NONLINEAR PATTERN FORMATION IN A MULTILAYER PHOTONIC PLATFORM

by

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Abstract

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An optical resonator containing a Kerr nonlinearity and supporting two coupled modes with the same polarization is investigated to reveal stable localized solutions beyond the usual formation mechanism involving a single mode. Paired breather dissipative Kerr solitons, in addition to temporally stable patterns, are demonstrated via numerical simulations that accounts for differential group delay and unequal second-order dispersions (in both magnitudes and signs). The soliton dynamics are explored from the viewpoint of modulational instability and the spatial dynamics. These results are put within the broader context of multilayer photonic platforms and address considerations relating to dispersion engineering, external coupling, and avoided crossing phenomena. Preliminary waveguide measurements show the feasibility of implementing such a structure for the stable generation of coherent optical frequency combs.
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## Contents

Acknowledgements iii  

Contents iv  

List of Figures vi  

Acronyms and Abbreviations ix  

1 Introduction 1  

1.1 Brief Historical Overview ................................................. 1  
1.2 Types of Optical Frequency Combs .................................................. 3  
1.3 Thesis Overview .............................................................. 4  
1.4 Thesis Outline .............................................................. 6  

2 Nonlinear Integrated Optics: Principles and Modelisation 7  

2.1 Origins ................................................................. 7  
2.2 The Wave Equation ................................................................. 9  
2.3 The Nonlinear Propagation Equation ............................................ 10  
2.4 The Ikeda Map ............................................................... 12  
2.5 The Lugiato-Lefever Equation ..................................................... 12  

3 Multilayer Integrated Photonic Platforms 15  

3.1 SiN-on-Si Platforms ............................................................... 15  
3.2 Dispersion Engineering of Multilayer Waveguides .......................... 18  
3.3 Coupling Between Multilayer Waveguides .................................... 21  
3.4 Avoided Crossings in a Multilayer Ring Resonator .......................... 23
4 Dynamics of a Multilayer Resonator

4.1 Conventional Pattern Formation Mechanisms in a Planar Kerr Resonator ................. 25
4.2 Description of System Under Study ................................................................. 26
4.3 Linear Stability Analysis ............................................................................... 28
4.4 Spatial Dynamics and Soliton Formation Mechanism ......................................... 32
4.5 Discussion ....................................................................................................... 38

5 Experimental Characterization of Optical Waveguides

5.1 Experimental Setup ......................................................................................... 41
5.2 Optical Losses ................................................................................................ 42
  5.2.1 Waveguide Backscatter ............................................................................. 44
5.3 Waveguide Dispersion .................................................................................... 46

6 Summary and Outlook

6.1 Summary ....................................................................................................... 49
6.2 Outlook .......................................................................................................... 50

A Adaptation of the Split-Step Method to Coupled Ikeda Maps

B Adaptation of the Split-Step Method to Coupled Lugiato-Lefever Equations

Bibliography
List of Figures

1.1 The different manners in which optical frequency combs are conventionally generated. (a) A mode-locked laser is realized with a passive saturable absorber and a gain medium that is pumped either optically or through carrier injection. They are surrounded by reflectors that partially leak optical power on at least one end. Taken from Ref. [19]. (b) An electro-optic comb is realized with a Mach-Zehnder interferometer. RF signals with varying amplitudes and relative phases are applied to each arm in order to modulate an optical carrier and create new frequencies. Taken from Ref. [17]. (c) A monochromatic input is coupled at close to resonance to a cavity containing a Kerr nonlinearity. The carrier is unstable and evolves into a set of new frequencies, that are then coupled out of the cavity. 4

1.2 Examples of multimode resonator designs. (a) A multilayer resonator as investigated in this thesis, comprised of two individual guiding layers located at different heights, that nevertheless interact due to their proximity. (b) A disk resonator supporting a family of higher-order modes. External coupling to these structures is not shown. 5

3.1 Schematic of the cross-section of the designed platform. Doping implantations are not shown. The platform has three waveguide layers (Si, SiN1, SiN2), two metal interconnect levels (M1, M2), vias, TiN thin film heaters, depletion modulators (MOD), and Ge photodetectors. Taken from Ref. [34]. 17

3.2 (a) Schematic of the cross-section, showing relevant geometric parameters. The SiN is deposited via a plasma-enhanced chemical vapor deposition (PECVD) process and has been characterized to have a refractive index of approximately 1.9 at a wavelength of 1.31 µm. (b) Dispersion parameter $D$ as a function of the separation $h_s$ for identical waveguides $w_1 = w_2 = 0.8 \, \mu \text{m}, h_1 = h_2 = 0.6 \, \mu \text{m}$. (c) Dispersion parameter $D$ as a function of the separation $h_s$ for asymmetric waveguides $w_1 = 0.6 \, \mu \text{m}, w_2 = 0.8 \, \mu \text{m}, h_1 = h_2 = 0.6 \, \mu \text{m}$. 18
3.3 Two sets of multilayer waveguides each supporting even and odd modes come into close proximity and exchange power through evanescent coupling. The light is incident in the port labeled “In” and exits in the “Out” waveguides, in the same modal basis, in a proportion that depends on the evanescent coupling strength and the length of the structure.

3.4 Fraction of cross-coupled power \((\text{In} \rightarrow \text{Out}_{\text{cross}})\) for the even and odd TE modes as a function of an interaction length that excludes the bent sections, based on the geometry of Fig. 3.3, for a wavelength of 1.31 \(\mu\)m. The waveguides have uniform widths of 0.74 \(\mu\)m, a vertical separation of 0.2 \(\mu\)m, and a lateral separation of 0.55 \(\mu\)m. The bends have a radius of 200 \(\mu\)m.

3.5 A microring resonator consisting of two guiding layers supports even and odd modes for TE and transverse magnetic (TM) polarizations. A bus coupler with an adiabatic transition implements a variable splitting ratio \(\zeta\) for the even and odd modes from a single input with power \(P_{\text{in}}\).

4.1 Intracavity power of the homogeneous solutions as a function of (a) input pump power \(P_{\text{in}} = X_u + X_v\), for \(X_v/X_u = 4\) and fixed detunings \(\theta_v = 2, \theta_u = \theta_v + 0.2\); and (b) input detuning \(\theta_v\), for \(\theta_u = \theta_v + 0.2\) and \(X_u = 3.24, X_v/X_u = 4\). Dashed lines indicate dynamically unstable solutions.

4.2 Impact on the MI power gain per normalized length of (a) intracavity power and (b) dispersion of the secondary symmetric mode \(u\); (c) detuning \(\theta\) of the primary anti-symmetric mode \(v\); and (d) group velocity mismatch \(\delta_g\). Nominal parameters are \(|u|^2 = 1/2, |v|^2 = 3, \eta_u = 2, \eta_v = -1, \theta_u = \theta_v = 3,\) and \(\delta_g = 0\).

4.3 Regions of the parameter space \((\theta_v, |v|^2)\) with qualitatively different eigenspectra for (a) \(|u|^2 = 0, \eta_u = 0\); (b) \(|u|^2 = 1/4, \delta_g = 10^{-5}, \eta_u = 2\); and (c) \(|u|^2 = 1/4, \delta_g = 10^{-5}, \eta_u = 10\). Other parameters are fixed as \(\eta_v = -1\) and \(\theta_u = 4\). The white region corresponds to the leading eigenvalues being complex, the light gray region to these eigenvalues being purely imaginary, and the dark gray region to these being purely real. As the eigenvalues are found numerically, a real or imaginary part below a tolerance of \(10^{-5}\) is considered null for the purpose of these figures.

4.4 (a) Temporally stable bright/dark soliton pair showing \(|u|^2\) and \(|v|^2\), and (b) \(\arg(u)\) and \(\arg(v)\). (c) Amplitude spectrum of the high intracavity power component \(v\), for \(\eta_u = 1.84, \eta_v = -0.16, \theta_u = \theta_v = 4, S_u = 1.76, S_v = 2.49, \) and \(\delta_g = 3.1 \times 10^{-6}\).
4.5 Temporal dynamics of a breather bright/dark soliton pair with characteristic periodic behavior, showing (a) $|u|^2$ and (b) $|v|^2$, respectively, for $\eta_u = 1.84$, $\eta_v = -0.16$, $\theta_u = 3$, $\theta_v = 2.98$, $S_u = 1.82$, $S_v = 2.03$, and $\delta_g = 2.5 \times 10^{-6}$. 36

4.6 Phase space showing the periodic orbit, for approximately 5 periods, of (a) symmetric mode $u$ and (b) anti-symmetric mode $v$ for the breather dynamics of Fig. 4.5, with $\tau \gg 0$. 37

5.1 Experimental setup showing the chip holder, in the middle, and the two translation stages that each support a lensed fiber. A top down microscope with in-line illumination, connected to a camera, is used to identify on-chip structures. 42

5.2 Linear fit of the optical transmissions obtained for waveguides of five different lengths, averaged over wavelengths within 10 nm of 1310 nm. (a) Si waveguides with a width of 500 nm and a height of 150 nm. (b) Si waveguides with a width of 380 nm and a height of 150 nm. (c) SiN waveguides with a width of 740 nm and a height of 450 nm. (d) Same as (c), but for a SiN waveguide at a different height within the layer stack (see Fig. 5.3). 43

5.3 Slope of the linear fit of the transmissions obtained for a set of waveguide of different lengths, for (a) the Si waveguides of Fig. 5.2a–b and (b) the SiN waveguides of Fig. 5.2c–d, for each wavelength individually. 44

5.4 (a) Schematic of the cross-section of the designed platform. The platform has three waveguide layers (Si, SiN1, SiN2). Taken from [38]. (b) Measured backscatter that includes, in its measurement path, a Si rib waveguide with a slab much larger than the optical mode and core width of 500 nm, surrounded by fiber patch cords. The blue line is the raw measurement and the orange one a moving average. (c) Similar to (b), but for a SiN strip waveguide with a core width of 900 nm. 45

5.5 (a) Schematic of the MZI used for the measurement of the waveguide dispersion, showing the path difference $\Delta L$. (b) Raw transmission spectrum of the MZI. (c) Measured dispersion parameter $D$ for a Si waveguide with a height of 150 nm and a width of 380 nm (black solid line) and simulated dispersion from a numerical mode solver for the same waveguide (dashed lines), as a function of wavelength. The orange line corresponds to a nominal waveguide with a width of 380 nm and a height of 150 nm. The purple line correspond to a reduction in width of 20 nm and the yellow to a reduction in height of 10 nm. 47
Acronyms and Abbreviations

$\mathcal{PT}$ parity-time.

CMOS complementary-metal-oxide-semiconductor.

CW continuous wave.

DC direct current (figuratively, a frequency of 0 Hz).

DKS dissipative Kerr soliton.

EME eigenmode expansion.

FCA free carrier absorption.

FSR free spectral range.

FWM four-wave mixing.

HSS homogeneous steady state.

I/O input/output.

LiDAR light detection and ranging.

LLE Lugiato-Lefever equation.

LPCVD low pressure chemical vapor deposition.

MI modulational instability.

MMS multiple mode spaced.
MZI  Mach-Zehnder interferometer.

NFOM  nonlinear figure of merit.

NLO  nonlinear optics.

NLSE  nonlinear Schrödinger equation.

NMS  natively mode spaced.

OFDR  optical frequency domain reflectometry.

OVA  optical vector analyzer.

PECVD  plasma-enhanced chemical vapor deposition.

RF  radio frequency.

Si  silicon.

SiN  silicon nitride.

SiO$_2$  silicon dioxide.

SiP  silicon photonics.

SVEA  slowly varying envelope approximation.

TE  transverse electric.

TM  transverse magnetic.

TPA  two-photon absorption.

WDM  wavelength division multiplexing.

XPM  cross-phase modulation.
Chapter 1

Introduction

1.1 Brief Historical Overview

Nonlinear optics (NLO) has been selected for important applications in optical communications. For example, the modulator is still to this day most often operated on the basis of the Pockels effect, a nonlinear mechanism. After the development of Raman lasers and optical parametric amplifiers, another notable development of nonlinear optics that only involves the light field itself, without an external input, is optical computing, or all-optical signal processing [1]. In this area, integrated photonics becomes a direct analog of integrated electronics, and another key area where integration provides tangible benefits compared to bulk free space implementations. Among those advantages, we note the high optical intensities achievable from waveguides with low propagation losses that considerably reduce the required threshold powers, the ability to resonantly enhance interactions at micro scales, and the vast array of materials with second and third order nonlinearities that are available from integrated platforms.

Today, there is a great deal of interest in using integrated photonics in attempts to improve the efficiency of optical networks via all-optical wavelength conversion [2]. Other applications include quantum optics – the generation of entangled photon pairs, e.g. for the purpose of quantum key distribution [3].

Beyond specific applications, NLO also functions as a generic framework for the study of nonlinear phenomena, including nonlinear dynamical systems, particularly those regarding chaos and nonlinear pattern formation. Optics was – and remains – a platform of choice for such studies. Optical rogue waves are a well-known example [4], among others. In a landmark paper [5], Ikeda extended the original studies of Kerr nonlinearities in optical resonators beyond their initial focus on optical bistability by showing that chaotic behavior was a fundamental property of such structures. Proper modeling of these structures
then quickly lead to the prediction of solitons, ultrashort pulses of light that propagate unperturbed in time, being able to arise from a continuous wave (CW) input, with a notable contribution to this modeling from Lugiato and Lefever [6]. Optical solitons were nevertheless known as a general feature of nonlinear waveguides prior to this development [7, 8]. Attention was given, after their theoretical prediction in single-mode fibers, to their potential use to optical communications – what remains their main application [9]. It was immediately noted that their immunity to dispersion-induced pulse spreading could translate into higher transmission rates, by extending the bandwidth-distance product. A lack of practicality in the excitation schemes of these solitons resulted instead in the dominant position of wavelength division multiplexing (WDM). Many options for dispersion compensation were then developed, either directly in-fiber, in an analog manner, or digitally at the receiver, that rendered soliton communication effectively redundant.

The connection of solitons within resonators to optical communications is more recent [10, 11]. A cavity supporting a soliton leads to an efficient soliton pulse train as this light leaves the cavity; and that its spectrum is discrete owing to the time-domain periodicity. These equally spaced spectral lines become good candidates for information carriers in a WDM links, if this spectral spacing can be controlled efficiently to match regular communication standards. This is the case for integrated microring resonators. In platforms with a high index contrast between the core and cladding, their free spectral range can vary from a few nm to below a nm depending on their radius. This represents an alternative to mode-locked laser diodes, that can have comparable repetition rates but are limited in bandwidth, pulse quality, and pulse energy. Overall, such a solution eliminates the need for a large number of individual lasers, and provides some additional benefits regarding correlated noise between the channels [12, 13]. Early experimental demonstrations of resonator-based soliton WDM links have been reported in recent years [14, 15].

The applications of frequency combs are not limited only to communications. They include optical clocks, frequency/time transfer, low-phase-noises microwaves, astronomical spectrograph calibration, comb-calibrated tuned laser, coherent light detection and ranging (LiDAR), and arbitrary optical/radio-frequency (RF) waveforms generation [16]. The 2005 Nobel prize of physics was awarded in part for developments surrounding the frequency comb and the absolute frequency reference technique. Integrated photonics retains a key role in that field by providing a small form factor, a potentially increased robustness to environmental conditions, and generally being a cost effective solution – if fabricated in a standard foundry process. Silicon photonics (SiP), including silicon (Si) compounds such as silicon nitride (SiN), is particularly well positioned to act as a disruptive technology in this space due to its compatibility with complementary-metal-oxide-semiconductor (CMOS) electronics, high modal confinement,
high nonlinear coefficient, and lack of two-photon absorption (in the case of SiN).

1.2 Types of Optical Frequency Combs

Frequency comb is a generic term for spectra that contain many discrete and equispaced spectral lines. Their generation traditionally follows one of three possible mechanisms.

The first way is through a mode-locked laser, a type of laser that emits pulsed coherent light at regular intervals that correspond to the cavity round-trip time. This is achieved either through an active modulation of the cavity characteristics, or passively through saturable absorbers. This leads to a self-reinforcing preference for pulsed steady-states.

Another way to generate a comb is through a direct phase modulation of a CW carrier at RF frequencies. This is similar to standard frequency modulation. If the modulation is periodic, then the generated sidebands are discrete. The decomposition into sinusoidal harmonics creates a sum of sidebands that follow a Bessel function of the first kind for each such harmonic, with spectral separation corresponding to the modulation rate. Combination of different phase modulations in each branch of an amplitude modulator such as a Mach-Zehnder interferometer (MZI) can create combs with interesting properties, such as spectral flatness [17].

A third way is through the confinement, in a resonator, of a nonlinear process such as four-wave mixing (FWM). The resonator enforces periodicity of the phenomena via its boundary conditions. From a spectral domain perspective, the resonator possesses a series of sharp resonant longitudinal modes that naturally enforces a spectrum with discrete, equally spaced lines, the spacing of which directly becomes the repetition of the combs. The resonator is most often coupled to a single CW input for simplicity, but also sometimes to several of them, to help initiate the FWM process and enforce coherence.

The third way represents a marked departure in that it does not require any optical amplification or electrical modulation. It is instead a purely dissipative phenomena, and an interesting example of self-organization in nature. Conceptually, the mode-locked laser can be viewed as another example of the conversion of a monochromatic input into a train of frequencies, in the case of optical pumping, but it is nevertheless considered an example of an active system. The nonlinear gain provided by the pump to the new frequencies to reach a steady state, in the third case, does remain mediated by a material, although no atomic transitions take place as in a laser. Certainly, the connection to mode-locking has been made [18]. These approaches are summarized visually in Fig. 1.1.
Chapter 1. Introduction

Figure 1.1: The different manners in which optical frequency combs are conventionally generated. (a) A mode-locked laser is realized with a passive saturable absorber and a gain medium that is pumped either optically or through carrier injection. They are surrounded by reflectors that partially leak optical power on at least one end. Taken from Ref. [19]. (b) An electro-optic comb is realized with a Mach-Zehnder interferometer. RF signals with varying amplitudes and relative phases are applied to each arm in order to modulate an optical carrier and create new frequencies. Taken from Ref. [17]. (c) A monochromatic input is coupled at close to resonance to a cavity containing a Kerr nonlinearity. The carrier is unstable and evolves into a set of new frequencies, that are then coupled out of the cavity.

1.3 Thesis Overview

This thesis is at the intersection of two major recent developments: SiP and microresonator-based optical frequency combs. The field of SiP is currently moving toward the inclusion of several heterogeneous layers within the same platform while maintaining CMOS compatibility. These added layers are useful to exploit the strengths of different materials for different applications, and allow for increased circuit density and routing flexibility via the additional vertical degrees of freedom. One of the materials that can be included is SiN, for its superior characteristics compared to Si for passive linear and nonlinear applications.

In this work, we propose a new kind device, in the form of a multilayer resonator, that can be implemented in a SiN-on-Si integrated photonic platform. A multilayer resonator is a multimode resonator in which the excitation of the modes as well as their basic characteristics are more easily controllable (via geometry) compared to other structures such as disks and spheres which contain many more modes.
Chapter 1. Introduction

Some examples of possible resonant multimode arrangements are shown in Fig. 1.2. Nonlinear processes such as FWM depend strongly on modal properties such as confinement and chromatic dispersion, leading to increased potential for control and tunability within a multilayer cavity. Another impairment to regular multimode resonators is the mode hybridization occurring at degenerate resonances, induced by the symmetry breaking of a bent mode profile, in the case of an implementation as a microring or microdisk. This problem is not one to occur for theoretical studies, but puts limits to the kind of multimode interactions that can be studied in a real-word setting, with the majority of the current literature choosing to exploit these effects rather than suppressing them. Here, it is further shown how a multilayer resonator allows for reduced mode mixing interactions, and thus allows the future probing of new, previously unattainable regimes.

Figure 1.2: Examples of multimode resonator designs. (a) A multilayer resonator as investigated in this thesis, comprised of two individual guiding layers located at different heights, that nevertheless interact due to their proximity. (b) A disk resonator supporting a family of higher-order modes. External coupling to these structures is not shown.

This thesis concerns the study of novel Kerr-based phenomena in multilayer resonators that allows for the formation of self-organized nonlinear patterns, most notably temporal solitons, that can exist on top of a homogeneous background in a driven regime, for the main purpose of frequency comb generation. The main contribution of this thesis is in demonstrating how such a regime is supported, from a linear analysis of the structure to a full dynamical modelisation, and how these new dynamics are promising for the generation of stable combs. In doing so, insights are also provided toward potentially similar coupled structures for other applications. Even though this thesis mainly derives theoretical predictions, an eye is kept toward real-world feasibility. Parameters that can translate directly into fabricable devices are used throughout the case studies.

The thesis differentiates itself from other related works by working simultaneously with the two co-resonant coupled modes, as opposed to Refs. [20] and [21] that use a structure similar to Fig. 1.2a although with only one excited mode, for the purpose of dispersion engineering only. Recent interest, without detailed analysis of the dynamics, has also been shown for the simultaneous generation of combs in both polarizations [22], where a similar cross-phase interaction mechanism exists, albeit at a reduced strength.
1.4 Thesis Outline

The thesis is organized as follows. Chapter 2 presents the fundamental assumptions behind the modelisation of nonlinear optics in integrated optics, from its microscopic origin within the polarization density of a dielectric material to wave propagation in waveguide structures and resonators. Chapter 3 covers linear multilayer photonics, a recent development within the broader field of integrated optics, and presents some intermediate results concerning such platforms that are useful for the next chapter. Chapter 4 concerns the detailed dynamical modeling of a multilayer ring resonator and analyses its dynamical regimes for the formation of patterned elements, most notably paired solitons. Chapter 5 covers the experimental measurement of key waveguide parameters with some examples. The thesis concludes in Chapter 6 with a review of the contributions and offers a perspective toward future directions of the research.
Chapter 2

Nonlinear Integrated Optics: Principles and Modelisation

This chapter lays the foundations and formalism of nonlinear optics to be used for the reminder of this thesis. The nonlinear properties of a dielectric medium are examined as a perturbation to the linear case, and their influence are described for progressively higher level models of wave propagation dynamics.

2.1 Origins

Nonlinear effects in electromagnetism originates from the dielectric polarization density field, \( P \) and magnetic polarization field, \( M \), that depend locally on the properties of the medium. More specifically, nonlinear effects occur when \( P \) and \( M \) loses their linear proportionality to the electric field, \( E \), and magnetic flux density, \( B \), respectively, and becomes functions of higher-order terms as described by a Taylor series expansion. From the definitions of the displacement field, \( D \), and magnetization field, \( H \),

\[ D(r, t) = \epsilon_o E(r, t) + P(r, t), \]  
\[ H(r, t) = \frac{1}{\mu_0} B(r, t) - M(r, t), \]

it is seen that \( D \) and \( H \) are proportional to \( E \) and \( B \) only if \( P \) and \( M \) are themselves proportional to \( E \) and \( B \). Similar to the linear case, the microscopic origin of this nonlinearity can be either treated semiclassically, with a Lorentz-Drude oscillator model to which anharmonic terms are added, or through a fully quantum mechanical formalism. In their usual formulations, both derivations contains phenomeno-
logical constants added to them. What are sometimes referred to as “turning on” parameters [23], as they “turn on the perturbations”, are present at both levels, with one choosing to include them either in a perturbed Hamiltonian representing the interaction of light with an atomic system, a semi-classical electronic potential well, or directly into the final polarization field $P$.

We choose to work with a classical approximation as it is sufficient for our purposes. Under such a model, the equation of motion for an electron subjected to a driving electric field $E_0 \cos \omega t$ is

$$\frac{d^2}{dt^2} X + \gamma \frac{d}{dt} X + \omega_0^2 + \sum_{k=1}^{n} D_k X^{(k+1)} = \frac{e E_0}{2m} (e^{i \omega t} + \text{c.c.})$$ (2.2)

where $mD_k$ are the anharmonic restoring forces of increasing orders $k$, $m$ is the mass of the electron, $\gamma$ is the damping term, $\omega_0$ is the undamped resonant frequency of the electron motion, and $e$ is the elementary charge. For materials lacking an inversion center in their crystal lattice structure, terms for odd $k$ are necessarily null, to maintain the same final result by inversion of the driving field. For the simplest case of $n = 1$, an interesting result is found [23], that is reproduced in parts here. We take $D_1$ as $D$ and consider a solution of the form

$$X = \frac{1}{2} \left( q_1 e^{i \omega t} + q_2 e^{2i \omega t} + \text{c.c.} \right).$$ (2.3)

With some assumptions regarding the magnitude of some coefficients, it is possible to separate the coefficients on each side of the equality for each of the oscillating terms. For $q_1$, this gives the usual linear susceptibility. Interestingly, $q_2$ takes the form

$$q_2 = -\frac{1}{2} Dq_1^2 \left( -4\omega^2 + 2i \omega \gamma + \omega_0^2 \right)^{-1}.$$ (2.4)

The scalar isotropic polarization density is, after substitution,

$$P(t) = \frac{Ne}{2} (q_1 e^{i \omega t} + q_2 e^{2i \omega t} + \text{c.c.}) = \frac{E_0}{2} \left( d^{(\omega)} e^{i \omega t} + d^{(2 \omega)} E_0 e^{2i \omega t} + \text{c.c.} \right),$$ (2.5)

where $N$ is the number of electrons per unit volume and where $d^{(\omega)} \propto q_1$ and $d^{(2 \omega)} \propto q_2$ are defined as the linear and nonlinear susceptibilities, respectively. In other words, $d^{(2 \omega)} \propto (d^{(\omega)})^2 D$, whereas if $D$ was found to be constant across materials, a further degree of proportionality would be implied with $d^{(2 \omega)} \propto d^{(\omega)}$. It has been empirically established to be the case for second-order materials, with $D$ staying within the same order of magnitude. This observation has been referred to as Miller’s rule [24]. A more general statement is that $d^{(s \omega)} \gg d^{((s+1)\omega)}$ for centrosymmetric material and $d^{(s \omega)} \gg d^{((s+2)\omega)}$ for non-
centrosymmetric materials, with \( s = 1, 2, 3, \ldots \), which is helpful for the selection of a material platform. This result is to be understood intuitively from the underlying Taylor series expansion of the potential well. If the potential of the electron, close to its equilibrium point, indeed deviates only slightly from that of a harmonic oscillator and remains well-behaved, it is expected that the deviation provided by higher order terms decreases progressively and correlates with the magnitude of the deviation provided by the term of the previous order.

### 2.2 The Wave Equation

From Maxwell’s equations, the wave equation for \( \mathbf{E}(r, t) \) in a medium with no free charges follows

\[
\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2},
\]

where \( \mathbf{P}(r, t) \) is the one found in the previous section, and can thus be separated into linear and nonlinear contributions \( \mathbf{P}_L(r, t) \propto \mathbf{E}(r, t) \) and \( \mathbf{P}_{NL}(r, t) \). For \( \mathbf{P}_{NL} = \mathbf{0} \), Eq. 2.6 is a linear partial differential equation with, as a general solution, the traveling wave of the form \( \mathbf{E}(r, t) = f(\omega t - \mathbf{k} \cdot \mathbf{r}) \) for a well-behaved function \( f \) and a dispersion relation \( \mathbf{k} = |\mathbf{k}| = \omega/c = 2\pi/\lambda \), where \( c \) is the speed of light, \( k \) the wavenumber, and \( \lambda \) the optical wavelength. In homogeneous, linear, and time-independent media, it corresponds to the standard plane wave solution of the form

\[
\mathbf{E}(r, t) = E_\omega(r) \Re\{e^{i\omega t - \mathbf{k} \cdot \mathbf{r}}\}.
\]

A consequence of \( \mathbf{P}_{NL} \neq \mathbf{0} \) is to further break the linearity of the differential equation. It is easier conceptually to treat \( \mathbf{P}_{NL} \) as a perturbation to the linear equation, that is solved first. A benefit of the linearity of Eq. 2.6 is the ability to switch between the temporal and Fourier domains, giving

\[
\nabla \times \nabla \times \tilde{\mathbf{E}}(r, \omega) = \epsilon(r, \omega) \frac{\omega^2}{c^2} \tilde{\mathbf{E}}(r, \omega),
\]

where \( \epsilon(\omega) = 1 + \tilde{\chi}(\omega) \), where \( \chi(\omega) \) is defined as the linear electric susceptibility, and \( \mathbf{P}_L(r, t) = \epsilon_0 \int_{-\infty}^{t} \chi(t - t') \cdot \mathbf{E}(r, \omega) dt' \). The permittivity \( \epsilon \) is complex in the general case, with its imaginary part relating to the optical losses. If a further approximation is made in assuming \( \Re\{\epsilon\} \gg \Im\{\epsilon\} \), the latter being eventually added again in a perturbative manner, then one is left with a real coefficient in front of \( \mathbf{E} \) on the right hand side of Eq. 2.7. If this coefficient is time-independent and isotropic, and if the solution to \( \mathbf{E} \) only depends locally on \( \epsilon \), then Eq. 2.7 becomes the following Helmholtz relation

\[
\nabla^2 \tilde{\mathbf{E}} + n(r, \omega) \frac{\omega^2}{c^2} \tilde{\mathbf{E}} = 0,
\]

where

\[
\nabla \times \nabla \times \tilde{\mathbf{E}}(r, \omega) = \epsilon(r, \omega) \frac{\omega^2}{c^2} \tilde{\mathbf{E}}(r, \omega),
\]

and

\[
\nabla \times \nabla \times \tilde{\mathbf{E}}(r, \omega) = \epsilon(r, \omega) \frac{\omega^2}{c^2} \tilde{\mathbf{E}}(r, \omega),
\]
with \( n(\omega) = \sqrt{\epsilon(\omega)} \), the refractive index. In a waveguide, this equation is solved in two parts. For an invariant x-y plane and propagation in the z axis, a general solution is \( E(x, y, z, t) = F(x, y)e^{i(\omega t - \beta z)} \). \( F(x, y) \) is found by locally solving the Helmholtz equation in two dimensions and then matching the boundary fields at interfaces. This matching gives rise to a discrete number of guided modes within a layered dielectric structures, with “guided mode” taking the meaning of a well-behaved solution within a layer of finite dimensions. At this stage, the propagation losses \( \alpha_i \) are reintroduced into the propagation constant \( \beta \), which takes the form \( \beta = \frac{2\pi}{\lambda} (n_{\text{eff}} + i\alpha_i) \), where \( n_{\text{eff}} \) is the real effective refractive index.

### 2.3 The Nonlinear Propagation Equation

With this solution in hand, we return to Eq. 2.7. For materials with third order nonlinearities such as silicon, which are the focus of this work, the refractive index takes the form \( n = n_L + n_2|E|^2 \) and the absorption becomes \( \alpha_i = \alpha_L + \alpha_{NL}|E|^2 \), with the permittivity rewritten as \( \epsilon(\omega) = (n + i\alpha_i/2k)^2 \). The coefficient \( n_2 \) links the perturbation of the refractive index to a field intensity—what is referred to as the Kerr effect. The intensity-dependent absorption originates from two-photon absorption (TPA), and thus depends on the value of the optical band gap of the material for a given \( \lambda \). The introduction of losses can be interpreted as the introduction of an envelope function that multiplies a carrier wave at \( \omega_0 \), that itself varies much faster along the propagation length. The separation of the solution into different scales is also useful for the description of nonlinear interactions, or for the description of pulse spreading due to refractive index dispersion, where this envelope takes a physical meaning as well. Essentially, Eq. 2.7 becomes \( \tilde{E} = F(x, y)\tilde{A}(z, \omega - \omega_0)e^{i(\beta_0 z)} \), where we are interested mainly in the envelope \( \tilde{A} \), as it captures the essence of the physical phenomena. \( F(x, y) \) is considered invariant to the added nonlinearity. It is possible to recalculate \( F(x, y) \) from Eq. 2.7 with the added perturbation to the real part. In practice, this step is often unnecessary in SiP due to the large index contrast between the core and cladding, that makes small perturbations of the the refractive index mostly insignificant to the profiles of the modes under normal conditions. This is an appropriate assumption for this work that is maintained throughout. Furthermore, any time dependence of the intensity would require a recalculation of the mode profiles at all time steps, which adds to the impracticality. In a single-mode waveguide, this change of the spatial profile has no direct consequence on the temporal evolution, but in a multi-mode setting, it is possible to introduce some amount of coupling between the modes due to the sudden change in the eigenmode basis.
From a separation of variables, for components that depend on $z$, Eq. 2.7 becomes
\[ 2i\beta_0 \frac{\partial \tilde{A}}{\partial z} + \left( \beta^2 - \beta_0^2 \right) \tilde{A} = 0, \] (2.9)
where the term in $\frac{\partial^2 \tilde{A}}{\partial z^2}$ has been neglected due to the assumption of a slow varying $\tilde{A}$, a manipulation appropriately known as the slowly varying envelope approximation (SVEA). This includes the effects of both the linear chromatic dispersion and the intensity-dependent refractive index change. If $\beta \sim \beta_0$, then Eq. 2.9 further simplifies to
\[ \frac{\partial \tilde{A}}{\partial z} = i \left( \beta(\omega) + \beta_{NL} + i \alpha_i/2 - \beta_0 \right) \tilde{A}, \] (2.10)
where the nonlinear perturbation has been added as $\beta_{NL}$, and where the optical losses have been reintroduced as being part of $\beta$. One can expand the contribution $\beta(\omega)$ around $\omega_0$ with a Taylor expansion of the form $\beta_0 + \sum_{k=1}^{n} \beta_k(\omega - \omega_0)^k/k!$. Taking the inverse Fourier transform of the expression, the terms $(\omega - \omega_0)^k$ becomes equivalent to the operator $i(\partial/\partial t)$ in the inverse domain. Writing the nonlinear perturbation and absorption as frequency and time independent quantities, Eq. 2.10 becomes, in the time domain,
\[ \frac{\partial A}{\partial z} = -\beta_1 \frac{\partial A}{\partial t} - i \beta_2 \frac{\partial^2 A}{\partial t^2} - \frac{\alpha}{2} A + i \gamma |A|^2 A, \] (2.11)
where only the first two terms of the Taylor expansion have been kept for simplicity, and where $\beta_{NL} = i \gamma |A|^2 = i \omega_0 n_2/(c A_{eff})$, with $A_{eff}$ an effective modal area that takes into account the field confinement provided by the waveguide. One can further do away with the term that contains $\beta_1$ by normalizing to a new time $\tau$ defined as $t - \beta_1 z$, corresponding to a reference frame moving at what is referred to as the group velocity.

The model provided by Eq. 2.11 is useful for the computation of envelopes $A$ that are monochromatic or possess continuous frequency extents, such as for pulses. For a limited set of discrete frequencies, it is also possible to write the interaction as a set of coupled differential equations in the frequency domain [25]. This is particularly useful for the so-called four-wave mixing (FWM) experiments and arguably provide more insights than a time domain model. Further manipulations such the non-depleted pump approximation, that allow one component to remain virtually constant while others grow, do permit analytical solutions, whereas Eq. 2.11 is most often solved through purely numerical means. Even though attempts were made to develop such models for Kerr-based optical frequency comb generation [26], they are in practice very computationally inefficient due to the large number of longitudinal modes interacting at the same time in a typical microresonator.
Eq. 2.11 often takes the name of the nonlinear Schrödinger equation (NLSE) for its ubiquity in different sub-field of physics and similarity to the original Schrödinger wave equation in quantum mechanics. The treatment of this section closely follows that of Ref. [25], where more details can be found.

2.4 The Ikeda Map

Resonators have been identified as a key structure for the emergence of a variety of nonlinear phenomena including that of coherent frequency combs as studied in this work. For guided optics (as opposed to diffracting optics), the simplest dynamical model is that of a propagation equation with a boundary where light is able to enter or exit. Mathematically, this gives [5]

\[ A_{m+1}(0, \tau) = \sqrt{\Gamma} A_m + \sqrt{1-\Gamma} e^{i\theta} A_m(L, \tau), \]  

\[ \frac{\partial A_m(z, \tau)}{\partial z} = -\frac{i \beta_2}{2} \frac{\partial^2 A_m(z, \tau)}{\partial \tau^2} - \frac{\alpha_i}{2} A_m(z, \tau) + i \gamma |A_m(z, \tau)|^2 A_m(z, \tau), \]  

where \( m \) enumerates the discrete round-trips, \( A_m(z, t) \) is the envelope of the intracavity field, \( A_m \) is the constant input field coupled to the cavity, \( \theta \) is the linear detuning between this input and the closest resonance, \( \Gamma \) is the power coupling coefficient, and \( L \) is the physical round-trip length of the cavity.

This model, referred to as the Ikeda map [5], is very accurate insofar as the lumped coupler approximation is an accurate one. It can be further refined to account for regions with different propagation characteristics in cases where it is warranted.

2.5 The Lugiato-Lefever Equation

From the Ikeda map, it is possible to construct a mean-field approximation with the main assumption being that the field envelope evolves slowly over the round-trip length. Writing a finite difference equation from Eq. 2.12b gives

\[ A_m(L, \tau) - A_m(0, \tau) = -i \frac{\beta_2 L}{2} \frac{\partial^2 A_m(0, \tau)}{\partial \tau^2} - \frac{\alpha_i L}{2} A_m(0, \tau) + i \gamma L |A_m(0, \tau)|^2 A_m(z, \tau). \]  

One can then substitute \( A_m(L, t) \) from the boundary condition of Eq. 2.12a, from which \( \sqrt{1-\Gamma} \) and
\( e^{i\theta} \) are also expanded to their first order variation in \( \Gamma \) and \( \vartheta \) respectively, to give
\[
A_{m+1}(0, \tau) - A_m(0, \tau) = -i \frac{\beta_2 L}{2} \frac{\partial^2 A_m(0, \tau)}{\partial \tau^2} - \left( \frac{\alpha_i L + \Gamma}{2} + i \vartheta \right) A_m(0, \tau) - i \gamma L |A_m(0, \tau)|^2 A_m(0, \tau) + \sqrt{\Gamma} A_{in}.
\] (2.14)

The left hand side of the equation above is similar to a finite difference equation done between successive round-trips. One can the reverse operation and define a new continuous time variable \( t \), referred to as a “slow” time variable, that would result in this finite difference equation if discretized over round-trips \( m \). With round-trip time \( t_r \), and by posing \( A_m(mt_r, \tau) = A_m(z = 0, \tau) \), one then obtains
\[
t_r \frac{\partial A_m(t, \tau)}{\partial t} = -i \frac{\beta_2 L}{2} \frac{\partial^2 A_m(t, \tau)}{\partial \tau^2} - \left( \frac{\alpha_i L + \Gamma}{2} + i \vartheta \right) A_m(t, \tau) - i \gamma L |A_m(t, \tau)|^2 A_m(t, \tau) + \sqrt{\Gamma} A_{in}.
\] (2.15)

Eq. 2.15, in this classical form, is the Lugiato-Lefever equation (LLE) [6], effectively a driven and damped NLSE. This model is by far the most common for the study of Kerr-based optical frequency combs. Its advantage lies in the much greater computational ease in using such an equation compared to the Ikeda map. The variable \( t \) can effectively be integrated over the scale of a single to multiple \( t_r \) instead of needing to be integrated several time within a round-trip because of the sharp boundary condition within Eqs. 2.12. Eq. 2.15 is effectively a multiscale equation where \( t \) and \( \tau \) can be treated as independent variables, despite both being time variables, and thus still be solved by pseudospectral methods, that handle \( \tau \) in the Fourier domain and \( t \) with an integration step in the time domain. More details concerning the numerical solutions to Eqs. 2.12 and 2.15 are contained in the appendices, where they are adapted to the particular applications of this thesis. Another way to solve Eq. 2.15 is by recognizing that a comb state is a steady state solution, meaning \( \partial / \partial t = 0 \), such that the problem becomes one of finding the roots of a discretized \( A(t, \tau) \) [27]. Despite being a time variable, \( \tau \) covers only the intracavity field, such that it is bounded over the round-trip time. All numerical methods already shares the discretization of the Eq. 2.15 in \( \tau \), otherwise effectively a (continuous) infinite-dimensional map that does not lend itself to numerical analysis. This strategy is not straightforwardly adapted for coupled equations such as those explored in this work due to the presence of two solutions that depend on each other, though efficient iterative algorithms could be imagined. The loss of any transient phenomena or breather dynamics is another consequence of this method.

One can certainly appreciate the degree of sophistication in the approximations, that considerably simplify the problem from its original formulation within Maxwell’s equations. More particularly, the requirements of the LLE over the Ikeda map are that the dispersion remains low, or more precisely, that the dispersion length, defined as \( \tau_0^2 / |\beta_2| \), where \( \tau_0 \) is the approximate temporal extent of a pattern,
remains much larger than the length of the resonator, and that the input field is close to being critically coupled to a resonance, with the optical losses not being too large such that $\vartheta$ also remains small. This is true in practice, for nonlinear effects leading to combs with considerable extent require large intensities that can only be achieved from inputs with limited power (due to fiber power handling), and thus depend on a high degree of field enhancement within the resonator, that can only be achieved with low losses and by being close to a resonance. A small dispersion is also needed to ensure proper phase matching over the cavity modes, that can cover ranges exceeding an octave.

The condition of a small detuning is with the linear resonance, not the one that takes into account the detuning due to the nonlinear phase shift, that moves the effective condition for high intracavity intensity to a different detuning value. The main consequence of exceeding this condition is to neglect possible multi-states arising from the coexistence of a nonlinearly shifted resonance and the next linear longitudinal resonant mode [28]. This is briefly revisited in Chapter 4, due the ease in achieving such a state for comparable powers from to the increase in nonlinear phase shift provided by coupled equations.
Chapter 3

Multilayer Integrated Photonic Platforms

This chapter covers the description of multilayer integrated photonics within a static linear optics framework, useful as a preamble to the dynamical model of Chapter 4.

3.1 SiN-on-Si Platforms

The purpose of a heterogeneous platform, realized with layers of different materials, is the combination of advantageous materials for specific applications within a single platform, as opposed to fully separate or flip-chip bonded optical dies. The benefits in integration are significant. The realization of efficient optical input/output (I/O) couplers (from the point of view of insertion losses, bandwidth and polarization dependence) remains an outstanding challenge, such that limiting their number by having all functionalities on the same platform is always beneficial. There also exist further challenges from the packaging side. In integrated photonics, in contrast to electronics, alignment tolerances pose high constraints on the efficiency of I/O couplers. The wire bonding of silica or polymer optical fibers between chips remains a research question [29] and an open problem within the community.

For transceivers, where the power budget has a strong impact on power consumption due to the direct relation between the bit-error rate and received power, the coupling losses of a semiconductor laser to another chip for modulation and multiplexing already represent a significant fraction of the total optical losses, especially for short-reach applications. For this reason, strong research efforts now push in the direction of III-V integration with silicon-based platforms, since the formers support laser oscillation...
Heteroepitaxial growth of III-V materials has proved to be a significant challenge due to the large mismatches in the lattice constants and thermal expansion coefficients [31].

The introduction of SiN-on-Si platforms is an interesting first step. SiN is a CMOS compatible material conventionally used in the microelectronics industry for masking, passivation, and strain engineering [32]. At optical telecommunication wavelengths, SiN does not possess a direct two-photon absorption mechanism owing to its higher optical band gap. Its nonlinear Kerr coefficient is approximately twenty times smaller than Si, but its lack of nonlinear losses yields an overall much better nonlinear figure of merit (NFOM) [33]. It is less thermally sensitive owing a lower thermo-optic coefficient. For single-mode waveguides, scattering losses are reduced compared to Si due to the lower index contrast with its silica cladding. This makes SiN generally much more suited to passive applications than Si. The main trade-off is a larger footprint for SiN devices, including waveguide bends. On the other hand, the higher thermo-optic coefficient of Si makes it useful for realizing thermo-optic phase-shifters. Its ability to be doped to realize PN junctions allows high-speed electro-optic modulation through the plasma dispersion effect. The abilities of foundries to grow germanium directly on top of Si allows for integrated photodetectors. This justifies the combination of the two materials. For NLO, Si is less directly useful as it possesses a prohibitively high TPA coefficient in the standard telecommunication bands. For this reason, only the two layers of SiN are used in this work.

Full SiN-on-Si platforms were recently demonstrated by our group in collaboration with the A*STAR Institute of Microelectronics in Singapore for one [32] and two [34] layers of SiN on top of an active Si layer containing high-speed modulators and thermo-optic phase-shifters. Fig. 3.1 shows the complete wafer stack for the latter. For simple transceivers that do not require elaborate passive devices, SiN remains relevant for improving the insertion losses of optical I/O couplers – either directly with an edge coupler in that layer or by using composite modes within more than one layer [35], or through surface gratings that make use of several layers to help improve directionality [36]. This development has been recognized as a key direction of the field, and is being adopted by existing foundries (IME, Lionix) as well as upcoming ones (AIM Photonics).

There remains many opportunities for fundamentally different and novel devices to be explored. For instance, the presence of composite modes within a vertical structure rather than a planar one allows one to more easily break a vertical symmetry, in order to couple the polarization into different modes for the purpose of rotation or splitting. An example of a polarization rotator-splitter is found in Ref. [37], where a transverse magnetic (TM) mode is able to be converted into a higher-order transverse electric (TE) mode, that is then reconverted to the fundamental TE mode of an adjacent waveguide, while the original TE mode remains in the same waveguide unaffected. Although not directly relevant to this
thesis, it is interesting to note that such a design can also be improved with a direct conversion to the desired TE mode, by recognizing that a TE mode could transfer from one layer to another, while a TM mode of a lower index could see its index increased by the vertically proximity of the waveguides, such that when these waveguide are then carefully separated, power can be coupled back to the TE mode of this first waveguide.

Another example of devices inherently tied to a multilayer photonic platform are interlayer waveguide crossings. Waveguide crossings in general do not only incur losses, but also induce crosstalk among crossing signals. Record level upper bounds of crosstalk have been reported in trilevel platforms measured by our group [34, 38], achieved from the large vertical separation (∼850 nm) and mismatch in effective refractive indices of the waveguides. These interlayer crossings are especially relevant for large-scale circuits if accompanied by tapering structures that allow the efficient transfer of light between layers. Overall, these improvements in both device performances and integration density pave the way for complex chip architectures comprising hundreds and thousands optoelectronic devices, such as those for optical switch fabrics, optical phased arrays, and coherent communications involving several degrees of multiplexing.

Here, we focus on the particular application of multilayer ring resonators. Such a ring is essentially a multi-mode resonator, comprised of an even and odd mode for each polarization, with the key difference being that these modes can be engineered to have vastly different characteristics compared to usual higher-order modes. We explore some of those differences and describe some key advantages in the next sections of this chapter.

![Figure 3.1: Schematic of the cross-section of the designed platform. Doping implantations are not shown. The platform has three waveguide layers (Si, SiN1, SiN2), two metal interconnect levels (M1, M2), vias, TiN thin film heaters, depletion modulators (MOD), and Ge photodetectors. Taken from Ref. [34].](image)
3.2 Dispersion Engineering of Multilayer Waveguides

The second-order dispersion of a waveguide, in units of $\text{ps nm}^{-1} \text{km}^{-1}$, is the combination of a material contribution and one that comes from the waveguide geometry itself. In a single-mode integrated waveguide with a strong index contrast, the impact of the waveguide geometry is much greater than the material one. For a three dimensional geometry, the wavenumber is itself found through numerical means, and as such there exists no reliable estimate of a waveguide dispersion besides a direct numerical computation.

![Diagram of cross-section](image)

Figure 3.2: (a) Schematic of the cross-section, showing relevant geometric parameters. The SiN is deposited via a plasma-enhanced chemical vapor deposition (PECVD) process and has been characterized to have a refractive index of approximately 1.9 at a wavelength of 1.31 µm. (b) Dispersion parameter $D$ as a function of the separation $h_s$ for identical waveguides $w_1 = w_2 = 0.8$ µm, $h_1 = h_2 = 0.6$ µm. (c) Dispersion parameter $D$ as a function of the separation $h_s$ for asymmetric waveguides $w_1 = 0.6$ µm, $w_2 = 0.8$ µm, $h_1 = h_2 = 0.6$ µm.

The dispersion parameter $D$, in the units expressed above, is defined as

$$D = \frac{2\pi c}{\lambda^2} \frac{d^2 \beta}{d\omega^2} = -\frac{\lambda}{c} \frac{d^2 n_{eff}}{d\lambda^2}. \quad (3.1)$$
Even and odd modes of a multilayer waveguide follows this general definition, as any other modes. For such modes, however, the description of $\beta$ or $n_{e,f}$ is not as constrained; and there exists in fact correlation between the odd and even modes. Consider two waveguides supporting each a transverse mode along with complex slowly-varying envelopes in the z-direction that we label $a(x, y, z)$ and $b(x, y, z)$. In a standard coupled-mode theory framework, the propagation along the propagation axis $z$ obey

$$\frac{\partial a}{\partial z} = -i\kappa_{ab} e^{(-i2\delta_p z)} b, \quad (3.2a)$$

$$\frac{\partial b}{\partial z} = -i\kappa_{ba} e^{(i2\delta_p z)} a, \quad (3.2b)$$

where $\kappa_{ab} = \kappa_{ba}^*$ are the mode coupling coefficients described by an integral of the two mode profiles over the perturbing waveguide, $\delta_p = (\beta_b - \beta_a)/2$, and where $\beta_a$ and $\beta_b$ are the propagation constants of the unperturbed waveguides. In this framework, $a$ and $b$ are not eigensolutions of the composite structure: they are in general oscillatory for a given set of initial conditions. Skipping the standard derivation, we are able to find uncoupled solutions $u$ and $v$, referred to as the even and odd (super-)modes, respectively, or, alternatively, as the symmetric and anti-symmetric solutions. Their propagation constant follows

$$\beta_u = \frac{\beta_a + \beta_b}{2} + \sqrt{\delta_p^2 + \kappa^2}, \quad (3.3a)$$

$$\beta_v = \frac{\beta_a + \beta_b}{2} - \sqrt{\delta_p^2 + \kappa^2}, \quad (3.3b)$$

with $\kappa = |\kappa_{ab}| = |\kappa_{ba}|$.

This leads to a few interesting observations. The propagation constants of $u$ and $v$ are reciprocal with regard to $(\beta_a + \beta_b)/2$. This implies that the dispersion also follow this reciprocity due to the linear combination that forms it. For $\delta_p = 0$, this dispersion is reciprocally centered on the dispersion of an uncoupled waveguide, with a deviation that depends directly on the dispersion of this coupling. It is now suitable to look more closely at $\kappa_{ab}$ and $\kappa_{ba}$, defined as

$$\kappa_{ij} = \frac{\omega \varepsilon_0}{4} \int (n_i^2 - n_j^2) e_i^* e_j ds, \quad (3.4)$$

where $e$ is the electric transverse mode profile, and where the index difference is the effective perturbation. For the dispersion, this gives

$$\frac{\partial^2 \kappa_{ij}}{\partial \omega^2} = 2 \frac{\partial \left( \kappa_{ij}/\omega \right)}{\partial \omega} + \omega \frac{\partial^2 \left( \kappa_{ij}/\omega \right)}{\partial^2 \omega}, \quad (3.5)$$

$\partial \left( \kappa_{ij}/\omega \right)/\partial \omega$ is the partial derivative of the overlap integral of an evanescent tail (that is very
frequency dependent) and a non-evanescent part. Because of the exponential nature of this tail, the derivative is thus going have a large magnitude as the waveguides are closer, or as $\kappa$ is larger. This is only countered if $\frac{\partial^2 (\kappa_{ij}/\omega)}{\partial^2 \omega}$ is particularly large. This coupling strength $\kappa$ being directly proportional to the waveguide separation, there exists potentially a great level of tunability in the value of the dispersion of $u$ and $v$, that exceeds the tunability provided by varying slightly the waveguide widths and heights while ensuring single-mode or close to single-mode operation.

This picture, however, holds only for weak perturbations. A more general description is given by a non-orthogonal coupled-mode theory framework, in which the variables of Eq. 3.3 are substituted as [39]

$$\left(\frac{\beta_a + \beta_b}{2}\right) \mapsto \frac{\beta_a + \beta_b}{2} + \frac{\kappa_{aa} + \kappa_{bb} - X (\kappa_{ab} + \kappa_{ba})}{2 (1 - X^2)},$$
$$\kappa \mapsto \frac{\kappa_{ab} + \kappa_{ba} - X (\kappa_{ab} + \kappa_{ba})}{2 (1 - X^2)^{1/2}},$$

(3.6a)

(3.6b)

where $X$ is a cross-power defined as

$$X = \frac{1}{4} \int \left( e_i^* \times h_j + e_j \times h_i^* \right) \cdot \hat{z} ds,$$

(3.7)

with $h$ as the transverse mode profile with the magnetic field. This cross-power becomes significant and eventually exceed $\kappa_{ab}$ or $\kappa_{ba}$ when the waveguides are close. Non-orthogonal coupled mode theory is a refinement over standard coupled mode theory for the determination of the propagation constants, but is less accurate for the prediction of the power exchange between coupled waveguides [39]. This theory is useful since we are only interested in the eigenvalues. The terms $\kappa_{aa}$ and $\kappa_{bb}$ are included in the standard formulation of coupled-mode theory, but are again here very small unless the waveguides are close, in which case $X$ is also significant.

A new prediction is that the dispersion can be steered away from the average of the two uncoupled waveguides. This is helpful to extend the tunability even further. The treatment so far remains qualitative, in that determining the dispersion can still only be done numerically for three-dimensional waveguides. Fig. 3.2 illustrates a numerical example of the dispersion of a double-layer waveguide, for a cross-section defined in Fig. 3.2a, first for the phase-matched case, and then with an asymmetry. The waveguide separation correlates with the strength of the linear coupling $\kappa$ and the cross-power $X$. At sufficiently high coupling strength, the dispersion goes beyond the framework of coupled-mode theory, which only remains an approximation of a linear perturbation in all its forms and is thus expected to fail at a certain point. Nevertheless, the insights developed above for rapidly exploring a parameter space by computing the dispersion of unperturbed waveguides, and allows an intuitive understanding of the
range of dispersion parameters that are achievable.

The impact of this dispersion on nonlinear processes is assessed more precisely in the next Chapter. One can still certainly appreciate the possibilities provided by dispersion engineering for applications that go beyond the formation of coherent patterns in Kerr resonators, or even nonlinear wave-mixing optics in general. These applications include pulse shaping, dispersion compensation, and chirped pulse amplification. Multilayer waveguides with engineered dispersion have been included in the platform of Fig. 3.1, but their characterization has proven challenging due to fabrication issues that are still being resolved.

### 3.3 Coupling Between Multilayer Waveguides

A set of multilayer waveguides approaching another set of multilayer waveguides represents an interesting three-dimensional generalization of the two-dimensional planar sets of coupled waveguides that have been extensively studied in the literature. The situation is schematically represented in Fig. 3.3 without loss of generality for two such sets of multilayer waveguides.

There are notable differences regarding the coupling of two waveguide supporting each higher-order modes and that of nominally single-mode multilayer waveguides. The odd mode supported by the coupled waveguides possess a sign change in the phase of its transverse profile that suppress inter-modal coupling with an even mode, that does not possess this change in sign, in a manner similar to the suppressed coupling between modes of different polarizations in geometries with an unbroken vertical symmetry, that have fields along different axes. This allows the precise coupling of one mode to the same mode in a different set of waveguides – a particularly useful characteristic for coupling to structures such as ring resonators, where coupling to non-resonant modes effectively becomes the same as radiative losses and thus lowers the quality factor dramatically. In the context of this work, where both more can still be excited, it prevents the interference of patterns in the bus coupler, that might otherwise disallow the formation of coherent combs.

The structure of Fig. 3.3 for arbitrary waveguides taken from the cross-section of Fig. 3.2a is simulated in a commercial eigenmode expansion (EME) software (FIMMPROP) as a proof of concept, the results of which are shown in Fig. 3.4. The relatively large radius of curvature (200 μm) is chosen to ensure there are no scattering losses prior to the coherent coupling. This assumption is reasonable for evanescent coupling with ring resonators. A wavelength of 1310 nm is chosen due to the O-band being chosen for the dynamical simulations of Chapter 4. Coupling to other modal families is not shown, but is, in this instance, effectively null beyond numerical noise. The presence of at least three degrees of freedom
(namely, width of the waveguides, coupler separation, and coupler length) makes it possible in theory to independently match the response of each mode to arbitrary coupling values, with consideration toward fabrication and wavelength sensitivities [40].

Figure 3.3: Two sets of multilayer waveguides each supporting even and odd modes come into close proximity and exchange power through evanescent coupling. The light is incident in the port labeled “In” and exits in the “Out” waveguides, in the same modal basis, in a proportion that depends on the evanescent coupling strength and the length of the structure.

Figure 3.4: Fraction of cross-coupled power (In → Out\text{cross}) for the even and odd TE modes as a function of an interaction length that excludes the bent sections, based on the geometry of Fig. 3.3, for a wavelength of 1.31 µm. The waveguides have uniform widths of 0.74 µm, a vertical separation of 0.2 µm, and a lateral separation of 0.55 µm. The bends have a radius of 200 µm.
3.4 Avoided Crossings in a Multilayer Ring Resonator

The previous sections established how to control the dispersion of a set of multilayer layers and how to transfer power between sets of coupled multilayer waveguides. From this, a ring such as the one shown in Fig. 3.5, supporting two coupled modes, can be constructed. Light launched into the wider (narrower) waveguide converts adiabatically to the even (odd) mode of the vertical bus coupler for selective modal excitation. The waveguides then couple evanescently to the resonator with low cross-talk due to the phase matching imposed by the directional coupler. The splitting ratio, $\zeta$, can be implemented either on- or off-chip. The optical input is monochromatic and has power $P_{in}$.

![Figure 3.5: A microring resonator consisting of two guiding layers supports even and odd modes for TE and transverse magnetic (TM) polarizations. A bus coupler with an adiabatic transition implements a variable splitting ratio $\zeta$ for the even and odd modes from a single input with power $P_{in}$.](image)

It is not immediately clear how the linear dynamics of such a ring are different from those of a ring supporting a fundamental and a higher-order mode, or even coupled polarization states. It is therefore potentially vulnerable to the general phenomenon of avoided crossings, also sometimes called anticrossings. We return to Eqs. 3.2 for their instructive nature on the origins of this phenomenon. As was mentioned, $a$ and $b$ are not eigensolutions of the set of coupled equations. Nevertheless, the original modes can still be resonant if coupling to the other mode is suppressed by virtue of the other mode being out of resonance. When both modes reach a degenerate resonance, the eigensolutions become a more instructive picture.

Eigensolutions of Eq. 3.2 are a combination of the original transverse modes, described by a ratio $a_t(x, y)/b_t(x, y)$, along with propagation constants that depend on the coupling strength and the phase matching condition. We now take interest in those ratios, which are, respectively, for regular coupled-
mode theory, \[41\]

\[a_t/b_t \mid_u = \kappa_{ba}/ \left( \delta_p + \sqrt{\delta^2_p + |\kappa_{ab}|^2} \right), \tag{3.8a}\]

\[a_t/b_t \mid_v = \kappa_{ba}/ \left( \delta_p - \sqrt{\delta^2_p + |\kappa_{ab}|^2} \right) . \tag{3.8b}\]

These ratios are general and apply to any sort of coupled modes, including polarization and higher-order hybridized modes such as those induced by bent mode profiles. Avoided crossings are a direct result of coupling between originally uncoupled modal families \[42, 43, 44\] – and even though great care is taken in exciting this uncoupled, eigenbasis, there can still be perturbations along the length of the resonator. The sensitivity of these ratios is therefore key to the issue of avoided crossings in a real device, even if it can be greatly reduced a priori by the choice of the basis. In their general form, these first order sensitivities are, for \(\kappa = |\kappa_{ab}| = |\kappa_{ba}| , \)

\[\partial (a_t/b_t \mid_u) / \partial \kappa = \delta_p/ \left( \delta_p \sqrt{\delta^2_p + \kappa^2 + \delta^2_p + \kappa^2} \right) , \tag{3.9a}\]

\[\partial (a_t/b_t \mid_u) / \partial \delta_p = \kappa/ \left( \delta_p \left( \kappa + \sqrt{\delta^2_p + \kappa^2} + \kappa^2 \right) \right) , \tag{3.9b}\]

\[\partial (a_t/b_t \mid_v) / \partial \kappa = \delta_p/ \left( -\delta_p \sqrt{\delta^2_p + \kappa^2 + \delta^2_p + \kappa^2} \right) , \tag{3.9c}\]

\[\partial (a_t/b_t \mid_v) / \partial \delta_p = -\kappa/ \left( -\kappa \sqrt{\delta^2_p + \kappa^2 + \delta^2_p + \kappa^2} \right) . \tag{3.9d}\]

For instance, for \(\delta_p = 0\), \(\partial (a_t/b_t) / \partial \kappa = 0\) and \(\partial (a_t/b_t) / \partial \delta_p = -2/\kappa\), with the expected outcome that strongly coupled modes are more robust to perturbations. Even then, the modes remain close to the right eigenbasis, whereas in the traditional case, with either \(a(z = 0)\) or \(b(z = 0)\) null, they are maximally far from it. For the reminder of this work, we assume that such effects are negligible. This is justified by the analysis presented above, that is nevertheless included for the sake of completeness.

Overall, the results of this section show the potential of a resonant structure with coupled modes, more particularly in the context of a multilayer photonic platform, in which an implementation is proposed. These results are useful for the nominal case of pattern formation within a single mode. One can go further by considering the dynamics of two co-resonant coupled modes to see how this affects the formation of coherent patterns. This is done in the next chapter.
Chapter 4

Dynamics of a Multilayer Resonator

This chapter covers the main results of this thesis relating to the dynamical behavior of a resonator containing a Kerr nonlinearity and supporting two coupled modes of a similar polarization. Both of these modes are simultaneously excited and interact with each other via incoherent cross-phase modulation. Stable localized solutions are revealed to exist beyond the usual formation mechanism involving a single mode. Periodic solutions from modulational instability (MI) are found to occur at a slight penalty on the nonlinear efficiency, but they stabilize the spatial dynamics, leading to the possibility of dissipative Kerr solitons (DKS) in previously unattainable regimes. While accounting for differential group delay and unequal second-order dispersions (in both magnitudes and signs), numerical simulations indicate paired breather solitons in addition to temporally stable solutions. The results demonstrate coupled modes can increase the stability of Kerr frequency comb generation.

4.1 Conventional Pattern Formation Mechanisms in a Planar Kerr Resonator

Optical frequency combs arising from Kerr microresonators are sometimes divided into so-called type-I and type-II combs [45, 46, 47]. A type-I comb originates from a periodic pattern sustained by modulational instability (MI) only, where the first sidebands appears at a distance of exactly one free spectral-range (FSR), and multiply from there. A related terminology is that of natively mode-spaced (NMS) combs [48]. A type-II comb, or multiple mode-spaced (MMS) comb, is one in which the first sidebands are created at a multiple of the FSR. As these sidebands grow in power, they create their own sidebands that are able to coherently interact with the sidebands created by other high-power sidebands.
In nominally single-mode resonators with anomalous dispersion, dissipative Kerr solitons are able to arise spontaneously from a single monochromatic continuous-wave (cw) input through such a process. Their dynamics are well documented [6, 49, 26]. In cavities with anomalous dispersion, dissipative solitons have been understood to emerge from the homoclinic orbit to a stable homogeneous steady state (HSS) passing asymptotically close, in phase space, to a stable periodic pattern, itself originating from MI. Compared to periodic MI patterns by themselves, solitons generally offer a broadened spectrum, making them preferred for frequency combs.

In a standard resonator, this soliton formation mechanism involves the lower intracavity power HSS in the bistable region of a nonlinear resonance. This metastability of the HSS causes stability issues in the practical implementations of Kerr combs [50, 51]. In the upper branch of the bistability, periodic patterns tend to progress into spatiotemporal chaos (sometimes also called optical turbulences) [52, 53] due to the high intracavity power. In the monostable regime, at lower intracavity power, spatial eigenvalues of the HSS are generally found unsuitable for this homoclinic orbit to occur, despite a HSS briefly co-existing with a subcritical periodic solution [54]. An interested reader can find more details concerning the terminology as it relates to the study of nonlinear dynamical systems in references such as [55].

Various methods have been put forward in recent years in order to stabilize this comb formation mechanism, including bichromatic inputs [56, 57, 58], parametric seeding [59], active wavelength or thermal stabilization [50, 60], and initialization procedures that match the timescale of the thermal nonlinearities to the soliton formation process [61]. This race towards stability illustrates the inherent fragility of Kerr combs, and the considerable interest for structures that minimizes the complexity and amount of external stabilization required for such coherent patterns to form.

There has also been resurgence of interest recently, motivated in parts by new theoretical understandings, in generation of optical frequency combs in single-mode resonators with normal dispersion. There, MI is not found to play a role (as it cannot be initiated), but it is discovered that certain dark DKS can still be sustained through switching waves between the co-existing high and low intracavity power HSS of the optical bistability [62, 63]. Although not exhaustively investigated, as opposed to the case with anomalous dispersion, no new insights are reported in this chapter within the context of two coupled modes experiencing normal dispersion.

4.2 Description of System Under Study

The system is that of Fig. 3.5, that we select for the convenience of an actual physical representation. Conceptually, the precise form of the resonator makes no difference to the main insights presented here.
Fig. 3.5 is convenient in that it provides, among other benefits, a framework for justifying the choice of input pumps of different powers. Such a device can be implemented in multilayer silicon nitride or silicon nitride-on-silicon integrated photonic platforms [34, 32].

Turning our attention to the dynamics of the system, we introduce the standard mean-field Lugiato-Lefever equation (LLE) [6] to describe the nonlinear propagation of the slowly varying field envelopes. Using the normalization of Ref. [64], modified to include the XPM interaction and mismatch in group velocities, the LLE becomes, for a symmetric mode \( u \) and anti-symmetric mode \( v \),

\[
\frac{\partial u}{\partial \tau} = - (\alpha_u + i \theta_u) u + i \left( f_{uu} |u|^2 + 2 f_{uv} |v|^2 \right) u + S_u - i \eta_u \frac{\partial^2 u}{\partial \tau^2} - \delta_g \frac{\partial u}{\partial \tau}, \quad (4.1a)
\]

\[
\frac{\partial v}{\partial \tau} = - (\alpha_v + i \theta_v) v + i \left( f_{vv} |v|^2 + 2 f_{uv} |u|^2 \right) v + S_v - i \eta_v \frac{\partial^2 v}{\partial \tau^2} + \delta_g \frac{\partial v}{\partial \tau}, \quad (4.1b)
\]

with \( t \mapsto \tau L / \tau_r \) as the slow time (on the scale of the optical round-trip time \( t_r \)), \( \tau \mapsto \tau [2 \alpha / (|\beta_2| L)]^{1/2} \) as the fast time (in a reference frame traveling at the average of the group velocities), \( \theta_{u,v} \mapsto \theta_{u,v} / \tau_r \), \( S_{u,v} \mapsto E_{in}^{u,v} (\gamma L / \tau_r)^{1/2} \), \( u \mapsto u (\gamma L / \tau_r)^{1/2} \) (similarly for \( v \)), \( \eta_{u,v} \mapsto \beta_2^{u,v} / |\beta_2| \), \( \alpha_{u,v} \mapsto \alpha_{u,v} / \tau_r \), \( \alpha_{u,v} \mapsto (\alpha_{u,v} L + \Gamma_{u,v}) / 2 \), and \( \delta_g \mapsto L^{1/2} (\eta_g^u - \eta_g^v) / [c (2 \pi |\beta_2|)^{1/2}] \). Quantities with an overline refers to an average of the absolute values of both modes. Within these normalized parameters, \( L \) is the cavity length, \( \vartheta \) is the linear phase detuning with the nearest resonance, \( \beta_2 \) is the second-order modal dispersion, \( \alpha^i \) is the linear power attenuation coefficient, \( \Gamma \) is the power coupling coefficient between the bus waveguide and the resonator, \( E_{in} \) is the amplitude of the cw pump, \( \gamma \) is the nonlinear coefficient, and \( c \) is the speed of light in vacuum. The coefficients \( f_{jk} \) refers to overlap integrals between the transverse modes, i.e.

\[
F_{jk} \propto \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |j(x,y)|^2 |k(x,y)|^2 \, dx \, dy}{\left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |j(x,y)|^2 \, dx \, dy \right) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k(x,y)|^2 \, dx \, dy \right)} \quad (4.2)
\]

that are themselves normalized to \( f_{jk} \mapsto 2 F_{jk} / (F_{uu} + F_{vv}) \). As the modes \( u \) and \( v \) are taken, in a first order approximation, as an equal weighting of the original transverse modes of the uncoupled waveguides, \( f_{jk} \approx 1 \) and \( \alpha_{u,v} \approx 1 \), except in regimes where \( \delta_p \) is significant. We keep the assumption of a small \( \delta_p \) throughout this work to simplify subsequent analyses. We further choose to work in a strongly coupled regime such that the coherent four-wave mixing term \( \exp(\pm i 4 \vartheta t) \) deliberately omitted from Eq. 4.1, oscillates rapidly with \( t \) to average to zero and can be neglected. The mechanism presented here therefore depends only on incoherent XPM.

Forms of the LLE similar Eq. 4.1 have been analyzed previously in the context of XPM coupling between polarizations [65, 66] and higher-order modes [67] for cases with no group velocity mismatch.
and with special forms of the continuous wave (cw) solutions. We extend these studies to a more general context with a special emphasis on the formation of coherent patterns.

### 4.3 Linear Stability Analysis

The homogeneous solutions of Eq. 4.1 are instructive as a starting point for the formation of patterned solutions. This is done by setting all derivatives to zero and leads to

\[
\begin{align*}
X_u &= Y_u^3 - 2(\theta_u - 2Y_u)Y_u^2 + \left[(\theta_u - 2Y_u)^2 + 1\right]Y_u, \\
X_v &= Y_v^3 - 2(\theta_v - 2Y_v)Y_v^2 + \left[(\theta_v - 2Y_v)^2 + 1\right]Y_v,
\end{align*}
\]

where \(X_{u,v} \equiv |S_{u,v}|^2\), the intensity of each cw input, and \(Y_u \equiv |u|^2\), \(Y_v \equiv |v|^2\). At first glance, the effect of XPM is to act as an additional effective detuning to \(\theta_{u,v}\). However, the dynamical nature of \(Y_u\) and \(Y_v\) means that this new control over the detuning cannot be achieved independently of either variable. Furthermore, as a system of two cubic polynomials, as many as nine simultaneous real solutions for given \(X_{u,v}\) are potentially accessible, as opposed to only three for the regular case of optical bistability from a single optical field (a result that follows directly from Bézout’s theorem in algebraic geometry).

Verifying the stability of \(\{Y_u, Y_v\}\) pairs is done by writing small perturbations \(\varepsilon \propto \exp(\lambda t + i\Omega \tau)\) in Eqs. 4.1a and 4.1b. Keeping only the linear terms from the simplified Eq. and solving for the non-trivial solutions of \([\varepsilon_u, \varepsilon_v, \varepsilon^*_u, \varepsilon^*_v]^T\) leads to

\[
\begin{vmatrix}
-\lambda_t - (1 + i\theta_u) + i(\eta_u \Omega^2 - \delta_u \Omega) + 2i(|u|^2 + |v|^2) & i|u|^2 \\
-i|u|^2 & -\lambda_t - (1 - i\theta_u) - i(\eta_u \Omega^2 + \delta_u \Omega) - 2i(|u|^2 + |v|^2) \\
2i|u||v| & 2i|u||v| \\
-2i|u||v| & -2i|u||v| \\
-2i|u||v| & -2i|u||v| \\
-2i|u||v| & -2i|u||v| \\
-\lambda_t - (1 + i\theta_v) + i(\eta_v \Omega^2 + \delta_v \Omega) + 2i(|u|^2 + |v|^2) & i|v|^2 \\
-i|v|^2 & -\lambda_t - (1 - i\theta_v) - i(\eta_v \Omega^2 - \delta_v \Omega) - 2i(|u|^2 + |v|^2)
\end{vmatrix} = 0,
\]

where \(|u|, |v|\) are the fixed points of Eqs. 4.3a and 4.3b. The resultant fourth order polynomial in \(\lambda_t\) is then solved to find roots with positive real parts (corresponding to gain) for certain normalized
frequencies \( \Omega \). The situation with gain and \( \Omega \neq 0 \) is consistent with the onset of MI, then an example of a Hopf bifurcation due to the periodicity enforced by the finite path length of the resonator, whereas \( \Omega = 0 \) checks for the dynamical stability to homogeneous perturbations.

It is in general disallowed to have, as a HSS, \( Y_u \) and \( Y_v \) of similar magnitudes in a resonant state, meaning \( X_{u,v} < Y_{u,v}(\Gamma/\alpha^2) \) and \( \theta_{u,v} > 2Y_{u,v} \). Both amplitude derivatives being generally large results in mode competition for the resonance, as both modes are acting as an effective detuning for the other. Consider the case where \( |u| = |v|, \theta_u = \theta_v, \) and \( \Omega = 0 \). The eigenvalues are

\[
\lambda_i^{(1)} = -1 \pm \left( -3|u|^2 - \theta_u \right) \left( 9|u|^2 - \theta_u \right)^{1/2}, \tag{4.5a}
\]

\[
\lambda_i^{(2)} = -1 \pm \left( -|u|^2 - \theta_u \right) \left( 3|u|^2 - \theta_u \right)^{1/2}. \tag{4.5b}
\]

With \( \theta_u > 2Y_u \), the requirements for a negative real part are \( 9|u|^2 < \theta_u \) and \( -3|u|^2 - \theta_u\) \( 9|u|^2 - \theta_u < 1 \). We take the limiting case \( 9|u|^2 = \theta_u \). The equation for optical bistability becomes

\[
X_u = 36Y_u^3 + Y_u, \tag{4.6}
\]
for which \( X_u < Y_u (\Gamma / \alpha^2) \) implies \( X_u < \left( \Gamma / \alpha^2 - 1 \right)^{1/2} / 6 \). However, reaching \( Y_u > 1 \), the usual threshold for patterned solutions, requires \( X_u > 37 \). This quantity is not very practical. Such high intracavity powers, in unnormalized units, are challenging to achieve experimentally [53]. It follows then that \( \Gamma / \alpha^2 > 222^2 + 1 \). For a critically coupled resonator, \( \Gamma = \alpha / 2 \). This gives a finesse value greater than 
\[
2(222^2 + 1) \pi\left(1 - (222^2 + 1)^{-1/2}\right),
\]
i.e. several hundred thousands. The required finesse remains close to five thousands for \( Y_u > 1 / 2 \). This has relevance for whispering gallery mode resonators, which can achieve these kind of values [68]. With \( \delta_g \approx 0 \), \( \eta_u = \eta_v \), \( \theta_{u,v} = 9|u|^2 \), and \( |u| = |v| \), the eigenvalues become
\[
\lambda^{(1)}_1 = -1 \pm \left( -\eta_u \Omega^2 + 6|u|^2 \right)^{1/2} \left( -\eta_u \Omega^2 + 8|u|^2 \right)^{1/2},
\]
\[
\lambda^{(2)}_1 = -1 \pm \Omega \left( \eta_u (-\eta_u \Omega^2 + 6|u|^2) \right)^{1/2},
\]
which indicate opportunities for MI to occur with both modes experiencing normal dispersion. The threshold power is \( |v|^2 = |u|^2 > 1/3 \) and the maximum gain at \( \Omega_{\text{max}} = \pm(3|u|^2 / \eta_u)^{1/2} \). This last result is in some ways a corollary of the one derived in Ref. [65], where the threshold is found to be unity due to the strength of the XPM coupling being three times weaker, owing to its origin from cross-polarized inputs. Allowing \( \delta_g \neq 0 \) and \( \eta_u \neq \eta_v \) does not change significantly the conclusions drawn here, although not all regimes were studied exhaustively.

Fig. 4.1 shows a typical example of one mode overwhelmingly dominating the resonance despite both inputs being close in magnitude. To otherwise gain some insights on a resonant regime with significant field enhancement at lower finesse values, solutions of Eq. 4.4 are therefore best treated as a perturbation of one mode to another of greater magnitude, without the expectation of very different qualitative features compared to the single-mode case. The impact of the main parameters influencing the MI gain \( \Re \{ \lambda_1 \} \) is shown in Fig. 4.2 for such intracavity power asymmetries. To allow a better comparison to the conventional single-mode case, the analysis is done with a high intracavity power mode experiencing a fixed anomalous dispersion \( \eta = \partial^2 \beta / \partial \omega^2 < 0 \), with \( \beta \) the propagation constant and \( \omega \) the optical angular frequency. We pick the anti-symmetric mode \( v \) as this mode, due to \( \partial^2 \kappa_{vu} / \partial \omega^2 = \partial^2 \kappa_{uv} / \partial \omega^2 > 0 \) being typical [20].

For at least one mode experiencing anomalous dispersion, the condition for MI, which nominally requires the normalized intracavity power to be unity, is thus only slightly perturbed, but the magnitude of the gain is generally lowered. While this indicates a reduction of the efficiency of the underlying nonlinear process, it simultaneously extends the range before the onset of spatiotemporal chaos. This regime tends to take over when the MI gain is too large and has created initial sidebands at multiples
of the free spectral range [48].

![Figure 4.2: Impact on the MI power gain per normalized length of (a) intracavity power and (b) dispersion of the secondary symmetric mode $u$; (c) detuning $\theta$ of the primary anti-symmetric mode $v$; and (d) group velocity mismatch $\delta_g$. Nominal parameters are $|u|^2 = 1/2$, $|v|^2 = 3$, $\eta_u = 2$, $\eta_v = -1$, $\theta_u = \theta_v = 3$, and $\delta_g = 0.$]
It is interesting to note that, for a dissipative system, an unstable pair of HSS is guaranteed to reach a stable state in a finite time. More specifically, this prevents the presence of a sustained periodic oscillation between coupled HSS (or in other words, a Hopf bifurcation in $t$ rather than $\tau$). For such a solution, the roots of the polynomial in $\lambda$ would need to be all purely imaginary. If $P(\lambda) = \sum_{k=0}^{4} a_k \lambda^{4-k}$ denotes the characteristic polynomial in $\lambda$ found from Eq. 4.4, then the Routh-Hurwitz criterion [69] imposes $a_3 = 0$ and $a_{k\neq3} > 0$ as sufficient conditions for all the roots to be purely imaginary.

Directly from the equation of the roots of a quartic equation, another possibility is hinted at with $\alpha_u + \alpha_v = 0$ (which cannot occur however from the assumption of a dissipative system with $\alpha_u > 0$ and $\alpha_v > 0$). In the nominal case of a single-mode resonator, this therefore give $\alpha = 0$, coinciding in this case with the Routh-Hurwitz criterion. For coupled waveguides, an additional solution seems enabled with a combination of gain and losses if it leads for instance to a broken $\mathcal{PT}$-symmetry, giving $\alpha_u = -\alpha_v$, but this solution is in fact unphysical as it would correspond to an optically injected laser above threshold with unsaturated gain. The solutions with $\alpha \sim 0$ for a single mode have in fact been studied in the context of dynamical semiconductor laser theory, where periodic oscillatory output have been observed, but the dynamics then depend strongly on those of the gain medium, that cannot be ignored or assumed instantaneous [70].

### 4.4 Spatial Dynamics and Soliton Formation Mechanism

An interesting question becomes the fate of dissipative solitons. For this, we turn our attention to spatial dynamics. As localized patterns are stationary with regard to the fast time variable $t$, we are able to set $\partial_t = 0$ in Eqs. 4.1a and 4.1b and apply perturbations of the form $\varepsilon \propto \exp(\lambda_s \tau)$ to study the possible interaction of co-existing solutions colliding with each other. With $[\varepsilon_u, \varepsilon_u^*, \varepsilon_v, \varepsilon_v^*]^T$, this leads to

\[
\begin{bmatrix}
- (1 + i\theta_u) - i\eta_u \lambda_s^2 - 2 \delta_\lambda \lambda_s + 2i(|u|^2 + |v|^2) & i|u|^2 & 2i|u||v| & 2i|u||v| \\
- i|u|^2 & - (1 - i\theta_u) + i\eta_u \lambda_s^2 - 2 \delta_\lambda \lambda_s - 2i(|u|^2 + |v|^2) & -2i|u||v| & -2i|u||v| \\
2i|u||v| & 2i|u||v| & (1 + i\theta_v) - i\eta_v \lambda_s^2 + 2i(|u|^2 + |v|^2) & i|v|^2 \\
-2i|u||v| & -2i|u||v| & -i|v|^2 & -(1 - i\theta_v) + i\eta_v \lambda_s^2 - 2i(|u|^2 + |v|^2)
\end{bmatrix} = 0.
\]

The roots of the resultant polynomial in $\lambda_s$, found numerically, are the eigenvalues of the spatial dynamics. Implicit in Eq. 4.8 is the assumption of a pattern encompassing both modes, as $\eta_{u,v}$ affects the values of $\lambda_s$ even if $u$ or $v$ are individually absent. For purely imaginary leading eigenvalues, the HSS oscillates and is unable to lock itself to a patterned state: this is precisely the case of the subcritical pattern in the monostable regime of the single-mode Kerr resonator. For purely real eigenvalues, single or
closely packed solitons with monotonic tails are expected to form. For complex eigenvalues, soliton trains are able to emerge due to the locking of their oscillatory tails that avoid their merging. Comprehensive discussions on the nature and significance of these eigenvalues can be found in Ref. [71, 54].

In Fig. 4.3, similar to Ref. [54], we show regions of the parameter space with qualitatively different eigenspectra – first for the nominal case $|v| = 0$, and then for the introduction of a second coupled mode with finite power and varying group velocity dispersions. A single bright soliton is able to form in the single-mode case due to the lower branch of the bistability occurring in the dark gray region for $\theta_v > 2$, where the leading eigenvalue are purely real. Following Ref. [54], leading eigenvalues are defined as those with a real part closest to zero. For very modest power injected in a second mode, the $\{|v|^2, \theta_v\}$ parameter space is shown here to be nearly fully stabilized – i.e. able to support the locking or homoclinic orbit of a HSS to a periodic pattern should they coexist for given external inputs $S_{u,v}$, as measured by the near complete absence of regions where the leading eigenvalues are purely imaginary. The assumption of a localized pattern involving both modes is a fair one considering that, from a calculation of the eigenvectors $\chi$ solving $(M + \lambda t I)\chi = \lambda \chi$, where $M$ is the matrix inside the determinant of Eq. 4.4, MI is seen to be initiated in both modes for $v$ and $u$ non-null. Physically, it would be unlikely for a pattern in one mode with a spatially varying XPM strength not to affect the resonance condition of another one.

In practice, spatio-temporal chaos can still be reached for relatively modest intracavity powers, depending on the full set of parameters of Eq. 4.1. It is therefore important to investigate realistic conditions that allow the formation of solitons. Though we expect the co-existing HSS and periodic patterns to be more numerous due to an increase in the potential number of multi-stabilities, exploring their density is complicated by the non-analytical solutions to and large dimensionality of Eq. 4.3. More rigorously, the onset of MI should be investigated for particular physical implementations to identify sub- or super-critical behaviors and their trajectories in phase space, these mechanisms having been identified as a necessary condition to soliton formation [54]. Even then, the linearization of the differential equations can only present a limited picture of the spatiotemporal dynamics, which are expected to be complex due to multiple bistabilities that can be triggered by large amplitude extents of the localized patterns. This severely limits the amount of intuition to be ultimately gained from such methods.

Nonetheless, by direct simulation of Eq. 4.1 with a split-step Fourier method [25], we find that soliton formation readily occurs for a large set of parameters that are robust to precise values of $\eta_{u,v}$. Figs. 4.4 and 4.5 show examples of such solutions in normalized units. Physically, Fig. 4.4 correspond to a microring with two SiN layers, a radius of 150 $\mu$m, a quality factor of $9.06 \times 10^5$, a differential group delay of 6.2 ps m$^{-1}$ (larger than a typical birefringent single-mode fiber by an order of magnitude),
Figure 4.3: Regions of the parameter space \((\theta_v, |v|^2)\) with qualitatively different eigenspectra for (a) \(|u|^2 = 0, \eta_u = 0\); (b) \(|u|^2 = 1/4, \delta_g = 10^{-5}, \eta_u = 2\); and (c) \(|u|^2 = 1/4, \delta_g = 10^{-5}, \eta_u = 10\). Other parameters are fixed as \(\eta_v = -1\) and \(\theta_u = 4\). The white region corresponds to the leading eigenvalues being complex, the light gray region to these eigenvalues being purely imaginary, and the dark gray region to these being purely real. As the eigenvalues are found numerically, a real or imaginary part below a tolerance of \(10^{-5}\) is considered null for the purpose of these figures.

and second-order dispersions of \(-510\ \text{ps nm}^{-1}\ \text{km}^{-1}\) and \(44\ \text{ps nm}^{-1}\ \text{km}^{-1}\) for the symmetric and anti-symmetric modes, respectively. These second-order dispersion values can be achieved, for example, with two PECVD SiN waveguide having a refractive index around 1.9 at a wavelength of 1310 nm with widths of 850 nm and heights of 600 nm, separated vertically by 400 nm of SiO\(_2\), which also form the cladding (see Fig. 3.2 (a) for a drawing). It should be noted, however, that such a cross-section does not have the required differential group delay, a general issue we address below. The external cw inputs are 50 mW and 100 mW, and the nonlinear coefficient, \(\gamma\), is \(1.645 \ \text{W}^{-1}\ \text{m}^{-1}\). The temporal phase deviation from the HSS background shows a correlation with the amplitude extent of the pattern. This is a general feature of dark (or gray) solitons [72]. The boundary condition of the resonator enforces the same amplitude
and phase for min(τ) and max(τ). In Fig. 4.5, the microring radius is decreased to 100 µm, and the input power of the symmetric mode increased to 80 mW, while the other parameters are kept the same. The temporal full width at half maximum of the soliton pair of Fig. 4.4a, in unnormalized units, is approximately 100 fs, extending the spectrum to several tens of THz.

Figure 4.4: (a) Temporally stable bright/dark soliton pair showing |u|^2 and |v|^2, and (b) arg(u) and arg(v). (c) Amplitude spectrum of the high intracavity power component v, for \( \eta_u = 1.84, \eta_v = -0.16 \), \( \theta_u = \theta_v = 4 \), \( S_u = 1.76 \), \( S_v = 2.49 \), and \( \delta_g = 3.1 \times 10^{-6} \).

The breather dynamics are unusual in that they show a sustained periodic oscillation of the DC levels, something that should have been disallowed from the linear stability analysis of the previous section. This oscillation disappears if the dispersion is turned off from the numerical integration of the coupled equations. This indicates that the DC oscillations are an accompanying feature of the subcriticality of the MI. The period is approximately 2.65 ns in unnormalized units, and does not vary with slight perturbation of the differential group delay or dispersion parameters, which classifies it as a stable limit.
Figure 4.5: Temporal dynamics of a breather bright/dark soliton pair with characteristic periodic behavior, showing (a) $|u|^2$ and (b) $|v|^2$, respectively, for $\eta_u = 1.84$, $\eta_v = -0.16$, $\theta_u = 3$, $\theta_v = 2.98$, $S_u = 1.82$, $S_v = 2.03$, and $\delta_g = 2.5 \times 10^{-6}$.

Cycle. Fig. 4.6a and 4.6b show the periodicity of dynamical orbit in the phase space comprised of the real and imaginary parts of $u$ and $v$, respectively, for a $\tau$ away from the soliton dynamics. Non-breather solitons are characterized by constant orbits in $t$ for all $\tau$. The strength of the coupling is sufficient to detune the modes to reach an almost null intracavity power, as shown by the proximity of the orbits to the origin. This slow ($\sim$ MHz) homogeneous self-oscillation with a significant amplitude extent is made interesting by the fact that it occurs in a driven system without gain (i.e., not an optically injected laser).

It is found numerically that the soliton formation mechanism strongly depend upon an upper threshold value for $\delta_g$, not unlike pulse formation in other structures with XPM coupling, most notably
birefringent fibers [73]. We confirm the empirical law of Ref. [74, 73] with, in non-normalized variables, $Z_0 |n^u_y - n^v_y|/c \lesssim \tau_p$ as a necessary condition for the locking of the solitons to each other, where $Z_0 \approx \pi (\tau_p/1.76)^2/(2 \min(|\beta_0^{u,v}|))$ is the soliton period and $\tau_p$ is the pulse duration (full width at half-maximum). Without this locking, if MI is otherwise supported, the system devolves into chaotic states as intracavity power is increased. Similar to Ref. [73], we also perturb the steady states with slight time delays and observe a return to a locked state, indicating the soliton pairing is stable for sufficiently small $\delta_g$. For differential group delays exceeding this requirement, the system devolves into chaos. This occurs simultaneously with the appearance of frequency lines outside the bandwidth of the main comb of the mode experience normal dispersion. This is a potential signature of Kelly sidebands [75], that are presumed to take over once the locking of the pulses become too weak and the MI of the mode experiencing anomalous dispersion is then seen as a periodic perturbation to the other mode.

In coupled structures, including double-layer resonators, $\delta_g$ is typically strongly positively correlated with the mode coupling term $\kappa$. A substantial value for $\kappa$ is itself needed to keep the coupled-mode approximation, prevent scattering-induced mode mixing, and perform dispersion engineering. This association between $\delta_g$ and $\kappa$ does not however hold unconditionally, especially if asymmetries ($\delta_p \neq 0$) are allowed. For instance, although the second-order dispersion is then not anomalous, a first LPCVD SiN waveguide (with close to a stoichiometric index of 2) with a height of 400 nm and a width of 600 nm, surrounded and separated by a SiO$_2$ cladding from another SiN waveguide 200 nm above or below with a similar height and a width of approximately 510 nm, is able to provide $\delta_g = 0$ for the coupled
TE modes. General procedures have been identified to reach $\delta_g = 0$ in the context of coupled multi-core fibers [76, 77, 78], to which the requirement of anomalous dispersion, also achieved through $\kappa$, is to be added.

An interpretation of soliton trapping in birefringent fibers is that the spectra of the two pulses shift equally and in opposite directions to accommodate for the group velocity difference. A simpler option is thus to use pumps of different wavelengths for each mode to extend this process beyond the region where trapping spontaneously occurs. By definition, the mode experiencing slightly positive anomalous dispersion has a fairly constant group velocity with regard to wavelength. On the other hand, the mode experiencing moderate normal dispersion has $\partial v_g/\partial \omega = (\partial v_g/\partial \lambda)(\partial \lambda/\partial \omega) < 0$ and thus $\partial v_g/\partial \lambda > 0$, where $v_g^{u,v} = c/n_g^{u,v}$. For a mode $u$ experiencing normal dispersion, the relationship $n_u < n_v$ generally holds. This means that the pump $S_u$ must be blueshifted in order to achieve $n_u \simeq n_v$.

It is still helpful for the group velocities to be relatively close to each other. For example, a dispersion of $-500$ ps nm$^{-1}$ km$^{-1}$ with a nominal group index close to 2 implies a change in group index of only $3.74 \times 10^{-5}$ per nm shift in wavelength. A change in wavelength of the pumps does not otherwise require a reformulation of Eq. 4.1 since the latter relies on incoherent XPM coupling.

### 4.5 Discussion

We have shown how the formation dynamics of periodic patterns and solitons in a double-layer Kerr resonator with two coupled modes interacting via XPM are modified compared to those from a single mode. The new degrees of freedom are promising for the practical generation of Kerr frequency combs from cavity solitons, achieved through a stabilization of spatial dynamics.

The largest obstacle to a physical realization, from a theoretical perspective, has been shown to come from the differential group delay, that is typically large in platforms with large to moderate index contrasts. There also remains many challenges of a technological nature that require overcoming for a satisfactory experimental demonstration to take place, among them the issue of resonator quality factors that remain quite poor in standard photonic foundry processes compared to in-house fabrication methods using, for instance, electron beam lithography.

Another practical challenge is the control over the relative detuning for a specific wavelength, that depends non-trivially on the input and intracavity powers of both modes though the thermo-optic effect and free carrier absorption (FCA), that changes the index and thus the detuning condition. This sort of detuning is hard to predict due to the uncertainty over the precise experimental conditions, but has been modeled for example in Ref. [61], as the time scale of these dynamics was there critical to
the soliton formation mechanism. Even if the stability to (small) time-dependent thermal effects is ultimately increased here by the self-balancing of the two modes, this scheme potentially restricts the range of wavelengths or input powers that can be used to achieve the desired relative detunings. The flatness of the dispersion is never infinite, nor is the tunable range of lasers, which often makes targeting a specific wavelength band crucial in practice.

The smaller the group detuning (another concurrent requirement), the longer it takes for the resonances to line up again on the wavelength axis. Neglecting the second-order dispersion, for a fixed detuning of the symmetric mode \( \vartheta_u \), the difference in unnormalized detunings \( \Delta \vartheta = \vartheta_u - \vartheta_v \) for successive resonances \( n \) is

\[
\Delta \vartheta_{n+1} = (\Delta \vartheta_n + 2\pi \left(n_y^n - n_y^u\right)/n_y^u) \mod 2\pi.
\]  

(4.9)

This further illustrates that the degree of control is simultaneously linked to the ability of targeting specific wavelengths for the frequency comb generation process. This issue is less relevant for two independent pumps of different wavelengths with independent detuning.

Besides thermal effects, other notable effects that were not included are higher order dispersion terms from the Taylor expansion and Raman interactions. Naturally, these inclusions are more recent, as the theoretical understanding matures and the models achieve progressively greater degrees of sophistication. Raman interactions have been shown to have influence over the final state of the combs in experimental studies, that can lead to stabilization [79, 80]. Higher order effects, on the other hand, have been shown to produce dispersive waves [81], represented in the spectrum as new peak far from the main comb, and that in some cases lead to optical frequency comb generation in new regimes, including visible light [82].

We also note in passing the recent interest in super-cavity solitons, that are generated by the excitation and coupling of subsequent longitudinal orders through the nonlinear tilt of the resonance [28, 83]. For this particular case, the LLE, as a mean-field approximation, is however unable to capture this behavior and must be extended or reverted to its original form as a NLSE adjoined to a boundary condition, namely the Ikeda map [28]. For a comparable nonlinearity, the co-resonance of coupled modes as presented in this work provides a simple way of achieving large cavity detunings at a lower power. This is achieved from the partial negative detuning compensation provided by the other mode, such that \( \theta_{u,v} \) can be increased equivalently to \( \theta_{u,v}\mid_{Y_{e,u}=0} + 2Y_{v,u} \). If this regime is not reached, then the LLE remains a valid approximation. It is thus not a limitation of the study presented in this thesis, but rather a potential avenue for future work.

Beyond the formation of nonlinear patterns, there are potentially many new applications for the engineered multistabilities of Eqs. 4.3. Applications of classical optical bistability abound, and include
switching and memory elements [84]. Accordingly, multistability can provide access to multilevel non-linear optical logic. The case presented here is made interesting by the fact that the multistability seen by one of the modes is itself tunable by the other one.

The opportunity of adding gain to modify the dynamics of multilayer resonators has yet to be explored. If the second-order dispersion is not exploited, then the dynamics are not predicted to be very different from optical injection in the two orthogonal polarizations of a single-mode laser. Nevertheless, the unification of the dynamics of dissipative optical cavities with those possessing gain has been the focus of recent attention within the research community. Both can be studied as the two extremes of a continuum within a unified framework developed from classical nonlinear dynamical system theory (see e.g., Ref. [85]).
Chapter 5

Experimental Characterization of Optical Waveguides

To investigate the feasibility of the micro-resonators described in the previous chapters, in this chapter, we present measurements of the Si and SiN waveguides realized in multilayer SiN-on-Si platform that were fabricated at the A*STAR IME Si photonic foundry on 8" wafers. Specifically, the Si and SiN waveguide propagation loss, backscatter, and second-order dispersion over the O-band are quantified, based on sample availability. This chapter gives general procedures for the characterization of optical waveguides, for the purpose of advancing toward an experimental demonstration of the structures proposed in the previous chapters.

5.1 Experimental Setup

The experimental setup is shown in Fig. 5.1. It consists of a temperature-controlled chip holder to which a vacuum pump is attached, surrounded on both sides by three-dimensional translation stages for the lensed fibers. The translation stages allow a coarse manual alignment that is then complemented by open-loop piezoelectric controllers for the final alignment. The lensed fibers are connected to a polarization controller and a tunable laser that covers the O-band (1262 nm to 1375 nm). The TE and TM polarizations are identified by maximizing and minimizing the transmission to a set of sharp waveguide bends in the Si layer, respectively.
5.2 Optical Losses

Optical losses relate to the quality factor of the resonator and thus to the field enhancement factor, on which the Kerr effect depends. Waveguide losses are best characterized by a series of waveguides of increasing lengths, that otherwise conserve the number of optical bends or the radius of curvature. A spectrum is acquired for each waveguide. A linear fit of the transmissions with regard to the propagation length differences directly give the propagation losses.

Fig. 5.2 shows an example of this linear fit for the losses averaged over $1300 \, \text{nm} \leq \lambda \leq 1320 \, \text{nm}$ for the main waveguides of the platform of Fig. 3.1 consisting of two PECVD SiN waveguide layers atop a Si waveguide layer, while Fig. 5.3 shows the resulting slope of this fit for the whole wavelength spectrum between 1262 nm and 1375 nm. This is done here with the TE polarization as it is the design polarization of the waveguides, except for the second SiN waveguide (SiN2). Deposition impurities were present to the side of this waveguide such that the losses are presented for a TM polarization, that doesn’t interact significantly with those impurities and is thus able to show relative uniformity between the two SiN waveguides.

The propagations losses for a 500 nm wide Si waveguide are approximately 1 dB/cm near the center wavelength. They are increased by approximately 3 dB for a waveguide narrowed down to a width of 380 nm. While a 500 nm wide Si waveguide is shown to have lowers losses, it is in fact slightly multimode at wavelengths in the O-band. Although it is then more sensitive to mode coupling effects (e.g., within bends), this indicates it is suitable for long waveguide sections that are used for routing, or
Figure 5.2: Linear fit of the optical transmissions obtained for waveguides of five different lengths, averaged over wavelengths within 10 nm of 1310 nm. (a) Si waveguides with a width of 500 nm and a height of 150 nm. (b) Si waveguides with a width of 380 nm and a height of 150 nm. (c) SiN waveguides with a width of 740 nm and a height of 450 nm. (d) Same as (c), but for a SiN waveguide at a different height within the layer stack (see Fig. 5.3).

for phase sensitive devices such as interferometers, since scattering losses are correlated to phase error.

The increased losses for a smaller width are expected, due to the increased interaction of the mode field with the roughness of the waveguide sidewalls. The two SiN waveguides are nominally similar, but located at different heights within the stack, and thus differs slightly depending on issues such as the quality and uniformity of the planarization steps. The worse fit of Fig. 5.2d can be explained in parts by the planarization and etching issues affecting this layer in particular, that reduce the uniformity of the propagation losses between different waveguides. The losses are fairly uniform at 1 dB/cm for most of the O-band, but start increasing sharply for wavelengths above 1360 nm. A possible explanation is the density of covalent hydrogen bonds caused by the PECVD fabrication process [86]. There exists an
absorption peak near 1520 nm due to a harmonic of the vibrational mode of nitrogen-hydrogen bonds. It is possible that this mechanism extends to the edges of the O-band.

Ignoring scattering-induced mode mixing effects, the impact of propagation losses is to increase the threshold power for nonlinearities to occur. At critical coupling, the quality factor of a ring resonator follows 

\[ \pi n_g L e^{\frac{\alpha L}{\lambda (1 - e^{\alpha L})}} \]  

[87]. In SiN, with \( n_g \sim 2 \), the quality factor of a resonator with a radius of 150 µm at a wavelength of 1310 nm with propagation losses near 1 dB/cm is approximately \( 2.08 \times 10^5 \), and increases exponentially for decreasing losses. Proof of concepts for new architectures routinely use large pump powers for the purpose of demonstration. For example, in the system demonstration of Ref. [14], an on-chip power of \( \sim 33 \) dBm was used due to the relatively low quality factor of \( 8 \times 10^5 \). These losses do not otherwise affect the qualitative nature of the results presented in the last chapter. In the C-band nm, SiN waveguides with high confinement and losses as low as 4.2 dB/m [88] and 0.4 dB/m [89] have been reported, yielding quality factors in the tens of millions for the latter. They make use of custom fabrication techniques that are not yet available or incompatible with semiconductor foundries with high throughput. It is expected that the losses of foundry fabricated waveguides will keep improving with times, reducing this discrepancy.

### 5.2.1 Waveguide Backscatter

Waveguide backscatter is the portion of scattered optical power that is coupled into the backward propagating modes instead of radiative modes. This has the consequence of adding a coupling factor
between the two counter-propagating longitudinal eigenmode that lifts the mode degeneracy and creates “resonance splitting” effects [90], which are another example of an avoided crossing phenomenon. Similar to avoided crossings due to polarization or higher-order mode coupling, this effect is more pronounced for resonators with high quality factors, and potentially perturb the dispersion and linear detuning parameters if significant. Characterization of this backscatter is thus important to diagnose such behavior and monitor waveguide quality.

Figure 5.4: (a) Schematic of the cross-section of the designed platform. The platform has three waveguide layers (Si, SiN1, SiN2). Taken from [38]. (b) Measured backscatter that includes, in its measurement path, a Si rib waveguide with a slab much larger than the optical mode and core width of 500 nm, surrounded by fiber patch cords. The blue line is the raw measurement and the orange one a moving average. (c) Similar to (b), but for a SiN strip waveguide with a core width of 900 nm.

Here, we compare the backscatter of a Si rib waveguide and a SiN strip waveguide. The layer stack is the one shown in Fig. 5.4a, from Ref. [38], as it allows for such as comparison. It is slightly different from that of Fig. 3.1 in terms of the interlayer separations and thicknesses, as with as the deposition process which is LPCVD. For the measured waveguides, the Si slab is virtually infinite in the lateral
direction, while the core, above it, has a width of 500 nm. The SiN waveguide has a core width of 900 nm. Both are surrounded by SiO$_2$ cladding. Waveguide backscatter is obtained from optical frequency domain reflectometry (OFDR), an interferometric technique that extracts backscatter from the period of the fringes produced by reflection events as the wavelength is swept. The experimental data is taken with an Optical Vector Analyzer™ (OVA) from Luna Innovations Inc. The results are presented in Fig. 5.4b and 5.4c. The two peaks correspond to two chip facets, and the data in-between to on-chip backscatter. Outside of this range, the backscatter is that of the fiber patch cords. The waveguide backscatter is approximately 25 dB above that of the optical fibers, around -74 dB/mm, and does not differ significantly between the two materials. This backscatter is correlated with scattering losses such that an improvement in waveguide quality would further reduce these losses. It is difficult to estimate precisely the effects of backscatter from a measured value since there exists no established model linking backscatter losses to a mode coupling coefficient. Resonance splitting has been observed experimentally, in unrelated work, for a quality factor close to a million in a silicon ring [3].

5.3 Waveguide Dispersion

Achieving the correct waveguide dispersion is critical to the initiation of the nonlinear process. Methods to measure it include (1) a measurement of the Jones matrix, followed by calculations involving the derivative of this matrix with respect to wavelength [91], and (2) the measurement of the deviation of the FSR of interferometric devices across a wavelength range [92, 93]. The former is not very robust to noise for short propagation lengths since it depends on a direct measurement of the group delay. For this reason, for the platform of Fig. 3.1, results are presented with the latter technique.

This method is however limited to the Si layer since no working interferometric devices exist on the SiN layers, due to layer planarization issues involving the second SiN layer [34]. The measured device is a MZI with a path length difference of 80 µm. The schematic of the MZI, for which its other purpose is to be used as a thermal switch, is shown in Fig. 5.5a. The optical waveguide in the unbalanced section consists of a 380 nm wide by 150 nm tall Si core surrounded by a SiO$_2$ cladding. Successive transmission minima $\nu_m$ define a FSR $\nu_{FSR} = \nu_{m+1} - \nu_m$. The mismatch between successive FSR is $\Delta \nu = \nu_{m-1} - 2\nu_m + \nu_{m+1}$. The dispersion parameter $\beta_2$ is then given by [92, 93]

$$\beta_2 = \frac{\partial^2 \beta}{\partial \omega^2} = \frac{-\Delta \nu}{2\pi \nu_{FSR}^3 \Delta L}.$$  \hspace{1cm} (5.1)

The dispersion parameter $D$ relates to $\beta_2$ through Eq. 3.1. This parameter is plotted in Fig. 5.5c.
Figure 5.5: (a) Schematic of the MZI used for the measurement of the waveguide dispersion, showing the path difference $\Delta L$. (b) Raw transmission spectrum of the MZI. (c) Measured dispersion parameter $D$ for a Si waveguide with a height of 150 nm and a width of 380 nm (black solid line) and simulated dispersion from a numerical mode solver for the same waveguide (dashed lines), as a function of wavelength. The orange line corresponds to a nominal waveguide with a width of 380 nm and a height of 150 nm. The purple line corresponds to a reduction in width of 20 nm and the yellow to a reduction in height of 10 nm.

alongside the simulated dispersion value. The spectrum used for this calculation is shown in Fig. 5.5b. The location of the transmission minima are found from a center of mass calculation and fitted to a polynomial function before a derivative is taken, to improve robustness to noise. For a given wavelength band, the technique is improved for small FSR, as it increases the number of data points available. The lack of relative fit between the measurements and simulations is explained by the high sensitivity of the waveguide to perturbations on its nominal dimensions due to the high degree of field confinement. For example, a reduction of only 10 nm in waveguide height is sufficient to approximately double the simulated values near the central wavelength and bring them closer to the measured ones. A simulation of angled sidewalls, including an angular deviation of up to 25°, does not show a significant change from the nominal case. Further combinations of width and height variations and other asymmetries can cause deviations from the simulated values that do not necessarily follow a linear combination of
the individual effects. The increase in dispersion compared to what was expected points toward the waveguide cross-section having a smaller area than designed.

Here, the dispersion is that of a regular waveguide that has not been engineered to be anomalous. It nevertheless illustrates the degree of correspondence (or lack thereof) between the expected and measured dispersion for optical waveguides in Si with high modal confinement. The requirement of an anomalous dispersion slightly above zero is, under regular circumstances, a strict requirement for the initiation of modulational instability and its evolution to a frequency comb with a broad spectrum. This therefore indicates that caution should be employed when designing waveguides with sensitive parameters to variations in width or height. It is expected that SiN waveguides are less sensitive to such fabrication errors due to their larger dimensions and their lower index contrast between the core and cladding.

The results presented in this chapter are nevertheless encouraging from the point of view of a practical realization in a foundry-compatible multilayer platform such as the one of Fig. 3.1. The losses are sufficient for a realization with reasonable (sub-Watt) input pumps. The dispersion, although sensitive, can be measured and estimated from the underlying sensitivity of the designed waveguides. The full range of behaviors described in Chapter 4 is then predicted to occur.
Chapter 6

Summary and Outlook

6.1 Summary

This thesis has described the nonlinear interaction, through FWM, of two coupled modes in multilayer resonators, and provided preliminary waveguide characterizations. The main results concern the investigation, from the viewpoint of nonlinear dynamical system theory and numerical analysis, of a new regime for nonlinear pattern formation, with co-resonant modes that do not experience significant hybridization from an avoided crossing, and with the benefit of a wide tunability of the modal parameters. Parameter spaces that support the formation of paired bright/dark solitons have been found, that are promising to for the generation of stable frequency combs in microresonators with coupled modes. This work is an attempt to show the rich dynamics provided by a multilayer platform by showcasing a few of its opportunities. Chapter 2 introduced the basic assumptions behind the modeling of the nonlinear interactions, starting from Maxwell’s equation. Chapter 3 introduced multilayer platforms and a specific implementation regarding multilayer ring resonators with coupled modes. Chapter 4 addressed the full dynamical simulations of the coupled differential equations and their analysis regarding linear stability and the spatial dynamics. Chapter 5 linked specific waveguide measurements of a fabricated multilayer platform to the potential realization of such results in practice.

The main contributions of this thesis are

1. A perspective toward the use of both coupled modes in a multilayer resonator; the impact of dispersion, modal avoided crossings, and coupling from a bus waveguide to such a structure;

2. The elaboration of the basic equations modeling the nonlinear interactions of coupled modes in a resonator, with the necessary assumptions;
3. The full dynamical analysis of these equations, alongside numerical simulations showing different solitonic regimes;

4. The choice of a protocol for the measurement of relevant waveguide characteristics (propagation losses, backscattering, and second-order dispersion), allowing the evaluation of the feasibility of an implementation.

6.2 Outlook

Optical frequency comb generation in resonators continues to be a research topic attracting considerable attention. From the early studies that aimed at a basic understanding of the pattern formation mechanism in relatively simple structures, the field has moved, as expected, to more complex structures exploiting more complex mechanisms in an attempt to improve the bandwidth, stability, or both simultaneously. As a natural progression of the technology, focus has recently shifted more sharply toward the manufacturability issues and cost associated with the generation of these combs. The bulkiness and stability requirement of fiber or free space mode-locked laser have prevented the widespread adoption of the technology. Similar to many other areas in integrated photonics, the foundry compatibility, and especially the CMOS compatibility, as addressed in this thesis, represent the most likely vector to achieve such goals. This history of the semiconductor laser and its impact is an example of a comparable optical technology. It is certainly possible nowadays to see this emphasis on manufacturability in many publications, including very recent work, e.g., Ref. [94].

Following Ref. [20] that showed the theoretical possibility of dispersion engineering with coupled waveguide, an experimental demonstration has been performed using planar concentric waveguides [21]. This thesis shows the potential for the excitation of both modes within the same device. Furthermore, the interest in multilayer platforms is growing, with more examples appearing, e.g. [95]. Academic and commercial foundries are progressively starting to offer multilayer photonic platforms combining Si and SiN as standard processes. This shall increase the interest in novel designs that exploit this new opportunity and bring attention to already existing designs making use of the multilayer structures, such as this work.

Future possible work in the specific area of this thesis includes, as previously mentioned, the inclusion of neglected effects such as Raman interaction or higher-order dispersion terms, that were not the focus of the analysis presented here. Perhaps more importantly, for there to be a true paradigm shift in the progress of on-chip microresonator-based frequency combs, there has to be serious considerations given to foundry-based fabrication methods, to ensure suitable quality factors that limit the input power
requirements. Eventually, systematic design procedures can then be elaborated for specific platforms, in a standardized manner, progressing ultimately to the offering of process design kits.

Another possibility provided by on-chip photonics is the combination of multiple optical functionalities together, in a true integrated fashion. Such integration, without the involvement of several chips, is still very much challenging, as shown in Ref. [14]. There is strong interest within the field for the demonstration of increasingly complex functionalities, that comes along with the overcoming of the unique challenges that such integration entails. For instance, there is no concept of signal buffering in integrated photonics. The resulting impact of imperfections on signals that reach later devices, that might themselves have static or time-varying parameter drifts uncorrelated with each other, can very easily lead to unpredictable behavior.

Linking back to the specific topic of this thesis, there remains so far a distinct lack of integrated combs within a broader system demonstration with a single chip, from academic or industrial researchers alike. The thesis demonstrates the feasibility of implementing photonic integrated circuits that incorporate frequency combs in foundry-compatible multilayer silicon photonic platforms. On-chip, foundry-manufactured comb sources have the potential to vastly expand the application of frequency combs at the telecommunication wavelength bands for communications and metrology.
Appendix A

Adaptation of the Split-Step Method to Coupled Ikeda Maps

The starting point is the expression of the Ikeda map of Eq. 2.12, that we rewrite for a set of coupled modes $u$ and $v$ as follow,

$$u_{m+1}(0, \tau) = \sqrt{\Gamma_u} u_{in} + \sqrt{1 - \Gamma_u} e^{i\phi_u} u_m(L, \tau), \quad (A.1a)$$

$$\frac{\partial u_m(z, \tau)}{\partial z} = -\frac{\alpha_i u}{2} + i\gamma \left( |u_m|^2 + 2|v_m|^2 \right) - \delta_g \frac{\partial u_m(z, \tau)}{\partial \tau} - i\frac{\beta_g}{2} \frac{\partial^2 u_m(z, \tau)}{\partial \tau^2}, \quad (A.1b)$$

$$v_{m+1}(0, \tau) = \sqrt{\Gamma_v} v_{in} + \sqrt{1 - \Gamma_v} e^{i\phi_v} v_m(L, \tau), \quad (A.1c)$$

$$\frac{\partial v_m(z, \tau)}{\partial z} = -\frac{\alpha_i v}{2} + i\gamma \left( |v_m|^2 + 2|u_m|^2 \right) + \delta_g \frac{\partial v_m(z, \tau)}{\partial \tau} - i\frac{\beta_g}{2} \frac{\partial^2 v_m(z, \tau)}{\partial \tau^2}. \quad (A.1d)$$

We pose the normalization $u \mapsto u(\gamma/\alpha)^{1/2}$ (similarly for $v$, $u_{in}$ and $v_{in}$), $z \mapsto \alpha_i z/2$, $\tau \mapsto \tau(2\pi_i/|\beta_2|)^{1/2}$, $\eta_{u,v} \mapsto \beta_{u,v}/|\beta_2|$, and $\delta_g \mapsto \delta_g/(2\pi_i|\beta_2|)$, to finally obtain

$$u_{m+1}(0, \tau) = \sqrt{\Gamma_u} u_{in} + \sqrt{1 - \Gamma_u} e^{i\phi_u} u_m(\alpha_i L/2, \tau), \quad (A.2a)$$

$$\frac{\partial u_m(z, \tau)}{\partial z} = -\frac{\alpha_i u}{\alpha_i} + i\gamma \left( f_{uu}|u_m|^2 + 2f_{uv}|v_m|^2 \right) - \delta_g \frac{\partial u_m(z, \tau)}{\partial \tau} - i\eta_u \frac{\partial^2 u_m(z, \tau)}{\partial \tau^2}, \quad (A.2b)$$

$$v_{m+1}(0, \tau) = \sqrt{\Gamma_v} v_{in} + \sqrt{1 - \Gamma_v} e^{i\phi_v} v_m(\alpha_i L/2, \tau), \quad (A.2c)$$

$$\frac{\partial v_m(z, \tau)}{\partial z} = -\frac{\alpha_i v}{\alpha_i} + i \left( f_{vv}|v_m|^2 + 2f_{uv}|u_m|^2 \right) + \delta_g \frac{\partial v_m(z, \tau)}{\partial \tau} - i\eta_v \frac{\partial^2 v_m(z, \tau)}{\partial \tau^2}, \quad (A.2d)$$

where the overlaps $f_{jk}$ have also been added for the sake of completeness, similarly to the LLE counterpart.
of Eq. 4.1.

The following set of coupled nonlinear equations is then solved with the regular split-step method for the NLSE, with the boundary conditions A.2a and A.2c re-applied at each step. In the time domain, the applied steps are

\[
\begin{align*}
    u_m(z + \Delta z, \tau) &= e^{\left(-\frac{\alpha_i}{\Delta_i} + i f_{uu} |u_m|^2 + i 2 f_{uv} |v_m|^2\right) \Delta z} u_m(z, \tau), \\
    v_m(z + \Delta z, \tau) &= e^{\left(-\frac{\alpha_i}{\Delta_i} + i f_{vu} |v_m|^2 + i 2 f_{uv} |u_m|^2\right) \Delta z} v_m(z, \tau).
\end{align*}
\] (A.3)

In the spectral domain, the steps are

\[
\begin{align*}
    \tilde{u}(z + \Delta z, \Omega) &= e^{i \Omega \Delta z \left[-\delta_g + (\eta_u / |\eta|) \Omega\right]} \tilde{u}(z, \Omega), \\
    \tilde{v}(z + \Delta z, \Omega) &= e^{i \Omega \Delta z \left[\delta_g + (\eta_v / |\eta|) \Omega\right]} \tilde{v}(z, \Omega),
\end{align*}
\] (A.4)

where \( \tilde{u} \) and \( \tilde{v} \) denote the Fourier transforms of \( u \) and \( v \), respectively.

**MATLAB Implementation**

```matlab
function U_out = runIkeda (ng, L, Aeff, n2, phi, alpha, D, lambda, P_in, theta, tau, net)

% runIkeda An implementation of the normalized Ikeda map for coupled modes.
% ng: group index, 1x2
% L: length of cavity [m], scalar
% Aeff: effective mode area [m^2], 1x2
% n2: nonlinear coefficient [m^2/W], scalar
% phi: linear detuning [rads], 1x2
% alpha: propagation losses [m^-1], 1x2
% D: second-order dispersion [ps/nm/km], 1x2
% lambda: optical wavelength [m], scalar
% P_in: input power in bus coupler [W], 1x2
% theta: power coupling coefficient of input field, 1x2
% tau: run time [s], scalar
% nt: number of discretized intracavity field points, scalar
```
c = 299792458;
delta_tau = -diff(ng)/c;
vg = c/mean(ng);
E_in = sqrt(P_in);

beta_2 = -D.*lambda^2/c/2/pi;
gamma = 2*pi*n2/lambda/Aeff;

% main loop
U_ring(:,ind_h+1) = sqrt(theta).*U_in;

% head
ind_h = 0;

% tail
ind_t = 1;

% main loop
U_ring(:,ind_h+1) = sqrt(theta).*U_in;
for ii=1:numel(U_out)
    uu(1,:) = fft(ifft(U_ring(1,:)).*gvd(1,:));
    uu(2,:) = fft(ifft(U_ring(2,:)).*gvd(2,:));
    U_ring(1,:) = uu(1,:).*exp((abs(uu(1,:)).^2 + 2.*abs(uu(2,:)).^2).*hhz).*loss;
    U_ring(2,:) = uu(2,:).*exp((abs(uu(2,:)).^2 + 2.*abs(uu(1,:)).^2).*hhz).*loss;
    U_out(:,ii) = sqrt(1-theta).*U_in - sqrt(theta).*exp(1i*phi).*U_ring(:,ind_t+1);
    U_ring(:,ind_h+1) = sqrt(theta).*U_in + ...
        sqrt(1-theta).*exp(1i*(phi-temp)).*U_ring(:,ind_t+1);
    ind_h = mod(ind_h+1, nt);
    ind_t = mod(ind_t+1, nt);
end
end
Appendix B

Adaptation of the Split-Step Method to Coupled Lugiato-Lefever Equations

We start from the normalized version of the coupled LLE of Eq. 4.1. For a singular step, quantities $|u|^2$ and $|v|^2$ are treated here as constants (that are independent of $u$ and $v$). Disregarding the terms containing partial derivatives with respect to $\tau$, the coupled equations possess analytical solutions

$$u(t + \Delta t, \tau) = e^{-\Delta t(\alpha_u + (\theta_u - f_{uu}|u|^2 - 2f_{uv}|v|^2))}$$

$$\left[u(t, \tau) \left(\theta_u - f_{uu}|u|^2 - 2f_{uv}|v|^2 - i\alpha_u\right) - iS_u \left(e^{\Delta t(\alpha_u + (\theta_u - f_{uu}|u|^2 - 2f_{uv}|v|^2))} - 1\right)\right]^{-1},$$

(B.1a)

$$v(t + \Delta t, \tau) = e^{-\Delta t(\alpha_v + (\theta_v - f_{vv}|v|^2 - 2f_{vu}|u|^2))}$$

$$\left[v(t, \tau) \left(\theta_v - f_{vv}|v|^2 - 2f_{vu}|u|^2 - i\alpha_v\right) - iS_v \left(e^{\Delta t(\alpha_v + (\theta_v - f_{vv}|v|^2 - 2f_{vu}|u|^2))} - 1\right)\right]^{-1},$$

(B.1b)

that are found from a symbolic computation software. We can then apply a separated step in the spectral domain

$$\tilde{u}(t + \Delta t, \Omega) = e^{i\Omega\Delta t[-\delta_u + (\eta_u/|\eta|)\Omega]}\tilde{u}(t, \Omega),$$

(B.2a)

$$\tilde{v}(t + \Delta t, \Omega) = e^{i\Omega\Delta t[\delta_v + (\eta_v/|\eta|)\Omega]}\tilde{v}(t, \Omega),$$

(B.2b)

where $\tilde{u}$ and $\tilde{v}$ denote the Fourier transforms of $u$ and $v$, respectively. It should be noted that it is also possible to solve Eq. 4.1 with direct finite difference schemes such as the Runge-Kutta methods; and it
is fact commonly done in the literature due to the non-trivial nature of B.1, that has not so far been reported in any publication.

Further refinements are possible to both of these methods (Ikeda and LLE) besides the addition of thermal, Raman effect, or higher-order dispersion effects, that have already been discussed in the literature. One easily notes the absence of wavelength dependence to the coupling values and propagation losses. This imperfection of the coupling or losses is more rarely tackled, even though integrated optical frequency combs are now exceeding octave spanning bandwidths. An example is found for multimode non-ideality [96], but not for wavelength dependence. Although not included in the numerical implementation of the LLE that follows, when required, a method to include this imperfection is the application of a linear filter to the intracavity field in \( \tau \), for example during the spectral step. There is no limit from linear filter theory to the precision to which a filter shape approximates an arbitrary frequency response, if the filter order is allowed to vary. It thus becomes possible to design a filter that interpolates loss values obtained from a separate simulation, or from experimental data itself.

MATLAB Implementation

```matlab
function U_ring = runLLE(ng, L, Aeff, n2, phi, alpha, D, lambda, P_in, theta, t, nt, dt)

% runLLE An implementation of the normalized LLE for coupled modes.
% ng: group index, 1x2
% L: length of cavity [m], scalar
% Aeff: effective mode area [m^2], 1x2
% n2: nonlinear coefficient [m^2/W], scalar
% phi: linear detuning [rads], 1x2
% alpha: propagation losses [m^-1], 1x2
% D: second-order dispersion [ps/nm/km], 1x2
% lambda: optical wavelength [m], scalar
% P_in: input power in bus coupler [W], 1x2
% theta: power coupling coefficient of input field, 1x2
% tau: run time [s], scalar
% nt: number of discretized intracavity field points, scalar
% dt: time step as a fraction of round-trip time, scalar

c = 299792458;
```
delta_tau = -diff(ng)/c; %* 5e-3;

a = .5.*(alpha+theta);

beta_2 = -D.*lambda^2/c/2/pi;

gamma = 2*pi*n2/lambda/Aeff;

t_rt = L/c*mean(ng); % s
dt = dt*t_rt;

steps = round(t/dt)+1;

dt_norm = dt*mean(a)/t_rt;

tau_rt = t_rt*sqrt(2*mean(a)/mean(abs(beta_2))/L);

omega = (pi/tau_rt) .* [(0:nt/2−1), (−nt/2:−1)];

S = sqrt(P_in).*sqrt(gamma*L.*theta./a.^3);

a_m = mean(a);

gvd = [exp(-1i*1*delta_tau/sqrt(mean(abs(beta_2)))/sqrt(2*mean(a)/L).*omega*dt_norm) .* 

    exp(1i*beta_2(1)/mean(abs(beta_2)).*omega.^2*dt_norm); 
    exp(+1i*1*delta_tau/sqrt(mean(abs(beta_2)))/sqrt(2*mean(a)/L).*omega*dt_norm) .* ... 
    exp(1i*beta_2(2)/mean(abs(beta_2)).*omega.^2*dt_norm)];

% main loop

for ii=1:steps

    uu(1,:) = fft(ifft(U_ring(1,:)).*gvd(1,:));
    uu(2,:) = fft(ifft(U_ring(2,:)).*gvd(2,:));

    nonlin = [abs(uu(1,:)).^2 + 2.*abs(uu(2,:)).^2; 
               abs(uu(2,:)).^2 + 2.*abs(uu(1,:)).^2];

    U_ring(1,:) = exp(-(a(1)/a_m+1i.*phi(1)−li*nonlin(1)).*dt_norm) .* ... 
            (uu(1,:).*(phi(1) − nonlin(1,:)) − li.*a(1)/a_m) − li.*S(1) .* ... 
            (−1 + exp((1i*phi(1)−li*nonlin(1)+a(1)/a_m).*dt_norm)) ./ ...
(phi(1) - nonlin(1,:)) - li.*a(1)/a_m);

U_ring(2,:) = exp(-(a(2)/a_m+li.*phi(2)-li*nonlin(2)).*dt_norm).* ... 
(uu(2,:).*(phi(2) - nonlin(2,:)) - li.*a(2)/a_m) - li.*S(2).* ... 
(-1 + exp((li*phi(2)-li*nonlin(2)+a(2)/a_m).*dt_norm)) ./ ... 
(phi(2) - nonlin(2,:) - li.*a(2)/a_m);

end
end
Bibliography


