Canadian Journal of Physics
Revue canadienne de physique

Approaching special relativity in complex time

Journal:	<i>Canadian Journal of Physics</i>
Manuscript ID	cjp-2017-0089
Manuscript Type:	Article
Date Submitted by the Author:	03-Feb-2017
Complete List of Authors:	Spears, James; University of Waterloo, Science
Keyword:	Special Relativity, Logic, set theory, and algebra, Mathematical Physics, Theoretical Physics, High level categorization
Please Select from this Special Issues list if applicable:	N/A

SCHOLARONE™
Manuscripts

Only

Approaching special relativity in complex time

PACS: 03.30.+p, 02.10.-v

by: James W. Spears

Faculty of Science,
University of Waterloo
200 University Avenue West
Waterloo, Ontario, Canada N2L 3G1,
Phone: (519) 888-4567
Fax: (519) 746-8115
email: jspears@uwaterloo.ca

April 6, 2017

Abstract

Algebraic techniques are employed to extend time in the complex plane. We then use the framework to examine the fundamental postulates of special relativity. Generalizing the notion of time introduces the reader to a unique perspective on the universality of light speed according to Einstein. In the context of complex time we will describe the mathematical conditions in which light speed universality arises. Using high level terminology we then develop a categorization scheme applicable to physical theories which assume light speed is universal.

1 Introduction

If interpretation of the founding postulates of a physical theory have influence on the interpretation of experimental results; it is of important consideration to illustrate founding principles in the clearest manner possible. Paying careful attention to definition we will create a vocabulary used to describe properties of time. And in doing so, facilitate an efficient discussion on the foundation of Einstein's theory of special relativity.

Our focus will be restating the postulates of special relativity upon extending time in the complex plane. Any generalizations to the foundations of special relativity will be accompanied by supporting mathematical arguments inspired by the geometry of complex numbers. Using vector notation we will provide more physically meaningful definitions and eliminate much of the abstraction before applying these ideas to time.

When referring to mathematical concepts defined herein, an *italicized* font will be used to indicate significance beyond the usual interpretation of the word. Except in quotations, where the original author may have used this font for separate reasons.

2 Developing complex time in vector notation

Consider now two orthogonal axes represented in Cartesian coordinates. Let the horizontal axis t_p represent the past, and the vertical axis t_f the future. Continue by defining the time vector $\vec{t}$ such that

$$\vec{t} = \begin{pmatrix} t_p \\ t_f \end{pmatrix} . \quad (1)$$

When these vector components satisfy

$$t_p < t_f , \quad (2)$$

we will say the time vector $\vec{t}$ is *future* oriented. When the vector components of $\vec{t}$ satisfy

$$t_p > t_f \quad (3)$$

we say $\vec{t}$ is *past* oriented. For vector components of $\vec{t}$ where

$$t_p = t_f , \quad (4)$$

we say the vector $\vec{t}$ is *present* oriented. If we have time elapse into the *future* it is sensible to reference the angular component ω of $\vec{t}$ from the *real* unit vector $\vec{e}_t$ where

$$\vec{e}_t = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} . \quad (5)$$

Any point in *real* time can then be described by scalar multiplication of the unit vector

$$p \cdot \vec{e}_t \mid p \in \mathbb{R} , \quad (6)$$

where we call p the component displacement in the *real* direction (it should be mentioned for clarity that $\mathbb{R}$ represents the set of real numbers and is not intended to imply *real* time). Let us now introduce the concept of *imaginary* time by defining the *imaginary* unit vector $\vec{e}_{it}$ such that

$$\vec{e}_{it} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (7)$$

And so any point in *imaginary* time can be described in a similar fashion

$$q \cdot \vec{e}_{it} \mid q \in \mathbb{R}, \quad (8)$$

where q is the component displacement in the *imaginary* direction. A vector with non-zero component of the *imaginary* unit vector will by definition have non-zero component displacement which is *present* oriented. *Presently* oriented component displacement will serve as our functional definition of *imaginary* time. We complete the definition of complex time ζ such that

$$\zeta := p \cdot \vec{e}_t + q \cdot \vec{e}_{it} \mid p, q \in \mathbb{R}, \quad (9)$$

where ζ may then represent any point in complex time by vector addition of the *real* and *imaginary* unit vectors and their respective scalar multiples p , and q . Now that we have a definition of complex time ζ , we may define *real* and *imaginary* operators on ζ . We define the *real* operator

$$\Re(\zeta) := p, \quad (10)$$

and the *imaginary* operator

$$\Im(\zeta) := q. \quad (11)$$

If we choose, we may write the *real* and *imaginary* operators in terms of component displacements p , q , and temporal angular component ω . These are

$$\Re(\zeta) = \sqrt{p^2 + q^2} \cdot \cos \omega , \quad (12)$$

and

$$\Im(\zeta) = \sqrt{p^2 + q^2} \cdot \sin \omega , \quad (13)$$

respectively. Now, we will need to define what the time interval Δt shall be represented by. So, let the time interval

$$\Delta t := \sqrt{p^2 + q^2} , \quad (14)$$

as the magnitude of ζ . By this definition Δt is a strictly positive quantity and that is fine. We have orientation dictated by the temporal angular component ω . For clarity of definition we will continue to use the form

$$\zeta = \Delta t [\cos \omega \cdot \vec{e}_t + \sin \omega \cdot \vec{e}_{it}] , \quad (15)$$

where we are explicitly stating the temporal angular component ω .

3 Degenerate vs. non-degenerate complex time

As we continue let us refer to ω as the *operational definition* of time, or synonymously, the temporal angular component.

3.1 Definition: degenerate complex time

Degenerate complex time is defined ζ , such that ζ satisfies the equation

$$\zeta = \Delta t [\cos(n\pi) \cdot \vec{e}_t + \sin(n\pi) \cdot \vec{e}_{it}] \mid n \in \mathbb{Z} . \quad (16)$$

In cases where the temporal angular component $\omega = n\pi$ for integers n , the complex time vector takes the value $\Delta t \cdot \pm \vec{e}_t$. In this case, time is said to *degenerate* such that orientations are either forward or reverse along the *real* line. For example, the formulation of special relativity which satisfies

$$\Re(\zeta) = \pm \Delta t , \quad (17)$$

for all possible *operational definitions* of time can be referred to as: special relativity in *degenerate* complex time. If all possible *operational definitions* of time satisfy Eq. (17), we say there exists an *absolute operational definition* of time.

3.2 Definition: non-degenerate complex time

If Eq. (17) is not satisfied for all *operational definitions* of time, then complex time is defined as being *non-degenerate*.

3.3 Definition: relatively degenerate case

Complex time is defined to be *relatively degenerate* for two arbitrary *operational definitions* if

$$\Delta\omega = n\pi \mid n \in \mathbb{Z} . \quad (18)$$

(All *operational definitions* in *degenerate* complex time are also *relatively degenerate*.)

3.4 Definition: relatively non-degenerate case

Complex time is defined to be *relatively non-degenerate* for two arbitrary *operational definitions* if

$$\Delta\omega \neq n\pi \mid n \in \mathbb{Z} . \quad (19)$$

(Not all *operational definitions* in *non-degenerate* complex time are also *relatively non-degenerate*.)

4 Reviewing the fundamental postulates of special relativity

The theory of special relativity according to Einstein can be deduced from two fundamental physical postulates.

1) *Principle of relativity* (Galileo, Newton) [1]: “If a system of coordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the *same* laws also hold good in relation to any other system of coordinates K' moving in uniform translation relatively to K .”

2) *Invariance of the speed of light en vacuo* (Einstein) [1]: “... light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body.”

If the relative motion of two systems of coordinates K , and K' is uniform translational, we will call these sets of coordinates: inertial frames of reference. If we are concerning ourselves with only inertial frames of reference; then the definite velocity of light en vacuo c , be measured by such observers to be equivalent, is called: universality of the speed of light.

As we continue we will use observers O and O' , and system of coordinates K and K' interchangeably.

Consider now a photon of light traveling either positively or negatively along the x -axis in *degenerate* complex time. We may define the speed of light

$$c = \left| \frac{\Delta x}{\Re(\zeta)} \right|. \quad (20)$$

Eq. (20) holds true for all *operational definitions* in *degenerate* complex time. This means universality of the speed of light is completely satisfied in *degenerate* complex time for all *operational definitions* and we will show why in §6. Identical logic can be extended to velocity in general (not only to that of light) and so may be applied in a similar fashion to describe the principle of relativity. We will elaborate in the first fundamental postulate in a later section, but for now let the reader be satisfied with the explanation that the first postulate is independent of any *operational definition* of time.

At the moment, we will continue as planned by showing the second fundamental postulate is not restricted to *degenerate* complex time and holds true for two arbitrary observers conditional on *operational definition*.

5 Space-time intervals in complex time

The Minkowski space-time has square interval of the form [2]

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2, \quad (21)$$

in coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, with metric signature $(-, +, +, +)$. By our definition the temporal interval is the magnitude of the complex time vector ζ . Therefore the time interval Δt in terms of the *real* and *imaginary*

operators is given

$$\Delta t = \sqrt{\Re(\zeta)^2 + \Im(\zeta)^2} . \quad (22)$$

We then make such a substitution and have in complex time, space-time interval Eq. (21) takes the form

$$\Delta s^2 = - [\cos^2 \omega + \sin^2 \omega] \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 . \quad (23)$$

6 Light speed measurement experiment in complex time

For two frames of reference O and O' writing square intervals in complex time we have

$$\Delta s^2 = - [\cos^2 \omega + \sin^2 \omega] \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 , \quad (24)$$

and

$$\Delta s'^2 = - [\cos^2 \omega' + \sin^2 \omega'] \Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 , \quad (25)$$

respectively. We will now stipulate these two frames remain in a state of relative rest and have origins which coincide. In a state of relative rest observers O , and O' will agree on the spatial intervals in their respective space-times. The observers O , and O' will also agree on the synchronization of their clocks. But this agreement of time intervals only extends so far as the magnitudes of complex time vectors ζ , and ζ' . That is, they may have non-identical *operational definitions* of time. The space-time intervals may then be written

$$\Delta s^2 = - [\cos^2 \omega + \sin^2 \omega] \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 , \quad (26)$$

and

$$\Delta s'^2 = - [\cos^2 \omega' + \sin^2 \omega'] \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 . \quad (27)$$

Have the observer in frame O measure a light beam traveling along the x -axis in his/her space-time. The square interval of this light beam is measured

$$\Delta s^2 = - [\cos^2 \omega + \sin^2 \omega] \Delta t^2 + \Delta x^2 = 0 , \quad (28)$$

according to O . By construction let us use the common configuration of the Minkowski space-time diagram (x, t) and entertain me by imagining our set up. Space-times for frames O and O' share a common spatial x -axis but have temporal axes rotated by $\Delta\omega$. This means the light-like interval in O has a geometric projection onto the interval

$$\Delta s'^2 = - \cos^2 (\Delta\omega) \Delta t^2 + \Delta x^2 \geq 0 , \quad (29)$$

in the space-time of O' . In *degenerate* complex time, we find O' will measure the interval Eq. (29) to be null and light speed by definition is universal for all inertial observers. Now consider the case of *non-degenerate* complex time for arbitrary *operational definitions*; where in general we find O' will measure the light beam to be traveling at the same speed and obtain a null square interval provided O' includes the *imaginary* part of O time which is normal to the space-time diagram of O' . But this is equivalent to O' adopting the *operational definition* of O . Eq. (29) becomes

$$\Delta s'^2 = - [\cos^2 (\Delta\omega) + \sin^2 (\Delta\omega)] \Delta t^2 + \Delta x^2 = 0 . \quad (30)$$

Given the geometric projection described by Eq. (29) is independent of uniform relative velocity between O and O' ; we have shown agreement on light speed for

inertial frames of reference is conditional on the *relative degeneracy of operational definitions*. Generalizing the definition of light speed to complex time for two observers with arbitrary *operational definitions* we find that each observer will measure the other's light beam to travel at

$$c = \left| \frac{\Delta x}{\Delta t \cos(\Delta\omega)} \right|, \quad (31)$$

using their respective *operational definitions* of time. We notice in *degenerate* complex time it happens to be $\cos(\Delta\omega) = \pm 1$ for all *operational definitions*. It is an immediate consequence of light speed invariance that upon extension of time into the complex plane, inertial frames O and O' will not agree on the value of light's speed when *operational definitions* are *relatively non-degenerate*. Applying a geometrization of units such that $c = 1$ via the usual prescription for some observer O , assuming the *operational definition* of O is *real*

$$c = \left| \frac{\Delta x}{\Re(\zeta)} \right| = 1. \quad (32)$$

But then for an observer O' with arbitrary *operational definition* in *non-degenerate* complex time, the speed of the same light beam is measured

$$c = \left| \frac{\Delta x}{\Re(\zeta')} \right| = \left| \frac{1}{\cos(\Delta\omega)} \right| \geq 1. \quad (33)$$

In the spirit of postulation, let the relativity of *real* and *imaginary* temporal components in complex time be raised to the status of postulate. Furthermore, let this principle be invoked on the basis we suppose there exists no preferred direction in complex time.

7 Generalization of the postulates of special relativity to general complex time

As mentioned in §4, the principle of relativity only superficially involves the velocity of observers. That is, invariance of physical laws under uniform translational motion of two coordinate systems K , and K' is independent of any *operational definition* of time. In the simplest of terms, whether or not observers agree on relative velocity has no influence on the same physical laws applying for each. As we have shown in the case of light, discrepancy of measurement provided by arbitrary *operational definitions* of K , and K' may arise solely as the result of invariant physical laws. For these reasons, extension of time into the complex plane offers no new perspective on the principle of relativity. Except that it may serve as an example of a physical principle which is independent of *operational definitions*.

However, we have found through the rigor of our definitions upon extending time into the complex plane, universality of the speed of light holds true provided all inertial observers involved have *operational definitions* of time which are *relatively degenerate*.

8 Conclusion

Universality of the speed of light, it seems, has guided the general opinion that there exists an *absolute operational definition* of time. And while *degenerate* complex time provides the conditions necessary for light speed universality by our definition, in general complex time this is the trivial case. Here it was shown that complex time may be *non-degenerate* and have light speed universality arise from the *relative degeneracy* of *operational definitions*. This opens the possibility our Universe may most accurately be described in *non-degenerate* complex

time, which is locally *relatively degenerate*. In the context of complex time as we have defined it, physical theories which adopt light speed universality as a fundamental postulate may be categorized either: (1) Having globally *degenerate* complex time. (2) Having globally *non-degenerate* complex time which is locally *relatively degenerate*.

References

- [1] H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity*, edited by A. Sommerfeld, p. 111 and p. 38, (Dover Publications, New York, 1952).
- [2] B. F. Schutz, *A First Course in General Relativity*, p. 9, (Cambridge University Press, New York, 2009).