Stability and Continuity in Robust Optimization

T. C. Y. Chan & P. A. Mar

Version Published


Additional publisher information The final version of this article is also available from the Society for Industrial and Applied Mathematics at https://doi.org/10.1137/16M1067512

Copyright/License © 2017, Society for Industrial and Applied Mathematics

How to cite TSpace items

Always cite the published version, so the author(s) will receive recognition through services that track citation counts, e.g. Scopus. If you need to cite the page number of the author manuscript from TSpace because you cannot access the published version, then cite the TSpace version in addition to the published version using the permanent URI (handle) found on the record page.

This article was made openly accessible by U of T Faculty. Please tell us how this access benefits you. Your story matters.
STABILITY AND CONTINUITY IN ROBUST OPTIMIZATION

TIMOTHY C.Y. CHAN† AND PHILIP ALLEN MAR†

Abstract. We consider the stability of robust optimization (RO) problems with respect to perturbations in their uncertainty sets. In particular, we focus on robust linear optimization problems, including those with an infinite number of constraints, and consider uncertainty in both the cost function and constraints. We prove Lipschitz continuity of the optimal value and \( \epsilon \)-approximate optimal solution set with respect to the Hausdorff distance between uncertainty sets. The Lipschitz constant can be calculated from the problem data. In addition, we prove closedness and upper semicontinuity for the optimal solution set with respect to the uncertainty set. In order to prove these results, we develop a novel transformation that maps RO problems to linear semi-infinite optimization (LSIO) problems in such a way that the distance between uncertainty sets of two RO problems correspond to a measure of distance between their equivalent LSIO problems. Using this isometry we leverage LSIO and variational analysis stability results to obtain stability results for RO problems.

Key words. robust optimization, linear semi-infinite optimization, stability, continuity, Lipschitz continuity, optimal value, optimal solution set, \( \epsilon \)-approximate optimal solution set

AMS subject classifications. 90C05, 90C31, 90C34

DOI. 10.1137/16M1067512

1. Introduction. The theory and applications of robust optimization (RO) have grown rapidly over the past two decades. However, with only a few notable exceptions, stability of RO problems has been largely unexplored. The stability of the optimal value and the optimal solution set, with respect to certain problem parameters, are examples of the types of stability that are of particular interest in optimization problems in general. For RO problems, one natural quantity with respect to which stability can be defined is the uncertainty set, which is our focus in this paper.

We first make a distinction between what we call quantitative and qualitative stability. Quantitative stability properties are those that are endowed with an explicit constant that describes the degree of stability. One such example is the Lipschitz continuity of the optimal value with some constant \( L \), which bounds the change in the optimal value by the product of \( L \) and some amount of perturbation in the parameters. On the other hand, qualitative stability properties are those that do not furnish an explicit constant measuring the degree of stability. One example is the upper semicontinuity of the optimal solution set, which describes the limiting behavior of the optimal solution set without providing any information about the rate of convergence.

In this paper, we present the first comprehensive study of stability of robust linear optimization problems with respect to the uncertainty set. We provide quantitative stability results for the optimal value and the \( \epsilon \)-approximate optimal solution set (hereafter abbreviated as the \( \epsilon \)-optimal solution set), and qualitative stability results for the optimal solution set. Our results hold for robust linear optimization problems with uncertainty in both the cost function and constraints. Our results also extend to robust linear semi-infinite optimization (LSIO) problems. In addition to minor technical assumptions, the main assumptions we make about the underlying RO problem

*Received by the editors March 24, 2016; accepted for publication (in revised form) March 20, 2017; published electronically April 27, 2017.

http://www.siam.org/journals/siopt/27-2/M106751.html

Funding: This research was funded by an Ontario Graduate Scholarship.

†University of Toronto, Toronto, ON M5S 3G8, Canada (tcychan@mie.utoronto.ca, philip.mar@mail.utoronto.ca).
are that its uncertainty set is convex and compact, that the strong Slater condition holds, and that its optimal solution set is nonempty and bounded. The specific contributions of this paper are the following:

1. **Lipschitz continuity of the optimal value.** This is the first quantitative stability result for the optimal value in RO with uncertainty in the constraints, providing an explicit Lipschitz constant that can be calculated from the problem parameters. The quantitative nature of this result contrasts with the qualitative results for the optimal value in previous papers [10, 29, 28].

2. **Closedness and upper semicontinuity of the optimal solution set.** Under the above outlined assumptions, our results relax restrictions such as the type of uncertainty set considered [10, 21] and the specific form or limit of its convergence [10, 29] that are found in previous papers. Our results hold for both uncertainty in the constraint set and cost function, in contrast to previous papers [21, 28] which only consider the cost function uncertainty.

3. **Lipschitz continuity of the $\epsilon$-optimal solution set.** This is the first quantitative stability result for the $\epsilon$-optimal solution set in RO, for which not even qualitative stability results have been established yet. Our proof leverages a similar result from variational analysis [23].

From an application viewpoint, stability of RO problems is essential in ensuring that the computational implementation of RO will not be overly sensitive to possible round-off errors, as alluded to in [12, 18]. Our stability results guarantee that the change in the optimal value is bounded linearly given mild perturbations in the uncertainty set, and hence may act as a sanity check to ensure that an RO problem, for which we have a robust reformulation, yields the true optimal value, by checking RO problems with similar uncertainty sets (of which we might know the optimal values exactly), and ensuring that the original problem’s optimal value is similar to these nearby problems’ optimal values.

The rest of the paper is organized as follows. In section 2, we summarize the relevant literature concerning stability. In section 3, we review relevant background and introduce our notation. The main stability results we prove require a mapping between RO and LSIO problems, which we develop in section 4. Finally, we prove the desired RO stability results: Lipschitz continuity of the optimal value in section 5, set convergence of the optimal solution set in section 6, and Lipschitz continuity of the $\epsilon$-optimal solution set in section 7. The appendix includes technical results with proofs, some of which were implicitly used in other papers [9].


Several recent papers have proved convergence properties of RO problems with respect to changes in the uncertainty set. It was shown that the optimal solution set (and value) of the robust problem converged to the optimal solution set (and value) of the nominal problem as a polyhedral uncertainty set converged to a singleton in robust linear programming [10]. An analogous result was shown for the case of general
convex uncertainty sets in robust conic programming [29]. These results have been extended to the case where a regularized convex uncertainty set converges to a nonsingleton set in robust quadratic programming [21]. Most recently, there has been some research [2] which approximates robust mixed-integer optimization problems (MIP) having ellipsoidal uncertainty sets with robust MIPs having polyhedral uncertainty sets.

Stability has been well studied in the related field of stochastic programming (see Schultz [26] for an extensive literature review). In particular, there exist results regarding Lipschitz continuity of the optimal value [19], of the optimal solutions [24, 27, 19] and approximate optimal solutions [25] with respect to the probability measure. Additionally, recent research in distributionally robust stochastic programming (DRSP) or distributionally RO (DRO) stability include convergence of optimal values (and optimal solution sets) as the ambiguity sets converge, focusing primarily on uncertainty in the cost function [28, 30, 13]. Quantitative convergence of the optimal value has also been established for a two-stage DRSP problem, including uncertainty in the constraints [22].

Stability has also been studied in LSIO problems [3]. C´anovas et al. [9] showed Lipschitz continuity of the optimal value while C´anovas et al. [5] showed convergence of the optimal solution set. Even though RO problems can be viewed as LSIO problems, the above-mentioned properties do not directly yield stability properties for RO due to the difference in their problem structures. In these LSIO results, the LSIO problems have constraints that are indexed relative to a predefined and common index set. On the other hand, RO problems have constraints that are typically defined by an uncertainty set, which may not constitute a common index set with other RO problems. However, as we will show, stability properties for RO can be established by taking advantage of the lack of explicit indexing in RO constraints. In order to obtain stability results for RO problems, we develop a method to transform RO problems to equivalent LSIO problems in such a way that distances between uncertainty sets of these RO problems correspond to distances of their equivalent LSIO problems. Finally, we obtain stability properties for these RO problems by applying LSIO results to their equivalent LSIO problems.

3. Preliminaries.

3.1. Notation. We write \(\text{int}(A), \text{bd}(A), \text{cl}(A),\) and \(\text{ext}(A)\) to denote the interior, boundary, closure, and exterior of a set \(A\). The appropriate topology or metric space for these definitions will be clear in context. We define \(\text{conv}(A)\) and \(\text{cone}(A)\) as the convex hull and conical hull of \(A\), respectively. Given a metric \(d\) on a metric space \(X\), we define the distance between a point \(x\) and a set \(A \subseteq X\) as \(d(x, A) := \inf_{y \in A} d(x, y)\). If \(X\) is additionally a normed vector space with norm \(\| \cdot \|\) and metric \(d\) is associated with the norm \(\| \cdot \|\), we can also analogously denote the metric \(d_*\) to be associated with the dual norm of \(\| \cdot \|\), as defined in [9]. Also, let \(\chi_A(x)\) denote the characteristic function with \(\chi_A(x) = 0\) if \(x \in A\) and \(\chi_A(x) = +\infty\) otherwise. Let \(B(x_0, \epsilon)\) denote the ball in the appropriate metric space of radius \(\epsilon\) centered at \(x_0\). Let \(B := B(0, 1)\) denote the unit ball. Define \((A)_r := A \cap rB\). For the convenience of the reader, Appendix A includes a helpful reference table of notation used throughout the paper.

3.2. LSIO problems. We now present the LSIO concepts that are most relevant to our development. Much of the concepts and notation are adapted from [9] and related papers [6, 8, 5, 20].
3.2.1. Formulation and related quantities. An LISO problem takes the form

\[
\inf_{x \in \mathbb{R}^n} \langle c, x \rangle,
\]

subject to \(\langle a_i, x \rangle \geq b_i \quad \forall t \in T,\)

where the \(\langle \cdot, \cdot \rangle\) denotes the dot product in \(\mathbb{R}^n\). The set \(T\) is possibly uncountably infinite, with no assumed topological structure. It is helpful to think of the coefficients and right-hand sides as functions, namely, \(a : T \to \mathbb{R}^n, t \mapsto a_t\) and \(b : T \to \mathbb{R}, t \mapsto b_t\).

We denote \(\pi := (c, \sigma)\) as the tuple that uniquely defines an LISO problem of the form (3.1). Further, we write \(\pi := (a_t, b_t)_{t \in T} = (c, \sigma)\). Different LISO problems will be indexed by \(j\) and written \(\pi_j := (c^j, (a^j_t, b^j_t)_{t \in T}) = (c^j, \sigma^j)\). We denote \(\Pi := \mathbb{R}^n \times \Sigma\) as the collection of all LISO problems, where \(\Sigma := (\mathbb{R}^n \times \mathbb{R})^T\) is the collection of all constraint systems (both feasible and infeasible). A problem \(\pi = (c, \sigma)\) has a feasible solution set (hereafter shortened to feasible set) denoted \(F(\sigma)\), an optimal value denoted \(\nu(\pi)\), an optimal solution set denoted \(F^{\text{opt}}(\pi)\), and, for some given \(\epsilon > 0\), an \(\epsilon\)-optimal solution set denoted \(F^{\epsilon\text{-opt}}(\pi) := \{x \in F(\sigma) : \langle c, x \rangle \leq \nu(\pi) + \epsilon\}\). For completeness, define \(\nu(\pi) = +\infty\) if \(F(\sigma) = \emptyset\). Last, LISO problems of the form (3.1) are said to satisfy the strong Slater condition with strong Slater constant \(\rho\) if there exists \(x_0 \in \mathbb{R}^n\) such that \(\langle a_t, x_0 \rangle \geq b_t + \rho\) for all \(t \in T\).

3.2.2. Distance metrics. Let \(\| \cdot \|\) be the Euclidean norm in the context-appropriate dimension. Let two constraint systems \(\sigma_1\) and \(\sigma_2\) be given. For a fixed \(t\), let \((a^1_t, b^1_t) \in \mathbb{R}^{n+1}\) and \((a^2_t, b^2_t) \in \mathbb{R}^{n+1}\) be the \(t\)th constraint of \(\sigma_1\) and \(\sigma_2\), respectively. Let \(d((a^1_t, b^1_t), (a^2_t, b^2_t)) := \| (a^1_t - a^2_t, b^1_t - b^2_t) \|\) be the distance between these constraints. Let \(\delta^\Sigma(\sigma_1, \sigma_2) := \sup_{t \in T} d((a^1_t, b^1_t), (a^2_t, b^2_t))\) be the distance between constraint systems \(\sigma_1\) and \(\sigma_2\). Also, let \(\delta^\Pi(\pi_1, \pi_2) := \max\{\| c^1 - c^2 \|, \delta^\Sigma(\sigma_1, \sigma_2)\}\) be the distance between problems \(\pi_1 := (c^1, \sigma_1)\) and \(\pi_2 := (c^2, \sigma_2)\).

3.2.3. Equivalence relations. We define equivalence relations between problems based on their constraint systems and their cost function vectors. Let \(\sigma_1 \sim^\Sigma \sigma_2\) (\("\Sigma\)-equivalence\)) denote an equivalence between the constraint systems of problems \(\pi_1 = (c^1, \sigma_1) = (c^1, (a^1_t, b^1_t)_{t \in T})\) and \(\pi_2 = (c^2, \sigma_2) = (c^2, (a^2_t, b^2_t)_{t \in T})\) when \(\{(a^1_t, b^1_t), (a^2_t, b^2_t), t \in T\} = \{(a^1_t, b^1_t), t \in T\}\). Under this equivalence, problems \(\pi_1\) and \(\pi_2\) are equivalent up to an ordering of the indices and redundancy in the constraints, which is a stronger condition than the feasible sets being the same. Let \(\pi_1 \sim^\Pi \pi_2\) (\("\Pi\)-equivalence\)) denote an equivalence between \(\pi_1\) and \(\pi_2\) when \(c^1 = c^2\) in addition to \(\sigma_1 \sim^\Sigma \sigma_2\). Note that \(\Pi\)-equivalence implies that \(\pi_1\) and \(\pi_2\) share the same optimal value, feasible set, optimal solution set, and \(\epsilon\)-optimal solution set.

3.2.4. Why indexing matters in LSIO problems. The following example illustrates an important subtlety when using the definitions of distance metrics and equivalence relations. The choice of indexing in LSIO problems affects the \(\delta^\Sigma\)-distance between them, as illustrated in the following example.

Example 3.1. Let \(\sigma_1 = (a^1, b^1)\) and \(\sigma_2 = (a^2, b^2)\) be \(\Sigma\)-equivalent constraint systems, with \(T = \{1, 2\}\), and \(a^1 := (a^1_1, a^1_2)\) and \(a^2 := (a^2_1, a^2_2)\) with \(a^1_1 = a^2_2 = (1, 0)\), \(a^1_2 = a^2_1 = (0, 1)\), \(b^1_t = b^2_t = 0 \forall t \in T\), then \(\delta^\Sigma(\sigma^1, \sigma^2) = 0\). If, however, we swap the indexing of two of the \(a^\Sigma\)-vectors, that is, \(a^1_1 = a^2_2 = (1, 0)\), \(a^1_2 = a^2_1 = (0, 1)\), \(b^1_t = b^2_t = 0 \forall t \in T\), then \(\delta^\Sigma(\sigma^1, \sigma^2) = \sqrt{2}\). Rearranging the indices generates a different \(\delta^\Sigma\)-distance, even though these constraint systems are \(\Sigma\)-equivalent.
3.2.5. Classifying LSIO problems. Let $\Sigma_I := \{ \sigma \in \Sigma : F(\sigma) \neq \varnothing \}$ denote the set of feasible constraint systems and $\Sigma_i := \Sigma \setminus \Sigma_I$ denote the set of infeasible constraint systems. Let $\Pi_I := \{ \pi \in \Pi : \sigma \in \Sigma_I \}$ denote the set of feasible problems and $\Pi_i := \Pi \setminus \Pi_I$ denote the set of infeasible problems. Furthermore, let $\Pi_s := \{ \pi \in \Pi : F^{\text{opt}}(\pi) \neq \varnothing \}$ denote the set of solvable problems. Finally, we define $\Sigma_\infty := \{ \sigma \in \Sigma : \delta^2(\sigma, bd(\Sigma_I)) = +\infty \}$ and $\Pi_\infty := \{ \pi \in \Pi : \delta^\Pi(\pi, bd(\Pi_I)) = +\infty \}$, denoting the set of infeasible constraint systems/problems that are “infinitely” far away from being feasible, so that changing a finite number of constraints of such a constraint system/problem will not make it feasible [6]. Useful facts about these sets of LSIO problems are stated and proved in Appendix B.

3.3. Robust optimization. Suppose $c$ is the cost function, $a$ is the coefficient vector, and $b$ is the right-hand side value. For now, suppose $c$ and $b$ are fixed, but $a$ is uncertain and lies in an uncertainty set $U \subseteq \mathbb{R}^n$, which is a nonempty, compact, and convex subset of $\mathbb{R}^n$. Then the single-constraint RO problem is

$$\inf_{x \in \mathbb{R}^n} \langle c, x \rangle$$

subject to $\langle a, x \rangle \geq b \quad \forall a \in U.$

We write $\text{RO}(U)$ to denote the single-constraint RO problem (3.2) which can be uniquely characterized by its parameters $(c, U, b)$. To avoid defining new notation, we denote the feasible set, optimal value, optimal solution set, and $\epsilon$-optimal solution set of RO problems in the same way as in the LSIO case: $F(\text{RO}(U))$, $\nu(\text{RO}(U))$, $F^{\text{opt}}(\text{RO}(U))$, and $F^{\epsilon-\text{opt}}(\text{RO}(U))$, respectively; the context will make it clear whether we are talking about LSIO or RO problems. Note that we write $F(\text{RO}(U))$ as the feasible set of an RO problem, without making a distinction between a “constraint system” $(U, b)$ and an RO “problem” $(c, U, b)$. The Hausdorff distance can be used to measure the distance between sets $U$ and $V$:

$$d_H(U, V) := \max \left\{ \sup_{u \in U} \| u - v \|, \sup_{v \in V} \inf_{u \in U} \| u - v \| \right\}.$$

Now we generalize formulation (3.2) to the multiple-constraint case, as well as the case of uncertainty in both the cost function and right-hand side values. Suppose $c$ is an uncertain cost function and lies in an uncertainty set $C \subseteq \mathbb{R}^n$. Suppose $I$ is an index set that is possibly uncountable. For each $\alpha \in I$, suppose $(a^\alpha, b^\alpha)$ is an uncertain constraint that lies in a nonempty, compact, and convex uncertainty set $U_\alpha \subseteq \mathbb{R}^{n+1}$. Each constraint, then, has an uncertainty set that is independent of the uncertainty sets in the other constraints. The multiple-constraint RO problem is

$$\inf_{x \in \mathbb{R}^n} \sup_{c \in C} \langle c, x \rangle$$

subject to $\langle a^\alpha, x \rangle \geq b^\alpha \quad \forall (a^\alpha, b^\alpha) \in U_\alpha, \alpha \in I,$

We can define the constraintwise uncertainty set

$$\hat{U} := \prod_{\alpha \in I} U_\alpha := \left\{ f : I \to \bigcup_{\alpha \in I} U_\alpha : f(\alpha) \in U_\alpha \right\}$$

as defined in [15]. Intuitively, for the cases when $I$ is finite, $\hat{U}$ can simply be thought
of as the Cartesian product between the sets $U_{\alpha}$. For example, if $U_1$ and $U_2$ are the uncertainty sets of some robust problem with only two constraints, then $\hat{U} := U_1 \times U_2 := \{(a, b) : a \in U_1, b \in U_2\}$. The formal definition of $\hat{U}$ in (3.4) is simply for rigor.

Without loss of generality, our stability results hold for the case without cost function uncertainty. As in the LSIO case, an RO problem satisfies the strong Slater condition with strong Slater constant $\rho$ if there exists $x_0 \in \mathbb{R}^n$ and $\rho > 0$ such that $\langle a^\alpha, x_0 \rangle \geq b^\alpha + \rho$ for all $(a^\alpha, b^\alpha) \in U_{\alpha}, \alpha \in I$. We can therefore assume, without loss of generality, that the uncertainty is only in the constraints and characterized completely by $\hat{U}$, with fixed cost function $c$. In other words, we need only examine problems of the form

$$\inf_{x \in \mathbb{R}^n} \langle c, x \rangle$$

subject to $\langle a^\alpha, x \rangle \geq b^\alpha, (a^\alpha, b^\alpha) \in U_{\alpha} \forall \alpha \in I,$

obtained from the formulation (3.3) through an epigraph reformulation. Also, if (3.3) satisfies the strong Slater condition, then (3.5) does as well, with the same Slater constant.

We write $\text{RO}(\hat{U})$ to denote the single-constraint RO problem (3.2), which can be uniquely characterized by its uncertainty set $\hat{U}$. When $I$ is uncountably infinite, formulation (3.5) is actually a robust LSIO problem. Thus, beyond simply generalizing the single-constraint problem, the results we derive for robust linear optimization using formulation (3.5) extend to robust LSIO problems by definition [17]. The quantities $F(\text{RO}(\hat{U})), \nu(\text{RO}(\hat{U})), F_{\text{opt}}(\text{RO}(\hat{U})), \text{and } F_{\epsilon_{\text{opt}}}(\text{RO}(\hat{U}))$ are defined analogously to the single-constraint case. Given two constraintwise uncertainty sets $\hat{U}$ and $\hat{V}$, we define a distance metric

$$d_{\hat{U}}(\hat{U}, \hat{V}) := \sup_{\alpha \in I} d_H(U_{\alpha}, V_{\alpha})$$

as defined on the metric space of constraintwise uncertainty sets where the uncertainty set on each constraint is a nonempty, compact, and convex subset of $\mathbb{R}^{n+1}$.

3.4. Key results from literature. Our main results parallel and leverage the following results on stability of LSIO and general convex optimization problems. First we provide a few key definitions.

**Definition 3.2 (from [9]).** Define the following set mappings for $\pi := (c, \sigma) = (c, (a_t, b_t)_{t \in T})$:

$$A(\sigma) := \text{conv}(\{a_t, t \in T\}),$$

$$R(\pi) := \{-b_t, t \in T; \nu(\pi)\},$$

$$Z^- (\pi) := \text{conv}(\{a_t, t \in T; -c\}),$$

$$H(\sigma) := \text{conv}\left(\left\{\left(\begin{array}{c} a_t \\ b_t \end{array}\right), t \in T\right\}\right) + \left\{\left(\begin{array}{c} 0_{n-x} \\ -\mu \end{array}\right), \mu \geq 0\right\}.$$
DEFINITION 3.3 (from [9]). For any $\pi_0 := (c^0, \sigma_0) \in \text{int}(\Pi_s)$, define the following quantities for $0 < \epsilon < \delta^\Pi(\pi_0, \text{bd}(\Pi_s))$:

\[
\begin{align*}
\varphi(\lambda) &= \varphi_*(\lambda) := \sqrt{1 + \lambda^2}, \\
\psi(\alpha) &= (1 + \alpha)\sqrt{1 + \alpha^2}, \\
\rho(\pi_0) &= \sup R(\pi_0) \\
\beta(\pi_0, \epsilon) &= \frac{\psi(\rho(\pi_0))}{\delta^{\Sigma}(\sigma_0, \Sigma_i) - \epsilon}, \\
\gamma(\pi_0, \epsilon) &= \varphi_*(0)(\rho(\pi_0) + \epsilon \beta(\pi_0, \epsilon)) + \|c^0\| \beta(\pi_0, \epsilon), \\
\mu(\pi_0, \epsilon) &= \frac{\psi(0)}{\sup R(\pi_0) + \epsilon} \max\{1, \gamma(\pi_0, \epsilon)\}, \\
L(\pi_0, \epsilon) &= \varphi_*(0) \left((\epsilon + \|c^0\|) \frac{\psi(\mu(\pi_0, \epsilon))}{\delta^{\Sigma}(\sigma_0, \Sigma_i) - \epsilon} + \mu(\pi_0, \epsilon)\right).
\end{align*}
\]

THEOREM 3.4 (from [9, Theorem 4.3]). Let $\pi_0 := (c^0, \sigma_0) \in \text{int}(\Pi_s)$, and let $0 \leq \epsilon < \delta^\Pi(\pi_0, \text{bd}(\Pi_s))$. If $\pi_1 = (c^1, \sigma_1)$ and $\pi_2 = (c^2, \sigma_2)$ are problems in $\Pi$ satisfying $\delta^\Pi(\pi_j, \pi_0) \leq \epsilon$ for $j = 1, 2$, then

\begin{equation}
|\nu(\pi_1) - \nu(\pi_2)| \leq L(\pi_0, \epsilon)\delta^\Pi(\pi_1, \pi_2),
\end{equation}

where

\begin{equation}
L(\pi_0, \epsilon) := \varphi_*(0) \left((\epsilon + \|c^0\|) \frac{\psi(\mu(\pi_0, \epsilon))}{\delta^{\Sigma}(\sigma_0, \Sigma_i) - \epsilon} + \mu(\pi_0, \epsilon)\right)
\end{equation}

with functions $\varphi_*(\cdot)$ and $\psi(\cdot)$, and constant $\mu(\cdot, \epsilon)$ as defined in Definition 3.3.

The above theorem establishes Lipschitz continuity of the optimal value for LSIO problems: the difference in the optimal value between two sufficiently close LSIO problems is bounded linearly by their distance in the $\delta^\Pi$-metric. Note that this is a local Lipschitz continuity result because the problems under consideration, $\pi_1$ and $\pi_2$, must reside in a neighborhood of a third problem, $\pi_0$.

Note that $F(\cdot)$ can be defined as a set mapping from the set of constraint systems $\Sigma$ to the power set of $\mathbb{R}^n$. Similarly, $\nu(\cdot)$, $F^\text{opt}(\cdot)$, and $F^\text{opt^*}(\cdot)$ can be defined as set mappings from the set of problems $\Pi$ to the set of (extended) real numbers, in the case of $\nu(\cdot)$, and the power set $\mathbb{R}^n$, in the case of $F^\text{opt}(\cdot)$ and $F^\text{opt^*}(\cdot)$.

THEOREM 3.5 (from [16, Theorem 3.1(i), (ii), and (vi)]). If $\pi := (c, \sigma) \in \Pi_f$, then the following are equivalent:

i. $F(\cdot)$ is lower semicontinuous at $\sigma$.
ii. $\pi \in \text{int}(\Pi_f)$.
iii. $\sigma$ satisfies the strong Slater condition.\footnote{In this paper, we will sometimes say that $\pi := (c, \sigma)$ satisfies the strong Slater condition, by which we mean, more precisely, that $\sigma$ satisfies the strong Slater condition.}

THEOREM 3.6 (from [5, Theorem 5.1(i) and (ii)]). If $\pi := (c, \sigma) \in \Pi_s$, then

i. $F^\text{opt}(\cdot)$ is closed at $\pi$ if and only if either $F(\cdot)$ is lower semicontinuous at $\pi$ or $F(\sigma) = F^\text{opt}(\pi)$;
ii. if $F^\text{opt}(\cdot)$ is upper semicontinuous at $\pi$, then $F^\text{opt}(\cdot)$ is closed at $\pi$. The converse is true if $F^\text{opt}(\pi)$ is bounded.
These two theorems are largely technical theorems that relate the different notions of qualitative stability in LSIO problems, including closedness, lower semicontinuity, and upper semicontinuity.

**Theorem 3.7** (from [23, Theorem 7.69]). Let $f$ and $g$ be proper, lower semicontinuous, convex functions on $\mathbb{R}^n$ with $\text{argmin} f$ and $\text{argmax} g$ nonempty. Let $r_0$ be large enough that $\text{argmin} f \cap r_0 B \neq \emptyset$ and $\text{argmax} g \cap r_0 B \neq \emptyset$, and also $\min f \geq -r_0$ and $\min g \geq -r_0$, where $B$ is the unit ball. Then, with $r > r_0$, $\epsilon > 0$, and $\eta = \frac{2r}{\eta + \epsilon/2}$,

\[
\hat{d}_r(\epsilon, \text{argmin} f, \epsilon, \text{argmax} g) \leq \eta \left(1 + \frac{2r}{\eta + \epsilon/2}\right) \leq (1 + 4r\epsilon^{-1}) \hat{\delta}_r^\ast(f, g),
\]

where $\hat{d}_r$ is defined in Definition 7.1 and $\hat{\delta}_r^\ast$ is defined in Definition 7.2.

This theorem establishes Lipschitz continuity of the $\epsilon$-optimal set of minimizers for proper, lower semicontinuous, convex functions. In turn, this result applies to LSIO problems because, as we shall see in section 7, an LSIO problem can be equivalently represented as the minimization of a proper, lower semicontinuous, convex function.

### 4. RO-LSIO Transformation

It is well known that RO problems can be considered as LSIO problems. However, LSIO problems are defined with respect to some index set $T$, while RO problems are defined without reference to any index set. In addition, for a given RO problem and for any choice of $T$ of appropriate cardinality, there are an infinite number of $T$-equivalent LSIO problems, defined with respect to $T$, that are equivalent to the RO problem. Thus, the challenge when comparing different RO problems (for the purpose of evaluating stability or continuity with respect to perturbations in the uncertainty set) is twofold. First, we must define a common index set $T$ so that for any two given RO problems, their respective LSIO problems generated with respect to $T$ have a well-defined $\delta$-distance. Second, using this $T$, we must define an RO-LSIO transformation that is an isometry between the space of RO problem uncertainty sets and the space of LSIO problems. That is, RO problems whose uncertainty sets are close in Hausdorff distance are mapped into LSIO problems that are close in the $\delta$-metric. The following example illustrates these challenges.

**Example 4.1.** Consider Figures 1(a) and 1(b), depicting two polyhedral uncertainty sets, $U$ and $V$. Since $U$ and $V$ are the same in both figures, $d_H(U, V)$ is the same as well. For polyhedral uncertainty sets, it suffices to index the vertices to transform an RO problem into an LSIO problem. Consider the common index set $T = \{1, \ldots, 6\}$. In Figures 1(a) and 1(b) the vertices of $V$ are indexed differently. Denote the LSIO problem generated by the indexing for $U$ by $\pi_U$, and the LSIO problems generated by indexing for $V$ as in Figures 1(a) and 1(b) by $\pi_V$ and $\pi'_V$, respectively. Thus, $\delta_\Pi(\pi_U, \pi'_V) > \delta_\Pi(\pi_U, \pi_V)$, even though $\pi_V \sim_\Pi \pi'_V$.

#### 4.1. Transformation theorem: Single-constraint case

To index all elements from general uncertainty sets $U$ and $V$, which are uncountable subsets of $\mathbb{R}^n$, we choose the index set $T$ to be $\mathbb{R}^n$ itself. The explicit RO-LSIO transformation for $\text{RO}(U)$ and $\text{RO}(V)$ that we formulate yields LSIO problems whose distance in the $\delta$-metric is precisely equal to $d_H(U, V)$.

**Theorem 4.2** (single-constraint case). Let $\text{RO}(U)$ and $\text{RO}(V)$ be RO problems of the form (3.2) with nonempty compact and convex uncertainty sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$, cost function $c$, and right-hand side value $b$. Let $\rho > 0$ be given. Define $T := \mathbb{R}^n$ as the common index set, with elements denoted by $t$. Define the LSIO...
problems $\pi_{U;V} := (c, \sigma_{U;V})$ and $\pi_{V;U} := (c, \sigma_{V;U})$ as follows:

$$
\sigma_{U;V}(t) := \begin{cases} 
(t, b) & \text{if } t \in U, \\
(\arg\min_{u \in U} \|u - t\|, b) & \text{if } t \in V \setminus U, \\
(0_n, -\rho) & \text{if } t \notin U \cup V,
\end{cases}
$$

$$
\sigma_{V;U}(t) := \begin{cases} 
(t, b) & \text{if } t \in V, \\
(\arg\min_{v \in V} \|v - t\|, b) & \text{if } t \in U \setminus V, \\
(0_n, -\rho) & \text{if } t \notin U \cup V.
\end{cases}
$$

Then $\pi_{U;V}$ and $\pi_{V;U}$ are well-defined and are equivalent to $\text{RO}(U)$ and $\text{RO}(V)$, respectively. Last, $\delta^H(\pi_{U;V}, \pi_{V;U}) = d_H(U, V)$.

Remark 4.3. For notational consistency with later theorems, we have written the indexing vector $t$ without boldface in this theorem. Both here and in later theorems, we write $\arg\min$ to refer to the unique element, rather than the set containing that unique element, that minimizes the function in question.

Proof. We first prove that $\pi_{U;V}$ is a well-defined LSIO problem and is equivalent to $\text{RO}(U)$. We omit the analogous proof for $\pi_{V;U}$ and $\text{RO}(V)$. Finally, we prove that $\delta^H(\pi_{U;V}, \pi_{V;U}) = d_H(U, V)$.

To show that $\pi_{U;V}$ is well-defined, we must show that $\sigma_{U;V}$ maps every element in $T := \mathbb{R}^n$ to exactly one constraint (i.e., vector in $\mathbb{R}^{n+1}$). When $t \in U$ or $t \notin U \cup V$, it is clear that $t$ maps to exactly one constraint, either $(t, b)$ or $(0_n, -\rho)$, respectively. When $t \in V \setminus U$, the existence and uniqueness of $\arg\min_{u \in U} \|u - t\|$ follows from $U$ being compact and convex, and $\| \cdot \|$ being the Euclidean norm, which becomes a strictly convex function when squared.

To show that $\pi_{U;V}$ is equivalent to $\text{RO}(U)$, it suffices to show that $\pi_{U;V}$ has exactly the same feasible set and cost function as $\text{RO}(U)$. First, if $t \in U$, then $\sigma_{U;V}(t) := (t, b)$, so every constraint in $\text{RO}(U)$ is listed in $\pi_{U;V}$. Second, since $\arg\min_{u \in U} \|u - t\| \in U$, if $t \in V \setminus U$, then $\sigma_{U;V}(t) := (\arg\min_{u \in U} \|u - t\|, b)$ is a redundant constraint. Last, if $t \notin U \cup V$, then $\sigma_{U;V}(t) := (0_n, -\rho)$, which is a trivial constraint. Thus, every constraint in $\text{RO}(U)$ is included in $\pi_{U;V}$ and all the other constraints of $\pi_{U;V}$ do not constrain the feasible set any further. Thus, $\pi_{U;V}$ and
RO(U) have the same feasible set, and since the cost function is also the same, the optimal value and optimal solution set of π_{U;V} and RO(U) are the same.

Last, we show that δ^H(π_{U;V}, π_{V;U}) = d_H(U, V). Observe that for every t ∈ T,

\begin{equation}
\|\sigma_{U;V}(t) - \sigma_{V;U}(t)\| = \begin{cases} 
0 & \text{if } t \in U \cap V, \\
\min_{v \in V} \|v - t\| & \text{if } t \in U \setminus V, \\
\min_{u \in U} \|u - t\| & \text{if } t \in V \setminus U, \\
0 & \text{if } t \notin U \cup V,
\end{cases}
\end{equation}

where the minimum when t ∈ U \setminus V follows because \|t - \text{argmin}_{v \in V} \|v - t\|\| = \min_{v \in V} \|v - t\| and, analogously, when t ∈ V \setminus U. Taking the supremum over t ∈ T,

\begin{equation}
\sup_{t \in T} \|\sigma_{U;V}(t) - \sigma_{V;U}(t)\| = \max \left\{ \sup_{t \in U \cap V} 0, \sup_{t \in U \setminus V} \min_{v \in V} \|v - t\|, \sup_{t \in V \setminus U} \min_{u \in U} \|u - t\|, \sup_{t \notin U \cup V} 0 \right\}.
\end{equation}

By definition, \sup_{t \in T} \|\sigma_{U;V}(t) - \sigma_{V;U}(t)\| = \delta^2(\sigma_{U;V}, \sigma_{V;U}). Since the cost function is the same for both π_{U;V} and π_{V;U}, it follows that \sup_{t \in T} \|\sigma_{U;V}(t) - \sigma_{V;U}(t)\| = \delta^H(\sigma_{U;V}, \sigma_{V;U}). Note that \sup_{t \in U \setminus V} \min_{v \in V} \|v - t\| = \sup_{t \in U} \min_{v \in V} \|v - t\| because if t ∈ V, then \min_{v \in V} \|v - t\| = 0. Analogously, \sup_{t \in V \setminus U} \min_{u \in U} \|u - t\| = \sup_{t \in V} \min_{u \in U} \|u - t\|. Finally, replacing the “min” with “inf” in (4.2),

\begin{equation}
\delta^H(\pi_{U;V}, \pi_{V;U}) = \sup_{t \in T} \|\sigma_{U;V}(t) - \sigma_{V;U}(t)\| = \delta^2(\sigma_{U;V}, \sigma_{V;U}) = \sup_{t \in U \cap V} 0, \sup_{t \in U \setminus V} \min_{v \in V} \|v - t\|, \sup_{t \in V \setminus U} \min_{u \in U} \|u - t\|, \sup_{t \notin U \cup V} 0 \right\}.
\end{equation}

Figure 2 gives an illustration of how π_{U;V} is defined in the proof.

4.2. Transformation theorem: Multiple-constraint case. To generalize Theorem 4.2, we consider the case of robust problems with multiple constraints under uncertainty. We also generalize to the case of uncertainty in the right-hand side as well as the cost function.
Theorem 4.4 (multiple-constraint case). Let \( \text{RO}(\hat{U}) \) and \( \text{RO}(\hat{V}) \) be RO problems with uncertainty sets \( \hat{U} \) and \( \hat{V} \) with compact and convex \( U_\alpha \subseteq \mathbb{R}^n \) and \( V_\alpha \subseteq \mathbb{R}^n \) for all \( \alpha \in I \) for some arbitrary index set \( I \). Let \( \rho > 0 \) be given. Define \( T := I \times \mathbb{R}^n \times \mathbb{R} \) as the common index set, with elements denoted by \( t \in T \) which are tuples \( t := (\alpha, t, s) \), where \( \alpha \in I \), \( t \in \mathbb{R}^n \), and \( s \in \mathbb{R} \). Define the LSIO problems \( \pi_{\hat{U};\hat{V}} := (c, \sigma_{\hat{U};\hat{V}}) \) and \( \pi_{\hat{V};\hat{U}} := (c, \sigma_{\hat{V};\hat{U}}) \) as follows:

\[
\sigma_{\hat{U};\hat{V}}(t) = \sigma_{\hat{V};\hat{U}}((\alpha, t, s)) = \begin{cases} (t, s) & \text{if } (t, s) \in U_\alpha, \\ \arg\min_{(u_\alpha, u_b) \in U_\alpha} d((u_\alpha, u_b), (t, s)) & \text{if } (t, s) \in V_\alpha \setminus U_\alpha, \\ (0_n, -\rho) & \text{if } (t, s) \notin U_\alpha \cup V_\alpha, \end{cases}
\]

\[
\sigma_{\hat{V};\hat{U}}(t) = \sigma_{\hat{U};\hat{V}}((\alpha, t, s)) = \begin{cases} (t, s) & \text{if } (t, s) \in V_\alpha, \\ \arg\min_{(v_\alpha, v_b) \in V_\alpha} d((v_\alpha, v_b), (t, s)) & \text{if } (t, s) \in U_\alpha \setminus V_\alpha, \\ (0_n, -\rho) & \text{if } (t, s) \notin U_\alpha \cup V_\alpha, \end{cases}
\]

where \( d(\cdot, \cdot) \) refers to the Euclidean metric. Then \( \pi_{\hat{U};\hat{V}} \) and \( \pi_{\hat{V};\hat{U}} \) are well-defined and are equivalent to \( \text{RO}(\hat{U}) \) and \( \text{RO}(\hat{V}) \), respectively. Last,

\[
\delta^\Pi(\pi_{\hat{U};\hat{V}}, \pi_{\hat{V};\hat{U}}) = d_h(\hat{U}, \hat{V}) := \sup_{\alpha \in I} d_H(U_\alpha, V_\alpha).
\]

Proof. The proof is analogous to that of Theorem 4.2. See Appendix C.

5. Lipschitz continuity of the optimal value. In order to prove Lipschitz continuity of the optimal value, we not only need the RO-LSIO transformation we developed in the previous section, but we also need to prove invariance of the Lipschitz constant with respect to \( \Pi \)-equivalent problems. This invariance result can be found in Appendix D.

5.1. Lipschitz continuity for single-constraint RO.

Theorem 5.1. Let \( \text{RO}(U) \) be of the form (3.2) with nonempty compact and convex uncertainty set \( U \subseteq \mathbb{R}^n \), fixed \( c \), and right-hand side value \( b \). Suppose that

i. \( \text{RO}(U) \) satisfies the strong Slater condition, with strong Slater constant \( \rho > 0 \), and

ii. \( F^\text{opt}(\text{RO}(U)) \) is nonempty and bounded.

Then, there exists an LSIO \( \pi_U := (c, \sigma_U) \in \text{int}(\Pi_s) \), where

\[
\sigma_U(t) := \begin{cases} (t, b) & \text{if } t \in U, \\ (0_n, -\rho) & \text{if } t \notin U, \end{cases}
\]

such that for any \( \epsilon \) satisfying \( 0 < \epsilon < \delta^\Pi(\pi_U, \text{bd}(\Pi_s)) \), and for all compact, convex uncertainty sets \( V \subseteq \mathbb{R}^n \) satisfying \( d_H(U, V) < \epsilon \),

\[
|\nu(\text{RO}(U)) - \nu(\text{RO}(V))| \leq L(U, \epsilon)d_H(U, V).
\]

The Lipschitz constant \( L(U, \epsilon) := L(\pi_U, \epsilon) \) can be calculated via Definition 3.3 using \( \pi_U := (c, \sigma_U) \) and the given \( \epsilon \).

Proof. Let \( T := \mathbb{R}^n \). The proof proceeds in seven steps.
1. Write \( \text{RO}(U) \) as the LSIO problem \( \pi_U \). First, we show that \( \pi_U \) and \( \text{RO}(U) \) are equivalent. Note that the cost functions for the two problems are the same, and that \( \pi_U \) contains all the constraints of \( \text{RO}(U) \) plus some additional trivial constraints. Thus \( \pi_U \) and \( \text{RO}(U) \) have the same feasible set, optimal value, and optimal solution sets. Since \( \text{RO}(U) \) satisfies the strong Slater condition with constant \( \rho \), and since \( (0_n, x) \geq -\rho + \rho = 0 \) for all \( x \), it follows that \( \pi_U \) satisfies the strong Slater condition with constant \( \rho \). Second, we show that \( \pi_U \in \text{int}(\Pi_s) \). By assumption, \( F^{\text{opt}}(\text{RO}(U)) \) is nonempty and bounded, so \( F^{\text{opt}}(\pi_U) \) is as well. Since \( \pi_U \) satisfies the strong Slater condition, by Theorem 3.5 (see [16, Theorem 3.1(i), (ii), and (vi)]) we have \( \pi_U \in \text{int}(\Pi_I) \).

By [8, Proposition 1(vi)], \( c \in \text{int}(\text{cone}(\{ a_t : t \in T \})) \), which, by (vi) of the same proposition, implies that \( \pi_U \in \text{int}(\Pi_s) \).

2. Choose \( V \) in an \( \epsilon \)-neighborhood of \( U \). Let \( \epsilon > 0 \) be given satisfying \( 0 < \epsilon < \delta^{\Pi}(\pi_U, \text{bd}(\Pi_s)) \). Such an \( \epsilon \) exists because \( \delta^{\Pi}(\pi_U, \text{bd}(\Pi_s)) > 0 \) by Lemma D.4. Let \( V \) be any compact and convex set in \( \mathbb{R}^n \) satisfying \( d_H(U, V) \leq \epsilon < \delta^{\Pi}(\pi_U, \text{bd}(\Pi_s)) \). Such a \( V \) exists, e.g., \( V := U + B(0, \epsilon/2) \).

3. Define \( \pi_{U:V} := (c, \sigma_{U:V}) \). Define the LSIO problem \( \pi_{U:V} := (c, \sigma_{U:V}) \) as in Theorem 4.2. By the same theorem, \( \pi_{U:V} \) is well-defined. Since for each \( t \in V \setminus U \), \( \arg\min_{u \in U} \|u - t\| \) is an element of \( U \), every constraint in \( \pi_{U:V} \) is in \( \pi_U \) and vice versa. Thus, \( \pi_{U:V} \sim_\Pi \pi_U \). Furthermore, since \( \pi_{U:V} \sim_\Pi \pi_U \) and \( \pi_U \) satisfies the strong Slater condition with Slater constant \( \rho \), it follows that \( \pi_{U:V} \) also satisfies the strong Slater condition with constant \( \rho \).

4. Define \( \pi_{V:U} := (c, \sigma_{V:U}) \). Define the problem \( \pi_{V:U} := (c, \sigma_{V:U}) \) as in Theorem 4.2. By the same theorem, \( \pi_{V:U} \) is well-defined and equivalent to \( \text{RO}(V) \).

5. By Theorem 4.2, \( \delta^{\Pi}(\pi_{U:V}, \pi_{V:U}) = d_H(U, V) < \epsilon \).

6. Apply Theorem 3.4 (see [9, Theorem 4.3]) with \( \pi_{U:V} \mapsto \pi_0, \pi_1 \) and \( \pi_{V:U} \mapsto \pi_2 \). We check that the assumptions of Theorem 3.4 are satisfied:
   (a) \( \pi_{U:V} \in \text{int}(\Pi_s) \). This proof is identical to the proof that \( \pi_U \in \text{int}(\Pi_s) \).
   (b) \( \epsilon < \delta^{\Pi}(\pi_{U:V}, \text{bd}(\Pi_s)) \). Recall that \( \pi_{U:V} \) and \( \pi_U \) are both in \( \text{int}(\Pi_s) \), have nonempty bounded optimal solution sets, and are \( \Pi \)-equivalent to each other. Thus, \( \epsilon < \delta^{\Pi}(\pi_{U:V}, \text{bd}(\Pi_s)) = \delta^{\Pi}(\pi_{U:V}, \text{bd}(\Pi_s)) \), where the inequality is by assumption and the equality is by Lemma D.5.
   (c) \( \delta^{\Pi}(\pi_{U:V}, \pi_{V:U}) \leq \epsilon \) and \( \delta^{\Pi}(\pi_{U:V}, \pi_{V:U}) = 0 \leq \epsilon \). The first inequality comes from Step 5, and the second inequality is trivial.

Thus, the assumptions of Theorem 3.4 are satisfied, so

\[
|\nu(\pi_{U:V}) - \nu(\pi_{V:U})| \leq L(\pi_{U:V}, \epsilon)\delta^{\Pi}(\pi_{U:V}, \pi_{V:U}),
\]

where \( L(\pi_{U:V}, \epsilon) \) is as defined in Definition 3.3.

7. Last, show that the Lipschitz constant is independent of the choice of \( V \). Applying Lemma D.5 to \( \Pi \)-equivalent problems \( \pi_U \) and \( \pi_{U:V} \), it follows that \( L(\pi_U, \epsilon) = L(\pi_{U:V}, \epsilon) \). Now define \( L(U, \epsilon) := L(\pi_U, \epsilon) \). Finally, since \( \nu(\pi_{U:V}) = \nu(\text{RO}(U)) \) and \( \nu(\pi_{V:U}) = \nu(\text{RO}(V)) \) and \( \delta^{\Pi}(\pi_{U:V}, \pi_{V:U}) = d_H(U, V) \), we obtain the final result:

\[
|\nu(\text{RO}(U)) - \nu(\text{RO}(V))| = |\nu(\pi_{U:V}) - \nu(\pi_{V:U})|
\]

(definition of \( \pi_{U:V} \) and \( \pi_{V:U} \))

\[
\leq L(\pi_{U:V}, \epsilon)\delta^{\Pi}(\pi_{U:V}, \pi_{V:U}) \quad \text{(by Theorem 3.4)}
\]

\[
= L(\pi_{U:V}, \epsilon)d_H(U, V) \quad \text{(by Theorem 4.2)}
\]

\[
= L(U, \epsilon)d_H(U, V) \quad \text{(since \( \pi_{U:V} \sim_\Pi \pi_U \))}
\]

as was to be shown. \( \Box \)

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
Figure 3 gives a conceptual illustration of the proof idea. We now provide an example explicitly calculating \( L(\bar{U}, \epsilon) = L(\bar{\pi}_U, \epsilon) \).

Example 5.2 (calculation of the Lipschitz constant). Write \( c = (-1, 0) \), \( a_1 = (-1, 0) \), \( a_2 = (-1, 0) + (-1/\sqrt{2}, 1/\sqrt{2}) \), and \( a_3 = (-1, 0) + (-1/\sqrt{2}, -1/\sqrt{2}) \), and \( x = (x, y) \), and \( b = -1 \). Consider the following RO problem

\[
\begin{align*}
\text{(5.2a)} & \quad \min_{x \in \mathbb{R}^2} c'x \\
\text{(5.2b)} & \quad \text{s.t. } a'x \geq b \quad \forall a \in B_1(a_1), \\
\text{(5.2c)} & \quad a_1'x \geq b, \\
\text{(5.2d)} & \quad a_3'x \geq b,
\end{align*}
\]

where \( B_1(a_1) := \{ a : \|a - a_1\|_2 \leq 1 \} \). In this example we have that \( a_2, a_3 \in B_1(a_1) \). Since the uncertainty set is a sphere, that is, an ellipsoid, (5.2) can be reformulated as a second order cone program [4, 2]. Let \( \sigma_U(t) \) be defined as in Theorem 5.3, which satisfies the hypotheses of the theorem because \( x = 0_2 \) is a strong Slater point with constant \( \rho = 1 \). Then in order to calculate \( L(\pi_U, \epsilon) \), we need to determine \( Z^- (\pi_U) \); in this case, it is

\[
\text{(5.3)} \quad Z^- (\pi_U) = \text{conv}(B_1(a_1) \cup \{-c\} \cup \{0_2\}),
\]

where the 0_2 comes from the \((0_2, -\rho) = (0_3, -1)\) that is chosen as the trivial constraint. The set \( Z^- (\pi_U) \) is illustrated in Figure 4, showing that \( d(0_2, \text{bd}(Z^- (\pi_U))) = d_*(0_2, \text{bd}(Z^- (\pi_U))) = 1/2 \) by using similar triangles and noting that the radius is 1. Since \( b_t = -1 \) for all \( t \), then since \( \delta_\Sigma (\sigma_0, \Sigma_0) = d(0_3, \text{bd}(H(\sigma_0))) \), we have that \( \delta_\Sigma (\sigma_0, \Sigma_0) = 1 \). We can now calculate the quantities in Definition 3.3:

\[
\begin{align*}
\text{(5.4a)} & \quad \delta_\Pi (\pi_U, \text{bd}(\Pi_*)) = \min\{1, 1/2\} = 1/2, \\
\text{(5.4b)} & \quad \varphi(0) = \varphi_*(0) = 1, \\
\text{(5.4c)} & \quad R(\pi_U) = \{-b_t, t \in T; \nu(\pi_U)\} = \{1, \nu(\pi_U)\}, \\
\text{(5.4d)} & \quad \hat{\rho}(\pi_U) = \frac{\sup R(\pi_U)}{d_*(0_2, \text{bd}(Z^- (\pi_U)))} = 2 \sup R(\pi_U) = 2,
\end{align*}
\]
\[
\psi(2) = (1 + 2)\sqrt{1 + 2^2} = 3\sqrt{5},
\]
(5.4c)

\[
\beta(\pi_0, \epsilon) = \frac{3\sqrt{5}}{1 - \epsilon},
\]
(5.4f)

\[
\gamma(\pi_0, \epsilon) = 1 \left( 2 + \epsilon \frac{3\sqrt{5}}{1 - \epsilon} \right) + 1 \frac{3\sqrt{5}}{1 - \epsilon},
\]
(5.4g)

\[
\mu(\pi_0, \epsilon) = \frac{1 + \epsilon \left( 1 + \frac{3\sqrt{5}}{1 - \epsilon} \right) + \frac{3\sqrt{5}}{1 - \epsilon}}{1/2 - \epsilon},
\]
(5.4h)

\[
L(\pi_0, \epsilon) = 1 \left( (\epsilon + 1) \frac{1 + (\mu(\pi_0, \epsilon))}{1 + (\mu(\pi_0, \epsilon))^2} + \mu(\pi_0, \epsilon) \right),
\]
(5.4i)

where (5.4a) comes from applying [7, Theorem 2], and we are using the Euclidean norm so \( \| \cdot \| = \| \cdot \|_2 \). Note that in (5.4c) we can replace the actual optimal value with the optimal value of an approximating superset of the original uncertainty set, because the approximating superset has a worse (i.e., higher) optimal value. In our case, we find that the objective value is always less than 0, so that \( R(\pi_0) = 1 \).

Figure 5 shows we can approximate the circle by the smallest regular \( M \)-gon containing the circle. This \( M \)-gon is given by the convex set of the vertices. So, if we have a polygon with \( M \) sides, then the vertices are:

\[
v_k = (-1 + r_M \cos(\theta_k), r_M \sin(\theta_k)), \quad k = 0, \ldots, M,
\]

where

\[
\theta_k = \frac{2\pi}{1/2} + k(2\pi/M), \quad k = 0, \ldots, M \quad \text{and} \quad r_M = \frac{1}{\cos \theta_0},
\]

where the \( 1/2 \cdot 2\pi/M \) in the definition of \( \theta_k \) guarantees that the orientation of the polygon is as we want (i.e., a vertical line segment that is perpendicular to the angle at 0 radians; cf. Figure 5). Let \( V_M \) be the polygon formed by taking the convex hull of the vertices \( v_k \) for \( k = 0, \ldots, M \). Consider the following formulation:

\[
\min_{x \in \mathbb{R}^2} c'x
\]
(5.5a)

\[
s.t. \quad a'x \geq b \quad \forall a \in V_M,
\]
(5.5b)

\[
a'_2x \geq b,
\]
(5.5c)

\[
a'_3x \geq b.
\]
(5.5d)
We can rewrite this as a simple linear program by enumerating the vertices:

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad c'x \\
\text{s.t.} & \quad v_k'x \geq b & \forall k \in \{0, \ldots, M\}, \\
& \quad a_2'x \geq b, \\
& \quad a_3'x \geq b.
\end{align*}
\]

Conveniently the distance between the sets is \(d_H(B_1(a_1), V_M) = r_M - 1\). Note that \(r_M \to 1\) as \(M \to \infty\). Figure 6 shows the estimated objective value and the actual objective value of the RO problems versus the number of sides of the approximating sets \(V_M\). Figure 7 shows the dependence of \(L(\pi_U, \epsilon)\) on \(\epsilon\).

5.2. Lipschitz continuity for multiple-constraint RO.

**Theorem 5.3.** Let \(\text{RO}(\hat{U})\) be an RO problem of the form (3.5) with nonempty compact and convex uncertainty set \(U_\alpha\) in each constraint, indexed by \(\alpha \in I\), with fixed cost function \(c\). Suppose that

i. \(\text{RO}(\hat{U})\) satisfies the strong Slater condition, with strong Slater constant \(\rho > 0\),

ii. \(F_{\text{opt}}(\text{RO}(\hat{U}))\) is nonempty and bounded,
Then, there exists an LSIO problem $\pi_\Omega \in \text{int}(\Pi_s)$, where

$$\sigma_\Omega(t) = \sigma_\Omega((\alpha, t, s)) := \begin{cases} (t, s) & \text{if } (t, s) \in U_\alpha, \\ (0_n, -\rho) & \text{if } (t, s) \notin U_\alpha, \end{cases}$$

such that for all $\epsilon > 0$ satisfying $\epsilon < \delta(\pi_\Omega, \text{bd}(\Pi_s))$, and for all $\widehat{V} = \Pi_{\alpha \in \Omega} V_\alpha$ with nonempty, compact, and convex $V_\alpha \subseteq \mathbb{R}^{n+1}$ satisfying $\sup_{\alpha \in I} d_H(U_\alpha, V_\alpha) < \epsilon$, we have

$$|\nu(\text{RO}(\widehat{U})) - \nu(\text{RO}(\widehat{V}))| \leq L(\widehat{U}, \epsilon) d_\natural(\widehat{U}, \widehat{V}).$$

The Lipschitz constant $L(\widehat{U}, \epsilon) := L(\pi_\Omega, \epsilon)$ can be calculated via Definition 3.3 using $\pi_\Omega := (c, \sigma_\Omega)$ and the given $\epsilon$.

Remark 5.4. If $\sup \{-b_\alpha, \alpha \in I\} = +\infty$, then the theorem is trivially true. For a given problem where $\sup \{-b_\alpha, \alpha \in I\} = +\infty$ is true, it is conceivable, though possibly very difficult, to use [9, Remarks 4.2 and 4.1] to establish nontrivial bounds using a modified Lipschitz constant on the right-hand side inequality of (5.6).

Proof. The proof is analogous to that of Theorem 5.1. See Appendix E.}

6. Closedness and upper semicontinuity of the optimal solution set.

First, we provide definitions of closedness, lower semicontinuity, and upper semicontinuity, as defined in [20]. Then, we present the main result of this section.

**Definition 6.1** (definitions from [20]). We define the following properties of set valued mappings. Suppose $K$ is a set valued mapping from $\Pi$ to the power set of $\mathbb{R}^n$. Suppose $K(\pi) \neq \emptyset$ for some given $\pi$. Then

i. $K$ is closed at $\pi$ if for any sequence $\{\pi_j\}_{j=1}^\infty \subseteq \Pi$ and $\{x_j\}_{j=1}^\infty \subseteq \mathbb{R}^n$ such that $x_j \in K(\pi_j)$ for all $j$, with $\pi_j \to \pi$ and $x_j \to x$ for some $x \in \mathbb{R}^n$, we have $x \in K(\pi)$;

ii. $K$ is lower semicontinuous at $\pi$ if for every open set $O \subseteq \mathbb{R}^n$ such that $K(\pi) \cap O \neq \emptyset$, there exists a neighborhood $N$ of $\pi$ such that for all $\pi \in N$ we have $K(\pi) \cap O \neq \emptyset$;

iii. $K$ is upper semicontinuous at $\pi$ if for each open set $O \subseteq \mathbb{R}^n$ such that $K(\pi) \subseteq O$, there exists a neighborhood $\mathcal{N}$ of $\pi$ such that for all $\pi \in \mathcal{N}$, we have $K(\pi) \subseteq \mathcal{N}$. 

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
The definitions in Definition 6.1 can be extended to any set valued mapping between a metric space and the set of subsets of a metric space, e.g., the metric space of all uncertainty sets of the form $\hat{U}$ with $d_2$ instead of the space of LSIO problems $\Pi$.

**Theorem 6.2.** Let $\text{RO}(\hat{U})$ be an RO problem of the form (3.5) with nonempty compact and convex $U_\alpha$, indexed by $\alpha \in I$, and with fixed cost function $c$. Suppose

i. $\text{RO}(\hat{U})$ satisfies the strong Slater condition, with strong Slater constant $\rho > 0$.

ii. $F^{\text{opt}}(\text{RO}(\hat{U}))$ is nonempty and bounded, and

iii. $\sup\{-b^\alpha, \alpha \in I\} < +\infty$.

Then, with respect to the metric $d_2(\hat{U}, \hat{V})$,

iv. $F^{\text{opt}}(\text{RO}(\cdot))$ is closed at $\hat{U}$,

v. $F^{\text{opt}}(\text{RO}(\cdot))$ is upper semicontinuous at $\hat{U}$.

**Proof.**

iv. $F^{\text{opt}}(\text{RO}(\cdot))$ is closed at $\hat{U}$.

Let $T := I \times \mathbb{R}^n \times \mathbb{R}$ with elements $(\alpha, t, s)$, where $\alpha \in I$, $t \in \mathbb{R}^n$, and $s \in \mathbb{R}$. Let a sequence $\{\hat{V}_j\}_{j=1}^\infty$ satisfying $\hat{V}_j = \Pi_{\alpha \in J} V_{j, \alpha}$ with nonempty, compact, and convex $V_{j, \alpha} \subseteq \mathbb{R}^{n+1}$ and a sequence $\{x_j\}_{j=1}^\infty \subseteq F^{\text{opt}}(\text{RO}(\hat{V}_j))$ with $\hat{V}_j \overset{d_2}{\to} \hat{U}$ and with $x_j \to \hat{x}$ for some $\hat{x} \in \mathbb{R}^n$ be given. By definition, $F^{\text{opt}}(\text{RO}(\cdot))$ is closed at $\hat{U}$ if $\hat{x} \in F^{\text{opt}}(\text{RO}(\hat{U}))$.

The idea of the proof is to show that, in the limit, $x_j$ comes arbitrarily close to $F(\text{RO}((\cdot)))$ as $j \to \infty$. Then, since $\nu(\text{RO}(\hat{V}_j)) \to \nu(\text{RO}(\hat{U}))$ as $j \to \infty$ by Theorem 5.3, with the assumption that $\sup\{-b^\alpha, \alpha \in I\} < +\infty$, it must be the case that $\hat{x} \in F^{\text{opt}}(\text{RO}(\hat{U}))$.

1. Define $\pi_0 \in \text{int}(\Pi_s)$, equivalent to $\text{RO}(\hat{U})$. Let $\pi_0 := (c, \sigma_0)$ be defined as in Theorem 5.3. Then $\pi_0 \in \text{int}(\Pi_s)$ is equivalent to $\text{RO}(\hat{U})$, and satisfies the strong Slater condition.

2. Define $\pi_{\hat{U}, \hat{V}}$, II-equivalent to $\pi_0$. For each $\hat{V}_j$, label the constraint uncertainty set associated with $\alpha \in I$ as $V_{j, \alpha}$, and define $\pi_{\hat{U}, \hat{V}_j} := (c, \sigma_{\hat{U}, \hat{V}_j}, \sigma_{\hat{U}, \hat{V}_j}, (\alpha, t, s))$

   $$
   \sigma_{\hat{U}, \hat{V}_j, \hat{V}_j, \hat{V}_j} := \begin{cases} 
   (t, s) & \text{if } (t, s) \in U_\alpha, \\
   \arg\min_{(u_n, u_b) \in U_\alpha} d((u_n, u_b), (t, s)) & \text{if } (t, s) \in V_{j, \alpha} \setminus U_\alpha, \\
   (0_n, -\rho) & \text{if } (t, s) \notin U_\alpha \cup V_{j, \alpha}.
   \end{cases}
   $$

   The proof that $\pi_0 :\hat{V}_j$ is well-defined, is II-equivalent to $\pi_0$, and that $\pi_{\hat{U}, \hat{V}_j} \in \text{int}(\Pi_s)$ is given in the proof of Theorem 5.3. Furthermore, since $\pi_{\hat{U}, \hat{V}_j} \sim_{\Pi_s} \pi_0$ and $\pi_0$ satisfies the strong Slater condition with constant $\rho$, it follows that $\pi_{\hat{U}, \hat{V}_j}$ also does, with constant $\rho > 0$.

3. Define $\pi_{\hat{V}_j, \hat{U}}$, equivalent to $\text{RO}(\hat{V})$. Define $\pi_{\hat{V}_j, \hat{U}} := (c, \sigma_{\hat{V}_j, \hat{U}})$, where

   $$
   \sigma_{\hat{V}_j, \hat{U}}^{\hat{V}_j, \hat{U}} := \begin{cases} 
   (t, s) & \text{if } (t, s) \in V_{j, \alpha}, \\
   \arg\min_{(v_n, v_b) \in V_{j, \alpha}} d((v_n, v_b), (t, s)) & \text{if } (t, s) \in U_\alpha \setminus V_{j, \alpha}, \\
   (0_n, -\rho) & \text{if } (t, s) \notin U_\alpha \cup V_{j, \alpha}.
   \end{cases}
   $$

   That $\pi_{\hat{V}_j, \hat{U}}$ is well-defined and equivalent to $\text{RO}(\hat{V})$ is given in the proof of Theorem 5.3.
4. Show that \(d(x_j, F(\sigma; \hat{U}, \hat{V}_j)) \leq \frac{\psi(||x_j||)}{\delta_\Sigma(\hat{U}, \Sigma_i)} d_s(\hat{U}, \hat{V}_j)\) for all \(j\). We first show that

\[
(6.1) \quad d(x_j, F(\sigma; \hat{U}, \hat{V}_j)) \leq \frac{\psi(||x_j||)}{\delta_\Sigma(\hat{U}, \Sigma_i)} \delta_\Sigma(\sigma; \Sigma_i).
\]

Corollary 4.1 from [9] states that for any \(\sigma_0 \in \Sigma_f\) and \(z^0 \in F(\sigma_0),\)

\[
d(z^0, F(\sigma)) \leq \frac{\psi(||z^0||)}{\delta_\Sigma(\sigma, \Sigma_i)} \delta_\Sigma(\sigma, \sigma_0)
\]

for all \(\sigma \in \mathbb{int}(\Sigma_f)\). Thus, inequality (6.1) follows from the application of that corollary with \(\sigma_{\hat{U}; \hat{V}_j} \rightarrow \sigma_0\) and \(\sigma_{\hat{U}; \hat{V}_j} \rightarrow \sigma\). What remains is to check that the assumptions of the corollary are satisfied, namely, that \(\sigma_{\hat{U}; \hat{V}_j} \in \Sigma_f\) and \(\sigma_{\hat{U}; \hat{V}_j} \in \mathbb{int}(\Sigma_f)\). By step 3, \(x_j \in F(\Sigma, \Sigma_i) \subseteq F(x_j, \hat{U}); \hat{V}_j), which implies that \(\pi_{\hat{U}; \hat{V}_j}\) has a nonempty feasible set. Thus, \(\sigma_{\hat{U}; \hat{V}_j} \in \Sigma_f\). Since \(\pi_{\hat{U}; \hat{V}_j}\) satisfies the strong Slater condition, by Theorem 3.5 (see [16, Theorem 3.1(i),(ii), and (vi)]), \(\sigma_{\hat{U}; \hat{V}_j} \in \mathbb{int}(\Sigma_f)\). Then, we note that

\[
(6.2) \quad \delta_\Sigma(\sigma_{\hat{U}; \hat{V}_j}, \sigma_{\hat{U}; \hat{V}_j}; \hat{V}_j)
\]

by Theorem 4.4, since \(\pi_{\hat{U}; \hat{V}_j}\) and \(\pi_{\hat{U}; \hat{V}_j}\) have the same cost function. Combining (6.1) and (6.2) yields the desired inequality

\[
(6.3) \quad d(x_j, F(\sigma; \hat{U}, \hat{V}_j)) \leq \frac{\psi(||x_j||)}{\delta_\Sigma(\sigma; \Sigma_i)} d_s(\hat{U}, \hat{V}_j).
\]

5. Show that \(x \in F(\sigma_{\hat{U}}). \) Since \(\pi_{\hat{U}; \hat{V}_j}\) is equivalent to \(\text{RO}(\hat{U})\), it follows that \(F(\pi_{\hat{U}; \hat{V}_j})\) is nonempty and bounded, which implies that \(\pi_{\hat{U}; \hat{V}_j} \in \mathbb{int}(\Pi_i)\) (cf. proof of Theorem 5.3). Also, since \(\pi_{\hat{U}; \hat{V}_j} \sim \Pi_i \pi_{\hat{U}}\) for all \(j\), then \(F(\pi_{\hat{U}; \hat{V}_j}) = F(\sigma_{\hat{U}})\) for all \(j\) and Lemma D.5 implies that \(\delta_\Sigma(\sigma_{\hat{U}; \hat{V}_j}, \Sigma_i) = \delta_\Sigma(\sigma_{\hat{U}}, \Sigma_i)\). Thus, for any \(j\), (6.3) can be rewritten

\[
(6.4) \quad d(x_j, F(\sigma_{\hat{U}})) \leq \frac{\psi(||x_j||)}{\delta_\Sigma(\sigma_{\hat{U}}, \Sigma_i)} d_s(\hat{U}, \hat{V}_j).
\]

Since the sequence \(\psi(||x_j||)\) is bounded, \(\psi(||\cdot||)\) is continuous, and \(x_j \rightarrow x\) implies that \(\psi(||x_j||) \rightarrow \psi(||x||)\) and \(d_s(\hat{U}, \hat{V}_j) \rightarrow 0\) as \(j \rightarrow \infty\) by assumption, it follows, due to (6.4), that \(d(x_j, F(\sigma_{\hat{U}})) \rightarrow 0\) as \(j \rightarrow \infty\). Next, we will use the closedness of \(F(\sigma_{\hat{U}})\) to show that \(x \in (\sigma_{\hat{U}})\). To that end, consider a subsequence of \(\{x_j\}_{j=1}^\infty\) labeled \(\{x_{j_k}\}_{k=1}^\infty\) and indexed by \(j_k\). For each \(k\), since \(d(x_j, F(\sigma_{\hat{U}})) \rightarrow 0\) as \(j \rightarrow \infty\), we can choose \(j_k\) large enough so that \(d(x_{j_k}, F(\sigma_{\hat{U}})) \leq 2^{-k+1}\). Now, we consider a different sequence \(\{x_k\}_{k=1}^\infty \subseteq F(\sigma_{\hat{U}})\) such that \(d(x_{j_k}, x_k) \rightarrow 0\) as \(k \rightarrow \infty\), which implies \(x_k \rightarrow x\). The sequence \(\{x_k\}_{k=1}^\infty\) exists because

\[
d(x_{j_k}, F(\sigma_{\hat{U}})) \leq 2^{-k+1} \implies \inf_{x' \in F(\sigma_{\hat{U}})} d(x_{j_k}, x') \leq 2^{-k+1}
\]
and for each $k$, we define a point $x'_k$ in $F(\sigma_{\hat{U}})$ such that $d(x_{jk}, x'_k) < 2^{-k}$.

Now we must show that $x'_k \to \bar{x}$ as $k \to \infty$. By the triangle inequality,

$$d(x'_k, \bar{x}) \leq d(x'_k, x_{jk}) + d(x_{jk}, \bar{x}) \to 0$$

as $k \to \infty$. Last, since $F(\sigma_{\hat{U}})$ is closed (it is the intersection of closed sets), and $x'_k \to \bar{x}$, we have that $\bar{x} \in F(\sigma_{\hat{U}})$.

6. Show that $\bar{x} \in F^{opt}(\pi_{\hat{U}}) = F^{opt}(RO(\hat{U}))$. Now we will show that $\bar{x}$ is indeed an optimal solution of $\pi_{\hat{U}}, \nu_{\hat{U}}$, and RO(\hat{U}). We are going to apply Theorem 5.3 as a convergence result. However, to do this, we must first check, for any $\epsilon$ satisfying $0 < \epsilon < \delta^I(\pi_{\hat{U}}, \partial(\Pi_{\hat{U}}))$, that we have $L(\hat{U}, \epsilon) < +\infty$. The case when $L(\hat{U}, \epsilon) < +\infty$ if and only if

(a) $\sup\{-b, t \in T; \nu(\pi_{\hat{U}})\} < +\infty$, which is guaranteed by the hypothesis, and that $\pi_{\hat{U}} \in \text{int}(\Pi_{\hat{U}})$;

(b) $d(0, \partial(Z^{-}(\pi_{\hat{U}}))) > 0$, which is guaranteed by Lemma D.4;

(c) $d(0, \partial(Z^{-}(\pi_{\hat{U}}))) > \epsilon$, which is guaranteed by [7, Theorem 2], since we assumed that $0 < \epsilon < \delta^I(\pi_{\hat{U}}, \partial(\Pi_{\hat{U}}))$;

d) $\Delta^s(\sigma_0, \Sigma_i) > \epsilon$, which is guaranteed by Lemma D.4.

Fix any $0 < \epsilon < \delta^I(\pi_{\hat{U}}, \partial(\Pi_{\hat{U}}))$, so that $L(\hat{U}, \epsilon) < +\infty$. Since $d_{\epsilon}(\hat{U}, V_{\hat{U}}) \to 0$, there exists $j$ large enough that $d_{\epsilon}(\hat{U}, V_{\hat{U}}) < \epsilon$ and Theorem 5.3 applies. Thus, $d_{\epsilon}(\hat{U}, V_{\hat{U}}) \to 0$ and $L(\hat{U}, \epsilon) < +\infty$ directly imply $\nu(RO(V_{\hat{U}})) \to \nu(RO(\hat{U}))$ as $j \to \infty$. Also, $x_j \to \bar{x}$, so that $\nu(c, x_j) \to \nu(c, \bar{x})$. Thus, since $x_j \in F^{opt}(RO(V_{\hat{U}}))$, we have that $\nu(c, x_j) = \nu(RO(V_{\hat{U}}))$, so that $(c, \bar{x}) = \nu(RO(\hat{U}))$, in other words, $\bar{x} \in F^{opt}(RO(\hat{U}))$.

v. $F^{opt}(RO(\cdot))$ is upper semicontinuous at $\hat{U}$.

Let $T := I \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with elements $(\alpha, \tau, \xi)$, where $\alpha \in I$, $\tau \in \mathbb{R}^{n+1}$, and $\xi \in \mathbb{R}^{n+1}$. Let $\mathcal{O}$ be an open set satisfying $F^{opt}(RO(\hat{U})) \subseteq \mathcal{O}$. By definition, $F^{opt}(RO(\cdot))$ is upper semicontinuous at $\hat{U}$ if there exists some $\epsilon > 0$ such that $F^{opt}(V) \subseteq \mathcal{O}$ for all $V$ satisfying $d_{\epsilon}(\hat{U}, V) < \epsilon$, where $V$ is a product of compact and convex $V_{\alpha}$. The idea of the proof is to use the fact that $F^{opt}(RO(\cdot))$ is upper semicontinuous for LSIO problems, then use another RO-LSIO transformation to gain upper semicontinuity for RO problems. In this proof, the metric $d(x, y) = \|x - y\|$ represents the Euclidean metric.

1. Define $\pi_{\hat{U}} := (c, \sigma_{\hat{U}})$. Define $\pi_{\hat{U}} := (c, \sigma_{\hat{U}})$, where

$$\sigma_{\hat{U}}(\alpha, \tau, \xi) := \begin{cases} \tau & \text{if } \tau \in U_{\alpha}, \xi \in \mathbb{R}^{n+1}, \\ 0_n, -\rho & \text{if } \tau \notin U_{\alpha}, \xi \in \mathbb{R}^{n+1}. \end{cases}$$

The proof that $\pi_{\hat{U}}$ is well-defined and equivalent to $RO(\hat{U})$ essentially follows the same proof as in Theorem 5.3.

2. $F^{opt}$ is upper semicontinuous at $\pi_{\hat{U}}$. Since $\pi_{\hat{U}}$ is equivalent to $RO(\hat{U})$, it follows that $\pi_{\hat{U}}$ satisfies the strong Slater condition and is in $\Pi_{\hat{U}}$. By Theorem 3.5 (see [16, Theorem 3.1(i), (ii), and (vi)]), $\pi_{\hat{U}}$ satisfying the strong Slater condition means that $F(\cdot)$ is lower semicontinuous at $\pi_{\hat{U}}$. By Theorem 3.6 (see [5, Theorem 5.1(i) and (ii)]), lower semicontinuity implies that $F^{opt}(\cdot)$ is closed at $\pi_{\hat{U}}$. Also note that $F^{opt}(\pi_{\hat{U}})$ is
nonempty and bounded because $\pi_\mathcal{O}$ shares the same optimal solution set as $\text{RO}(\hat{U})$. Thus, by Theorem 3.6, $F^\text{opt}(\cdot)$ is upper semicontinuous at $\pi_\mathcal{O}$.

3. With the given $\mathcal{O}$, choose $\epsilon > 0$ such that if $F^\text{opt}(\pi_\mathcal{O}) \subseteq \mathcal{O}$, then for all $\pi$ satisfying $\delta^H(\pi, \pi_\mathcal{O}) \leq \epsilon$, we have $F^\text{opt}(\pi) \subseteq \mathcal{O}$. Such an $\epsilon$ exists by definition of upper semicontinuity at $\pi_\mathcal{O}$.

4. Choose $\hat{V}$ satisfying $d_\delta(\hat{U}, \hat{V}) < \epsilon$ and $\hat{V} = \Pi_{\alpha \in I} V_\alpha$, where, for all $\alpha \in I$, the set $V_\alpha$ is nonempty, compact, and convex.

5. Define $\pi_{\hat{V}, \hat{U}}$, equivalent to $\text{RO}(\hat{V})$. Define $\pi_{\hat{V}, \hat{U}} := (c, \sigma_{\hat{V}, \hat{U}})$, where

$$
\sigma_{\hat{V}, \hat{U}}(\alpha, \tau, \xi)
:= \begin{cases} 
\xi & \text{if } \tau = \text{argmin}_{u \in U_\alpha} d(\xi, u), \xi \in V_\alpha, \\
(0_n, -\rho) & \text{if } \tau \notin U_\alpha, \xi \in \mathbb{R}^{n+1}, \\
\text{argmin}_{v \in V_\alpha} d(v, \tau) & \text{otherwise}. 
\end{cases}
$$

Note that this is a well-defined mapping, since both the argmin in the first and third case exist and are unique due to $U_\alpha$ and $V_\alpha$ being compact and convex. Note that every constraint in $\text{RO}(\hat{V})$ is included in $\sigma_{\hat{V}, \hat{U}}$ (first case), and that all other constraints are either trivial (second case) or redundant (third case). Thus, $\pi_{\hat{V}, \hat{U}}$ is equivalent to $\text{RO}(\hat{V})$.

6. $\sup_{\alpha, \tau, \xi} \|\sigma_{\hat{O}}(\alpha, \tau, \xi) - \sigma_{\hat{V}, \hat{U}}(\alpha, \tau, \xi)\| \leq d_\delta(\hat{U}, \hat{V})$. Rewriting $\sigma_{\hat{O}}(\alpha, \tau, \xi)$,

$$\sigma_{\hat{O}}(\alpha, \tau, \xi) := \begin{cases} 
\tau & \text{if } \tau = \text{argmin}_{u \in U_\alpha} d(\xi, u), \xi \in V_\alpha, \\
(0_n, -\rho) & \text{if } \tau \notin U_\alpha, \xi \in \mathbb{R}^{n+1}, \\
\tau & \text{otherwise}. 
\end{cases}
$$

Then

$$\|\sigma_{\hat{O}}(\alpha, \tau, \xi) - \sigma_{\hat{V}, \hat{U}}(\alpha, \tau, \xi)\| = \begin{cases} 
\min_{u \in U_\alpha} d(\xi, u) & \text{if } \tau = \text{argmin}_{u \in U_\alpha} d(\xi, u), \xi \in V_\alpha, \\
0 & \text{if } \tau \notin U_\alpha, \xi \in \mathbb{R}^{n+1}, \\
\min_{v \in V_\alpha} d(v, \tau) & \text{otherwise}. 
\end{cases}
$$

Notice the “otherwise” condition occurs only if $\tau \in U_\alpha$. Finally,

$$\delta^H(\pi_{\hat{V}, \hat{U}}, \pi_\mathcal{O}) := \sup_{\alpha, \tau, \xi} \|\sigma_{\hat{O}}(\alpha, \tau, \xi) - \sigma_{\hat{V}, \hat{U}}(\alpha, \tau, \xi)\| \leq \sup_{\alpha \in I} \left\{ \sup_{\xi \in V_\alpha} \min_{u \in U_\alpha} d(\xi, u), \sup_{\tau \in U_\alpha} \min_{v \in V_\alpha} d(v, \tau) \right\}$$

$$= \sup_{\alpha \in I} d_H(U_\alpha, V_\alpha) =: d_\delta(\hat{U}, \hat{V}) < \epsilon.$$

This implies that $\delta^H(\pi_{\hat{V}, \hat{U}}, \pi_\mathcal{O}) \leq \epsilon$, in turn implying $F^\text{opt}(\pi_{\hat{V}, \hat{U}}) \subseteq \mathcal{O}$ by step 3. Since $F^\text{opt}(\pi_{\hat{V}, \hat{U}}) = F^\text{opt}(\text{RO}(\hat{V}))$, the proof is complete. \(\Box\)
Note that we used a different RO-LSIO transformation in the proof of the second part of Theorem 6.2 than the one given in Theorems 4.2 and 4.4. This new RO-LSIO transformation would have also worked for the proofs of Theorems 5.1 and 5.3, however, we would have had to use the definition \( T := I \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \), and lose some of the geometric intuition of those proofs. On the other hand, we cannot use the RO-LSIO transformation from Theorems 4.2 and 4.4 in the proof of the second part of Theorem 6.2. The reason is because the definition of upper semicontinuity for LSIO problems is fairly weak in that it does not provide estimates for convergence rates. In the proofs of Theorems 5.1 and 5.3, we exploited the fact that Theorem 3.4 (see [9, Theorem 4.3]) had Lipschitz estimates that were invariant for \( \Pi \)-equivalent LSIO problems. That is, given some \( \eta \) with \( \pi_0 \sim \pi_0' \), we can choose \( \epsilon \) sufficiently small so that \( |\nu(\pi) - \nu(\pi_0)| \leq L(\pi_0) \epsilon < \eta \) for all \( \eta \). In other words, the optimal value function \( \nu(\cdot) \) is, in some sense, “uniformly Lipschitz continuous” on \( \Pi \)-equivalent LSIO problems.

In contrast, LSIO upper semicontinuity at \( \pi_0 \) states that, for any given open set \( O \), there exists a neighborhood \( \delta(\pi, \pi_0) \leq \epsilon \) for which each point \( \pi \) in that neighborhood satisfies \( F^\text{opt}(\pi) \subseteq O \) if \( F^\text{opt}(\pi_0) \subseteq O \). Thus, given \( O \) and that \( \pi_0 \sim \pi_0' \), and \( F^\text{opt}(\pi_0') \subseteq O \), even if (from the definition of upper semicontinuity of \( F^\text{opt} \) at \( \pi_0 \) ) guarantees that \( F^\text{opt}(\pi) \subseteq O \) for all \( \pi \) satisfying \( \delta(\pi, \pi_0) \leq \epsilon \), it might not be the case that \( F^\text{opt}(\pi) \subseteq O \) for all \( \pi \) satisfying \( \delta(\pi, \pi_0') \leq \epsilon \). In other words, the optimal solution set \( F^\text{opt}(\cdot) \) is, in some sense, “uniformly continuous,” even on \( \Pi \)-equivalent LSIO problems. Thus, to overcome the lack of an explicit convergence rate in the definition of upper semicontinuity, the new RO-LSIO transformation transfers the machinery of Theorem 4.4 into the second \( \mathbb{R}^{n+1} \) in the index set \( T \).

**Corollary 6.3.** Theorem 6.2 holds for the set \( F \) in place of \( F^\text{opt} \), e.g., if the set \( F(\text{RO}(\hat{U})) \) is nonempty and bounded.

**Proof.** Replace \( c \leftrightarrow 0 \) for any problem to yield the desired continuity results. \( \square \)

**Remark 6.4.** Corollary 6.3 holds for RO problems in which the feasible region is bounded. We may derive more general qualitative continuity results for the feasible region, using the results from [12]. Proving such a stability result for the feasible region, however, would require a careful treatment, including some subtleties due to the differences in the paper [12] and the other papers whose theorems we use in our work. First, we would need to ensure that the RO-LSIO mapping yielded problems within the parameter spaces which are studied in [12]. Second, we would need to show the equivalence of convergence in the metric that we use in our work, and the convergence in the special metric used in [12]. Last, ensuring that the hypotheses of the theorems in [12] are satisfied, we would need to apply these theorems to our setting, or specialize the theorems to LSIO problems and then apply them to our setting.

7. Lipschitz continuity of the \( c \)-optimal solution set.

7.1. Set and function distance notation. The notation in this subsection is taken from [1] and [23]. For any given \( r > 0 \), define the set \( (A)_r := (A \cap rB) \); recall that \( B \) is the unit ball. For two sets \( C \) and \( D \), subsets of Euclidean space, define the “excess of \( C \) on \( D \)” as \( e(C, D) := \sup_{x \in C} \inf_{y \in D} d(x, y) \), where \( d(x, y) \) is the Euclidean metric. Note that \( d_H(C, D) = \max \{ e(C, D), e(D, C) \} \). The following two definitions are taken from [23]: the first one is an extension of the Hausdorff distance, the second one is related to the epigraph distance.
Definition 7.1. Let $C$ and $D$ be two nonempty sets and $r > 0$. Define the following metrics $d_r(C, D)$ and $\hat{d}_r(C, D)$:

\begin{align}
(7.1a) \quad d_r(C, D) &:= \max_{\|x\| \leq r} |d(x, C) - d(x, D)|, \\
(7.1b) \quad \hat{d}_r(C, D) &:= \inf \left\{ \eta \geq 0 \left| C \cap rB \subseteq D + \eta B \right. \right. \\
&\left. \left. D \cap rB \subseteq C + \eta B \right\} \\
(7.1c) \quad &= \max \{ e((C)_r, D), e((D)_r, C) \},
\end{align}

where we recall that $(C)_r = C \cap rB$, where $B$ is the unit ball.

Definition 7.2. Let $f_1$ and $f_2$ be two extended real-valued functions. Define

\begin{align*}
\delta_r(f_1, f_2) &:= d_r(epi f_1, epi f_2), \\
\hat{\delta}_r(f_1, f_2) &:= \hat{d}_r(epi f_1, epi f_2), \\
\hat{\delta}_r^+(f_1, f_2) &:= \inf \left\{ \eta \geq 0 \left| \min_{B(x, \eta)} f_2 \leq \max \{ f_1(x), -r \} + \eta \right. \right. \\
&\left. \left. \min_{B(x, \eta)} f_1 \leq \max \{ f_2(x), -r \} + \eta \right. \forall x \in rB \right\},
\end{align*}

where $epi f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq f(x)\}$.

Definition 7.3. Define the set of $\epsilon$-minimizers of a function $f$ as

\[ \epsilon\text{-argmin} f := \{ x \in \mathbb{R}^n : f(x) \leq \inf f + \epsilon \}. \]

In order to prove the stability results for $\epsilon$-optimal solution sets, we first derive Theorem 7.4, by combining variational analysis and LSIO results from the literature. This theorem provides new insight into the quantitative continuity of $\epsilon$-optimal solution sets for LSIO problems, complimenting the results for the quantitative continuity of the optimal value given in [9].

Theorem 7.4. Let $\pi_0 := (c, \sigma_0)$ such that $\pi_0 \in \text{int}(\Pi_f)$ be given. Let $\eta$ satisfying $0 < \eta < \delta^2(\sigma_0, \Sigma_i)$ be given. Suppose $\pi_1 := (c, \sigma_1)$ and $\pi_2 := (c, \sigma_2)$ are LSIO problems with the following properties:

i. $\delta^2(\sigma_1, \sigma_0) \leq \eta$ and $\delta^2(\sigma_2, \sigma_0) \leq \eta$.

ii. There exists an $r_0 > 0$ such that $r_0 B \cap F^{opt}(\pi_1) \neq \emptyset$ and $r_0 B \cap F^{opt}(\pi_2) \neq \emptyset$ and $\nu(\pi_1) > -r_0$ and $\nu(\pi_2) > -r_0$.

Define the extended real valued functions $f_1(x) := \langle c, x \rangle + \chi_{F(\sigma_1)}(x)$ and $f_2(x) := \langle c, x \rangle + \chi_{F(\sigma_2)}(x)$. Then we have, for all $r > r_0$,

\[ d_r(\epsilon\text{-argmin} f_1, \epsilon\text{-argmin} f_2) \leq (1 + 4r\epsilon^{-1})(1 + \|c\|) \left( \frac{(1 + r)\sqrt{1 + r^2}}{\delta^2(\sigma_0, \Sigma_i) - \eta} \right) \delta^{\Pi}(\pi_1, \pi_2). \]

Proof. The result follows from Lemmas F.1 and F.2 in Appendix F.

We now prove Lipschitz continuity of the $\epsilon$-optimal solution set for RO problems.

Theorem 7.5. Let $\text{RO}(\hat{U})$ be an RO problem of the form (3.5) with nonempty compact and convex uncertainty set $U_\alpha$ in each constraint, indexed by $\alpha \in I$, with fixed cost function $c$. Suppose that

i. $\text{RO}(\hat{U})$ satisfies the strong Slater condition, with strong Slater constant $\rho > 0$,

ii. $F^{opt}(\text{RO}(\hat{U}))$ is nonempty and bounded.
Define the LSIO problem $\pi_{\tilde{U}} \in \text{int}(\Pi_s)$, where

$$\sigma_{\tilde{U}}(t) = \sigma_{\tilde{U}}((\alpha, t, s)) := \begin{cases} (t, s) & \text{if } (t, s) \in U_{\alpha}, \\ (0, -\rho) & \text{if } (t, s) \notin U_{\alpha}. \end{cases}$$

Let $\eta > 0$ satisfying $0 < \eta < \delta^{\text{U}}(\pi_{\tilde{U}}, \text{bd}(\Pi_s))$ be given. Let $r_0 > 0$ satisfy $r_0 B \cap \text{argmin}(\text{RO}(\tilde{U})) \neq \emptyset$ and $\nu(\text{RO}(\tilde{U})) > -r_0$. Suppose $\tilde{V}$, with nonempty, compact, and convex uncertainty sets $V_{\alpha}$, satisfies the following conditions:

1. $d_2(\tilde{U}, \tilde{V}) < \eta$,
2. $r_0 B \cap \text{argmin}(\text{RO}(\tilde{V})) \neq \emptyset$,
3. $\nu(\text{RO}(\tilde{V})) > -r_0$.

Then, for all $r > r_0$,

$$\tilde{d}_r(\epsilon, \text{argmin}(\text{RO}(\tilde{U})), \epsilon, \text{argmin}(\text{RO}(\tilde{V}))) \leq (1 + 4r\epsilon^{-1})(1 + ||c||)(1 + r \frac{\sqrt{1 + r^2}}{\delta^{\text{U}}(\sigma_{\tilde{U}}, \Sigma_i) - \eta}) d_2(\tilde{U}, \tilde{V}).$$

Remark 7.6. We may replace the set of conditions on $\tilde{V}$ with the two conditions

i. $d_2(\tilde{U}, \tilde{V}) < \min\{\eta, \frac{\nu(\text{RO}(\tilde{U}))) + r_0}{L(\pi_{\tilde{U}}, \eta)}\}$,
ii. $r_0 B \cap \text{argmin}(\text{RO}(\tilde{V})) \neq \emptyset$,

if we add the condition that $\sup\{-b^0, \alpha \in I\} < +\infty$. The condition $d_2(\tilde{U}, \tilde{V}) < \min\{\eta, \frac{\nu(\text{RO}(\tilde{U}))) + r_0}{L(\pi_{\tilde{U}}, \eta)}\}$ replaces conditions i and iii in Theorem 7.5 because we can apply Theorem 5.3 to guarantee $\nu(\text{RO}(\tilde{U}))) - \nu(\text{RO}(\tilde{V})) < L(\pi_{\tilde{U}}, \eta)d_2(\tilde{U}, \tilde{V}) \leq \nu(\text{RO}(\tilde{U})) + r_0$ and, hence, $\nu(\text{RO}(\tilde{V})) > -r_0$. Note that we need the assumption that $\sup\{-b^0, \alpha \in I\} < +\infty$, in order to allow us to apply Theorem 5.3 with a finite constant $L(\tilde{U}, \epsilon) = L(\pi_{\tilde{U}}, \eta)$. More details about the necessity of this assumption to guarantee the finiteness of $L(\tilde{U}, \epsilon) = L(\pi_{\tilde{U}}, \eta)$ is given in the proof of Theorem 6.2.

Remark 7.7. Note that unlike Theorem 5.3, the existence of such a $\tilde{V}$ satisfying the conditions of Theorem 7.4 is not guaranteed for any given $\tilde{U}$ and $r_0$. However, assuming the existence of any $\tilde{V}$ that does satisfy the conditions, the results of the theorem hold. In Theorem 7.5, the conditions on $\tilde{V}$ are there because they were technical requirements in [23]. Looking at the proof of [23, Theorem 7.69], however, we may be able to relax the conditions in Theorem 7.5, in particular, condition ii, by using Theorem 5.3. The proof of this conjecture is left for future work.

Proof. The proof is almost identical to that of Theorem 5.3, except that we apply Theorem 7.4 and Lemma D.5 in place of Theorem 3.4 (see [9, Theorem 4.3]).

8. Conclusion. We have showed the continuity of robust linear optimization with respect to perturbations in the uncertainty set. By constructing an explicit RO-LSIO transformation, we showed that the optimal value and $\epsilon$-optimal solution set mappings are Lipschitz continuous, which are novel quantitative stability results for RO. We also showed that the optimal solution set mapping is closed and upper semicontinuous under less restrictive conditions on the uncertainty set compared with previous results in the literature. Our results also extend to the case of robust LSIO problems, which have an infinite number of constraints, and thus may serve as a stepping stone to understanding stability in robust convex problems. These quantitative stability results are important when computing solutions to RO problems, and may
be used as a first step in understanding the effect of computer round-off errors when solving RO problems. Furthermore, as the bounds on the optimal value seem loose (see Example 5.2), an important next step would be to either prove the tightness of our Lipschitz constant, or tighten it.

Acknowledgments. We would like to thank the associate editor and the reviewers for their helpful comments. The second author would like to thank God for His mercy in the course of the research.

REFERENCES