# Evolution of Inhomogeneous LTB Geometry with Tilted Congruence and Modified Gravity

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<tr>
<th>Journal:</th>
<th>Canadian Journal of Physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript ID</td>
<td>cjp-2017-0214.R1</td>
</tr>
<tr>
<td>Manuscript Type:</td>
<td>Article</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>10-Apr-2017</td>
</tr>
<tr>
<td>Complete List of Authors:</td>
<td>Yousaf, Z.; Punjab University, Department of Mathematics</td>
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<td>Bhatti, Muhammad; Punjab University, Mathematics</td>
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<td></td>
<td>Rafaqat, A; Punjab University, Department of Mathematics</td>
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<tr>
<td>Keyword:</td>
<td>Dissipative fluids, Relativistic interiors, f(R) gravity, Self-gravitating systems, Modified gravity</td>
</tr>
<tr>
<td>Is the invited manuscript for consideration in a Special Issue?:</td>
<td>N/A</td>
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Evolution of Inhomogeneous LTB Geometry with Tilted Congruence and Modified Gravity

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Abstract
The goal of this paper is to shed some light on the significance of congruence of observers which seems to affect the dynamics of the universe under Palatini $f(R)$ formalism. Starting by setting up the formalism needed, we have explored the field equations using Lemaitre-Tolman-Bondi geometry as an interior metric. We have formulated the relationship between the matter variables as seen by the observers in both comoving and non-comoving frames. The dynamical equations are evaluated to study the dynamics of inhomogeneous universe by exploring conservation equations along with the Ellis equations. We have also explored a collapsing factor describing the bouncing phenomena via transport equation and conclude the stability region.

Keywords: $f(R)$ gravity; Structure scalars; Relativistic dissipative fluids.

PACS: 04.20.Cv; 04.40.Nr; 04.50.Kd.

1 Introduction
During last decades, theoretical as well as observational indications revealed that Einstein’s gravity theory (GR) cannot demonstrate the flat rotation
curves of galaxies in the absence of dark matter (DM) and the present expansion rate of the universe without both DM and dark energy (DE), which is summed up to 96% of the universe’s total energy content. Also, the recent cosmological observations for late-time universe indicate the dominance of cosmological constant due to mysterious form of DE. However, inclusion of such constant mimics the behavior of standard model of GR and its microphysical origin remains a mystery. Different puzzling properties have been came up with recent released information of 2.7 full sky survey through Planck data [1] which changed the concept of universe modeling.

Since the universe evolution is mostly governed by the gravitational field on large scales, so there exist many alternatives addressing DM and DE during the modification of theory of gravity. The simplest modification in GR is the inclusion of non-linear interaction of Ricci scalar R in its action and such theories are named as f(R) theories. It is well established that GR can be derived from Einstein-Hilbert (EH) action by varying it with respect to connection and metric independently as well as by standard metric variation (for reviews on the late-time cosmic acceleration, i.e., DE problem, and modified gravity theories, see, e.g., [2–6]). During the modification of GR to f(R), one can vary the action in the same manner, which leads to different versions of f(R) theory. The version in which the action is varied by keeping the connection and metric as independent, is known as Palatini variational approach while the other one is named as metric approach (which experience gravitational instabilities [7]). Moreover, the field equations are second order in Palatini formulism which are fourth order with metric variation.

Einstein [8] himself introduces this Palatini approach to examine the nature of dynamical universe in gravitational theories, however, due to a historical misconception, it is named as Palatini approach [9]. In this paper, we focussed our attention on this particular modification of GR, i.e., Palatini formulism. Different f(R) models have already been proposed with Palatini approach in the literature with the focus on the late-time universe due to inverse powers of the curvature scalar. Sotiriou [10] discussed that the positive powers of the curvature scalar in gravitational action can also be used to study the late-time universe evolution. An extensive literature is available describing the cosmologically viable f(R) models consistent with the solar system constraints even with very light scalar field [11]. Amarzguioui et al. [12] investigated a class of DE models due to the modification in EH Lagrangian by using the current cosmological data.

Kainulainen et al. [13] explored the modified Tolman-Oppenheimer-Volkoff
equation for interior spherical geometry under hydrostatic equilibrium which does not affect the form of vacuum solution. This is one of the main reasons of the viability of Solar system’s constraints in Palatini formulism. Also, most of the \( f(R) \) theories in metric formulism are ruled out due to the failure in finding any interior geometry which can be matched with Schwarzschild de-Sitter metric \([14]\). Olmo \([15]\) studied the possibility of curvature singularities in locality of low curvature spherically symmetric matter configurations within the Palatini formulism and polytropic equation of state. Barragán et al. \([16]\) found that Palatini version of \( f(R) \) gravity have unusual properties to study the collapse process and early universe. They also explored the conditions which avoid the big bang singularity and responsible for the emergence of isotropic and homogeneous models.

Barrow and Ottewill \([17]\) examined the nature of isotropic and homogeneous solutions within the \( f(R) \) gravity theory and explored the constraints for which these solutions correspond to those in GR for late and early cosmic epochs. Popławski \([18]\) applied the field equations obtained under Palatini formalism to a homogeneous and flat isotropic dust universe and check the viability of this Palatini formalism to reveal present cosmic acceleration. Allemandi et al. \([19]\) investigated the equivalence between GR and non-minimally coupled higher order theories in Palatini approach and studied conformal invariance due to the conjecture of isotropy and homogeneity. Koivisto and Mota \([20]\) explored the anisotropic and homogeneous solutions using covariant approach on the cosmological equations. Böhmer et al. \([21]\) explored the effects of in/homogeneous perturbations on the Einstein static universe to examine its stability in the background of Palatini gravitational theory. They show that the respective stable regimes exist with certain parametric space of equation of state.

The inhomogeneities in energy density distribution are described as spherical epochs where the geometry takes the form of Lemaître-Tolman-Bondi (LTB) spacetime. It has been observed that inhomogeneous matter distribution is necessary for the applicability of supernova data in the context of LTB metric. There exist a common scale factor which can demonstrate the expanding regions of the universe. The role of inhomogeneous matter density has broadly been studied in the discussion of self-gravitating collapsing stars \([22]\). Herrera et al. \([23]\) explored the role of density irregularities on the structure and evolution of anisotropic spherical stars. Herrera \([24]\) discussed inhomogeneity factors for adiabatic and non-adiabatic relativistic interiors and claimed that the system must satisfy these constraints to achieve stable
configurations. There have been interesting results about the inhomogeneity parameters in comoving, non-comoving charged and conformally flat [25] relativistic systems. Yousaf and his collaborators found [26] modified gravity could be considered as a viable platform to describe new solutions with some extra degrees of freedom that were not possible in GR. They have also investigated those factors that are responsible to disturb the stability of homogeneous universe [27].

Here, our aim is to compute the relationship of material variables between comoving and non-comoving reference frames and discuss its dynamics in the background of Palatini $f(R)$ gravity. In the following section, we explored the basic formulism including the field equations with a particular form of $f(R)$ model. Some particular relations have been explored with and without heat flux as well as in the absence of radial pressure. In order to study the congruence of observers, we have investigated the kinematical quantities in section 3 which have enough physical significance in the study of relativistic astrophysics. The zero and non-zero values of these kinematical quantities demonstrate different realistic regimes. Section 4 is dedicated to the exploration of some dynamical variables which have significance to explore the evolution of the universe. These dynamical quantities are related with material variables by using the contracted form of Bianchi identities. We have also formulated a collapsing factor in the light of comoving observer. The last section summarizes our main findings.

2 Palatini $f(R)$ Formalism

In the universe, different stellar systems are bound together with the help of gravitational interactions. These celestial bodies provide a powerful tool to study galaxy and inter galactic structure formation. A well-consistent gravitational theory is required to understand these populations. In recent times, astrophysicists found MGTs as a powerful tool to trace the cosmic structures evolution. For $f(R)$ gravity, the usual Einstein-Hilbert action is modified as [13]

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M,$$

where $S_M$ is action for matter fields, $\kappa$ is a constant number with appropriate dimensions, for instance $\kappa = 8\pi G$ for GR action and $f(R)$ is any viable Ricci invariant function. In order to attain Palatini $f(R)$ field equation, we shall
ponder that $\Gamma^\mu_{\gamma\delta} \neq \Gamma^\mu_{\delta\gamma}$. Making variation on Eq.(1) with $g_{\gamma\delta}$ and $\Gamma^\mu_{\gamma\delta}$, we have

$$f_R(\hat{R})\hat{R}_{\gamma\delta} - [g_{\gamma\delta} f(\hat{R})]/2 = \kappa T_{\gamma\delta},$$

(2)

$$\hat{\nabla}_\mu(g^{\gamma\delta}\sqrt{-g} f_R(\hat{R})) = 0,$$

(3)

where $T_{\gamma\delta}$ is the standard energy-momentum tensor. Solving Eq.(2) for connections and then using this value in Eq.(3), one can achieve the following Palatini field equation

$$\frac{1}{f_R} \left( \hat{\nabla}_\gamma \hat{\nabla}_\delta - g_{\gamma\delta} \hat{\Box} \right) f_R + \frac{1}{2} g_{\gamma\delta} \hat{R} + \frac{\kappa}{f_R} T_{\gamma\delta} + \frac{1}{2} g_{\gamma\delta} \left( \frac{f}{f_R} - R \right)$$

$$+ \frac{3}{2f_R^2} \left[ \frac{1}{2} g_{\gamma\delta} (\hat{\nabla}_f R)^2 - \hat{\nabla}_\gamma f_R \hat{\nabla}_\delta f_R \right] - \hat{R}_{\gamma\delta} = 0.$$  

(4)

It is worthwhile to note that above equation does not contain connection terms rather it comprises of matter and metric tensors as variable quantities. The trace of Eq.(4) yields

$$R f_R(R) - 2 f(R) = \kappa T.$$  

(5)

This indicates that connections Ricci scalar can be manipulated through trace of standard stress-energy tensor, i.e., $T$. This consequently gives $R = R(T)$ and $f_R = f_R(T)$, i.e., $R$ and $f_R$ are the function of $T$ that eventually make them dependent on metric variables not on connections.

The vacuum case, i.e., $T_{\gamma\delta} = 0$ would necessarily leads the differential equation to have a constant solution that would secure connections to be well-known Levi-Civita. Further, this would also assign constant value to $f_R$. In this scenario, Eq.(5) boils down to

$$G_{\gamma\delta} = - \Lambda g_{\gamma\delta},$$

where $\Lambda$ is a constant term that could be regarded as an affective cosmological constant, this suggests the validity of Birkhoff’s theorem in this modified gravity. Here, the value of $\Lambda$ is $- \left( \frac{I - Bf_R}{2f_R^2} \right)$, where breve shows that the corresponding values are evaluated with vacuum environment. Equation (4) can be rewritten through Einstein tensor as

$$\hat{G}_{\gamma\delta} = \frac{\kappa}{f_R} (T_{\gamma\delta} + T_{\gamma\delta}),$$

(6)
where
\[
\mathcal{T}_{\gamma\delta} = \frac{1}{\kappa} \left( \hat{\nabla}_\gamma \hat{\nabla}_\delta - g_{\gamma\delta} \hat{\Box} \right) f_R - \frac{f_R}{2\kappa} g_{\gamma\delta} \left( R - \frac{f}{f_R} \right) + \frac{3}{2\kappa f_R} \left[ \frac{1}{2} g_{\gamma\delta} (\hat{\nabla} f_R)^2 - \hat{\nabla}_\gamma f_R \hat{\nabla}_\delta f_R \right],
\]

while \( \hat{G}_{\gamma\delta} \equiv \hat{R}_{\gamma\delta} - \frac{1}{2} g_{\gamma\delta} \hat{R} \) and \( \hat{\Box} \) is a de Alember operator, expressed with the help of independent connection dependent covariant derivative, \( \hat{\nabla}_\gamma \), as \( \hat{\Box} = \hat{\nabla}_\gamma \hat{\nabla}_\delta g_{\gamma\delta} \).

We wish to consider LTB spherical spacetime with its generic form as [28]
\[
\begin{align*}
\int ds^2 &= dt^2 - A_0^2 (h + \varepsilon) dr^2 - C^2 (d\theta^2 + \sin \theta^2 d\phi^2),
\end{align*}
\]

(7)

where \( \varepsilon \) can have 0 or \( \pm 1 \) constant numerical value, \( h = h(r) \) obeying the condition \( h + \varepsilon \geq 0 \) and prime means radial partial derivative of the corresponding quantity. This model could be considered as a power and effective tool to understand many inhomogeneous aspect of this accelerating cosmos. Besides many other well-known cosmological benefits [29], the analysis of exact inhomogeneous models such as LTB may provide us better platform for comprehending the influence of irregularities on the existence of many stellar objects [30]. By considering \( B = A' \) and \( h + \varepsilon = 1 \), one can find the following form of diagonal non-static irrotational LTB line element
\[
\begin{align*}
\int ds^2 &= -dt^2 + B^2 dr^2 + C^2 (d\theta^2 + \sin \theta^2 d\phi^2).
\end{align*}
\]

(8)

The geometry of any stellar objects exist due to the gravitational interaction of relativistic matter sources. These sources are closely linked with their 4-velocities, thus giving importance to the choice of fluid 4-velocities. The general dynamical description of the gravitational source along with its congruence kinematics could be dissimilar, if the two feasible relativistic explanations of a given spacetime are related to the boost of one of the observer congruences regarding to the other one. For example, Friedmann-Robertson-Walker is an ideal fluid solution for the rest observer with reference to time-like congruence. In the same time, this metric can be interpreted as the solution of viscous radiating gravitating source, if the observer is moving with relative velocity respecting the first previous frame. Motivated from
this concept, we first assume the comoving coordinate frame under which the 4-velocity for dust energy momentum tensor

\[ T_{\gamma\delta} = \hat{\rho}u_\gamma u_\delta \]  

is

\[ u^\gamma = (1, 0, 0, 0), \]

where \( \hat{\rho} \) is the fluid’s energy density configuring in comoving congruences.

Apart from that, we now suppose that matter content has some radial velocity \( \omega \) with respect to a new frame of reference. Let our system is subjected to Lorentz boost from locally Minkowskian frame hosting perfect fluid to this new frame. This makes us possible to have a tilted congruences, supported by the following 4-vector field

\[ U^\gamma = \left( \frac{1}{\sqrt{1 - \omega^2}}, \frac{\omega}{B\sqrt{1 - \omega^2}}, 0, 0 \right). \]

We consider that observer, in tilted frame, has noted locally anisotropic radiating matter source for LTB geometry with the above mentioned 4-velocity. Its stress-energy tensor is

\[ T_{\gamma\delta} = (\rho + P_\perp)U_\gamma U_\delta + \epsilon l_\gamma l_\delta - P_\perp g_{\gamma\delta} + q_\gamma U_\delta + (P_r - P_\perp)S_\gamma S_\delta + q_\delta U_\gamma, \]

where \( \rho, \epsilon, q_\gamma, P_\perp \) and \( P_r \) are energy density, radiation density, heat flux vector, tangential and radial pressure, respectively. The quantity \( l_\gamma \) is a null 4-vector defined as follows

\[ l^\gamma = \left( \frac{1 + \omega}{\sqrt{1 - \omega^2}}, \frac{1 + \omega}{B\sqrt{1 - \omega^2}}, 0, 0 \right), \]

while \( q_\gamma \) can be defined through another 4-vector, \( S^\gamma \) as

\[ q^\gamma = qS^\gamma. \]

with

\[ S^\gamma = \left( \frac{\omega}{\sqrt{1 - \omega^2}}, \frac{1}{B\sqrt{1 - \omega^2}}, 0, 0 \right). \]

All tilted 4-vectors are obeying

\[ U^\gamma U_\gamma = -1 = l_\gamma U_\gamma, \quad S^\gamma S_\gamma = 1 = l_\gamma S_\gamma, \quad l^\gamma l_\gamma = 0 = S^\gamma U_\gamma = U^\gamma q_\gamma. \]
2.1 Tilted and Non-tilted Relations with Positive and Negative Ricci Scalar Corrections

The Palatini $f(R)$ field equations for tilted environment are obtained as follows

$$
G_{00} = \frac{\kappa}{f_R} \left[ \hat{\rho} + \hat{\rho}_r \omega^2 + 2 \hat{q} \omega - \frac{1}{2} \left( f_R - \frac{f}{R} \right) - \hat{f}_R \left( \frac{\dot{B}}{B} + \frac{9 f_R}{4 f_R^2} + \frac{2 C'}{C} \right) \right]
- \left( \frac{B'}{B} - \frac{2 C''}{C} + \frac{f_R'}{4 f_R^2} \right) \frac{f'}{B^2} \left( \frac{f'}{B^2} + \frac{f''}{B^2} \right),
$$

(16)

$$
G_{11} \frac{B^2}{B^2} = \frac{\kappa}{f_R} \left[ \hat{\rho}_r \omega^2 + \hat{\rho}_r + 2 \hat{q} \omega - \frac{1}{2} \left( f_R - \frac{f}{R} \right) - \hat{f}_R \left( \frac{9 f_R'}{4 f_R^2} + \frac{2 C'}{C} \right) \right]
- \left( \frac{f_R'}{4 f_R^2} - \frac{2 C'}{C} \right) \frac{f'}{B^2} \left( \frac{f'}{B^2} + \frac{f''}{B^2} \right),
$$

(17)

$$
G_{22} \frac{C^2}{C^2} = \frac{\kappa}{f_R} \left[ \hat{p} \hat{p} + \hat{p}_r (1 + \omega^2) - \frac{R}{2} \left( f_R - \frac{f}{R} \right) + \hat{f}_R R + \left( \frac{\dot{B}}{B} + \frac{f_R'}{4 f_R^2} + \frac{C'}{C} \right) \frac{f'}{R} \right]
+ \left( \frac{B'}{B} + \frac{f_R'}{4 f_R^2} - \frac{C'}{C} \right) \frac{f'}{B^2} \left( \frac{f'}{B^2} + \frac{f''}{B^2} \right),
$$

(18)

$$
G_{01} = \frac{\kappa}{f_R} \left[ \hat{B} \left( \frac{\hat{\rho} + \hat{\rho}_r}{1 - \omega^2} \right) \right]
+ \frac{1}{\kappa} \left\{ \frac{f_R'}{2 f_R} \left( \frac{f_R'}{2 f_R} - \frac{f''}{2 f_R} \right) \right\},
$$

(19)

where $\tilde{x} \equiv x + \epsilon$ and the components of $G_{\gamma\delta}$ are mentioned in [31].

In order to present $f(R)$ gravity as an acceptable model, one should consider observationally viable $f(R)$ model. A viable model not only helps to shed light over current cosmic acceleration but also obeys the requirements imposed by terrestrial and solar system experiments with relativistic background. In this paper, we wish to find some relations that could highlight inflationary and current acceleration of the universe. It is well-known that Ricci scalar function, in $f(R)$ action, incorporating negative and positive powers endorse recent cosmic acceleration and its inflationary eras, respectively [32]. We now consider two models, one is associated with the source (9) and other with the tilted congruence (12). This would eventually gives us relation between parameters depicting inflationary and recent accelerating
universe. For non-tilted congruence we consider the following $f(R)$ model \[33\]

$$f(R) = R + \alpha R^2,$$  \hspace{1cm} (20)

where $\alpha$ is any positive constant number. This model could be taken for the description of both DM (with $\alpha = \frac{1}{6M^2}$ \[34\]) and of DE. For former purpose, $M$ is constrained to be $2.7 \times 10^{-12} GeV$ with $\alpha \leq 2.3 \times 10^{22} Ge/V^2$ \[3\].

Next, we suppose $f(R)$ model widely known as Carroll-Duvvuri-Trodden-Turner (CDTT) model \[35\], i.e.,

$$f(R) = R + \rho \delta^4,$$  \hspace{1cm} (21)

where $\delta > 0$ and $\rho = +1$. For the specific value of $\delta$, i.e., $\delta^{-1} \sim (10^{33} eV)^{-1} \sim 10^{18}$sec, this model could support current accelerating cosmic regimes. For vacuum case, this model boils down to present de-Sitter solution.

Since the Einstein tensor for both tilted and non-tilted reference frames are equal, therefore we obtain some important relations

\[
\frac{\kappa \dot{\rho}}{1 + 2\alpha R} + \frac{\alpha R^2}{2(1 + 2\alpha R)} = \frac{\kappa R^2(\dot{R} + \dot{P}_r \omega^2 + 2\dot{q}\omega)}{(R^2 - \delta^2)(1 - \omega^2)} - \frac{R\delta^2}{R^2 - \delta^2}, \hspace{1cm} (22)
\]

\[
(\dot{\rho} + \dot{P}_r)\omega + \dot{q}(1 + \omega^2) = 0, \hspace{1cm} (23)
\]

\[
\frac{\kappa R^2(\dot{\rho} \omega^2 + \dot{P}_r + 2\dot{q}\omega)}{(R^2 - \delta^2)(1 - \omega^2)} = \frac{-\alpha R^2}{2(1 + 2\alpha R)} - \frac{R\delta^2}{R^2 - \delta^2}, \hspace{1cm} (24)
\]

\[
P_\perp = \frac{-\alpha(R^2 - \delta^4)}{2(1 + 2\alpha R)\kappa} - \frac{\delta^4}{R\kappa}, \hspace{1cm} (25)
\]

It is noted that due to Palatini $f(R)$ gravity, the tangential pressure is non-zero and purely depends upon the degrees of freedom coming from inflationary as well as late time cosmic acceleratory terms.

The relations between tilted congruence with late time accelerating corrections and non-tilted congruence with inflationary degrees of freedom can be achieved through Eqs.(9) and (12). These are

\[
\epsilon = \frac{(R^2 - \delta^4)}{1 + 2\alpha R} \left\{ \dot{\rho} R(1 - \omega^2) + \frac{\alpha}{2\kappa} \right\} + \frac{\delta^4}{R\kappa} - \rho, \hspace{1cm} (26)
\]

\[
P_r = \rho - \frac{(R^2 - \delta^4)\dot{\rho}}{R^2(1 + 2\alpha R)} - \frac{(R^2 - \delta^4)\alpha}{\kappa(1 + 2\alpha R)} - \frac{2\delta^4}{R\kappa}. \hspace{1cm} (27)
\]
\[ q = \rho - \frac{\dot{\rho}(R^2 - \delta^4)}{R^2(1 - \omega)(1 + 2\alpha R)} - \frac{\alpha(R^2 - \delta^4)}{2\kappa(1 + 2\alpha R)} - \frac{\delta^4}{R\kappa}, \] (28)
\[ = \frac{P_r - \rho \omega}{(1 - \omega)} + \frac{\alpha(R^2 - \delta^4)(1 + \omega)}{2\kappa(1 + 2\alpha R(1 - \omega))} + \frac{\delta^4(1 + \omega)}{R\kappa(1 - \omega)}, \] (29)
\[ = -\frac{\dot{\rho}(R^2 - \delta^4)\omega}{R^2(1 + 2\alpha R)(1 - \omega^2)} - \epsilon. \] (30)

It is worthy to mention that, these relations have been found by taking into account constant Ricci scalar condition. Under the limit \( f(R) \to R \), all our relations reduce to that of GR [28]. For bounded laboratory distributions, it is found that, like GR, the junction conditions are not satisfied thereby indicating the existence of thin shell over the hypersurface. In the following subsections, we shall discuss some particular cases.

### 2.1.1 \( \epsilon = 0 \) with Non-zero Heat Flux

In this scenario, Eqs.(26)-(30) yield

\[ P_r = \rho \omega^2 - \frac{\alpha(R^2 - \delta^4)(1 + \omega^2)}{2\kappa(1 + 2\alpha R)} - \frac{\delta^4(1 + \omega^2)}{\kappa R}, \] (31)
\[ \rho = \frac{(R^2 - \delta^4)}{(1 + 2\alpha R)} \left\{ \frac{\dot{\rho}}{R^2(1 - \omega^2)} + \frac{\alpha}{2\kappa} \right\} + \frac{\delta^4}{R\kappa}, \] (32)
\[ q = -\frac{\dot{\rho}(R^2 - \delta^4)\omega}{R^2(1 + 2\alpha R)(1 - \omega^2)}. \] (33)

### 2.1.2 \( \epsilon \neq 0 \) with Zero Heat Flux

For this case, it is evident from Eqs.(26)-(30) that

\[ P_r = \rho \omega - \frac{\alpha(R^2 - \delta^4)(1 + \omega)}{2\kappa(1 + 2\alpha R)} - \frac{\delta^4(1 + \omega)}{\kappa R}, \] (34)
\[ \rho = \frac{(R^2 - \delta^4)}{(1 + 2\alpha R)} \left\{ \frac{\dot{\rho}}{R^2(1 - \omega)} + \frac{\alpha}{2\kappa} \right\} + \frac{\delta^4}{R\kappa}, \] (35)
\[ \epsilon = -\frac{\dot{\rho}(R^2 - \delta^4)\omega}{R^2(1 + 2\alpha R)(1 - \omega^2)}. \] (36)
2.1.3 \( P_r = 0 \)

In this scenario, we have from Eqs.(26)-(30) that

\[
\rho = \frac{\dot{\rho}(R^2 - \delta^4)}{R^2(1 + 2\alpha R)} + \frac{\alpha(R^2 - \delta^4)}{\kappa(1 + 2\alpha R)} + \frac{2\delta^4}{\kappa R}, \tag{37}
\]

\[
q = -\frac{\rho\omega}{1 - \omega} + \frac{\alpha(R^2 - \delta^4)(1 + \omega)}{2\kappa(1 + 2\alpha R)(1 - \omega)} + \frac{\delta^4(1 + \omega)}{R\kappa(1 - \omega)}, \tag{38}
\]

\[
\epsilon = \frac{\rho\omega^2}{(1 - \omega^2)} - \frac{\alpha(R^2 - \delta^4)(1 + \omega^2)}{2\kappa(1 + 2\alpha R)(1 - \omega^2)} - \frac{\delta^4(1 + \omega^2)}{R\kappa(1 - \omega^2)}. \tag{39}
\]

3 Kinematical Quantities

For the complete description of non-rotating relativistic matter distribution, three kinematical quantities, i.e., 4-acceleration, shear tensor and expansion scalar occupy enticing importance. One can find four acceleration through its general formula

\[
a^\gamma = U^\gamma_\delta U^\delta. \tag{40}
\]

This kinematical variable has been found to be expressed by means of 4-vector field and Palatini dark source terms as

\[
a^\gamma = aS^\gamma - g^\gamma_\delta \frac{\partial f_R}{\partial f_R}, \tag{41}
\]

where

\[
a = \frac{1}{\sqrt{1 - \omega^2}} \left[ \dot{\omega} + \frac{\omega\omega'}{B} + (1 - \omega^2) \left( \frac{\omega B}{f_R} + \frac{\omega f_R}{B f_R} + \frac{f_R}{B f_R} \right) \right]. \tag{42}
\]

It is noted that in tilted Palatini \( f(R) \) frame, the congruences always behave non-geodesically. By making \( a = 0 \), one can achieve non-trivial solution for a particular LTB geometry. In other words, a special kind of constraint can be obtained that would express velocity fields with the help of dark source terms coming from Palatini \( f(R) \) gravity.

Further, the mathematical expression for expansion scalar is given by

\[
\Theta = U^\mu_\nu, \tag{43}
\]
which, for our tilted system, turns out to be

\[
\Theta = \frac{1}{\sqrt{1-\omega^2}} \left[ \omega \dot{\omega} + \frac{\omega'}{B} + (1 - \omega^2) \left( \frac{\dot{B}}{B} + 2 \frac{\dot{f}_R}{f_R} + \frac{2 \dot{C}}{C} + \frac{2 \omega C'}{CB} + \frac{\omega f'_R}{B f_R} \right) \right].
\] (44)

The shear tensor is defined as

\[
\sigma_{\gamma \delta} = U_{(\gamma \delta)} + a_{(\gamma \delta)} - \frac{1}{3} \Theta h_{\gamma \delta},
\] (45)

where \( h_{\gamma \delta} \) is a projection tensor. The non-vanishing components of the above equation are

\[
\sigma_{00} = \frac{2 \omega^2}{3(1-\omega^2)} \left[ \sigma + \frac{3 \dot{f}_R}{f_R \sqrt{1-\omega^2}} + \frac{3 \omega f'_R}{4 B f_R \omega \sqrt{1-\omega^2}} \right],
\] (46)

\[
\sigma_{11} = \frac{2 B^2}{3(1-\omega^2)} \left[ \sigma + \sqrt{1-\omega^2} \left( \frac{3 \dot{f}_R}{4 f_R} - \frac{3 \omega f'_R}{2 B f_R} \right) \right],
\] (47)

\[
\sigma_{22} = \frac{-C^2}{3} \left[ \sigma + \frac{1}{\sqrt{1-\omega^2}} \left( \frac{3 \dot{f}_R}{2 f_R} - \frac{\omega f'_R}{2 B f_R} \right) \right],
\] (48)

where

\[
\sigma = \frac{1}{\sqrt{1-\omega^2}} \left[ \omega \dot{\omega} + \frac{\omega'}{B} + (1 - \omega^2) \left\{ \frac{\dot{B}}{B} - \frac{\dot{f}_R}{f_R} - \frac{\dot{C}}{C} - \frac{\omega C'}{CB} + \frac{\omega f'_R}{B f_R} \right\} \right].
\] (49)

By taking \( \omega = 0 \) along with \( f(R) = R \), the kinematical quantities for non-tilted LTB spacetime can be recovered.

4 Some Basic Auxiliary Equations

In this section, we shall evaluate some basic expressions required to understand the complete general discussion of the tilted LTB relativistic matter configurations. The equations of motion describing the dynamics of the evolving cosmic system occupy alluring importance as such equations could efficiently provide the energy variation of the stellar population gradients respecting time and proximate surfaces. These are generally found via contracted Bianchi identities, i.e.,

\[
X^\gamma_{\delta, \gamma} = 0, \quad \text{where} \quad X^\gamma_\delta = T^\gamma_\delta + T^\gamma_\delta.
\]
Using above relation along with Eqs. (16)-(19), (42), (44) and (49), we get

\[
\begin{align*}
\dot{\rho} + \dot{\rho} \Theta + \dot{q} \left\{ \omega \Theta + \frac{\sqrt{1 - \omega^2}}{B} \left( \frac{2C'}{C} + \frac{f'_R}{f_R} \right) + \frac{2\omega}{\sqrt{1 - \omega^2}} \right\} + \frac{\dot{\rho} f'_R}{f_R} \\
+ \frac{q f'_R}{f_R} + \frac{2\omega P'_R}{B} + P_\perp \left( \Theta + \frac{f'_R}{f_R} + \frac{\omega f'_R}{f_R} \right) + D_0 = 0, \\
\left( \ddot{P}_r + a(\ddot{P} + \ddot{\rho}) + \frac{2q}{3} \left[ 2\Theta + \sigma - 3\omega (\ln C) \dot{\Theta} \right] \right) + \ddot{q} + \frac{\omega f'_R}{f_R} (\ddot{\rho} + P_\perp) - \dot{q} \sqrt{1 - \omega^2} \\
\times \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) + \frac{1}{f_R \sqrt{1 - \omega^2}} \left( \dot{q} \omega^2 \ddot{f}_R - \frac{\dot{\rho} f'_R}{f_R} - \frac{P_\perp f'_R}{f_R} \right) - \sqrt{1 - \omega^2} (P_\perp \omega) \\
- \frac{\omega^2 P'_R}{B \sqrt{1 - \omega^2}} + \frac{\omega}{\sqrt{1 - \omega^2}} \left[ \dot{\mu} + \left( \omega \dot{q} \right) \right] + D_1 = 0,
\end{align*}
\]

where

\[ z^\dagger = \gamma \gamma^\mu S^\mu, \quad z^* = \gamma_{\mu} U^\mu, \]

while \( D_0 \) and \( D_1 \) are terms containing Palatini \( f(R) \) corrections in general form and are mentioned in Appendix.

Now, we would find factors disturbing the energy density inhomogeneity of the tilted LTB system coupled with anisotropic dissipative relativistic matter. We found the well known equation that has related the Weyl tensor with fluid source variables. The detail for the evaluation of such expressions (for various non-tilted frames) has been found in literature with both GR [24] and modified gravity [36] backgrounds. For tilted observer, we found

\[
\left[ \mathcal{E} - \frac{\kappa}{2f_R} \left( \ddot{\rho} - \ddot{\rho}_r + P_\perp + T_{00} - \frac{T_{11}}{B^2} + \frac{T_{22}}{C^2} \right) \right]' = \frac{3C'}{C} \left[ \mathcal{E} + \frac{\kappa}{2f_R} \left( \ddot{P}_r - P_\perp \\
+ \frac{T_{11}}{B^2} - \frac{T_{22}}{C^2} \right) \right] + \frac{3\kappa C}{2Cf_R} \left\{ (\ddot{\rho} + \ddot{P}_r) \omega + \ddot{q} (1 + \omega^2) \right\} \left( \frac{1 - \omega^2}{(1 - \omega^2)} - \frac{T_{01}}{B} \right).
\]

This equation is in complex form whose general analytic equation is a difficult task. But one can find its analytic solution by taking some constraints. Here, we wish to check the influence of pressure gradient on the evolution of tilted LTB geometry. The dynamics of tilted environment can be examined through locally anisotropic source material. In this context, we consider \( q = \epsilon = 0 \).
along with $P_r = P_\perp = P$. Under this framework, the solution of above differential equation is

$$\mathcal{E}' + \frac{3C''}{C}\mathcal{E} = \frac{\kappa}{2} \left[ \frac{1}{f_R} \left( \hat{\rho} + T_{00} - \frac{T_{11}}{B^2} + \frac{T_{22}}{C^2} \right) \right]' + \frac{3\kappa C'}{2Cf_R} (\hat{\rho} + \hat{P}) (1 - \omega^2) \omega. \quad (53)$$

In GR, the Weyl scalar is responsible for disturbing homogeneities in the isotropic system, but here the situation is quite different. The two important parameters are trying to produce hindrances for the system to enter in the inhomogeneous phase. First one is Palatini dark source terms and second one is the fluid velocity $\omega$, emerging due to tilted congruences. From the above expression, it is clear that the tilted congruence has given birth to pressure for helping the system to be remain in the regular stage. If one assumes, $\omega = 0$, then the contribution of pressure will be vanished. On taking $f(R) = R$ and $\omega = 0$ in the above equation, we get

$$\hat{\rho}' = 0 \iff \mathcal{E} = 0. \quad (54)$$

It is pertinent to mention that various interesting aspects about the evolution of the cosmic system, dissipating in the form of diffusion approximation, can be highlighted by exploring transport equation. Such mathematical interpretation can be achieved with the help of casual dissipative theory. In this framework, relaxation time, $\tau$, corresponding to radiating phenomena occupy basic prominence. For example, in the field thermodynamics, this definite positive term can not be ignored for the discussion of system’s transient and hydrodynamical epochs. For heat conducting vector field, the transport equation is given by

$$\tau h^{\gamma \delta} U^\mu q_{\gamma \mu} + q^{\gamma} + K h^{\gamma \delta}(\Delta \delta + \Delta a_\delta) + \frac{K}{2} \Delta^2 \left( \frac{\tau U^\delta}{K \Delta^2} \right)_{\delta} q^\gamma = 0, \quad (55)$$

where $\Delta$ and $K$ stand for temperature and thermal conductivity, respectively. Using Eqs.(16)-(19) and (55), we get the following single independent component of the above equation

$$\tau \left( \hat{q} - \frac{q f_R}{2f_R} + \frac{\omega B}{B} q' \right) + q\sqrt{1 - \omega^2} = -\frac{\tau}{2} q(1 - \omega^2)^{1/2} \Theta - K \left( \hat{\Delta} \omega + \frac{\Delta'}{B} + \Delta \right)\right)$$

$$\times \left( a\sqrt{1 - \omega^2} - \frac{\omega f_R}{2f_R} - \frac{f_R'}{2Bf_R} \right) - \frac{K \Delta^2}{2} q \left\{ \left( \frac{\tau}{K \Delta^2} \right)' + \frac{\omega}{B} \left( \frac{\tau}{K \Delta^2} \right) + \frac{\delta}{\delta A} \right\}. \quad (56)$$
This equation shows that relaxation time for tilted radiating source is influenced by dark source terms induced by Palatini $f(R)$ gravity.

Now, we shall talk about the existence of non-tilted congruence against perturbations. We assume that our non-tilted distributions is subjected to small oscillations at initial time, $t = 0$. We would analyze oscillating non-tilted relativistic distribution with the time duration smaller than taken by hydrostatic and thermal relaxation processes. Upon employing perturbation, we found that

$$\omega = q = 0, \quad \dot{\omega} \approx \dot{q} \neq 0.$$  \hspace{1cm} (57)

In order to lessen down the order of complexity in our system, we have assumed that radiations stem from streaming out approximations are negligible, i.e., $\epsilon = 0$. The second dynamical equation provides

$$\rho \ddot{\omega} + \dot{q} + \frac{\rho f_R'}{B f_R} - \frac{\rho f_R'}{B f_R} = 0, \Rightarrow \dot{\omega} \rho + \dot{q} = 0.$$  \hspace{1cm} (58)

Since, initially, the system was in complete geodesically thermal equilibrium, therefore, we can take $\Delta^\prime = 0$. Then, Eq.(56) reduces to

$$\tau \dot{q} = -K \Delta \dot{\omega} - \frac{K \Delta f_R'}{2 B f_R}.$$  \hspace{1cm} (59)

Equations (58) and (59) yield

$$\omega \left( \rho - \frac{K \Delta}{\tau} \right) - \frac{K \Delta f_R'}{2 B f_R \tau} \equiv \dot{\omega} (\rho - \alpha_1 \rho) - \frac{K \Delta f_R'}{2 B f_R \tau},$$  \hspace{1cm} (60)

where $\alpha_1 \equiv K \Delta / (\rho \tau)$. If $\alpha_1 = 1$, then perturbation will induce instability to the system evolving with non-tilted congruences. Infact, this scenario implies impart zero value to inertial mass density of the system. The detail analysis in this context under the variety of cosmological applications has been discussed in literature [37,38].

5 Summary

In relation with recent observations of supernova type Ia, a renewed interest of researchers have been developed in LTB geometry to study the expansion
history of the accelerating universe. We have defined and revisited the basic equations describing evolution of the universe in the framework of Palatini $f(R)$ theory from the congruence of different observers. A congruence may be tilted or non-tilted depending upon the motion of fluid and observer. The idea is to expose the role of observers in the depiction of any physical process. The physical interpretation of both tilted and non-tilted models is quite different, thus one can ask that which model is better than the other one? However, one cannot characterize what is the correct interpretation as both are physically viable and each analysis is related to a specific congruence of observers.

We explore the cosmologies from that point where the universe is inhomogeneously expanding by a field of anisotropic and dissipative fluid. Our search will depend upon two different types of $f(R)$ models. For the observer comoving with the fluid, we consider quadratic $f(R)$ model while for non-comoving observer, i.e., in tilted frame, we choose CDTT model to construct our systematic analysis. A better understanding of these models in inflationary and late time cosmic epochs could help us deepen into such situations. We have formulated a link between the matter variables of both tilted and non-tilted congruences in the background of Palatini $f(R)$ gravity with LTB metric as interior geometry. The matter distribution is dust cloud for non-tilted observer which becomes anisotropic for tilted observer when one apply the Lorentz boost (rotation free transformation) to a locally Minkowskian frame in which the fluid element has radial velocity. The effects of dark source terms are also explored in these relations.

We have explored the kinematical quantities under the influence of extra degrees of freedom of $f(R)$ gravity with the reference of tilted observer. The four acceleration for LTB geometry takes the general form as obtained in Eq.(41) with $f(R)$ extra curvature terms. We have investigated the non-vanishing values of the shear tensor and expansion scalar which have significance in the light of relativistic astrophysics. We have formulated the conservation equations in a precise manner to distinguish the role of dark source terms as well as the congruence of observers. The evolution equation have been established to explore the inhomogeneities appearing due to the existence of Weyl tensor. The necessary and sufficient condition for a homogeneous system is found as the system should be conformally flat for non-tilted observer. We have obtained a differential equation describing the heat transfer phenomena for a collapsing matter distribution. We found a similar factor $\alpha_1$ which describe the stability of non-tilted congruence in the frame-
work of GR. It is pertinent to mention that one can identify more irregularity factors except that one which we have formulated here using Eqs.(50)-(52). All results obtained here are well consistent with the literature.

Appendix

The parts of Eqs.(50) and (51) are

\[
\mathcal{D}_0 = -\dot{T}_{00} + \left(\frac{T_{10}}{B^2}\right)' - T_{00} \left(\frac{\dot{B}}{B} + \frac{3\dot{f}_R}{2f_R} + \frac{2\dot{C}}{C}\right) + \frac{T_{01}}{B^2} \left(\frac{B'}{B} + \frac{2f_R'}{f_R} + \frac{2C''}{C}\right) \\
- \frac{T_{11}}{B^2} \left(\frac{\dot{B}}{B} + \frac{\dot{f}_R}{2f_R}\right) - 2T_{22} \left(\frac{\dot{C}}{C} + \frac{\dot{f}_R}{2f_R}\right),
\]

(A1)

\[
\mathcal{D}_1 = -\dot{T}_{10} + \left(\frac{T_{11}}{B^2}\right)' + T_{00} \frac{f_R'}{2f_R} - T_{10} \left(\frac{\dot{B}}{B} + \frac{2\dot{f}_R}{f_R} + \frac{2\dot{C}}{C}\right) + \frac{T_{11}}{B^2} \left(\frac{2C''}{C} + \frac{3f_R'}{2f_R}\right) \\
- \frac{2T_{22}}{C^2} \left(\frac{\dot{C}}{C} + \frac{f_R'}{2f_R}\right).
\]

References


