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On molecular topological properties of diamond like networks^{*}

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Abstract. The Randić (product) connectivity index and its derivative called the sum-connectivity index are well known topological indices and these both descriptors correlate well among themselves and with the π -electronic energies of benzenoid hydrocarbons. The general n -connectivity of a molecular graph G is defined as ${}^n\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} \dots v_{i_{n+1}}} \frac{1}{\sqrt{d_{i_1} d_{i_2} \dots d_{i_{n+1}}}}$ and the n -sum connectivity of a molecular graph G is defined as ${}^nX(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} \dots v_{i_{n+1}}} \frac{1}{\sqrt{d_{i_1} + d_{i_2} + \dots + d_{i_{n+1}}}}$, where the paths of length n in G are denoted by $v_{i_1}, v_{i_2} \dots v_{i_{n+1}}$ and the degree of each vertex v_i is denoted by d_i . In this paper, we discuss third connectivity and third sum-connectivity indices of diamond like networks and compute analytical closed results of these indices for diamond like networks.

Keywords: Generalizes Aztec diamond, Extended Aztec diamond; Third connectivity index; Third sum-connectivity-index.

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1 Introduction and preliminary results

Graph theory is proficient for modeling and designing of chemical structures and complex networks and to predict their properties. The chemical graph theory applies the tools from graph theory to mathematical modeling of molecular phenomena, which is helpful for the study of molecular structure and molecular modeling. This theory play a vital role in the field of theoretical chemical sciences. Chemical compounds have a variety of applications in chemical graph theory, drug design, and in nanotechnology etc. The manipulation and examination of chemical structural information is made conceivable by using molecular descriptors. A great variety of topological indices are studied and used in theoretical chemistry, pharmaceutical researchers. A chemical structure can be represented by using graph theory, where vertices denote atoms and edges denote molecular bond. A topological index is a numeric number which indicates some useful information about molecular structure. It is the numerical invariants of a molecular graph and are useful to correlate with their bioactivity and physico-chemical properties. Researchers have found topological index to be powerful and useful tool in the description of molecular structure. A graph $G(V, E)$ is a pair of $V(G)$ and $E(G)$. Where $V(G)$ represent the set of vertices and $E(G)$ represent the set of edges. The order of a graph is n and it is denoted as $|V(G)| = n$. The size of a graph is m and it is denoted as $|E(G)| = m$. The general n -connectivity index of a graph G is denoted as

$${}^n\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} \dots v_{i_{n+1}}} \frac{1}{\sqrt{d_{i_1} d_{i_2} \dots d_{i_{n+1}}}}. \quad (1)$$

Where the paths of length n in G are denoted by $v_{i_1}, v_{i_2} \dots v_{i_{n+1}}$ and the degree of vertex v_i is denoted by d_i . The first connectivity index is described as

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}. \quad (2)$$

In 1971, Milan Randić introduced ³¹ first connectivity index. Now it is known as Randić index. Many variations of Randić connectivity index have been discussed in ^{4,5,18,21,23,24,28,29,31,32}.

The second connectivity index is expressed as

$${}^2\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3}}}. \quad (3)$$

Similarly, third connectivity is expressed as

$${}^3\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}}. \quad (4)$$

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In 2008, B. Zhou and N. Trinajstić see ^{1,12,13,29,35}. developed the sum-connectivity index, and it is defined as

$$X(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}. \quad (5)$$

Where the degree of vertices u and v are represented as d_u and d_v respectively. Similarly, second-sum-connectivity is described as

$${}^2X(G) = \sum_{v_{i_1} v_{i_2} v_{i_3}} \frac{1}{\sqrt{d_{i_1} + d_{i_2} + d_{i_3}}}. \quad (6)$$

Where the paths of length 2 in graph G are denoted by $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ and degree of each vertex v_i is denoted by d_{v_i} where $1 \leq i \leq 3$. Similarly, third sum-connectivity is expressed as

$${}^3X(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} + d_{i_2} + d_{i_3} + d_{i_4}}}. \quad (7)$$

In general, the n sum-connectivity index of graph G is given by

$${}^nX(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} \dots v_{i_{n+1}}} \frac{1}{\sqrt{d_{i_1} + d_{i_2} + \dots + d_{i_{n+1}}}}. \quad (8)$$

The further study of the m -connectivity index is explained in ^{6,7,8,13,15,16,23,26,27,33}. The further study of the m sum-connectivity index is clarified ^{9,10,11,15,16,25,32,34}. The topological descriptors of certain networks and nanotubes are studied in ^{3,19,20}. The diamond like networks have been examined in different ways. The relation of domino tilings and Aztec diamond theorem has been argued in ^{14,22}. In this paper, we examined third connectivity and third sum-connectivity of diamond like networks.

2 Main results for generalized Aztec diamond

An Aztec diamond with order n containing all squares lattice whose centre (x, y) satisfy $|x| + |y| \leq n$. Here n is fixed, and square lattice contains unit squares with the origin as vertex of degree 4, so that both x and y are half integer. Consider a path L_i with i vertices, where $i = 1, 2, 3, 4, \dots$. The tensor product of

Fig. 1: Aztec diamond of different dimensions.

two paths L_n and L_m denoted by $L_n * L_m$ is the graph on $n \times m$ vertices $\{(x, y) : 1 \leq x \leq n, 1 \leq y \leq m\}$. Any two vertices are adjacent if and only if $|x - x'| + |y - y'| = 1$. Clearly, this graph is a disconnected

Fig. 2: Generalized Aztec diamond $L_{11} * L_{11}$.

graph having two connected components. One component is denoted as $O(L_n * L_m)$ has the vertices $\{(x, y) : x + y \text{ is odd}\}$. The other component is denoted as $E(L_n * L_m)$ has the vertices $\{(x, y) : x + y \text{ is even}\}$. First of all, we discuss $O(L_{2n+1} * L_{2m+1})$ and it is called odd generalized Aztec diamond.

Odd generalized Aztec diamond

Fig. 3: Types of three edges path connectivity of odd generalized Aztec diamond $L_{11} * L_{11}$.

Fig. 4: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set A.

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Three edges path connectivity is obtained by twice the set A or ${}^3\chi(O(L_n * L_m)) = 2A$.

Table 1: Edge partition of edge set A of ${}^3\chi(O(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{2242}	d_{2424}	d_{2442}	d_{2444}
Cardinality of d_{ijkl}	4	$2n - 2$	12	$12n$

Fig. 5: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set B .

Similarly, ${}^3\chi(O(L_n * L_m)) = 2B$.

Table 2: Edge partition of edge set B of ${}^3\chi(O(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{2242}	d_{2424}	d_{2244}	d_{44444}	d_{24444}
Cardinality of d_{ijkl}	4	4	8	16	$16n - 16$

Fig. 6: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set C .

${}^3\chi(O(L_n * L_m)) = 2C.$

Table 3: Edge partition of edge set C of ${}^3\chi(O(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{4224}	d_{4242}	d_{4424}	d_{4422}	d_{4444}
Cardinality of d_{ijkl}	4	4	$2n - 4$	4	$16n - 16$

Fig. 7: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set D .

${}^3\chi(O(L_n * L_m)) = 2D.$

Table 4: Edge partition of edge set D of ${}^3\chi(O(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{2422}	d_{2442}	d_{2444}	d_{4444}
Cardinality of d_{ijkl}	4	8	16	$20n - 28$

Fig. 8: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set E .

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$${}^3\chi(O(L_n * L_m)) = (m - 3)E, \text{ where } m = n \text{ and } m, n \geq 6.$$

Table 5: Edge partition of edge set E of ${}^3\chi(O(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{4242}	d_{4442}	d_{4444}
Cardinality of d_{ijkl}	$4n - 12$	$4n - 12$	$12mn - 24m - 16n + 12$

Fig. 9: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set F .

$${}^3\chi(O(L_n * L_m)) = (m - 4)F \text{ where } m = n \text{ and } m, n \geq 6.$$

Table 6: Edge partition of edge set F of ${}^3\chi(O(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	Cardinality of $d_G(i), d_G(j), d_G(k), d_G(l)$
d_{2424}	$4n - 16$
d_{2444}	$4n - 16$
d_{4424}	$2n - 8$
d_{4444}	$12mn - 64m + 64$

The following theorems present analytically closed formulas of third connectivity and third sum-connectivity indices for ${}^3\chi(O(L_{2n+1} * L_{2m+1}))$.

Theorem 1 Let ${}^3\chi(O(L_{2n+1} * L_{2m+1}))$ be the graph, then third connectivity of the graph is given by

$$\frac{1}{2} \left[\frac{3}{2}mn - \left(\frac{11}{2}\right)m + \left(\frac{17}{4} + \frac{3}{4\sqrt{2}} + \sqrt{2}\right)n + \left(\frac{9}{4} + \frac{6}{4\sqrt{2}}\right) \right].$$

Proof. Using the above Tables [1-6] and Equation(4) we have

$${}^2\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}}.$$

Since each path repeated twice therefore, we take

$$\begin{aligned}
& \frac{1}{2} \left[\sum_{a \in A} \frac{1}{\sqrt{d_a \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \right. \\
& \left. \sum_{d \in D} \frac{1}{\sqrt{d_d \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f \times d_{i_2} \times d_{i_3} \times d_{i_4}}} \right] = \\
& \frac{1}{2} \left[\left(\frac{d_{2242}}{\sqrt{2 \times 2 \times 4 \times 2}} + \frac{d_{2424}}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{d_{2442}}{\sqrt{2 \times 4 \times 4 \times 2}} + \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{2242}}{\sqrt{2 \times 2 \times 4 \times 2}} + \frac{d_{2424}}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{d_{2244}}{\sqrt{2 \times 2 \times 4 \times 4}} + \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} \right) + \right. \\
& \left(\frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{4224}}{\sqrt{4 \times 2 \times 2 \times 4}} + \frac{d_{4242}}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{d_{4424}}{\sqrt{4 \times 4 \times 2 \times 4}} + \frac{d_{4422}}{\sqrt{4 \times 4 \times 2 \times 2}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{2422}}{\sqrt{2 \times 4 \times 2 \times 2}} + \frac{d_{2442}}{\sqrt{2 \times 4 \times 4 \times 2}} + \right. \\
& \left. \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{4242}}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{2424}}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left. \left(\frac{d_{4242}}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) \right] = \\
& \frac{1}{2} \left[\left(\frac{4}{\sqrt{2 \times 2 \times 4 \times 2}} + \frac{2n-2}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{12}{\sqrt{2 \times 4 \times 4 \times 2}} + \frac{12n}{\sqrt{2 \times 4 \times 4 \times 4}} \right) + \left(\frac{4}{\sqrt{2 \times 2 \times 4 \times 2}} + \frac{4}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{8}{\sqrt{2 \times 2 \times 4 \times 4}} + \frac{16}{\sqrt{2 \times 4 \times 4 \times 4}} + \right. \right. \\
& \left. \frac{16n-16}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{4}{\sqrt{4 \times 2 \times 2 \times 4}} + \frac{4}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{2n-4}{\sqrt{4 \times 4 \times 2 \times 4}} + \frac{4}{\sqrt{4 \times 4 \times 2 \times 2}} + \frac{16n-16}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{4}{\sqrt{2 \times 4 \times 2 \times 2}} + \frac{8}{\sqrt{2 \times 4 \times 4 \times 2}} + \right. \\
& \left. \frac{16}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{20n-28}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{4n-12}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{4n-12}{\sqrt{4 \times 4 \times 4 \times 2}} + \frac{12mn-24m-16n+12}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{4n-16}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{8n-32}{\sqrt{2 \times 4 \times 4 \times 4}} + \right. \\
& \left. \frac{2n-8}{\sqrt{4 \times 4 \times 2 \times 4}} + \frac{12mn-64m+64}{\sqrt{4 \times 4 \times 4 \times 4}} \right) \right].
\end{aligned}$$

After simplification, we get

$$= \frac{1}{2} \left[\frac{2}{3} mn - \left(\frac{11}{2} \right) m + \left(\frac{17}{4} + \frac{3}{4\sqrt{2}} + \sqrt{2} \right) n + \left(\frac{9}{4} + \frac{16}{4\sqrt{2}} \right) \right].$$

Theorem 2 Let ${}^3X(O(L_{2n+1} * L_{2m+1}))$ be the graph, then third sum connectivity of the graph is given by

$$\frac{1}{2} [6mn - (22)m + \left(\frac{24}{\sqrt{14}} + 2\sqrt{3} + 9 \right) n + \left(14 + \frac{10}{\sqrt{12}} - \frac{24}{\sqrt{14}} + \frac{12}{\sqrt{10}} \right)].$$

Proof. Using the above Tables [1-6] and Equation(7) we have

$${}^2\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}}.$$

Since each path repeated twice therefore, we take

$$\begin{aligned}
& \frac{1}{2} \left[\sum_{a \in A} \frac{1}{\sqrt{d_a + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c + d_{i_2} + d_{i_3} + d_{i_4}}} + \right. \\
& \left. \sum_{d \in D} \frac{1}{\sqrt{d_d + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f + d_{i_2} + d_{i_3} + d_{i_4}}} \right] = \\
& \frac{1}{2} \left[\left(\frac{d_{2242}}{\sqrt{2+2+4+2}} + \frac{d_{2424}}{\sqrt{2+4+2+4}} + \frac{d_{2442}}{\sqrt{2+4+4+2}} + \frac{d_{2444}}{\sqrt{2+4+4+4}} \right) + \left(\frac{d_{2242}}{\sqrt{2+2+4+2}} + \frac{d_{2424}}{\sqrt{2+4+2+4}} + \frac{d_{2244}}{\sqrt{2+2+4+4}} + \frac{d_{2444}}{\sqrt{2+4+4+4}} \right) + \right. \\
& \left(\frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{4224}}{\sqrt{4+2+2+4}} + \frac{d_{4242}}{\sqrt{4+2+4+2}} + \frac{d_{4424}}{\sqrt{4+4+2+4}} + \frac{d_{4422}}{\sqrt{4+4+2+2}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{2422}}{\sqrt{2+4+2+2}} + \frac{d_{2442}}{\sqrt{2+4+4+2}} + \right. \\
& \left. \frac{d_{2444}}{\sqrt{2+4+4+4}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{4242}}{\sqrt{4+2+4+2}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{2424}}{\sqrt{2+4+2+4}} + \frac{d_{2444}}{\sqrt{2+4+4+4}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left. \left(\frac{d_{4242}}{\sqrt{4+2+4+2}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) \right] = \\
& \frac{1}{2} \left[\left(\frac{4}{\sqrt{2+2+4+2}} + \frac{2n-2}{\sqrt{2+4+2+4}} + \frac{12}{\sqrt{2+4+4+2}} + \frac{12n}{\sqrt{2+4+4+4}} \right) + \left(\frac{4}{\sqrt{2+2+4+2}} + \frac{4}{\sqrt{2+4+2+4}} + \frac{8}{\sqrt{2+2+4+4}} + \frac{16}{\sqrt{2+4+4+4}} + \right. \right. \\
& \left. \frac{16n-16}{\sqrt{4+4+4+4}} \right) + \left(\frac{4}{\sqrt{4+2+2+4}} + \frac{4}{\sqrt{4+2+4+2}} + \frac{2n-4}{\sqrt{4+4+2+4}} + \frac{4}{\sqrt{4+4+2+2}} + \frac{16n-16}{\sqrt{4+4+4+4}} \right) + \left(\frac{4}{\sqrt{2+4+2+2}} + \frac{8}{\sqrt{2+4+4+2}} + \right. \\
& \left. \frac{16}{\sqrt{2+4+4+4}} + \frac{20n-28}{\sqrt{4+4+4+4}} \right) + \left(\frac{4n-12}{\sqrt{4+2+4+2}} + \frac{4n-12}{\sqrt{4+4+4+2}} + \frac{12mn-24m-16n+12}{\sqrt{4+4+4+4}} \right) + \left(\frac{4n-16}{\sqrt{2+4+2+4}} + \frac{8n-32}{\sqrt{2+4+4+4}} + \right. \\
& \left. \frac{2n-8}{\sqrt{4+4+2+4}} + \frac{12mn-64m+64}{\sqrt{4+4+4+4}} \right) \right].
\end{aligned}$$

After simplification, we get

$$= \frac{1}{2} [6mn - (22)m + \left(\frac{24}{\sqrt{14}} + 2\sqrt{3} + 9 \right) n + \left(14 + \frac{10}{\sqrt{12}} - \frac{24}{\sqrt{14}} + \frac{12}{\sqrt{10}} \right)].$$

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3 Third connectivity and third sum-connectivity indices of even generalized Aztec diamond

Fig. 10: Types of three edges path of even generalized Aztec diamond $L_{11} * L_{11}$.

In third connectivity the paths of length 4 in G are denoted by $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ and the degree of each vertex v_i is denoted by d_i . Where $i = 1, 2, 3, 4$.

$${}^3\chi(E(L_n * L_m)) = 2A, \text{ where } m = n \text{ and } m, n \geq 6.$$

Fig. 11: Types of three edges paths of even generalized Aztec diamond $L_{11} * L_{11}$ that start from set A .

Table 7: Edge partition of edge set A of ${}^3\chi(E(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{1424}	d_{1444}	d_{2424}	d_{2444}
Cardinality of d_{ijkl}	8	12	$2n - 2$	$12n - 12$

$${}^3\chi(E(L_n * L_m)) = 2B, \text{ where } m = n \text{ and } m, n \geq 6.$$

$${}^3\chi(E(L_n * L_m)) = 2C, \text{ where } m = n \text{ and } m, n \geq 6.$$

$${}^3\chi(E(L_n * L_m)) = 2D, \text{ where } m = n \text{ and } m, n \geq 6.$$

Fig. 12: Types of three edges paths of even generalized Aztec diamond, that start from set B .

Table 8: Edge partition of edge set B of ${}^3\chi(E(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{4242}	d_{4444}
Cardinality of d_{ijkl}	8	$12n$

Fig. 13: Types of three edges paths of even generalized Aztec diamond $L_{11} * L_{11}$ that start from set C .

Table 9: Edge partition of edge set C of ${}^3\chi(E(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{2424}	d_{2444}	d_{4444}
Cardinality of d_{ijkl}	4	12	$20n - 8$

Fig. 14: Types of three edges paths of even generalized Aztec diamond $L_{11} * L_{11}$ that start from set D .

Table 10: Edge partition of edge set D of ${}^3\chi(E(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{4241}	d_{4242}	d_{4442}	d_{4444}
Cardinality of d_{ijkl}	4	4	$12n - 12$	$20n - 8$

${}^3\chi(E(L_n * L_m)) = (m - 3)E$, where $m = n$ and $m, n \geq 6$.

Table 11: Edge partition of edge set E of ${}^3\chi(E(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	d_{2424}	d_{2444}	d_{4424}	d_{4444}
Cardinality of d_{ijkl}	$4n - 12$	$8n - 24$	$2n - 6$	$12mn + 12n - 32$

Fig. 15: Types of three edges paths of even generalized Aztec diamond that start from set E .

$${}^3\chi(E(L_n * L_m)) = (m - 4)F, \text{ where } m = n \text{ and } m, n \geq 6.$$

Fig. 16: Types of three edges paths of even generalized Aztec diamond that start from set F .Table 12: Edge partition of edge set F of ${}^3\chi(E(L_n * L_m))$, where $m = n$ and $m, n \geq 6$.

d_{ijkl}	Cardinality of $d_G(i), d_G(j), d_G(k), d_G(l)$
d_{4242}	$2n - 4$
d_{4442}	$2n - 4$
d_{4444}	$12mn - 28m - 24n + 16$

Following theorems present analytically closed formula of third connectivity and third sum-connectivity even generalized Aztec diamond.

Theorem 3 Let ${}^3\chi(E(L_{2n+1} * L_{2m+1}))$ be the graph, then third connectivity of the graph is given by

$$\frac{1}{2} \left[\frac{3}{2} mn - \left(\frac{58}{16} \right) m + \left(\frac{13}{8} + \frac{7}{2\sqrt{2}} \right) n + \left(\frac{5}{2} - \frac{7}{4\sqrt{2}} \right) \right].$$

Proof. Using the above Tables [7-12] and Equation(4) we have

$${}^2\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}}.$$

Since each path repeated twice therefore, we take

$$\frac{1}{2} \left[\sum_{a \in A} \frac{1}{\sqrt{d_a \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{d \in D} \frac{1}{\sqrt{d_d \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f \times d_{i_2} \times d_{i_3} \times d_{i_4}}} \right] =$$

$$\begin{aligned} & \frac{1}{2} \left[\left(\frac{d_{1424}}{\sqrt{1 \times 4 \times 2 \times 4}} + \frac{d_{1444}}{\sqrt{1 \times 4 \times 4 \times 4}} + \frac{d_{2424}}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{4242}}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{2424}}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} + \right. \right. \\ & \left. \left. \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{4241}}{\sqrt{4 \times 2 \times 4 \times 1}} + \frac{d_{4242}}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{d_{4442}}{\sqrt{4 \times 4 \times 4 \times 2}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{2424}}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 2}} \right) + \right. \\ & \left. \left(\frac{d_{4242}}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{d_{2444}}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{d_{4242}}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{d_{4442}}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) \right] = \\ & \frac{1}{2} \left[\left(\frac{8}{\sqrt{1 \times 4 \times 2 \times 4}} + \frac{12}{\sqrt{1 \times 4 \times 4 \times 4}} + \frac{2n-2}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{12n-12}{\sqrt{2 \times 4 \times 4 \times 4}} \right) + \left(\frac{8}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{12n}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{4}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{12}{\sqrt{2 \times 4 \times 4 \times 4}} + \right. \right. \\ & \left. \left. \frac{20n-8}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{4}{\sqrt{4 \times 2 \times 4 \times 1}} + \frac{4}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{12n-12}{\sqrt{4 \times 4 \times 4 \times 2}} + \frac{20n-8}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left(\frac{4n-12}{\sqrt{2 \times 4 \times 2 \times 4}} + \frac{8n-24}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{2n-6}{\sqrt{4 \times 4 \times 2 \times 4}} + \right. \right. \\ & \left. \left. \frac{12m-30m-18n+36}{\sqrt{4 \times 4 \times 4 \times 2}} \right) + \left(\frac{2n-4}{\sqrt{4 \times 2 \times 4 \times 2}} + \frac{2n-4}{\sqrt{2 \times 4 \times 4 \times 4}} + \frac{12mn-28m-24n+16}{\sqrt{4 \times 4 \times 4 \times 4}} \right) \right]. \end{aligned}$$

After simplification, we get

$$= \frac{1}{2} \left[\frac{3}{2} mn - \left(\frac{58}{16} \right) m + \left(\frac{13}{8} + \frac{7}{2\sqrt{2}} \right) n + \left(\frac{5}{2} - \frac{7}{4\sqrt{2}} \right) \right].$$

Theorem 4 Let ${}^3X(E(L_{2n+1} * L_{2m+1}))$ be the graph, then third sum connectivity of the graph is given by

$$\frac{1}{2} [6mn - \frac{58}{4} m + \left(\frac{5}{2} + \frac{10}{\sqrt{3}} + \frac{23}{\sqrt{14}} \right) n + \left(9 - \frac{7}{\sqrt{3}} + \frac{12}{\sqrt{13}} + \frac{12}{\sqrt{11}} - \frac{46}{14} \right)].$$

Proof. Using the above Tables [7-12] and Equation(7) we have

$${}^2\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}}.$$

Since each path repeated twice therefore we take

$$\begin{aligned} & \frac{1}{2} \left[\sum_{a \in A} \frac{1}{\sqrt{d_a + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c + d_{i_2} + d_{i_3} + d_{i_4}}} + \right. \\ & \left. \sum_{d \in D} \frac{1}{\sqrt{d_d + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f + d_{i_2} + d_{i_3} + d_{i_4}}} \right] = \\ & \frac{1}{2} \left[\left(\frac{d_{1424}}{\sqrt{1+4+2+4}} + \frac{d_{1444}}{\sqrt{1+4+4+4}} + \frac{d_{2424}}{\sqrt{2+4+2+4}} + \frac{d_{2444}}{\sqrt{2+4+4+4}} \right) + \left(\frac{d_{4242}}{\sqrt{4+2+4+2}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{2424}}{\sqrt{2+4+2+4}} + \frac{d_{2444}}{\sqrt{2+4+4+4}} + \right. \right. \\ & \left. \left. \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{4241}}{\sqrt{4+2+4+1}} + \frac{d_{4242}}{\sqrt{4+2+4+2}} + \frac{d_{4442}}{\sqrt{4+4+4+2}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{2424}}{\sqrt{2+4+2+4}} + \frac{d_{2444}}{\sqrt{2+4+4+4}} + \frac{d_{4444}}{\sqrt{4+4+4+2}} + \right. \right. \\ & \left. \left. \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) + \left(\frac{d_{4242}}{\sqrt{4+2+4+2}} + \frac{d_{4442}}{\sqrt{4+4+4+2}} + \frac{d_{4444}}{\sqrt{4+4+4+4}} \right) \right] = \\ & \frac{1}{2} \left[\left(\frac{8}{\sqrt{1+4+2+4}} + \frac{12}{\sqrt{1+4+4+4}} + \frac{2n-2}{\sqrt{2+4+2+4}} + \frac{12n-12}{\sqrt{2+4+4+4}} \right) + \left(\frac{8}{\sqrt{4+2+4+2}} + \frac{12n}{\sqrt{4+4+4+4}} \right) + \left(\frac{4}{\sqrt{2+4+2+4}} + \frac{12}{\sqrt{2+4+4+4}} + \right. \right. \\ & \left. \left. \frac{20n-8}{\sqrt{4+4+4+4}} \right) + \left(\frac{4}{\sqrt{4+2+4+1}} + \frac{4}{\sqrt{4+2+4+2}} + \frac{12n-12}{\sqrt{4+4+4+2}} + \frac{20n-8}{\sqrt{4+4+4+4}} \right) + \left(\frac{4n-12}{\sqrt{2+4+2+4}} + \frac{8n-24}{\sqrt{2+4+4+4}} + \frac{2n-6}{\sqrt{4+4+2+4}} + \right. \right. \\ & \left. \left. \frac{12m-30m-18n+36}{\sqrt{4+4+4+4}} \right) + \left(\frac{2n-4}{\sqrt{4+2+4+2}} + \frac{2n-4}{\sqrt{4+4+4+2}} + \frac{12mn-28m-24n+16}{\sqrt{4+4+4+4}} \right) \right]. \end{aligned}$$

After simplification, we get

$$= \frac{1}{2} [6mn - \frac{58}{4} m + \left(\frac{5}{2} + \frac{10}{\sqrt{3}} + \frac{23}{\sqrt{14}} \right) n + \left(9 - \frac{7}{\sqrt{3}} + \frac{12}{\sqrt{13}} + \frac{12}{\sqrt{11}} - \frac{46}{14} \right)].$$

4 Main results of extended Aztec diamond

An Aztec diamond of order n containing all squares lattice whose center (x, y) satisfy $|x| + |y| \leq n$. Here n is fixed and square lattice contains unit squares with the origin as vertex 4 of them, so that both p and q are half integer. An extended Aztec diamond is denoted by $EAZ(n)$. An extended Aztec diamond is gained by joining all vertices of degree 2. The subsequent graph is non-isomorphic to the generalized Aztec diamond having vertices of degree 3 also. There are three types of partite set in the edge partition of edge set of $EAZ(n)$. We compute third connectivity and third sum-connectivity indices of extended Aztec diamond.

Fig. 17: Extended Aztec diamond of different dimension.

Fig. 18: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set A .

Three edges path connectivity is obtained by twice the set A or ${}^3\chi(EAZ(n)) = 2A$.

Table 13: Edge partition of edge set A of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{3334}	d_{3344}	d_{3444}
Cardinality of d_{ijkl}	4	8	24

Fig. 19: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set B .

Similarly, ${}^3\chi(EAZ(n))=2B$.

Table 14: Edge partition of edge set B of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{4333}	d_{4344}	d_{4433}	d_{4444}
Cardinality of d_{ijkl}	4	8	4	48

Fig. 20: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set C .

Similarly, ${}^3\chi(EAZ(n)) = 2C$.

${}^3\chi(EAZ(n)) = 2D$.

${}^3\chi(EAZ(n)) = (n - 4)E$.

${}^3\chi(EAZ(n)) = 2F$.

${}^3\chi(EAZ(n)) = 2G$.

Table 15: Edge partition of edge set C of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{4433}	d_{4444}
Cardinality of d_{ijkl}	4	116

Fig. 21: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set D .Table 16: Edge partition of edge set D of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{4443}	d_{4444}
Cardinality of d_{ijkl}	4	150

Fig. 22: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set E .Table 17: Edge partition of edge set E of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{4433}	d_{4443}	d_{4444}
Cardinality of d_{ijkl}	4	8	$16n^2 - 20n - 466$

Fig. 23: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set F . Fig. 24: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set G .

Table 18: Edge partition of edge set F of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{4333}	d_{4433}	d_{4443}	d_{4444}
Cardinality of d_{ijkl}	4	4	4	$34n - 30$

Table 19: Edge partition of edge set G of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{3334}	d_{3344}	d_{3444}	d_{4444}
Cardinality of d_{ijkl}	4	4	4	$34n - 30$

Fig. 25: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set H .

Table 20: Edge partition of edge set H of $EAZ(n)$, where $n \geq 8$.

d_{ijkl}	d_{3344}	d_{3444}	d_{4444}
Cardinality of d_{ijkl}	4	4	$34n - 30$

Following theorems present analytically closed formula of third connectivity and third sum-connectivity extended Aztec diamond.

Theorem 5 Let ${}^3\chi(EAZ(n))$ be the graph, then third connectivity of the graph is given by

$$\frac{1}{2}\left[n^2 + \left(\frac{41}{8}\right)n + \left(\frac{48}{8\sqrt{3}} + \frac{16}{6\sqrt{3}} + \frac{32}{12} - \frac{242}{16}\right)\right].$$

Proof. Using the above Tables[13-19] and Equation (4) we have

$$^2\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}}.$$

Since each path repeated twice therefore, we take

$$\begin{aligned} & \frac{1}{2} [\sum_{a \in A} \frac{1}{\sqrt{d_a \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \\ & \sum_{d \in D} \frac{1}{\sqrt{d_d \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \\ & \sum_{g \in G} \frac{1}{\sqrt{d_g \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{h \in H} \frac{1}{\sqrt{d_h \times d_{i_2} \times d_{i_3} \times d_{i_4}}}] = \\ & \frac{1}{2} [(\frac{d_{3334}}{\sqrt{3 \times 3 \times 3 \times 4}} + \frac{d_{3344}}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{d_{3444}}{\sqrt{3 \times 4 \times 4 \times 4}}) + (\frac{d_{4333}}{\sqrt{4 \times 3 \times 3 \times 3}} + \frac{d_{4344}}{\sqrt{4 \times 3 \times 4 \times 4}} + \frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 2}}) + (\frac{d_{4333}}{\sqrt{4 \times 3 \times 3 \times 3}} + \frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{3334}}{\sqrt{3 \times 3 \times 3 \times 4}} + \frac{d_{3344}}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{d_{3444}}{\sqrt{3 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{3344}}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{d_{3444}}{\sqrt{3 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}})] = \\ & \frac{1}{2} [(\frac{4}{\sqrt{3 \times 3 \times 3 \times 4}} + \frac{8}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{24}{\sqrt{3 \times 4 \times 4 \times 4}}) + (\frac{4}{\sqrt{4 \times 3 \times 3 \times 3}} + \frac{8}{\sqrt{4 \times 3 \times 4 \times 4}} + \frac{4}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{48}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{4}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{116}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{4}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{150}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{4}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{8}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{16n^2 - 20n - 466}{\sqrt{4 \times 4 \times 4 \times 2}}) + (\frac{4}{\sqrt{4 \times 3 \times 3 \times 3}} + \frac{4}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{4}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{34n-30}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{4}{\sqrt{3 \times 3 \times 3 \times 4}} + \frac{4}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{4}{\sqrt{3 \times 4 \times 4 \times 4}} + \frac{34n-30}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{4}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{4}{\sqrt{3 \times 4 \times 4 \times 4}} + \frac{4}{\sqrt{4 \times 4 \times 4 \times 4}})] \end{aligned}$$

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$$\frac{34n-30}{\sqrt{4 \times 4 \times 4 \times 4}})].$$

After simplification, we get

$$= \frac{1}{2}[n^2 + (\frac{41}{8})n + (\frac{48}{8\sqrt{3}} + \frac{16}{6\sqrt{3}} + \frac{32}{12} - \frac{242}{16})].$$

Theorem 6 Let ${}^3X(EAZ(n))$ be the graph, then third sum connectivity of the graph is given by

$$\frac{1}{2}[4n^2 - (\frac{41}{2}n + \frac{16}{\sqrt{13}})n + (\frac{32}{\sqrt{14}} + \frac{52}{\sqrt{15}} + \frac{121}{2})].$$

Proof. Using the above Tables [13-19] and Equation (7) we have

$${}^3\chi(G) = \sum_{v_{i_1}v_{i_2}v_{i_3}v_{i_4}} \frac{1}{\sqrt{d_{i_1}d_{i_2}d_{i_3}d_{i_4}}}.$$

Since each path repeated twice therefore, we take

$$\begin{aligned} & \frac{1}{2}[\sum_{a \in A} \frac{1}{\sqrt{d_a + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c + d_{i_2} + d_{i_3} + d_{i_4}}} + \\ & \sum_{d \in D} \frac{1}{\sqrt{d_d + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f + d_{i_2} + d_{i_3} + d_{i_4}}} + \\ & \sum_{g \in G} \frac{1}{\sqrt{d_g + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{h \in G} \frac{1}{\sqrt{d_h + d_{i_2} + d_{i_3} + d_{i_4}}}] = \\ & \frac{1}{2}[(\frac{d_{3334}}{\sqrt{3 \times 3 \times 3 \times 4}} + \frac{d_{3344}}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{d_{3444}}{\sqrt{3 \times 4 \times 4 \times 4}}) + (\frac{d_{4333}}{\sqrt{4 \times 3 \times 3 \times 3}} + \frac{d_{4344}}{\sqrt{4 \times 3 \times 4 \times 4}} + \frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \\ & \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{4443}}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \frac{d_{4443}}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 2}}) + (\frac{d_{4333}}{\sqrt{4 \times 3 \times 3 \times 3}} + \frac{d_{4433}}{\sqrt{4 \times 4 \times 3 \times 3}} + \\ & \frac{d_{4443}}{\sqrt{4 \times 4 \times 4 \times 3}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{3334}}{\sqrt{3 \times 3 \times 3 \times 4}} + \frac{d_{3344}}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{d_{3444}}{\sqrt{3 \times 4 \times 4 \times 4}} + \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}}) + (\frac{d_{3344}}{\sqrt{3 \times 3 \times 4 \times 4}} + \frac{d_{3444}}{\sqrt{3 \times 4 \times 4 \times 4}} + \\ & \frac{d_{4444}}{\sqrt{4 \times 4 \times 4 \times 4}})] = \\ & \frac{1}{2}[(\frac{4}{\sqrt{3+3+3+4}} + \frac{8}{\sqrt{3+3+4+4}} + \frac{24}{\sqrt{3+4+4+4}}) + (\frac{4}{\sqrt{4+3+3+3}} + \frac{8}{\sqrt{4+3+4+4}} + \frac{4}{\sqrt{4+4+3+3}} + \frac{48}{\sqrt{4+4+4+4}}) + (\frac{4}{\sqrt{4+4+3+3}} + \\ & \frac{116}{\sqrt{4+4+4+4}}) + (\frac{4}{\sqrt{4+4+4+3}} + \frac{150}{\sqrt{4+4+4+4}}) + (\frac{4}{\sqrt{4+4+3+3}} + \frac{8}{\sqrt{4+4+4+3}} + \frac{16n^2-20n-466}{\sqrt{4+4+4+2}}) + (\frac{4}{\sqrt{4+3+3+3}} + \frac{4}{\sqrt{4+4+3+3}} + \\ & \frac{4}{\sqrt{4+4+4+3}} + \frac{34n-30}{\sqrt{4+4+4+4}}) + (\frac{4}{\sqrt{3+3+3+4}} + \frac{4}{\sqrt{3+3+4+4}} + \frac{4}{\sqrt{3+4+4+4}} + \frac{34n-30}{\sqrt{4+4+4+4}}) + (\frac{4}{\sqrt{3+3+4+4}} + \frac{4}{\sqrt{3+4+4+4}} + \\ & \frac{34n-30}{\sqrt{4+4+4+4}})]. \end{aligned}$$

After simplification, we get

$$= \frac{1}{2}[4n^2 - (\frac{41}{2}n + \frac{16}{\sqrt{13}})n + (\frac{32}{\sqrt{14}} + \frac{52}{\sqrt{15}} + \frac{121}{2})].$$

The graphical representation of third connectivity and third sum-connectivity for odd generalized Aztec diamond is depicted in Figure 26. The graphical representation of third connectivity and third sum-connectivity for even generalized Aztec diamond is depicted in Figure 27. By increasing the values of m and n , the value of indices increases.

The graphical representation of third connectivity and third sum-connectivity for extended Aztec diamond is depicted in Figure 28. By increasing the values of n , the value of indices increase and third sum connectivity index approaches towards the third connectivity index. The comparison of third connectivity and sum third connectivity index of both even and odd generalized Aztec diamond is depicted in Figure 29 and 30 respectively. The graphical representation shows that the third and sum third connectivity index of generalized Aztec diamond for even and odd cases respectively are very close to each other.

Fig. 26: The comparison of third connectivity and third sum-connectivity for odd generalized Aztec diamond, third connectivity index $\rightarrow$ Green, third sum connectivity index $\rightarrow$ Blue.

Fig. 27: The comparison of third connectivity and third sum-connectivity for even generalized Aztec diamond, third connectivity index $\rightarrow$ Red and third sum connectivity index $\rightarrow$ Green.

Fig. 28: The comparison of third connectivity and third sum-connectivity for extended Aztec diamond, third connectivity index $\rightarrow$ Red and third sum connectivity index $\rightarrow$ Green.

Fig. 29: The comparison of third connectivity of generalized Aztec diamond for both even and odd cases, third connectivity index for odd $\rightarrow$ Blue and third connectivity index for even $\rightarrow$ Green.

Fig. 30: The comparison of sum third connectivity index of generalized Aztec diamond for both even and odd cases, sum third connectivity index for odd $\rightarrow$ Red and third sum connectivity index for even case $\rightarrow$ Green.

5 Conclusion and general remarks

In this paper, we considered the conclusion third connectivity and third sum-connectivity for generalized Aztec diamond and extended diamond. We derived the close formulas of third connectivity and third sum-connectivity for generalized Aztec diamond and extended diamond. General connectivity and general sum-connectivity for these diamonds can be considered for future study.

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Figure Captions

Fig.1: Aztec diamond of different dimensions.

Fig.2: Generalized Aztec diamond $L_{11} * L_{11}$.

Fig.3: Types of three edges path connectivity of odd generalized Aztec diamond $L_{11} * L_{11}$.

Fig.4: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set A .

Fig.5: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set B .

Fig.6: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set C .

Fig.7: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set D .

Fig.8: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set E .

Fig.9: Types of three edges paths of odd generalized Aztec diamond $L_{11} * L_{11}$ that start from set F .

Fig.10: Types of three edges path of even generalized Aztec diamond $L_{11} * L_{11}$.

Fig.11: Types of three edges paths of even generalized Aztec diamond $L_{11} * L_{11}$ that start from set A .

Fig.12: Types of three edges paths of even generalized Aztec diamond, that start from set B .

Fig.13: Types of three edges paths of even generalized Aztec diamond $L_{11} * L_{11}$ that start from set C .

Fig.14: Types of three edges paths of even generalized Aztec diamond $L_{11} * L_{11}$ that start from set D .

Fig.15: Types of three edges paths of even generalized Aztec diamond that start from set E .

Fig.16: Types of three edges paths of even generalized Aztec diamond that start from set F .

Fig.17: Extended Aztec diamond of different dimension.

Fig.18: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set A .

Fig.19: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set B .

Fig.20: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set C .

Fig.21: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set D .

Fig.22: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set E .

Fig.23: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set F .

Fig.24: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set G .

Fig.25: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set H .

Fig.26: The comparison of third connectivity and third sum-connectivity for odd generalized Aztec diamond, third connectivity index $\rightarrow$ Green, third sum connectivity index $\rightarrow$ Blue.

Fig.27: The comparison of third connectivity and third sum-connectivity for even generalized Aztec diamond, third connectivity index $\rightarrow$ Red and third sum connectivity index $\rightarrow$ Green.

Fig.28: The comparison of third connectivity and third sum-connectivity for extended Aztec diamond, third connectivity index $\rightarrow$ Red and third sum connectivity index $\rightarrow$ Green.

Fig.29: The comparison of third connectivity of generalized Aztec diamond for both even and odd cases, third connectivity index for odd $\rightarrow$ Blue and third connectivity index for even $\rightarrow$ Green.

Fig.30: The comparison of sum third connectivity index of generalized Aztec diamond for both even and

odd cases, sum third connectivity index for odd $\rightarrow$ Red and third sum connectivity index for even case $\rightarrow$ Green.

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