On molecular topological properties of diamond like networks

<table>
<thead>
<tr>
<th>Journal:</th>
<th>Canadian Journal of Chemistry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript ID</td>
<td>cjc-2017-0206.R1</td>
</tr>
<tr>
<td>Manuscript Type:</td>
<td>Article</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>28-Apr-2017</td>
</tr>
<tr>
<td>Complete List of Authors:</td>
<td>Imran, Muhammad; United Arab Emirates University, Department of Mathematical Sciences; National University of Sciences and Technology, School of Natural Sciences Baig, A. Q.; COMSATS Institute of Information Technology, Attock, Pakistan, Mathematics Siddiqui, Hafiz Muhammad Afzal; COMSATS Institute of Information Technology - MA Jinnah Campus, Department of Mathematics Sarawar, Rabia; Government College University Faisalabad</td>
</tr>
<tr>
<td>Is the invited manuscript for consideration in a Special Issue?:</td>
<td>N/A</td>
</tr>
<tr>
<td>Keyword:</td>
<td>Generalizes Aztec diamond, Extended Aztec diamond, Third connectivity index, Third sum-connectivity-index</td>
</tr>
</tbody>
</table>

https://mc06.manuscriptcentral.com/cjc-pubs
On molecular topological properties of diamond like networks*

*Muhammad Imran, Abdul Qudair Baig, Hafiz Muhammad Afzal Siddiqui, Rabia Sarwar,

Department of Mathematical Sciences, United Arab Emirates University, P. O. Box 15551, Al Ain, United Arab Emirates

School of Natural Sciences, National University of Sciences and Technology, H-12, Islamabad, Pakistan

Department of Mathematics, COMSATS Institute of Information Technology, Attock, Pakistan

Department of Mathematics, COMSATS Institute of Information Technology Lahore, Pakistan

Department of Mathematics, Government College University Faisalabad (GCUF), Pakistan

Email: {imrandhab, aqbaig1, hmasiddiqui, rabiasarwar80}@gmail.com

* This research is supported by the Start Up Research Grant 2016 of United Arab Emirates University, Al Ain, United Arab Emirates via Grant No. G00002233 and by the grant of Higher Education Commission of Pakistan Ref. No. 21-808/SRGP/R&D/HEC/2016.
Abstract. The Randić (product) connectivity index and its derivative called the sum-connectivity index are well known topological indices and these both descriptors correlate well among themselves and with the π-electronic energies of benzenoid hydrocarbons. The general $n$-connectivity of a molecular graph $G$ is defined as $n\chi(G) = \sum_{v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{n+1}} \frac{1}{\sqrt{d_{v_1}d_{v_2} \ldots d_{v_{n+1}}}}$ and the $n$-sum connectivity of a molecular graph $G$ is defined as $nX(G) = \sum_{v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{n+1}} \frac{1}{\sqrt{d_{v_1} + d_{v_2} + \ldots + d_{v_{n+1}}}}$, where the paths of length $n$ in $G$ are denoted by $v_{i1}, v_{i2}, \ldots, v_{in+1}$ and the degree of each vertex $v_i$ is denoted by $d_i$. In this paper, we discuss third connectivity and third sum-connectivity indices of diamond like networks and compute analytical closed results of these indices for diamond like networks.

Keywords: Generalizes Aztec diamond, Extended Aztec diamond; Third connectivity index; Third sum-connectivity-index.
1 Introduction and preliminary results

Graph theory is proficient for modeling and designing of chemical structures and complex networks and to predict their properties. The chemical graph theory applies the tools from graph theory to mathematical modeling of molecular phenomena, which is helpful for the study of molecular structure and molecular modeling. This theory play a vital role in the field of theoretical chemical sciences. Chemical compounds have a variety of applications in chemical graph theory, drug design, and in nanotechnology etc. The manipulation and examination of chemical structural information is made conceivable by using molecular descriptors. A great variety of topological indices are studied and used in theoretical chemistry, pharmaceutical researchers. A chemical structure can be represented by using graph theory, where vertices denote atoms and edges denote molecular bond. A topological index is a numeric number which indicates some useful information about molecular structure. It is the numerical invariants of a molecular graph and are useful to correlate with their bioactivity and physico-chemical properties. Researchers have found topological index to be powerful and useful tool in the description of molecular structure. A graph $G(V,E)$ is a pair of $V(G)$ and $E(G)$. Where $V(G)$ represent the set of vertices and $E(G)$ represent the set of edges. The order of a graph is $n$ and it is denoted as $|V(G)| = n$. The size of a graph is $m$ and it is denoted as $|E(G)| = m$. The general $n$-connectivity index of a graph $G$ is denoted as

$$n\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} \ldots v_{i_{n+1}}} \frac{1}{\sqrt{d_{i_1} d_{i_2} \cdots d_{i_{n+1}}}}.$$ (1)

Where the paths of length $n$ in $G$ are denoted by $v_{i_1}, v_{i_2}, \ldots v_{i_{n+1}}$ and the degree of vertex $v_i$ is denoted by $d_i$. The first connectivity index is described as

$$\chi(G) = \sum_{w \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$ (2)

In 1971, Milan Randić introduced 31 first connectivity index. Now it is known as Randić index. Many variations of Randić connectivity index have been discussed in 4,5,18,21,23,24,28,29,31,32.

The second connectivity index is expressed as

$$2\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3}}}.$$ (3)

Similarly, third connectivity is expressed as

$$3\chi(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{i_1} d_{i_2} d_{i_3} d_{i_4}}}.$$ (4)
In 2008, B. Zhou and N. Trinajstić see \(^1\),\(^12\),\(^13\),\(^29\),\(^35\). developed the sum-connectivity index, and it is defined as

$$X(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}. \quad (5)$$

Where the degree of vertices \(u\) and \(v\) are represented as \(d_u\) and \(d_v\) respectively. Similarly, second-sum-connectivity is described as

$$2X(G) = \sum_{v_i v_2 v_3} \frac{1}{\sqrt{d_{v_1} + d_{v_2} + d_{v_3}}}. \quad (6)$$

Where the paths of length 2 in graph \(G\) are denoted by \(\{v_1, v_2, v_3\}\) and degree of each vertex \(v_i\) is denoted by \(d_{v_i}\) where \(1 \leq i \leq 3\). Similarly, third sum-connectivity is expressed as

$$3X(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} v_{i_4}} \frac{1}{\sqrt{d_{v_{i_1}} + d_{v_{i_2}} + d_{v_{i_3}} + d_{v_{i_4}}}}. \quad (7)$$

In general, the \(n\) sum-connectivity index of graph \(G\) is given by

$$nX(G) = \sum_{v_{i_1} v_{i_2} v_{i_3} ... v_{i_{n+1}}} \frac{1}{\sqrt{d_{v_{i_1}} + d_{v_{i_2}} + ... + d_{v_{i_{n+1}}}}}. \quad (8)$$

The further study of the \(m\)-connectivity index is explained in \(^6\),\(^7\),\(^8\),\(^13\),\(^15\),\(^16\),\(^23\),\(^26\),\(^27\),\(^33\). The further study of the \(m\) sum-connectivity index is clarified \(^9\),\(^10\),\(^11\),\(^15\),\(^16\),\(^25\),\(^32\),\(^34\). The topological descriptors of certain networks and nanotubes are studied in \(^3\),\(^19\),\(^20\). The diamond like networks have been examined in different ways. The relation of domino tilings and Aztec diamond theorem has been argued in \(^14\),\(^22\). In this paper, we examined third connectivity and third sum-connectivity of diamond like networks.

### 2 Main results for generalized Aztec diamond

An Aztec diamond with order \(n\) containing all squares lattice whose centre \((x, y)\) satisfy \(|x| + |y| \leq n\). Here \(n\) is fixed, and square lattice contains unit squares with the origin as vertex of degree 4, so that both \(x\) and \(y\) are half integer. Consider a path \(L_i\) with \(i\) vertices, where \(i = 1, 2, 3, 4, \ldots\). The tensor product of

![Fig. 1: Aztec diamond of different dimensions.](https://mc06.manuscriptcentral.com/cjc-pubs)
two paths $L_n$ and $L_m$ denoted by $L_n \ast L_m$ is the graph on $n \times m$ vertices $\{(x, y) : 1 \leq x \leq n, 1 \leq y \leq m\}$. Any two vertices are adjacent if and only if $|x - x'| + |y - y'| = 1$. Clearly, this graph is a disconnected graph having two connected components. One component is denoted as $O(L_n \ast L_m)$ has the vertices $\{(x, y) : x + y \text{ is odd}\}$. The other component is denoted as $E(L_n \ast L_m)$ has the vertices $\{(x, y) : x + y \text{ is even}\}$. First of all, we discuss $O(L_{2n+1} \ast L_{2m+1})$ and it is called odd generalized Aztec diamond.

Fig. 2: Generalized Aztec diamond $L_{11} \ast L_{11}$.

Odd generalized Aztec diamond

Fig. 3: Types of three edges path connectivity of odd generalized Aztec diamond $L_{11} \ast L_{11}$.

Fig. 4: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $A$. https://mc06.manuscriptcentral.com/cjc-pubs
Three edges path connectivity is obtained by twice the set $A$ or $3\chi(O(L_n \ast L_m)) = 2A$.

Table 1: Edge partition of edge set $A$ of $3\chi(O(L_n \ast L_m))$, where $m = n$ and $m, n \geq 6$. 

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{2242}$</th>
<th>$d_{2424}$</th>
<th>$d_{2442}$</th>
<th>$d_{2444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>$4$</td>
<td>$2n - 2$</td>
<td>$12$</td>
<td>$12n$</td>
</tr>
</tbody>
</table>

Fig. 5: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $B$. 

Similarly, $3\chi(O(L_n \ast L_m)) = 2B$.

Table 2: Edge partition of edge set $B$ of $3\chi(O(L_n \ast L_m))$, where $m = n$ and $m, n \geq 6$.  

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{2242}$</th>
<th>$d_{2424}$</th>
<th>$d_{2444}$</th>
<th>$d_{2444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>$4$</td>
<td>$4$</td>
<td>$8$</td>
<td>$16$</td>
</tr>
</tbody>
</table>

Fig. 6: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $C$. 

$3\chi(O(L_n \ast L_m)) = 2C$.

Table 3: Edge partition of edge set $C$ of $3\chi(O(L_n \ast L_m))$, where $m = n$ and $m, n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{4224}$</th>
<th>$d_{4242}$</th>
<th>$d_{4424}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>4</td>
<td>2$n - 4$</td>
<td>4</td>
</tr>
</tbody>
</table>

Fig. 7: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $D$.

$3\chi(O(L_n \ast L_m)) = 2D$.

Table 4: Edge partition of edge set $D$ of $3\chi(O(L_n \ast L_m))$, where $m = n$ and $m, n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{2422}$</th>
<th>$d_{2442}$</th>
<th>$d_{2444}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>20$n - 28$</td>
</tr>
</tbody>
</table>

Fig. 8: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $E$. 

https://mc06.manuscriptcentral.com/cjc-pubs
\[3\chi(O(L_n \ast L_m)) = (m - 3)E, \text{ where } m = n \text{ and } m, n \geq 6.\]

Table 5: Edge partition of edge set \(E\) of \(3\chi(O(L_n \ast L_m))\), where \(m = n\) and \(m, n \geq 6\).

<table>
<thead>
<tr>
<th>(d_{ijkl})</th>
<th>(d_{4242})</th>
<th>(d_{4442})</th>
<th>(d_{4444})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of (d_{ijkl})</td>
<td>(4n - 12)</td>
<td>(4n - 12)</td>
<td>(12mn - 24m - 16n + 12)</td>
</tr>
</tbody>
</table>

Fig. 9: Types of three edges paths of odd generalized Aztec diamond \(L_{11} \ast L_{11}\) that start from set \(F\).

\[3\chi(O(L_n \ast L_m)) = (m - 4)F, \text{ where } m = n \text{ and } m, n \geq 6.\]

Table 6: Edge partition of edge set \(F\) of \(3\chi(O(L_n \ast L_m))\), where \(m = n\) and \(m, n \geq 6\).

<table>
<thead>
<tr>
<th>(d_{ijkl})</th>
<th>Cardinality of (d_G(i), d_G(j), d_G(k), d_G(l))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_{2424})</td>
<td>(4n - 16)</td>
</tr>
<tr>
<td>(d_{2444})</td>
<td>(4n - 16)</td>
</tr>
<tr>
<td>(d_{4424})</td>
<td>(2n - 8)</td>
</tr>
<tr>
<td>(d_{4444})</td>
<td>(12mn - 64m + 64)</td>
</tr>
</tbody>
</table>

The following theorems present analytically closed formulas of third connectivity and third sum-connectivity indices for \(3\chi(O(L_{2n+1} \ast L_{2m+1}))\).

**Theorem 1** Let \(3\chi(O(L_{2n+1} \ast L_{2m+1}))\) be the graph, then third connectivity of the graph is given by

\[
\frac{1}{2} \left[ \frac{3}{2} mn - \left( \frac{11}{2} \right) m + \left( \frac{17}{4} + \frac{3}{\sqrt{2}} + \sqrt{2} \right) n + \left( \frac{9}{4} + \frac{6}{4\sqrt{2}} \right) \right].
\]

**Proof.** Using the above Tables [1-6] and Equation(4) we have

\[2\chi(G) = \sum_{w_1w_2w_3w_4} \frac{1}{\sqrt{d_{i1} d_{i2} d_{i3} d_{i4}}}.\]

Since each path repeated twice therefore, we take

https://mc06.manuscriptcentral.com/cjc-pubs
\[ \frac{1}{2} \left( \sum_{a \in A} \sqrt{d_a \times d_{i_a} \times d_{i_b} \times d_{i_c}} + \sum_{b \in B} \sqrt{d_b \times d_{i_2} \times d_{i_3} \times d_{i_4}} + \sum_{c \in C} \sqrt{d_c \times d_{i_1} \times d_{i_2} \times d_{i_4}} + \sum_{d \in D} \sqrt{d_d \times d_{i_2} \times d_{i_3} \times d_{i_4}} + \sum_{e \in E} \sqrt{d_e \times d_{i_1} \times d_{i_2} \times d_{i_3}} + \sum_{f \in F} \sqrt{d_f \times d_{i_2} \times d_{i_3} \times d_{i_4}} \right) = \]
\[ \frac{1}{2} \left( \left( \frac{d_{i_1}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_2}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_3}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} \right) + \left( \frac{d_{i_2}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_3}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} \right) + \left( \frac{d_{i_3}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_1}}{\sqrt{2\times2\times2\times2}} \right) + \left( \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_1}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_2}}{\sqrt{2\times2\times2\times2}} \right) \right) = \]
\[ \frac{1}{2} \left( \frac{4}{\sqrt{2\times2\times2\times2}} + \frac{2n-2}{\sqrt{2\times2\times2\times2}} + \frac{12}{\sqrt{2\times2\times2\times2}} + \frac{12n}{\sqrt{2\times2\times2\times2}} + \frac{4}{\sqrt{2\times2\times2\times2}} + \frac{4}{\sqrt{2\times2\times2\times2}} + \frac{8}{\sqrt{2\times2\times2\times2}} + \frac{16}{\sqrt{2\times2\times2\times2}} + \frac{4n-4}{\sqrt{2\times2\times2\times2}} + \frac{4n-28}{\sqrt{2\times2\times2\times2}} + \frac{4n-12}{\sqrt{2\times2\times2\times2}} + \frac{4n-12}{\sqrt{2\times2\times2\times2}} + \frac{12m-n-16m+12}{\sqrt{2\times2\times2\times2}} + \frac{4n-16}{\sqrt{2\times2\times2\times2}} + \frac{8n-32}{\sqrt{2\times2\times2\times2}} + \frac{2n-8}{\sqrt{2\times2\times2\times2}} + \frac{12m-n-64m+64}{\sqrt{2\times2\times2\times2}} \right). \]

After simplification, we get:
\[ \frac{1}{2} \left[ 6nm - (22) + \left( \frac{24}{\sqrt{14}} + 2\sqrt{3} + 9 \right) + (14 + \frac{10}{\sqrt{12}} - \frac{24}{\sqrt{14}} + \frac{12}{\sqrt{10}}) \right]. \]

**Theorem 2** Let \( 3X(O(L_{2n+1} \ast L_{2m+1})) \) be the graph, then third sum connectivity of the graph is given by
\[ \frac{1}{2} \left[ 6nm - (22) + \left( \frac{24}{\sqrt{14}} + 2\sqrt{3} + 9 \right) + (14 + \frac{10}{\sqrt{12}} - \frac{24}{\sqrt{14}} + \frac{12}{\sqrt{10}}) \right]. \]

**Proof.** Using the above Tables [1-6] and Equation(7) we have
\[ 2\chi(G) = \sum_{v_1,v_2,v_3,v_4} \frac{1}{\sqrt{d_{i_1}d_{i_2}d_{i_3}d_{i_4}}}. \]

Since each path repeated twice therefore, we take
\[ \frac{1}{2} \left( \sum_{a \in A} \frac{1}{\sqrt{d_a + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{d \in D} \frac{1}{\sqrt{d_d + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e + d_{i_2} + d_{i_3} + d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f + d_{i_2} + d_{i_3} + d_{i_4}}} \right) = \]
\[ \frac{1}{2} \left( \frac{d_{i_1}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_2}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_3}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} \right) + \left( \frac{d_{i_2}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_3}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} \right) + \left( \frac{d_{i_3}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_1}}{\sqrt{2\times2\times2\times2}} \right) + \left( \frac{d_{i_4}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_1}}{\sqrt{2\times2\times2\times2}} + \frac{d_{i_2}}{\sqrt{2\times2\times2\times2}} \right) \right) = \]
\[ \frac{1}{2} \left( \frac{4}{\sqrt{2\times2\times2\times2}} + \frac{2n-2}{\sqrt{2\times2\times2\times2}} + \frac{12}{\sqrt{2\times2\times2\times2}} + \frac{12n}{\sqrt{2\times2\times2\times2}} + \frac{4}{\sqrt{2\times2\times2\times2}} + \frac{4}{\sqrt{2\times2\times2\times2}} + \frac{8}{\sqrt{2\times2\times2\times2}} + \frac{16}{\sqrt{2\times2\times2\times2}} + \frac{4n-4}{\sqrt{2\times2\times2\times2}} + \frac{4n-28}{\sqrt{2\times2\times2\times2}} + \frac{4n-12}{\sqrt{2\times2\times2\times2}} + \frac{4n-12}{\sqrt{2\times2\times2\times2}} + \frac{12m-n-16m+12}{\sqrt{2\times2\times2\times2}} + \frac{4n-16}{\sqrt{2\times2\times2\times2}} + \frac{8n-32}{\sqrt{2\times2\times2\times2}} + \frac{2n-8}{\sqrt{2\times2\times2\times2}} + \frac{12m-n-64m+64}{\sqrt{2\times2\times2\times2}} \right). \]

After simplification, we get
\[ \frac{1}{2} \left[ 6nm - (22) + \left( \frac{24}{\sqrt{14}} + 2\sqrt{3} + 9 \right) + (14 + \frac{10}{\sqrt{12}} - \frac{24}{\sqrt{14}} + \frac{12}{\sqrt{10}}) \right]. \]
3 Third connectivity and third sum-connectivity indices of even generalized Aztec diamond

Fig. 10: Types of three edges path of even generalized Aztec diamond $L_{11} \ast L_{11}$.

In third connectivity the paths of length 4 in $G$ are denoted by $v_{i_1}$, $v_{i_2}$, $v_{i_3}$, $v_{i_4}$ and the degree of each vertex $v_i$ is denoted by $d_i$. Where $i = 1, 2, 3, 4$.

$3\chi(E(L_n \ast L_m)) = 2A$, where $m = n$ and $m, n \geq 6$.

Fig. 11: Types of three edges paths of even generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $A$.

Table 7: Edge partition of edge set $A$ of $3\chi(E(L_n \ast L_m))$, where $m = n$ and $m, n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{1424}$</th>
<th>$d_{1444}$</th>
<th>$d_{2444}$</th>
<th>$d_{2444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>8</td>
<td>12</td>
<td>$2n - 2$</td>
<td>$12n - 12$</td>
</tr>
</tbody>
</table>

$3\chi(E(L_n \ast L_m)) = 2B$, where $m = n$ and $m, n \geq 6$.

$3\chi(E(L_n \ast L_m)) = 2C$, where $m = n$ and $m, n \geq 6$.

$3\chi(E(L_n \ast L_m)) = 2D$, where $m = n$ and $m, n \geq 6$. 
Fig. 12: Types of three edges paths of even generalized Aztec diamond, that start from set $B$.

Table 8: Edge partition of edge set $B$ of $3\chi(E(L_n \ast L_m))$, where $m = n$ and $n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{4242}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>8</td>
<td>$12n$</td>
</tr>
</tbody>
</table>

Fig. 13: Types of three edges paths of even generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $C$.

Table 9: Edge partition of edge set $C$ of $3\chi(E(L_n \ast L_m))$, where $m = n$ and $n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{2424}$</th>
<th>$d_{2444}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>12</td>
<td>$20n - 8$</td>
</tr>
</tbody>
</table>

Fig. 14: Types of three edges paths of even generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $D$.

Table 10: Edge partition of edge set $D$ of $3\chi(E(L_n \ast L_m))$, where $m = n$ and $n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{2424}$</th>
<th>$d_{2444}$</th>
<th>$d_{4444}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>$4n - 12$</td>
<td>$8n - 24$</td>
<td>$2n - 6$</td>
<td>$12mn + 12n - 32$</td>
</tr>
</tbody>
</table>

$3\chi(E(L_n \ast L_m)) = (m - 3)E$, where $m = n$ and $n \geq 6$.

Table 11: Edge partition of edge set $E$ of $3\chi(E(L_n \ast L_m))$, where $m = n$ and $n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{2424}$</th>
<th>$d_{2444}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>$4n - 12$</td>
<td>$8n - 24$</td>
<td>$2n - 6$</td>
</tr>
</tbody>
</table>
Fig. 15: Types of three edges paths of even generalized Aztec diamond that start from set $E$.

$3\chi(E(L_n \ast L_{m})) = (m - 4)F$, where $m = n$ and $m, n \geq 6$.

Fig. 16: Types of three edges paths of even generalized Aztec diamond that start from set $F$.

Table 12: Edge partition of edge set $F$ of $3\chi(E(L_n \ast L_m))$, where $m = n$ and $m, n \geq 6$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>Cardinality of $d_G(i), d_G(j), d_G(k), d_G(l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{1242}$</td>
<td>$2n - 4$</td>
</tr>
<tr>
<td>$d_{1442}$</td>
<td>$2n - 4$</td>
</tr>
<tr>
<td>$d_{1444}$</td>
<td>$12mn - 28m - 24n + 16$</td>
</tr>
</tbody>
</table>

Following theorems present analytically closed formula of third connectivity and third sum-connectivity even generalized Aztec diamond.

**Theorem 3** Let $3\chi(E(L_{2n+1} \ast L_{2m+1}))$ be the graph, then third connectivity of the graph is given by

$$
\frac{1}{2} \left[ \frac{7}{2} mn - \left( \frac{35}{12} \right) m + \left( \frac{12}{5} + \frac{7}{2\sqrt{2}} \right) n + \left( \frac{5}{2} - \frac{7}{4\sqrt{2}} \right) \right].
$$

**Proof.** Using the above Tables [7-12] and Equation(4) we have

$$
2\chi(G) = \sum_{v_{i_1}v_{i_2}v_{i_3}v_{i_4}} 1 \sqrt{d_{i_1}d_{i_2}d_{i_3}d_{i_4}}.
$$

Since each path repeated twice therefore, we take

$$
\frac{1}{2} \left[ \sum_{a \in A} \frac{1}{\sqrt{d_a \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{b \in B} \frac{1}{\sqrt{d_b \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{c \in C} \frac{1}{\sqrt{d_c \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{d \in D} \frac{1}{\sqrt{d_d \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{e \in E} \frac{1}{\sqrt{d_e \times d_{i_2} \times d_{i_3} \times d_{i_4}}} + \sum_{f \in F} \frac{1}{\sqrt{d_f \times d_{i_2} \times d_{i_3} \times d_{i_4}}} \right] =
$$

https://mc06.manuscriptcentral.com/cjc-pubs
1 \left( \frac{d_{1243}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{1444}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2324}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2444}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left( \frac{d_{1231}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{1434}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2312}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2414}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left( \frac{d_{1234}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{1432}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2312}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2414}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left( \frac{d_{1234}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{1432}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2312}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2414}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left( \frac{d_{1234}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{1432}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2312}}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{d_{2414}}{\sqrt{4 \times 4 \times 4 \times 4}} \right) = \\
\frac{1}{2} \left( \frac{8}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{12}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{2n-2}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{12n-12}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left( \frac{8}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{12}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{20n-8}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{8n-24}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{2n-6}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{12n-30m-18n+36}{\sqrt{4 \times 4 \times 4 \times 4}} \right) + \left( \frac{2n-4}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{2n-4}{\sqrt{4 \times 4 \times 4 \times 4}} + \frac{12n-28n-24n+16}{\sqrt{4 \times 4 \times 4 \times 4}} \right) = \\
\frac{1}{2} \left[ 6mm - \frac{58}{3} m + \left( \frac{13}{3} + \frac{7}{2 \sqrt{3}} \right)n + \left( \frac{5}{2} - \frac{7}{4 \sqrt{3}} \right) \right].

Theorem 4 Let $3X(E(L_{2n+1} * L_{2m+1}))$ be the graph, then third sum connectivity of the graph is given by

$$2 \chi^3(G) = \sum_{v_i v_j v_k v_l} \sum_{1}^{1} \frac{1}{\sqrt{d_{i1} + d_{i2} + d_{i3} + d_{i4}}} \times \frac{1}{\sqrt{d_{j1} + d_{j2} + d_{j3} + d_{j4}}} \times \frac{1}{\sqrt{d_{k1} + d_{k2} + d_{k3} + d_{k4}}} \times \frac{1}{\sqrt{d_{l1} + d_{l2} + d_{l3} + d_{l4}}}.$$

Proof. Using the above Tables [7-12] and Equation(7) we have

After simplification, we get

$$= \frac{1}{2} \left[ 6mn - \frac{58}{3} m + \left( \frac{13}{3} + \frac{7}{2 \sqrt{3}} \right)n + \left( \frac{5}{2} - \frac{7}{4 \sqrt{3}} \right) \right].$$

4 Main results of extended Aztec diamond

An Aztec diamond of order $n$ containing all squares lattice whose center $(x, y)$ satisfy $|x| + |y| \leq n$. Here $n$ is fixed and square lattice contains unit squares with the origin as vertex 4 of them, so that both $p$ and $q$ are half integer. An extended Aztec diamond is denoted by $E AZ(n)$. An extended Aztec diamond is gained by joining all vertices of degree 2. The subsequent graph is non-isomorphic to the generalized Aztec diamond having vertices of degree 3 also. There are three types of partite set in the edge partition of edge set of $E AZ(n)$. We compute third connectivity and third sum-connectivity indices of extended Aztec diamond.

https://mc06.manuscriptcentral.com/cjc-pubs
Fig. 17: Extended Aztec diamond of different dimension.

Fig. 18: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set $A$. 
Three edges path connectivity is obtained by twice the set $A$ or $3\chi((EAZ(n))) = 2A$.

Table 13: Edge partition of edge set $A$ of $EAZ(n)$, where $n \geq 8$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{3334}$</th>
<th>$d_{3344}$</th>
<th>$d_{3444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>8</td>
<td>24</td>
</tr>
</tbody>
</table>

Fig. 19: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set $B$.

Similarly, $3\chi(EAZ(n)) = 2B$.

Table 14: Edge partition of edge set $B$ of $EAZ(n)$, where $n \geq 8$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{4333}$</th>
<th>$d_{4344}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>

Fig. 20: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set $C$.

Similarly, $3\chi(EAZ(n)) = 2C$.

$3\chi((EAZ(n))) = 2D$.

$3\chi(EAZ(n)) = (n - 4)E$.

$3\chi(EAZ(n)) = 2F$.

$3\chi(EAZ(n)) = 2G$.
Table 15: Edge partition of edge set $C$ of $EAZ(n)$, where $n \geq 8$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{4433}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>116</td>
</tr>
</tbody>
</table>

Fig. 21: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set $D$.

Table 16: Edge partition of edge set $D$ of $EAZ(n)$, where $n \geq 8$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{4433}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>150</td>
</tr>
</tbody>
</table>

Fig. 22: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set $E$.

Table 17: Edge partition of edge set $E$ of $EAZ(n)$, where $n \geq 8$.

<table>
<thead>
<tr>
<th>$d_{ijkl}$</th>
<th>$d_{4433}$</th>
<th>$d_{4443}$</th>
<th>$d_{4444}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cardinality of $d_{ijkl}$</td>
<td>4</td>
<td>8</td>
<td>$16n^2 - 20n - 466$</td>
</tr>
</tbody>
</table>

Fig. 23: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set $F$.

Fig. 24: Types of three edges paths of extended Aztec diamond $EAZ(n)$ that start from set $G$. 
Since each path repeated twice therefore, we take

\[ \sum_{i,j,k,l} d_{ijkl} \]

Theorem 5

Let \( \chi(EAZ(n)) \) be the graph, then third connectivity of the graph is given by

\[ \frac{1}{2} [n^2 + (\frac{11}{8})n + (\frac{48}{8\sqrt{3}} + \frac{16}{6\sqrt{3}} + \frac{12}{\sqrt{2}} - \frac{242}{16})] \]

Proof. Using the above Tables [13-19] and Equation (4) we have

\[ 2\chi(G) = \sum_{v_i,v_j,v_k,v_l} \frac{1}{\sqrt{d_{ijkl}}} \]

Since each path repeated twice therefore, we take

\[ \frac{1}{2} [\sum_{a \in A} \sqrt{d_a \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{b \in B} \sqrt{d_b \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{c \in C} \sqrt{d_c \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{d \in D} \sqrt{d_d \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{e \in E} \sqrt{d_e \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{f \in F} \sqrt{d_f \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{g \in G} \sqrt{d_g \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{h \in H} \sqrt{d_h \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{i \in I} \sqrt{d_i \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{j \in J} \sqrt{d_j \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{k \in K} \sqrt{d_k \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{l \in L} \sqrt{d_l \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{m \in M} \sqrt{d_m \times d_{ij} \times d_{kl} \times d_{mn}} + \sum_{n \in N} \sqrt{d_n \times d_{ij} \times d_{kl} \times d_{mn}}]

Following theorems present analytically closed formula of third connectivity and third sum-connectivity extended Aztec diamond.
Proof. Using the above Tables [13-19] and Equation (7) we have

\[ 3\chi(G) = \sum_{v_1, v_2, v_3, v_4} \frac{1}{\sqrt{d_{v_1}d_{v_2}d_{v_3}d_{v_4}}} \]

Since each path repeated twice therefore, we take

\[ \frac{1}{2} \left[ \sum_{a \in A} \sqrt{d_a + d_{a_2} + d_{a_3} + d_{a_4}} + \sum_{b \in B} \sqrt{d_b + d_{b_2} + d_{b_3} + d_{b_4}} + \sum_{c \in C} \sqrt{d_c + d_{c_2} + d_{c_3} + d_{c_4}} + \sum_{f \in F} \sqrt{d_f + d_{f_2} + d_{f_3} + d_{f_4}} \right] \]

After simplification, we get

\[ = \frac{1}{2} \left[ 2\left(n^2 + \left( \frac{11}{2} \right)n + \left( \frac{16}{3} \sqrt{\frac{16}{3}} + \frac{22}{12} - \frac{243}{10} \right) \right) \right] \]

\[ \frac{3\chi(EAZ(n))}{[34n - 30]} \left\{ \left( 1 \right) \frac{1}{\sqrt{d_1d_2d_3d_4}} \right\} \]

After simplification, we get

\[ = \frac{1}{2} \left[ 4n^2 - \left( \frac{11}{2} n + \frac{16}{3} \sqrt{\frac{16}{3}} \right) n + \left( \frac{22}{12} - \frac{243}{10} \right) \right] \]

Theorem 6 Let \( 3\chi(EAZ(n)) \) be the graph, then third sum connectivity of the graph is given by

\[ \frac{1}{2} \left[ 4n^2 - \left( \frac{11}{2} n + \frac{16}{3} \sqrt{\frac{16}{3}} \right) n + \left( \frac{22}{12} - \frac{243}{10} \right) \right] \]

The graphical representation of third connectivity and third sum-connectivity for odd generalized Aztec diamond is depicted in Figure 26. The graphical representation of third connectivity and third sum-connectivity for even generalized Aztec diamond is depicted in Figure 27. By increasing the values of \( m \) and \( n \), the value of indices increases.

The graphical representation of third connectivity and third sum-connectivity for extended Aztec diamond is depicted in Figure 28. By increasing the values of \( n \), the value of indices increase and third sum connectivity index approaches towards the third connectivity index. The comparison of third connectivity and sum third connectivity index of both even and odd generalized Aztec diamond is depicted in Figure 29 and 30 respectively. The graphical representation shows that the third and sum third connectivity index of generalized Aztec diamond for even and odd cases respectively are very close to each other.
Fig. 26: The comparison of third connectivity and third sum-connectivity for odd generalized Aztec diamond, third connectivity index → Green, third sum connectivity index → Blue.

Fig. 27: The comparison of third connectivity and third sum-connectivity for even generalized Aztec diamond, third connectivity index → Red and third sum connectivity index → Green.

Fig. 28: The comparison of third connectivity and third sum-connectivity for extended Aztec diamond, third connectivity index → Red and third sum connectivity index → Green.

Fig. 29: The comparison of third connectivity of generalized Aztec diamond for both even and odd cases, third connectivity index for odd → Blue and third connectivity index for even → Green.
Fig. 30: The comparison of sum third connectivity index of generalized Aztec diamond for both even and odd cases, sum third connectivity index for odd $\rightarrow$ Red and third sum connectivity index for even case $\rightarrow$ Green.

5 Conclusion and general remarks

In this paper, we considered the conclusion third connectivity and third sum-connectivity for generalized Aztec diamond and extended diamond. We derived the close formulas of third connectivity and third sum-connectivity for generalized Aztec diamond and extended diamond. General connectivity and general sum-connectivity for these diamonds can be considered for future study.

References

23. Li, X.; Gutman, I.; *Mathematical Aspects of Randic-Type Molecular Structure Descriptors* 2006, VI + 330, Hardcover.
**Figure Captions**

Fig.1: Aztec diamond of different dimensions.

Fig.2: Generalized Aztec diamond $L_{11} \ast L_{11}$.

Fig.3: Types of three edges path connectivity of odd generalized Aztec diamond $L_{11} \ast L_{11}$.

Fig.4: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $A$.

Fig.5: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $B$.

Fig.6: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $C$.

Fig.7: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $D$.

Fig.8: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $E$.

Fig.9: Types of three edges paths of odd generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $F$.

Fig.10: Types of three edges path of even generalized Aztec diamond $L_{11} \ast L_{11}$.

Fig.11: Types of three edges paths of even generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $A$.

Fig.12: Types of three edges paths of even generalized Aztec diamond, that start from set $B$.

Fig.13: Types of three edges paths of even generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $C$.

Fig.14: Types of three edges paths of even generalized Aztec diamond $L_{11} \ast L_{11}$ that start from set $D$.

Fig.15: Types of three edges paths of even generalized Aztec diamond that start from set $E$.

Fig.16: Types of three edges paths of even generalized Aztec diamond that start from set $F$.

Fig.17: Extended Aztec diamond of different dimension.

Fig.18: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $A$.

Fig.19: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $B$.

Fig.20: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $C$.

Fig.21: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $D$.

Fig.22: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $E$.

Fig.23: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $F$.

Fig.24: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $G$.

Fig.25: Types of three edges paths of extended Aztec diamond $E AZ(n)$ that start from set $H$.

Fig.26: The comparison of third connectivity and third sum-connectivity for odd generalized Aztec diamond, third connectivity index → Green, third sum connectivity index → Blue.

Fig.27: The comparison of third connectivity and third sum-connectivity for even generalized Aztec diamond, third connectivity index → Red and third sum connectivity index → Green.

Fig.28: The comparison of third connectivity and third sum-connectivity for extended Aztec diamond, third connectivity index → Red and third sum connectivity index → Green.

Fig.29: The comparison of third connectivity of generalized Aztec diamond for both even and odd cases, third connectivity index for odd → Blue and third connectivity index for even → Green.

Fig.30: The comparison of sum third connectivity index of generalized Aztec diamond for both even and
odd cases, sum third connectivity index for odd $\rightarrow$ Red and third sum connectivity index for even case $\rightarrow$ Green.