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Multiobjective Optimization by Decision Diagrams

David Bergman¹ and Andre A. Cire²

¹ Department of Operations and Information Management, University of Connecticut
`david.bergman@business.uconn.edu`

² Department of Management, University of Toronto Scarborough
`acire@utsc.utoronto.ca`

Abstract. In this paper we present a technique for solving multiobjective discrete optimization problems using decision diagrams. The proposed methodology is related to an algorithm designed for multiobjective optimization for dynamic programming, except utilizing decision diagram theory to reduce the state space, which can lead to orders of magnitude performance gains over existing algorithms. The decision diagram-based technique is applied to knapsack, set covering, and set partitioning problems, exhibiting improvements over state-of-the-art general-purpose multiobjective optimization algorithms.

Keywords: Decision diagrams; Multiobjective optimization; Multicriteria decision making; Multicriteria shortest path

Keywords: Discrete Multiobjective Optimization, Decision Diagrams.

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1 Introduction

Automated decision making, by its very nature, requires the consideration of a multitude of objectives. Multiobjective optimization, also known as multiobjective programming, vector optimization, multi-criteria optimization, multi-attribute optimization or Pareto optimization, has a rich history, dating back to the emergence of rigorous mathematical programming [21]. Several books have been written on the topic [11, 28, 14], and many in-depth surveys [34, 37, 12, 13, 15].

The present paper concerns *multiobjective discrete optimization problems* (MODO), where the variables of the problem are discrete and the number of objectives is $p \geq 2$. An *efficient solution* to a MODO is one in which there is no other solution that improves, simultaneously, on each of the objectives. The vector corresponding to the objectives of an efficient solution is called a *nondominated solution*. The set of efficient solutions is known as the *efficient set* and the set of all nondominated solution is known as the *nondominated set*. The goal of a MODO, for the purposes of this paper, is to enumerate the nondominated set.

Perhaps the most commonly studied technique for identifying all nondominated solutions for MODOs is the ϵ -constraint method [18], having well-known properties for more than two objectives [11]. The ϵ -constraint method was first used by Laumanns et al. [24], where an adaptive search over weighted objectives was proposed. Özlen et al. [27,29] and Kirlik and Sayın [23] provide algorithmic improvements and are the basis for the comparison in the computational results presented in this paper. Other utilized scalarization techniques include Benson’s method [3] (a combination of the weight-sum technique and the ϵ -constraint method) and the augmented weighted Chebychev method [32]. The main drawback of these techniques is that a discrete optimization problem, often NP-hard in practice, must be solved several times so as to enumerate the necessary scalarizations, leading to the bottleneck of their procedures.

In this paper, we propose an alternative method that creates a single *binary decision diagram* (BDD) [25,1,9,10] which represents all feasible solutions to the problem as a directed acyclic graph, and then employs *multicriteria shortest path problem* (MSP) algorithms to enumerate the nondominated set. This technique therefore transforms the problem into that of (1) finding the exact BDD for the constraint set of the MODO, and (2) using MSP algorithms to find the set of nondominated solutions.

The utilization of decision diagrams in optimization is recent and focusses on the use of BDDs for a variety of purposes [5]. In this research stream, top-down compilation methods for set covering problems [7,6] and set packing problems [6,4] have been investigated, along with methods for creating BDDs for knapsack problems [2]. On the other hand, published articles on MSP are abundant. Research on MSP started in the 1970s [35,19] and has typically concentrated on two objectives (see Raith and Ehrgott [30] for details). A survey on exact methods for the MSP is provided by Garroppo et al. [17].

Our methodology is primarily based on the work by Loui [26], who proposes a multidimensional labeling technique for dynamic programming (DP) models with stochastic or multidimensional weights. Such techniques have been considered in several papers [30] and are known in the DP community (see, e.g., Bertsekas [8], §2.3.4), often being applied to multiobjective knapsack algorithms, as by Figueira et al. [31]. Nonetheless, labeling algorithms are typically prohibitive in practice due to the curse of dimensionality of DP models, as the state space often grows too quickly in practice.

BDDs are closely related to dynamic programming and, as in DPs, they also reduce discrete optimization to shortest path problems. However, it has been shown that the state-space associated with a BDD can be much more compact than that of a similar DP model [20], which is key to our methodology. Our major contribution, hence, is the introduction of a MSP labeling algorithms for decision diagrams. This yields an alternative, simple method for enumerating the nondominated set of a multiobjective problem. The technique is applicable to a number of discrete optimization problems, and our numerical study indicates that it can be orders of magnitude faster than state-of-the-art techniques.

2 Multiobjective Discrete Optimization

A MODO $\mathcal{M}$ is specified by a set of p objective functions $f^j : \mathbb{R}^n \rightarrow \mathbb{R}$, for $j = 1, \dots, p$, and a *feasible set* $\mathcal{X}$. In this paper it is assumed that each objective function is additively separable or, more simply, linear, so that $f^j(x) = \sum_{i=1}^n c_i^j x_i$. The feasible set $\mathcal{X}$ is discrete — this paper will focus on *binary optimization problems* so that $\mathcal{X} \subseteq \mathbb{B}^n$, although the techniques are easily generalizable. Each $x \in \mathcal{X}$ is said to be a *feasible solution*.

Let f be the set of objective functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ (assumed to be maximized unless otherwise specified). Each solution $x \in \mathcal{X}$ is mapped through the function f into a corresponding objective vector $y = f(x) \in \mathbb{R}^p$. The set of objective vectors $\mathcal{Y} = \{f(x) : x \in \mathcal{X}\}$ resulting from feasible solutions is the *objective space*. A solution x^* is called an *efficient solution* if there exists no other feasible solution x' such that, for all j , $f^j(x') \geq f^j(x^*)$ with $f^{j'}(x') > f^{j'}(x^*)$ for some j' . For an efficient solution x^* the vector $f(x^*)$ is referred to as a *nondominated solution*. The *efficient set*, denoted by $\mathcal{X}_E$, is the set of all efficient solutions and its image, denoted by $\mathcal{Y}_N = \{y : y = f(x) \text{ for some } x \in \mathcal{X}_E\}$, is the *nondominated set*. The (typical) goal of MODO and the focus of this paper is to obtain $\mathcal{Y}_N$.

3 Binary Decision Diagrams

A *binary decision diagrams* BDD $B = (n, U, A, \ell, d)$ is a *layered-acyclic digraph* composed of node set U and arcs A . The mapping $\ell : U \rightarrow \{1, 2, \dots, n+1\}$ partitions the nodes into $n+1$ layers $L_i := \{u \in U : \ell(u) = i\}$, $i = 1, 2, \dots, n+1$. Layers L_1 and L_{n+1} have cardinality one, with the single nodes in these layers denoted by *root* $\mathbf{r}$ and *terminal* $\mathbf{t}$, respectively. Each arc $a \in A$ leaves a *tail* node $t(a)$ and enters a *head* node $h(a)$, where $t(a), h(a) \in U$. It is assumed that $\ell(h(a)) = \ell(t(a)) + 1$, so that each arc connects nodes in adjacent layers. Also, each node u has at most one out-directed arc a with *arc-domain* $d \in \{0, 1\}$. An arc leaving node u is denoted by $a_0(u)$ if $d = 0$ and by $a_1(u)$ otherwise. The *width* of a layer is $w(L_i) := |L_i|$, and the *width* of B is $w(B) := \max_{i \in \{1, \dots, n+1\}} w(L_i)$. Finally, the *size* of a BDD is $|U|$.

A BDD represents a set of binary vectors in the following way. Each arc-specified path $p = (a_1, a_2, \dots, a_k)$ represents the vector $x(p) = (d(a_1), d(a_2), \dots, d(a_k))$. Any path from $\mathbf{r}$ to $\mathbf{t}$ thereby corresponds to a vector in $\mathbb{B}^n$. Let $\mathcal{P}(B)$ be the set of arc-specified paths from $\mathbf{r}$ to $\mathbf{t}$. Define $\mathbf{Sol}(B)$ as the set of binary vectors (called *solutions*) corresponding to arc-specified $\mathbf{r} - \mathbf{t}$ paths:

$$\mathbf{Sol}(B) = \{x(p) \in \mathbb{B}^n : p \in \mathcal{P}(B)\}.$$

BDDs can be used to represent the feasible set of a MODO $\mathcal{M}$ through relating $\mathbf{Sol}(B)$ with the set of feasible solutions $\mathcal{X}$ of $\mathcal{M}$. BDD B is said to be an *exact BDD* for $\mathcal{M}$ if $\mathbf{Sol}(B) = \mathcal{X}$.

Fix $u, v \in U$ for which $\ell(u) < \ell(v)$. Let $B_{u,v}$ be the BDD obtained by removing from B any nodes and arcs that do not lie on any directed path from

u to v . A BDD is said to be *reduced* if for all $i = 1, \dots, n$ and any two nodes $u, u' \in L_i$, the BDDs $B_{u,t}$ and $B_{u',t}$ are such that $\mathbf{Sol}(B_{u,t}) \neq \mathbf{Sol}(B_{u',t})$.

It is well-known that for any set of solutions $\mathcal{X}$, with a specified ordering of the variables given, there is a unique reduced BDD [9]. The reduction of a BDD can be performed efficiently by a simple bottom-up $O(|U| \log(|U|))$ algorithm [36]. It often has dramatic effects on the size of the BDD and the calculation of the nondominated set.

Suppose $\tilde{\mathcal{X}} \subseteq \mathbb{B}^4$ is given by the following set of solutions

$$\begin{aligned} \tilde{\mathcal{X}} = \{ & (0, 0, 0, 0)(0, 0, 0, 1)(0, 0, 1, 0)(0, 1, 0, 0)(0, 1, 0, 1)(0, 1, 1, 0) \\ & (1, 0, 0, 0)(1, 0, 0, 1)(1, 0, 1, 0)(1, 1, 0, 0)(1, 1, 1, 0) \} \end{aligned}$$

Consider the BDD $\tilde{B}$, depicted in Figure 1. Dashed/solid arcs correspond to arcs with arc-domain 0/1. $\tilde{B}$ is an exact BDD for $\tilde{\mathcal{X}}$ — each path from $\mathbf{r}$ to $\mathbf{t}$ corresponds to a solution in $\tilde{\mathcal{X}}$ and vice versa so that $\mathbf{Sol}(\tilde{B}) = \tilde{\mathcal{X}}$.

We refer the reader to the work by Bergman et al. [5] for BDD compilation procedures for general optimization problems.

Fig. 1. Exact BDD for $\tilde{\mathcal{X}}$

4 Determining the Nondominated Set

In this section we propose a technique which generalizes previous works on single-objective optimization problems using BDDs [5]. Namely, suppose a MODO $\mathcal{M}$ has one objective function (i.e., $p = 1$) with $f^1(x) = \sum_{i=1}^n c_i^1 x_i$, and assume $\mathcal{X}$ is represented by a BDD B . To optimize f^1 , we associate an *arc-value* $v(a)$ with each arc $a \in L_i$, $i = 1, \dots, n$, where $v(a) = c_i^1$ if $d(a) = 1$, and $v(a) = 0$ otherwise. Any longest $\mathbf{r}$ to $\mathbf{t}$ path p with respect to v yields an optimal solution $x(p)$, with the corresponding optimal value equal to the length of the path.

Let $\mathcal{M}$ be a MODO with $p \geq 2$. Given an exact BDD B for $\mathcal{X}$, we apply Algorithm 1 to determine $\mathcal{Y}_N$. The algorithm traverses B in a top-down fashion and associates a *state* s with each node u , representing the set of nondominated solutions in $\mathbb{R}^{\ell(u)-1}$ contained in $B_{\mathbf{r},u}$; at the end of the execution, $s(\mathbf{t}) = \mathcal{Y}_N$.

Algorithm 1 Find $\mathcal{Y}_N$ for MODO $\mathcal{M}$

```

1: procedure FINDNDS( $\mathcal{M}, B$ )                                 $\triangleright \mathcal{X}$  is the feasible set for  $\mathcal{M}$  and
2:                                                                objectives  $f^j, j = 1, \dots, p$ 
3:   for all  $u \in U$  do
4:      $s(u) \leftarrow \emptyset$ 
5:    $s(\mathbf{r}) \leftarrow \{\mathbf{0}\}$                                  $\triangleright \mathbf{0}$  is the  $p$ -dimensional 0 vector
6:   for  $i = 1, \dots, n$  do
7:     for all  $u \in L_i$  do
8:       if  $a_0(u)$  exists then
9:         for all  $s' \in s(u)$  do
10:          if  $s'$  is not dominated in  $s(h(a_0(u)))$  then
11:             $s(h(a_0(u))) \leftarrow s(h(a_0(u))) \cup s'$ 
12:          for all  $s'' \in s(h(a_0(u)))$  dominated by  $s'$  do
13:             $s(h(a_0(u))) \leftarrow s(h(a_0(u))) \setminus \{s''\}$ 
14:       if  $a_1(u)$  exists then
15:         for all  $s' \in s(u)$  do
16:           Increase state values of  $s'$  by  $c_i^j$ , for each  $j = 1, \dots, p$ 
17:           if  $s'$  is not dominated in  $s(h(a_1(u)))$  then
18:              $s(h(a_1(u))) \leftarrow s(h(a_1(u))) \cup s'$ 
19:           for all  $s'' \in s(h(a_1(u)))$  dominated by  $s'$  do
20:              $s(h(a_1(u))) \leftarrow s(h(a_1(u))) \setminus \{s''\}$ 
21:   return  $s(\mathbf{t})$ 

```

The algorithm initializes $s(u)$ for each node $u \in U \setminus \{\mathbf{r}\}$ to $\emptyset$, and initializes $s(\mathbf{r})$ to $\{\mathbf{0}\}$ (the p -dimensional vector with 0s in each coordinate). Having set the states of each node in L_i the algorithm proceeds to determine the states for the nodes in L_{i+1} . For each node in L_i and each arc directed out of u , the algorithm checks whether or not a nondominated solution will arise from extending solutions ending at u with the arc domain of the arc.

For each layer L_i , if the arc under consideration is a 0-arc, then any non-dominated solution will not be affected by setting $x_i = 0$. Therefore, each non-dominated solution in $s(u)$ is considered as a possible nondominated solution in $s(h(a))$. If it is nondominated, the solution is added to $s(h(a))$, and otherwise omitted. The process is analogous for the case of a 1-arc, except that any potential nondominated solution will have its objective value increased by c_i^j for each $j = 1, \dots, p$. Finally, if a solution s' is added to the state of a node, we must verify if any solutions s'' in that same state are now dominated by s' , removing them if that is the case.

Suppose, for example, that a MODO $\mathcal{M}'$ is specified with $\tilde{\mathcal{X}}$ defined as above and 2 objective functions, $f^1(x) = 2x_1 + 5x_2 + x_3 + 7x_4$, $f^2(x) = 4x_1 + x_2 + 4x_3 + 4x_4$. Figure 1 shows the result of applying Algorithm 1 to the BDD in Figure 1. Each node u is labeled with a set of vectors in the objective space. Those marked with a $\checkmark$ compose $s(u)$ and those marked with a $\times$ are those candidates for $s(u)$ that are determined to be dominated by some solution in $s(u)$. As a concrete example, consider the node u marked with the three vectors $(0, 0)$, $(5, 1)$ and

$(2, 4)$. $(0, 0)$ is first added to $s(u)$ because of the zero-arc directed at u from the node immediately above it. Then, when the one-arc from that same node is considered, it generates the vector $(0, 0) + (5, 1) = (5, 1)$, eliminating $(0, 0)$ because $(5, 1)$ dominates it. Then, the zero-arc from the right-most node on the previous layer is examined, generating vector $(2, 4) + (0, 0) = (2, 4)$. The set $\{(5, 1), (2, 4)\}$ is a collection of vectors, none of which dominate any other, and so $(2, 4)$ is added to $s(u)$. At the conclusion of the algorithm, $\mathbf{t}$ is labeled with $s(\mathbf{t}) = \{(12, 5), (9, 8), (8, 9)\}$, the set $\mathcal{Y}_N$ for $\mathcal{M}'$. The proof of correctness of Algorithm 1 follows immediately, e.g., from the results of Figueira et al. [31] and Loui [26].

5 Numerical Study

In this section we evaluate the empirical performance of the proposed multi-objective methodology on three classical optimization problems: *knapsack*, *set covering*, and *set packing*. Our key performance metric is the time to enumerate the complete set of nondominated solutions. We compare our approach with another general-purpose MODO solver developed by Kirlik and Sayın [23], an ϵ -constraint scalarization method which is currently regarded as one of the state of the art approaches for MODOs. We note in passing that we have also investigated other MODO solvers, such as the one proposed by Özlen et al. [29], but the method by Kirlik and Sayın was the best performing and numerically stable method across all solvers tested in our empirical setting.

The experiments ran on an Intel(R) Xeon(R) CPU E5-2640 v3 at 2.60GHz with 128 GB RAM. The BDD method was implemented in C++ and compiled with GCC 4.8.4. Our source code and all tested instances are available at <http://www.andrew.cmu.edu/user/vanhoeve/mdd/>. The source code for the technique by Kirlik and Sayın was downloaded from <http://home.ku.edu.tr/~moolibrary/> and linked with ILOG CPLEX 12.6.3 [22]. A time limit of 3,600 seconds was allotted in all cases.

Multicriteria Knapsack. Given n items, a capacity $W > 0$, and for each item i a weight $w_i > 0$ and p profits $v_i^1, v_i^2, \dots, v_i^p > 0$, the multicriteria knapsack problem (MKP) is: $\max \left\{ \sum_{i=1}^n v_i^j x_i, j = 1, \dots, p : \sum_{i=1}^n w_i x_i \leq W, x \in \{0, 1\}^n \right\}$.

We generated random MKP instances following the procedure by Kirlik and Sayın [23]. The values v_i^j and w_i were drawn uniformly at random from the set $\{1, \dots, 1000\}$, and $W = \lceil 0.5 \sum_{i=1}^n w_i \rceil$. Due to the growth of the nondominated set, for $p = 3$, we considered $n = \{10, \dots, 100\}$; for $p = 4$, $n \in \{10, \dots, 70\}$; for $p = 5, 6$, and 7 , $n \in \{20, 30, 40\}$. We generated 10 instances per pair (n, p) .

Table 1 presents the average cardinalities of the nondominated sets ($|\mathcal{Y}_N|$) and average solution times (within solved instances) for $p = 3, 4, 5$, where **Kirlik** and **BDD** denotes the method by Kirlik and Sayın [23] and the BDD technique, respectively. Figure 2(a) depicts a scatter plot comparing solution times for all instances, where the size of a point is proportional to n . The BDD method is substantially faster and more robust than **Kirlik**, in particular when p is large.

Table 1. Average solution times (in seconds) for knapsack problems for $p = 3, 4, 5$. Number in parentheses indicate instances unsolved within the time limit (out of 10).

n	$p = 3$			$p = 4$			$p = 5$		
	$ \mathcal{Y}_N $	Kirlik	BDD	$ \mathcal{Y}_N $	Kirlik	BDD	$ \mathcal{Y}_N $	Kirlik	BDD
10	9	0.10	0.01	14	0.28	0.01	22	0.04	0.01
20	37	0.85	0.03	79	7.22	0.04	241	1,319.48 ⁽⁷⁾	0.04
30	113	4.95	0.17	397	466.40 ⁽¹⁾	0.26	972	(10)	0.47
40	370	24.71	0.95	1,278	217.38 ⁽⁹⁾	2.39	4,943	(10)	18.06
50	598	50.47	3.64	3,374	(10)	27.87			
60	1,080	120.21	12.52	6,624	(10)	166.34			
70	1,325	154.50	32.69	14,696	(10)	1,164			
80	2,575	454.46	120						
90	3,847	912.47	350						
100	4,248	1,070 ⁽¹⁾	551 ⁽¹⁾						

In all cases, the time to create the BDD is only 5% of the total BDD time, and therefore the bottleneck is the computation of Algorithm 1. Finally, Figure 2(b) depicts a scatter plot comparing solution times of reduced and nonreduced BDDs for $n \in \{40, 50, 60\}$ and $p = 4$, emphasizing the importance of reducing the number of nodes of the BDD before performing Algorithm 1 (which, on average, take less than a second on all tested cases). This saves approximately a half of an order of magnitude on the total computation time.

Fig. 2. (a) Solution time comparison between Kirlik and BDD (in logarithmic scale), and (b) Solution times for reduced and non-reduced BDDs for the MKP.

Multicriteria Set Covering and Set Packing. Let A be a 0–1 $m \times n$ constraint matrix, and let $c^1, \dots, c^p$ be the p cost vectors in $\mathbb{R}^n$. The *multicriteria set covering problem* (MSCP) is defined as $\min \{(c^j)^T x, j = 1, \dots, p : Ax \geq 1, x \in \{0, 1\}^n\}$. The *multicriteria set packing problem* (MSPP) is similar to the MSCP and is written as $\max \{(c^j)^T x, j = 1, \dots, p : Ax \leq 1, x \in \{0, 1\}^n\}$. The multiobjective variants pose a considerably more difficult challenge [16].

We performed experiments on random instances generated as in Stidsen et al. [33]. Specifically, we considered $n \in \{100, 150\}$, $m = n/5$, and in our case we fixed 10 variables per constraint, i.e., for every $k = 1, \dots, m$, 10 elements of the k -th row of A were chosen to be equal to one uniformly at random. 10 instances were created per pair (n, p) . The objective coefficients were generated as they were for the MKP. The reduced BDDs for the MSPP and MSCP were compiled according to Bergman et al. [7, 5].

Table 2 presents the average cardinalities of the nondominated sets and the average solution times. As before, BDD is substantially faster and more robust than Kirlik, especially as the number of objective function increases. The BDDs performed particularly well for set packing problems, since they are typically much more compact when compared to the BDD representing MSCP instances. The time to compile the BDDs was also less than 5% of the total BDD time, as in the MKP case.

Table 2. Average solution times (in seconds) for MSCP and MSPP problems. Number in parentheses indicate instances unsolved within the time limit (out of 10).

n	p	Set Covering (MSCP)			Set Packing (MSPP)		
		$ \mathcal{N} $	Kirlik	BDD	$ \mathcal{N} $	Kirlik	BDD
100	3	117	5.91	5.46	164	5.61	5.07
	4	428	282.19	7.03	551.10	321.64	5.12
	5	1,171	82.79 ⁽⁹⁾	7.87	1,211.90	(10)	5.35
150	3	305	23.99	110.50	336.00	12.81	8.37
	4	1,178	1,228.18 ⁽²⁾	248.44 ⁽²⁾	4,557	3,025.52 ⁽⁸⁾	22.24
	5	4,711	(10)	329.49 ⁽⁵⁾	9,213	(10)	49.18

6 Conclusions

This paper proposes an algorithm for solving general multiobjective discrete optimization problems using decision diagrams. We utilize decision diagrams to represent, exactly, the feasible set of the problem, and then uses a multicriteria shortest path algorithm for finding the set of nondominated solutions. The algorithm is applied to three classical discrete optimization problems, and com-

putational methods indicate that the proposed method is superior to a state-of-the-art multiobjective technique, often providing orders of magnitude speedups.

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