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Deriving compatible taper functions from volume ratio equations based on upper-stem height

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2 ratio equations based on upper-stem height

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17 Abstract

18 A very common model for prediction of tree stem volumes to upper-stem height or di-
19 ameter limits is the use of a merchantable-to-total volume ratio function multiplied by
20 a total stem volume function. Many users of these prediction systems also desire taper
21 equations which can predict heights to upper-stem diameters. While taper equations
22 compatible with volume ratio equations available for many years, compatible taper equa-
23 tions from volume ratio equations that are functions of upper-stem height have been used
24 infrequently. Yet many studies have indicated that height-based ratio equations perform
25 well and frequently have statistics of fit that are comparable to diameter-based volume
26 ratio equations. Compatible taper equations derived from height-based ratio equations
27 are presented here. The methodology that uses height-based merchantable to total vol-
28 ume ratios does not require the solution of a differential equation after differentiating
29 the height-based volume ratio, as is necessary when using the method of deriving taper
30 equations from diameter-based merchantable-to-total volume ratios. This could be an
31 advantage depending on the complexity of the ratio function. Example taper equations
32 fitted to loblolly pine data from the southeastern USA and the state of Oklahoma, USA
33 indicate good fit to these data, whether fitted directly to taper data or implicitly by using
34 parameters fitted to volume ratio data.

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36 KEYWORDS: stem weight estimation, merchantable volume, weight ratio, upper-stem
37 diameter, stem profile, stem form

38

Introduction

Clutter (1980) derived a taper equation that was mathematically compatible with a volume-to-upper-stem-diameter ratio equation of Burkhart (1977). Jordan et al. (2005) used a similar approach to obtain a compatible taper equation from the “exponential ratio” volume-to-upper-stem-diameter equation originally proposed by Van Deusen et al. (1981) and later used with slight modifications by several authors including Parresol et al. (1987) and Bullock and Burkhart (2003). Several researchers including Cao and Burkhart (1980), Cao et al. (1980), Parresol et al. (1987), Bullock and Burkhart (2003) and most recently Zhao and Kane (2017) have used weight and/or volume-to-upper-height ratio functions. Van Deusen et al. (1982) and Reed and Green (1984) derived taper functions compatible with merchantable-to-total height volume ratio functions. Zhao and Kane (2017) described the following conditions that should be satisfied by a volume ratio function based on height, where the ratio function is $R_z(p) = v/V_t$, v (m^3) is stem cumulative volume from the butt to height h (m) above ground, V_t (m^3) is the total stem volume, and $p = \frac{h}{H}$ where H (m) is total tree height with $0 \leq h \leq H$:

- (I) $R_z(p) = 0$, if $p = 0$;
- (II) $R_z(p) = 1$, if $p = 1$;
- (III) $\frac{\partial R_z(p)}{\partial p} \geq 0$, for $0 \leq p \leq 1$;
- (IV) $\frac{\partial^2 R_z(p)}{\partial^2 p} \leq 0$, for $0 \leq p \leq 1$.

Condition I states that the merchantable-to-total volume ratio should be zero when the merchantable-to-total height ratio is zero. This naturally follows from the fact that the volume of a geometric solid is a three-dimensional quality that requires non-zero height as well as cross-sectional area. Condition II states that the merchantable-to-total volume ratio should be equal to one when the merchantable-to-total height is equal to one. This condition follows from the fact that if the merchantable height is set equal to total

height, the numerator of the ratio function $R(p)$ must be equal to its denominator, total volume. Condition III states that the rate of change in the merchantable-to-total volume ratio with respect to increase in the merchantable-to-total height ratio should be greater than or equal to zero. This results from the fact that any infinitesimal increase in the merchantable-to-total height ratio p results in the addition of a “thin disk” of merchantable volume to the numerator of the volume ratio $R(p)$. Condition IV states that the rate of this latter increase in $R(p)$ with respect to p actually decreases with increasing p , because the diameter of each additional thin disk decreases with increasing p due to stem taper. Although the conditions (especially I-III) might be regarded as evident given the nature of the situation and graphs of cumulative relative stem volume data presented by Zhao and Kane (2017), a mathematical proof of condition IV is given in Appendix A. The proof highlights the role of the taper function which will be significant in this article.

The procedure for developing compatible taper equations from these upper-stem height based volume ratio prediction systems is similar to the approach used by Clutter (1980) but differs in an important way because when the height-based ratio system is used it is not necessary to solve a differential equation to obtain a compatible taper equation. This can be a significant advantage depending on the mathematical complexity of the ratio function.

Basic derivation of taper function

We can equate volumes to upper-stem height limits with the volume ratio function to the classic tree stem “slices” volume integral as follows:

$$V_t R(h) = \int_s^h k \times d^2(h) \partial h \quad (1)$$

where V_t is total stem volume outside bark in cubic meters (cubic feet, English unit), $R(h) = v/V_t$ is the ratio of outside bark volume in cubic meters up to upper-stem height h in meters (feet, English unit) to total outside bark volume in cubic meters, $d(h)$ is

outside bark diameter in centimeters (inches, English unit) at upper-stem height h in meters from groundline, $k = \frac{\pi}{40,000}$ for use with metric units ($k = \frac{\pi}{576}$ for use with English units) and s is stump height in meters. Since s can be considered a constant with respect to changes in h and the derivative of a constant is zero, the value of s does not affect the derivation of compatible taper equations. Now we may differentiate both sides with respect to upper-stem height:

$$V_t \frac{\partial R(h)}{\partial h} = k \times d^2(h) \quad (2)$$

Solving for $d(h)$ yields the taper function:

$$d(h) = \sqrt{\frac{V_t}{k} \frac{\partial R(h)}{\partial h}} + \varepsilon \quad (3)$$

We have added the error term ε having zero mean ($E(\varepsilon) = 0$) and variance σ^2 because this model would of course be fitted to individual tree data subject to natural variation. Subsequently we will add ε to taper functions after their mathematical derivation is complete. This derivation is essentially similar to derivations found in Van Deusen et al. (1982) and Reed and Green (1984). As an example let us derive a taper function compatible to the volume ratio function termed the “modified Burkhart” by Cao et al. (1980), and also used by Cao and Burkhart (1980) and for weight ratio estimation by Bullock and Burkhart (2003). Cao et al. (1980) found that this equation was superior among several alternatives including taper functions available at that time (e.g. Max and Burkhart 1976) for the purpose of estimating volume to a specified upper-stem height limit. The equation has the following form where H is total height in meters (feet, English unit) and β_i are parameters for $i = 0, 1, 2$

$$R(h) = 1 + \beta_0 \left(\frac{(H - h)^{\beta_1}}{H^{\beta_2}} \right) \quad (4)$$

108

$$\frac{\partial R(h)}{\partial h} = -\beta_0\beta_1 \left(\frac{(H-h)^{\beta_1-1}}{H^{\beta_2}} \right) \quad (5)$$

109 Equation 5 meets Condition III because the fitted value for β_0 must be negative. According
 110 to equation 3 the taper function then becomes;

$$d(h) = \sqrt{-\frac{V_t}{k}\beta_0\beta_1 \left(\frac{(H-h)^{\beta_1-1}}{H^{\beta_2}} \right)} + \varepsilon \quad (6)$$

111 This is similar to derivations by Van Deusen et al. (1982) and Reed and Green (1984). If
 112 it is desired to find the upper-stem height $h(d)$ corresponding to upper-stem diameter d
 113 this taper function can be inverted to obtain:

$$h(d) = H - \left[\frac{-kd^2 H^{\beta_2}}{V_t\beta_0\beta_1} \right]^{\frac{1}{\beta_1-1}} + \varepsilon \quad (7)$$

114 The advantage of equating the “slices” volume integral of equation 1 to the height-
 115 based volume ratio is that it does not require the solution of a differential equation after
 116 differentiation to derive the taper function as would be required by the Clutter (1980)
 117 approach using volume ratios to an upper-stem diameter limit. When using the function
 118 we need to predict total volume with a volume equation such as the well-known and
 119 frequently applied Schumacher and Hall (1933) equation:

$$V_t = a_0 D^{a_1} H^{a_2} \quad (8)$$

120 where H is total height (m), D is DBH (cm) and a_i are parameters for $i = 0, 1, 2$. Perhaps
 121 the first to estimate parameters of this equation in the nonlinear form shown here were
 122 Moser and Beers (1969). With this total volume equation the complete taper equation
 123 would be:

$$d(h) = \sqrt{-\frac{a_0}{k} D^{a_1} H^{a_2} \times \beta_0\beta_1 \left(\frac{(H-h)^{\beta_1-1}}{H^{\beta_2}} \right)} + \varepsilon \quad (9)$$

124 or, inverted to predict upper-stem height at a specified upper-stem diameter:

$$h(d) = H - \left[\frac{-kd^2 H^{\beta_2}}{a_0 D^{a_1} H^{a_2} \beta_0 \beta_1} \right]^{\frac{1}{\beta_1 - 1}} + \varepsilon \quad (10)$$

125 **Taper equations compatible with new ratio equations of**
126 **Zhao and Kane (2017)**

127 The best fitting of the new height ratio models recently proposed by Zhao and Kane
128 (2017) were based on equations numbered A.11 and A.6 in their paper. Equation A.11 of
130 Zhao and Kane (2017) is

$$R_z(p) = 1 - \beta (1 - p)^\alpha \quad (11)$$

131 where $p = h/H$, $0 \leq p \leq 1$ and $R_z(p) = R(pH) = R(h)$ is the ratio of volume up to height
133 $h = pH$ to total volume and $\alpha > 1$ and $\beta \geq 0$ are parameters. This implies:

$$\frac{\partial R(h)}{\partial h} = \frac{\alpha \beta}{H} (1 - p)^{\alpha-1} \quad (12)$$

134 The resulting taper function is:

$$d(h) = \sqrt{V_t \frac{\alpha \beta}{kH} \left(1 - \frac{h}{H}\right)^{\alpha-1}} + \varepsilon \quad (13)$$

135 or inverted to predict upper-stem height for a given upper-stem diameter:

$$h(d) = H \left\{ 1 - \left[\frac{kd^2 H}{V_t \alpha \beta} \right]^{\frac{1}{\alpha-1}} \right\} + \varepsilon \quad (14)$$

138 As indicated by Zhao and Kane (2017) ratio function equation 11 implicitly includes
139 the ratio function equation 4 used by Cao and Burkhart (1980), as well as several later
140 authors. In fact with a re-parameterization the equivalence of equations 11 and 4 can be
141 demonstrated algebraically (available from the authors on request). Equation 13 implies
143 the following stem diameter at ground line d_g when $h = 0$:

$$d_g = \sqrt{V_t \frac{\alpha \beta}{kH}} + \varepsilon \quad (15)$$

144 However, this may not be reliable because the form of the ratio function (11) does not
145 equal zero when $h = 0$. Naturally, groundline diameter is not generally synonymous with

146 stump diameter unless heights are defined as above stump.

147 Equation A.6 of Zhao and Kane (2017) is:

$$R_z(p) = [1 - (1 - p)^\alpha]^\beta \quad (16)$$

148 where $0 < \beta \leq 1$, $\alpha > 1$ and $0 \leq p \leq 1$. For this equation the change rate in the volume
149 ratio is:

$$\frac{\partial R(h)}{\partial h} = \frac{\alpha\beta}{H} \left[1 - \left(1 - \frac{h}{H} \right)^\alpha \right]^{\beta-1} \left(1 - \frac{h}{H} \right)^{\alpha-1} \quad (17)$$

150 The resulting taper function is:

$$d(h) = \sqrt{\frac{V_t \alpha \beta}{kH} \left[1 - \left(1 - \frac{h}{H} \right)^\alpha \right]^{\beta-1} \left(1 - \frac{h}{H} \right)^{\alpha-1}} + \varepsilon \quad (18)$$

152 In order to obtain upper-stem heights $h(d)$ to a specified upper-stem diameter d the
153 inverse function must be solved using numerical methods. This can easily be done using
154 SOLVER in EXCEL (Flystra 1998) or by using one of several numerical routines such as
155 the secant method, Newton's method (e.g. Conti and de Boor 1972 p. 32-34) or others
156 available in various programming languages. This taper function is able to model butt
157 swell.

158 Based on an analysis of equation 16 Zhao and Kane (2017, eq. A.6) proposed the
159 following model (equation 8 in Zhao and Kane 2017) where θ and ϕ are parameters in
160 addition to those defined above:

$$V_t R(h) = a_0 D^{a_1} H^{a_2} \left[1 - \left(1 - \frac{h}{H} \right)^\alpha \right]^{\{1 - \theta \exp[-\exp(\phi D^{a_1} H^{a_2})]\}} \quad (19)$$

161 Differentiating this expression with respect to upper-stem height h yields:

$$\begin{aligned} 162 \quad & V_t \frac{\partial R(h)}{\partial h} = \\ 163 \quad & \frac{\alpha}{H} a_0 D^{a_1} H^{a_2} \left(1 - \frac{h}{H} \right)^{\alpha-1} \{1 - \theta \exp[-\exp(\phi D^{a_1} H^{a_2})]\} \left[1 - \left(1 - \frac{h}{H} \right)^\alpha \right]^{\{-\theta \exp[-\exp(\phi D^{a_1} H^{a_2})]\}} \end{aligned} \quad (20)$$

164 The corresponding compatible taper function is then:

165

$$d(h) =$$

166

$$\sqrt{\frac{\alpha}{kH} a_0 D^{a_1} H^{a_2} \left(1 - \frac{h}{H}\right)^{\alpha-1} \{1 - \theta \exp[-\exp(\phi D^{a_1} H^{a_2})]\}} \left[1 - \left(1 - \frac{h}{H}\right)^{\alpha}\right]^{\{-\theta \exp[-\exp(\phi D^{a_1} H^{a_2})]\}} + \varepsilon \quad (21)$$

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Zhao and Kane developed the following equation which could be expressed as a special case of equation 11 (Zhao and Kane 2017 equation 11) where c_i , $i = 0, 1, 2$ are parameters in addition to those previously defined:

$$V_t R(h) = a_0 D^{a_1} H^{a_2} - c_0 D^{c_1} H^{c_2} \left(1 - \frac{h}{H}\right)^{\alpha} \quad (22)$$

170

Differentiation with respect to h gives:

$$V_t \frac{\partial R(h)}{\partial h} = \frac{\alpha}{H} c_0 D^{c_1} H^{c_2} \left(1 - \frac{h}{H}\right)^{\alpha-1} \quad (23)$$

171

This results in the following compatible taper equation:

$$d(h) = \sqrt{\frac{\alpha}{kH} c_0 D^{c_1} H^{c_2} \left(1 - \frac{h}{H}\right)^{\alpha-1}} + \varepsilon \quad (24)$$

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This equation 24 can be solved algebraically to obtain an equation that can be used to predict the upper-stem height corresponding to a specified upper-stem diameter d :

$$h(d) = H \left\{ 1 - \left(\frac{d^2 k H}{\alpha c_0 D^{c_1} H^{c_2}} \right)^{\frac{1}{\alpha-1}} \right\} + \varepsilon \quad (25)$$

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Taper functions corresponding to additional new height-based volume ratio functions presented in Zhao and Kane (2017) are given in Appendix B.

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Methods

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178

Parameter estimates for taper equations 21 and 24 can be obtained in three ways. The first way is to extract parameter estimates from the corresponding fitted cumulative volume

equations. For example, parameter estimates for taper equations 21 and 24 are available from Zhao and Kane (2017) in which the outside bark volume ratio equations were fitted to cumulative volume data of 480 trees sampled from loblolly pine plantations in the southeastern (SE) USA. Parameter estimates for equation 21 are located in Table 3 of Zhao and Kane (2017) while parameter estimates suitable for equation 24 can be obtained in Table 4 of Zhao and Kane (2017). The second way is to fit the taper equations directly to taper measurement data. We used nonlinear ordinary least squares to accomplish this. The third way is to simultaneously fit the taper equation and the corresponding cumulative volume equation with the taper measurement data and cumulative volume data of the sampled trees. In this study, we fitted taper equations 21 and 24 to taper measurement data of the same sample trees as used by Zhao and Kane (2017) obtained from loblolly pine plantations in Georgia, Alabama, Florida, Mississippi, and South Carolina (hereafter, referred to as SE dataset). We also fitted these equations to taper data of 158 trees sampled from loblolly pine plantations in the state of Oklahoma, USA ranging in *DBH* from 6.7 cm to 53 cm and in total height from 6.5 m to 24.3 m (hereafter, referred to as OK dataset). We compared these results to results from fitting the well-known Max-Burkhart (1976) taper equation. The Max-Burkhart (1976) segmented polynomial taper equation consists of three polynomial segments designed to model stem taper at the base of the bole, the mid-bole and the upper bole and was initially fitted to loblolly pine taper data. Since its initial development, the Max-Burkhart (1976) equation and variants have been applied successfully to taper modeling of several tree species (Burkhart and Tomé 2012, p. 16-20). The taper equation of Max and Burkhart (1976) was rewritten in the following form:

$$d(h) = D \sqrt{b_1 \left(\frac{h}{H} \right) + b_2 \left(\left(\frac{h}{H} \right)^2 - 1 \right) + b_3 \left(a_1 - \frac{h}{H} \right)^2 I_1 + b_4 \left(a_2 - \frac{h}{H} \right)^2 I_2 + \varepsilon} \quad (26)$$

where:

$$I_1 = \begin{cases} 1 & \text{if } \frac{h}{H} \leq a_1 \\ 0 & \text{if } \frac{h}{H} > a_1 \end{cases}, \quad I_2 = \begin{cases} 1 & \text{if } \frac{h}{H} \leq a_2 \\ 0 & \text{if } \frac{h}{H} > a_2 \end{cases}$$

We will refer to Equation 26 as MB76 hereafter. We fitted this form to both the SE dataset and OK dataset. Three statistical criteria: average bias (E) where bias is observed minus predicted, root mean square error ($RMSE$), and the proportion of total variation in the dependent variable explained (i.e. the coefficient of determination (R^2)) were employed to evaluate the goodness of fit for the new fitting. We also evaluated the average bias from each taper equation at various points on the stem and presented the results graphically. Kozak and Kozak (2003) questioned the usefulness of data splitting techniques for validation of tree volume prediction equations. Therefore, we believe a better validation can be obtained by validating equations fitted to the SE dataset on the OK dataset, and vice versa. The OK dataset comes from just north of the extreme northwest corner of the loblolly pine natural range, so it is geographically different from the locations associated with the SE dataset. For the purpose of validation, the same statistics (E , $RMSE$ and R^2) were used to evaluate the fitted equations for the SE data on the OK dataset, and the fitted taper equations for the OK dataset on the SE dataset. They are also used to assess the performance of taper equations 21 and 24 (designated as equations 21z and 24z) with the parameter estimates obtained from the fitted cumulative volume equations (Zhao and Kane 2017). The behavior of these taper equations were also compared graphically both on the fitting and validation datasets.

Results and Discussion

Figure 1 illustrates the typical taper curves for a loblolly pine tree having DBH of 30 cm and total height of 21 m, provided by the equations 21 and 24 according to the parameter values in Zhao and Kane (2017). Comparing the two curves indicates that equation 21 is capable of modeling butt swell at the base of the tree while the mathematically simpler form of equation 24 does not account for butt swell. The new parameter estimates of equations 21 and 24, when fitted to taper measurement data, were shown in Table 1. There was little difference in the parameter values between the new estimates (from the second way) and the estimates extracted from the fitted cumulative volume equations

(i.e., from the first way; parameter values not shown here but given in Zhao and Kane 2017). This suggests that it is reasonable to use the third way to estimate the parameters. That is, fitting the taper equations and the corresponding cumulative volume equations simultaneously leads to exactly compatible taper functions with volume ratio equation in both model form and parameter estimates. This third approach has been used for a taper equation compatible with an upper-stem diameter-based volume ratio model by Jordan et al. 2005 using the nonlinear seemingly unrelated regressions technique. We did not use the third way, simultaneous fitting, in this paper. Compared with the taper equations with parameters extracted from the fitted cumulative volume equations (21z and 24z), the new fitted taper equations had higher values of R^2 and smaller values E and $RMSE$ (Table 2). No matter how the parameters were estimated, equation 21 was better than equation 24 in terms of E , $RMSE$ and R^2 (Table 2). Average biases of estimated diameter outside bark at different stem portions also confirmed that equation 21 was better than equation 24, especially from ground to half total height (Figure 2). The widely-used MB76 (equation 26) performed slightly better on the fitting data for the SE dataset than either equation 21 or equation 24.

When fitted to the OK data the taper equations performed similarly to the SE data (Table 1) with $R^2 > 0.97$ for all three equations. The absolute value of mean bias E was smaller than 0.07cm for all three OK equations. Figure 3 indicates that the pattern of biases at various stem heights was similar to that of the SE data in Figure 2 with equation 21 having generally smaller biases than equation 24 except at heights in excess of 80 percent of total height. As with the SE data, $RMSE$ and R^2 for the MB76 equation 26 were superior to equations 21 or 24 but the value of E associated with MB76 (equation 26) was slightly higher than the other two equations on the OK fitting data.

When validated using the OK dataset taper equation 21 fitted to the SE data performed slightly better than MB76 (equation 26) in terms of E , $RMSE$ and R^2 (Table 2, Figure 3). However, when fitted to the OK dataset, MB76 (equation 26) performed better in a validation with the SE data than either equations 21 or 24. All taper equations had an $R^2 \geq 0.96$ for both the fitting and validation datasets. Figure 3 indicates that equation

260 21 performed better in terms of bias than MB76 (equation 26) when fitted to SE data and
261 validated on the OK data for the bole below 1.37m and for the portion of the bole between
262 20% and 70% of total height. The latter range of the bole can be important for estimation
263 of sawlog merchantable height. On its own SE fitting data however, MB76 (equation 26)
264 performed better than the other two taper equations in terms of bias on most portions
265 of the bole (Figure 2). The MB76 (equation 26) fitted to the OK data performed better
266 than the other two equations in terms of bias on most portions of the bole but equation
267 21 was very competitive and actually had the lowest bias below 0.5m. Figure 1 illustrates
268 that equation 21 can represent butt swell and MB76 (equation 26) contains a quadratic
269 sub-model which models butt swell (Max and Burkhart 1976). This study suggests that
270 equation 21 can be competitive with MB76 (equation 26) for modeling loblolly pine taper,
271 and equation 21 is mathematically compatible with a volume ratio model that performed
272 well in the study of Zhao and Kane (2017). Both MB76 (equation 26) and equation 21
273 have six parameters. Equation 21 appears to have a rather complex nonlinear form while
274 MB76 (equation 26) is a function of first and second degree polynomial terms, but MB76
275 (equation 26) additionally contains join points at segment boundaries. Thus, MB76 (equa-
276 tion 26) and equation 21 are about equally complex. Although it has a relatively complex
277 mathematical form equation 21 can readily be implemented in a spreadsheet, and might
278 be easier to implement in spreadsheet environments for users who are not professional
279 biometricians than MB76 (equation 26) because it lacks the indicator variables and join
280 points that are required by MB76. The simpler mathematical form of equation 24 which
281 contains only four parameters does not permit precise modeling of butt swell at the base
282 of the tree stem (Figure 1). Figures 2 and 3 indicate that equation 24 did not do as well
283 as MB76 (equation 26) and equation 21 on the lowest portion of the stem for the OK and
284 SE fitting data and the SE validation data, although surprisingly it did better than MB76
285 (equation 26) on the OK validation data at the very base of the stem. Although taper
286 model 24 does not perform quite as well as the other two alternatives tested, it would still
287 likely be acceptable for many purposes, such as determination of heights to merchantable
288 tops on the upper-stem, where excellent performance at the base of the stem may not be

essential.

The method presented here could be applied to derive a compatible taper equation from a weight ratio equation if the ratio function in equation 2 (where total stem weight would replace V_t in the equation) is adjusted by a reasonable density factor or possibly a density function depending on stem height h . Following equation 2 above but using a density factor to convert cubic volume to green or dry weight, let W_t be total tree weight (kg) and $\delta(h)$ be density in units of kg/m^3 as a function of height on the tree stem h .

$$W_t R(h) = \int_s^h k \delta(h) \times d^2(h) \partial h \quad (27)$$

In some cases a constant value may suffice for $\delta(h)$, especially for green weight. Differentiation of both sides and algebraic rearrangement results in the taper function:

$$d(h) = \sqrt{\frac{W_t}{k \delta(h)} \frac{\partial R(h)}{\partial h}} + \epsilon \quad (28)$$

The importance of condition IV for volume ratio equations, described in the Introduction, is relevant to the proper function of compatible taper functions. The proof of condition IV in Appendix A indicates that condition IV insures that the compatible taper function will be monotonically decreasing as h increases from 0 to total tree height H . We have fitted ratio equations that did not conform to condition IV (results not presented in this article but available from senior author on request) and observed that the corresponding taper functions in these instances contained sections that increased rather than monotonically decreasing as h goes from 0 to H , a condition sometimes referred to as “reverse taper.” While reverse taper is sometimes observed on short sections of actual forest trees it is not a logical property of a taper model which is intended to represent average taper for all trees of a given total height and DBH within a population from which sample trees have been selected.

310 Conclusions

311 Two stem taper equations that are mathematically compatible with volume ratio equations
 312 based on upper-stem height have been developed and tested on loblolly pine data from
 313 the Southeastern USA and Oklahoma USA. When fitted to data the pair of equations
 314 (19, 21) and the pair of equations (22, 24) each provide a system of compatible taper
 315 and volume ratio equations that can be used to predict volumes to upper-stem height
 316 limits and to provide stem profile models. If it is desired to predict volume to a specified
 317 upper-stem diameter limit, the taper functions can be used to predict the upper-stem
 318 height corresponding to that diameter limit and then the compatible height ratio model
 319 can be used to predict the volume to that upper-stem diameter and height location. These
 320 compatible taper equations provided a good fit to taper data for loblolly pine from the
 321 southeastern USA and Oklahoma USA with $R^2 > 0.96$ for both equations fitted either
 322 directly to taper data or implied by using coefficients from fitting volume ratio equations.
 323 Comparisons of taper shapes generated by equations 21 and 24 suggest that equation 21
 324 might be more suitable for datasets that exhibit substantial butt swell. This is borne
 325 out by the fact that equation 21 fitted the SE and OK data better in terms R^2 and
 326 $RMSE$ than equation 24 with smaller biases on the lower portion of the bole. Equation
 327 21 was competitive in performance with the well-known MB76 equation 26, performing
 328 somewhat better than MB76 on the OK validation data but slightly inferior to MB76
 329 on the SE and OK fitting data and the SE validation data. Alternatively, the simple
 330 power taper equation (equation 24) also performed well for both the SE and OK data
 331 and similar equations reported in the taper modeling literature have often been used and
 332 have frequently been sufficiently accurate for many applications.

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397

398 Appendix A: Proof of Conditions III and IV

399 We desire to prove the necessity of Condition III: $\frac{\partial R(p)}{\partial p} > 0$ and Condition IV: $\frac{\partial^2 R(p)}{\partial^2 p} < 0$.

400 The proof follows from differentiating the ratio function twice as indicated below and
401 utilizes a taper function defined as $d_m = f(h)$ where d_m is tree stem diameter as a
402 function $f(h)$ of height on the stem above ground h .

403

$$R(p) = \frac{v(h_m)}{V_t} = \frac{1}{V_t} \left\{ \int^{h_m} k \times d_m^2 dh \right\} = \frac{k}{V_t} \left\{ \int^{h_m} [f(h)]^2 dh \right\}$$

$$\Rightarrow \frac{\partial R(p)}{\partial p} = \frac{k}{V_t} \left\{ [f(h_m)]^2 \frac{\partial h_m}{\partial p} \right\} = \frac{kH}{V_t} [f(h_m)]^2 > 0$$

404

405 verifying condition III

$$\Rightarrow \frac{\partial^2 R(p)}{\partial^2 p} = \frac{kH}{V_t} 2f(h_m) \frac{\partial f(h_m)}{\partial h_m} \frac{\partial h_m}{\partial p} = \frac{kH^2}{V_t} 2f(h_m) \frac{\partial f(h_m)}{\partial h_m} < 0$$

406 verifying condition IV

407 The last inequality holds because it is reasonable to model the taper function $d_m =$
408 $f(h_m)$ where h_m is upper stem height corresponding to upper-stem diameter d_m as a
409 continuous differentiable function that is monotonically decreasing with increasing h_m so
410 that the first derivative of $f(h_m)$ has to be negative $\frac{\partial f(h_m)}{\partial h_m} < 0$. In the derivation above
411 V_t is total volume, $v(h_m)$ is volume to merchantable height $h_m = Hp$, implying $\frac{\partial h_m}{\partial p} = H$,
412 H is total height, $k = \pi/40,000$ for the usual metric units with diameter in centimeters,
413 height in meters, and volume in cubic meters, and other symbols are as defined in the body
414 of the manuscript. The second line in the derivation above proves condition III although
415 it is fairly obvious. Another simple way to think about this proof is that as p increases
416 from zero to 1, thin disks of volume (or weight) are being added to the numerator of $R(p)$.
417 That implies condition III. But the disks are getting smaller in diameter as p increases
418 toward 1 at the tip of the tree, so a smaller amount is being added to the numerator of

419 $R(p)$ per unit of p as p increases to 1 and that implies condition IV. This says in words
420 essentially what the mathematical proof above says.

421

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422

423 Appendix B: taper equations corresponding to new Zhao 424 and Kane (2017) volume ratio functions

425 In addition to the examples above Zhao and Kane (2017) presented several other new
426 height-based ratio functions. In this section are presented the taper functions corre-
427 sponding each of these volume ratio functions. Each taper function has been derived in
428 the manner described in the main body of the paper above using equation 3. Zhao and
429 Kane 2017 equation A.5 is:

$$430 \quad R(h) = 1 - \left(1 - \frac{h}{H}\right)^\alpha; \quad \alpha > 1, \quad 0 \leq p \leq 1 \quad (\text{A.1})$$

431 with compatible taper function:

$$432 \quad d(h) = \sqrt{\frac{\alpha V_t}{Hk} \left(1 - \frac{h}{H}\right)^{\alpha-1}} + \varepsilon \quad (\text{A.2})$$

433 The taper function corresponding to Zhao and Kane (2017) equation A.6 was derived in
434 the body of the paper above. The ratio function for Zhao and Kane (2017) equation A.7
435 is:

$$436 \quad R(h) = 1 - \left(1 - \frac{h}{H}\right) \exp\left(-\alpha \frac{h}{H}\right); \quad \alpha > 0, \quad 0 \leq p \leq 1 \quad (\text{A.3})$$

437 with compatible taper function:

$$438 \quad d(h) = \sqrt{\frac{V_t}{Hk} \exp\left(-\alpha \frac{h}{H}\right) \left\{1 + \alpha - \frac{\alpha h}{H}\right\}} + \varepsilon \quad (\text{A.4})$$

439 The ratio function A.8 of Zhao and Kane (2017) is:

$$440 \quad R(h) = \frac{h}{H} \exp\left(\alpha \left(1 - \frac{h}{H}\right)\right); \quad 0 < \alpha \leq 1, \quad 0 \leq p \leq 1 \quad (\text{A.5})$$

441 with corresponding taper function:

$$442 \quad d(h) = \sqrt{\frac{V_t}{Hk} \exp\left(\alpha \left(1 - \frac{h}{H}\right)\right) \left\{1 - \frac{\alpha h}{H}\right\}} + \varepsilon \quad (\text{A.6})$$

443 Zhao and Kane (2017) ratio equation A.9 is:

444 $R(h) = 1 - \left(1 - \frac{h}{H}\right)^\alpha \exp\left(-\beta \frac{h}{H}\right); \alpha \geq 1, \beta > 0, 0 \leq p \leq 1$ (A.7)

445 with compatible taper equation:

446 $d(h) = \sqrt{\frac{V_t}{Hk} \exp\left(-\beta \frac{h}{H}\right) \left(1 - \frac{h}{H}\right)^{\alpha-1} \left\{\alpha + \beta \left(1 - \frac{h}{H}\right)\right\}} + \varepsilon$ (A.8)

447 Ratio equation A.10 of Zhao and Kane (2017) is:

448 $R(h) = 1 - \left(1 - \frac{h}{H}\right) \frac{\alpha}{\alpha + \frac{h}{H}}; \alpha > 0, 0 \leq p \leq 1$ (A.9)

449 the compatible taper equation is:

450 $d(h) = \sqrt{\frac{V_t(\alpha^2 + \alpha)}{Hk\left(\alpha + \frac{h}{H}\right)^2}} + \varepsilon$ (A.10)

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451

452 Table 1. Asymptotic parameter estimates and their stand errors and p-values for Equa-
453 tions 21, 24, and taper equation 26 of Max and Burkhart (MB76). Equations 21 and 24
454 were refitted and MB76 was fitted to taper measurement data of the same sample trees
455 as used by Zhao and Kane (2017) from loblolly pine plantations in the southeastern USA
456 (SE); three equations were also fitted directly to taper data from loblolly pine plantations
457 in OK.

Eq.	Param.	Est. SE	Est. OK	Std. err. SE	Std. err OK	p-value SE & OK
21	$\hat{a}_0$	4.18E-5	4.09E-5	9.14E-7	1.66E-6	<0.0001
	$\hat{a}_1$	1.19159	1.9030	0.0075	0.0142	<0.0001
	$\hat{a}_2$	1.0556	1.0790	0.0116	0.0185	<0.0001
	$\hat{\alpha}$	2.2683	2.3500	0.0064	0.0112	<0.0001
	$\hat{\theta}$	0.3422	0.3727	0.0064	0.0132	<0.0001
	$\hat{\phi}$	6.01E-6	8.68E-6	7.04E-7	1.18E-6	<0.0001
24	$\hat{c}_0$	4.39E-5	4.30E-5	1.18E-6	2.02E-6	<0.0001
	$\hat{c}_1$	1.9153	1.8901	0.0094	0.0173	<0.0001
	$\hat{c}_2$	1.0463	1.0872	0.0145	0.0230	<0.0001
	$\hat{\alpha}$	2.5231	2.6244	0.0058	0.0102	<0.0001
MB76	$\hat{b}_1$	-5.1970	-4.3703	0.1799	0.1561	<0.0001
	$\hat{b}_2$	2.5884	2.1681	0.0978	0.0851	<0.0001
	$\hat{b}_3$	115.7393	63.2916	4.9027	4.7052	<0.0001
	$\hat{b}_4$	-2.5021	-2.1950	0.0913	0.0780	<0.0001
	$\hat{a}_1$	0.0773	0.1025	0.0015	0.0036	<0.0001
	$\hat{a}_2$	0.7898	0.7418	0.0070	0.0120	<0.0001

459

460 Table 2. Fit statistics of Equations 21, 24 and 26 (MB76) fitted directly to taper mea-
 461 surement data from loblolly pine plantations in the southeastern USA (SE) and OK;
 462 Validation statistics of Equations 21 and 24 (21z and 24z) with parameter estimates ob-
 463 tained from the fitted merchantable volume equations of Zhao and Kane (2017) and OK
 464 taper equations (21OK, 24OK, MB76OK) on the SE data, and SE taper equations (21SE,
 465 24SE, MB76SE) on the OK data.

Data	Equation	E	$RMSE$	R^2
.....Fitting.....				
SE	21	-0.012	1.100	0.982
	24	0.008	1.381	0.972
	MB76	0.062	0.962	0.986
OK	21	-0.043	1.583	0.981
	24	0.010	1.930	0.971
	MB76	0.065	1.404	0.985
.....Validation.....				
SE	21z ¹	-0.175	1.39	0.972
	24z ¹	0.325	1.611	0.962
	21OK	-0.033	1.144	0.981
	24OK	-0.085	1.428	0.970
	MB76OK	0.300	1.012	0.985
OK	21SE	-0.076	1.628	0.979
	24SE	0.045	1.978	0.970
	MB76SE	-0.882	2.266	0.960

467

468 $E = \frac{\sum(y-\hat{y})}{n}$, $RMSE = \sqrt{\frac{\sum(y-\hat{y})^2}{n}}$, $R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2}$

469 ¹parameter estimates for equations 21z and 24z were obtained from Zhao and Kane (2017)

470

471 **Figures**

472 Figure 1. Typical taper curves for a loblolly pine tree with DBH 30 cm and total height
473 21 m, provide by taper equations 21 and 24 with parameter estimates obtained from the
474 commercial volume equations of Zhao and Kane (2017).

475

476

477 Figure 2. Average biases of estimated diameter outside bark by height from ground using
 478 the taper equations 21 (Model 21) 24 (Model 24) and 26 (MB 76) that were fitted to taper
 479 measurement data of 480 trees sampled from loblolly pine plantations in the southeastern
 480 USA (Zhao and Kane 2017), and validating the OK taper equations on that same data
 481 (Model 21OK, Model 24OK, and MB76OK are equations 21, 24, and 26 fitted to the OK
 482 data).

483

484
485 Figure 3. Average biases of estimated diameter outside bark by height from ground using
486 the taper equations 21 (Model 21) 24 (Model 24) and 26 (MB 76) that were fitted to taper
487 measurement data of 158 trees sampled from loblolly pine plantations in Oklahoma, USA,
488 and validating the SE taper equations on that same data (Model 21SE, Model 24SE and
489 MB76SE are equations 21, 24, and 26 fitted to the SE data).