
\end{array}
\]

Let \((P, \Sigma, \gamma)\) be a metaplectic-c prequantization for \((M, \omega)\). By exactly the same process as in Section 4.2.2, we construct the following three-level structures.

\[
\begin{array}{c}
(P, \gamma) \leftarrow \text{incl.} \quad \left(\begin{array}{c}
P^S, \gamma^S \\
\end{array}\right) \xrightarrow{\hat{\nu}} (P_S, \gamma_S) \\
\downarrow \Sigma \\
\text{Sp}(M, \omega) \leftarrow \text{incl.} \quad \left(\begin{array}{c}
\text{Sp}(M, \omega; S) \\
\end{array}\right) \xrightarrow{\nu} \text{Sp}(TS/TS^\perp) \\
\downarrow \rho \\
(M, \omega) \leftarrow \text{incl.} \quad S = \text{incl.} \quad S
\end{array}
\]

The second column is obtained by restriction, and the third column is obtained by taking associated bundles. The Hamiltonian vector fields \(\xi^1, \ldots, \xi^k\) lift to \(\hat{\xi}^1, \ldots, \hat{\xi}^k\) on the level of symplectic frame bundles, and to \(\hat{\xi}^1, \ldots, \hat{\xi}^k\) on the level of metaplectic-c bundles.

With these constructions established, we can now state the generalized version of our quantized energy condition.

**Definition 6.1.1.** Let \(E = (E^1, \ldots, E^k) \in \mathbb{R}^k\) be a regular value for the family \(H = (H^1, \ldots, H^k)\) of Poisson-commuting functions on \(M\). If the one-form \(\gamma_S\) has trivial holonomy over any closed path \(u(t)\) in \(\text{Sp}(TS/TS^\perp)\) that satisfies \(\dot{u}(t) \in \text{span}\{\hat{\xi}^1(u(t)), \ldots, \hat{\xi}^k(u(t))\}\) for all \(t\), then the value \(E\) is a **quantized energy level** for the system \((M, \omega, H)\).

In Section 6.2, we state and prove a generalized version of the dynamical invariance theorem. The proof follows essentially the same steps as in Section 4.3.2, and as such, we omit some of the details of the calculations. Then, in Section 6.3, we examine the special case of a completely
integrable system. When $k = n$, we will see that the quantization condition reduces to a Bohr-Sommerfeld condition.

### 6.2 Generalized Dynamical Invariance

Let $H = (H^1, \ldots, H^k)$ and $L = (L^1, \ldots, L^k)$ be two families of Poisson commuting functions on $M$. Suppose $E = (E^1, \ldots, E^k)$ and $F = (F^1, \ldots, F^k)$ are regular values of $H$ and $L$, respectively, such that $H^{-1}(E) = L^{-1}(F)$. We will show that the quantization condition is identical for $H$ at $E$ and for $L$ at $F$.

Let $\xi^1, \ldots, \xi^k$ be the Hamiltonian vector fields corresponding to $H^1, \ldots, H^k$, and let these vector fields have flows $\phi^{1t}, \ldots, \phi^{kt}$. Similarly, let $\eta^1, \ldots, \eta^k$ be the Hamiltonian vector fields corresponding to $L^1, \ldots, L^k$, and let these vector fields have flows $\rho^{1t}, \ldots, \rho^{kt}$.

Let $S = H^{-1}(E) = L^{-1}(F)$. Then $S$ is a $(2n - k)$-dimensional submanifold of $M$. For all $s \in S$, $T_s S^\perp$ is a $k$-dimensional subspace of $T_s S$, and

$$T_s S^\perp = \text{span} \{ \xi^1(s), \ldots, \xi^k(s) \} = \text{span} \{ \eta^1(s), \ldots, \eta^k(s) \}.$$

The above relationship implies that there is a matrix-valued function $C : S \to \mathbb{R}^{k \times k}$ such that

$$\eta^j(s) = \sum_{l=1}^{k} C^{jl}(s) \xi^l(s), \quad \forall s \in S.$$

For any $1 \leq j \leq k$, $\xi^j(s)$ and $\eta^j(s)$ are elements of $T_s S$, which is naturally identified with $T_\zeta T_s S$ for any $\zeta \in T_s S$. Since $T_\zeta T_s S \subset T_\zeta TS$, we can think of $\xi^j(s)$ or $\eta^j(s)$ as an element of $T_\zeta TS$. The following lemma is stated in terms of that identification.

**Lemma 6.2.1.** For all $s \in S$ and all $\zeta \in T_s S$,

$$\left. \frac{d}{dt} \right|_{t=0} (\rho^j|_s \zeta) = \sum_{l=1}^{k} \left( C^{jl}(s) \left. \frac{d}{dt} \right|_{t=0} (\phi^l|_s \zeta) + \left( \zeta C^{jl} \right) \xi^l(s) \right),$$

for any $1 \leq j \leq k$.

**Proof.** We work on the $(2n - k)$-dimensional manifold $S$. Let $U \subset S$ be a coordinate neighborhood for $s \in S$, and let $X = (X_1, \ldots, X_{2n-k})$ be local coordinates on $U$. With respect to these
coordinates, we write \( \phi^j_1 = (\phi^j_1, \ldots, \phi^j_{2n-k}) \) near \( s \), and likewise for \( \rho^j \).

Since \( \eta^j = \sum_{l=1}^k \sum_{l=1}^k C_{jl}(s) \xi^l(s) \) on \( S \), we have \( \eta^j = \sum_{l=1}^k C_{jl}(s) \xi^l(s) \) on \( U \) for each \( 1 \leq a \leq 2n-k \), and so

\[
\begin{align*}
\frac{d}{dt} \bigg|_{t=0} \rho^j_a &= \sum_{l=1}^k C_{jl}(s) \frac{d}{dt} \bigg|_{t=0} \phi^l_a(s) \\
&= 0
\end{align*}
\]

on \( U \).

The pushforward \( \phi^j_\ast \) is a \( (2n-k) \times (2n-k) \) matrix with entries given by \( \left( \phi^j_\ast \right)_{ac} = \partial \phi^j_a / \partial x_c \bigg|_{X=s} \). For any \( \zeta \in T_s S \), \( \left( \phi^j_\ast \right)_a \zeta = \zeta \phi^j_a \). The same relationship holds for \( \rho^j_\ast \zeta \). An analogous calculation to that in the proof of Lemma 4.3.3 yields

\[
\frac{d}{dt} \bigg|_{t=0} (\rho^j_\ast \zeta) = \sum_{l=1}^k \sum_{l=1}^k C_{jl}(s) \frac{d}{dt} \bigg|_{t=0} \phi^l_\ast \zeta + \left( \zeta C^j \right) \xi^l(s), \tag{6.2.1}
\]

for \( 1 \leq j \leq k \) and \( 1 \leq a \leq 2n-k \).

Given the coordinates \( X \) on \( U \), we get natural coordinates for \( TS \big|_U \), and thence coordinates for \( T_\zeta TS \). In terms of these,

\[
\begin{align*}
\frac{d}{dt} \bigg|_{t=0} (\rho^j_\ast \zeta) &= \left( \eta^j_1(s), \ldots, \eta^j_{2n-k}(s), \frac{d}{dt} \bigg|_{t=0} \left( \rho^j_\ast \zeta \right)_1, \ldots, \frac{d}{dt} \bigg|_{t=0} \left( \rho^j_\ast \zeta \right)_{2n-k} \right) \\
\frac{d}{dt} \bigg|_{t=0} (\phi^j_\ast \zeta) &= \left( \xi^j_1(s), \ldots, \xi^j_{2n-k}(s), \frac{d}{dt} \bigg|_{t=0} \left( \phi^j_\ast \zeta \right)_1, \ldots, \frac{d}{dt} \bigg|_{t=0} \left( \phi^j_\ast \zeta \right)_{2n-k} \right), \\
\end{align*}
\]

\[
\xi^j(s) = (0, \ldots, 0, \xi^j_1(s), \ldots, \xi^j_{2n-k}(s)).
\]

If we apply Equation (6.2.1) in the first line and compare with the second two lines, recalling that \( \eta^j = \sum_{l=1}^k \sum_{l=1}^k C_{jl}(s) \xi^l \), we conclude that

\[
\begin{align*}
\frac{d}{dt} \bigg|_{t=0} (\rho^j_\ast \zeta) &= \sum_{l=1}^k \sum_{l=1}^k C_{jl}(s) \frac{d}{dt} \bigg|_{t=0} \phi^l_\ast \zeta + \left( \zeta C^j \right) \xi^l(s).
\end{align*}
\]

Now we use Lemma 6.2.1 to establish a relationship between the vector fields \( \tilde{\eta}^1, \ldots, \tilde{\eta}^k \) and \( \tilde{\xi}^1, \ldots, \tilde{\xi}^k \) on \( \mathrm{Sp}(TS/T S^\perp) \).
Lemma 6.2.2. For all $s \in S$ and all $b' \in \text{Sp}(TS/TS^\perp)_s$, 

$$\tilde{\eta}^j(b') = \sum_{l=1}^{k} C_{jl}^i(s) \tilde{\xi}^j(b'), \quad 1 \leq j \leq k.$$ 

Proof. For $V$, choose a symplectic basis $\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n$. Let

$$W = \text{span}\{\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_{k+1}, \ldots, \hat{y}_n\}.$$ 

Consequently,

$$W^\perp = \text{span}\{\hat{x}_1, \ldots, \hat{x}_k\}, \quad W/W^\perp = \text{span}\{[\hat{x}_{k+1}], \ldots, [\hat{x}_n], [\hat{y}_{k+1}], \ldots, [\hat{y}_n]\}.$$ 

For convenience, let

$$(\hat{z}_1, \ldots, \hat{z}_{2n}) = (\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_{k+1}, \ldots, \hat{y}_n, \hat{y}_1, \ldots, \hat{y}_k)$$

so that

$$W = \text{span}\{\hat{z}_1, \ldots, \hat{z}_{2n-k}\}, \quad W^\perp = \text{span}\{\hat{z}_1, \ldots, \hat{z}_k\}, \quad W/W^\perp = \text{span}\{[\hat{z}_{k+1}], \ldots, [\hat{z}_{2n-k}]\}.$$ 

For any $m \in M$ and any $b \in \text{Sp}(M, \omega)_m$, identify $b$ with the $2n$-tuple $(\zeta_1, \ldots, \zeta_{2n}) \in (T_m M)^{2n}$ where $\zeta_c = b\hat{z}_c$ for $1 \leq c \leq 2n$. Similarly, for any $s \in S$ and any $b' \in \text{Sp}(TS/TS^\perp)_s$, identify $b'$ with the $(2n - 2k)$-tuple $([\zeta_{k+1}], \ldots, [\zeta_{2n-k}]) \in (T_s S/T_s S^\perp)^{2n-2k}$, where $[\zeta_c] = b'[\hat{z}_c]$ for $k + 1 \leq c \leq 2n - k$.

Let $s \in S$ and $b \in \text{Sp}(M, \omega; S)_s$ be arbitrary, and let $\zeta_c = b\hat{z}_c$ for $1 \leq c \leq 2n$. Note that $\zeta_c \in T_s S$ for all $1 \leq c \leq 2n - k$ and $\zeta_c \in T_s S^\perp$ for all $1 \leq c \leq k$. The lifted flows $\tilde{\rho}^j$ on $\text{Sp}(M, \omega)$ act on $b \in \text{Sp}(M, \omega; S)$ by

$$\tilde{\rho}^j(b) = \rho^j_{s}|_{s} \circ b = (\rho_{s}|_{s}\zeta_1, \ldots, \rho_{s}|_{s}\zeta_{2n}).$$

The element $b$ descends to the element $b' \in \text{Sp}(TS/TS^\perp)$ that is identified with $([\zeta_{k+1}], \ldots,$
\( \zeta_{2n-k} \) \( \in \) \( (T_s S/T_s S^\perp)_{2n-2k} \). The induced flows \( \tilde{\rho}^j \) on \( \text{Sp}(T S/T S^\perp) \) act on \( b' \) by

\[
\tilde{\rho}^j(b') = ([\rho^j_s | s \zeta_{k+1}], \ldots, [\rho^j_s | s \zeta_{2n-k}] ),
\]

which is a path in \( (T S/T S^\perp)_{2n-2k} \). If we take the time derivative, we get

\[
\tilde{\eta}^j(b') = \frac{d}{dt} \bigg|_{t=0} \tilde{\rho}^j(b') = \left( \frac{d}{dt} \bigg|_{t=0} [\rho^j_s | s \zeta_{k+1}], \ldots, \frac{d}{dt} \bigg|_{t=0} [\rho^j_s | s \zeta_{2n-k}] \right)
\in T_{\zeta_{k+1}}(T S/T S^\perp) \times \ldots \times T_{\zeta_{2n-k}}(T S/T S^\perp).
\]

The pushforward of the projection map \( T S \to T S/T S^\perp \), based at \( \zeta_c \), is a linear surjection \( T_{\zeta_c} T S \to T_{[\zeta_c]}(T S/T S^\perp) \), with kernel equal to the tangent space \( T_{\zeta_c} T_s S^\perp \). Identifying this tangent space with the vector space \( T_{\zeta_c} T_s S^\perp \times \ldots \times T_{\zeta_{2n-k}} T S/T_s S^\perp \). Using this, we get

\[
\tilde{\eta}^j(b') = \left( \frac{d}{dt} \bigg|_{t=0} \rho^j_s | s \zeta_{k+1}, \ldots, \frac{d}{dt} \bigg|_{t=0} \rho^j_s | s \zeta_{2n-k} \right)
\in T_{\zeta_{k+1}} T S/T_s S^\perp \times \ldots \times T_{\zeta_{2n-k}} T S/T_s S^\perp.
\]

Now, since \( \zeta_c \in T_s S \) for all \( k + 1 \leq c \leq 2n - k \), Lemma 6.2.1 applies and we have

\[
\frac{d}{dt} \bigg|_{t=0} \rho^j_s | s \zeta_c = \sum_{l=1}^k \left( C^{jl}(s) \frac{d}{dt} \bigg|_{t=0} \phi^j_s | s \zeta_c + (\zeta_c C^{jl}) \xi^j_l(s) \right)
\]

for all \( c \). Recall that \( \xi^j_l(s) \in T S^\perp \) for all \( 1 \leq l \leq k \). Therefore, upon taking equivalence classes with respect to the quotient by \( T_s S^\perp \), the terms of the form \( (\zeta_c C^{jl}) \xi^j_l(s) \) all vanish and we are left with

\[
\frac{d}{dt} \bigg|_{t=0} \rho^j_s | s \zeta_c = \sum_{l=1}^k \left( C^{jl}(s) \frac{d}{dt} \bigg|_{t=0} \phi^j_s | s \zeta_c \right), \quad k + 1 \leq c \leq 2n - k.
\]

It follows that

\[
\tilde{\eta}^j(b') = \sum_{l=1}^k C^{jl}(s) \tilde{\xi}^j_l(b'),
\]

as desired. \( \square \)

We can now prove the generalized version of the dynamical invariance theorem.
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**Theorem 6.2.3.** If $H = (H^1, \ldots, H^k)$ and $L = (L^1, \ldots, L^k)$ are two families of Poisson-commuting functions on $M$ such that $H^{-1}(E) = L^{-1}(F)$ for some regular values $E$ of $H$ and $F$ of $L$, then $E$ is a quantized energy level for $(M, \omega, H)$ if and only if $F$ is a quantized energy level for $(M, \omega, L)$.

**Proof.** From Lemma 6.2.2, each of the Hamiltonian vector fields $\tilde{\eta}^1, \ldots, \tilde{\eta}^k$ on $\text{Sp}(TS/TS^\perp)$ is a linear combination of the vector fields $\tilde{\xi}^1, \ldots, \tilde{\xi}^k$. Therefore, if there is a closed curve in $\text{Sp}(TS/TS^\perp)$ whose tangent at every point is in the subspace spanned by $\{\tilde{\eta}^1, \ldots, \tilde{\eta}^k\}$, then that tangent is also in the subspace spanned by $\{\tilde{\xi}^1, \ldots, \tilde{\xi}^k\}$, and vice versa. The desired result follows from our definition of a quantized energy level. \(\square\)

### 6.3 Completely Integrable Systems and Bohr-Sommerfeld Conditions

In this section, we consider the special case in which the Poisson-commuting family is of maximal size: namely, $k = n$. As we will show, the quantization condition simplifies to a Bohr-Sommerfeld condition in this case. First, we briefly review Bohr-Sommerfeld conditions in the context of Kostant-Souriau quantization. Our summary is based on the more detailed treatments available in [12, 20, 21, 26].

#### 6.3.1 Kostant-Souriau quantization and Bohr-Sommerfeld conditions

A completely integrable system on the $2n$-dimensional manifold $(M, \omega)$ is a family of $n$ Poisson-commuting functions $H = (H^1, \ldots, H^n)$ whose differentials are linearly independent almost everywhere. Given a completely integrable system, outside a closed set of measure zero, the corresponding Hamiltonian vector fields $\xi^1, \ldots, \xi^n$ span a real polarization: that is, an involutive, Lagrangian subbundle of the tangent bundle. A regular leaf of the polarization is a Lagrangian submanifold of the tangent bundle. Assume that $(M, \omega)$ is equipped with a Kostant-Souriau prequantization line bundle $(L, \nabla)$. A leaf of a real polarization $F$ is called **Bohr-Sommerfeld** provided there exists a global, non-vanishing section of $L$ over $S$ that is horizontal in the directions of $F$, with respect to the
connection $\nabla$. Equivalently, a leaf $S$ is Bohr-Sommerfeld if the connection has trivial holonomy over every closed curve in $S$. Since the connection is flat over a Lagrangian submanifold, homotopic curves have equal holonomy. Therefore it suffices to check the holonomy of $\nabla$ over the generators of the fundamental group $\pi_1(S)$.

Now assume that $(M, \omega)$ also admits a metaplectic structure, which implies that we can construct the half-form bundle $\wedge^{1/2}F$. Then we modify the Bohr-Sommerfeld condition so that the Bohr-Sommerfeld leaves are those over which there is a horizontal section of $L \otimes \wedge^{1/2}F$. This corrected Bohr-Sommerfeld condition is necessary in order to obtain the quantum mechanical energy levels for the system of $n$ independent one-dimensional harmonic oscillators [21]. In the next sections, we will show that our quantization condition reduces to a Bohr-Sommerfeld condition when $k = n$, and specifically to one that replicates the half-form correction for the system of harmonic oscillators.

### 6.3.2 Quantized energy condition for $k = n$

Let $H = (H^1, \ldots, H^n)$ be a family of $n$ Poisson-commuting functions on $M$, with Hamiltonian vector fields $\xi^1, \ldots, \xi^n$. As usual, let $E = (E^1, \ldots, E^n)$ be a regular value of $H$, and let $S = H^{-1}(E)$. The crucial observation is that for all $s \in S$, $T_sS^\perp = T_sS = \text{span}\{\xi^1(s), \ldots, \xi^n(s)\}$, which implies that $\text{Sp}(TS/TS^\perp)$ has trivial fiber and can be identified with $S$. The bundle $(P_S, \gamma_S)$ is a principal circle bundle with connection one-form over $\text{Sp}(TS/TS^\perp)$, and can now be viewed as a principal circle bundle with connection one-form over $S$. Since $S \subset M$ is Lagrangian, we have $d\gamma_S = 0$.

Per our definition, $E$ is a quantized energy level of $(M, \omega, H)$ if $\gamma_S$ has trivial holonomy over all closed paths in $\text{Sp}(TS/TS^\perp)$ whose tangents are in the span of the lifted Hamiltonian vector fields $\tilde{\xi}^1, \ldots, \tilde{\xi}^n$. But now $\text{Sp}(TS/TS^\perp)$ can be identified with $S$, which means that $\tilde{\xi}^j$ on $\text{Sp}(TS/TS^\perp)$ can be identified with $\xi^j$ on $S$, $1 \leq j \leq n$. Further, the vector fields $\xi^1, \ldots, \xi^n$ span all of $TS$. Thus $E$ is a quantized energy level of $(M, \omega, H)$ if $\gamma_S$ has trivial holonomy over all closed paths in $S$. This is a Bohr-Sommerfeld condition, as described in the previous section.

Our primary examples in previous chapters have been the harmonic oscillator and the hydrogen atom. In both cases, the Hamiltonian energy function can be made part of a completely
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integrable system. In the remainder of this chapter, we revisit each example and apply the generalized quantization condition to a regular level set of the completely integrable system.

6.3.3 The $n$-dimensional harmonic oscillator

Let $M = \mathbb{R}^{2n}$, with Cartesian coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic polar coordinates $(s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$. We use all of the same definitions that were established in Sections 4.4.1 and 4.4.2. In particular, the metaplectic-c prequantization for $(M, \omega)$ is $(P, \Sigma, \gamma)$, where $P = M \times \text{Mp}^c(V)$, $\Sigma : P \to \text{Sp}(M, \omega)$ is defined with respect to the global trivialization of $\text{Sp}(M, \omega)$ given by the symplectic frame $(\partial/\partial p_1, \ldots, \partial/\partial p_n, \partial/\partial q_1, \ldots, \partial/\partial q_n)$, and

$$\gamma = \frac{1}{i\hbar} \Pi^* \beta + \frac{1}{2} \eta_* \vartheta_0,$$

where $\vartheta_0$ is the trivial connection on $P$ and

$$\beta = \frac{1}{2} \sum_{j=1}^n (p_j dq_j - q_j dp_j)$$

on $M$.

Let $H = (H^1, \ldots, H^n)$, where

$$H^j = \frac{1}{2} (p_j^2 + q_j^2) = s_j$$

for each $1 \leq j \leq n$. In Cartesian coordinates, the corresponding Hamiltonian vector fields are

$$\xi^j = q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j}.$$

These vector fields clearly Poisson commute, and they are linearly independent unless $q_j = p_j = 0$ for some $j$. Thus a regular value of the family takes the form $E = (E^1, \ldots, E^n) \in \mathbb{R}^k$, where $E^j > 0$ for all $j$. Assume such a regular value $E$ has been fixed, and consider the regular level set $S = H^{-1}(E)$. Then $S$ is an $n$-torus. In symplectic polar coordinates, which are defined everywhere on $S$, a point $S$ takes the form $(E^1, \ldots, E^n, \theta_1, \ldots, \theta_n)$.

We know from Section 4.4.2 that we can change coordinates from Cartesian to symplectic
polar over the open subset of $U \subset M$ where $s_j > 0$ for all $j$. In terms of these coordinates, we have the Hamiltonian vector fields $\xi^j = -\frac{\partial}{\partial \theta^j}$ for all $j$. Let $\phi^j_t$ be the flow for the vector field $\xi^j$. It is clear that all orbits in the level set $S$ are closed, with period $2\pi$. If we fix a starting point in $S$, then the orbits of the Hamiltonian vector fields $\xi^j$ through that point generate the fundamental group $\pi_1(S)$. Therefore we need to find the values of $E$ for which $\gamma_S$ has trivial holonomy over each of these orbits.

Our procedure is exactly the same as that in Sections 4.4.2 and 4.4.3: we locally change variables from Cartesian to symplectic polar, lift that change of variables to the level of metaplectic-c structures, and construct local trivializations for all of the relevant bundles over an orbit $C$. The only difference is that the curve $C$ is now the orbit for $\xi^j$. We omit the details of the calculations when they are repetitions of those given in Section 4.4.

Without loss of generality, fix the starting point $m_0 = (E^1, \ldots, E^n, 0, \ldots, 0) \in S$. Let $C$ be the orbit for $\xi^j$ through $m_0$. Then a point in $C$ takes the form

$$m(\tau) = (E^1, \ldots, E^n, 0, \ldots, \tau, \ldots, 0),$$

where $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ is the $j$th angle coordinate.

For all $m \in U$, we construct the matrix of partial derivatives $G(m)$ that corresponds to the change of variables, just as in Section 4.4.2, and then we restrict it to lie over $C$. Let $G(\tau) = G(m(\tau))$. In order to lift the change of variables to the level of metaplectic-c structures, we calculate

$$\text{Det}_C C_{G(\tau)} = \frac{1}{2} \left( \sqrt{2E^j} + \frac{1}{\sqrt{2E^j}} \right) e^{i\tau},$$

which we denote by $Ke^{i\tau}$ where $K$ is a positive real constant. Then a lift of $G(\tau)$ to $\text{Mp}(V)$ has parameters

$$\tilde{G}(\tau) \mapsto \left( G(\tau), \frac{1}{\sqrt{K}} e^{-i\tau/2} \right).$$

The matrix $G(\tau)$ is single-valued with respect to $\tau$, but $e^{-i\tau/2}$ is not. As $\tau$ ranges from 0 to $2\pi$, $\tilde{G}(\tau)$ ranges from $(I, 1)$ to $(I, -1)$. We use $\tilde{G}(\tau)$ to perform the change of variables over the subset $\tilde{C} = \{ m(\tau) : \tau \in (0, 2\pi) \}$, then manually correct for the change in sign of the $\mu$ parameter when we close the loop from $\tilde{C}$ to $C$. 
Let \((\hat{x}_1, \ldots, \hat{x}_n, \hat{y}_1, \ldots, \hat{y}_n)\) be the usual symplectic basis for \(V\). Let \(W = \text{span}\{\hat{y}_1, \ldots, \hat{y}_n\}\), so \(W^\perp = W\) and \(W/W^\perp = \{0\}\). Using the local trivializations over \(U\) given by \(\hat{x}_j \mapsto \frac{\partial}{\partial s_j}\) and \(\hat{y}_j \mapsto \frac{\partial}{\partial \theta_j}\), we obtain the identifications \(\text{Sp}(M, \omega)|_U = U \times \text{Sp}(V)\), \(\text{Sp}(M, \omega; S)|_U = U \times \text{Sp}(V; W)\), and \(\text{Sp}(TS/TS^\perp)|_U = U \times \{I\} = U\).

Over \(\dot{C}\), the lifted change of variables gives us the identifications \(P|_\dot{C} = \dot{C} \times \text{Mp}^c(V)\), \(P^S|_\dot{C} = \dot{C} \times \text{Mp}^c(V; W)\), and \(P_S|_\dot{C} = \dot{C} \times U(1)\). The one-form \(\gamma_S\) takes the form

\[
\gamma_S|_\dot{C} = \frac{1}{i \hbar} \sum_{j=1}^n E_j d\theta_j + \frac{1}{2} \eta^* \vartheta_0.
\]

Since \(\text{Sp}(TS/TS^\perp)|_U\) is identified with \(U\), the lift of \(\xi^j(m(\tau))\) to \(\text{Sp}(TS/TS^\perp)\) is simply

\[
\tilde{\xi}^j(m(\tau), I) = \xi^j(m(\tau)), \quad \forall m(\tau) \in \dot{C}.
\]

Then the horizontal lift to \(P_S\) with respect to \(\gamma_S\) is

\[
\tilde{\xi}^j(m(\tau), I) = \left(\xi^j(m(\tau)), -\frac{1}{i \hbar} E_j \right), \quad \forall m(\tau) \in \dot{C}.
\]

Since the Lie algebra component is constant over the orbit, we obtain the lifted flow by exponentiating:

\[
\tilde{\phi}^{jt}(m_0, I) = \left(\phi^{jt}(m_0), e^{-E_j t/i \hbar} \right).
\]

On \(S\), \(\phi^{jt}\) has period \(2\pi\). Because of the change in sign introduced by the definition of \(\tilde{G}(\tau)\), the \(U(1)\) component is closed with period \(2\pi\) if \(e^{-2\pi E_j t/i \hbar} = -1\). Using these observations and the fact that \(E_j\) is positive, we find that the quantization condition is

\[
\frac{2\pi E_j}{i \hbar} = -2\pi i \left(N_j + \frac{1}{2}\right)
\]

for some \(N_j \in \mathbb{Z}, N_j \geq 0\), which implies that

\[
E_j = \hbar \left(N_j + \frac{1}{2}\right), \quad N_j \in \mathbb{Z}, \quad N_j \geq 0.
\]

Note that the Hamiltonian energy function for the \(n\)-dimensional harmonic oscillator is
just \( H^1 + \ldots + H^n \). Using the generalized dynamical invariance property, we see immediately that if we describe \( S \) as a level set of the completely integrable system \( (H^1 + \ldots + H^n, H^2, \ldots, H^n) \), the values of \( H^1 + \ldots + H^n \) on the quantized level sets are \( E^1 + \ldots + E^n = h \left( N_1 + \frac{1}{2} + \ldots + N_n + \frac{1}{2} \right) = h \left( N + \frac{n}{2} \right) \), for some \( N \in \mathbb{Z}, N \geq 0 \). This result reproduces the \( \frac{n}{2} \)-shift in the quantized energy levels that was seen in the analysis from Section 4.4. In addition, the quantized energy levels begin at \( N = 0 \), which is consistent with the quantum mechanical calculation.

### 6.3.4 The 2-dimensional hydrogen atom

For simplicity, we restrict our attention to the two-dimensional version of the hydrogen atom. Our initial definitions are special cases of those that appear in Section 5.2.

Let \( M = \mathbb{R}^2 \times \mathbb{R}^2 \) with Cartesian coordinates \((q_1, q_2, p_1, p_2)\) and symplectic form \( \omega = \sum_{j=1}^{2} dq_j \wedge dp_j \). As before, the energy function for the hydrogen atom is

\[
H = \frac{1}{2m_e} |p|^2 - \frac{k}{|q|},
\]

where \( k, m_e > 0 \). Let \((P, \Sigma, \gamma)\) be the metaplectic-c prequantization for \( M \) in which \( P = M \times M_p c(V_4), \Sigma : P \to Sp(M, \omega) \) is defined with respect to the global trivialization of \( Sp(M, \omega) \) given by the Cartesian coordinate vector fields \( \left( \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right) \), and

\[
\gamma = \frac{1}{i\hbar} \Pi^* \beta + \frac{1}{2} \eta_* \vartheta_0,
\]

where \( \vartheta_0 \) is the trivial connection on \( P \) and

\[
\beta = \sum_{j=1}^{2} (q_j dp_j + d(q_j p_j))
\]

on \( M \).

Let \((r, \theta)\) be polar coordinates for \( \mathbb{R}^2 \), with conjugate momenta \((p_r, p_\theta)\) on \( \mathbb{R}^2 \). The change
of coordinates is given by

\[ r = \sqrt{q_1^2 + q_2^2}, \]
\[ \theta = \tan^{-1} \left( \frac{q_2}{q_1} \right), \]
\[ p_r = \frac{1}{r} (q_1 p_1 + q_2 p_2), \]
\[ p_\theta = q_1 p_2 - q_2 p_1, \]

and the inverse transformation is

\[ q_1 = r \cos \theta, \]
\[ q_2 = r \sin \theta, \]
\[ p_1 = p_r \cos \theta - \frac{1}{r} p_\theta \sin \theta, \]
\[ p_2 = p_r \sin \theta + \frac{1}{r} p_\theta \cos \theta. \]

Note that polar coordinates are defined everywhere on \( M \), so we could have defined \( \Sigma : P \to \text{Sp}(M, \omega) \) in terms of the global trivialization for \( \text{Sp}(M, \omega) \) given by the symplectic frame \( \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial p_r}, \frac{\partial}{\partial p_\theta} \right) \), and let \( \gamma \) be as given above, with \( \beta \) written in polar coordinates. However, since \( (M, \omega) \) is not simply connected, it is not obvious that the metaplectic-c prequantization so constructed is isomorphic to the one that we obtain using Cartesian coordinates. We therefore begin with the Cartesian trivialization, and lift the change of coordinates to the metaplectic-c level.

At any point \( m \in M \), the matrix of partial derivatives representing the transformation from Cartesian to polar coordinates is

\[
G(m) = \begin{pmatrix}
\frac{\partial q_1}{\partial r} & \frac{\partial q_2}{\partial r} & \frac{\partial q_1}{\partial p_r} & \frac{\partial q_2}{\partial p_r} \\
\frac{\partial q_1}{\partial \theta} & \frac{\partial q_2}{\partial \theta} & \frac{\partial q_1}{\partial p_\theta} & \frac{\partial q_2}{\partial p_\theta} \\
\frac{\partial p_1}{\partial r} & \frac{\partial p_2}{\partial r} & \frac{\partial p_1}{\partial p_r} & \frac{\partial p_2}{\partial p_r} \\
\frac{\partial p_1}{\partial \theta} & \frac{\partial p_2}{\partial \theta} & \frac{\partial p_1}{\partial p_\theta} & \frac{\partial p_2}{\partial p_\theta}
\end{pmatrix}
\]
To determine a lift of $G(m)$ to $\text{Mp}(V_4)$, we calculate $C_{G(m)}$, convert to a complex matrix, and take the determinant. The result is

$$\text{Det}_C C_{G(m)} = r + \frac{1}{r} + \frac{1}{4r^3} p_\theta^2.$$  

While this is not necessarily constant, it is real and positive everywhere on $M$. Therefore, over any closed orbit in $M$, we can consistently take $\sqrt{\text{Det}_C C_{G(m)}}$. Thus a lift of $G(m)$ to $\text{Mp}(V_4)$ is given by

$$\hat{G}(m) = \left( G(m), \frac{1}{\sqrt{\text{Det}_C C_{G(m)}}} \right),$$

which is single-valued over any closed orbit in $M$. This demonstrates that the metaplectic-c prequantization that we obtain using the symplectic polar global trivialization is, in fact, isomorphic to $(P, \Sigma, \gamma)$.

From now on, we use the global trivializations of $P$ and $\text{Sp}(M, \omega)$ given by the symplectic frame $(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial p_r}, \frac{\partial}{\partial p_\theta})$. The one-form $\gamma$ is given by

$$\gamma = \frac{1}{i\hbar} \Pi^* \beta + \frac{1}{2} \vartheta_0,$$

where

$$\beta = 2r dp_r + p_r dr - p_\theta d\theta$$

on $M$.

In terms of the polar coordinates,

$$H = \frac{1}{2m_e} \left( p_r^2 + \frac{1}{r^2} p_\theta^2 \right) - \frac{k}{r},$$
and \( \omega = dp_r \wedge dr + dp_\theta \wedge d\theta \). A calculation establishes that

\[
\xi_H = \left( \frac{k}{r^2} - \frac{1}{m_e r^2} P_\theta^2 \right) \frac{\partial}{\partial p_r} - \frac{1}{m_e} p_r \frac{\partial}{\partial r} - \frac{1}{m_e r^2} P_\theta \frac{\partial}{\partial \theta}.
\]

Now consider \( p_\theta \). Physically, \( p_\theta \) is the angular momentum of the system. Let \( E < 0 \) be a fixed negative energy value. As discussed in Section 5.2.1, the angular momentum can take values in \([0, \sqrt{-m_e k^2/E}]\). The value \( p_\theta = 0 \) represents a degenerate elliptical orbit in which the electron collapses into the proton in finite time. The value \( p_\theta = \sqrt{-m_e k^2/E} \) represents a circular orbit.

Observe that

\[
\xi_{p_\theta} = -\frac{\partial}{\partial \theta},
\]

from which we see that \( \{H, p_\theta\} = 0 \). Therefore the system \((H, p_\theta)\) Poisson commutes. Further, \( \xi_H \) and \( \xi_{p_\theta} \) are linearly independent unless \( p_r = 0 \) and \( \frac{k}{r^2} - \frac{1}{m_e r^2} P_\theta^2 = 0 \). These two conditions imply that \( r = -\frac{k}{2E} \) and \( P_\theta^2 = -\frac{m_e k^2}{2E} \): that is, \( \xi_H \) and \( \xi_{p_\theta} \) are linearly independent everywhere except on the circular orbit.

Let \((E, L) \in \mathbb{R}^2\) be such that \( E < 0 \) and \( 0 < L < \sqrt{-m_e k^2/2E} \). Then \((E, L)\) is a regular value of the family \((H, p_\theta)\). Let \( S \) be the corresponding level set. Then, from Section 5.2.1, all orbits of \( \xi_H \) in \( S \) are closed with period \( \frac{2\pi}{N} \), where \( N = \sqrt{-\frac{8E^3}{m_e k^2}} \). All orbits of \( \xi_{p_\theta} \) in \( S \) are closed with period \( 2\pi \).

Given a starting point \( m_0 \in S \), we argue that the orbits of \( \xi_H \) and \( \xi_{p_\theta} \) through \( m_0 \) generate \( \pi_1(S) \). By fixing \( E \) and \( L \), we determine the lengths of the major and minor axes of the elliptic orbit of the electron in \( q \)-space. Within \( S \), we can either fix a particular elliptical orbit and let \( m_0 \) revolve around it, or we can fix the position of \( m_0 \) on its orbit and rotate the orbit itself about its focal point at the origin. The latter rotation has the effect of rotating \( m_0 = (q_0, p_0) \) about the origin at the same rate in \( q \)-space and \( p \)-space, keeping the relative orientation of the position and momentum vectors constant. The flow of \( \xi_H \) clearly represents the revolution about a particular elliptic orbit. The flow of \( \xi_{p_\theta} \), on the other hand, represents the rotation of
an orbit about the origin. This is most easily seen in Cartesian coordinates:

\[
\xi_{pq} = q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1},
\]

which has flow

\[
\phi_{pq}^t(q,p) = \begin{pmatrix}
\cos t & -\sin t & 0 & 0 \\
\sin t & \cos t & 0 & 0 \\
0 & 0 & \cos t & -\sin t \\
0 & 0 & \sin t & \cos t
\end{pmatrix} \begin{pmatrix}
q_1 \\
q_2 \\
p_1 \\
p_2
\end{pmatrix}
\]
as needed.

Since \( S \) is a Lagrangian submanifold of \( M \), we have \( TS^\perp = TS \), and so \( Sp(TS/TS^\perp) \) is naturally identified with \( S \). We must therefore evaluate the holonomy of \( \gamma_S \) over the orbits \( \xi_H \) and \( \xi_{pq} \) through \( m_0 \). Let \( \xi_H \) have flow \( \phi_{H}^t \), and let \( \xi_{pq} \) have flow \( \phi_{pq}^t \).

First, we consider \( \xi_{pq} \). Let \( m \) be any point on the orbit of \( \xi_{pq} \) through \( m_0 \). On \( Sp(TS/TS^\perp) \), we simply have

\[
\tilde{\xi}_{pq}(m,I) = \xi_{pq}(m).
\]

Using the expression for \( \beta \) in polar coordinates, we calculate that the horizontal lift to \( P_S \) is

\[
\tilde{\xi}_{pq}(m,I) = \left( \xi_{pq}(m), -\frac{L}{\iota h} \right).
\]

Upon exponentiating, we obtain the integral curve through \( (m_0,I) \):

\[
\tilde{\phi}_{pq}^t(m_0,I) = \left( \phi_{pq}^t(m_0), e^{-\frac{Lt}{\iota h}} \right).
\]

Since the orbit in \( S \) has period \( 2\pi \), we see immediately that the quantization condition for the angular momentum is

\[
L = Nh, \quad N \in \mathbb{Z}.
\]

This is consistent with the Bohr model of the hydrogen atom, in which angular momentum is assumed to be quantized in units of \( h \).

Now we apply the same process to \( \xi_H \). Let \( m \) be any point on the orbit of \( \xi_H \) through \( m_0 \).
Since
\[ \xi_{H} \beta = \frac{2k}{r} - \frac{1}{m_e} \left( p_r^2 + \frac{1}{r^2} p_\theta^2 \right) = -2E, \]
we see that the horizontal lift to \( P_S \) is
\[ \tilde{\xi}_H(m, I) = \left( \xi_H(m), \frac{2E}{i\hbar} \right). \]

Then the integral curve through \( m_0 \) is
\[ \tilde{\phi}_H^t(m_0, I) = \left( \phi_H^t(m_0), e^{2Et/i\hbar} \right). \]

Using the fact that the period of the orbit in \( S \) is \( \frac{2\pi}{\Lambda} \) where \( \Lambda = \sqrt{-\frac{8E^3}{m_ek^2}} \), we find that the quantization condition for the energy is
\[ E = -\frac{m_ek^2}{2\hbar^2N^2}, \quad N \in \mathbb{N}. \]

This agrees with our result from Chapter 5.
Bibliography


