MEROMORPHIC CONTINUATION OF EISENSTEIN SERIES

by

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Abstract

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The Eisenstein series associated to the cusp $a$ of a discrete subgroup $\Gamma < \text{SL}(2; \mathbb{R})$ is defined by $E_a(z; s) := \sum_{\Gamma_a \backslash \Gamma} \text{Im}(\sigma_a^{-1} \gamma z)^s$, where $\sigma_a$ is a certain fractional linear transformation. These sums are originally defined for $\text{Re}(s) > 1$, and generally diverge for $\text{Re}(s) \leq 1$. However they can be extended to all of $\mathbb{C} \setminus \{a \text{ discrete set} \}$ if they are reinterpreted in the sense of distributions. In fact, we shall prove that the distribution valued function $s \mapsto E_a(\cdot; s)$ is meromorphic.

To prove this, we begin by investigating modified function spaces on $\Gamma \backslash \text{SL}(2; \mathbb{R})$. These spaces are obtained by truncating Fourier expansions of regular automorphic functions near the cusp $a$. Versions of the hyperbolic Laplacian on truncated space have discrete spectra, while truncated approximations to $E_a$ are expressible as unique solutions to equations in these operators. Since $E_a$ differs from the truncated approximations by a holomorphic function, meromorphicity of $s \mapsto E_a(z; s)$ follows.
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Chapter 1

Introduction

In [11], Gel’fand and Shilov consider the problem of regularizing certain divergent integrals. For instance, they consider the functional

$$\langle \varphi, x_+^{\lambda} \rangle := \int_{\mathbb{R}} \varphi(x) x_+^{\lambda} \, dx$$  \hspace{1cm} (1.1)

where

$$x_+^{\lambda} = \begin{cases} 0 & \text{for } x \leq 0 \\ x^{\lambda} & \text{for } x > 0. \end{cases}$$

When Re(\(\lambda\)) > -1 these integrals converge, producing a well-defined functional. Since \(\varphi\) has compact support, we can differentiate under the integral sign to see \(\lambda \mapsto \langle \varphi, x_+^{\lambda} \rangle\) is even analytic for Re(\(\lambda\)) > -1:

$$\frac{d}{d\lambda} \langle \varphi, x_+^{\lambda} \rangle = \int_0^\infty \varphi(x) x^{\lambda} \log(x) \, dx$$

$$\vdots$$

$$\frac{d^n}{d\lambda^n} \langle \varphi, x_+^{\lambda} \rangle = \int_0^\infty \varphi(x) x^{\lambda} \log(x)^n \, dx.$$

In order to make sense of \(x_+^{\lambda}\) for \(\lambda\) with real part \(-2 < \text{Re}(\lambda) < -1\), the defining equation (1.1) is rewritten using the identity

$$\int_{\mathbb{R}} \varphi(x) x^{\lambda} \, dx = \int_0^1 (\varphi(x) - \varphi(0)) x^{\lambda} \, dx + \int_1^\infty \varphi(x) x^{\lambda} \, dx + \frac{\varphi(0)}{\lambda + 1}.$$
the identities
\[ \int_\mathbb{R} \varphi(x) x^\lambda \, dx = \int_0^1 \left[ \varphi(x) - \sum_{j=1}^{n-1} \frac{\varphi^{(j)}(0)}{j!} x^j \right] x^\lambda \, dx + \int_1^\infty \varphi(x) x^\lambda \, dx + \sum_{j=1}^n \frac{\varphi^{(j-1)}(0)}{(j-1)!(\lambda + j)}. \] (1.2)

In effect, we subtracting the Taylor expansion of the test function \( \varphi \) near zero to obtain a convergent integral, then add an appropriate constant to undo the error caused by subtracting. If we adopt the perspective of \( x^\lambda \) as an element of the dual to \( C_\infty^c(\mathbb{R}) \) (ie. we forget the dependence on the specific function \( \varphi \)), we can instead view this operation as a truncation of the space \( C_\infty^c(\mathbb{R}) \). Gel'fand and Shilov demonstrate the utility of this technique in [11] by regularizing various divergent quantities, like the generalized functions
\[ (x \pm i \cdot 0)^\lambda \quad \text{and} \quad \log(x \pm i \cdot 0) \]
whose integrals are often important in applications.

Similar ideas make an appearance in [8] as Lax and Phillips explore spectral properties of the hyperbolic Laplacian on quotients \( \Gamma \backslash \mathbb{H} \) of the upper half plane by discrete subgroups \( \Gamma < \text{SL}(2; \mathbb{R}) \). Yves Colin de Verdière makes use of these ideas in [2] to regularize the Eisenstein series associated to \( \Gamma \backslash \mathbb{H} \). The simplest case is \( \Gamma = \text{SL}(2; \mathbb{Z}) \), where the Eisenstein series is given by
\[ E(z; s) := \sum_{(c,d)=1} \frac{y^s}{|cz + d|^{2s}}. \]

In regularizing \( E \), the Fourier expansion of an automorphic function around the cusp \( \infty \) plays the role of the Taylor expansion in the previous example. Instead of truncating functions in \( C_\infty^c(\mathbb{R}) \) near 0 using their Taylor expansion, we truncate functions in \( C_\infty^c(\Gamma \backslash \mathbb{H}) \) by removing the constant term of the Fourier expansion in a neighbourhood of the cusp \( \infty \). In a more general setting, an arbitrary cusp \( \mathfrak{a} \) takes the place of \( \infty \). It was relatively easy to deduce that the continuation of \( x^\lambda \) given by (1.2) had a discrete set of poles (negative integers) was meromorphic. For the Eisenstein series, the process is more involved and will depend both on spectral properties of the Laplacian as an operator on truncated \( C_\infty^c(\Gamma \backslash \mathbb{H}) \), and on a characterization of \( E \) as an eigenfunction of the Laplacian.

To this end, in chapter 2 we review the basics of automorphic functions and the domains \( \Gamma \backslash \mathbb{H} \) on which they are defined. In particular, we define Eisenstein series for a broad class of subgroups \( \Gamma < \text{SL}(2; \mathbb{R}) \). In chapter 3, we describe the truncated version of some important function spaces, including \( C_\infty^c(\Gamma \backslash \mathbb{H}) \). Friedrichs-Sobolev spaces, which are important for the later spectral theory are introduced here. In chapter 4, we first study properties of the Laplacian as it operates on truncated \( C_\infty^c(\Gamma \backslash \mathbb{H}) \), and on a characterization of \( E \) as an eigenfunction of the Laplacian.

The resolvent is shown to be a holomorphic operator in a neighbourhood of any point \( \lambda \) where it is defined. Finally, in chapter 5, we re-express the Eisenstein series in terms of the resolvent \( (\Delta - \lambda)^{-1} \). Since the truncated Laplacian has discrete spectrum, and the resolvent is locally holomorphic, we are able to conclude in 5.3.2 that the extended Eisenstein series is meromorphic on \( \mathbb{C} \). A formula expressing \( E \) in terms of known meromorphic functions is provided by theorem 5.3.1. The meromorphicity follows
from this formula. Coupled with results like 4.2.2 this formula provides information on the location of poles of $E^Y$. 
Chapter 2

Automorphy and Eisenstein Series

In this section we review the elementary theory of automorphic functions and distributions. In particular, we discuss a procedure for automorphic functions, and investigate their Fourier theory. Further details about harmonic analysis on $\mathbb{H}$ can be found in [7] and [5]. Fourier theory, especially of Eisenstein series, is well presented in [6]. The theory of the group $\text{SL}(2; \mathbb{R})$ and its subgroups, along with quotients of $\mathbb{H}$ is excellently explained [8]. Material on distributions was inspired by [11].

2.1 Hyperbolic Surfaces

A standard representation of the hyperbolic plane is the metric space $(\mathbb{H}, d\mu)$, where

$$\mathbb{H} = \{ z = x + iy \mid y > 0 \} \quad \text{and} \quad ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The group $G = \text{SL}(2; \mathbb{R})$ acts on the extended half-plane $\mathbb{H} = \mathbb{H} \cup \mathbb{R} \cup \{ \infty \}$ by Möbius transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

In particular, $\mathbb{H}$ is a homogeneous space for the Möbius $G$-action. Since $G$ factors as the product $N A K$, where

$$N = \{ [\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}] : x \in \mathbb{R} \}, \quad A = \left\{ \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} : y > 0 \right\}, \quad K = \text{SO}(2; \mathbb{R})$$

and since the stabilizer of $i \in \mathbb{H}$ is $K$, we have the identifications

$$\mathbb{H} \cong G/K \cong NA.$$

Observing that

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} i = x + iy^2$$

gives an explicit correspondence $\mathbb{H} \rightarrow NA$:

$$z = x + iy \mapsto \begin{bmatrix} 1 & n(x) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a(y) & 0 \\ 0 & a(y)^{-1} \end{bmatrix}$$ (2.1)
where \( n(z) = x \) and \( a(z) = y^{1/2} \).

Direct computation shows the metric tensor \( d\mu \) is \( G \)-invariant. It gives rise to the invariant volume element

\[
d\mu = \frac{dx dy}{y^2}
\]

used for computing hyperbolic plane integrals. It is often useful to compute such integrals on \( NA \) instead of \( \mathbb{H} \). From equation (2.1) it follows that

\[
dx = dn \quad \text{and} \quad dy = 2 ada
\]

and hence

\[
\int_{\mathbb{H}} f(x, y) \frac{dx dy}{y^2} = \int_{NA} f(x, y) \frac{dn 2 ada}{a^4} = 2 \int_{NA} f(na \cdot i) \frac{dnda}{a^3}.
\]

The Laplacian on \( \mathbb{H} \) is defined by

\[
\Delta := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).
\]

It is \( G \)-invariant in the sense that

\[
\Delta(f \circ \gamma)(z) = (\Delta f) \circ \gamma(z) \quad \forall f \in C^\infty(\mathbb{H}), \forall \gamma \in G
\]

We will need to investigate quotients \( X_\Gamma := \Gamma \backslash \mathbb{H} \) of \( \mathbb{H} \) by discrete subgroups \( \Gamma < G \). The action of such a subgroup on \( \mathbb{H} \) is always properly discontinuous, meaning the spaces \( X_\Gamma \) are again manifolds. If \( F \subset \mathbb{H} \) is a fundamental domain for \( \Gamma \), and if \( q: \mathbb{H} \to X_\Gamma \) is the quotient map, then the restricted map \( q|_F \) gives a coordinate system on \( X_\Gamma \) via

\[
p = q|_F(z) \mapsto z \quad \text{for all } z \in F.
\]

Discontinuity also means the orbits \( \Gamma z \), for \( z \in \mathbb{H} \), have no accumulation points in \( \mathbb{H} \); however, they may still accumulate on the boundary \( \partial \mathbb{H} = \hat{\mathbb{R}} \).

**Definition 2.1.1.** If \( \Gamma < G \) is discrete, and if each point in \( \hat{F} \) is an accumulation point of some orbit \( \Gamma z \), then \( \Gamma \) is said to be a **finite volume group**. If \( \Gamma \backslash \mathbb{H} \) is compact then we say also that \( \Gamma \) is **co-compact**. ■

### 2.2 Cusps

Suppose \( \Gamma < G \) is of finite volume. The fundamental domain \( F \) can be chosen to be a hyperbolic polygon such that \( \partial F \) consists of finitely many geodesic arcs, intersecting at common vertices. If \( \Gamma \) is not co-compact then the closure \( \bar{F} \) in \( \hat{\mathbb{H}} \) must have a vertex \( a \) on \( \hat{\mathbb{R}} \).

**Definition 2.2.1.** A vertex for \( \bar{F} \) on \( \hat{\mathbb{R}} \) is called a **cusp**. If \( U_a \) is an open set in \( \hat{\mathbb{C}} \) not containing any other cusp of \( \Gamma \), then the open set \( V_a := q(U_a \cap F) \subset X_\Gamma \) is called a **cuspidal region**. ■

Provided that \( \Gamma \) is of finite volume, we can always choose \( F \) so that the cusps are all \( \Gamma \)-inequivalent. If \( F \) is such a domain, with cusps \( a, b, \ldots \), the cuspidal regions \( V_a, V_b, \ldots \) are all non-compact. The remaining region \( X_\Gamma \setminus (V_a \cup V_b \cup \cdots) \) is compact and simply connected. As a result, we shall be mostly interested in the cuspidal regions.
In order to study such a function in a cusp neighbourhood $V_a$, we need to first introduce a convenient change of coordinates on $H$. The stabilizer $\Gamma_a$ of the cusp $a$ of $F$ is an infinite cyclic group

$$\Gamma_a = \langle \gamma_a \rangle$$

(2.2)
generated by a parabolic element $\gamma_a \in G$. There exists $\sigma_a \in G$ such that

$$\sigma_a(\infty) = a, \quad \text{and} \quad \sigma_a^{-1} \gamma_a \sigma_a = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right).$$

(2.3)
If we apply the automorphism $\sigma_a^{-1}$ to $H$, the action of $\Gamma$ on $H$ is taken that of $\sigma_a^{-1} \Gamma \sigma_a$ on $\sigma_a^{-1}(H)$. In particular, the cusp neighbourhood $U_a$ is mapped into a strip of the form

$$S(Y) = \{ x + iy : 0 < x < 1, y > Y \} \quad \text{for some } Y > 0.$$ 

The point $a$ itself corresponds to $\infty$ in $\sigma_a^{-1}(H)$.

**Definition 2.2.2.** We refer to the coordinates on $X_\Gamma$ defined by

$$p_a = q|_F \circ \sigma_a(\zeta) \mapsto \zeta \quad \text{for all } \zeta \in \sigma_a^{-1}(F)$$

as *cusp coordinates* for $a$. ■

Certain cuspidal regions, defined through cusp coordinates, are particularly useful.

**Definition 2.2.3.** For $Y > \sqrt{3}$, set $H^Y := \{ x + iy \in H \mid y > Y \}$. We define the cusp neighbourhoods $V^Y_a$ by

$$V^Y_a := p_a(H^Y) \subset X_\Gamma.$$ ■

### 2.3 Automorphic Functions

**Definition 2.3.1.** A function $f : \mathbb{H} \to \mathbb{C}$ is said to be *automorphic* for the finite volume group $\Gamma < G$ if it satisfies the following transformation rule:

$$f(\gamma z) = f(z) \quad \text{for all } \gamma \in \Gamma.$$ 

Clearly any automorphic function descends to a function on $X_\Gamma$. As such, the set of automorphic functions on $X_\Gamma$ is denoted $A(X_\Gamma)$. ■

**Definition 2.3.2.** In the special case that $f$ is smooth and $\text{supp}(f)$ is a compact, we say $f$ is an *automorphic test function*. The space of automorphic test functions is denoted $D(X_\Gamma)$. ■

In order to topologize $D(\Gamma \backslash \mathbb{H})$, we first describe it as the direct limit

$$D(\Gamma \backslash \mathbb{H}) = \lim_{\rightarrow} C^\infty(K_n).$$

where $K_1 \subset K_2 \subset \cdots$ an exhaustion of $\Gamma \backslash \mathbb{H}$ by compacts (i.e. $\Gamma \backslash \mathbb{H} = \bigcup_n K_n$, and each $K_n$ is compact). We then equip $D(\Gamma \backslash \mathbb{H})$ with the associated locally convex inductive limit topology.
**Definition 2.3.3.** Elements of the dual space $D'(\Gamma \backslash \mathbb{H})$ are called automorphic distributions. ■

We could equivalently characterize the automorphic distributions as those $\mu \in D'(\mathbb{H})$ satisfying

$$\langle \varphi, \mu \rangle = \langle \varphi \circ \gamma, \mu \rangle$$

for all automorphic test functions $\varphi$, and all $\gamma \in \Gamma$. Parameterized families $\{\mu_s\}$ of automorphic distributions, where the parameter $s$ ranges in some domain $\Omega \subseteq \mathbb{C}$, are of particular importance. For any such family, and any automorphic test function $\varphi$, we can define the $\varphi$-evaluation function $\Omega \to \mathbb{C}$ by

$$M_\varphi(s) := \langle \varphi, \mu_s \rangle.$$

**Definition 2.3.4.** If $M_\varphi$ is a holomorphic (resp. meromorphic) function for all $\varphi \in D(\Gamma \backslash \mathbb{H})$ then we say $\mu_s$ is a holomorphic (resp. meromorphic) family of distributions. ■

**Definition 2.3.5.** Suppose $\mu_s$, $\nu_s$ are a meromorphic families parameterized by $s \in \Omega_\mu$ and $s \in \Omega_\nu$ respectively. If $\Omega_\mu \subset \Omega_\nu$ we say that $\nu_s$ is a meromorphic continuation of $\mu_s$ to $\Omega_\nu$. ■

Suppose we are given an automorphic function $f$. In cusp coordinates $\zeta = \sigma_a^{-1}(z)$ on $\mathbb{H}$, $f$ is represented by $f_a$, where

$$f_a(\zeta) := f(\sigma_a(\zeta)) = f(z)$$

Our observations in section 2.2 show that $f_a$ is $\sigma_a^{-1}\Gamma_a\sigma_a$-invariant. In particular, $f_a$ must be $\sigma_a^{-1}\Gamma_a\sigma_a = B(\mathbb{Z})$-invariant, meaning

$$f_a(x + k + iy) = f_a((\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}) \cdot \zeta) = f_a(\zeta) = f_a(x + iy) \quad (2.4)$$

for all $k \in \mathbb{Z}$. As an immediate consequence of equation (2.4) we have

**Proposition 2.3.6.** Any automorphic function $f \in L^2(\mathcal{X}_\Gamma)$ admits a Fourier expansion at the cusp $a$ of the form

$$f_a(\zeta) = \sum_{k \in \mathbb{Z}} f_k^a(y)e^{2\pi ikx}$$

The coefficient functions $f_k^a(y)$, which are defined on $N \backslash \mathbb{H}$ (which we identify with $(0, \infty)$), are given by

$$f_k^a(y) = \int_0^1 f_a(x + iy)e^{-2\pi ikx} \, dx.$$  

The $0^{th}$ order Fourier coefficient plays an important and recurring role, so it gets some special notation:

**Definition 2.3.7.** The constant term projection $c_a: L^2(\mathcal{X}_\Gamma) \to C^\infty(N \backslash \mathbb{H})$ at $a$ is defined by

$$c_a[f](y) = \int_0^1 f_a(x + iy) \, dx.$$  

Making the convention $c_a[f](x + iy) := c_a[f](y)$, we recover an $N$-invariant function on $\mathcal{X}_\Gamma$. We can therefore extend $c_a$ to distributions $\mu$ by setting

$$\langle \varphi, c_a[\mu] \rangle := \langle c_a[\varphi], \mu \rangle \quad \forall \varphi \in D(\mathcal{X}_\Gamma).$$
It is worthwhile to record how the constant term map interacts with the Laplacian in this context.

**Proposition 2.3.8.** The constant term map commutes with the Laplacian in the sense that

\[ \Delta c_a[f] = c_a[\Delta f], \]

where \( f \) is smooth, and \( c_a[f] \) is interpreted as an \( N \)-invariant function.

**Proof.** Direct computation:

\[
\begin{align*}
\Delta c_a[f](x+iy) &= y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \int_0^1 f^a(t+iy) \, dt \\
&= y^2 \int_0^1 \frac{\partial^2 f^a}{\partial y^2}(t+iy) \, dt \\
&= y^2 \int_0^1 \frac{\partial^2 f^a}{\partial x^2}(t+iy) \, dt + y^2 \int_0^1 \frac{\partial^2}{\partial y^2} f^a(t+iy) \, dt \\
&= \int_0^1 \Delta f^a(t+iy) \, dt \\
&= c_a[\Delta f](x+iy)
\end{align*}
\]

\[\blacksquare\]

**Corollary 2.3.9.** The distributional versions of the constant term map and Laplacian also commute:

\[ \Delta c_a[\mu] = c_a[\Delta \mu] \]

for all distributions \( \mu \).

**Proof.** Immediate from proposition 2.3.8.

\[\blacksquare\]

### 2.4 Automorphization

The use of cusp coordinates \( \zeta \) around a cusp \( a \) gives rise to the process of automorphization at \( a \). Automorphization allows us to quickly construct functions on \( \mathbb{H} \) that descend to functions on \( X_\Gamma \).

**Definition 2.4.1.** Given a function \( g \) satisfying

\[ g(x+iy) = g(\zeta) = g \left( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \zeta \right) = g(x+n+iy) \]

for all \( n \in \mathbb{Z} \)

the *automorphization* of \( g \) at \( a \), denoted \( A_a[g] \), is defined in cusp coordinates by

\[
A_a[g](\zeta) = \sum_{\gamma \in B(\mathbb{Z}) \backslash \Gamma \sigma_a^{-1} \Gamma \sigma_a} g(\gamma \zeta). \tag{2.5}
\]

That we need to sum over cosets by \( B(\mathbb{Z}) \) follows from equations (2.2) and (2.3).

Two alternative representations of \( A_a[g] \) frequently prove useful:
1. In standard $z$-coordinates $A_a[g]$ takes the form

$$A_a[g](z) = \sum_{\gamma \in \Gamma_a \setminus \Gamma} g(\sigma_a^{-1} \gamma \sigma_a^{-1} z) = \sum_{\gamma \in \Gamma_a \setminus \Gamma} g(\sigma_a^{-1} \gamma z).$$

2. In cusp coordinates $\zeta$ for some cusp $b \neq a$

$$A_a^b[g](\zeta) = \sum_{\gamma \in B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b} f(\gamma \zeta). \quad (2.6)$$

The fact that the coset sum takes the form $B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b$ follows from the identity

$$\sigma_a^{-1} \Gamma a \gamma \sigma_b = \sigma_a^{-1} \Gamma a \sigma_a^{-1} \gamma \sigma_b = B(\mathbb{Z}) \sigma_a^{-1} \gamma \sigma_b$$

which holds for all $\gamma \in \Gamma$.

In order to compute explicit automorphizations, it is necessary to gain a more explicit description of the set $B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b$ of cosets. In order to do this, we shall decompose $\sigma_a^{-1} \Gamma \sigma_a$ into double cosets $B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b / B(\mathbb{Z})$. Computing sums of the form in equation (2.6) requires the case $a \neq b$, while equation (2.5) corresponds to the special case $a = b$. We proceed in two steps: first we investigate the left cosets $B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b$, then we build on this to describe the double coset situation.

**2.4.1 The cosets $B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b$**

Two matrices $(a_0\ b_0 \ c_0\ d_0)$ and $(a_1\ b_1 \ c_1\ d_1)$ are in the same coset of $B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b$ if and only if $c_0 = c_1$ and $d_0 = d_1$, or $c_0 = -c_1$ and $d_0 = -d_1$. Indeed, computing directly for arbitrary $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in B(\mathbb{Z})$,

$$\left( \begin{smallmatrix} 1 & k \\ 0 \end{smallmatrix} \right) \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \left( \begin{smallmatrix} a + ck & b + dk \\ c & d \end{smallmatrix} \right).$$

In other words, left multiples by $\gamma$ all have the same bottom row. On the other hand, suppose $\gamma_0 = \left( \begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix} \right)$ and $\gamma_1 = \left( \begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix} \right)$ are elements of $\sigma_a^{-1} \Gamma \sigma_b$ with the same bottom row. Since $\gamma_1^{-1} = \left( \begin{smallmatrix} d & -b_1 \\ -c & a_1 \end{smallmatrix} \right)$, we have

$$\gamma_0 \gamma_1^{-1} = \left( \begin{smallmatrix} 1 & -a_0b_1 - a_1b_0 \\ 0 & 1 \end{smallmatrix} \right) \in \Gamma \cdot \infty$$

which means that $\gamma_0 \equiv \gamma_1 \mod B(\mathbb{Z})$. Finally, we remark that $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ and $(\begin{smallmatrix} c & -a \\ d & -b \end{smallmatrix})$ are left $B(\mathbb{Z})$-equivalent since $(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}) \in B(\mathbb{Z})$. So we have shown the cosets $B(\mathbb{Z}) \setminus \sigma_a^{-1} \Gamma \sigma_b$ are parameterized by pairs $(c, d)$ of numbers such that $c \geq 0$ and $\sigma_a^{-1} \Gamma \sigma_b$ contains $(\begin{smallmatrix} c & -a \\ d & -b \end{smallmatrix})$.

It is useful to note that the set

$$\Omega_\infty := \{ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \sigma_a^{-1} \Gamma \sigma_b \}$$

is non-empty if and only if $a$ and $b$ are equivalent. Indeed, if $\sigma_a^{-1} \gamma \sigma_b \in \sigma_a^{-1} \Gamma \sigma_b$, then

$$\gamma \cdot b = \sigma_a(\sigma_a^{-1} \gamma \sigma_b) \cdot \infty = \sigma_a \cdot \infty = a.$$

Conversely, if $a$ is equivalent to $b$ choose $\gamma \in \Gamma$ such that $\gamma b = a$ and note that

$$\sigma_a^{-1} \gamma \sigma_b \cdot \infty = \sigma_a^{-1} \gamma \cdot b = \sigma_a^{-1} \cdot a = \infty.$$
In other words, \( \gamma \) stabilizes \( \infty \), whence it has the form \((\alpha^* \ y)\).

### 2.4.2 The double cosets \( B(\mathbb{Z})\backslash \sigma_a^{-1}\Gamma \sigma_b \backslash B(\mathbb{Z}) \)

For any \((c \ d) \in \sigma_a^{-1}\Gamma \sigma_b\) we compute directly that
\[
(c \ d) \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = (c \ d + cn)
\]
From this we deduce that specifying a double \( B(\mathbb{Z})\)-coset requires two choices of parameter. First we must choose \( c > 0 \) such that \((c^* \ 0) \in \sigma_a^{-1}\Gamma \sigma_b\). With \( c \) fixed, we must then choose \( 0 \leq d < c \) such that \((c^* \ d) \in \sigma_a^{-1}\Gamma \sigma_b\).

### 2.5 Eisenstein & pseudo-Eisenstein series

\( N \)-invariant functions are evidently \( B(\mathbb{Z}) \)-invariant, so the space \( C^\infty(\mathbb{N} \setminus \mathbb{H}) \cong C^\infty(0, \infty) \) ends up being a prime candidate for the automorphization process.

**Definition 2.5.1.** The space \( PEisen_a \) of pseudo-Eisenstein series attached to \( a \) is obtained by automorphising \( C^\infty(0, \infty) \) at \( a \):

\[
PEisen_a = \left\{ f \mid f(z) = \sum_{\gamma \in \Gamma_a \setminus \Gamma} \varphi(\text{Im}(\sigma_a^{-1} \gamma z)) \text{ for some } \varphi \in C^\infty(0, \infty) \right\}.
\]

The automorphization process can be used to construct smooth automorphic \( \Delta \)-eigenfunctions. It is easy to find a \( \Delta \)-eigenfunction on \( \mathbb{H} \); indeed, the function
\[
z \mapsto (\text{Im}(z))^s = y^s
\]
has eigenvalue \( \lambda = s(s - 1) \). It is clearly \( N \)-invariant, hence \( B(\mathbb{Z}) \)-invariant too, making it a viable candidate for automorphization.

**Definition 2.5.2.** The Eisenstein series attached to the cusp \( a \) is the function \( E_a(z, s) \) obtained by automorphising \( z \mapsto (\text{Im}(z))^s = y^s \):

\[
E_a(z, s) = \sum_{\gamma \in \Gamma_a \setminus \Gamma} \text{Im}(\sigma_a^{-1} \gamma z))^s. \tag{2.7}
\]

In cusp coordinates around \( b \), equation \( (2.7) \) looks like

\[
E_a^b(\zeta, s) = \sum_{\gamma \in B(\mathbb{Z}) \setminus \sigma_a^{-1}\Gamma \sigma_b} \text{Im}(\gamma \zeta))^s.
\]

**Lemma 2.5.3.** The Eisenstein series \( E_a(z, s) \) is convergent for \( \text{Re}(s) > 1 \).
Proof. See e.g. [6], theorem 2.1.1.

Since \( \text{Im}(z)^s = y^s \) is a \( \Delta \)-eigenfunction and \( \Delta \) is \( G \)-invariant, the Eisenstein series \( E_a(z, s) \) is an automorphic \( \Delta \)-eigenfunction with eigenvalue \( \lambda = s(s - 1) \). This means the Eisenstein series have Fourier expansions at each cusp. By the techniques in section 2.4, we can write

\[
E^b_a(\zeta, s) = \sum_{\gamma \in B(\mathbb{Z}) \setminus \Gamma} \text{Im}(\gamma(\zeta))^s
= \delta_{ab} (\text{Im}(\zeta))^s + \sum_{c > 0} \sum_{d \pmod{c} \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \left( \text{Im} \left( \frac{a}{c} - \frac{1}{c(c(\zeta + n) + d)} \right) \right)^s
\]

where \( \delta_{ab} = 1 \) if \( a = b \), otherwise \( \delta_{ab} = 0 \). To develop the Fourier expansion, we concentrate on the inner sum over \( \mathbb{Z} \). Applying Poisson summation and the change of variables \( t \mapsto t - x - \frac{2}{c} \) (where \( \zeta = x + iy \)) yields

\[
\sum_{n \in \mathbb{Z}} \left( \text{Im} \left( \frac{a}{c} - \frac{1}{c(c(\zeta + n) + d)} \right) \right)^s = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \left( \text{Im} \left( \frac{a}{c} - \frac{1}{c(c(\zeta + t) + d)} \right) \right)^s e^{-2\pi i n t} dt
= \sum_{n \in \mathbb{Z}} e^{2\pi i n(x + \frac{2}{c})} \int_{-\infty}^{\infty} \left( \text{Im} \left( \frac{a}{c} - \frac{1}{c^2(t + iy)} \right) \right)^s e^{-2\pi i n t} dt
\]

An explicit computation shows \( \text{Im} \left( \frac{a}{c} - \frac{1}{c^2(t + iy)} \right) = \frac{y}{c^2(t^2 + y^2)} \), whence

\[
\int_{-\infty}^{\infty} \left( \text{Im} \left( \frac{a}{c} - \frac{1}{c^2(t + iy)} \right) \right)^s e^{-2\pi i n t} dt = \frac{y^s}{c^{2s}} \int_{-\infty}^{\infty} \left( \frac{1}{t^2 + y^2} \right)^s e^{-2\pi i n t} dt
\]

This last integral can be evaluated explicitly:

\[
\int_{-\infty}^{\infty} \left( \frac{y}{t^2 + y^2} \right)^s e^{-2\pi i n t} dt = \begin{cases} \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} y^{1-2s} & \text{if } n = 0 \\ \frac{2\pi^n}{\Gamma(s)} |n|^{-\frac{1}{2}} y^{-s+\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) & \text{otherwise} \end{cases}
\]

Substituting back into equation (2.8), we get

\[
E^b_a(\zeta, s) = \delta_{ab} y^s + \sum_{c > 0} \frac{1}{c^2} \sum_{d \pmod{c} \in \mathbb{Z}} \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} y^{1-s}
+ \sum_{c > 0} \frac{1}{c^2} \sum_{d \pmod{c} \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{n \neq 0} e^{2\pi i n(x + \frac{2}{c})} \frac{2\pi^n}{\Gamma(s)} |n|^{-\frac{1}{2}} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y)
\]

where \( \delta_{ab} = 1 \) if \( a = b \), otherwise \( \delta_{ab} = 0 \).
Chapter 3

Truncation at a cusp

In this section we modify the space $L^2(X\Gamma)$ and the operator $\Delta$ to obtain a new space $\Lambda^Y_\alpha L^2(X\Gamma)$ equipped with a new densely defined operator $\Delta^Y$. Later, by extending the domain of $\Delta^Y$ slightly, we will obtain an operator $\Delta^Y_{\text{Fr}}$ with discrete spectrum. Because the Eisenstein series can be expressed in terms of $\Delta^Y_{\text{Fr}}$, the meromorphic continuation will follow from the discrete spectrum property. The main references for this section were [3] and [10]. A detailed study of truncated function spaces on $X\Gamma$ and associated truncated Laplacians is given, in context of the hyperbolic wave equation, in [8]. An excellent explanation of the truncation process, complete with pictures, can be found in [1].

3.1 Truncation of function spaces

Heuristically, we will construct the $\Lambda^Y_\alpha L^2(X\Gamma)$ spaces by subtracting off constant terms of functions near to the cusp $\alpha$. While the constant term projection might seem useful for such a purpose, it is easier to construct $\Lambda^Y_\alpha L^2(X\Gamma)$ as an orthogonal complement of pseudo-Eisenstein series supported entirely in the cuspidal regions.

Definition 3.1.1. For $Y > 0$ we define the spaces $\text{PEisen}^Y_\alpha \subseteq \text{PEisen}_a$ according to

$$\text{PEisen}^Y_\alpha := \{ A_\alpha[\varphi] : \varphi \in C^\infty_c(Y, \infty) \}$$

Definition 3.1.2. The truncated space $\Lambda^Y_\alpha L^2(X\Gamma)$ is the orthogonal complement of $\text{PEisen}^Y_\alpha$ in $L^2(X\Gamma)$:

$$\Lambda^Y_\alpha L^2(X\Gamma) = (\text{PseudoE}^Y_\alpha)^\perp.$$ 

To truncate subspaces of $L^2(X\Gamma)$ we simply take intersections; for instance,

$$\Lambda^Y_\alpha C^\infty_c(X\Gamma) = C^\infty_c(X\Gamma) \cap \Lambda^Y_\alpha L^2(X\Gamma)$$

Notice that the truncation $\Lambda^Y_\alpha C^\infty_c(X\Gamma)$ is a subspace of $\mathcal{D}(X\Gamma)$, the space automorphic test functions.
3.2 The truncated Laplacian and its extension

In this section we first define the truncated Laplacian. We then extend the domain of this new operator using the Friedrichs procedure. The resulting operator will be very useful later on due to its spectral properties.

**Definition 3.2.1.** The truncated Laplacian $\Delta^Y$ is defined to be the restriction

$$\Delta^Y := \Delta_{|_{\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma)}}.$$

$\Delta^Y$ is a densely defined defined operator $\Lambda_a^Y L^2(\mathcal{X}_\Gamma) \to \Lambda_a^Y L^2(\mathcal{X}_\Gamma)$ in light of the following result:

**Proposition 3.2.2.** $\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma)$ is dense in $\Lambda_a^Y L^2(\mathcal{X}_\Gamma)$.

**Proof.** See, e.g. [3] page 2. \hfill \Box

Now that the truncation has been defined, we need some minor machinery for the extension process. Define the map $\langle -,- \rangle_{Fr}$ taking $\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma) \times \Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma) \to \mathbb{C}$ according to

$$\langle f,g \rangle_{Fr} := \langle f,g \rangle_2 - \langle \Delta^Y f,g \rangle_2 = \langle (1-\Delta^Y)f,g \rangle_2.$$

The pairing $\langle -,- \rangle_{Fr}$ clearly respects scalar multiplication and addition of functions. Since $-\Delta^Y$ is a positive operator, we see that

$$\langle f,f \rangle_{Fr} = \langle f,f \rangle_2 - \langle \Delta^Y f,f \rangle_2 \geq 0.$$

Symmetry of $\Delta^Y$ yields

$$\langle f,g \rangle_{Fr} = \langle f,g \rangle_2 - \langle \Delta^Y f,g \rangle_2 = \langle g,f \rangle_2 - \langle \Delta^Y g,f \rangle_2 = \langle g,f \rangle_{Fr}.$$

It follows that $\langle -,- \rangle_{Fr}$ is an inner product on $\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma)$.

**Definition 3.2.3.** The inner product $\langle -,- \rangle_{Fr}$ specified above is called the Friedrichs-Sobolev inner product. It determines a norm and hence a metric on $\Lambda_a^Y L^2(\mathcal{X}_\Gamma)$ which we denote by $\| - \|_{Fr}$ and $d_{Fr}$ respectively. \hfill \Box

**Definition 3.2.4.** The completion of $\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma)$ in the $d_{Fr}$-metric is denoted by $\Lambda_a^Y H^1(\mathcal{X}_\Gamma)$. Spaces like $\Lambda_a^Y H^1(\mathcal{X}_\Gamma)$ are called Friedrichs-Sobolev spaces. \hfill \Box

There is a natural identification of $\Lambda_a^Y H^1(\mathcal{X}_\Gamma)$ with a subset of $\Lambda_a^Y L^2(\mathcal{X}_\Gamma)$. Indeed, strictly speaking, elements of $\Lambda_a^Y H^1(\mathcal{X}_\Gamma)$ are equivalence classes of $d_{Fr}$-Cauchy sequences in $\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma)$. If $\{f_i\}$ is any such $d_{Fr}$-Cauchy sequence, then the inequality

$$\|f_i - f_j\|_2 \leq \|f_i - f_j\|_{Fr}$$

implies $\{f_i\}$ is Cauchy in the $L^2$-metric as well. Thus there exists $f \in \Lambda_a^Y L^2(\mathcal{X}_\Gamma)$ to which $\{f_i\}$ converges in the usual topology. We make the association

$$\{f_i\} \to f.$$ (3.1)
Limits in $\Lambda^Y_a L^2(\mathcal{X}_T)$ are unique, and Cauchy sequences that are $\Lambda^Y_a H^1(\mathcal{X}_T)$-equivalent converge to the same $\Lambda^Y_a L^2(\mathcal{X}_T)$ limit since $\|f_i - g_i\|_2 \leq \|f_i - g_i\|_{F^r}$. Therefore the identification in equation (3.1) is well-defined.

For each $g \in \Lambda^Y_a L^2(\mathcal{X}_T)$ define the linear functional $\lambda_g: \Lambda^Y_a H^1(\mathcal{X}_T) \rightarrow \mathbb{C}$ by

$$\lambda_g(f) := \langle g, f \rangle_2.$$ 

Note that we use the standard $L^2$ inner product above. An application of Cauchy-Schwarz shows that

$$\lambda_g(f) \leq \|f\|_2 \|g\|_2 \leq \|f\|_{F^r} \|g\|_2,$$

from which we deduce $\lambda_g$ is continuous on $H^1(\mathcal{X}_T)$ with operator norm $\|\lambda_g\|_{F^r}$ bounded above by $\|g\|_2$.

In light of the above, the Riesz representation theorem furnishes a linear operator $B: \Lambda^Y_a L^2(\mathcal{X}_T) \rightarrow \Lambda^Y_a H^1(\mathcal{X}_T)$ satisfying

$$\lambda_g(f) = \langleBg, f\rangle_{F^r} \quad \forall f \in \Lambda^Y_a H^1(\mathcal{X}_T) \quad (3.2)$$

and

$$\|Bg\|_{F^r} = \|\lambda_g\|_{F^r} \leq \|g\|_2 \quad \forall g \in \Lambda^Y_a L^2(\mathcal{X}_T). \quad (3.3)$$

Equation (3.2) by definition says

$$\langle g, f \rangle_2 = \langle(1 - \Delta^Y)Bg, f\rangle_2.$$ 

The idea is therefore to invert the operator $B$ in order to obtain an extension of $(1 - \Delta^Y)$, then subtract off the 1 and multiply by $-1$ to obtain an extension of $\Delta^Y$.

First let us collect some useful properties of $B$.

**Proposition 3.2.5.** The operator $B$ defined above satisfies the following properties: i. $B$ is a bounded operator; ii. $B$ is positive; iii. $B$ is injective; iv. $B$ has dense image; v. $B$ is symmetric.

**Proof.** i. This is an immediate consequence of equation (3.3).

ii. Follows from the identity

$$0 \leq \langle Bg, Bg\rangle_2 \leq \langle Bg, Bg\rangle_{F^r} = \lambda_g(Bg) = \langle g, Bg\rangle_2.$$

iii. Suppose $Bf = 0$ for some $f \in \Lambda^Y_a L^2(\mathcal{X}_T)$. Then for all $g \in \Lambda^Y_a H^1(\mathcal{X}_T)$ we have

$$0 = \langle g, 0 \rangle_{F^r} = \langle g, Bf\rangle_{F^r} = \langle Bf, g\rangle_{F^r} = \lambda_f(g) = \langle f, g\rangle_2$$

Density of $\Lambda^Y_a H^1(\mathcal{X}_T)$ in $\Lambda^Y_a L^2(\mathcal{X}_T)$ implies $f \equiv 0$.

iv. Suppose $f \in \Lambda^Y_a H^1(\mathcal{X}_T)$ is such that $\langle Bg, f\rangle_{F^r} = 0$ for all $g \in \Lambda^Y_a L^2(\mathcal{X}_T)$. Then in particular

$$0 = \langle Bf, f\rangle_{F^r} = \lambda_f(f) = \langle f, f\rangle_2.$$

Since $\langle - , - \rangle_2$ is an inner product, this means $f \equiv 0$ and hence $B(\Lambda^Y_a L^2(\mathcal{X}_T))$ is dense in $\Lambda^Y_a H^1(\mathcal{X}_T)$. 


v. Direct computation:
\[
\langle g, Bf \rangle_2 = \lambda_g(Bf) = \langle Bg, Bf \rangle_{Fr} \\
= \langle Bf, Bg \rangle_{Fr} = \lambda_f(Bg) = \langle f, Bg \rangle_2 \\
= \langle Bg, f \rangle_2.
\]

We can exploit proposition 3.2.5 to get our inverse to \(B\), and hence the candidate extension of \((1 - \Delta^Y)\).

**Corollary 3.2.6.** The operator \(B: \Lambda_a^Y L^2(\mathcal{X}_\Gamma) \to \Lambda_a^Y H^1(\mathcal{X}_\Gamma)\) defined above admits a positive, potentially unbounded inverse \(B^{-1}: \text{dom}(B^{-1}) \to \Lambda_a^Y L^2(\mathcal{X}_\Gamma)\), where \(\text{dom}(B^{-1}) = \text{range}(B)\) is a dense subset of \(\Lambda_a^Y H^1(\mathcal{X}_\Gamma)\).

**Corollary 3.2.7.** The operator \(B: \Lambda_a^Y L^2(\mathcal{X}_\Gamma) \to \Lambda_a^Y H^1(\mathcal{X}_\Gamma)\) defined above is self-adjoint.

In a complex Hilbert space properties (i) - (iv) of proposition 3.2.5 are enough to ensure corollary 3.2.7. The explicit symmetry of proposition 3.2.5 (v), while illustrative, is only necessary if we restrict our attention to \(\mathbb{R}\)-valued functions.

Let us now check that this construction actually worked.

**Proposition 3.2.8.** The operator \(B^{-1}\) constructed above is an extension of \((1 - \Delta^Y)\) in the sense that
\[
\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma) = \text{dom}(1 - \Delta^Y) \subseteq \text{dom}(B^{-1}) \subset \Lambda_a^Y H^1(\mathcal{X}_\Gamma).
\]

**Proof.** First we prove \(\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma)\) is contained in \(\text{dom}(B^{-1})\), then we show the inclusion is strict. Fix any \(f, g \in \Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma)\). By definition we have
\[
\langle g, f \rangle_{Fr} = \langle (1 - \Delta^Y)g, f \rangle_2 = \lambda_{(1 - \Delta^Y)}g(f) = \langle B(1 - \Delta^Y)g, f \rangle_{Fr}.
\]

Subtracting one side from the other yields
\[
\langle g - B(1 - \Delta^Y)g, f \rangle_{Fr} = 0 \quad \forall f, g \in \Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma).
\]

It follows that \(g - B(1 - \Delta^Y)g = 0\), and hence \(g = B(1 - \Delta^Y)g\). In particular, \(g \in \text{range}(B) = \text{dom}(B^{-1})\), establishing the inclusion
\[
\Lambda_a^Y C_c^\infty(\mathcal{X}_\Gamma) = \text{dom}(1 - \Delta^Y) \subseteq \text{dom}(B^{-1}).
\]

To see that the inclusion is strict, we remark that \(B\) injects \(\Lambda_a^Y L^2(\mathcal{X}_\Gamma)\) into \(\Lambda_a^Y H^1(\mathcal{X}_\Gamma)\). Therefore \(B^{-1}\) must surject from its domain onto \(\Lambda_a^Y L^2(\mathcal{X}_\Gamma)\). This is a strictly larger set than the range of \((1 - \Delta^Y)\), so the domain of \(B^{-1}\) must be strictly larger than that of \((1 - \Delta^Y)\). \(\square\)

As we saw above, proposition 3.2.8 gives the desired extension of \(\Delta^Y\) since
\[
B^{-1} \text{ extends } (1 - \Delta^Y) \implies -(B^{-1} - 1) \text{ extends } \Delta^Y.
\]
**Definition 3.2.9.** Denote the Friedrichs extension of the truncated Laplacian by

\[
\Delta_{Fr}^{Y} := -(B^{-1} - 1)
\]

\[\Delta_{Fr}^{Y}\] is negative, since it is negative on a dense subset of its domain. It will also be useful to know that \(\Delta_{Fr}^{Y}\) is self-adjoint. In order to prove this it is sufficient to show \(B^{-1}\) is self-adjoint, which is more convenient.

**Proposition 3.2.10.** The operator \(B^{-1}\) defined above is self-adjoint.

**Proof.** i. Define \(U: \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma) \times \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma) \to \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma) \times \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma)\) by \(U(f,g) := (-g, f)\). Then

\[
\text{graph}(B^{-1*}) = U(\text{graph}(B^{-1}))^\perp.
\]

Indeed, suppose \((h, g) \in U(\text{graph}(B^{-1}))^\perp\). Then for all \((f, B^{-1}f) \in \text{graph}(B^{-1})\),

\[
0 = \langle (h, g), (-B^{-1}f, f) \rangle_{\text{graph}} = \langle h, -B^{-1}f \rangle_2 + \langle g, f \rangle_2
\]

\[= (-B^{-1*}f, h)_2 + (g, f)_2;\]

in other words, \(g = B^{-1*}h\). So we have shown

\[(h, g) \in U(\text{graph}(B^{-1}))^\perp \implies (h, g) = (h, B^{-1*}h) \in \text{graph}(B^{-1*}).\]

For the reverse inclusion, suppose \((-B^{-1}f, f) \in U(\text{graph}(B^{-1}))\) and \((g, B^{-1*}g) \in \text{graph}(B^{-1*})\) are arbitrary. Directly,

\[
\langle (-B^{-1}f, f), (g, B^{-1*}g) \rangle_{\text{graph}} = -\langle B^{-1}f, g \rangle_2 + \langle f, B^{-1*}g \rangle_2
\]

\[= -\langle B^{-1}f, g \rangle_2 + \langle B^{-1}f, g \rangle_2
\]

\[= 0.\]

ii. Use (i) to prove \(B^{-1}\) is self-adjoint. First we must define another operator:

\[
S: \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma) \times \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma) \to \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma) \times \Lambda_{a}^Y L^2(\mathcal{X}_\Gamma)
\]

\[S: (f, g) \mapsto (g, h).\]

The operator \(S\) relates the graphs of \(B\) and \(B^{-1}\) by \(\text{graph}(B^{-1}) = S(\text{graph}(B))\), and when composed with \(U\) satisfies \(U \circ S = -S \circ U\). Moreover, we get \((S(W))^\perp = S(W^\perp)\) for any subset \(W\). Thus a computation yields

\[
\text{graph}(B^{-1*}) = U(\text{graph}(B^{-1}))^\perp = U(S(\text{graph}(B)))^\perp
\]

\[= -S(U(\text{graph}(B)))^\perp = -S(U(\text{graph}(B)))^\perp
\]

\[= -S(\text{graph}(B^*)) = -S(\text{graph}(B))
\]

\[= \text{graph}(B^{-1}) = \text{graph}(B^{-1}).\]
3.3 Extending the usual Laplacian

In section 3.2 we constructed an operator $B^{-1}$ which extended $(1 - \Delta Y)$; we then defined the extension $\Delta_{Fr}^Y$ in terms of $B^{-1}$. The same procedure can be used to define an extension $\Delta_{Fr}$ of the usual Laplacian. We outline the process below.

1. Define $H^1(\mathcal{X}_\Gamma)$ to be the completion of $C^\infty_c(\mathcal{X}_\Gamma)$ with respect to norm derived from
   \[ \langle f, g \rangle_{H^1} := \langle (1 - \Delta)f, g \rangle_2. \]

2. Use the linear functional $\lambda_g: H^1(\mathcal{X}_\Gamma) \to \mathbb{C}$ given by
   \[ \lambda_g(f) := \langle g, f \rangle_2, \]

along with Riesz representation to obtain $B: L^2(\mathcal{X}_\Gamma) \to H^1(\mathcal{X}_\Gamma)$ such that

   \[ \lambda_g(f) = \langle Bg, f \rangle_{H^1}. \]

3. Prove that $B$ is bounded, positive, injective, symmetric, and has dense image. The proofs of these five properties proceed exactly in section 3.2.

4. Use the previous point to deduce that $B$ admits an inverse $B^{-1}: \text{dom}(B^{-1}) \to L^2(\mathcal{X}_\Gamma)$, where $\text{dom}(B^{-1})$ is a dense subset of $H^1(\mathcal{X}_\Gamma)$.

5. Define the Friedrichs extension of the usual Laplacian by $\Delta_{Fr} := -(B^{-1} - 1)$.

As before, $\Delta_{Fr}$ is negative and self-adjoint.

3.4 Localized Sobolev spaces

Fix any compact $K \subset (0, \infty) \sim N \setminus \mathbb{H}$ and define the seminorm $\rho_K$ on functions $f \in C^\infty(\mathbb{H})$ by

\[ \rho_K(f) := \int_{K \times [0,1]} (1 - \Delta)f \cdot \frac{dxdy}{y^2}. \]

A function $f \in C^\infty(N \setminus \mathbb{H})$ can be considered as an $N$-invariant element of $C^\infty(\mathbb{H})$, and by restricting to such functions $\rho_K$ can instead be considered as a seminorm $C^\infty(N \setminus \mathbb{H}) \to \mathbb{R}$.

**Definition 3.4.1.** The local Sobolev space $H^1_{loc}(N \setminus \mathbb{H})$ is the completion of the space $C^\infty(N \setminus \mathbb{H})$ with respect to the initial topology induced by the family

\[ \{ \rho_K \mid K \subset N \setminus \mathbb{H} \text{ is compact} \} \]
of seminorms. We can also understand $H^1_{\text{loc}}(N \setminus \mathbb{H})$ if we note that
\[ f \in H^1_{\text{loc}}(N \setminus \mathbb{H}) \iff f \in H^1(K) \quad \forall K \Subset (0, \infty) \cong N \setminus \mathbb{H}. \]
Here $H^1(K)$ is the usual Sobolev space on the compact subset $K$ of the real line.

These localized Sobolev spaces are important here mostly because they appear as the codomain of the constant term map:

**Proposition 3.4.2.** The constant term map $c_a(f)$ is a continuous function mapping
\[ c_a(f): \Lambda^Y_a H^1(\mathcal{X}_\Gamma) \longrightarrow H^1_{\text{loc}}(N \setminus \mathbb{H}). \]

**Proof.** Indeed, suppose $\{f_n\}$ is a sequence in $\Lambda^Y_a H^1(\mathcal{X}_\Gamma)$ converging to $f$ and set $U_n := f_n - f$. Then $U_n \to 0$ in $\Lambda^Y_a L^2(\mathcal{X}_\Gamma)$, and since $U_n$ has a Fourier expansion
\[ U_n^a(\zeta) = \sum_{k \in \mathbb{Z}} u_{n,k}^a(y)e^{2\pi inx} \]
we see that $u_n^a(y) \to 0$ for each $n$ and a.e. $y$. In particular,
\[ c_a(f_n - f)(y) = u_{n,0}^a(y) \longrightarrow 0 \quad \text{for a.e. } y. \]
It follows that $\|c_a(f_n - f)\|_{H^1(K)} \to 0$ for all $K \Subset N \setminus \mathbb{H}$, so $c_a$ is continuous.

It is well known that
\[ H^1_{\text{loc}}(N \setminus \mathbb{H}) \cong H^1_{\text{loc}}(0, \infty) \subseteq C^0(0, \infty), \]
so we have the useful corollary:

**Corollary 3.4.3.** If $f \in \Lambda^Y_a H^1(\mathcal{X}_\Gamma)$ then $c_a(f): N \setminus \mathbb{H} \to \mathbb{C}$ is continuous.

It is also useful to note a criterion for when an element of $\text{PEisen}^Y_a$ is in $\Lambda^Y_a H^1(\mathcal{X}_\Gamma)$.

**Proposition 3.4.4.** Let $\varphi \in C^\infty(Y, \infty)$ be smooth. Then the pseudo-Eisenstein series
\[ A_a[\varphi](z) = \sum_{\Gamma_a \setminus \Gamma} \varphi \left( \text{Im}(\sigma_a^{-1} \gamma z) \right) \]
is contained in $H^1(\mathcal{X}_\Gamma)$ if
\[ \int_0^\infty |\varphi|^2 + \left| \frac{\partial \varphi}{\partial y} \right|^2 \frac{dy}{y^2} < \infty. \]

**Proof.** This result follows from the Lie theory of $\text{SL}(2; \mathbb{R})$. See e.g. [3].
Chapter 4

Spectrum of $\Delta_{F_1}^Y$

The main goal of this chapter is to prove the spectrum $\text{sp}(\Delta_{F_1}^Y)$ is a discrete subset of $\mathbb{C}$. We will also see that the resolvent $(\lambda - \Delta_{F_1}^Y)^{-1}$, which only exists for $\lambda \notin \text{sp}(\Delta_{F_1}^Y)$, is a compact operator for all valid $\lambda$. The main references for this section were [3] and [4].

4.1 The domain of $\Delta_{F_1}^Y$

Of central importance in describing the domain of $\Delta_{F_1}^Y$ is the linear functional $T^Y : \Lambda^Y a H^1(X_1) \to \mathbb{R}$ defined by

$$T^Y[f] = c_a[f](Y).$$

Recall that in section 3.4 we showed that $c_a[\Lambda^Y a H^1(X_1)] \subset C^0(0, \infty)$, so the evaluation $c^Y_a[f](Y)$ in the definition of $T^Y$ is well-defined. Moreover, $T^Y$ is a continuous functional. Indeed, we saw also in section 3.4 that $c_a : \Lambda^Y a H^1(X_1) \to H^1_{\text{loc}}(X_1)$ is continuous, and since the evaluation-at-$Y$ map $C^0(0, \infty) \to \mathbb{R}$ is continuous, so is the composition

$$T^Y : \Lambda^Y a H^1(X_1) \longrightarrow H^1_{\text{loc}}(X_1) \longrightarrow \mathbb{R}.$$

Proposition 4.1.1. The domain of $\Delta_{F_1}^Y$ satisfies

$$\text{dom}(\Delta_{F_1}^Y) \subseteq \{ f \in \Lambda^Y a L^2(X_1) \mid \Delta f \in \Lambda^Y a L^2(X_1) + \mathbb{C} \cdot T^Y \}$$

where all derivatives are taken in the distributional sense.

Proof. i. First let us suppose $f \in \text{dom}(\Delta_{F_1}^Y)$ and find $F \in \Lambda^Y a L^2(X_1)$ such that

$$\Delta f = F + \alpha \cdot T^Y, \quad \alpha \in \mathbb{C}. \quad (4.1)$$

This will prove $\text{dom}(\Delta_{F_1}^Y)$ is contained in the set described in the proposition. We may assume

$$f = (1 - \Delta_{F_1}^Y)^{-1} f_0 \quad \text{for some } f_0 \in \Lambda^Y a H^1(X_1)$$
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Since $\text{dom}(\Delta_{Fr}^Y) = \text{dom}(1 - \Delta_{Fr}^Y)$. Appealing to proposition [3.2.10] we see that

$$\langle (1 - \Delta^Y)g, (1 - \Delta_{Fr}^Y)^{-1}f_0 \rangle_2 = \langle g, f_0 \rangle_2 \quad \forall g \in \Lambda_a^Y C^\infty_c(X_\Gamma), \forall f_0 \in \Lambda_a^Y L^2(X_\Gamma).$$

Treating $(1 - \Delta_{Fr}^Y)^{-1}f_0$ as an element of $\mathcal{D}'(X_\Gamma)$ we compute the distributional derivative to obtain

$$\langle g, f_0 \rangle_2 = \langle (1 - \Delta^Y)g, (1 - \Delta_{Fr}^Y)^{-1}f_0 \rangle_2 = \langle g, (1 - \Delta^Y)(1 - \Delta_{Fr}^Y)^{-1}f_0 \rangle_2 = \langle g, (1 - \Delta^Y)f \rangle_2$$

at least for test functions $g \in \Lambda_a^Y C^\infty_c(X_\Gamma)$. Equation (4.2) shows that the distribution

$$\nu_f := (f - f_0) - \Delta f$$

annihilates $\Lambda_a^Y C^\infty_c(X_\Gamma)$. It follows that $\text{supp}(\nu_f)$ is contained in the closed cuspidal neighbourhood $\overline{\Gamma_a^Y}$.

Elements of $\Lambda_a^Y C^\infty_c(X_\Gamma)$ are smooth functions with compact support on $X_\Gamma$ whose $0^{th}$ Fourier coefficient vanishes above the line $y = Y$. However, the $n^{th}$ coefficient for $n \neq 0$ need not vanish anywhere. The fact that $\nu_f$ annihilates $\Lambda_a^Y C^\infty_c(X_\Gamma)$ therefore implies that $\nu_f$ is a distribution whose $n^{th}$ Fourier coefficients all vanish on $[Y, \infty)$ for $n \neq 0$.

ii. If we suppose instead that $g \in \text{PEisen}_a^Y$ then certainly $\langle g, f_0 \rangle_2 = 0$; indeed $f_0 \in \Lambda_a^Y L^2(X_\Gamma)$, which is perpendicular to $\text{PEisen}_a^Y$. The identity

$$0 = \langle g, f_0 \rangle_2 = \langle g, (1 - \Delta^Y)f \rangle_2,$$

derived similarly to equation (4.2), shows $\nu_f$ also annihilates $\text{PEisen}_a^Y$. We conclude $\text{supp}(\nu_f) = \{Y\}$, and that all Fourier coefficients of $\nu_f$ at $Y$ vanish except for the constant term. Restrictions on the order of distributions like $C_a[\nu_f]$ in the dual space to $H_{loc}^\infty(N \setminus X_\Gamma)$ now imply that $C_a[\nu_f]$ is some constant multiple of the delta distribution $\delta_Y$. In other words, as distributions $\nu_f = \alpha \cdot T^Y$ for some $\alpha \in \mathbb{C}$ since

$$\langle \varphi, \nu_f \rangle = \langle \varphi, C_a[\nu_f] \rangle = \langle C_a[\varphi], \alpha \cdot \delta_Y \rangle = \langle \varphi, \alpha \cdot T^Y \rangle$$

(4.3)

Rearranging the definition of $\nu_f$ and applying (4.3) yields

$$\Delta f = (f - f_0) - \alpha \cdot T^Y.$$

Taking $F = f - f_0$ in equation (4.4), we are done.

Proposition [4.1.1] tells us that, for each $f \in \text{dom}(\Delta_{Fr}^Y)$, the distribution $\Delta f$ can be uniquely decomposed as

$$\Delta f = F + \alpha \cdot T^Y \quad \text{for } F \in \Lambda_a^Y L^2(X_\Gamma) \text{ and } \alpha \in \mathbb{C}.$$  

(4.4)

This is useful for computing $\Delta_{Fr}^Y f$ since it turns out that $\Delta_{Fr}^Y f = F$. In other words, $\Delta_{Fr}^Y f$ is the $L^2$-part of the decomposition 4.4.

\footnote{Note that equations (3.4) and (4.2) are different since the orders of application of the operators $(1 - \Delta)$ and $(1 - \Delta_{Fr}^Y)^{-1}$ vary. The computation in equation (4.2) necessarily occurs in the context of $\mathcal{D}'(X_\Gamma)$.}
**Proposition 4.1.2.** If \( f \in \text{dom}(\Delta_{Fr}^Y) \) then \( \Delta_{Fr}^Y f \in \Lambda_a^Y L^2(\mathcal{X}_r) \), and \( \Delta f \) decomposes uniquely as:

\[
\Delta f = \Delta_{Fr}^Y f + \alpha \cdot T^Y
\]

for some \( \alpha \in \mathbb{C} \).

**Proof.** Fix any \( f \in \text{dom}(\Delta_{Fr}^Y) \), and choose \( f_0 \) such that \( f = (1 - \Delta_{Fr}^Y)^{-1} f_0 \) In the proof of proposition 4.1.1 we saw that \( \Delta f = f - f_0 - \alpha \cdot T^Y \); substitute \( f_0 = (1 - \Delta_{Fr}^Y) f \) to obtain

\[
\Delta f = f - (1 - \Delta_{Fr}^Y) f - \alpha \cdot T^Y = \Delta_{Fr}^Y f - \alpha \cdot T^Y.
\]

Solving for \( \Delta_{Fr}^Y f \) establishes the desired identity. To see that \( \Delta_{Fr}^Y f \in \Lambda_a^Y L^2(\mathcal{X}_r) \), combine the two equations above to deduce

\[
\Delta_{Fr}^Y f = f - f_0.
\]

Since \( f, f_0 \in \Lambda_a^Y L^2(\mathcal{X}_r) \), so is their difference.

\[
\square
\]

### 4.2 The resolvent operator \((\Delta_{Fr}^Y - \lambda)^{-1}\)

**Definition 4.2.1.** The resolvent operator \( R_\lambda : \Lambda_a^Y L^2(\mathcal{X}_r) \to \Lambda_a^Y L^2(\mathcal{X}_r) \) is defined by

\[
R_\lambda := (\Delta_{Fr}^Y - \lambda)^{-1}
\]

for those values \( \lambda \in \mathbb{C} \) such that the given inverse exists.

**Proposition 4.2.2.** For \( \lambda \notin (-\infty,0] \), the resolvent \( R_\lambda \) is everywhere defined on \( \Lambda_a^Y L^2(\mathcal{X}_r) \).

**Proof.** Fix \( \lambda = \xi + i\eta \in \mathbb{C} \setminus \mathbb{R} \). First we show \( R_\lambda \) is everywhere defined for such \( \lambda \). In (ii) we extend to the case \( \lambda \in (0,\infty) \).

For any \( f \in \text{dom}(\Delta_{Fr}^Y) \), a direct computation yields

\[
\|(\Delta_{Fr}^Y - \lambda)f\|_2^2 = \langle (\Delta_{Fr}^Y - \xi)f, (\Delta_{Fr}^Y - \xi)f \rangle - \langle i\eta f, (\Delta_{Fr}^Y - \xi)f \rangle - \langle (\Delta_{Fr}^Y - \xi)f, i\eta f \rangle + \xi^2 \| f \|^2.
\]

The middle two terms cancel since \( \Delta_{Fr}^Y \) self-adjoint and \( \xi \in \mathbb{R} \) imply

\[-\langle (\Delta_{Fr}^Y - \xi)f, i\eta f \rangle = -i\eta \langle f, (\Delta_{Fr}^Y - \xi)f \rangle = i\eta \langle f, (\Delta_{Fr}^Y - \xi)f \rangle.
\]

We are left with

\[
\|(\Delta_{Fr}^Y - \lambda)f\|_2^2 = \|(\Delta_{Fr}^Y - \xi)f\|_2^2 + \xi^2 \| f \|^2 \geq \xi^2 \| f \|^2
\]

and hence

\[
\|(\Delta_{Fr}^Y - \lambda)f\|_2 \geq |\xi| \| f \|_2.
\]
So if \( \xi \neq 0 \) and \( f \neq 0 \) then \( \Delta_Y^\ast f = \lambda f \neq 0 \). In other words, \( \Delta_Y^\ast \) is injective, hence potentially invertible, whenever \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

In light of the previous paragraph, to show \( R_\lambda \) is everywhere defined for \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), it is enough to demonstrate the set equality

\[
(\Delta_Y^\ast - \lambda) \text{dom}(\Delta_Y^\ast) = \Lambda^\ast L^2(\mathcal{X}_\Gamma).
\]  

Clearly \( (\Delta_Y^\ast - \lambda) \text{dom}(\Delta_Y^\ast) \subseteq \Lambda^\ast L^2(\mathcal{X}_\Gamma) \). Suppose the inclusion were strict. Then there must exist \( f \in \Lambda^\ast L^2(\mathcal{X}_\Gamma) \) such that \( f \perp (\Delta_Y^\ast - \lambda) \text{dom}(\Delta_Y^\ast) \); ie

\[
\langle (\Delta_Y^\ast - \lambda)g, f \rangle = 0 \quad \forall g \in \text{dom}(\Delta_Y^\ast).
\]

This implies \( f \in \text{dom}((\Delta_Y^\ast - \lambda)^*) = \text{dom}(\Delta_Y^\ast - \bar{\lambda}) \), and

\[
\langle g, (\Delta_Y^\ast - \bar{\lambda})f \rangle = 0 \quad \forall g \in \text{dom}(\Delta_Y^\ast), \quad \text{whence} \quad \Delta_Y^\ast f = \bar{\lambda} f.
\]

In other words, \( \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R} \) is an \( \Delta_Y^\ast \)-eigenvalue. But \( \Delta_Y^\ast \) is negative, so its eigenvalues are non-positive reals. This is a contradiction, which proves equation (4.6).

ii. Now we extend to the case \( \lambda \in (0, \infty) \). Recall from section 3.2 that \( \Delta_Y^\ast \) is a negative self-adjoint operator, whence \( -\Delta_Y^\ast \) is positive and self-adjoint. If \( \lambda \in (-\infty, 0) \) then \( -\Delta_Y^\ast - \lambda \) is also positive, allowing us to compute

\[
\|(\Delta_Y^\ast - \lambda)f\|^2 = \|\Delta_Y^\ast f\|^2 + |\lambda|^2 \|f\|^2 + 2|\Re(\lambda)|\langle \Delta_Y^\ast f, f \rangle \\
\geq |\lambda|^2 \|f\|^2.
\]

From this we deduce that if \( \lambda \in (0, \infty) \) then \( (\Delta_Y^\ast - \lambda) \) is injective. The exact same argument as in part (i) then shows that \( (\Delta_Y^\ast - \lambda) \text{dom}(\Delta_Y^\ast) = \Lambda^\ast L^2(\mathcal{X}_\Gamma) \).

\[\square\]

**Corollary 4.2.3.** If \( \lambda \in \mathbb{C} \setminus (-\infty, 0] \) then

\[\|R_\lambda\| \leq \frac{1}{|\Im(\lambda)|}.\]

**Proof.** Fix \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and choose \( f \in \Lambda^\ast L^2(\mathcal{X}_\Gamma) \). By proposition 4.2.2 we can write \( f = (\Delta_Y^\ast - \lambda)f_0 \) for some \( f_0 \in \text{dom}(\Delta_Y^\ast) \). From equation (4.5):

\[
\|f\|_2 = \|(\Delta_Y^\ast - \lambda)f_0\|_2 \geq |\Im(\lambda)| \cdot \|f_0\|_2 \\
= |\Im(\lambda)| \cdot \|R_\lambda(\Delta_Y^\ast - \lambda)f_0\|_2 \\
= |\Im(\lambda)| \cdot \|R_\lambda f\|_2,
\]

whence \( \|R_\lambda\| \leq 1/|\Im(\lambda)| \) as claimed. If instead \( \lambda \in (-\infty, 0) \) we appeal to equation (4.7) to compute

\[
\|f\|_2 = \|(\Delta_Y^\ast - \lambda)f_0\|_2 \geq |\lambda| \cdot \|f_0\|_2 \\
= |\lambda| \cdot \|R_\lambda(\Delta_Y^\ast - \lambda)f_0\|_2 \\
= |\lambda| \cdot \|R_\lambda f\|_2;
\]

this time we deduce \( \|R_\lambda\| \leq 1/|\lambda| = 1/|\Im(\lambda)| \).

\[\square\]
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Proposition 4.2.4. Fix $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. If the resolvent $R_{\lambda}$ exists, it is holomorphic at $\lambda$.

Proof. We want to show there exists a linear operator $L$ on $\Lambda_a^Y L^2(\mathcal{X}_\Gamma)$ such that

$$\lim_{\epsilon \to 0} \left\| \frac{R_{\lambda+\epsilon} - R_{\lambda}}{\epsilon} - L \right\|_2 = 0.$$ 

Consider the identity

$$1 - (\Delta_{Fr}^Y - (\lambda + \epsilon))R_{\lambda} = 1 - (\Delta_{Fr}^Y - \lambda)R_{\lambda} - \epsilon R_{\lambda} = \epsilon R_{\lambda}.$$ 

Applying $R_{\lambda+\epsilon}$ to both sides of this equation yields

$$R_{\lambda+\epsilon}(1 - (\Delta_{Fr}^Y - (\lambda + \epsilon)))R_{\lambda} = R_{\lambda+\epsilon} - R_{\lambda}$$

on the left-hand side, and $\epsilon R_{\lambda+\epsilon} R_{\lambda}$ on the right-hand side. It follows that $R_{\lambda+\epsilon} - R_{\lambda} = \epsilon R_{\lambda+\epsilon} R_{\lambda}$; that is, $\frac{R_{\lambda+\epsilon} - R_{\lambda}}{\epsilon} = R_{\lambda+\epsilon} R_{\lambda}$.

This suggests $R_{\lambda+\epsilon}^2$ as a candidate for $L$. Indeed, apply the above equality a few times to produce

$$\frac{R_{\lambda+\epsilon} - R_{\lambda}}{\epsilon} - R_{\lambda}^2 = R_{\lambda+\epsilon} R_{\lambda} - R_{\lambda}^2 = (R_{\lambda+\epsilon} - R_{\lambda}) R_{\lambda} = \epsilon R_{\lambda+\epsilon} R_{\lambda}^2.$$ 

Finally, compute the limit with an appeal to corollary 4.2.3 for bounds on the operator norm:

$$\lim_{\epsilon \to 0} \left\| \frac{R_{\lambda+\epsilon} - R_{\lambda}}{\epsilon} - R_{\lambda}^2 \right\|_2 = \lim_{\epsilon \to 0} \left\| \epsilon R_{\lambda+\epsilon} R_{\lambda}^2 \right\|_2 \leq \lim_{\epsilon \to 0} \frac{|\epsilon|}{|\text{Im}(\lambda + \epsilon)| \cdot |\text{Im}(\lambda)|^2} = 0.$$ 

$\square$

4.3 Compact embedding $\Lambda_a^Y H^1(\mathcal{X}_\Gamma) \hookrightarrow \Lambda_a^Y L^2(\mathcal{X}_\Gamma)$

Proposition 4.3.1. The inclusion $\iota: \Lambda_a^Y H^1(\mathcal{X}_\Gamma) \to \Lambda_a^Y L^2(\mathcal{X}_\Gamma)$ is a compact map.

By definition this means that $\overline{\iota(K)}$ is compact in $\Lambda_a^Y L^2(\mathcal{X}_\Gamma)$ whenever $K$ is compact in $\Lambda_a^Y H^1(\mathcal{X}_\Gamma)$. In practice it suffices to show $\overline{\iota(B_1(0))}$ is compact, where $B_1(0) \subset \Lambda_a^Y H^1(\mathcal{X}_\Gamma)$ is the unit ball. If for any $\epsilon > 0$ we can exhibit a finite set of functions $\{\eta_1, \ldots, \eta_N\}$ such that

$$\iota(B) \subseteq B_\epsilon(\eta_1) \cup B_\epsilon(\eta_2) \cup \cdots \cup B_\epsilon(\eta_N)$$ (4.8)

we get total boundedness of $\iota(B_1(0))$, from which compact closure follows. The condition in equation 4.8 is equivalent to showing

$$\forall f \in \iota(B_1(0)), \quad \|f - \eta_i\|_2 < \epsilon \text{ for at least one } \eta_i.$$ 

This is the approach we take.
Proof of proposition 4.3.1. i. Setup.

Assume $\epsilon > 0$ is given, and suppose $X_\Gamma$ has cusps $a_1, \ldots, a_k$ (so that $a \in \{a_i\}$). Fix some $Y > \sqrt{3}/2$, and consider cusp neighbourhoods $V_{a_1}^Y, \ldots, V_{a_k}^Y$ about these cusps. Let $V_0$ be an open set such that

$$X_\Gamma \setminus (V_{a_1}^Y \cup \cdots \cup V_{a_k}^Y) \subseteq V_0.$$ 

With these definitions $X_\Gamma \subseteq V_{a_1}^Y \cup \cdots \cup V_{a_k}^Y \cup V_0$ is an open covering, and $V_0$ is a bounded set.

ii. Handle the $V_0$ part. Let $\{\varphi_i\} \cup \{\varphi_0\}$ be a partition of unity on $X_\Gamma$ subordinate to the cover $V_{a_1}^Y \cup \cdots \cup V_{a_k}^Y \cup V_0$ such that

$$\text{supp}(\varphi_i) \subseteq V_{a_i}^Y \quad \text{and} \quad \text{supp}(\varphi_0) \subseteq V_0.$$ 

In cusp coordinates around $a_i$ we can assume $\varphi_i(x + iy) = \psi\left(\frac{y}{Y}\right)$ where $\psi: (0, \infty) \to \mathbb{R}$ is such that

$$
\begin{cases}
\psi(y) = 0 & \text{if } y \in (0, 1] \\
0 \leq \psi(y) \leq 1 & \text{if } y \in (1, 2) \\
\psi(y) = 1 & \text{if } y \in [2, \infty)
\end{cases}
$$

Arbitrary $f \in \Lambda_a Y^1(X_\Gamma)$ can be decomposed against our partition as

$$f = \varphi_0 \cdot f + \sum_{i=1}^k \varphi_i \cdot f.$$ 

$V_0$ is bounded, so Rellich compactness implies the smaller inclusion $\tilde{\iota}: H^1(V_0) \hookrightarrow \Lambda_a Y^1 L^2(V_0)$ is a compact map. In particular $\tilde{\iota}(B_1(0))$ is totally bounded, and since

$$\varphi_0 \cdot \tilde{\iota}(B_1(0)) = \tilde{\iota}(\varphi_0 \cdot B_1(0)) \subset \tilde{\iota}(B_1(0)),$$

it follows all of these sets are totally bounded. We use this to choose $\{\eta_1, \ldots, \eta_N\} \subseteq \Lambda_a Y^1 L^2(V_0)$ such that

$$\forall \varphi_0 \cdot f \in \varphi_0 \cdot \tilde{\iota}(B_1(0)), \quad \|\varphi_0 \cdot f - \eta_i\|_2 < \frac{\epsilon}{2} \text{ for at least one } \eta_i.$$ 

The $\eta_i$ can be extended to all of $X_\Gamma$ by setting $\eta_i \equiv 0$ on the complement of $V_0$.

Now, if $f \in \iota(B_1(0))$ it follows that $\varphi_0 \cdot f \in \tilde{\iota}(B_1(0)) \subset H^1(V_0)$. Thus, for some $\eta_i$, we have a chain of inequalities

$$\|f - \eta_i\|_2 = \left\|\left(\varphi_0 \cdot f + \sum_{i=1}^k \varphi_i \cdot f\right) - \eta_i\right\|_2 \leq \|\varphi_0 \cdot f - \eta_i\|_2 + \sum_{i=1}^k \|\varphi_i \cdot f\|_2 \leq \frac{\epsilon}{2} + \sum_{i=1}^k \|\varphi_i \cdot f\|_2.$$ 

The above argument holds for arbitrary $Y > \sqrt{3}/2$. Therefore we are done once we prove that choosing
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$Y$ sufficiently large implies

$$\sum_{i=1}^k \| \varphi_i \cdot f \|_2 \leq \frac{\epsilon}{2} \quad \forall f \in \iota(B_1(0)). \quad (4.9)$$

iii Handle the $V_{a_i}$ parts. Fix any $f \in \iota(B_1(0))$ and any $Y > \sqrt{3}/2$. For convenience set $F_i := \varphi_i \cdot f$. One way to prove the bound (4.9) is to show

$$\lim_{y \to \infty} \| F_i \|_2^2 = \lim_{y \to \infty} \int_{V_{a_i}} |F_i|^2 \frac{dx dy}{y^2} = 0$$

at each cusp $a_i$. To make computations easier, we change into cusp coordinates $\zeta = x + iy$, apply the Tonelli theorem, and simplify the result:

$$\| F_i \|_2^2 \leq \int_{V_{a_i}} |F_i|^2 \frac{dx dy}{y^2} = \int_{y>Y} \int_0^1 |F^{a_i}_i(\zeta)|^2 \frac{dx dy}{y^2}$$

$$\leq \frac{1}{Y^2} \int_{y>Y} \int_0^1 |F^{a_i}_i(\zeta)|^2 \, dx \, dy.$$

The trick is to express this last integral in terms of $\Delta F_i$. First, liberal application of the Parseval identity for $F_x$, the Fourier transform in $x$, along with a scaling of the $n^{th}$ Fourier coefficient by $(2\pi n)^2$ brings us to

$$\frac{1}{Y^2} \int_{y>Y} \int_0^1 |F^{a_i}_i(\zeta)|^2 \, dx \, dy = \frac{1}{Y^2} \int_{y>Y} \int_0^1 \left| \sum_{n \in \mathbb{Z}} F_x [F^{a_i}_i(-,y)](n) \right|^2 \frac{dx dy}{y^2}$$

$$\leq \frac{1}{Y^2} \int_{y>Y} \sum_{n \in \mathbb{Z}} (2\pi n)^2 \left| F_x [F^{a_i}_i(-,y)](n) \right|^2 \frac{dx dy}{y^2}$$

$$= \frac{1}{Y^2} \int_{y>Y} \sum_{n \in \mathbb{Z}} n^2 \left| F_x \left[ \frac{\partial F^{a_i}_i}{\partial x}(-,y) \right](n) \right|^2 \frac{dx dy}{y^2}$$

$$= \frac{1}{Y^2} \int_{y>Y} \int_0^1 \left| \frac{\partial F^{a_i}_i}{\partial x}(\zeta) \right|^2 \, dx \, dy.$$

Now, notice

$$0 < \int_{y>Y} \left| \frac{\partial F^{a_i}_i}{\partial y} \right|^2 \, dy = - \int_{y>Y} \frac{\partial^2 F^{a_i}_i}{\partial y^2} \cdot F^{a_i}_i \, dy; \quad \text{similarly} \quad 0 < - \int_{y>Y} \frac{\partial^2 F^{a_i}_i}{\partial x^2} \cdot F^{a_i}_i \, dy.$$

Therefore changing the order of integration yields

$$\| F_i \|_2^2 \leq \frac{1}{Y^2} \int_{y>Y} \int_0^1 \frac{\partial^2 F^{a_i}_i}{\partial y^2}(\zeta) \cdot F^{a_i}_i(\zeta) + \frac{\partial^2 F^{a_i}_i}{\partial x^2} \cdot F^{a_i}_i \, dx \, dy$$

$$\leq \frac{1}{Y^2} \int_{y>Y} \int_0^1 -\Delta F^{a_i}_i \cdot \frac{F^{a_i}_i}{y^2} \, dx \, dy$$

$$\leq \frac{\| F_i \|_2^2}{Y^2}. \quad (4.10)$$
Suppose we knew that for some constant $C$,
\[
\|F_i\|_{Fr}^2 = \langle F_i, F_i \rangle_{Fr} - \langle \Delta^Y F_i, F_i \rangle_{Fr} \leq C \|f\|_{Fr}^2.
\] (4.11)

Since $\|f\|_{Fr}^2 < 1$ by hypothesis, equation (4.10) would say
\[
\lim_{Y \to \infty} \|F_i\|_{Fr}^2 \leq \lim_{Y \to \infty} \frac{C}{Y^2} = 0
\]
and we would be done. The immediate fact that $\|F_i\|_{Fr}^2 \leq \|f\|_{Fr}^2$ reduces a proof of (4.11) to demonstrating
the inequality
\[
-\langle \Delta^Y F_i, F_i \rangle \leq C \|f\|_{Fr}^2.
\]

Proceeding directly,
\[
-\langle \Delta^Y F_i, F_i \rangle = I_1 + I_2
\]
where $I_1$ and $I_2$ are integrals which have representations in cusp coordinates $\zeta = x + iy$ around $a_i$ of the form
\[
I_1 = -\int_{y > Y} \int_0^1 \psi \left( \frac{y}{Y} \right)^2 \left[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right] f + \psi \left( \frac{y}{Y} \right) \psi'' \left( \frac{y}{Y} \right) \frac{f^2}{Y^2} \, dx \, dy
\]
\[
I_2 = 2 \int_{y > Y} \int_0^1 \psi \left( \frac{y}{Y} \right) \psi' \left( \frac{y}{Y} \right) f \frac{\partial f}{\partial y} \, dx \, dy.
\]

Let’s get a bound on $I_1$ first. Using the fact that $\Delta^Y$ is negative definite and $\psi''$ vanishes above $y = 2Y$,
\[
|I_1| \leq \int_{y > Y} \int_0^1 \psi'' \left( \frac{y}{Y} \right) \psi \left( \frac{y}{Y} \right) \frac{f^2}{Y^2} \, dx \, dy - \int_{y > Y} \int_0^1 \psi \left( \frac{y}{Y} \right)^2 \Delta^Y f \cdot f \frac{dx \, dy}{y^2}
\]
\[
\leq \int_{y > Y} \int_0^1 \frac{(2Y)^2 f^2}{Y^2} \, dy \, dx - \int_{y > Y} \int_0^1 \psi \left( \frac{y}{Y} \right)^2 \Delta^Y f \cdot f \frac{dx \, dy}{y^2}
\]
\[
\leq C_{1,\psi} \int_{y > Y} \int_0^1 f^2 - \Delta^Y f \cdot f \frac{dx \, dy}{y^2}
\]
\[
\leq C_{1,\psi} \|f\|_{Fr}^2 - \int_{y > Y} \Delta^Y f \cdot f \frac{dx \, dy}{y^2} = C_{1,\psi} \|f\|_{Fr}^2.
\]

Here $C_{1,\psi}$ is a constant that depends only on $\psi$, the cutoff function. For $I_2$ we can use an integration by parts and the fact that $\psi'(y) = 0$ for $y > 2Y$ to obtain
\[
I_2 = \int_0^1 \int_Y^{2Y} \psi \left( \frac{y}{Y} \right) \psi' \left( \frac{y}{Y} \right) \frac{\partial f^2}{\partial y} \frac{dy}{Y}
\]
\[
= -\int_0^1 \int_Y^{2Y} \frac{\partial}{\partial y} \left( \frac{1}{Y} \psi \left( \frac{y}{Y} \right) \psi' \left( \frac{y}{Y} \right) \right) f^2 \, dy.
\]
Applying absolute values, we obtain a bound:

$$
|I_2| = \left| \int_0^1 \int_Y^{2Y} \left\| \frac{\partial}{\partial y} \left( \frac{1}{Y} \psi \left( \frac{y}{Y} \right) \psi' \left( \frac{y}{Y} \right) \right) \right\| f^2 \, dx \, dy \right| \\
\leq \int_0^1 \int_Y^{2Y} \left\| \frac{\partial}{\partial y} \left( \frac{1}{Y} \psi \left( \frac{y}{Y} \right) \psi' \left( \frac{y}{Y} \right) \right) \right\| f^2 (2Y)^2 \, dx \, dy \, y^2 \\
= 4 \int_0^1 \int_Y^{2Y} \left\| \psi'' \left( \frac{y}{Y} \right) \psi \left( \frac{y}{Y} \right) + \psi' \left( \frac{y}{Y} \right) \right\|^2 f^2 \, dx \, dy \, y^2 \\
\leq C_{2,\psi} \|f\|^2
$$

where again $C_{2,\psi}$ is some constant that depends on $\psi$.

Choosing $C := C_{1,\psi} + C_{2,\psi}$ completes the proof of equation (4.11), and hence the proposition. \(\square\)

**Corollary 4.3.2.** The spectrum $\text{sp}(\Delta_{Fr}^Y)$ is discrete, and the operator $(\Delta_{Fr}^Y - \lambda)^{-1}$ exists and is compact for $\lambda \notin \text{sp}(\Delta_{Fr}^Y)$. Moreover, $(\Delta_{Fr}^Y - \lambda)^{-1}$ is meromorphic in $\lambda$.

**Proof.** Proposition 3.2.5 and corollary 3.2.7 imply that $(\Delta_{Fr}^Y)^{-1}$ is continuous and self-adjoint respectively, when considered as a map $\Lambda_a^Y \rightarrow H^1(\mathcal{X}_r)$. We just proved (proposition 4.3.1) that the inclusion $\Lambda_a^Y \rightarrow \Lambda_a^X L^2(\mathcal{X}_r)$ is compact. It follows that the composition

$$
\iota \circ (\Delta_{Fr}^Y)^{-1} : \Lambda_a^Y L^2(\mathcal{X}_r) \rightarrow \Lambda_a^Y H^1(\mathcal{X}_r) \rightarrow \Lambda_a^X L^2(\mathcal{X}_r)
$$

is also a compact map. For convenience we abuse notation and denote $\iota \circ (\Delta_{Fr}^Y)^{-1}$ simply by $(\Delta_{Fr}^Y)^{-1}$.

The regular theory of compact, self-adjoint operators now says that $(\Delta_{Fr}^Y)^{-1}$ has only point spectrum; points of $\text{sp}((\Delta_{Fr}^Y)^{-1})$ can only accumulate at 0. We claim there is a bijection between $\text{sp}(\Delta_{Fr}^Y)$ and $\text{sp}((\Delta_{Fr}^Y)^{-1}) \setminus \{0\}$ taking $\lambda \mapsto \lambda^{-1}$. Indeed, the point spectra are the same: from

$$
(\Delta_{Fr}^Y)^{-1} - \lambda^{-1} = (\Delta_{Fr}^Y)^{-1} (\lambda - \Delta_{Fr}^Y) \lambda^{-1} = - (\Delta_{Fr}^Y)^{-1} (\Delta_{Fr}^Y - \lambda) \lambda^{-1}
$$

we can deduce

$$
(\Delta_{Fr}^Y)^{-1} - \lambda^{-1} \text{ injective } \implies \Delta_{Fr}^Y - \lambda \text{ injective}.
$$

Similarly, from

$$
\Delta_{Fr}^Y - \lambda = \Delta_{Fr}^Y (\lambda^{-1} - (\Delta_{Fr}^Y)^{-1}) \lambda = - \Delta_{Fr}^Y ((\Delta_{Fr}^Y)^{-1} - \lambda^{-1}) \lambda
$$

we see that

$$
\Delta_{Fr}^Y - \lambda \text{ injective } \implies (\Delta_{Fr}^Y)^{-1} - \lambda^{-1} \text{ injective}.
$$

Together these imply there is a bijection $\lambda \mapsto \lambda^{-1}$ at least between the point spectra of $\Delta_{Fr}^Y$ and $(\Delta_{Fr}^Y)^{-1}$. Moreover, we know that $0 \notin \text{sp}(\Delta_{Fr}^Y)$ simply because $(\Delta_{Fr}^Y)^{-1}$ exists and is continuous.

If $\lambda^{-1} \notin \text{sp}((\Delta_{Fr}^Y)^{-1})$, then $(\Delta_{Fr}^Y)^{-1} - \lambda^{-1}$ is injective with continuous, everywhere defined inverse denoted by

$$
((\Delta_{Fr}^Y)^{-1} - \lambda^{-1})^{-1} = - (\lambda^{-1} - (\Delta_{Fr}^Y)^{-1}).
$$

Inverting the identity $\Delta_{Fr}^Y - \lambda = - \Delta_{Fr}^Y ((\Delta_{Fr}^Y)^{-1} - \lambda^{-1}) \lambda$ then allows us to write

$$
(\Delta_{Fr}^Y - \lambda)^{-1} = \lambda^{-1} (\lambda^{-1} - (\Delta_{Fr}^Y)^{-1})^{-1} (\Delta_{Fr}^Y)^{-1}
$$
from which we deduce that \((\Delta^Y_{Fr} - \lambda)^{-1}\) is also continuous and everywhere defined. In other words,

\[
\lambda^{-1} \not\in \text{sp}(\Delta^Y_{Fr}) \quad \Longrightarrow \quad \lambda \not\in \text{sp}(\Delta^Y_{Fr}).
\]

Taking the contrapositive of this statement, we conclude \(\text{sp}(\Delta^Y_{Fr})\) is in bijection with \(\text{sp}(\Delta^Y_{Fr})^{-1} \setminus \{0\}\).

The meromorphy follows immediately from proposition 4.2.4.
Chapter 5

Meromorphic Continuation

Throughout this section we make the notational convention that

\[ \lambda = \lambda_s = s(s - 1) \quad \text{where} \; s \in \mathbb{C}, \]

and set

\[ \Omega := \left\{ s \in \mathbb{C} \mid \Re(s) > \frac{1}{2} \text{ and } \lambda_s \notin \text{sp}(\Delta_{Fr}) \right\}. \]

Since \( \Delta_{Fr} \) is a positive operator, we know \( \Omega \cap \left[ \frac{1}{2}, 1 \right] = \emptyset \).

The main references for this section were [2] and [3].

5.1 Another description of \( E(z, s) \)

This section presents an alternative characterization of the Eisenstein series \( E(z, s) \). This characterization will be valid for \( s \in \Omega \). The key ingredients are a certain pseudo-Eisenstein series obtained by truncating \( y^s \), and its images under the operators \( (\Delta - \lambda) \) and \( (\Delta_{Fr} - \lambda)^{-1} \).

More precisely, recall from section 4.3 the cutoff function \( \psi: (0, \infty) \rightarrow \mathbb{R} \) given by

\[
\begin{cases}
\psi(y) = 0 & \text{if } y \in (0, 1] \\
0 \leq \psi(y) \leq 1 & \text{if } y \in (1, 2) \\
\psi(y) = 1 & \text{if } y \in [2, \infty).
\end{cases}
\]

Set \( \varphi(y) := \psi \left( \frac{y}{Y} \right) \) and form a pseudo-Eisenstein series \( h(\zeta, s) \) in \( a \)-coordinates by automorphising the product \( y \mapsto \varphi(y) \cdot y^s \):

\[ h(\zeta, s) := \sum_{\gamma \in B(\mathbb{Z}) \setminus \sigma_a \setminus \Gamma a} \varphi(\text{Im}(\gamma \zeta)) \cdot \text{Im}(\gamma \zeta)^s \]  \hspace{1cm} (5.1)

Since \( \varphi \) is supported in the cuspidal region \( V_a^{Y} \), there can only be one non-zero summand in equation (5.1). Thus the sum defining \( h(\zeta, s) \) converges everywhere on \( \mathcal{X}_\Gamma \). In fact, it can be checked that \( s \mapsto h(-, s) \) is a holomorphic distribution on \( \mathcal{X}_\Gamma \).
For convenience set \( H(z,s) := (\Delta - \lambda)h(z,s) \). Define \( K : \mathcal{X}_\Gamma \times \Omega \to \mathbb{C} \) according to the formula

\[
K(\zeta,s) := h(\zeta,s) - (\Delta_{\text{Fr}} - \lambda)^{-1}H(\zeta,s).
\]

It is valid to apply \((\Delta_{\text{Fr}} - \lambda)^{-1}\) to \( H \), as a simple computation shows that \( H \in L^2(\mathcal{X}_\Gamma) = \text{dom}(\Delta_{\text{Fr}} - \lambda)^{-1} \). We restrict to \( \operatorname{Re}(s) > 1/2 \) since in this case we know \( K \) does not vanish. Indeed, if \( \operatorname{Re}(s) > 1/2 \):

- \( h(-,s) \not\in L^2(\mathcal{X}_\Gamma) \)
- \( (\Delta_{\text{Fr}} - \lambda)^{-1}H(\zeta,s) \not\in \text{dom}(\Delta_{\text{Fr}} - \lambda) \subset L^2(\mathcal{X}_\Gamma) \).

Thus, when \( \operatorname{Re}(s) > 1/2 \), \( K \) is the difference of an \( L^2 \) and non-\( L^2 \) function and so cannot vanish. Our claim is that \( K(z,s) \) is the same as the usual Eisenstein series \( E(z,s) \), at least for \( s \in \Omega \).

**Proposition 5.1.1.** For \( s \in \Omega \), the function \( K(z,s) - h(z,s) \) is the unique element of \( \text{dom}(\Delta_{\text{Fr}}) \) satisfying

\[
(\Delta_{\text{Fr}} - \lambda)[K(z,s) - h(z,s)] = -H(z,s). \tag{5.2}
\]

**Proof.** Suppose \( K - h \) satisfies equation \( (5.2) \). Then uniqueness follows from the fact that \( (\Delta_{\text{Fr}} - \lambda) \) is an injective map. With this in mind we focus on proving \( (5.2) \).

First, use the definition of \( K \) to obtain the identity

\[
K(z,s) - h(z,s) = h(z,s) - (\Delta_{\text{Fr}} - \lambda)^{-1}H(z,s)) - h(z,s)
= -(\Delta_{\text{Fr}} - \lambda)^{-1}H(z,s).
\]

This equation implies \( K - h \in \text{dom}(\Delta_{\text{Fr}}) \). Applying \( (\Delta_{\text{Fr}} - \lambda) \) to both sides therefore yields

\[
(\Delta_{\text{Fr}} - \lambda)[K(z,s) - h(z,s)] = H(z,s). \tag{5.3}
\]

On the other hand, if \( \operatorname{Re}(s) > 1 \) then the series defining \( E(z,s) \) converges, and \( E(z,s) - h(z,s) \in \text{dom}(\Delta_{\text{Fr}}) \). Notably, the function \( E - h \) is smooth, so an application of \( (\Delta_{\text{Fr}} - \lambda) \) can be replaced by an application of the resolvent \( (\Delta - \lambda) \) of the usual Laplacian. We get

\[
(\Delta_{\text{Fr}} - \lambda)[E(z,s) - h(z,s)] = (\Delta - \lambda)[E(z,s) - h(z,s)]
= 0 - (\Delta - \lambda)h(z,s)
= -H(z,s). \tag{5.4}
\]

Combining equations \( (5.3) \) and \( (5.4) \) with the uniqueness result shows that \( K(z,s) \) is merely the usual Eisenstein series \( E(z,s) \) for \( \operatorname{Re}(s) > 1 \). Moreover, \( K \) provides a continuation of \( E \) (in the \( s \)-parameter) to the domain \( \Omega \). 

\[\square\]
5.2 The functions $E^Y(z, s)$

5.2.1 Definitions

Let $h(z, s)$ be the same pseudo-Eisenstein series as in section 5.1, and again set $H(z, s) = (\Delta - \lambda)h(z, s)$. Inspired by the characterisation of $E(z, s)$ via the identity $E = h - (\Delta_{Fr} - \lambda)^{-1}H$, let us set

$$E^Y(z, s) := h(z, s) - (\Delta_{Fr}^Y - \lambda)^{-1}H(z, s).$$

This time we use the fact that $H(z, s) \in \Lambda^Y L^2(\mathbb{H}) = \text{dom}(\Delta_{Fr}^Y - \lambda)$ to justify the application of $(\Delta_{Fr}^Y - \lambda)^{-1}$ to $H$.

**Proposition 5.2.1.** For $s$ not contained in the discrete set $\text{sp}(\Delta_{Fr}^Y)$, $E^Y(z, s) - h(z, s)$ is the unique element of $\text{dom}(\Delta_{Fr}^Y)$ satisfying

$$(\Delta_{Fr}^Y - \lambda)[E^Y(z, s) - h(z, s)] = -H(z, s).$$

**Proof.** Clearly $E^Y(z, s) - h(z, s) \in \text{dom}(\Delta_{Fr}^Y)$ when defined. The function $h(z, s)$ is entire in $s$, and $(\Delta_{Fr}^Y - \lambda) = (\Delta_{Fr}^Y - s(s-1))$ is a meromorphic operator-valued function of $s$. It follows that $s \mapsto E^Y(z, s)$ is meromorphic. Thus the expression $E^Y(z, s) - h(z, s)$ is valid for $s$ not contained $\text{sp}(\Delta_{Fr}^Y)$. For such $s$, unravelling definitions gives

$$(\Delta_{Fr}^Y - \lambda)[E^Y(z, s) - h(z, s)] = (\Delta_{Fr}^Y - \lambda)[h(z, s) - (\Delta_{Fr}^Y - \lambda)^{-1}H(z, s) - h(z, s)]$$

$$= -H(z, s).$$

Uniqueness follows from the fact that $(\Delta_{Fr}^Y - \lambda)$ is an injective map. \hfill \Box

5.2.2 Constant terms

In order to understand the constant term map $c_a[E^Y(-, s)] : N \backslash \mathbb{H} \to \mathbb{C}$ it is useful to think of $N \backslash \mathbb{H} \sim (0, \infty)$ as the union

$$N \backslash \mathbb{H} = (0, Y) \cup \{Y\} \cup (Y, \infty).$$

Let’s investigate what happens on each of these three regions.

i. $z = x + iy$ with $y \in (Y, \infty)$: The constant term of $(\Delta_{Fr}^Y - \lambda)^{-1}H(z, s)$ vanishes above $Y$ since it’s an element of $\Lambda^Y L^2(\mathbb{H})$. Therefore

$$c_a[E^Y(-, s)] = c_a[h(-, s)] + c_a[(\Delta_{Fr}^Y - \lambda)^{-1}H(-, s)]$$

$$= c_a[h(-, s)]$$

$$= y^s.$$

ii. $y \in (0, Y)$: Proposition 4.1.2 enables the following computation, valid in the sense of distributions:

$$-H(z, s) = (\Delta_{Fr}^Y - \lambda)(E^Y(z, s) - h(z, s))$$

$$= (\Delta - \lambda)(E^Y(z, s) - h(z, s)) + C(s) \cdot T^Y$$

$$= (\Delta - \lambda)E^Y(z, s) - H(z, s) + C(s) \cdot T^Y.$$
The \( H \) terms cancel to produce \((\Delta - \lambda)E^Y(z, s) = -C(s) \cdot T^Y\), which restricts to the identity \((\Delta - \lambda)E^Y(z, s) = 0\) for \( z = x + iy \) with \( y \in (0, Y) \). In proposition 2.3.8 we saw that \( c_a \) and \( \Delta \) commute, so the above implies

\[
c_a[(\Delta - \lambda)E^Y(z, s)] = (\Delta - \lambda)c_a[E^Y(z, s)] = 0.
\]

In other words, \( c_a[E^Y(z, s)] \) solves the distributional differential equation

\[
\left( y^2 \frac{\partial^2}{\partial y^2} - s(s-1) \right) u = 0.
\]

Solutions of these equations are of the form:

\[
c_a[E^Y(z, s)] = A(s)y^s + B(s)y^{1-s}.
\]

In this case, since \( c_a[E^Y(z, s)] \) is a meromorphic distribution in \( s \), \( A \) and \( B \) are meromorphic functions of \( s \).

iii. \( y \in \{Y\} \): Since \( E^Y - h \in \text{dom}(\Delta^Y_{\Gamma_s}) \), proposition 3.4.2 can be applied to deduce \( c_a[E^Y - h] \in H^1_{\text{loc}}(\mathcal{X}_r) \). Functions in \( H^1_{\text{loc}}(\mathcal{X}_r) \) are continuous, so \( c_a[E^Y - h] \) is continuous. Parts (i) and (ii) respectively show

\[
c_a[E^Y(z, s) - h(z, s)] = \begin{cases} 
0 & \text{for } y \in (Y, \infty) \\
A(s)y^s + B(s)y^{1-s} - y^s & \text{for } y \in (0, Y).
\end{cases}
\]

These can be equated at \( y = Y \) to produce expressions for the constant term:

\[
Y^s = c_a[E^Y(x + iy, s)] = A(s)Y^s + B(s)Y^{1-s}.
\]

5.3 The meromorphic continuation formula

We saw in section 5.2 that the constant term \( c_a[E^Y(\zeta, s)] \) behaves differently depending on whether \( \zeta = x + iy \) satisfies \( y \in (0, Y) \) or \( y \in (Y, \infty) \). The idea behind the continuation is that \( E(\zeta, s) \) can be recovered from \( E^Y(\zeta, s) \) by “correcting” the constant term \( c_a[E^Y(\zeta, s)] \) so that it equals \( A(s)y^s + B(s)y^{1-s} \) on all of \( (0, \infty) \).

More precisely, set

\[
\varphi(z, s) := \chi_{(Y, \infty)} : (A(s)y^s + B(s)y^{1-s} - y^s)
\]

where \( \chi_{(Y, \infty)} \) is the indicator function for the interval \( (Y, \infty) \). Denote the derived pseudo-Eisenstein series by \( \Phi(z, s) \):

\[
\Phi(z, s) := \sum_{\gamma \in \Gamma \setminus \Gamma_a} \varphi(\text{Im}(\gamma z), s).
\]

**Theorem 5.3.1.** For \( \text{Re}(s) > 1 \), the follow identity holds in the sense of distributions:

\[
A(s) \cdot E(z, s) = E^Y(z, s) + \Phi(z, s).
\]
Proof. i. Suppose we can demonstrate

\[(\Delta_{F_1} - \lambda)[E^Y(z, s) + \Phi(z, s) - A(s)h(z, s)] = -A(s)H(z, s).\]  \hspace{1cm} (5.5)

It has already been shown that \(A(s)(E(z, s) - h(z, s))\) is the unique solution in \(H^1(\chi_1)\) to

\[(\Delta_{F_1} - \lambda)[A(s)E(z, s) - A(s)h(z, s)] = -A(s)H(z, s).\]

Indeed, this equation is obtained from the identity in proposition 5.1.1 after multiplying through by \(A(s)\). Therefore a proof that \(E^Y + \Phi - A \cdot h \in H^1(\chi_1)\) and satisfies (5.5) will also establish

\[A(s)E(z, s) - A(s)h(z, s) = E^Y(z, s) + \Phi(z, s) - A(s)h(z, s);\]

in other words, \(A(s)E(z, s) = E^Y(z, s) + \Phi(z, s)\) as claimed. This is the approach we take.

ii. For the inclusion \(E^Y + \Phi - A \cdot h \in H^1(\chi_1)\) it is useful to rewrite

\[E^Y + \Phi - A(s) \cdot h = (E^Y - h) + (\Phi - A(s) \cdot h + h),\]

as \(E^Y - h \in H^1(\chi_1)\) then reduces the matter to showing

\[F := \Phi - A(s) \cdot h + h \in H^1(\chi_1).\]

Note that \(F\) is supported in \(V^Y_a\), where it has a simpler representation obtained by undoing automorphization, namely

\[F = \varphi(z, s) + (1 - A(s)) \cdot \psi \left( \frac{y}{Y} \right) y^a.\]

\(F\) depends only on \(y\), so we can consider it an element of \(C^\infty(Y, \infty)\). Moreover, \(\lim_{y \to Y} F(y) = 0\) means \(F\) can be extended to \((0, \infty)\) by setting \(F \equiv 0\) on \((0, Y]\). An appeal to proposition 3.4.4 then reveals \(F \in H^1(\chi_1)\) if

\[\varphi(z, s) + (1 - A(s)) \cdot \psi \left( \frac{y}{Y} \right) y^a \in L^2(Y, \infty)\]  \hspace{1cm} (5.6)

and

\[y \frac{\partial}{\partial y} \left[ \varphi(z, s) + (1 - A(s)) \cdot \psi \left( \frac{y}{Y} \right) y^a \right] \in L^2(Y, \infty).\]  \hspace{1cm} (5.7)

Let’s tackle equation (5.6) first. \(F\) is continuous on \((Y, \infty)\) and vanishes at \(Y\), so \(\int_Y^{2Y} F \, dy < \infty\). In other words, it is sufficient to show \(F \in L^2(2Y, \infty)\). On this set we have an even simpler representation:

\[F(y, s) = \varphi(y, s) + y^a - A(s) \cdot y^a = (A(s)y^a + B(s)y^{1-s} - y^a) + y^a - A(s)y^a = B(s)y^{1-s}.\]

Given that \(B(s)y^{1-s} \in L^2(2Y, \infty)\) for \(\text{Re}(s) > 1\), we conclude \(F \in L^2(2Y, \infty)\) as desired.
For equation (5.7), let us first get an explicit form for the derivative:

\[
\frac{y}{y} \frac{\partial}{\partial y} \left[ \varphi(z, s) + \left(1 - A(s)\right) \cdot \psi \left(\frac{y}{Y}\right) y^s \right]
\]

\[
= \frac{y}{y} \frac{\partial}{\partial y} \left[ A_s y^s + B_s y^{1-s} - y^s + \left(1 - A_s\right) \psi \left(\frac{y}{Y}\right) y^s \right]
\]

\[
= \left( \psi \left(\frac{y}{Y}\right) - 1 \right) \left(1 - A(s)\right) s y^s + \left(1 - s\right) B(s) y^{1-s} + \left(1 - A(s)\right) \frac{\psi'(\frac{y}{Y})}{Y} y^{s+1}.
\]

The final term in (5.8) has finite integral on \((Y, \infty)\), so we reduce again to showing this term is in \(L^2(2Y, \infty)\). On \((2Y, \infty)\), equation (5.8) takes a simpler form:

\[
\frac{y}{y} \frac{\partial}{\partial y} \left[ \varphi(z, s) - \left(A(s) + 1\right) \cdot \psi \left(\frac{y}{Y}\right) y^s \right] = \left(1 - s\right) B(s) y^{1-s}.
\]

Like before, \((1 - s)B(s)y^{1-s} \in L^2(2Y, \infty)\) for \(\text{Re}(s) > 1\).

We have now shown that \(F \in H^1(\mathcal{X}_\Gamma)\), and hence \(E^Y + \Phi - A \cdot h \in H^1(\mathcal{X}_\Gamma)\). By our earlier remarks, this completes the proof.

We have now shown that \(F \in H^1(\mathcal{X}_\Gamma)\), and hence \(E^Y + \Phi - A \cdot h \in H^1(\mathcal{X}_\Gamma)\). By our earlier remarks, this completes the proof.

Since \(A\) and \(E^Y\) are meromorphic in \(s\), while \(\Phi\) is holomorphic, immediately implies that \(E_a\) is meromorphic in \(s\) as well. Moreover, proposition 4.2.2 we know that poles of \(E\) contributed by \(E^Y\) correspond to values of \(\lambda_s = s(s - 1) \in (-\infty, 0)\); i.e. \(s \in (0, 1)\). The meromorphic continuation is our main result:

**Theorem 5.3.2.** The Eisenstein series \(E_a(z, s)\) can be meromorphically continued in \(s\) to the entire complex plane.
Bibliography


