Key Ideas In Proof
In Undergraduate Mathematics Classrooms

by

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Abstract

The mathematics education literature reveals an ongoing interest in fostering the ability of students to construct and reconstruct proofs. One promising tool to this end is the concept of “key idea”. Observing groups of undergraduate mathematics students, the study examined in some detail how such students understand the concept of key idea and how well they identify the key ideas in a proof and use them in reconstructing it.

The study used a mixed methods design, driven primarily by qualitative methods, while incorporating a complementary quantitative component. Participants in the study were first-year and mixed-year (2nd to 4th) university mathematics students. Data were collected from four sources: (1) an online survey, (2) classroom observations in the form of audio recordings and field notes, (3) semi-structured interviews with students, and (4) students’ work on identifying key ideas in proofs and reconstructing a proof.

The study results form a detailed portrayal of students’ perceptions of key ideas as well as the ways they go about identifying and using key ideas in reconstructing a proof. The
picture is complex. While most of the students reported that they consciously identified key ideas in proofs they were studying, they varied widely in their understanding of the concept itself. Key ideas were perceived by the students as “important points”, as “blueprints”, as “landmarks and pathways” or as “steps to follow”. When asked to identify key ideas in specific proofs, most of the students were unable to articulate clear responses. Very few students were able to use precise language and point to an idea that helped them both understand the proof and reconstruct it.

These findings are a significant addition to our present knowledge of undergraduate mathematics students’ ability to see the important ideas embedded in proofs. They imply that mathematics educators, in their desire to see students enhance their understanding of proof and proving by the use of key ideas, will need to extend considerable support to students by actively intervening to draw their attention to features of proofs that are candidates for key ideas.
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Chapter 1

Introduction

This research focuses on an important and promising approach to the effective teaching of proof at the undergraduate level. More specifically, the purpose of this research is to investigate in some detail how and how well undergraduate mathematics students identify key ideas embodied in proofs and how they might use these key ideas to reconstruct a proof. The research also reports on students’ previous experience with proof.

There is a widespread belief among mathematics educators that proof is an elusive concept for many mathematics students, and that it is difficult to teach and learn proof (Healy & Hoyles, 2000; Senk, 1985). Over the past five decades, partly as a result of this belief, educators have tended to downplay proof, as evinced by periods of significant decline in the appearance of the word “proof” in North American mathematics curricula (NCTM, 1989; 2000). However, as research on the potential role of proof has expanded in the most recent two decades, there has been an increasing emphasis on the teaching and learning of proof at all grade levels (Bass, 2011; Fischbein, 1982, 1999; Hanna, 1995; Hoyles & Jones, 1998; NCTM, 2009; Reid & Knipping, 2010; Stylianides, 2007). Much of the recent discussion on the role of proof in mathematics education has focused on improving students’ mathematical understanding through the practice of proof, in particular by fostering their ability to construct and reconstruct proofs (Stylianides & Stylianides, 2008; Weber & Mejía-Ramos, 2013).
1.1 Definition of Terms

1.1.1 Proof

In this study, the term “proof” refers to what is often called “informal proof,” as opposed to “formal proof.” Formal proofs, also known as axiomatic proofs, consist of derivations of propositions from premises which are taken as axioms assumed to be true. These derivations are syntactical manipulations of signs through the application of rules of logical inference, with no necessary appeal to the meaning of the concepts. By contrast, informal proofs (also called conceptual or real proofs) do make appeal to the meaning of the concepts and formulae employed in the argument, while nevertheless observing a degree of rigour acceptable to mathematicians. Though a conceptual proof does not have a precise mathematical definition, mathematicians readily understand the overall structure of such a proof and can verify the correctness of each of its steps. Most proofs in mathematical practice – those that are published in scholarly journals of mathematics – are informal, conceptual proofs (Cellucci & Gillies, 2005; Corfield, 2003; Mancosu, 2008; Manin, 1998; Rav, 1999).

1.1.2 Proof construction and reconstruction

The term “proof construction”, as it relates to classroom practice, refers to the production of a valid argument that provides conclusive evidence for the truth of a statement new to the student, while the term “proof reconstruction” refers to the production by a student of a valid argument for a statement that is not new to the student and for which a proof has already been taught to the student and perhaps also discussed in class with the instructor.
1.1.3 Key ideas

The term “key ideas” refers to the most important ideas contained in a proof, those that help generate further steps in the argument so as to arrive at the full proof in a relatively efficient way. According to Gowers (2007), it is these important ideas that enable a mathematician to understand and remember a proof: “Instead of remembering the details of a proof, it is much more efficient to remember a few important ideas [Italics mine] and develop the technical skill to convert them quickly into a formal proof. And it is still better if the ideas themselves are not so much memorized as understood so that one feels that they arise naturally” (p. 40).

1.2 Context

Since Hanna (1990) highlighted the distinction between proofs that prove and proofs that explain, it has been widely accepted that the purpose of using proof in the classroom is not merely to convince students that a mathematical statement is true, but also, and more importantly, to provide them with mathematical insights (e.g., Hanna, 1995; Hersh, 1993; Mejia-Ramos et al., 2012; Thurston, 1994). Closely associated with the notion of “proofs that explain” is that of a key idea in a proof. For some scholars this term refers to a central mathematical idea, method, or strategy used in a proof, whereas for others it connotes the outline, overview or architecture of a proof. Despite these different interpretations of key idea, this notion has had pedagogical implications that are consistently aligned with the promotion of students’ ability to undertake proof construction and reconstruction on the basis of richer understanding (Deltlefsen, 2008; Duval, 2007; Gowers, 2007; Hanna & Mason, 2014; Leron, 1983; Raman, 2003; Robinson, 2000).

From a mathematician’s perspective, Gowers (2007) ascribes great importance to the
main ideas (or key ideas) in a proof, and relates the ability to reconstruct a proof to the number of main ideas it contains. He defines the “width” of a proof as the number of pieces of information or main ideas (key ideas) one must carry in one’s head. In his opinion, the fewer main ideas needed for a proof, the more memorable that proof will be; he then maintains that memorability fosters proof reconstruction (Hanna & Mason, 2014). In the interest of reconstructability, Gowers suggests, then, that one must first make an effort to identify the key idea(s) in a proof.

Despite the increasing interest among mathematics educators in fostering the ability of students to construct and reconstruct proofs, and in proof reconstructability in particular, my review of the literature does not reveal any studies that investigated how university mathematics students go about identifying the key idea(s) of a proof, consciously or unconsciously, or how key idea(s) are actually used to help students grasp the overall structure of a proof or the mathematical insights embedded in it. Of those empirical studies that have investigated students’ comprehension of proof, to my knowledge none have even probed the identification of key ideas as a factor that promotes the ability of undergraduate university mathematics students to construct and reconstruct proofs. This study sought to close some of the gaps in past research by addressing questions related to the effective teaching of mathematical proof at the undergraduate level, with specific reference to the use of key ideas.

This study adds to the literature by examining how and how well mathematics students identify key ideas embodied in proofs and how they might use these key ideas to reconstruct a proof. This study also advances our understanding of the role that key ideas might play in fostering students’ “conceptual understanding and procedural fluency”.
Conceptual understanding refers to implicit or explicit command of the principles that govern a particular domain. The term “procedural fluency” defined by NCTM (2000; 2014) as:

The ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another. […] Procedural fluency builds on a foundation of conceptual understanding, strategic reasoning, and problem solving. (p. 42)

There are many challenges to advancing students’ proof-related competences such as conceptual understanding and procedural fluency. Among them is the variety of complex causes and conditions that help or hinder students’ ability to construct and reconstruct proofs while gaining mathematical understanding.

In what follows, I discuss the basis for the research questions, with a focus on their originality, theoretical relevance, and pedagogical implications, referring to pertinent conceptual frameworks: (a) the idea of proofs that promote mathematical understanding, and (b) the notion of key idea as a crucial component of mathematical explanation. This shows the rationale for this study as well as the significant gaps in the literature on its topic.

1.3 Research Questions

This study intended to address to some degree the significant lack of information in the mathematics education literature on the degree to which students pay attention to key ideas in proofs and use them in the reconstruction of proofs at the undergraduate level. This gap in the literature is particularly unfortunate in light of the concern that mathematics educators have expressed over the lack of competence of mathematics majors in matters of proof. In what follows, I expand upon the reasons for the choice of the topic and the research questions.

As mentioned, the literature of mathematics education reveals an ongoing interest in
fostering the ability of students to construct and reconstruct proofs. But I have been unable to locate any studies that have investigated how or how well university mathematics students identify the key idea(s) of a proof, or how and to what extent such key idea(s) actually help students grasp the overall structure of a proof and the mathematical insights embedded in it. In this study, I examined research on promoting the ability of university mathematics students to construct and reconstruct proofs while gaining mathematical understanding. I sought to address some of the gaps in past research by investigating how students go about identifying the key ideas in a proof and use them in reconstructing it, and examining to what extent the notion of key idea is related to students’ conceptual understanding and procedural fluency. Accordingly, this study addresses the following questions about the practices of undergraduate mathematics students in the use of proof:

1. (a) What are students’ perceptions of the role of key idea in a proof?
   
   (b) What are students’ interpretations of key ideas of a proof?

2. From the students’ perspective, is the notion of key idea in proof associated more with conceptual understanding or with procedural fluency?

3. (a) What is the evidence that students identify and use key ideas in reconstructing a proof?
   
   (b) Which features of a proof do students identify as key ideas?

1.4 Rationale and Significance

1.4.1 Originality

Despite the ongoing interest in the reconstructability of proofs, I have been unable to locate, as mentioned, any studies that have investigated how or how well university mathematics students, consciously or unconsciously, identify the key idea(s) of a proof or
how and to what extent the key idea(s) actually help students grasp the overall structure of a proof as well as the mathematical insights embedded in it. Thus this study addresses a topic which has been explored insufficiently, with the aim of gaining new insights.

1.4.2 Theoretical and social relevance

The conceptual framework that shapes this study consists of: (1) the idea of proofs that promote mathematical understanding, (2) the notion of key idea as a crucial component in mathematical explanation. Based on this framework, the research questions were formulated to best link to the relevant theory and related literature. More importantly, this study aimed to advance knowledge of the field by exploring the factors that influence how students construct and reconstruct proofs while gaining mathematical understanding. These factors include the degree to which students are aware of their own grasp of proof, their interest and ability in constructing proofs, and, first and foremost, the role that key ideas might play in fostering students’ conceptual understanding and procedural fluency. Answering the research questions might help revise or build upon existing assumptions informing research and practice on teaching proof.

1.4.3 Pedagogical implications

The notion of key idea itself has pedagogical implications. It has the potential to improve the ability of students to undertake proof construction and reconstruction, because students’ attention can be directed effectively to locating key ideas and then using them to reconstruct proofs. The outcomes of this investigation are expected to be useful to mathematics educators in formulating instructional approaches and identifying forms of intervention that are worth implementing in undergraduate classrooms.
1.5 Research Approach

Aiming to bring new insights to understand students’ perception of proof, approach to proof, and ability to “see” key ideas, the research approach is closest to an exploratory study. Creswell (2013) notes that:

One of the chief reasons for conducting a qualitative study is that the study is exploratory. This usually means that not much has been written about the topic or the population being studied, and the researcher seeks to listen to participants and build an understanding based on what is heard. (p. 29)

The qualitative research approach was chosen because it allowed a good measure of flexibility with respect to data collection methods. Multiple sources of information were used in this study: classroom observations, field notes, audiotape, an online survey, structured student interviews, and a collection of students’ work. The data analysis methods, such as constant-comparative method (Kolb, 2012), developing categories, portraying relationships, and descriptive statistics, were used where appropriate.

1.6 Research Design Overview

This study is driven primarily by qualitative methods, while incorporating a complementary quantitative component. The participants were in three mathematics classes at an urban university in Ontario. The majority of the students were in the program of honours mathematics or mathematics education. A quantitative survey with a sample size of 59 participants was administered first, and then classroom observations and in-depth student interviews were conducted throughout the semester. At the end of the semester, a follow-up question sheet was also administered. The quantitative survey had two purposes. First and foremost, its goal was to gather data on the use of key idea(s) in proof construction and reconstruction and on students’ awareness of their own development of proof schemes. Second, by gathering information on students’ general perception of proof and on their
demographics, it aimed at obtaining a representative qualitative sample (Hesse-Biber, 2010) for the purpose of enhancing the quantitative findings. This two-phased approach allowed me to review and analyze the survey results and to tailor the subsequent in-depth interview instrument.

To gather firsthand data on students’ learning experiences and proving practices, face-to-face student interviews and classroom observations proved to be the best methods. Specifically, semi-structured student interviews allowed me to collect detailed information about students’ perception of key idea as well as the challenges that the students have faced in their proving practices, while classroom observations allowed me to observe and interact closely with the participants so as to establish an insider’s identity without controlling and interfering with the classroom activities (Denzin & Lincoln, 2013).

In addition to interviews and classroom observations, student work samples were collected from all three classes throughout the entire study, including worksheets, individual assignments, group work, mid-term tests, final exams, as well as snapshots of students’ proving products on a board or on paper. Taking into account that student work could be a major source for investigating how students might go about identifying key ideas in a proof, small interventions were conducted for a few classroom exercises and test/exam questions. Based on the teaching materials of each course, 5 proofs were used for students to identify and formulate key ideas. In this thesis, I investigate and discuss two proofs.

1.7 Background of the Researcher

Proof and proving has been an important part of my academic and professional teaching life. My interests in proofs can be traced back to when I was in the middle school in Beijing, China. Because the mathematics curriculum put a special emphasis on Euclidean
I was exposed to Euclid’s axioms, postulates, and to theorems proving at an early age. Before I entered high school, I had developed a good understanding and deep appreciation of Euclidean geometry and mathematical elegance.

My interests in proofs in geometry as well as linear algebra reached a new level during my study of mathematics in University. After I became a high school mathematics teacher, I shifted my attention from proof itself to the teaching of proof.

1.8 Plan of the Thesis

The remaining chapters of the thesis are organized as follows:

Chapter 2 is the literature review which examines the various aspects of proof that are directly or indirectly related to this study, namely, factors that promote proof construction and reconstruction, current research on argumentation and proof, and types of proof that promote mathematical understanding.

Chapter 3 describes the methods and procedures employed in gathering information for the empirical research. It also reports on the kind of instruments used to collect data, and explains the methods of data analysis.

Chapter 4 gives an account of students’ background information and on their interests and previous learning experiences of proof and proving. It then addresses the research questions and reports and discusses the findings of the study.

Chapter 5 revisits the research questions and discusses the possible answers to the research questions. It also examines potential implications for the teaching of proof and suggests some directions for further research.
Chapter 2

Literature Review

The purpose of this study, as discussed, is to investigate in some detail how and how well undergraduate mathematics students identify key ideas embodied in proofs and how they might use these key ideas to reconstruct a proof. To carry out this study, I conducted a critical review of the current literature pertaining to the teaching of proof. In particular, I focused on studies investigating the teaching of proof at the undergraduate level that addressed (a) proof construction, (b) proof reconstruction, and (c) proofs that promote mathematical understanding.

In this section, I begin with studies of the fundamental elements of proof construction, followed by proof reconstruction, with focused attention on the notion of key idea as a crucial component of mathematical explanation. I then review studies that describe the explanatory and communicative nature of proof and its relation to the growth of mathematical understanding.

2.1 Proof Construction

Previous studies have considered a number of aspects of proof construction, including the structure of proofs (Leron, 1983), cognitive processes of proof developments (Tall et al., 2012), different types of reasoning (Tall & Mejia-Ramos, 2010), and the logic involved (Dubinsky & Yiparaki, 2000; Selden & Selden, 1995). In what follows, I describe the fundamental elements of proof construction: proof schemes, proof and argumentation, diagrammatic reasoning, and the width of a proof.
2.1.1 Proof and proof schemes

Throughout the 1980s and 1990s several classroom studies at the secondary level documented students’ difficulties when faced with proof tasks, students’ misconceptions related to the role of proof, and students’ frequent failure to distinguish between an empirical argument and a valid mathematical proof (Mariotti, 2006; Reid & Knipping, 2010; Selden & Selden, 1995; 2003). Harel and Sowder (1998) analyzed several classroom studies and came to the conclusion that “a major reason that students have serious difficulties understanding, appreciating, and producing proof is that we, their teachers, take for granted what constitutes evidence in their eyes” (p. 237). They thought to remedy the situation by focusing on a classification of current students’ conceptions (and/or misconceptions) of proof. They proposed a system of categories that purports to account for students’ behavior when confronted with proof tasks. They named this system of categories “Proof Schemes”.

Harel and Sowder (1998; 2007) defined proof schemes as ways in which an individual (or community) assures oneself or convinces others of the truth of a mathematical assertion. (In other words, proof schemes are the ways of thinking associated with the proving act.) Based on how students ascertain for themselves or persuade others of the truth of a mathematical statement, Harel and Sowder (1998) proposed a classification that consists of seven proof schemes, organized into three major categories (see Figure 1). These categories suggest that a developmental sequence of students’ proof schemes begins with a dependence on external conviction schemes (e.g., the professor’s assertions), progresses to empirical schemes (e.g., studying examples and counterexamples), and culminates in the goal of instruction, which are analytical proof schemes.
The proof schemes suggested by Harel and Sowder (1998) have given impetus to a large number of empirical studies that have helped researchers uncover sources of students’ misconceptions. Most studies showed that the pervasive proof schemes students were relying on were “external conviction” and “empirical proof schemes” (Lester, 2007, p. 820). The students’ persistent struggles with going beyond empirical proof schemes so as to reach the desired goal of analytical proof schemes remains in need of further investigations (Inglis & Alcock, 2012; Mejia-Ramos et al., 2012; Selden & Selden, 2013, 2015; Weber, 2014).

2.1.2 Argumentation and proof

Several researchers have perceived a strong though complex relationship between argumentation and proof and went as far as considering “argumentation essential to teaching and learning of proof” (Reid & Knipping, 2010, p. 153). In their view, much mathematical practice takes place in the argumentation structure (Aberdein, 2013) elaborated by Toulmin’s (1958) model of argumentation. The model may be read as follows: The model assumes that there is a Claim to be established; Data or reasons provide information to justify or support the claim; a Warrant acts as a bridge between the Data and the Claim; aQualifier presents
the relative strength of the *Warrant*; a *Backing* may be needed to establish the validity of the *Warrant*; a *Rebuttal* may be needed to set the limitations of the argument or to indicate conditions where the *Warrant* is not applicable and the conclusion rendered invalid (see Figure 2).

![Figure 2. The Standard Toulmin’s Model of Argumentation.](image)

*Note.* Adapted from “The uses of argument”, by S. E. Toulmin, 1958, p. 104.

The model shown in Figure 2, originally designed and developed to reconstruct arguments used in all types of rational debate, has become prevalent in mathematics education, and has been used in the analysis, assessment and construction of mathematical arguments and in proof and proving (Banegas, 2013; Douek, 1999, 2007; Knipping, 2008; Mariotti, 2006; Pedemonte, 2007). Douek (1999, 2007) in particular emphasizes the similarities between argumentation in general and mathematical proof. In her view, the common features between the two include the validation of a statement, the reference to an established corpus of knowledge, and the requirement of a chain of reasoning.
The relationship between argumentation and mathematical proof is highlighted as cognitive unity (Boero et al., 1996; Garuti et al., 1998; Mariotti et al., 1997). Boero et al. (1996) wrote:

During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organising some of the previously produced arguments according to a logical chain. (p. 345)

The theoretical construct of cognitive unity has been suggested as a tool to predict the level of difficulty in proving. Garuti et al. (1998) point out that the cognitive unity is broken when there is a gap between the arguments for the plausibility of the conjecture that are produced during its initial exploration, and the arguments that are actually used when constructing its proof; they have also shown that the greater this gap is, the harder it is for students to construct the proof.

However, there are significant differences between argumentation and mathematical proof in (1) status of assumptions, (2) epistemic value, and (3) deduction (Balacheff, 1987; Boero et al., 2010; Duval, 1991), are presented in Table 1.

Table 1

<table>
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<tr>
<th>Differences between Argumentation and Mathematical Proof</th>
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<td>Status of assumptions</td>
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<td>Unclear</td>
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<td>Epistemic value depends on</td>
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<td>Chain of deduction</td>
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The results of hundreds of empirical studies on argumentation and proof were inconclusive despite the claim that “argumentation is essential to teaching proof”. Sommerhoff et al. (2015) who surveyed over 700 Psychology of Mathematics Education (PME) studies published between 2010 and 2014 found that most studies were carried out in middle and high-school settings, covered “a broad range of processes, sub-skills and knowledge facets” (p. 4-194) that make it difficult to “gather a comprehensive understanding” (p. 4-196) of mathematical argumentation and proof and the extent of its impact on teaching proof.

2.1.3 Visual and diagrammatic reasoning

If linguistic representation is sequential or linear, then diagrammatic representation is planar (Larkin & Simon, 1987). In many geometric as well as some algebraic proofs, figures and diagrams are often created and used not only in presenting a problem, but also in illuminating a solution (e.g., Alsina & Nelsen, 2010; Nelsen, 1993). Davis (1993) reminds us that the concept of visual proof is an ancient one that proved its usefulness but was unfortunately overshadowed by the rise of formal logic and in his view deserves to regain its important place in mathematics.

In geometry, the synergy of visual representation and reasoning ability is considered to be of the essence (Hershkowitz, 1998). In proof construction particularly, visual means are much more than an aid to understanding; they can be resources for discovery and justification, and even for proving (Arzarello et al., 1998; Giaquinto, 2007). Noting that the various relationships embedded in a good diagram represent real mathematics awaiting recognition and verbalization, a recent study on proof without words (PWWs) suggests that argumentation and thinking can be triggered and enhanced by proof without words (PWWs),
for both pre-service and in-service mathematics teachers (Katz, Stupel, & Segal, 2016).

Because visual proofs illuminate and explain without words, mathematicians have had great interests in discovering visual proofs especially in proving typical algebraic results. For example, as shown in Figure 3a, the proof that the sum of the first $n$ odd natural numbers is $n^2$ can be approached visually by arranging color-coded dots to create squares. Similarly, the Pythagorean theorem has been proven by a very large number of visual configurations. One of the various visual proofs is shown in Figure 3b; the proof stems from a rearrangement of the pieces showing clearly that the total area of $a^2 + b^2$ is equal to $c^2$.

![Figure 3a](image1.png) ![Figure 3b](image2.png)

*Figure 3. (a) A Visual Proof of the Sum of Odd Numbers. (b) A Visual Proof of Pythagorean Theorem.*

Note. The graphs adopted from
http://www.suffolkmaths.co.uk/pages/SoW/1SoWModule14_TeacherGuide.htm
http://mathworld.wolfram.com/PythagoreanTheorem.html

On the one hand, what makes visual representation and diagrammatic reasoning effective is the directness of its interpretation – the relatively great capacity a user has to read off key features of the target structure from the appearance of the diagram (Stenning & Lemon, 2001). On the other hand, diagrammatic reasoning is also considered deceptive (in
particular when it causes one to treat an apparent relationship as being valid in general). Due to its unreliability and particularity, a diagram has for some time been considered merely as a heuristic device, a useful instrument for the discovery, formulation and/or intuitive comprehension of a proof. However, even though diagrammatic reasoning is widely thought to lack a justificative role in proof, Detlefsen (2008) has argued that, “[...] insightful cases have been made for the significance of diagrammatic reasoning as justificative (as distinct from purely heuristic)” (p. 27).

In fact, from a philosophical perspective, visual representations can be seen as adjuncts to proofs, as an integral part of proof, or as proofs (Hanna & Sidoli, 2007). The two visual proofs mentioned above fall into the third type – legitimate and valid visual proofs. In her research, Nardi (2014) found that some teachers have started using graph-based argumentation as part of the learning trajectory towards the construction of proof. Despite the fact that even after acknowledging the beauty of proof without words (PWWs), some people will continue to seek the “traditional logical proof” (Katz, Stupel, & Segal, 2016), a suggested optimal approach is to use logic/proof and visualization simultaneously and to acknowledge them as different ways of thinking (Nardi, 2014).

2.1.4 Formal-rhetorical aspects of proof construction

As mentioned above, in the case of visual proofs or proof without words (PWWs), figures, graphs or diagrams not only explain themselves, but also, by their very communicative nature, invite verbalization and communication. In other types of proof, the proof comes to existence only after a claim for the validity of an utterance (Balacheff, 2008). Selden and Selden (2003) see proof validation as a complex process involving evaluating statements, posing and answering questions, constructing sub-proofs, and recalling
definitions and theorems, followed by a process of validation that is fundamental to proof construction. They argue that “constructing or producing proofs is inextricably linked to the ability to validate them reliably, and a proof that could not be validated reliably would not provide much of a warrant” (p. 9).

Selden and Selden (2013, 2015) also argue that a proof could be divided into a formal-rhetorical part and a problem-centered part. In their view the formal-rhetorical part of a proof depends only on unpacking and using the logical structure of a theorem and its associated definitions and assumptions, while the problem-centered part depends on exploration, genuine problem solving, intuition, and a deeper understanding of the concepts involved in the proof. Therefore exploration associated with problem solving is essential to proof construction. A major feature of the formal-rhetorical part is what Selden and Selden have called a proof framework. Indeed, since every proof can be constructed using a proof framework, they consider constructing proof frameworks as a reasonable place to start.

They call the remaining part of a proof the problem-centered part. It is the part that does depend on genuine problem solving, intuition, and a deeper understanding of the concepts involved (Selden & Selden, 2009; 2011). According to Inglis and Alcock (2012), students often have insufficient understanding of what constitutes a proof. The difficulties they have with proof validation may be associated with a tendency to overvalue algebraic manipulations (the local aspects of proof) and thus to allocate too little attention to the logical relationships between the various components of the proof (the global aspects of proof).

Weber and Mejia-Ramos (2011) propose two broad strategies that can be adopted when reading proofs: zooming in and zooming out. When a mathematician zooms out they do not focus on the logical detail of the argument, but on what Rav (1999) has called
methodological moves: encapsulated strings of logical derivations that together form coherent chunks of the whole argument. Based on introspective reports given by practicing mathematicians, Weber and Mejia-Ramos (2013) reported that they would adopt a zooming-out strategy specifically at the start of an attempted proof validation. Participants in his study suggested that adopting such a strategy allowed them to gain an overview of the structure of the proof, before moving on to a zooming-in strategy.

Indeed, the expectation of “zooming-in and zooming-out” is that a zooming-in strategy would involve large numbers of shifts of attention between consecutive lines, but few shifts of longer distances, whereas a zooming-out strategy would involve large numbers of shifts of attention of distance greater than one line (Inglis & Alock, 2012).

2.2 Proof Reconstruction

Practicing mathematicians have paid considerable attention to the concepts of understanding, remembering, and reconstructing proofs (Byers, 2010; Manin, 1998; Thurston, 1995), but there is limited literature on the application of these concepts in mathematics education. Gowers (2007) proposes a critical dimension of proof, which he terms the width of a proof, and underscores the relationship between width of a proof and its memorability and reconstructability.

2.2.1 Width of a proof

Once a proof is constructed, it can clearly be seen as having length and depth, where length refers to the total number of words, symbols and numerals, and depth refers to the richness of its connections to other mathematical ideas and in particular to other mathematical domains (Hanna & Mason, 2014). Gowers (2007) suggests a third dimension,
the “width” of a proof, referring to the number of distinct main ideas one has to keep in mind (or memorize) in order to remember a proof.

This study focuses on the concept of width of a proof, given that the length and depth of a proof are often evident, whereas the width of a proof (Gowers, 2007) is often unclear and unknown. More importantly, from a cognitive and educational perspective both, the most productive focus when dealing with a proof is on the distinct key ideas that must be impressed on students’ minds to foster their mathematical understanding (Hanna & Mason, 2014).

The notion of width introduced by Gowers (2007), is borrowed from theoretical computer science, where it represents the amount of storage space needed to run an algorithm on a computer. Applying it to proof, Gowers defines the width of a proof as the number of steps or step-generating thoughts that one has to hold in one’s head at any one time. When a proof is communicated, in either written or oral form, the proof could have a high or low width. Analogically speaking, as artificial intelligence would benefit greatly from a new paradigm of searching for ultra-low-width computations (Gowers, 2007), a proof of low width, with fewer pieces of information to carry in one’s head, would be more useful than one of high width (Hanna, 2013). The advantages of a low-width proof lie at the heart of proof presentation and reconstruction, by making the proof more transparent and more easily remembered (Gowers, 2007).

2.2.2 Memorability and proof reconstruction

The most influential theory of education at present is undoubtedly constructivism in its various forms. To create opportunities for constructing knowledge for learners, many or most mathematics educators have focused on problem-solving, instruction by discovery, and
cooperative learning, yet shying away from rote memorization, drill and practice. Gowers (2007) reminds us that, “The notion of understanding a piece of mathematics is closely related to how we memorize it” (p. 48). There is a great difference between “rote memorization” and “memorization with understanding”, the latter having several virtues (Fried, 2010). Nevertheless, it is fair to ask, coming from a constructivist viewpoint, whether memorization deserves a place in the teaching of proof. The simple answer is that, in order to retrieve and reconstruct information, one has to gather and store first. A more sophisticated answer to the question has been offered by Gowers (2007), “It is obviously easier to remember a proof if one understands the argument; it is less obvious, but true, that one’s understanding of an argument is greatly advanced if one commits the proof to memory” (p. 40).

Gowers then reiterates that the degree to which we understand a piece of mathematics is closely related to how we memorize it. In proof reconstruction, what makes some mathematical proofs much easier to remember than others is that they have low width (Gowers, 2007). As Hanna and Mason (2014) put it, when the width of a proof is lower, its memorability is greater, and its reconstruction will thus be easier. This makes the width of a proof pedagogically significant – and width is highly linked with the notion of key idea.

### 2.2.3 Key idea

The notion of key idea in proof construction and reconstruction stems from both mathematical and pedagogical considerations. Gowers (2007) states that:

Some mathematics problems have the interesting property of being very hard, until one is given a hint that suddenly makes them very easy. The solution to such a problem, when fully written out, may be quite long, but if all one actually needs to remember, or to communicate to another person, is the hint, then one can have the sensation of grasping it all at once. (p. 55)
In this context, the “hint” is an idea, such as a mathematical fact or concept, or a specific proving approach or technique. Such ideas do not spring from nowhere (Gowers, 2007), rather they may come, in fact, from a part of mathematics not initially thought to be related to the proof under consideration.

Lai and Weber (2014) reveal that mathematicians value and emphasize the presentation of the main ideas of a proof over that of a completely rigorous proof. Similarly, mathematics educators have discovered that to help students understand and remember proofs it is more beneficial to put an emphasis on the main ideas contained in the proofs than to teach the students how to build valid sequences of logical steps (Durand-Guerrier et al., 2012; Hanna & Mason, 2014; Hemmi, 2008; Knipping, 2008; Malek & Movshovitz-Hadar, 2011). Furthermore, Duval (2007) suggests using main ideas to overcome students’ mental blocks in proof construction. Such ideas described above are “key ideas” which carry the flow of information in mathematical proof (Detlefsen, 2008) and capture the gist of a proof (Robinson, 2000). Though Gowers and many researchers use “main ideas”, “big ideas”, or “critical ideas” to describe “the hints” in a proof, they are simply using different terms, and for the sake of consistency and simplicity this thesis uses the term key ideas.

Nevertheless, the term key ideas may mean somewhat different things to different people. Some scholars refer by that term to the most important mathematical ideas, methods, or strategies used in a proof, whereas others have in mind an outline, overview or architecture of a proof. The former maintain that a proof can foster understanding more successfully if it is constructed on the basis of well understood and internalized key mathematical ideas (Gowers, 2007; Hanna & Mason, 2014; Mason et al., 1985; Yang & Lin, 2008). The latter, while also focusing on the central constructive idea, propose a general
approach in which a proof task is broken into chunks to highlight its overall structure (Leron, 1983, 1985; Mejia-Ramos et al., 2012; Robinson, 2000; Selden & Selden, 2015).

Specifically, for the purpose of obtaining high-level understanding, Robinson (2000) suggests that it is essential to ignore low-level details while highlighting the overall structure of the proof. Selden and Selden (2015) suggest the construction of a proof framework or outline specifically to reduce the burden on the working memory of the mathematician or student.

Raman (2003) understands the term key idea differently. For her, it refers to a heuristic idea that one can map to a formal proof with an appropriate degree of rigor. This concept of key idea is related to that of explanatory proofs, and brings together a private aspect (engendering understanding) and a public aspect (containing sufficient rigor) (Hanna & Mason, 2014), and therefore offers a way to both understanding and conviction.

Concerning the relation of a key idea of a proof to constructing a formal proof, Raman et al.’s view, is that “while a key idea engenders a sense of understanding, it does not always provide a clue about how to write up a formal proof” (p. 2-156), whereas for Gowers (2007) an idea (key idea) does help in the construction of a proof. According to him, an idea may seem arbitrary at first, but upon reflection mathematicians “come to see why it is not arbitrary” (p. 48) and proceed to construct a proof. He also clearly states, “So perhaps when mathematicians talk of proofs containing ideas what they are referring to is demonstrations of how to generate a step that would otherwise not have sprung to mind” (p. 48).

2.2.3.1 Key ideas, conceptual understanding, and procedural fluency

Procedural knowledge is often defined as the ability to execute action sequences to solve problems, whereas conceptual knowledge is defined as implicit or explicit
understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain (Rittle-Johnson, Siegler, & Alibali, 2001). Regardless of the traditional dichotomy in mathematics education between conceptual understanding and procedural skills, recent research suggests both that, “Conceptual understanding (i.e., the comprehension and connection of concepts, operations, and relations) establishes the foundation, and is necessary, for developing procedural fluency (i.e., the meaningful and flexible use of procedures to solve problems)” (NCTM, 2014, p. 7), and that procedural knowledge is a good and reliable predictor of improvements in conceptual knowledge (Rittle-Johnson, Schneider, & Star, 2015). Rittle-Johnson et al. (2001) discuss the beneficial relationship of mutual reinforcement between conceptual and procedural knowledge, showing that increased conceptual knowledge improves procedural knowledge, which in turn can lead to further improvements in the first.

Acknowledging the bidirectional relation between the two, Kieran (2013) argues that procedures have conceptual aspects that, historically and pedagogically, have been insufficiently appreciated: procedures are in themselves conceptual in nature, and procedures are regularly being updated, revised and extended by means of conceptual elements. Kieran suggests that procedures are not static entities that need be executed mindlessly and clerically without comprehension, but rather that procedures can be executed mindfully with comprehension.

As mentioned earlier, in proof construction and reconstruction, the notion of key idea could refer to the mathematical ideas, methods or strategies used in a proof, which are associated with conceptual understanding; it could also refer to an outline or architecture of a proof, which is associated with procedural knowledge. Baroody et al. (2007) argue that “big
ideas” are integral to achieving a deep understanding of both concepts and procedures, and therefore are particularly important in promoting all aspects of mathematical proficiency.

2.2.4 Generic proof

A number of scholars have discussed the notion of generic example, generic proof, and generic proving. According to Leron and Zaslavsky (2009), “A generic proof, is roughly, a proof carried out on a generic example” (p. 2-53). They went on to state that one of the many strengths of generic proof is that “they enable students to engage with the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issues of full generality, formalism and symbolism” (p. 2-56). They also stated that among the weaknesses of a generic proof is the fact that “it doesn’t really prove the theorem” (p. 2-56). Notwithstanding any weakness generic proofs might have, a consensus has been reached among researchers that generic proofs and proving with the aid of an example can help students both understand and create a proof through a guided discovery process. More importantly, the advantages lie in the heart of generic proofs’ capacity to render the main ideas of a general proof accessible to a particular audience (Balacheff, 1988; Kempen & Biehler, 2015; Leron & Zaslavsky, 2009, 2013; Malek & Movshovitz-Hadar, 2011; Mason & Pimm, 1984; Rowland, 1998).

Indeed several studies have shown that providing appropriate examples to students has great potential in terms of helping them gain insights into the main idea(s) of a proof, thereby helping in moving towards acceptable deductive arguments and eventually formal proof (Biehler & Kempen, 2013; Leron & Zaslavsky, 2013; Zaslavsky et al., 2016).
2.2.5 Reflection and reconstruction

Students learn by doing and by thinking about what they did (Papert, 1980). Several scholars call on students’ reflections on the development of their informal or formal proofs so as to gain insights from their own proving practices. Healy and Hoyles (2000) who carried out research on high school students’ understanding of proof inferred that it would be beneficial to students if mathematics educators were to engage students with proof while discussing with them the idea of proof at a meta-level, in terms of its meaning, generality and purposes. Bartlo (2013) explored ways in which teaching students proof and proving can contribute to the learning of mathematical content. She reached the conclusion that engaging students in identifying and reflecting on the key idea of a proof can create opportunities for learning mathematics.

In summary, it is evident that, in proof construction and reconstruction, a balance is needed between how much to focus on proof ideas and how much to deal with rigorous detailed presentations of proofs (Hemmi, 2008). In the teaching and learning of proof, much attention has been drawn to proofs that promote and deepen students’ mathematical understanding. The next section focuses on the pedagogical aspects of proof.

2.3 Proofs That Promote Mathematical Understanding

The functions and roles of proof and proving have been extensively discussed since the early 1990s. Proofs are more than syntactic derivations, statement-form of theorems, or guarantees of truth (Gower & Neilson, 2009; Hanna, 2000; Rav, 1999); they are seen as sources of understanding and insight, or as repositories of knowledge (Zaslavsky, et al., 2012). For mathematics educators, then, proof and proving have the potential to deepen mathematical understanding among students (Hanna & de Villiers, 2008; 2012).
Bell (1976), De Villiers (1990), and Hanna and Jahnke (1996) present a richly differentiated view of proof and proving:

- **Verification** (concerned with the truth of a statement)
- **Explanation** (providing insight into why it is true)
- **Systematisation** (the organisation of various results into a deductive system of axioms, major concepts and theorems)
- **Discovery** (the discovery or invention of new results)
- **Communication** (the transmission of mathematical knowledge)
- **Construction of an empirical theory**
- **Exploration of the meaning of a definition or the consequences of an assumption**
- **Incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective.**

Based on this view of proof and proving, every student entering the world of mathematics starts with the fundamental functions of verification and explanation, to see not only that a statement is true, but also why it is true (Hanna, 2000).

From a mathematician’s perspective, Auslander (2008) singles out three functions of proof and proving – certification, explanation, and exploration. Exploration has also been advocated by Lakatos (1976) who argued that a proof goes through a process, which is a “zig-zag of discovery” that “cannot be discerned in the end product” (p. 42). By certification, Auslander refers not only to the verification aspect of proof, but also to the fact that mathematics could not be a coherent discipline without an additional process of certification. He adds that explanation and exploration are often intertwined. Many researchers have also emphasized explanatory, exploratory and communicatory role of proof (Aigner & Ziegler, 2010; Chazan, 1993a, 1993b; de Villiers, 1990; Hanna, 1989; Hersh, 1993; Hoyles & Jones, 1998; Mudaly & de Villiers, 2000; Steiner, 1978; Thurston, 1994; Volmink, 1990). In what follows, I focus on the explanatory and communicatory functions of proof.
2.3.1 Proofs that explain

In proof and proving, “verification is proof, but verification might not give reason” (Rota, 1997, p. 186-187). The search for reason or “explanations” plays an important role in mathematical practice (Mancosu, 2001). Hanna (1990) distinguishes between proofs that prove and proofs that explain, putting more weight on explanatory proofs that provide a rationale based upon the mathematical ideas involved, as opposed to proofs that are syntactical, mechanical and merely follow formal logic. Weber (2014) advises mathematics educators in particular to assign more weight to the explanatory than to the justificatory role of proof. As Hersh (1997) stated, “Proof can convince, and it can explain. In research convincing is primary. In high-school or undergraduate class, explaining is primary” (p. 164).

But practicing mathematicians also value the explanatory function of proof. Hanna (2013) points out that most mathematicians see proofs less as correct syntactical derivations and more as conceptual entities that lead to understanding. For instance, the mathematician Robinson (2000) argues that the glory of mathematics is the peculiarly magical explanatory power and psychological effectiveness of elegant and convincing proofs, which he calls “proofs-as-explanations” as opposed to just “proofs-as-guarantee”. His notion of “real proofs” as opposed to “formal proof” is nicely aligned with the concept of “conceptual proofs” that has been widely accepted in the mathematics community and in mathematics classrooms:

To observe how real proofs work, then, we must attend the performances of them, or perform them ourselves, if we can. Proofs are like stories: we must listen to them as they are told and follow the plot. They are dramas: they must be acted. A formal proof is only the score, the script, only the instructions for producing the real proof. The real proof is the series of cognitive events called for in the script. (p. 281)

For Robinson, proofs are essentially performances rather than pure text; and the
explanatory process leading to the cognitive internalization of a proof seems to be far more important, far more interesting and far more challenging than proof-as-guarantees.

### 2.3.2 Proofs that communicate additional mathematical knowledge

Sfard (2008) suggests that thinking can be thought of as internalized communication, while learning can be thought of as the process of changing one’s discursive ways in a certain well-defined manner. In other words, learning is an activity aimed at lasting modification of communicational practices (Zazkis, 2011). Mathematical definitions, proofs, and theorems, as endorsed narratives, are a form of mathematical discourse and communication (Sfard, 2008).

Thurston (1994) calls for greater effort into communicating mathematical ideas in proofs, not only with definitions and theorems, but also with our ways of thinking. In discussing the pedagogical role of proof, Lai and Weber (2014) point out that some proofs transform mathematical knowledge into ways of “representing ideas so that the unknowing can come to know, those without understanding can comprehend and discern, and the unskilled can become adept” (Shulman, 1987, p. 7). Hanna and Barbeau (2010) observe that mathematics educators have often overlooked the fact that a proof may display fresh methods, tools, strategies and concepts, rather than merely demonstrating a result. As Rav (1999) explained:

> The whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving problems, the establishment of interconnections between theories, the systemisation of results – the entire mathematical know-how is embedded in proofs. (p. 20)

Robinson (2000) also emphasizes the communication of new knowledge through proving. He describes proofs and the messages embedded in them as “stories and dramas” (p. 280) and sees such methods, tools, strategies, and concepts as indeed the real meat of
mathematics.

### 2.3.3 Promotion of mathematical understanding

A major concern among mathematics educators is to teach proofs for the purpose of developing in students a degree of mathematical insight and thereby fostering their mathematical understanding (e.g. Hanna, 1990; Hersh, 1993; Thurston, 1994). Understanding is generally taken to refer to the development of connections between ideas, facts or procedures (Burton, 1984; Davis, 1984; Hiebert & Carpenter, 1992). Skemp’s (1987) influential work separated understanding from knowledge and distinguished between “relational understanding” and “instrumental understanding”. Relational understanding means knowing both what and why, whereas instrumental understanding is described as “rules without reasons” (p. 153). In mathematics, each of these understandings has its own set of advantages (Meel, 2003). To improve understanding and reduce the need for re-learning, Skemp (1987) emphasizes the importance of learning relationships and connections, as opposed to a focus on procedural rules.

Skemp’s categories of relational and instrumental understanding have spawned a variety of other categorizations: (a) procedural and conceptual, (b) concrete and symbolic, and (c) intuitive and formal (Meel, 2003). Though mathematics educators have not reached total agreement on the meaning of “understanding”, the consensus is that all knowledge is personally constructed and organized (Pirie & Kieren, 1994). Accordingly, no one can give a student understanding; the student must create her or his own (Cobb, 1988; Kieren & Steffe, 1994; Pirie & Kieren, 1991; von Glasersfeld, 1983). Asiala et al. (1998) suggest that, “The growth of understanding is highly non-linear with starts and stops, and the student develops partial understandings, repeatedly returns to the same piece of knowledge, and periodically
summarizes and ties ideas together” (p. 13).

Based on these beliefs, a variety of instructional practices that relate the classroom environment to the promotion of understanding have been proposed. In seeking new ways of teaching proof, most proposals suggest shifting away from rigour and formality while attempting to develop conceptual understanding by focusing on communication and social processes (Alibert, 1988; Hanna, 1990; Leron, 1983; Movshovitz-Hadar, 1988; Volmink, 1990). Mason and Johnston-Wilder (2004) suggest that the human power of sense making is a valuable resource to be developed by the teacher of mathematics. In fact, students’ learning and thinking can explicitly inform teaching practice (Goos, 2004). Fennema et al. (1996) provide strong evidence that knowledge of children’s thinking is a powerful tool that enables teachers to transform this knowledge and use it to change instruction.

2.4 Summary

The research reviewed so far indicates that understanding the difficulties that undergraduate students encounter when they learn to prove and teaching them effectively how to construct and reconstruct proofs, is a complex and multifaceted enterprise. In the last two decades, in an effort to put an emphasis on the learning of proof with understanding, mathematics educators have attempted to build learning environments that include: (a) a focus on visual interpretation and spatial reasoning, (b) opportunities for collaboration and communication, (c) an emphasis on exploration, (d) the development of conjecturing skills, and (e) an emphasis on explaining why a particular result is true and why it must always be true.

This literature review has also shown that the teaching of proof is challenging as it depends on a variety of factors that are not all well understood. It has also shown that with
the intention of promoting undergraduate students’ proof comprehension, some studies have focused on several concepts of proof, such as, turning students’ attention to the overall form of a proof, emphasizing the structure of a proof, “zooming in” and “zooming out” strategies, relating a proof to specific examples, and being aware of main ideas. All these concepts are closely related to the concept of key idea.

In addition, the literature review has shown that the concept of key idea itself, though mentioned by a few researchers, has not been explicitly investigated to date. My study aims at filling this gap in the research literature. In particular, the study intended to advance the knowledge in the field by providing a characterization of undergraduate mathematics students’ perceptions of key ideas and by presenting the findings of an investigation into the features of proofs that undergraduate mathematics students identify as key ideas and how they might use these key ideas to reconstruct a proof.
Chapter 3

Methods and Procedures

The purpose of this study is to investigate in some detail how and how well undergraduate mathematics students identify key ideas embodied in proofs and how they might use these key ideas to reconstruct a proof. In this section, I begin with a justification of my choice of the methods for conducting research in the classroom, and then present a detailed account of the procedure and methods for data collection and analysis. I also include discussions of the limitations of the study and ethical considerations.

3.1 Research Design

Both qualitative and quantitative methods were used in this exploratory study on the key ideas of proofs. Greene and Caracelli (1997) argue that combining these two method types can expand the scope or breadth of research, offset the weakness inherent in using either approach alone, and provide a fuller understanding of the research problem. In this study, I used primarily qualitative methods while incorporating a complementary quantitative component, as shown in Table 2. First I administered a quantitative survey on a sample of the target population, and then carried out classroom observations and in-depth student interviews.

The quantitative survey had two purposes. First, its goal was to gather data on the use of key idea(s) in proof construction and reconstruction and on students’ awareness of their own development of proof schemes. Second, purposive sampling guided by information on students’ general perception of proof and on their demographics not only allowed me to locate potential interviewees, but also allowed me to modify and refine the subsequent in-
depth interview instrument based on the review and analysis of the survey results.

Table 2

*Study Design*

<table>
<thead>
<tr>
<th>Research questions</th>
<th>Quantitative Data</th>
<th>Qualitative Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Online survey</td>
<td>Student interviews</td>
</tr>
<tr>
<td>1a</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>1b</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3b</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Research questions:
1. (a) What are students’ perceptions of the role of key idea in a proof?  
   (b) What are students’ interpretations of key ideas of a proof?
2. From the students’ perspective, is the notion of key idea in proof associated more with conceptual understanding or with procedural fluency?
3. (a) What is the evidence that students identify and use key ideas in reconstructing a proof?  
   (b) Which features of a proof do students identify as key ideas?

I chose classroom observation and interviews with students as the principal methods of qualitative, face-to-face data collection. Classroom observations allowed me to capture and document students’ work on proof and proving. Semi-structured interviews with students enabled me to collect detailed information about how and why they identified key ideas of a proof and about the factors that might have major impacts on their key idea formulation.

3.2 Participants and Settings

The participants were first-year (n=17) and mixed-year (2nd to 4th) (n=42) undergraduate mathematics students at an urban university in Ontario. The majority of the students were in the program of honours mathematics or mathematics education; they came from three different classes, class A, class B, and class C (See Table 3).
### Table 3

**Characteristics of Participants**

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Class A (n=17)</th>
<th>Class B (n=18)</th>
<th>Class C (n=24)</th>
<th>Total (n=59)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gender</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>11</td>
<td>10</td>
<td>15</td>
<td>36</td>
</tr>
<tr>
<td>Female</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>22</td>
</tr>
<tr>
<td>Subtotal</td>
<td>16</td>
<td>18</td>
<td>24</td>
<td>58</td>
</tr>
<tr>
<td>Age Range</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>≤25</td>
<td>13</td>
<td>14</td>
<td>20</td>
<td>47</td>
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<tr>
<td>&gt;25</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>Subtotal</td>
<td>16</td>
<td>18</td>
<td>22</td>
<td>56</td>
</tr>
<tr>
<td>Program</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Pure Mathematics</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>22</td>
</tr>
<tr>
<td>Applied Mathematics</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Mathematics Education</td>
<td>2</td>
<td>2</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>Mathematics for Commerce</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>Other</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>5</td>
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<tr>
<td>Subtotal</td>
<td>16</td>
<td>18</td>
<td>24</td>
<td>58</td>
</tr>
<tr>
<td>Where completed high school</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Canada</td>
<td>7</td>
<td>10</td>
<td>21</td>
<td>38</td>
</tr>
<tr>
<td>Outside of Canada</td>
<td>10</td>
<td>8</td>
<td>3</td>
<td>21</td>
</tr>
<tr>
<td>Subtotal</td>
<td>17</td>
<td>18</td>
<td>24</td>
<td>59</td>
</tr>
</tbody>
</table>

*Note.* 1. One student skipped the question of gender.  
2. Three students skipped the question of age.  
3. One student skipped the question of program.

The first-year mathematics students in class A were taking a proof-in-transition summer course “Problems, Conjectures, and Proofs”, attended by both mathematics majors and future mathematics educators. The age range was from 19 to 38, consisting of 11 males and five females, with seven out of the 17 having completed high school outside of Canada. The mixed-year (2\textsuperscript{nd} to 4\textsuperscript{th}) mathematics students consisted of 17 males and 25 females, with
11 out of the 42 having completed high school outside of Canada, and were taking two different courses. The age range of the mixed-year students was from 19 to 34.

The mixed-year (2\textsuperscript{nd} to 4\textsuperscript{th}) mathematics students in class B were also taking a proof-in-transition course “Problems, Conjectures, and Proofs” (n=18) that was offered in the Fall because they had not taken this course in the first year of their studies. The mixed-year (2\textsuperscript{nd} to 4\textsuperscript{th}) students in class C (n=24) were taking a geometry course “Advanced Geometry” that included proof and proving as appropriate. It was a fall/winter/spring course.

### 3.2.1 Course information and classroom settings

Because it is important to understand the context and settings of the participants (Marshall & Rossman, 2006), I gathered additional information on each of the three selected courses. Each course was equivalent to a full-year course that ran through two academic terms, containing 19 to 20 sessions of various lengths (See Table 4).

<table>
<thead>
<tr>
<th>Course Information</th>
<th>Proof-in-transition</th>
<th>Proof-in-transition</th>
<th>Advanced geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time offered</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hours per session</td>
<td>1.5</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>Total number of sessions</td>
<td>20</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>Total hours of instruction</td>
<td>30</td>
<td>28.5</td>
<td>60</td>
</tr>
<tr>
<td>Total hours of tutorial</td>
<td>10</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>Number of credits</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>Code</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

*Note.* 1. The summer course was offered from May 16 to August 24.
2. The fall/winter/spring course was offered from September 10 to April 6.

The three classes were selected for three reasons. First, the topics and materials covered in the three courses were highly associated with deductive and inductive reasoning,
the conceptualization of proof, and the methods of proof including proof by cases, proof by contradiction, and proof by contrapositive. Secondly, the mix of novice and mature (2nd to 4th) mathematics students, as well as the mix of mathematics and education majors, allowed me to gather distinctive perspectives of students’ learning experiences and proving practices. Finally, the course instructors favoured an inquiry-based classroom approach that allowed students to work both individually and collaboratively and gave them the opportunity to experience a meaningful cognitive engagement with the content. The course setting offered purposeful learning activities that were designed to help the students understand important ideas and concepts. A lecture-based classroom with little teacher-student or student-student interactions and with passive or limited active listening on the part of the students would not have allowed me to investigate the active participation of the students in the proving processes.

3.2.1.1 Proof-in-transition course

The proof-in-transition summer course and the Fall/winter/Spring course “Problems, Conjectures, and Proofs” were taught by the same instructor. The goals of these two courses were to provide experiences for students to learn and read proofs, learn to communicate their ideas, and learn to cope with and resolve difficulties in dealing with novel situations. The methodology of the courses included the use of a guided discovery approach. Class sizes were kept small to facilitate communication.

The classroom settings of these two courses were very similar: lecture rooms with tables and seats in rows, blackboards at front or two sides of the rooms. The instructor intended to use lecture rooms with multiple boards for small groups to present their work.
Although such a lecture room was unavailable for the summer course, the students still had opportunities to work together and communicate their ideas on the board with limited space.

3.2.1.2 Advanced geometry course

The advanced geometry course focused on explorations of plane geometry, spherical geometry, and some hyperbolic geometry, and the critical role of transformations (symmetries) in all of these geometries. A related goal of the course was to help students develop and expand their capacity in spatial reasoning, as well as general performance in mathematics, and general creativity and problem solving. Students were invited to reflect on the teaching and learning of geometry and to generate their own project throughout the course. The students were assessed by every-other week assignments, project presentations with topics and contents generated by students themselves, as well as project reflection and overall course reflection. There were no mid-term or final exams.

The classroom setting of the geometry course was unconventional: seven large round tables with chairs around; five large boards mounted onto the three walls of the room. In addition, the classroom had many manipulatives and visual aids associated with geometry.

3.2.1.3 Codes for courses and participants

To ensure confidentiality, codes were given to each class and participant. Particularly, all participants were coded starting with a class code, followed by a number assigned to each participant. The proof-in-transition summer class “Problems, Conjectures, and Proofs” was coded as A, the proof-in-transition fall/winter/spring class “Problems, Conjectures, and Proofs” was coded as B, and the advanced geometry class was coded as C. For instance, C2 refers to a student numbered 2 in the advanced geometry class.
3.2.2 Instructors’ pedagogical perspectives

3.2.2.1 The instructor of the “Problem, Conjectures and Proofs” course

The instructor of the “Problem, Conjectures and Proofs” course has been teaching various mathematics courses in the university for 30 years, working one on one, with small, medium and large groups. He was one of the two initiators of the proof-in-transition course and has been teaching one or two sessions each year. When he reflected on the teaching, he revealed that he used to think of teaching using an image of theatre and performance before an audience. “I enjoyed (and from time to time continue to enjoy) the thrill of having the rapt attention of an audience but don’t any longer aspire to this kind of performance as a means of teaching.” He now consciously aims for a conversation in the classroom. In the interest of mathematical communication, he delves into honest exchange with students sharing their questions and concerns and calling on knowledge and personal experience in trying to address them, in and outside of the classroom.

Frequently, the instructor invites the students to engage in the tasks and asks them to collaborate in the development of the mathematical ideas. He also identifies what he found difficult and invites the students to confront what is unclear to them. He believes that the question of why a student should make the effort to learn specific topics needs to be addressed fully and class presentations which answer that question are of particular value.

3.2.2.2 The instructor of the advanced geometry course

The instructor of the course is a geometer and mathematics educator. He has been teaching various mathematics courses in the university for more than 30 years with special interests in the teaching of geometry. He believes that students should learn mathematics by doing and communicating mathematics, including by asking their own questions, and then
seeking ways to answer them. “Lectures are a poor way for most students to learn.” To seek alternatives, he works to provide a safe space for students to explore the ideas, make conjectures, and ‘own’ the material with their questions. Particularly, he is interested in using The Geometer’s Sketchpad, manipulatives and visual aids to allow students to touch, play, and observe so as to develop ownership of their learning. With the awareness of students’ struggles with the material and learning, he also provides space and time for rewrites and late submissions of assignments to avoid cutting off students’ interests and efforts.

When it comes to proofs, he refers himself as ‘recovered logician’ given that he switched advisors multiple times – moving from a sequence of logicians to a generalist with broad interests. The overall axiomatic approach, to him, is not a reasonable way to first learn a subject and perhaps not even the best way to re-learn. As he put it through one of the email discussions, “‘formal proof’, while important, is over-rated.” He prefers that students carry out local explorations of a network of properties and implications and make the connections between them.

3.3 Data Collection Procedures

To enhance generalizability and usefulness of the study (Marshall & Rossman, 2016), my data were collected from multiple sources: (a) an online survey, (b) classroom observations in the form of audio recordings and field notes, (c) structured interviews with students of 45 to 60 minutes that were audio recorded, (d) student work including group work samples, individual assignments, quizzes, and written tests, (e) follow-up questions and email discussions, and (e) my own research journal. A summary of the research questions and the data collection methods can be found in Table 2. A description of each data collection technique follows.
3.3.1 Online student survey

The online survey consists of 26 Likert-scale items, from strongly disagree to strongly agree, and six questions to gather demographic data (see Appendix A). For the students who attended the advanced geometry class, one item was added to the survey to gather data on use of geometric software. The survey was developed to gather information about five major issues (see Table 5): (1) perception and use of key ideas of a proof (five items, e.g., “When I read a proof, I try to identify its key ideas”), (2) students’ perceptions of proof (five items, e.g., “Mathematical proofs often include ideas, methods, and strategies”), (3) students’ proving practices (11 items, “I had enough opportunities to practice writing proofs in university”), (4) students’ awareness of their proof schemes (four items, e.g., “Examples convince me that a mathematical result may be true”); and (5) use of the Geometer’s Sketchpad (one item).

Table 5

<table>
<thead>
<tr>
<th>Survey Statements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aspect</td>
</tr>
<tr>
<td>On key ideas of a proof</td>
</tr>
<tr>
<td>On students’ perception of proof</td>
</tr>
<tr>
<td>On students’ proving practices</td>
</tr>
<tr>
<td>On students’ proof schemes</td>
</tr>
<tr>
<td>On GSP*</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

*Note. *This item was administered only to class C.

The statements used to gather students’ perceptions of proof were adapted from previous studies (Almeida, 2000; Hemmi, 2006), while the rest of the statements were
developed by me and tested with experienced researchers as well as with a small group of university mathematics students before its use during the study.

The survey was created using the online survey software, SurveyMonkey. At the beginning of the term, all participants were informed of the purpose of the survey and the instruments that were used to complete it. The survey was administered in the classroom and completed on the same occasion. All 59 participants completed the online survey. Of the 26 survey items, five items on key ideas in proof are relevant to address the research questions. Another five items on students’ interests and learning experiences of proof and proving are also relevant as background information about students’ learning of proof in high school and university.

3.3.2 Classroom observations

Observation offers a firsthand account of the situation under study and, when combined with interviewing and document analysis, allows for a holistic interpretation of the phenomenon being investigated (Merriam, 2009). I conducted classroom observations throughout the study on a weekly or bi-weekly basis, depending on how frequently the classes met. The observations entailed the systematic noting and recording of events, behaviors, and objects in the classroom settings (Creswell, 2013). Being a participant observer allowed me to observe and interact closely with the students to establish an insider’s identity while not controlling or interfering with the activities (Denzin & Lincoln, 2013). The observational record, i.e. field notes (Hesse-Biber & Leavy, 2010), was maintained throughout the study. In the early stages of the qualitative inquiry, I observed the classrooms to discover recurring patterns. After the patterns were identified through early analysis of field notes, focused observation were used at later stages of the study.
Classroom observations in this study primarily focused on how students’ thinking unfolded in a proving process and how proving products were constructed. Special attention was paid to group discussions in which students bounced ideas off each other at the outset of a proving process and to discussions in which students recapped and debriefed a final proving product. In addition to written and audio records of the conversations and discussions that took place in the classrooms, checklists (see Appendix B) were used to keep the observations focused and consistent. Snapshots were taken to document instructors’ revisions and corrections on students’ proof presentations on boards.

3.3.3 Interviews

To achieve a high degree of standardization between interviews (Hesse-Biber & Leavy, 2010), I used a semi-structured approach. Inspired by Almeida (2000) and many others, I designed the interview instrument (see Appendix C) consisting of 20 open-ended questions (including sub-questions) that aimed at exploring students’ perspectives in four aspects: (1) perceptions of proof (e.g., “How do axioms and theorems differ?”); (2) learning experiences and proving practices (e.g., “How do you begin when you encounter a proving task?”); (3) challenges in proof construction and reconstruction (e.g., “What challenges have you faced in proof construction?”); and (4) supports needed in proof construction and reconstruction in and out of classrooms (e.g., “What supports would you expect from course instructors to advance your skills and techniques in proof construction and reconstruction?”). Of the 20 interview questions, five questions were designed to address the use and importance of key ideas of a proof.

The interview questions were modified and finalized based upon the analysis of the survey data. Each interview was 45-60 minutes in length and audio-recorded. In total, 12
student interviews were conducted, including five students in the first-year class and seven students in the mixed-year classes.

3.3.4 Student work

I collected student work including worksheets, individual assignments, group work, mid-term tests, final exams, as well as snapshots of students’ proving products on a board or on paper. Student work was collected from all three classes throughout the entire study. All relevant student work samples were photocopied and organized by course title.

Because student work could be a major source for investigating how students might go about identifying key ideas in a proof, small interventions were conducted for a few classroom exercises and test/exam questions. Based on the teaching materials of each course, five complete and correct proofs were used for students to identify and formulate key ideas. In this thesis, I focus on two proofs for investigation and discussion: (1) The carpet proof of the irrationality of $\sqrt{2}$, and (2) The proof of the irrationality of $\sqrt{k}$ for non-square $k$. In what follows, I explain how each proof was used for the purpose of examining how and how well students identified key ideas in a proof.

3.3.4.1 The carpet proof of the irrationality of $\sqrt{2}$

The carpet proof of the irrationality of $\sqrt{2}$ was discovered by Stanley Tennenbaum in the 1950’s but was made widely known by John Conway around 1990. The proof is beautifully constructed by finding a contradiction in the constructed geometric configuration (see Appendix H). The instructor of the “Problem, Conjectures and Proofs” summer course (class A) was interested in knowing how well students would understand the proof. He incorporated the carpet proof in the final exam and placed it at the end as the last question. The students were not asked to construct the carpet proof during the exam, but rather, they
were asked to read the actual proof and respond to five prompt questions designed by the instructor. The last prompt question was my intervention, “What is the key idea of the proof?” 15 out of 100 points were assigned to the proof in the exam.

The carpet proof was also used in the advanced geometry class (class C). In contrast to the use in the “Problem, Conjectures and Proofs” summer class (class A), the first four designed prompt questions of the proof were used for group investigation and discussion. After the students handed in the group worksheets, they were then asked to work individually to identify the key idea of the proof. The given time for the entire activity was approximately one hour.

Having students from two different classes identify the key ideas of the proof allowed me to examine both the first year and mixed year students’ key idea identification and formulation, and to compare the results from the two sets of the data.

3.3.4.2 The proof of the irrationality of $\sqrt{k}$ for non-square $k$

The proof of the irrationality of $\sqrt{k}$ for non-square $k$ was selected from Charming Proofs: A Journey into Elegant Mathematics (Alsina & Nelsen, 2010, see Appendix G). It was given to the students who attended “Problem, Conjectures and Proofs” summer course (class A) as an in-class exercise. It consisted of three components: 1) reading the proof, 2) responding to the seven prompt questions including identifying the key idea of the proof, and 3) reconstructing the proof. The students were given approximately 40 minutes to read the proof and to respond to the prompt questions designed by the instructor. After the students handed in the worksheets, they were given approximately 30 minutes to reconstruct the proof. For all the three components of the exercise, the students were asked to work individually only.
Due to tight schedule and fast progression of undergraduate mathematics courses, the proof of the irrationality of $\sqrt{k}$ for non-square $k$ was the only proof used for students to identify the key ideas of the proof and also to reconstruct it.

### 3.3.5 Follow-up questions and email discussions

Three follow-up questions (See Appendix D) were designed as part of the study for mixed-year students in “Problem, Conjectures and Proofs” fall/winter/spring class (class B, $n=18$) and for the advanced geometry class (class C, $n=24$). The purpose of creating the follow-up question sheet was to gather information about students’ perspectives on the proofs that have been done during the course and particularly to gather more information about students’ perceptions of the notion of key idea. The follow-up questions included open-ended questions that invite students to reflect on the process of constructing a proof in the class, as well as questions that have appeared in the online survey to identify patterns or changes.

The follow-up question sheets were distributed at the end of the course and collected on the same occasion. 35 responses were collected with 17 responses from class A and 18 responses from class B. Email discussions were also conducted with three participants.

### 3.4 Data Analysis

The survey data, consisting of 26 Likert-scale items in total (including an item on GSP for class C only) and six demographical questions, was analyzed on www.surveymonkey.com. By merging the extremes of five response categories (from strongly disagree to strongly agree) into three categories, responses to the five items about the notion of key idea were analyzed and organized by agree, neutral, and disagree. Responses between first-year and mixed-year ($2^{nd}$ to $4^{th}$) mathematics students on the use of key ideas (five items) and on their proving practices (five items) were compared. In
particular, the disagree responses between first-year and mixed-year (2nd to 4th) students on the five proving practice items were also compared.

Students’ responses to the interviews were analyzed using NVivo 10 software for qualitative analysis to explore themes and patterns of responses. Given that data collection and data analysis are iterative process (Hesse-Biber & Leavy, 2010), coding was started as soon as I began the data collection. To seek associations, NVivo 10 was also used to quantify qualitative codes identified in interview responses (Driscoll et al., 2007; Hesse-Biber, 2010) and to integrate them with the associated survey responses. For example, three interpretations of key ideas from students’ perspective were identified, supported by both the survey and interview data.

The analysis of student work on the two proofs started separately yet ended simultaneously. The unit of analysis was a proof statement. Each participant’s work was divided into proof statements and grouped in categories using the constant-comparative method (Kolb, 2012). This process continued until a set of categories was formed. For example, for the carpet proof of the irrationality of $\sqrt{2}$, by breaking participants’ responses into proof statements, three categories were formed: the key ideas that focus on methods, the key ideas focused on ideas, and the key ideas focused on details. Each unit was compared with other units or with properties of a category. Analysis began during data collection and continued after its completion. The comparison of units was adapted, changed, and redesigned as the analysis proceeded.

3.5 Validity

To address issues of internal validity, the analysis of student responses into proof statements and categories was first carried out by myself and subsequently discussed with the
two instructors and an additional professor of mathematics. This thorough peer debriefing (Creswell, 2013) of the assignment of student proof statements to any one category helped to enhance the credibility of the findings and reduce any research bias. As advocated by Creswell (2013), I also kept detailed records of participants’ responses to provide a clear decision trail and thus ensure that interpretations of data are transparent. Since validity is not absolute but rather a matter of degree, I believe that the procedures of having independent investigators look over many aspects of the study and of keeping detailed records added to the overall validity of the findings.

3.6 Limitations of the Study

This study was conducted in one university where only three undergraduate classes were investigated and only two proofs were explored. Therefore the results may not be generalizable to every undergraduate mathematics class and to every form of proof. Nevertheless the results from the classes and the proofs that were investigated do provide insights into students’ perceptions of key ideas, and to their ability to identify key ideas in proofs and to re-construct proofs. These insights may be helpful and useful to mathematics educators.

3.7 Ethical Considerations

Each participant received an information letter with a consent form (see Appendix E). All participants had the option, without negative consequences, to withdraw from the study as well as to decline to answer any specific questions or to participate in any parts of the study.

The research has been reviewed and approved for compliance to research ethics protocols by the Research Oversight and Compliance Office – Human Research Ethics
Program of University of Toronto, and the Human Participants Review Subcommittee (HPRC) of a large urban University. To ensure confidentiality, all identifying information received from participants has been kept on a secure server and used only for the purpose of initiating contact. All participants were referenced by a code or pseudonym. The hard copy of identifiable or confidential data has been kept securely under lock and key at the University of Toronto. I have controlled access to the data during and after the research and work with de-identified data at all times.
Chapter 4

Findings and Discussion

The purpose of this study is to investigate in some detail how and how well undergraduate mathematics students identify key ideas embodied in proofs and how they might use these key ideas to reconstruct a proof. In this section, I first report on students’ backgrounds and previous experience with proof, then organize and present the results of the analysis by research questions with the mix of the qualitative and quantitative data, followed by discussion of the findings.

4.1 Research Question 1a: What are undergraduate mathematics students’ perceptions of the role of key ideas in a proof? 1b: What are undergraduate mathematics students’ interpretations of key ideas of a proof?

To gather background information about students’ interests and learning experiences of proof and proving, five of the 26 survey items, I1 to I5, examined students’ proving practices in high school and in university (see Table 6). For the convenience of presenting the results, the five statements are re-numbered and re-ordered, and therefore, the numbering is different from the actual survey (Appendix A).

Compared to the students from class C, more students from classes A and B reported that constructing a mathematical proof interests them (I1). Although 53% of class A students would like to see more mathematical proofs in university (I2), only 17% of class C students reported so. While as few as about a quarter of the students from all classes responded that they had opportunities to learn about different kinds of proof in high school (I3), even fewer students claimed that they had enough opportunities to give oral explanations of
mathematical results in university (I5). However, 34% more class C than class A students reported that they had opportunities practice writing proofs in university (I4).

Table 6

*Responses (%) to the Statements about Proving Practice*

<table>
<thead>
<tr>
<th>Item</th>
<th>Response</th>
<th>Class A (n=17)</th>
<th>Class B (n=18)</th>
<th>Class C (n=24)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1: Constructing a mathematical proof interests me.</td>
<td>Agree</td>
<td>47</td>
<td>56</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>41</td>
<td>22</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>12</td>
<td>22</td>
<td>54</td>
</tr>
<tr>
<td>I2: I would like to see more mathematical proofs in university.</td>
<td>Agree</td>
<td>53</td>
<td>28</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>41</td>
<td>61</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>6</td>
<td>11</td>
<td>44</td>
</tr>
<tr>
<td>I3: I had opportunities in high school to learn about different kinds of proof.</td>
<td>Agree</td>
<td>23</td>
<td>28</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>12</td>
<td>22</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>65</td>
<td>50</td>
<td>67</td>
</tr>
<tr>
<td>I4: I had enough opportunities to practice writing proofs in university.</td>
<td>Agree</td>
<td>12</td>
<td>22</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>41</td>
<td>28</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>47</td>
<td>50</td>
<td>21</td>
</tr>
<tr>
<td>I5: I had opportunities to give oral explanations of mathematical results in university.</td>
<td>Agree</td>
<td>12</td>
<td>22</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>29</td>
<td>28</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>59</td>
<td>50</td>
<td>54</td>
</tr>
</tbody>
</table>

*Note.* * One respondent from Class C did not respond to item I2.

4.1.1 Students’ perceptions of the role of key ideas

Five of the 26 survey statements, I6 to I10, examined to what extent key ideas might promote proof construction and reconstruction, from students’ perspective (Items were renumbered as mentioned above). As shown in Table 7, when reading a proof (I6), most of the students from all three classes agreed that they attempted to identify its key ideas. More students from class C disagreed than from the other two classes. After constructing a proof (I7), about half of the students from all classes reported that they reflected back on the key
ideas used. More of the class C than class A and B students disagreed. Almost three quarters of the students across classes agreed they were less likely to be lost when reconstructing a proof if they remembered the key ideas (I8), a dominantly conceptual approach. Fewer, but still a majority of students, said that they needed to have taken notes on the key ideas in order not to get lost (I9), a predominantly procedural approach. When asked whether seeing the overall structure of a proof is more important than following the details in each step (I10), class A and B students seemed to prefer overall structure while class C students seemed to favour details.

Table 7

Responses (%) to the Statements about the Role of Key Ideas

<table>
<thead>
<tr>
<th>Item</th>
<th>Response</th>
<th>Class A (n=17)</th>
<th>Class B (n=18)</th>
<th>Class C (n=24)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>I6: When I read a proof, I try to identify its key ideas.</td>
<td>Agree</td>
<td>76</td>
<td>61</td>
<td>67</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>24</td>
<td>33</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>0</td>
<td>6</td>
<td>25</td>
</tr>
<tr>
<td>I7: After constructing a proof, I tend to identify the key ideas used in it.</td>
<td>Agree</td>
<td>59</td>
<td>50</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>35</td>
<td>44</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>6</td>
<td>6</td>
<td>26</td>
</tr>
<tr>
<td>I8: When I reconstruct a proof, I am less likely to get lost if I already know the key idea used in the proof.</td>
<td>Agree</td>
<td>65</td>
<td>72</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>23</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>12</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>I9: When I reconstruct a proof, I would likely get lost unless I took notes of the major steps when I saw it proved in class.</td>
<td>Agree</td>
<td>53</td>
<td>55</td>
<td>79</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>18</td>
<td>28</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>29</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td>I10: It is more important for me to see the overall structure of a proof than to follow the details in each step.</td>
<td>Agree</td>
<td>47</td>
<td>50</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Neutral</td>
<td>35</td>
<td>33</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>Disagree</td>
<td>18</td>
<td>17</td>
<td>42</td>
</tr>
</tbody>
</table>

* One respondent from class C did not respond to item I7.
The comparisons of the agree and disagree responses between the three classes are shown in Figures 4a and 4b. It is interesting to note that as many as 75% of the students in class A (the first year students) claimed they tried to identify key ideas (I6); 25% were neutral, and none disagreed with the statement. In contrast, more students in class C (the advanced geometry class) seemed to disagree.

![Comparison of the Agree Responses to I6 to I10](image)

*Figure 4. (a) Comparison of the Agree Responses to I6 to I10. (b) Comparison of the Disagree Responses to I6 to I10.*

The comparisons on statement 1 and 2 between the three classes are shown in Figure 5. Despite that the differences across classes were very small, more students in class A (first year) than class B and C (both mixed-year (2nd to 4th)) claimed that they attempted to identify the key ideas when reading a proof. More students in class A also claimed that they reflected back on the key ideas after constructing a proof. This suggests that more of the first-year mathematics students seemed to pay attention to the key ideas of a proof than the mature students.
Figure 5. Percentage of Agree Responses to the Use of Key Ideas between the Students from the Three Classes.

Note. I6: When I read a proof, I try to identify its key ideas.  
I7: After constructing a proof, I tend to identify the key ideas used in it.

4.1.2 Students’ interpretations of key ideas in a proof

During the interviews and in the follow-up question sheets, in response to what key ideas meant to their learning, some students considered key ideas as “a cognitive tool to make sense of a proof”, as student B7 put it, whereas others look at key ideas as “a script for restating and explaining a proof to others”, as student B8 stated. Despite the variations of these responses, the majority of the students across the three classes acknowledged the importance of the key ideas of a proof in their learning. As student A5 put it, “They [key ideas] were the thing, central to understand the proof in the first place.” To be more specific, I categorized the students’ interpretations of key ideas as follows.

Key ideas are important points. The following responses indicate that several students viewed key ideas as important to their understanding of a proof and their subsequent ability to communicate it. Student B9 stated that key ideas highlight important points in a proof so
as to make the proof clear in one’s mind. Similarly, student C6 claimed that key ideas helped her retain the proof by remembering the most important information, while student A5 outlined them in his work so he could better communicate with the instructor. For student B8, key ideas are “my check if I can make sense of all the work I did especially in the proofs that are quite lengthy”.

*Key ideas are blueprints.* Some students perceived key ideas as an overall design of a proof. For example, student C22 described key ideas of a proof as blueprints. With the blueprints, student C7 stated that she was able to build a proof around them, and student A4 said that the blueprint enabled him to build blocks in order to solve the “puzzle”. Student C21 asserted that, “I can use the blueprints as prior knowledge to build on new proofs.”

*Key ideas are landmarks and pathways.* Other students credited key ideas with helping them finding a path to arrive at a proof. Student C20 believed that key ideas could be seen as landmarks when driving or heading a place because “they act as directions for thoughts that allow me to think and reason between these landmarks.” This resonates with student B1’s description, “Without landmarks, I would get lost or even hit a dead end.” Likewise, student A2 sees key ideas as landmark or even pathway, “It is always important to have ideas of what direction to go at one point. If you know the path from A to B, and C is right beside B, you can use the same path from A to B to get to C.”

*Key ideas are steps to follow.* Finally, there were students for whom key ideas were simply procedures to be followed. Student A14 put it this way: “Key ideas are the steps to follow. My high school teacher often summarized key points in a proof in this way.” Similarly, student A18 considered key ideas as the procedure of completing a proof. With
this belief, both students A14 and A18 just listed the steps of a proof when asked to identify the key ideas.

4.2 Research Question 2: From undergraduate mathematics students’ perspective, is the notion of key idea in proof associated more with conceptual understanding or with procedural fluency?

The two items I8, “When I reconstruct a proof, I am less likely to get lost if I already know the key idea used in the proof”, and I9 “When I reconstruct a proof, I would likely get lost unless I took notes of the major steps when I saw it proved in class”, which were meant to examine students’ perceptions of the extent to which key ideas helped them with conceptual understanding and procedural fluency.

![Graph](image)

**Figure 6.** Percentage of Agree Responses to Conceptual Understanding and Procedural Fluency.

*Note. I8: When I reconstruct a proof, I am less likely to get lost if I already know the key idea used in the proof (interpreted as conceptual understanding). I9: When I reconstruct a proof, I would likely get lost unless I took notes of the major steps when I saw it proved in class (interpreted as procedural fluency).*
As shown in Figure 6, the majority of the students reported that both conceptual and procedural approaches are important to proof reconstruction (I8 and I9). While 26% more students in class C than class A claimed to use a predominantly procedural approach (I8), 7% more class B than class A students claimed to use a dominantly conceptual approach.

Table 8

Responses to I8 and I9 Class Cross Tabulation

<table>
<thead>
<tr>
<th>Class</th>
<th>I8rec</th>
<th>1.00 Disagree</th>
<th>2.00 Neutral</th>
<th>3.00 Agree</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18rec</td>
<td>1.00 Disagree</td>
<td>Count</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>.0</td>
<td>.0</td>
<td>45.5</td>
</tr>
<tr>
<td></td>
<td>2.00 Neutral</td>
<td>Count</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>.0</td>
<td>50.0</td>
<td>9.1</td>
</tr>
<tr>
<td></td>
<td>3.00 Agree</td>
<td>Count</td>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>100.0</td>
<td>50.0</td>
<td>45.5</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>Count</td>
<td>2</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18rec</td>
<td>1.00 Disagree</td>
<td>Count</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>50.0</td>
<td>.0</td>
<td>15.4</td>
</tr>
<tr>
<td></td>
<td>2.00 Neutral</td>
<td>Count</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>.0</td>
<td>100.0</td>
<td>15.4</td>
</tr>
<tr>
<td></td>
<td>3.00 Agree</td>
<td>Count</td>
<td>1</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>50.0</td>
<td>.0</td>
<td>69.2</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>Count</td>
<td>2</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18rec</td>
<td>1.00 Disagree</td>
<td>Count</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>33.3</td>
<td>.0</td>
<td>.0</td>
</tr>
<tr>
<td></td>
<td>2.00 Neutral</td>
<td>Count</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>.0</td>
<td>.0</td>
<td>23.5</td>
</tr>
<tr>
<td></td>
<td>3.00 Agree</td>
<td>Count</td>
<td>2</td>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>66.7</td>
<td>100.0</td>
<td>76.5</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>Count</td>
<td>3</td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18rec</td>
<td>1.00 Disagree</td>
<td>Count</td>
<td>2</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>28.6</td>
<td>.0</td>
<td>17.1%</td>
</tr>
<tr>
<td></td>
<td>2.00 Neutral</td>
<td>Count</td>
<td>0</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>.0</td>
<td>45.5</td>
<td>17.1%</td>
</tr>
<tr>
<td></td>
<td>3.00 Agree</td>
<td>Count</td>
<td>5</td>
<td>6</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>71.4</td>
<td>54.5</td>
<td>65.9</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>Count</td>
<td>7</td>
<td>11</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>% within I9rec</td>
<td></td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Note. I8: When I reconstruct a proof, I am less likely to get lost if I already know the key idea used in the proof (interpreted as conceptual understanding).
I9: When I reconstruct a proof, I would likely get lost unless I took notes of the major steps when I saw it proved in class (interpreted as procedural fluency).
As shown in Table 8, students in class A seemed to perceive the relationship between I8 and I9 differently than most in classes B and C. While 50% or more of B and C students (mature students in years 2 to 4) value both procedure and concept, amongst those who agreed with conceptual understanding in class A, there is an almost even split between those who agreed and disagreed with the importance of remembering procedure. This indicates that the teaching or class makeup of class A might have impacts on how students associated the notion of key idea with conceptual understanding or procedural fluency.

4.3 Research Question 3a: What is the evidence that students identify and use key ideas in reconstructing a proof?

4.3.1 Students’ self-reports on the use of key ideas in a proof

With respect to identification and use of key ideas in the learning of proof, the students’ responses were gathered through student interviews and follow-up question sheets. The follow-up question sheets distributed to the students from class B and C consisted of two questions:

1. You were asked to summarize the key idea of this proof in the class. Do you consciously use or identify key ideas in a proof when you study on your own?  
   Always  Sometimes  Never

2. Do you think the identification of key ideas is an important practice that would help you reproduce a proof that you have seen before? Please provide a few sentences to support your answer.

The results are based on 33 responses, consisting of 16 (out of 18) students from class B and 17 (out of 24) students from class C. When students were asked question 1, almost all reported that they always or sometimes use key ideas in a proof, whereas only two students reported that they never did (See Table 9).
Table 9

*Responses to the Conscious Use of Key Ideas from Classes B and C*

<table>
<thead>
<tr>
<th></th>
<th>Always</th>
<th>Sometimes</th>
<th>Never</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class B</td>
<td>6</td>
<td>9</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>Class C</td>
<td>7</td>
<td>9</td>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>Total</td>
<td>13</td>
<td>18</td>
<td>2</td>
<td>33</td>
</tr>
</tbody>
</table>

The responses to question 2 are grouped based on the responses to question 1. Examples of the responses can be found in Table 10. For students C21, C22, and C19 who chose “Always”, the reason is that key ideas of a proof can be used to build new proofs, theorems, and corollaries, while for student B5, key ideas help her go through the steps of a proof.

Table 10

*“Always” Responses to Why Use Key Ideas from Classes B and C*

<table>
<thead>
<tr>
<th>Always</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Because I can use that prior knowledge to build on new proofs. I have used key aspects of proofs in the past to prove other theorems for both the proof course and this geometry course (C21).</td>
</tr>
<tr>
<td>2. Key ideas in proofs can be used to provide a proof to other theorems and corollaries (C22).</td>
</tr>
<tr>
<td>3. Because I can use it later on for another proof (C19, C13, and C24).</td>
</tr>
<tr>
<td>4. It helps me go through the steps of a proof (B5).</td>
</tr>
</tbody>
</table>

Of those who responded “sometimes” (see Table 11), students B10, C3, and B12 revealed that they only sometimes identified key ideas of a proof due to difficulties that they encountered, including the difficulty to narrow down a few key ideas to one, the lack of explicit guidance, and the lack of understanding of a proof. Student C8 said he identified key ideas of a proof only sometimes because of the tendency of focusing on the proving process,
while student B7 only tried to identify the key ideas for the part of the proof that she didn’t understand. This resonates with student A2:

Sometimes [I identify key ideas of a proof]. If I am interested in what is going on or if it is something that entirely new to me and I feel like I understand it, I would go back and go over it because I want to make sure I understand it. But if it is something that I feel I already understood before going through and writing it out, then I would way less likely to go back and go over it.

As shown in Table 11, the two students (C5 and B3) who claimed they never identified the key ideas of a proof, added that this was so because they either didn’t develop a habit of identifying key ideas of a proof, or they didn’t find it was useful.

Table 11

“Sometimes” and “Never” Responses to Why Use Key Ideas from Classes B and C

<table>
<thead>
<tr>
<th>Sometimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Only sometimes because I still have trouble figuring out what the exact key idea is since most times during a proof I feel like there’s more than one important idea and it’s hard narrowing down the exact key idea (B10).</td>
</tr>
<tr>
<td>2. Because sometimes it is hard to identify them without explicit guidance (C3).</td>
</tr>
<tr>
<td>3. It is not easy to find key ideas since lack of understanding of knowledge (B12).</td>
</tr>
<tr>
<td>4. I tend to focus on just proving instead of thinking about key ideas (C8).</td>
</tr>
<tr>
<td>5. I naturally go through the proof first. For the steps that I don’t understand, I go back and identify the key ideas (B7).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Never</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. I haven’t found a use for it (C5).</td>
</tr>
<tr>
<td>2. Not in the habit (B3).</td>
</tr>
</tbody>
</table>

4.3.2 Evidence of students’ identification of key ideas in a proof

In addition to students’ self-reporting, evidence on students’ identification of key ideas in a proof was also obtained from an in-class activity and assignments given to the proof-in-transition summer course (class A), the first year students. Although the students
were not asked to identify key ideas, two indicated that they paid special attention to: 1) outlining the procedures of a proof presentation as the “key ideas”; 2) highlighting the mathematical ideas used in a proof as the “key ideas”.

Table 12

*Examples of How the Students Identified the Key Ideas of a Proof*

<table>
<thead>
<tr>
<th>Example</th>
<th>The procedure</th>
</tr>
</thead>
</table>
| 1. Student A5 indicated the procedure followed when approaching a proving task. | 1. Given a rectangle with integer dimensions, first **check to see if it is square**.  
2. If it is not a square, call the long side \( n \) and the short side \( m \)  
   2.1. split the rectangle into:  
      - a square with side \( m \)  
      - a rectangle with sides \( m \) and \( (n-m) \)  
   2.2. repeat the process with the rectangle \((m \times (n-m))\)  
3. If it is a square stop. |
| 2. Student A2 read a proof and attempted to list its key ideas. | ![Image](image.png) |

Student A5 is one example of identification of the key ideas in a proof as a procedure (see Table 12). In one of his assignments, he outlined the procedures of one of his proofs as the “key ideas” in order to better communicate with the instructor. Indeed, in all of his assignments, when applicable, he consistently summarized the procedures of his proofs or solutions at the front page of his assignment papers so that the instructor could follow the overall structure of his work. In the procedure, the student indicated how he began with and when he was finished with the problem. In details, he also indicated his point of departure, the details of his investigation of the problem, and the end point of his solution to the problem.
Student A2 is an example of identification of three key ideas in a proof (see Table 12). In class, the students were asked to read a solution to the “cutting log” problem and to discuss why the solution was correct. The problem:

A ten-foot pole is dropped into a milling saw and randomly cut into three shorter poles. What is the probability that these three pieces will form a triangle? (Krantz, 1997, p. 21)

The solution of the problem was one and half pages long and given to students for discussion (See Appendix F). After reading the proof, student A2 jotted down three identified key points in the solution. As shown in Table 12, she successfully identified the use of the property of inequality of triangle that leads to the restriction of any cut poles to be less than five feet.

4.4 Research Question 3b: Which features do undergraduate mathematics students identify as key ideas in a proof?

4.4.1 The proof of the irrationality of $\sqrt{k}$ for non-square $k$

In this section I investigate the students’ work on the proof of the irrationality of $\sqrt{k}$ for non-square $k$ by examining the relationships between how well students identified and formulated key ideas of the proof and their subsequent ability to reconstruct the proof. The students’ work is organized and presented by category. The categories are further explained in the next section. Discussion on each category follows.

The proof of the irrationality of $\sqrt{k}$ for non-square $k$ was given to the students in class A ($n=17$) as an in-class exercise. It consisted of three components: 1) reading the proof, 2) responding to the seven prompt questions including identifying the key idea of the proof, and 3) reconstructing the proof. The students were given approximately 40 minutes to read the proof (see Figure 7) and to respond to the seven prompt questions (see Table 13). After the
students handed in the worksheets (see Appendix G), they were given approximately 30
minutes to reconstruct the proof. For all the three components of the exercise, the students
were asked to work individually only.

The irrationality of $\sqrt{k}$ for non-square $k$

In this proof, we interpret $\sqrt{k}$ as the slope of a line through the origin, as illustrated below.

![Graph showing the irrationality of \(\sqrt{k}\) for non-square \(k\).]

**Theorem:** If \(k\) is not the square of an integer, then $\sqrt{k}$ is irrational.

**Proof.** Assume $\sqrt{k} = \frac{m}{n}$ in lowest terms. Then the point on the line $y = \sqrt{k}x = \frac{m}{n}x$ closest to the origin with integer coordinates is $(n, m)$. However, if we let $p$ be the greatest integer less than $\sqrt{k}$ so that $p < \sqrt{k} < p + 1$, then the point with integer coordinates $(m - pn, kn - pm)$ lies on the line and is closer to the origin since $\frac{m}{n}(m - pn) = \frac{m^2}{n} - pm = kn - pm$, and $p < \frac{m}{n} < p + 1$ implies $0 < m - pn < n$ and $0 < kn - pm < m$. Thus we have a contradiction and $\sqrt{k}$ is irrational. ■

*Figure 7.* The Proof of the Irrationality of $\sqrt{k}$ for Non-square $k$.

Table 13

**Prompt Questions for Learning the Proof of the Irrationality of $\sqrt{k}$**

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>
4. Explain how to use \( p < m/n < p + 1 \) to prove that \( 0 < m - np < n \) and \( 0 < kn - pm < m \). What is the relevance of these inequalities?

5. Verify that \( n(\sqrt{k} - p) = m - pn \) and \( m(\sqrt{k} - p) = kn - pm \). This explains the choice of the point \((m - pn, kn - pm)\).

6. What is the method of the proof?

7. What is the idea of the proof? Limit yourself to three or four sentences.

4.4.1.1 Results and discussions

The results are based on the work of 15 students for whom there is complete data.

The students’ work on this proof is organized by the relationship between how well the students identified and formulated the key ideas of the proof and how well they subsequently used them to reconstruct the proof (see Table 14). The students’ responses were grouped into three categories:

1) Clear key ideas (well-formulated mathematical ideas useful in the proof)
2) Unclear key ideas (imprecisely formulated mathematical ideas with no clear way to use them in the proof)
3) Unhelpful key ideas (well-formulated mathematical ideas irrelevant to the proof)

Table 14

<table>
<thead>
<tr>
<th>Proof Reconstruction</th>
<th>Identification of Key Idea</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Clear</td>
<td>Unclear</td>
</tr>
<tr>
<td>Successful</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Unclear</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Problematic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Following the categories describe above, four categories of relationships between how well students identified and formulated key ideas of the proof and their subsequent
reconstruction the proof are formed: 1) clearly identified key ideas followed by successful reconstruction (A2 and A15); 2) unclearly identified key ideas followed by unclear reconstruction (A3, A5, and A10); 3) unhelpful key ideas followed by problematic reconstruction (9 students), and 4) unhelpful key ideas followed by successful reconstruction (A6).

4.4.1.1.1 Category 1: A clearly identified key idea followed by a successful reconstruction

The responses of students A2 and A15 display a clearly identified key idea of the proof followed by a successful reconstruction. As shown in Figure 8, student A2 offered a clearly identified and well-articulated key idea of the proof by breaking down the key idea into three related elements: 1) an assumption being made – “assume \( \sqrt{k} \) can be presented as a rational number \( \frac{m}{n} \) in lowest terms”, 2) the consequence of the assumption – “then point \((n, m)\) should be the closest point to the origin on the line \( y = \frac{m}{n} x \) for \( x \) and \( y \) integers”, and 3) what contradicts to the assumption – “point \((m-pn, k-nm)\) is also on the line and that \((m-pn)\) and \((km-pn)\) are both integers less than \( n \) and \( m \) respectively, so there is a contradiction”. The logical progression of the three elements was clearly stated and well presented.

Even though the instructor commented on the redundancy of her reconstruction (see Figure 9), this student’s proof reconstruction was successful and seamlessly aligned with the key ideas identified in Figure 8. The same structure and wording of the key ideas appeared in the reconstruction – “assume… then… but”. In fact, the reconstruction was an expanded version of the key ideas with adequate details of reasoning and verifying.
Figure 8. Student A2’s Key Idea of the Proof of the Irrationality of $\sqrt{k}$.

Assume $\sqrt{k}$ can be represented as a rational number, $\frac{m}{n}$, which is in lowest terms.
- Then point $(n, m)$ should be the closest point to the origin on the line $y = \frac{m}{n}x$ for any integers.
- But for an integer $p$ less than $\frac{m}{n}$ the point $(m - pn, kn - pm)$ is also on the line and $(m - pn)$ and $(kn - pm)$ are both integers less than $n$ and $m$ respectively, so there's a contradiction and $\sqrt{k}$ cannot be expressed in lowest terms.

Figure 9. Student A2’s Reconstruction of the Proof of the Irrationality of $\sqrt{k}$.

Assume $\sqrt{k}$ rational, then $\sqrt{k} = \frac{m}{n}$ for some integers $m$ and $n$, $n \neq 0$, and $\frac{m}{n}$ in lowest terms.
- Then for the line through the origin with the slope $\frac{m}{n}$, $(n, m)$ should be the closest integral point on the line to the origin.
- But for an integer $p$, where $p < \frac{m}{n} < p + 1$ and the point $(m - pn, kn - pm)$ is also on the line $y = \frac{m}{n}x$ and from $p < \frac{m}{n} < p + 1$, $0 < m - pn < n$ and $0 < kn - pm < m$ and since $m$, $p$, $k$, and $n$ are all integers, the point $(m - pn, kn - pm)$ has integer coordinates, and is closer to the origin than $(n, m)$.
- Since $y = \frac{m}{n}x$ is linear and passes through the origin, then $\frac{m}{n} - \frac{m - pn}{m - pn}$ is in lower terms than $\frac{m}{n}$.
- But $\frac{m}{n}$ is in lowest terms, so there is no lowest terms for $\frac{m}{n}$ and $\sqrt{k}$ cannot be rational.
Notice that, from the instructor’s perspective, the proof could have stopped when the student reached the point where “the point \((m-pn, kn-pm)\) has integer coordinates and is closer to the origin than \((n, m)\)”.

However, the student continued on to explain that “\(y = \frac{m}{n} x\)” is linear and passes through the origin then \(\frac{kn-pm}{m-np} = \frac{m}{n}\) and \(\frac{kn-pm}{m-np}\) is in lower terms than \(\frac{m}{n}\)”, which did not appear in the original proof presented to the students. She drew the conclusion by stating that, “\(\frac{m}{n}\) was supposed to be in lowest terms, so there is no lowest terms for \(\frac{m}{n}\) and \(\sqrt{k}\) cannot be rational.” It seemed that in order to come to the conclusion, the student needed to see the integer coordinates of the two points in the form \(\frac{kn-pm}{m-np} = \frac{m}{n}\) to argue that now a new point was found such that \(\sqrt{k} = \frac{kn-pm}{m-np}\) in lower terms than \(\frac{m}{n}\). This intention and action to be more explicit and transparent than the original proof shows that the student reconstructed a proof that she understood, and needed to communicate with herself and the instructor about what she considered critical to reach the contradiction.

I asked this student to reflect on what led to the success in understanding and communicating the key ideas of the proof; she stated that:

I have to break the proof into related steps so that I can understand it when I’m reading it, and communicating that is just the process of putting the ideas of those steps back into words. When doing this, I choose phrasing that would help me to understand the proof better. So I think that I did well on those questions because they are similar to my own process of understanding the proofs presented.

As she put it, this in-class exercise asked the questions that she was good at answering due to her own process of understanding proofs, which suggests that her process of understanding and learning proofs was effective. It is important to note that the student
was also comfortable with using words and articulating what she understood, in contrast to some mathematics students who tend to rely on only numbers and symbols when expressing and presenting their mathematical thinking.

Student A15 also clearly identified the key idea of the proof: the assumption that $\sqrt{k}$ is in lowest terms, and the need to search for a contradiction (see Figure 10). Notice that the student made an error in the last sentence “This contradicts the idea $\sqrt{k}$ is irrational”. What he meant was “This contradicts the idea $\sqrt{k}$ is rational”.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Student A15’s Key Idea of the Proof of the Irrationality of $\sqrt{k}$.}
\end{figure}

Despite the error, the student made two points: 1) the contradiction was found by manipulating the coordinates; and 2) new points with integer coordinates were found on the line, which contradicts the assumption. The first point stressed the act of algebraic manipulations, whereas the second point highlighted the verification of the new integral coordinates. Clearly, the student paid attention to the choice and discovery of the new point.

The reconstruction of the proof by student A15 was also successful and well aligned with the identified key ideas of the proof. As shown in Figure 11, he first showed that the goal of the proof is to find a contradiction based on the assumption. He then continued to reconstruct the pieces that lead to the contradiction. With adequate details, the student stated
what the assumption was and worked though the algebraic manipulation to make his way to the contradiction. His reconstruction ended by showing that the coordinates of the new point were integers and “below m and n”. The expression “below m and n” shows his unique way to interpret “a closer point to origin” and to link back to the straight-line \( y = \frac{m}{n} x \).

*Figure 11.* Student A15’s Reconstruction of the Proof of the Irrationality of \( \sqrt{k} \).

4.4.1.1.2 Discussion of category 1: A clearly identified key idea followed by a successful reconstruction

Category 1 shows that two out of 15 students were able to formulate clearly identified and well-articulated key ideas of a proof and to reconstruct the proof successfully. These
outcomes resulted from 1) a good understanding of the proof, supported by an effective learning process and strategy; and 2) the effective communication with the self and others.

Student A2’s process of understanding proofs involves breaking a proof into smaller pieces, phrasing each piece so as to capture what is understood and most critical, and then joining the pieces together through logical connections. Obviously, this process and strategy of understanding and learning proofs was effective. This process – how she managed to understand a proof habitually – also demonstrated an effective way of identifying and formulating key ideas of a proof.

In terms of mathematical communication, the need to convince herself seemed to come first and the need to communicate with the instructor followed. This approach resonates with the three stages of convincing suggested by Mason (1985): convince yourself, convince your friend, and convince your enemy (or sceptic). The student at least convinced herself and her instructor in this learning context, despite the fact that she did not seem to have an enemy. She also seemed comfortable dancing around with words and articulating what she understood, perhaps due to her background in arts.

In terms of student A2’s reconstruction of the proof, despite that the last few lines were being considered redundant, from the student’s perspective, she needed the “unnecessary verification” as a necessary self-explanation to make a strong argument. Similarly, student A15 interpreted “a closer point to the origin” presented in the original proof as the integral coordinates “below m and n”. In fact, in order to describe a mathematical result or idea, students tend to choose a form of an expression that speaks to them the most or could best express what they understand. When students are able to use
their own words to describe and communicate what is in their mind mathematically, they are more likely to understand the proof at a deeper level.

Both students A2 and A15’s proof reconstructions, compared to the original proof, demonstrated some variations. These variations deserve some pedagogical attention given the fact that, for the same proof, students would focus on different parts of it. Indeed, people usually pay focus on what is least understood. For some students one part of the proof might be understood easily and read through quickly, whereas for others, it might be exactly where they get stumped. And sometimes the part gone through quickly might not come back to mind during reconstruction. In student A2’s case, she deliberately added some details that did not appear in the original proof. The “unnecessary verification” reveals where her attention was given. Perhaps the original proof was not explanatory and communicative enough by her standard.

Student A15 obviously paid considerable attention to the choice of the new point and the analysis of the algebraic expressions and manipulations, as opposed to the translation from “$$\sqrt{k} = \frac{m}{n}$$ in the lowest terms” to “it is the closest point to the origin”, and the translation from “the coordinates of the point $$(m-pn, kn-pm)$$ are integers” to “a closer point to the origin is found.” This suggests that, in order to recognize where students struggle and to understand those struggles, both individually and collectively, instructors could attend to students’ various versions of proof reconstructions to analyze what thinking might be going on and to identify the strengths and weaknesses.
4.4.1.1.3 Category 2: Unclearly identified key ideas followed by unclear reconstructions

Unclearly identified key ideas followed by unclear reconstructions were found in the work of three students (A3, A5, and A10). Given that student A3 and A10’s works contain much similarity, I focus on the work of two students (A3 and A5) to discuss the consistency between unclearly identified key ideas and unclear proof reconstructions.

All three students attempted to identify key ideas and to reconstruct the proof, but with limited success. The key ideas they provided were unclearly or poorly stated, and the subsequent proof reconstructions also lacked clarity. For example, student A3 attempted to formulate the key idea of the proof in one sentence, “The idea of the proof is showing that if $\sqrt{k}$ is a square of an integer then we run into a contradiction of having a point on a [the] line even more closer to an [the] origin than initially assumed” (See Figure 12). Despite an error he made stating that, “$\sqrt{k}$ is a square of an integer” in which $\sqrt{k}$ should be $k$, the key idea was well structured and stated by explaining how “we run into a contradiction”. However, what he missed was the assumption. That is to say, he did not specify the exact cause of the contradiction - that the idea is to find a point closer to the origin than any chosen point. As the instructor commented, the key idea could have been expressed more completely.

![Figure 12. Student A3’s Key Idea of the Proof of the Irrationality of $\sqrt{k}$](image-url)
Student A3’s proof reconstruction displayed the components needed to complete the proof. However, the components seemed independent of each other and did not follow a logical progression.

Compared to the two students mentioned above, student A5’s work also had an issue of clarity but what makes it an interesting case is that the student explained where the lack of clarity came from. In part of the key ideas he identified, it reads:

The proof assumes that \( \sqrt{k} \) is rational when \( k \) is not a square and therefore \( \sqrt{k} \) is not an integer. If this is the case then the point \((n, m)\) should be the closest point on the line to the origin. The proof continues to prove that the point \((m-pn, kn-pm)\) is on the line and that \(m-pn\) and \(km-pm\) are integers, which is a contradiction.

Notice that the student stated what the assumption was and what followed based on the assumption. He also stated that the new point was on the line and its coordinates were integers. However, what he missed was that, to contradict the assumption that \((n, m)\) is the closest to the origin, the new point could not be just any integral coordinates, but needed to be closer to the origin.

In his work of proof reconstruction, he first nicely stated the assumption and the consequence of the assumption as he did for the key ideas of the proof. He then clearly defined integer \( p \) as the greatest integer smaller than \( \frac{m}{n} \). Unexpectedly, the proof did not proceed but instead, the student made a written admission, as shown in Figure 13, in fact, the entire proof revolves around the choice of the new point \((m-pn, kn-pm)\) – where it came from (multiplying \( p < \frac{m}{n} < p + 1 \) by \( n \)), where it was located (between \((n, m)\) and the origin), and how it led to a contradiction (it is closer to the origin than \((n, m)\)). The student’s admission is irrelevant to the proof but explains why he did not explain how the existence of the new point contradicted to the in the key idea that he identified. Because the student “completely failed
to find the reason they chose point B [the new point] in the proof”, he had to skip this part of 
the proof and jumped to the conclusion, as shown in Figure 14, “somehow $p < \sqrt{k} < p + 1$
$\implies (x, y)$ which in on the line and $x, y \in \mathbb{Z}$, which is a contradiction.”

Figure 13. Student A5’s Admission.

Figure 14. A Part of Student A3’s Reconstruction of the Proof of the Irrationality of $\sqrt{k}$.

4.4.1.1.4 Discussion of category 2

Category 2 shows that in three out of 15 students’ attempts to identify and formulate 
the key ideas of the proof. They did not manage to state them clearly; and their 
reconstructions of the proof tended to lack clarity in the same fashion. A couple of obstacles 
in the learning of proofs seem to thwart their ability to reconstruct proof: 1) students did not 
express the key ideas of the proof with the intention of using it in future reconstruction; and 
2) students had difficulties in formulating the key idea of the proof both succinctly and 
clearly.
As mentioned earlier, the work of student A3 and others showed that students had difficulties in formulating the key idea of the proof both succinctly and clearly. The students tended to have one and lose the other. One possible reason for this struggle is that identifying key ideas of a proof requires mathematical knowledge and understanding, whereas formulating key ideas of a proof requires argumentation skills and language proficiency. Not all of the students have these skills and competencies. Even for those who have them, the mathematical and language skills and competencies might not be equally strong.

The other possible reason for this struggle is that students are not aware of the lack of clarity in their proof presentations. I do not mean to suggest that criteria concerning clarity should be established and added to assessment of students’ work, but rather to raise the issue that students tend not to consider their readers. They are not accustomed to telling a complete story about a proof (again, convince yourself, a friend and enemy). Students may need to see and discuss what a well-articulated key idea and well-argued proof reconstruction look like. More importantly, students could be invited to revise and polish their work after the first or even second attempt so that they would gradually develop a sense of what clarity could do for them.

Although the three students in category 2 share some commonalities – both the key ideas and the reconstructions lacking clarity, the difference between student A3’s case and the rest is that A3 did not understand the choice of the new point, while the rest seemed to understand most of the proof. This shows that lack of clarity in students’ work might result from lacking understanding of the proof, partially understanding the proof, or simply the struggling with expression. As mathematical educators, it is important to be aware of these possibilities to avoid false judgements through a final product.
4.4.1.1.5 Category 3: Unhelpful key ideas followed by problematic reconstructions

Nine students gave unhelpful key ideas followed by problematic reconstructions. The inappropriately identified key ideas can be grouped into three types (see Table 15). The first two types reflect students’ misunderstandings of the proof: 5 out of 9 students (A1, A7, A8, A9, and A12) mistakenly considered that the key idea of the proof involved the properties of straight lines or linear functions; while 3 out of 9 students (A4, A11, and A13) claimed that the Well Ordering Property was used in the proof. The third type “mechanical procedures” shows a misunderstanding of what key idea refers to: Student A14 simply listed the procedures to complete the proof without addressing the mathematical ideas used.

Table 15

Types of Unhelpful Key Ideas

<table>
<thead>
<tr>
<th>Type</th>
<th>What key ideas focused on</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Properties of straight lines or linear function</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>Well ordering property</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>Mechanical procedures</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

In type 1, student A7 claimed that the key idea of the proof was that “the slope of the straight line cannot be irrational”. However, her proof reconstruction did not reflect the key idea she identified but attempted to follow the original proof. The reconstruction foundered when she was dealing with the inequalities. In contrast to student A7, student A8 did not claim that the slope of a line cannot be irrational, but she stated that, “Unless rise/run (slope) of the line squared equals an integer, the slope must be an irrational number” (See Figure 15). This version of the key idea of the proof lacked sufficient content given that the student
simply restated the premise and conclusion of the proof in a graphical context. The proof reconstruction was incomplete and unsuccessful because partly due to the fact that incorrect integer coordinates \((m-nmp, nm-nmp). (m-pn, kn-pm)\) were used.

Figure 15. Student A8’s Key Idea of the Proof of the Irrationality of \(\sqrt{k}\).

Unlike the students mentioned above, student A9 considered that the key idea was to make a comparison between two straight lines \(y = \sqrt{k}x\) and \(y = (\sqrt{k}-p)x\), where \(p\) is the greatest integer smaller than \(\sqrt{k}\). Consistent with the key idea identified, the student attempted to reconstruct the proof by comparing the two straight lines. The attempt was unsuccessful.

In type 2, three students attempted to make the use of the Well Ordering Property fit in the proving context and considered it as the key idea of the proof. For example, student A4 stated that:

The idea is to prove that \(\sqrt{k}\) is irrational by assuming that it is rational. Using well ordering he proves that there are integer terms lower than the assumed lowest terms \(m\) and \(n\) which when expressed as a ratio equal \(k\). Since he proves there is a set smaller than \((m, n)\) which equals \(k\), then \(k\) is irrational. (See Figure 16)

This student consistently talked about \(“k”\) when he actually meant was \(\sqrt{k}\). He also made use of the Well Ordering Property in his interpretation of the reason why the new point \((m-pn, kn-pm)\) was smaller than \((m, n)\). The Well Ordering Property states that every non-
empty set of positive integers contains a least element. What the student misunderstood was
1) \((m, n)\) represents a point, not a set of positive integers even though both \(m\) and \(n\) are
positive integers; 2) the new point \((m-pn, kn-pm)\) was found by multiplying \(p < \frac{m}{n} < p + 1\) by
\(n\), without using the Well Ordering Property to warrant its existence.

\[
\text{The idea is to prove that } \sqrt{k} \text{ is irrational by assuming it is need rational, using well ordering he proves there are terms lower than the assumed lowest common terms } m \text{ and } n \text{ which when expressed as a ratio equal } k. \text{ Since he proves there is a set smaller than } (m,n) \text{ which equals } k, \text{ then } \sqrt{k} \text{ is irrational.}
\]

\textit{Figure 16. Student A4’s Key Idea of the Proof of the Irrationality of } \sqrt{k} \text{ – The Well Ordering Property.}

In the proof reconstruction, however, student A4 did not mention the Well Ordering Property. This was also the case for students A11 and A13. From my discussion with the instructor, it turned out that the misuse of the Well Ordering Property was possibly due to the fact that this property along with some of its applications had been taught in the previous class. In other words, the students seemed to focus on using recently learned properties that might fit the current proof, rather than working through the process for understanding.

In type 3 “mechanical procedures”, student A14 listed the procedures to complete the proof:

1) Use well-ordering method to let \(\sqrt{k} = \frac{m}{n}\) in lowest terms;
2) Then to get two points \((n, m)\) and \((m-pn, kn-pm)\);
3) To compare the inequalities; and
4) The proof has a contradiction, then \(\sqrt{k}\) is irrational”. (See Figure 17)
Assume \( k \) is not the square of an integer, then \( \sqrt{k} \) is irrational.

1. Use well-ordering method to let \( \sqrt{k} = \frac{m}{n} \) in lowest terms.
2. Then get two points \((n, m)\) and \((m-pn, kn-pm)\).
3. To compare the inequalities.
4. The proof have a contradiction, then \( \sqrt{k} \) is irrational.

*Figure 17. Student A14’s Key Idea of the Proof of the Irrationality of \( \sqrt{k} \).*

First, the student claimed that the Well Ordering Property was used to make the assumption. Given that the major issue of her work was about what key idea means as opposed to what property is misused, she wasn’t included in category 2. Secondly, the student did not explain where the two points \((m, n)\) and \((m-pn, kn-pm)\) came from. It was not simply to “get two points”, but rather, one particular point in relation to a given point. This suggests that the student focused on the final product rather than the thinking process. Lastly, as the third step, the student mentioned, “compare the inequalities”. What the student did not understand was that it was not the inequalities that were compared but the coordinates of the two points, which leads to two sets of inequalities.

When I asked the student what “key ideas” meant to her, she responded that, “Key ideas are the steps to follow. My high school teacher often summarized key points in this way” (A14). The student’s proof reconstruction did follow the steps listed but lacked sufficient detail and valid verifications.
4.4.1.1.6 Discussion of category 3: Unhelpful key ideas followed by problematic reconstructions

Category 3 shows that nine students identified the key ideas of the proof inappropriately and were unsuccessful in reconstructing. The wrong directions they were taking indicate that these students did not understand the original proof.

As shown in type 3, student A14 simply listed the procedures to complete the proof. It is not surprising that for some students the key ideas of a proof are meant to be procedural and mechanical. Nevertheless, conceptual components are likely to be reflected in mechanical procedures to make them meaningful. As Kieran (2013) suggested, procedures are conceptual in nature and procedures are regularly updated, revised and extended by means of conceptual elements. In the case of A14’s work, the student focused solely on procedures without displaying appreciation or comprehension of how the proof proceeded. In fact, she seemed to observe a sequence of actions taken by the proof but some of the observations were inaccurate.

The instructor’s insights could help us understand the students’ responses. He had an interesting discussion about the key ideas of the proof “\(\sqrt{2} + \sqrt{3}\) is irrational”. He found a difference between being conceptual and procedural. One way to identify and formulate its key ideas is that 1) assume that \(\sqrt{2} + \sqrt{3}\) is rational; 2) square \(\sqrt{2} + \sqrt{3}\) to obtain \(5 + 2\sqrt{6}\); and 3) contradiction. The other way to do it is that 1) assume that \(\sqrt{2} + \sqrt{3}\) is rational; 2) because squaring preserves rationality, \((\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}\) is rational; and 3) contradiction. Notice that the first way describes what has been done without addressing why it can be done, and therefore it is procedural. The second way gives an explanation of why squaring can be done, even though the key ideas were presented in a procedural manner. Concerning student A14’s
work, the issue was not that she listed the steps to follow, but that she listed the steps without actual mathematical substance.

4.4.1.1.7 Category 4: Unhelpful key idea followed by a successful reconstruction

An incorrect key idea followed by a successful reconstruction was found in student A6’s work. He stated that, “I think the main idea of this proof is we won’t have a line when the slope of it is irrational. The graph will be a curve. But when the slope is rational the graph will be a line. We used geometrical shapes to prove √k is irrational” (see Figure 18). However, the reconstruction of the proof was complete and clear, inconsistent with the key idea identified.

Figure 18. Student A6’s Key Idea of the Proof of the Irrationality of √k.

4.4.1.1.8 Discussion of category 4

Category 4 shows an interesting inconsistency between the identified key ideas of the proof and the reconstruction of it. The proof reconstruction showed that student A15 grasped the proof with a fairly good comprehension, while the key ideas of the proof did not reflect much of it. One possible explanation is that the student did not understand the question “what
is the key idea of the proof?” It is possible that this student, like some of the others, had never come across this question throughout his academic life. Without knowing what was being asked, he might 1) have guessed that he was asked to locate some mathematical ideas related but beyond the proof; or 2) have made up an answer merely to fill the blank. To know what exactly was going on in the student’s mind, attempts were made to discuss the work with him. Unfortunately, the student was not interested in such discussions.

4.4.1.2 Summary

This section investigates the students’ work on the proof of the irrationality of $\sqrt{k}$ for non-square $k$ by examining the relationships between how well students identified and formulated key ideas and their subsequent ability to reconstruct the proof. The students’ work were organized and presented in four categories:

- A clearly identified key idea followed by a successful reconstruction
- Unclearly identified key ideas followed by unclear reconstructions
- Unhelpful key ideas followed by problematic reconstructions
- Unhelpful key idea followed by a successful reconstruction

These four categories show the complexity and subtlety of the relationships between students’ key idea identification and formulation and their subsequent ability to reconstruct the proof, particularly in category 2 and 3. Drawing from students’ work samples, only two of the 15 students were able to identify and formulate key ideas of the proof appropriately and reconstruct the proof successfully, while nine students did not manage to do so. The four students in between these two groups largely struggled with clarity in proof reconstruction. Despite the considerable discrepancies between categories, the analysis of each category offers fresh insights as to the role that key ideas play in the learning of proof.
4.4.2 The carpet proof of the irrationality of $\sqrt{2}$

The carpet proof of the irrationality of $\sqrt{2}$ (see Figure 19 and Appendix H) was discovered by Stanley Tennenbaum in the 1950’s but was made widely known by John Conway around 1990. It is also referred to as Tennenbaum’s “Covering” Proof. This proof was chosen to investigate students’ ability to identify key ideas because it was part of the curriculum of course A and also had a geometric component that fitted well into the curriculum of course C. Interestingly, it was the proof of the same proposition that had been used by Gowers (2007) to demonstrate the power of key ideas.

Proof:
Assume that there exist positive integers $a$ and $b$ such that $a/b = \sqrt{2}$ or equivalently, $a^2 = 2b^2$. Let $a$ be minimal positive integer with the property that there exists $b$ such that $a^2 = 2b^2$.

Begin with one square of side $a$ and two squares of side $b$. Since $a^2 = 2b^2$, the area of the large square is equal to the sum of the areas of the two smaller squares.

The area of the overlap equals the sum of the areas of the two unshaded squares. But the overlap is a square with side of integer length less than $a$ and the two unshaded squares have sides of positive integer length as well. This contradicts the choice of $a$ as the smallest integer with the property that there exists an integer $b$ with $a^2 = 2b^2$.

*Figure 19. The Carpet Proof the Irrationality of $\sqrt{2}$.*

In class A the carpet proof was used in a final exam, whereas in class C it was used as an in-class activity. In this section, I examine students’ work on the carpet proof of the irrationality of $\sqrt{2}$ in class A and C by focusing on the ways that students identified and
formulated the key ideas of the carpet proof. I present and discuss the students’ work in each course separately, and then underscore the similarities and differences between the two groups of students.

4.4.2.1 Class A “Proof-in-Transition” (Summer)

Number of students: 18

Final exam: Carpet proof (given 15 points out of 100)

Prompt questions:

1. The second diagram shows that the two $b \times b$ squares overlap. Prove that this must occur.
2. Explain why the area of the central square (overlap) equals the sum of the areas of the two shaded squares.
3. Explain why the central square and the two shaded squares each have sides of integer length.
4. The Well Ordering Property for the positive integers states that a nonempty subset of the positive integers has a smallest element. Where is this used in the proof?
5. What is the key idea (essence) of the proof? Please be concise. Your answer should fit in the space provided.

In the proof-in-transition summer class (class A), the carpet proof and the five prompt questions were incorporated in the final exam and placed at the end as the last question, with weight of 15 points out of 100. The results of how and how well the students identified the key ideas of the proof are based on the work of 16 students.

The test papers were collected and photocopied for analysis prior to being graded, for two reasons: 1) the analysis was intended to focus on the quality of students’ responses to the questions as opposed to the marks received; and 2) the analysis was intended to be independent, that is not influenced by the instructor’s feedback and judgments.

As shown in Table 16, one student (A6) reconstructed the entire proof without identifying a key idea, while two students (A12 and A17) stated without further explanation that “proof by contradiction” was the key idea. The other three students (A2, A5, and A16)
are discussed in the section “clearly identified the key ideas”, while the remaining 10 students who did not identify a key idea in a concise and clear way are discussed in the section “unclearly identified key ideas”. Thus in the case of the carpet proof the key ideas identified fell into only two of the categories discussed earlier (page 65): Clear key ideas, and Unclear key ideas.

Table 16

Responses to What Are the Key Ideas of the Carpet Proof in the Final Exam (class A)

<table>
<thead>
<tr>
<th>Response</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entire proof</td>
<td>1</td>
</tr>
<tr>
<td>Proof by contradiction</td>
<td>2</td>
</tr>
<tr>
<td>Clearly identified key ideas</td>
<td>3</td>
</tr>
<tr>
<td>Minor flaws of key idea identification</td>
<td>5</td>
</tr>
<tr>
<td>Incomplete justification of key ideas</td>
<td>5</td>
</tr>
<tr>
<td>Total</td>
<td>16</td>
</tr>
</tbody>
</table>

In what follows, I first take a closer look at the three successful responses (A2, A5, and A16); I then examine the work of five out of 10 students whose responses lacked clarity (A3, A4, A7, A9, and A13).

4.4.2.1.1 Clearly identified key ideas

Students A2, A5, and A16 identified and stated the proof’s key ideas clearly. In the following, I discuss these three students’ work to illustrate how they identified the key ideas of the proof and the common features of their responses.

Student A5 wrote in response to the prompt question, “What is the key idea (essence) of the proof”:

The proof proceeds as if \( \sqrt{2} \) is a rational number until a contradiction appears which shows that it is not. The contradiction is that the number \( a \), though specified to be the
smallest integer for which \( a^2 = 2b^2 \) is found not to be, proving that there is no rational representation of \( \sqrt{2} \). (See Figure 20)

Figure 20. Student A5’s Key Idea of the Carpet Proof.

The student first indicated that the assumption was “\( \sqrt{2} \) is a rational number” and that the method of the proof was proof by contradiction. He then described that the contradiction was reached by finding an integer smaller than the “specified smallest” integer \( a \). He concluded that, “there is no rational representation of \( \sqrt{2} \).”

Student A2 offered a more detailed answer. She started with the assumption, as shown in Figure 21, that “\( \sqrt{2} \) is rational and can be represented with integer \( a \) and \( b \) as \( \frac{a}{b} \)” and “since \( \sqrt{2} = \frac{a}{b} \) can be manipulated to show \( a^2 = 2b^2 \), it is possible to represent \( a^2 = 2b^2 \) as three squares of side length \( a \), \( b \) and \( b \).” She continued on and explained the geometric configuration of the integer-side squares:

The two \( b \) squares, when placed in opposite corners of the \( a \) square, must overlap. When they do they form another set of three squares where the two smaller squares is equal to that of the larger, and their sides are integers...

Her conclusion was “therefore \( \sqrt{2} \) is not in lowest terms, and by contradiction, \( \sqrt{2} \) cannot be presented rationally.” The highlight of student A2’s work was that she explicitly
stated the rationale and context of the geometric construction to show that she understood the process of this construction. In addition, her conclusion “√2 is not in lowest terms” speaks back to the assumption “√2 is rational and can be represented with integer $a$ and $b$ as $\frac{a}{b}$, where $\frac{a}{b}$ is in lowest terms.”

**Figure 21.** Student A2’s Key Idea of the Carpet Proof.

The third student A16, a non-native speaker, struggled with sentence structure and English grammar, nevertheless presented the key ideas clearly:

It use [uses] “if √2 is rational, $a$ will be the smallest integer with the property that [there] exists an integer $b$ with $a^2 = 2b^2$, but the side of the overlap square is also integer and smaller than $a$” as a contradiction to show that √2 is irrational. (See Figure 22)
Figure 22. Student A16’s Key Idea of the Carpet Proof.

She put quotation marks around what she identified as the basic structure of the key idea: the proof “used” a contradiction to show that $\sqrt{2}$ is irrational; she also restated the assumption and showed how the contradiction was arrived at.

### 4.4.2.1.2 Discussion of clearly identified key ideas

The common features of the three students’ work are completeness and coherence. Collectively, the three students’ responses include the pillars that constitute the proof – the assumption, the geometric construction, the contradiction discovered, and the conclusion. Their responses also demonstrate the importance of coherence: the purpose of discussing one mathematical object is to make connections with the other. Without linking one with the other, the analysis of the proof, for the purpose of understanding or communication, would be pointless. For instance, student A5 started off the assumption that “$\sqrt{2}$ is rational” and ended with “there is no rational representation of $\sqrt{2}$”, while student A2’s argument revolved around whether or not “$\sqrt{2}$ is in lowest terms”.

However, although the three students all clearly identified the key ideas of the proof, concerning lower terms, they were not explicit about a new set of rational representation or a smaller pair of integers. For instance, student A2 stated that lower terms than $\frac{a}{b}$ were found yet implicitly, while students A5 and A16 only focused on finding an integer smaller than $a$.
(not paying attention to the property attached to \( a \) namely that it is the smallest integer such as \( a^2 = 2b^2 \)). One may wonder why the students did not directly address a smaller pair of integers, but rather discussed \( a \) without \( b \).

It would be adequate to formulate the key idea of the proof by discussing a newly found smaller integer than \( a \) to show the contradiction, as indicated in the original proof. Nevertheless a good question for students to consider is that whether or not the assumption “\( a \) is assumed to be the minimal integer with the property that \( a^2 = 2b^2 \)” is equivalent to “\( \sqrt{2} = \frac{a}{b} \) is in lowest terms”. Perhaps due to the way that the original proof was presented – with an assumption that focuses on minimal \( a \) without addressing integer \( b \), the students might be under the impression that it is adequate to discuss \( a \) without \( b \).

### 4.4.2.1.3 Unclearly identified key ideas

The responses given by 10 students were unclear and incomplete. Their attempts to identify the key ideas of the proof were unclear for two reasons: 1) there were some problems in their articulation which caused minor flaws (A3, A4, A5, A13, and A14); and 2) the important information of the proof was missing (A7, A9, A10, A11, and A15). In what follows, I take three responses (A3, A4, and A13) as examples for the first group of work and two responses (A9 and A7) for the second group to discuss the lack of clarity.

*Key ideas that contain minor flaws*

Five students’ key ideas of the proof (A3, A4, A5, A13, and A14) were unclear due to minor flaws in their articulation. Due to the common flaws found in the responses, I examine three representative responses (A3, A4, and A13).

In student A3’s response (see Figure 23), he first acknowledged the method of the proof was proof by contradiction, he then went on, “showing at first that if \( a^2 = 2b^2 \) then it is
possible to construct different size squares which will correspond to the same statement $a^2 = 2b^2$ as to show that initial assumption $a^2$ being the minimal is not true as one can find a smaller value.” The structure of his key idea was that if we assume that $a^2 = 2b^2$ then it is possible to construct a geometric configuration for $a^2 = 2b^2$. However, the initial assumption is found not true because one can find a smaller integer than $a$.

Figure 23. Student A3’s Key Idea of the Carpet Proof.

Regardless of the sentence arrangement of his response, the key ideas covered all of the components of the proof: the method used, the assumption being made, the geometric configuration constructed, and the contradiction discovered. The flaw in his work is the claim that “it is possible to construct different size squares”. The contradiction was reached by constructing smaller size squares rather than “different size squares”. The student could have expressed it more precisely.
In a similar fashion, student A13 explicitly stated how the proof proceeded (See Figure 24):

- Firstly, we assume \( \sqrt{2} \) is rational and \( \sqrt{2} = \frac{a}{b} \) for integer \( a \) & \( b \), hence it implies \( 2b^2 = a^2 \).
- Secondly, applying the Well Ordering Property to our assumption. It says \( a \) be [is] minimal with the property \( a^2 = 2b^2 \).
- Next, after overlapping two smaller squares with the larger one, we can still find a square with side of integral length less than \( a \) and the two unshaded squares have sides of integral length as well.
- Finally, the area of two unshaded squares = the area of central one. This situation contradicts to our assumption that \( a \) is the minimum. Therefore, \( \sqrt{2} \) is irrational.

The student intentionally specified each step of the proof to allow the key idea to flow. In the first and second step in particular, he made it clear that the assumption contained two points. One is that if \( \sqrt{2} \) is rational then \( \sqrt{2} = \frac{a}{b} \) for integer \( a \) & \( b \), whereas the other is
that $a$ is minimal with the property $a^2 = 2b^2$ by the Well Ordering Property. Very few students explicitly stated them.

The key ideas continued to flow until he reaches the point “Finally, the area of two unshaded squares = the area of central one (the overlap). This situation contradicts to our assumption that $a$ is the minimum.” The flaw in this argument is that “the area of two unshaded squares = the area of central one” does not contradict to the assumption, but rather, it supports the previous point that the overlap and two unshaded squares all have integral-side length. The student somehow reversed the order of the “next” and “finally”, and therefore misplaced what contradicts to the assumption.

In contrast to students A2 and A13, student A4’s response manifested a sense of conciseness. He stated that, “To prove the irrationality of $\sqrt{2}$ by claiming that $\sqrt{2}$ can be represented by a minimal value $a$ which is an integer and then showing that it can be then represented by an even smaller value” (See Figure 25). Despite that the student implicitly stated the assumption and the contradiction, his articulation did uncover the key idea of the proof. One flaw in his work was the claim that “$\sqrt{2}$ can be represented by a minimal value $a$ which is an integer”.

\[
\text{To prove the irrationality of } \sqrt{2} \text{ by claiming that } \sqrt{2} \text{ can be represented by a minimal value } a \text{ which is an integer and then showing that it can be then represented by an even smaller value.}
\]

\text{Figure 25. Student A4’s Key Idea of the Carpet Proof.}
Without the context, the claim itself is mathematically incorrect given the fact that $\sqrt{2}$ CANNOT be represented by any integer. What the student meant might have been that “$\sqrt{2}$ can be represented by a minimal value $a$ which is an integer with the property $a^2 = 2b^2$,” he did not, however, state it precisely.

**Key ideas that miss important information**

Five students’ key ideas of the proof (A7, A9, A10, A11, and A15) were unclear caused by missing the important information of the proof in their articulation. Taking into account the similarities in their work, I examine students A9 and A7’s responses to illustrate the obscurity.

Student A9’s response had a good structure that supports a clear articulation (see Figure 26), “If $\sqrt{2}$ is rational, there should be a fraction of 2 integers without common factor […]. However, […], this statement is proven to be wrong and that we can prove that $\sqrt{2}$ can only be irrational.” Nevertheless, rather than adequately stating where $a$ and $b$ come from and how a smaller integral-side length than $a$ was found, the student addressed it by “forming a few squares” and quickly concluded that the assumption was proven false.

![Figure 26. Student A9’s Key Idea of the Carpet Proof.](image)
Similarly, student A11’s key idea of the proof reads, “It proves $\sqrt{2}$ is irrational using [the] well ordering property and it is finally done by leading to a contradiction”. Despite the student acknowledging the relationship of the Well Ordering Property to the contradiction in the proof, further explanation of how the two things are connected and related was missing.

In contrast to student A9 and A11, student A7 attempted to address the contradiction in the proof, her words, however, were unclear and therefore it is difficult to follow. As shown in Figure 27, “$\sqrt{2}$ cannot be rational because largest the side of the square $a$ is not the square that can be formed by $a$ instead $b$ is, since $b$ overlaps its squares in $a$.” What student A7 attempted to express might have been that “$\sqrt{2}$ cannot have a rational representation”.

Another possibility could be that the student left the statement unfinished due to time constraint on the final exam. In any case, important information was missing in the response.

After the final exam, when discussing with the student about her response, she stated that she had forgotten what she was trying to express during the exam but stressed that she struggled to identify the key ideas of the proof because she had difficulty to catch what was going on in the proof while she was reading it.

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Figure 27. Student A7’s Key Idea of the Carpet Proof.
4.4.2.1.4 Discussion of unclearly identified key ideas

Ten students’ responses displayed some difficulties when they attempted to identify the key ideas of the proof. Of the 10 students, half of the students had minor flaws in the articulation, while the other half did not address important information of the proof.

For those who had minor flaws in the responses, they did not accurately state what they understood about the proof even though they seemed to have a good understanding of it. Some of the flaws in the articulation caused ambiguity or confusion, while other led to mathematically incorrect statements. In the case of student A4’s response, for example, he stated that, “√2 can be represented by a minimal value of a which is an integer.” What was expressed might not be what was in the students’ mind. Nevertheless, it reflects their struggles when they dealt with mathematical content and mathematical language at the same time.

For those who missed important information in the responses, they seemed to touch upon what happens in the proof but overlook the core issues. Student A9 stated that the contradiction was found “by forming squares”, while student A11 claimed that the proof “is finally done by leading to a contradiction”. In reading these key ideas of the proof, it is difficult to judge how well the students understood the proof due to the lack of essential information, and how these key ideas might help with the proof reconstruction.

4.4.2.2 Class C “Advanced Geometry”

Number of students: 24

One-hour in-class activity: Carpet proof

Prompt questions for group discussion:
1. The second diagram shows that the two $b \times b$ squares overlap. Prove that this must occur.
2. Explain why the area of the central square (overlap) equals the sum of the areas of the two shaded squares.
3. Explain why the central square and the two shaded squares each have sides of integer length.
4. The Well Ordering Property for the positive integers states that a nonempty subset of the positive integers has a smallest element. Where is this used in the proof?

Question for individual work:

What is the key idea (essence) of the proof? Please be concise. Your answer should fit in the space provided.

The 19 students taking the course were divided into eight groups of two students each and one group of three students. They were asked to read the carpet proof, reflect on it and discuss it in these small groups without the instructor’s intervention. Each group was given about 40 minutes to read and discuss the carpet proof, to respond to prompt questions and to hand in group worksheets. The students were then given about 15 minutes to work individually on re-examining the carpet proof and identifying its key idea.

At the end of the session, the instructor collected nine group worksheets and 17 (out of 19) individual worksheets. At the end of the term, the students were asked to complete a follow-up question sheet that included the question: “Do you remember the carpet proof of the irrationality of $\sqrt{2}$?”

The students attended to different aspects of the proof when they responded to the prompt question, “What is the key idea (essence) of the proof? Please be concise. Your answer should fit in the space provided.” I sorted students’ individual work in three categories, using the constant-comparative method as mentioned in Chapter 3, based on the focus of the identified key ideas (see Table 17): 1) focusing on the methods used in the proof (students C3, C4, C10, and C13); 2) focusing on the ideas used in the proof (10 students);
and 3) focusing on the details of the proof (students C7, C20, and C22). In the following, I discuss these three categories with students’ work samples along with the survey responses.

Table 17

*Students’ Foci on the Key Ideas of the Carpet Proof in Response to Prompt Question 6*

<table>
<thead>
<tr>
<th>Focus</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>On ideas</td>
<td>10</td>
</tr>
<tr>
<td>On methods</td>
<td>4</td>
</tr>
<tr>
<td>On details</td>
<td>3</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>17</strong></td>
</tr>
</tbody>
</table>

4.4.2.2.1 Category 1: Key ideas that focus on ideas

As shown in Table 18, 10 students focused on the mathematical ideas used in the carpet proof when they attempted to identify its key idea. However, only two (C8 and C21) out of 10 students clearly identified a key idea of the proof; six students did not express any key ideas of the proof completely and clearly; and the remaining two students gave unhelpful responses (C19 and C27). In what follows, I discuss these three types of response: clear, unclear and unhelpful.

Table 18

*Quality of Articulation of Key Ideas Identified*

<table>
<thead>
<tr>
<th>Identification of Key ideas</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clear</td>
<td>2</td>
</tr>
<tr>
<td>Unclear</td>
<td>6</td>
</tr>
<tr>
<td>Unhelpful</td>
<td>2</td>
</tr>
<tr>
<td><strong>Number of students</strong></td>
<td><strong>10</strong></td>
</tr>
</tbody>
</table>

4.4.2.2.1.1 Clearly identified key ideas

Students C8 and C21 worked in different groups but both clearly identified and adequately stated a key idea of the proof. Student C8 wrote:
Assume that $\sqrt{2}$ is rational. We can find an $\frac{a}{b}$ where $a$ is the smallest integer with $a$, $b$. However, we found out that the shaded square is an integer as well. Since we found a smaller integer, there is a contradiction. $\sqrt{2}$ is irrational.

Student C8’s work demonstrated the ability to identify a pattern in the proof (see Figure 28). The student correctly started with the assumption that $\sqrt{2}$ is rational, with $a$ being the smallest integer, then discovered a smaller integer than $a$ that contradicted the original assumption that “$a$ is the smallest”, and ended with the conclusion $\sqrt{2}$ is irrational.

Figure 28. Student C8’s Key Idea of the Carpet Proof.

Similarly, student C21 stated a key idea of the proof with the same structure and completeness, as shown in Figure 29:

We begin with the assumption that $a$ is a minimal positive integer (so $A = a^2$ is a “smallest square”). However, we are able to create smaller squares whose length is clearly smaller than $a$ & $b$’s, thus the square with side length $a$ can’t be the smallest.
Notice that, at the right-upper corner, student C21 also highlighted the “start” and “end” as an outline of the proof. If translate the symbols into words, it reads:

- **Start**: $a$ is an minimal integer, then there exists an integer $b$ such that $a^2 = 2b^2$
- **End**: the area of the overlap is equal to the sum of areas of two unshaded squares. But the overlap is a square with the length smaller than $a$.

Despite the fact that the student almost reconstructed the entire proof in the process of formulating the key ideas, her presentation indicated not only an attempt to sketch the outline of the proof from the start to the closure, but also an interest in sharing the thinking process by labeling the graph and translating the labels into plain language.

Students C8 and C21’s work shared some similarities in the sense of completeness. Student C8 started with the assumption that $\sqrt{2}$ is rational” and arrived at the conclusion that $\sqrt{2}$ is irrational”, whereas student C21 started with the assumption that “$a$ is a minimal positive integer” and arrived at the conclusion that “$a$ cannot be the smallest”. This explicitly
shows the completeness of a proof by contradiction argument – start with an assumption and always end with addressing the assumption. In the follow-up survey, both of these two students reported that they remembered the carpet proof.

4.4.2.2.1.2 Unclearly identified key ideas

Six students attempted to identify the key ideas of the proof but struggled to formulate them completely and precisely. Particularly, one of the six students, (C5), paid attention to one aspect of the proof as opposed to the whole picture, while the other five students’ responses (C1, C11, C12, C23, and C24) focused on the whole picture but displayed a lack of clarity in their presentation.

![Image of handwritten note]

*Figure 30. Student C5’s Key Idea of the Carpet Proof.*

As shown in Figure 30, student C5 stated, “The key idea to this proof is understanding that $a$ is not the smallest positive integer by the Well Ordering Property, depending on the elements of the subset, a smallest element can be found.” Clearly, the student acknowledged the use of the Well Ordering Property in the proof and even stated what it was. However, he did not specify how it was used. In fact, he erroneously attributed the conclusion that “$a$ is not the smallest positive integer” to the Well Ordering Property.
“Where is the Well Ordering Property used in the proof” was one of the prompt questions for group discussion. Tracking back to student C5 and his partner C6’s shared group worksheet, it shows that students C5 and C6 skipped this question. Indeed, the Well Ordering Property was used in the proof to provide a rationale for the assumption that the minimal positive integer $a$ did exist. The contradiction was reached by finding “a smaller integer” with no further reference to the Well Ordering Property. Since we know that the Well Ordering Property holds for any non-empty subsets, finding a smaller integer than $a$ (which is assumed to be the smallest) violates the Well Ordering Property.

Unfortunately, student C5 was unclear about how the Well Ordering Property played out in the proof. In the follow-up survey, student C5 reported that he did not remember the carpet proof.

The responses of the five students work (C1, C11, C12, C23, and C24) that displayed limited clarity, had two flaws in common: 1) a key idea of the proof was somewhat identified but unclearly stated; and 2) some of the arguments were obscure and confusing. For example, student C12 stated that:

If the area of two smaller squares is the same as the area of a bigger square, there will exist two new smaller squares with areas that equal the area of the overlap square of the original squares. If we do this infinitely, we will obtain fractions when the algebra contains only integer. This is impossible. Integer has no fractions. (See Figure 31)

The student had a good start by employing an “if – there will (then)” statement to show the assumption and its consequence. He then intended to repeat the geometric construction infinitely and discussed the conflict between “fractions and the algebra”, which was irrelevant to the contradiction found in the proof. In addition, the statement “Integer has no fractions” was obscure and confusing.
Students C1 and C11, although worked in different groups, identified a key idea of the proof in the same fashion. Taking student C11’s work as an example (see Figure 32), he first stated the assumption of the proof, “Since it is a proof by contradiction, we assume that \( a \) is the smallest length for a square is the smallest #”. He then jumped to the contradiction by stating, “we keep finding smaller integers”.

It is true that one positive integer smaller than \( a \) and \( b \) was found but it is insufficient to argue that “We keep finding” them. In fact, the student missed the point that as long as one smaller integer is found, the proof is done. His conclusion was that “We know that irrational numbers cannot get smaller which means \( \sqrt{2} \) is irrational” (Possibly, what he intended to express was that a rational number cannot be represented by a smaller set of \( a \) and \( b \)).
Similar to student C11’s response that “we keep finding smaller integers”, student C23 also stated, “we can continuously find a smaller and smaller integer. We know by contradiction that \( \sqrt{2} \) has to be irrational”. Students C12, C11, and C2 seemed to attribute the cause of contradiction to the Well Ordering Property. According to Toulmin’s model of argumentation (1958), the Well Ordering Property was used as a warrant to support the assumption that there exists a minimal positive integer \( a \). Rather than taking it into account as a \textit{Warrant}, students C12, C11, and C23 all misapplied the Well Ordering Property as a \textit{Rebuttal} to create contradiction of the proof.

Although the five students’ work contained confusing messages, three students confirmed that they remembered the carpet proof at the end of the term, while the other two did not submit the follow-up question sheets.

4.4.2.1.3 \textit{Unhelpful key ideas}

Students C19 and C27 worked in different groups but both claimed that “regression” was the key that led to the contradiction in the proof. As shown in Figure 33 and 34, student C19 stated that “rational numbers cannot undergo infinite regression”, while student C27 claimed, “by integral regression we know it cannot be a rational number because it will eventually be less than 0”. The two students’ consideration of “infinite regression” and “integral regression” likely resulted from the same reasoning that “we can continuously find a smaller and smaller integer” (C23).

\[
\text{The Key idea of the 'carpet proof' is the fact that rational numbers cannot undergo infinite regression.}
\]

\textit{Figure 33.} Student C19’s Key Idea of the Carpet Proof.
After consulting with the instructor, it turned out that the students might confuse “infinite regression” with “infinite descent”. The gist of infinite descent is that it is impossible to find an infinite strictly decreasing sequence of natural numbers so a process that would lead to such a result is impossible, while “infinite regression” is completely irrelevant to the proof. As the instructor put it, “Using infinite descent allows one here to avoid the Well Ordering – if any configurations exist then smallest configurations exist but oops here’s a smaller one – in presenting the argument.”

There is a possibility that what the two students meant by “infinite regression” was proof by infinite descent as explained by the instructor, despite the fact that they neither named it properly nor explained how it worked in the proof. In fact, in the follow-up question sheets, both students reported that they remembered the carpet proof at the end of the term. This may indicate that the students understood the proof and captured the key idea used in it but due to their struggle with the articulation and terminology, their work did not reflect what they knew.

4.4.2.2 Discussion of category 1 – Focus on ideas

Category 1 consists of the work of 10 students that focused on “the ideas” used in the proof, in contrast to the students who focused on “the methods” and “the details”. Although
all of the 10 students attended to “the ideas”, in terms of how clearly they identified a key idea of the proof, it fell in a broad range. Only two out of 10 students clearly identified and well stated a key idea of the proof. Six out of 10 students formulated a key idea incompletely or unclearly. Another two students’ key ideas contained the incorrect terms and thus seemed irrelevant.

Although student C8 and C21 worked in different groups, they created the same structure when they formulated a key idea of the proof:

- start with the assumption and its consequence;
- discover a result that contradicts to the assumption; and
- end with the conclusion.

In particular, it seemed natural for these two students to use the proper conjunction words so that the ideas that they attempted to convey were connected and flowed from one to the next – we begin with an assumption A (usually opposite to the conclusion), however we discovered a result B that contradicts A, therefore the assumption is false and the conclusion can be made.

Such a flow however was by no means achieved easily for the majority of the students. A logical chain, as demonstrated above, is rarely applied and reflected in daily arguments and conversations. Most of the students felt bewildered when asked to arrange their thoughts and communicate them coherently in a complete proof. This suggests that in the learning of proof, students might need explicit instruction on how to move from premises to conclusion and make correct inferences. This is not suggesting going back to teaching logical operations and truth tables, but rather using plain language to make sense of the logic embedded.
In addition to supports for students to make logical progression, the structure of mathematical arguments could also be helpful with students need to see Toulmin’s (1958) model of argumentation, for instance, indicates that warrants are often implicit in arguments. This was the case when making the assumption in the carpet proof. The assumption “a is minimal positive integer such that \(\frac{a}{b} = \sqrt{2}\)” is an argument, supported by the Warrant “the Well Ordering Property”. Without seeing the structure of the argument, students mistook one thing for another which created confusions in logic.

It would be inappropriate to argue that the work of the six students who did not express the key ideas of the proof completely and clearly, was an indication that they did not understand the proof. Indeed, the majority of this group, in one way or another, were close to capturing a key idea of the proof. What perhaps happened was that when asked, “What is the key idea of the proof?” the students might have chosen to present the most relevant and important idea to comprehend the proof, from their perspectives. For student C26, the most important idea was the assumption that “\(a \text{ is minimal and } b \text{ exists such that } a^2 = 2b^2\)” while for student C5, it was how the Well Ordering Property worked out in the given context.

4.4.2.2.3 Category 2: Key ideas that focus on methods

Students C3, C10, and C13 worked in the same group and perceived the geometric approach used in the proof the irrationality of \(\sqrt{2}\) (a typical algebraic proof from their perspective) as being the key idea. When they identified a key idea of the proof individually, they all attended to the method used in the proof. As shown in Figure 35, student C13 stated the key idea of the proof with a broad brush, “The key idea of this proof is too [to] show that there is a geometric approach to prove that \(\sqrt{2}\) is irrational”, while student C10 stated it more explicitly:
I believe what the proof is trying to show that proofs that don’t look geometric really can be. Using squares [squaring] an irrational [number] it is shown that literal squares can sub [substitute] algebraic ones like \(a^2 = 2b^2\). Looking at a problem differently or breaking it down to things like geometry helps to see the problem better. (See Figure 36)

The key idea of this proof is too show that \(\sqrt{2}\) is irrational.

**Figure 35.** A Key Idea that Focuses on Methods (C13).

Although student C10 did not mention what contradiction the geometrical representation of \(a^2 = 2b^2\) brings about, he attempted to highlight the connection being made between geometry (the configuration of the squares) and algebra (the equation \(a^2 = 2b^2\)) and to indicate that this geometric approach used in the proof brought him a new perspective.

Similarly, student C3 also pointed out the geometric component in the proof, “There is a geometrical along with an algebraic analysis that needs to occur simultaneously” (see Figure 37). In particular, the student used “simultaneously” to indicate the importance of
thinking geometrically and algebraically in this particular context and of blending the two approaches simultaneously.

Figure 37. A Key Idea that Focuses on Methods (C3).

Student C4 who worked in a different group from the students above identified the key idea of the proof as a general method: “If we have got something solid first, everything that comes after comes easy since it is produced from something that is true. And it ensures our prediction by erasing another possibility by using contradiction.”

The student explained in further discussions over emails that by “something solid” she meant the equation $a^2 = 2b^2$ and the configuration of squares corresponding to the equation; and by “our prediction” she meant the conclusion that “$\sqrt{2}$ cannot be rational”. Even though her writing did not clearly reflect her thinking, her intention was to highlight deductive inference in a philosophical sense while avoiding a detailed account of the particular proof. “I thought the key ideas of a proof would be something beyond the proof itself,” she explained.

In the follow-up survey, the three students (C13, C10, and C3) who worked together reported that they remembered the carpet proof of the irrationality of $\sqrt{2}$, while student C4 who highlighted deductive inference said she did not.

4.4.2.2.4 Discussion of category 2 – Focus on methods

Category 2 consists of four students who focused on the methods used in the proof when they identified a key idea of the carpet proof. Student C4 paid attention to the general
method used in any proofs—“deductive inference”, while students C3, C10, and C13 agreed that the key idea was “the geometric approach along with the algebraic analysis” (C3). More specifically, student C10 seemed to imply that the connection being made between the configuration of the squares and the equation $a^2 = 2b^2$ had the power to make him “see the problem better”. Student C3 emphasized the importance of blending the geometrical and algebraic approaches by switching back and forth. This indicates that, when the students’ attention is directed to reflecting on their task and summarizing their work, they were able to ponder about their thinking and review the proof at a metacognitive level.

Some proofs are amenable to connecting one mathematical field to another. The carpet proof is one of such proofs; it is a typical algebraic or rather number theory proof, but it can also be proven geometrically. In fact, during the after-class discussion, students C3, C6, C17, and C20 revealed that they favored the carpet proof over the proof taught in number theory courses using the principle of parity, due to its geometric nature. This resonates with the comment made by Peter Tennenbaum when he reflected on his father’s (Stanley Tennenbaum) carpet proof, “My dad thought that MOST of math will be reduced, at some point, to proofs of this nature, where the essential ideas are crystal clear and the proofs are radically simplified” (Tennenbaum, 2009).

On the other hand, simply knowing which general method is used in a proof (algebra or geometry), might not give sufficient clues about the reconstruction of the proof. The four students did not offer an explanation on how the geometric approach was used to find the contradiction to the original assumption, or how the construction of the geometrical representation works along with the assumption that $a^2 = 2b^2$ to arrive at the contradiction, and complete the proof. Without understanding the mathematical ideas behind the use of a
general method, (algebra or geometry in this case) the method itself seems to be an empty label, and thus may be unhelpful.

4.4.2.2.5 Category 3: Key ideas that focus on details

Students C7, C20, and C22 focused on the overlap square when they attempted to identify a key idea of the proof. As student C7 stated:

There is a square of side length \(a\) which has area \(a^2\). There are two squares of side length \(b\) and area \(b^2\). Then \(a^2 = b^2 + b^2 \Rightarrow a^2 = 2b^2 \Rightarrow a = \sqrt{2}b\). The two squares of side length \(b\) overlap within square of side length \(a\). The overlap is equal to the empty space within the square of side length \(a\). (See Figure 38)

Figure 38. Student C7’s Key Idea of the Carpet Proof.

The student first discussed the big square with side length \(a\) and the two smaller squares with side length \(b\), then described how the overlap square was formed and ended with the equality of the area of the overlap and the uncovered.

Students C20 and C22 worked in the same group. Student C20 stated:

The key idea is to use the assumption \(a^2 = 2b^2\). If you use this in the context of the ‘carpet proof’ it means understanding that \(2b^2 = b^2 + b^2\) must cover ALL of \(a^2\). This idea is important to being able to make the proof work. (See Figure 39)
The key idea is to use the assumption $a^2 = 2b^2$. If you use this in the context of the "carpet proof" it means understanding that $2b^2 = b^2 + b^2$ must cover ALL of $a^2$. This idea is important to being able to make the proof work.

Figure 39. Student C20’s Key Idea of the Carpet Proof.

The student stated what has to happen once the assumption is made and seemed to understand the importance of this consequence – “$2b^2 = b^2 + b^2$ must cover ALL of $a^2$”. He then claimed that two smaller squares must cover all of the bigger square is an important idea to the proof. Similarly, as shown in Figure 40, student C22 stated, “The key idea of the proof is to show that $2b^2 < a^2 < 4b^2$ and that if you put two squares of side length $b$, then will always be some sort of overlap.”

The reason that C20 emphasized on “$2b^2 = b^2 + b^2$ must cover ALL of $a^2$” and C22 brought up $2b^2 < a^2 < 4b^2$ is probably due to the first prompt question for the group work which was a sub-proof of the carpet proof: “The second diagram shows that the two $b \times b$ squares overlap. Prove that this must occur.”

Figure 40. Student C22’s Key Idea of the Carpet Proof.
Tracking back to the group worksheet, as shown in Figure 41, students C20 and C22 successfully constructed a proof that the two $b \times b$ squares overlap, using proof by contradiction. They first assumed that the two $b \times b$ squares do not overlap, and therefore, $b$ has to be less than half of $a$, namely, $b < \frac{a}{2}$. Then they used $a^2 = 2b^2$ to arrive at $b = \frac{a}{\sqrt{2}} > \frac{a}{2}$, which contradicts the assumption $b < \frac{a}{2}$ and therefore the overlap must occur. When students C20 and C22 were asked to identify a key idea of the proof, they chose this very proof, a sub-proof of the carpet proof, to be the key idea.

Answers:

1. Assume there is no overlap. Then $b < \frac{a}{2}$ is necessary.

\[
\begin{align*}
  a^2 &= 2b^2 \\
  b &= \sqrt{\frac{a^2}{2}} \\
  &= \frac{a}{\sqrt{2}} \\
  \sqrt{2} < 2 &\Rightarrow \frac{1}{\sqrt{2}} > \frac{1}{2}, \text{ contradiction} \quad \Rightarrow b > \frac{a}{2} \text{ and the squares must overlap}
\end{align*}
\]

Figure 41. The Proof of the Two $b \times b$ Squares Overlap (C20 & C22).

Similarly, student C7 who also focused on the overlap in her key idea of the proof, also successfully constructed a proof with her group partner C1, using proof by cases to show the two $b \times b$ squares must overlap (see Figure 42). They explained what “don’t overlap” means: Case 1 was the situation in which the two smaller squares are touching, therefore $a = b + b$, while case 2 was the two smaller squares are not touching, and therefore $a = b + b + x$. For each case, they presented the case graphically with sufficient verification to reach the contradiction. “The overlap must occur” was nicely proven. Consequently, “the overlap must occur” became the key idea of the proof.
4.4.2.6 Discussion of category 3 – Focus on details

Category 3 consists of three students’ work (C7, C20 and C22) that focused on the details of the proof when they identified key ideas. Mathematically, their work contains no errors or flaws. However, by reading the key ideas that they identified, the message one could get out of it is that the overlap must occur. This is inadequate to summarize what the carpet proof was about and more specifically, how the overlap has to do with the irrationality of $\sqrt{2}$.

By examining the sub-proof of the two $b \times b$ squares overlap, it can be seen that the three students successfully constructed the sub-proof. It is important to understand each detail of a proof. It is equally important to see the full picture. In fact, the sub-proof was designed purposefully by the instructor for zooming in the details, whereas the question about a key idea of the proof was to ask students to step back and zoom out. The three students’ responses show that, in the process of learning a proof, they have not developed the flexibility of zooming in and zooming out as appropriate.

This indicates that students tended to pay attention to the proof at the micro level but lost sight at the macro level. In this context, micro means the details of one part of a proof.

Figure 42. The Proof of the Two $b \times b$ Squares Overlap (C1 & C7).
whereas macro means the overall structure of a proof. The concern for “forgetting details” crossed all levels of university students, from junior to senior (e.g., A15, B2, C6), and led to a immediate consequence: they did not see the forest for the trees. No matter what is being studied, a forest or a proof, a more effective approach is to go back and forth between zooming out and zooming in to see both the details and the structure, as Weber & Mejia-Ramos (2011) and many others suggest. In this sense, in the teaching of proofs, instructors could advise students to pay attention to proofs at the macro level, namely, the key ideas of the proof, as a supplement and an alternative approach to the detail-oriented approach as advocated by Gowers (2007).

4.4.2.3 Comparisons between class A and C

The carpet proof of the irrationality of √2 (Appendix H) was used in the final exam of class A to evaluate student’s understanding of the proof. It was also used in class C as an in-class activity to invite the student to read, think and discuss in small groups, without the instructor’s facilitation or intervention. The results are based on the work of 16 students from class A and 16 from class C for whom there is complete data.

In what follows, I compare the two sets of results with special attention to the features that students identified a key idea of the proof. Taking into account the nature of the carpet proof, I also compare approaches that the students used to justify the area equality of the central square and the areas of the two unshaded squares.

4.4.2.3.1 Comparison of key ideas identified

For class A, I organized students’ individual work on the key ideas of the proof in 4 categories: 1) those who gave the entire proof (one student); 2) those who claimed that proof
by contradiction was the key idea (two students); 3) those who clearly identified key ideas (three students); and 4) those who unclearly identified key ideas (10 students).

For class C, I sorted students’ individual work in three categorizes, based on the focus of the identified key ideas: 1) focusing on the methods used in the proof (three students); 2) focusing on the ideas used in the proof (10 students); and 3) focusing on the details of the proof (three students). Under the second category (focusing on the ideas), I also sorted students’ responses by the level of the clarity: 1) clearly identified key ideas (two students); 2) unclearly identified key ideas (six students); and 3) unhelpful key ideas (two students).

Table 19

Comparisons of the Two Sets of Results

<table>
<thead>
<tr>
<th>Identification of Key ideas</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Clear</td>
</tr>
<tr>
<td>Responses from class A</td>
<td>3</td>
</tr>
<tr>
<td>Responses from class C</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>5</td>
</tr>
</tbody>
</table>

In order to compare the two set of results, I reorganized the students’ responses to “what is the key idea of the proof” by the level of the clarity. As shown in Table 19, of the 16 responses from the in-class activity, two students clearly identified a key idea, 12 did not identify a key idea clearly, while another two students’ key ideas were unhelpful. Of the 16 responses from the final exam, three students clearly identified the key ideas, while 13 did not identify the key idea clearly.

Considering the background of the two groups of students, those who wrote the final exam in class A were on their first year study, while those who participated in the in-class activity in class C were the 2\textsuperscript{nd} to 4\textsuperscript{th} year mathematics students. The results show that, in
terms of the clarity of the identified key ideas of the proof, no substantial discrepancies were found between the novice and mature students.

Of the 32 students from the two settings, five students clearly identified a key idea of the proof. The common features of these five responses are clarity, completeness and coherence. Collectively, their works include the major pillars that constitute the proof – the assumption, the geometric construction, the contradiction discovered, and the conclusion. They also used the proper conjunctions consistently to allow the ideas to flow from one to the next – we begin with an assumption A (usually opposite to the conclusion), however we discovered a result B that contradicts A, therefore the assumption is false and the conclusion can be made. Their successful responses also indicate the importance of coherence: the purpose of discussing one mathematical object is to make connections with the other.

The majority of the students from both courses, 25 out of 32, did not identify the key ideas of the proof in a clear and concise sense. The major issues were conceptual misunderstanding, logical confusion, and level of clarity. Of the 25 students, three students’ work displayed logical confusion (A13, A14, and C5), eight students’ work reflected conceptual misunderstanding, and 14 students’ work lacked clarity. Pedagogically, this suggests that in the teaching and learning of proof, 1) students could be given explicit instruction on how to make correct logical inferences in a proof; 2) students may need help and support to develop appropriate language to make sense of the logic embedded; and 3) students could be exposed to both formal and informal structure of mathematical arguments.

4.4.2.3.2 Comparison of the justifications of the area equality

In class A, the set of five prompt questions were given to the students and intended to evaluate their understanding of the proof, whereas for the in-class activity of course C, the set
of five prompt questions were given to the students to invite them to think, discuss and reflect the proof presented. Although the final question for students to identify the key idea of the proof was the main focus of the data analysis, the second prompt question that “Explain why the area of the central square (overlap) equals the sum of the areas of the two shaded squares” was essential to the understanding of the proof. In the following, I compare two groups of students’ justifications of the area equality of the squares and discuss the results.

Table 20

*Comparisons of the Approaches Used in the Justification of the Area Equality*

<table>
<thead>
<tr>
<th>Approach Used in the Justification</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebraic</td>
<td>Geometric</td>
</tr>
<tr>
<td>Responses from class A</td>
<td>18</td>
</tr>
<tr>
<td>Responses from class C</td>
<td>9</td>
</tr>
<tr>
<td>Total</td>
<td>27</td>
</tr>
</tbody>
</table>

Of the 18 students who participated in the final exam in class A, 18 responses to question 2 were collected. Of the 19 students who participated in the in-class activity in class C, nine responses to question 2 were collected from nine group discussions. As shown in Table 20, in class A, 11 out of 18 students used an algebraic approach, while seven used geometry. In class C, six out of nine groups used an algebraic approach to justify the area equality, while three groups used a geometric approach. The ratios of algebra to geometry are 11:7 and 2:1, respectively. Somewhat more first-year students used geometry to solve the problem than the mature students, even though the mature students were taking an advanced geometry course.

The algebraic approach to justify the area equality is shown in Figure 43. 17 out of 27 students used this approach. Students C21 and C26 labelled the sides of the squares and
calculated the areas of three squares labelled as I, II, and III. Clearly, their goal was to work towards an identical algebraic expression that indicates the areas are equal. A flaw in their work was that they took the conclusion as the assumption by saying “now \( i = ii \).” As a consequence, they had \( 2(a - b)^2 = 4b^2 - 4ab + a^2 \). From there, they found that \( a^2 = 2b^2 \), which was given.

![Diagram of squares with areas labeled]

*Figure 43. An Algebraic Approach to Justify the Area Equality (C21 & C26).*

Similarly, student A10’s work also reflected this method. These two responses were mathematically incorrect due to a logical flaw. They could have worked backwards: starting from \( a^2 = 2b^2 \), though a few steps of algebraic manipulation, they could arrive at the equality of \( 2(a - b)^2 \) and \( (2b - a)^2 \). Alternatively, they could have subtracted \( 2(a - b)^2 \) from \( (2b - a)^2 \), as student A16 approached it, if the result is 0, it indicates that the areas are equal.

With respect to the geometric approach to justify the equality of the areas, the students discovered two ways to arrive at the solution. As shown in Figure 44, student C5 and C6 labelled the sides of the squares by \( a, b, c, \) and \( d \), and used them to represent the area of each square by \( a^2, b^2, c^2, \) and \( d^2 \). Considering the relationships between them, they came up with \( a^2 = b^2 + b^2 - c^2 + d^2 + d^2 \). From there, they then proceeded to arrive at \( 2d^2 = c^2 \),
which shows the overlap is equal to the two unshaded. Of the 10 students who used the geometric approach, three (A4, A11, and C5 & C6 as a group) responses reflected this method.

![Diagram](image)

Now, we know:

\[ a^2 = b^2 + b^2 \quad \text{therefore,} \]

\[ a^2 = b^2 + b^2 \quad - c^2 + d^2 + d^2 = 0 \]

\[ \text{if} \quad - c^2 + d^2 + d^2 = 0 \quad \text{then,} \quad d^2 + d^2 = c^2 \implies 2d^2 = c^2 \]

*Figure 44. A Geometric Approach to Justify the Area Equality (C5 & C6).*

Similar but a more direct approach was explained by student A6, as shown in Figure 45, “The squares in top right and bottom left remain uncovered but \( c^2 \) appears twice (since it’s an overlap). Therefore, \( c^2 \) must be the sum of the uncovered regions since \( 2b^2 \) covers all \( a^2 \).” The student made the best use of the given \( a^2 = 2b^2 \) and quickly figured out the area equality. Of the nine students who used the geometric approach, six responses reflected this method.

![Diagram](image)

*Figure 45. A Geometric Approach to Justify the Area Equality (A6).*
An interesting case worth mentioning is an approach that “blends” algebra and geometry (See Figure 46). The group labelled the overlap square as $c^2$ and calculated the area of the overlap to yield $4b^2 - 4ab + a^2$. They also calculated the two squares labelled as D to get $2a^2 - 4ab + 2b^2$, in spite of an error $D^2 = 2a^2 - 4ab + 2b^2$ that ought to be $2D^2 = 2a^2 - 4ab + 2b^2$. What becomes interesting is that the students were aware that “due to the overlap: $2D^2 = C^2$”, which should be the final argument. However, they went on testing the argument by equalling $2a^2 - 4ab + 2b^2$ and $4b^2 - 4ab + a^2$ until they found $a^2 = 2b^2$ which was given.

![Figure 46. An Interesting Approach to Justify the Area Equality (C3, C10, & C13).](image)

When reflected their work after the class, the group members revealed that they understood that the overlap must be equal to the uncovered but struggle to express the idea. In fact, they expressed the idea well enough but did not consider it as a valid proof. In their mind, algebraic expressions with equal signs are more convincing to justify equality.

**4.4.3 Comparisons between the two proofs**

In contrast to the proof of the irrationality of $\sqrt{k}$ for non-square $k$ mentioned in the previous section, the carpet proof focuses on a geometric construction as opposed to
algebraic manipulations to reach its contradiction. However, fundamentally, the two proofs share a similar approach and strategy: a contradiction is reached by finding lower terms than assumed.

Figure 47. A Labeled Carpet Proof by Miller & Montague (2012).

Along this line, knowing that if $\sqrt{2}$ is rational then it can be represented as $\frac{a}{b}$ with integer $a$ and $b$ in lowest terms, a direct explanation to show that there exist lower terms than $\frac{a}{b}$ would be to spell out the lower terms $\sqrt{2} = \frac{2b-a}{a-b}$ algebraically, based on the given geometric configuration. In Miller & Montague’s paper on “picturing irrationality” (2012) (see Figure 47), they intentionally presented and labeled a smaller pair of integers found in the configuration, $2b-a$ and $a-b$, which would bring readers’ attention to the claim $\sqrt{2} = \frac{a}{b} = \frac{2b-a}{a-b}$. Similarly, when Gowers (2007) discussed the carpet proof from the point of view of memorization, he also stresses the particular choice of $\frac{2b-a}{a-b}$, although he arrived at this step from a different approach.
Chapter 5

Conclusions and Implications

The purpose of this study was to investigate in some detail how and how well undergraduate mathematics students identify key ideas embodied in proofs and how they use these key ideas to reconstruct a proof. In the first chapter, I described the present use of proof and proving in undergraduate mathematics, to put this study in context and show its originality and significance. I then critically reviewed in chapter 2 the various aspects of proof that are directly or indirectly related to this study in order to identify a gap in the educational literature. In chapter 3, I made the case for my choice of methods used for conducting research in classrooms and presented a detailed account of the procedure and methods for data collection and analysis. In the chapter on findings and discussions, I first reported on students’ backgrounds and previous experience with proof, and then presented and discussed the results of the analysis.

In this section, I begin with conclusions from the findings reported in the previous chapter by addressing each research question in turn. I then identify the contributions and pedagogical implications of the study, and conclude by discussing the possible directions for future research.

5.1 Conclusions

As discussed in the first chapter, the study sought answers to the following questions about the practices of undergraduate mathematics students in the use of proof:

1. (a) What are students’ perceptions of the role of key idea in a proof?
   (b) What are students’ interpretations of key ideas of a proof?
2. From the students’ perspective, is the notion of key idea in proof associated more with conceptual understanding or with procedural fluency?

3. (a) What is the evidence that students identify and use key ideas in reconstructing a proof?

   (b) Which features of a proof do students identify as key ideas?

In the course of the study, the three classes of 59 undergraduate mathematics students were surveyed regarding their perceptions of the role of key ideas in a proof, and 42 of them were asked to read two proofs and to identify the key ideas of the two proofs. In addition, 14 students agreed to be interviewed regarding the use and identification of key ideas in a proof. In what follows, I summarize the conclusions from the results presented in the previous chapter by research question.

5.1.1 What are students’ perceptions of the role of key ideas in a proof?

The online survey aimed to investigate how undergraduate mathematics students perceive the role of key ideas in a proof, from three aspects: 1) the role of key ideas in reading a proof, 2) the role of key ideas in constructing a proof, and 3) the role of key ideas in reconstructing a proof.

The findings show that the majority of the mathematics students, from year I to year IV of their undergraduate studies, acknowledged the importance of key ideas in a proof when reading it, and claimed that they attempted to identify key ideas of a proof. About half of the students recognized the value of key ideas in the two proofs and reported that they reflected back on the key ideas when constructing and reconstructing these proofs.

The analysis of the survey results and student interviews indicate that the students paid attention to key ideas when reading a proof more often than when constructing one.
Given that the text of a proof at the undergraduate level is often of some length, students tended to extract key ideas out of the proof’s text, or from other sources such as their previous knowledge. While constructing a proof, however, students seemed to pay less attention to the key ideas of the proof, simply because they had to cope with other aspects of the proof, such as logic structure, mathematical language, English grammar, and graphs and diagrams. After having constructing a proof, half of the students did not even look back upon the key ideas of the proof. Their mindset at that point appeared to be, as one student put it, “The mission is accomplished and I am done” (C6).

Concerning the role of key ideas in reconstructing a proof, the students agreed that key ideas in a proof could play an important role in helping proof reconstruction. Nevertheless, a large number of students also stated that they needed to have taken notes on the details of each step of a proof in order to be able to reconstruct it. In fact, more than half of the students claimed that both the key ideas and details in a proof were equally important for reconstruction. Interestingly enough, the more senior mathematics students seemed to favour paying attention to the details of each proof step, while the more junior students seemed to prefer knowing the key ideas used in a proof.

5.1.2 What are students’ interpretations of key ideas of a proof?

In the students’ own words, key ideas were interpreted as “a cognitive tool to make sense of a proof”, “a script for restating and explaining a proof to others”, or “the thing, central to understand the proof in the first place.” The interpretations and analogies that emerged more than once are as follows:

- Key ideas are important points.
- Key ideas are blue prints.
- Key ideas are landmarks and pathways.
- Key ideas are steps to follow.
These interpretations are well aligned with the existing literature on the notion of key idea. The phrasing of “important points” can be seen as a parallel to idea of the gist of a proof (Robinson, 2000); the analogy “blue prints” reflects what Leron (1983; 1985) and Majia-Ramos et al. (2012) call overall structure or overview of a proof; the analogies “landmarks and pathways” can be understood as “hints” that allow one to have a sense of grasping a proof all at once (Gowers, 2007); and knowing “steps to follow” is akin to what Detlefsen (2008) refers to as carrying the flow of information in proof.

Clearly, since the term “key ideas” seems to mean somewhat different things to different established scholars, it is understandable that the students’ interpretations of key ideas of a proof would also display some nuances. When students were asked to identify a key idea of a proof, they looked for a key idea based on how they understand the concept. This explains why different key ideas emerged regarding the two proofs used in the study. Those who interpreted key ideas as “landmark” were more likely to discuss a landmark discovered in a proof, and those who interpreted key ideas as “blueprints” would be more likely to focus on the structure of the proof, while those who interpreted key ideas as “steps to follow” were more likely to list the procedures of a proof.

5.1.3 From the students’ perspective, is the notion of key idea in proof associated more with conceptual understanding or with procedural fluency?

The majority of the students claimed that conceptual and procedural approaches are equally important to proof reconstruction. As discussed in chapter two, the literature in mathematics education has shifted from an emphasis on the traditional dichotomy between conceptual understanding and procedural skills (in which procedures were seen as static
entities executed mindlessly), to recognition of the beneficial relationship of mutual reinforcement between conceptual and procedural knowledge (Kieran, 2013). The findings of the study support the literature on this issue. The students clearly agreed that identifying key ideas in a proof could be useful for both conceptual understanding and procedural competence.

5.1.4 What is the evidence that students identify and use key ideas in reconstructing a proof?

The self-reported evidence of students’ identification of key ideas in a proof was found in mixed-year students’ responses to the question “Do you consciously use or identify key ideas in a proof when you study on your own?” Almost all students reported that they always or sometimes used key ideas in a proof.

In addition to self-reporting, evidence on students’ identification of key ideas in a proof was also obtained from in-class activity and assignments given to the first-year students (class A). Although the students were not asked to identify the key ideas, two of the students’ work samples indicated that they paid special attention to: 1) outlining the procedures of a proof presentation as the “key ideas”; 2) highlighting the mathematical ideas used in a proof as the “key ideas”.

5.1.5 Which features of a proof do students identify as key ideas?

Three foci on key ideas were identified through analysis of students’ work: 1) focusing on the mathematical ideas used in the proof, 2) focusing on the methods used in the proof, and 3) focusing on the details of the proof. The findings show that most students seemed to focus on mathematical ideas when they attempted to identify a key idea in a proof;
a few students focused on the methods, while very few students focused on details of a proof, such as algebraic manipulation.

The students who succeeded in identifying and formulating a structural key idea in a proof created a similar structure for the proofs they reconstructed:

- start with the assumption and its consequence;
- discover a result that contradicts the assumption; and
- end with the conclusion.

In particular, it seemed natural for these students to use the proper conjunction words so that the ideas they attempted to convey were connected and flowed naturally from one to the next. Other common features of the successful key idea identifications are completeness and coherence. Collectively, the students included the pillars that constitute the proof in their work – the assumption, the geometric construction, the contradiction discovered, and the conclusion. They also seemed to recognize the importance of coherence: i.e. that the purpose of discussing one mathematical object is to make connections with the other.

Those students who were able to formulate well-articulated key ideas of a proof and to reconstruct the proof successfully using those key ideas appeared to be eager to reflect on their results and to communicate with others. They seemed comfortable using words creatively and articulating what they had understood. To convince themselves of their results, they often used their own words to describe what was in their mind mathematically, as if to explain it to themselves, and when seeking to convince others they did their best to keep the logical progression transparent and clear.

The findings indicate that many students seemed to struggle between the micro level and macro level of a proof. In this context, “micro” means the details of one part of a proof, while “macro” means the overall structure of a proof. Those who tended to pay attention to
the local details of a proof certainly did acknowledge the importance of seeing the big picture as well, but they nevertheless were more worried about “forgetting details” when reconstructing the proof. The concern with “forgetting details” crossed all levels of university students. The fact that students had difficulties in formulating the key idea of the proof both clearly and concisely may have been also due in part to a lack of strong argumentation skills and language proficiency, but the study was not able to establish whether this was the case.

The findings show that a key idea may have been judged as such by students for a variety of reasons, such as its value as a conceptual explanation, its procedural usefulness, its usefulness for self-clarification, and its association with previous learning. In other words, the status and content of a key idea of a proof may depend upon why it is judged to be a key idea. In this sense, the choice of key idea may simply be based on pragmatic reasons and depend on a variety of contexts, and therefore the idea of key idea may best be seen as a multi-faceted concept.

It is important to note how differently a key idea might be judged by mathematicians and mathematics students. The findings show that students did not necessarily choose a key idea and identify it as such in order to facilitate proof construction or reconstruction. Rather they made their choices due to the aspect of the idea that attracted their attention at the moment and seemed most memorable. In contrast, practicing mathematicians most often identify key ideas to improve their understanding of a proof and, beyond that, for the purpose of effectively storing, retrieving, and reconstructing the proof. This suggests that future studies could profitably investigate further the perspectives of mathematicians on key ideas.
and how they identify key ideas in a proof in various mathematical contexts, with a view to constructing guidance appropriate to students of mathematics.

Although the identification of key idea may be subjective and learner dependent, as discussed above, the findings indeed indicate that the key ideas identified by the students did overlap to some extent. The overlaps do not appear to be coincidental but rather suggest that certain key ideas such as the methods used are seen as salient.

This study’s findings in showing students’ success in the proof reconstructions of the irrationality of $\sqrt{k}$ for a non-square $k$, was somewhat in line with Raman et al. (2009) in that a proof should not be thought as having only one particular key idea “We refer to “a” key idea rather than “the” key idea, because it appears that some proofs have more than one key idea” (p. 2-156), and did support Gowers’ view that a key idea provides a clue how to remember a proof and to write it. In fact, the findings of this study show that the students’ proof reconstructions were closely aligned with the key ideas they identified – varied in type as those key ideas may have been.

Perhaps due to the nature of this particular proof, its key ideas seemed to provide students with something striking to remember and to rely upon later in rewriting the proof. Different proofs could be considered in future studies to seek other possibilities of what a key idea could lend to a proof that meets both explanatory persuasion (Gowers, 2014) and the standard of mathematical rigor.

5.2 Contribution of the Study

An important contribution of this study is that it has provided a detailed portrayal of students’ perception of key ideas as well as the ways they go about identifying and using key ideas in reconstructing a proof. The picture is complex; it constitutes a significant addition to
the current discourse on the teaching of proof as it has also provided useful empirical insights into the learning of proof at the undergraduate level.

Another contribution of this study is that the identification of a key idea has been shown to be somewhat more subjective than initially anticipated at the start of this research. Unlike what might have been assumed from a reading of previous literature, the results of this study indicate that the identification of key ideas is not only context dependent but also learner dependent. This insight stands to contribute to the theoretical discussion of the concept of key idea, since the study findings may call into question the assumption advanced by Raman (2003) and Raman et al. (2009) that key ideas are actually inherent properties or features of a proof. This study brings evidence that there is a need to give this assumption more though. In arguing that key ideas of a proof are independent of the audience, Raman et al. (2009) stress that “The key idea is actually a property of the proof” (p. 2-155), and also concede that “psychologically it appears as a property of an individual” (p. 2-155). The study findings indicate, however, that considering key ideas to be properties of a proof (part and parcel of a proof) that can be picked out by examining the text of the proof may overlook the interplay between the proof and the reader, either a student or a mathematician. For students, the perception of a key idea of a proof might depend not only on their past experience and mathematical maturity, but also on their perception of what is most relevant and important for their own comprehension of a proof. For mathematicians, a key idea in a proof might be a feature of it that helps them remember the proof, retrieve it successfully and reconstruct it easily (Gowers, 2007). It could also be some dominant feature that is valued because it includes a surprise element, or a visual element, or because it affords generality, or even because it is notable for its brevity, its elegance or its beauty (Conway & Shipman, 2014).
5.3 Pedagogical Implications

The key ideas of a proof identified by students reflected not only their understanding of the concept of key idea as it had been presented to them, but also, and perhaps even to a greater degree, which feature of the proof most attracted their attention. This may be one reason for the lack of clarity in some students’ work, but other reasons could range from an insufficient understanding of the proof and its mathematical subject matter to a simple struggle with poor language proficiency. To avoid erroneous judgements of students’ work, it is important for mathematics educators to be aware of these possibilities.

Students’ work on proof reconstructions demonstrated some variations. These variations stemmed in part from the different interpretations students gave to the concept of key idea, as discussed above. Though some students struggled with identifying key ideas, the majority of the students, in one way or another, came very close to capturing what clearly were the key ideas of the two proofs used in the study. However, some students focused on the methods used in a proof, while others focused on the details and major steps. Indeed, when asked, “What is the key idea of the proof?” the students gave answers that were based on how they had interpreted the notion of key idea.

These results suggest that, in order to help students in their struggle to succeed in proof and proving, instructors would do well to create more opportunities for students to reconstruct a proof and, just as importantly, be attentive to students’ inevitably differing identifications of the aspects of a proof and their differing approaches to proof reconstruction. In other words, despite the considerable overlap in what students see as key ideas, there is room for more pedagogical attention to the variations in key-idea identification.
The findings also imply that mathematics educators, in their desire to see students enhance their understanding of proof and proving by the use of key ideas, will need to extend considerable support to students by very actively intervening to draw their attention to features of proofs that are candidates for key ideas.

5.4 Suggestions for Further Research

This study investigated only two proofs of proof by contradiction. Therefore the results may not be generalizable to every form of proof. Further research could explore this matter in several directions as follows: 1) Interview mathematicians to find out how they remember proofs and to investigate what importance they attribute to key ideas in proofs; 2) Investigate the conjecture that helping undergraduate students more actively to identify key ideas in various forms of proofs would enhance their understanding and would help them remember a proof; and 3) Mount a longitudinal study to investigate the conjecture that identifying key ideas in proof leads students to remember proofs a year later.

Another suggestion is to investigate other proofs that use a variety of proof techniques such as, proofs by regrouping numbers, by folding conics, by dissecting figures, by extending lines, or proofs by mathematical induction and contraposition, especially with first year undergraduate students. In addition, perhaps investigating several different proofs of one theorem might help shed light on more perspectives on key ideas.
References


Biehler, R., & Kempen, L. (2013). Students’ use of variables and examples in their transition from generic proof to formal proof. In *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (pp. 86-95). Antalya, Turkey.


Truth in mathematics (pp. 147-159). Oxford, UK: Clarendon.


relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review*, 27(4), 587-597.


Appendices

Appendix A

Student Survey: An Investigation About Proof Perceptions and Practices

Please indicate the extent to which you agree or disagree with each statement, by completing filling in the appropriate box.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Strongly disagree</th>
<th>Disagree</th>
<th>Neutral</th>
<th>Agree</th>
<th>Strongly agree</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Mathematical proofs are different from other kinds of proofs.</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>2. Mathematical proofs explain why a mathematical result is true.</td>
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<td>☐</td>
<td>☐</td>
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<tr>
<td>3. Mathematical proofs often include ideas, methods, and strategies.</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>4. Examples convince me that a mathematical result may be true.</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>5. A mathematical proof depends on other results in mathematics.</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
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<td>☐</td>
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<tr>
<td>6. Constructing a mathematical proof interests me.</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>7. It is easier for me to understand that a mathematical statement is true</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>after seeing examples than after seeing its proof.</td>
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<td>8. I cannot see the point of doing proofs: all formulas in the course have</td>
<td>☐</td>
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<td>already been proved beyond doubt by mathematicians.</td>
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<tr>
<td>9. I had opportunities in high school to learn about different kinds of</td>
<td>☐</td>
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<td>proof.</td>
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<tr>
<td>10. I often find it hard to create a proof that convinces others.</td>
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<td>☐</td>
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<tr>
<td>11. Mathematical proofs are irrelevant to processes of discovery or</td>
<td>☐</td>
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<tr>
<td>invention.</td>
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<tr>
<td>12. I usually need to convince myself of the correctness of a mathematical</td>
<td>☐</td>
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<tr>
<td>result.</td>
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<tr>
<td>13. When I read a proof, I try to identify its key ideas.</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>14. I often work on investigations (alone or in a group) that lead to</td>
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<td>☐</td>
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<tr>
<td>conjectures and sometimes to proofs.</td>
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<tr>
<td>15. When I reconstruct a proof, I would likely get lost unless I took</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>notes of the major steps when I saw it proved in class.</td>
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<tr>
<td>16. I had enough opportunities to practice writing proofs in university.</td>
<td>☐</td>
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</tr>
<tr>
<td>17. I had opportunities to give oral explanations of mathematical results</td>
<td>☐</td>
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<tr>
<td>in university.</td>
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<tr>
<td>18. I would like to see more mathematical proofs in university.</td>
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<tr>
<td>19. When I reconstruct a proof, I am less likely to get lost if I already</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
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<td>☐</td>
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<tr>
<td>know the key idea used in the proof.</td>
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<tr>
<td>20. It is more important for me to see the overall structure of a proof</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
</tr>
<tr>
<td>than to follow the details in each step.</td>
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<tr>
<td>21. Discussing with my peers the key ideas used in a proof helps me</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>reconstruct the proof later on.</td>
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</tr>
<tr>
<td>22. Many proofs seem boring and distant to me.</td>
<td>☐</td>
<td>☐</td>
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</tr>
<tr>
<td>23. After constructing a proof, I tend to identify the key ideas used in</td>
<td>☐</td>
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<td>it.</td>
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<tr>
<td>24. To convince myself that a mathematics result is true, I need to see a</td>
<td>☐</td>
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<tr>
<td>proof.</td>
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<tr>
<td>25. I enjoy working through a proof of a mathematical result in a group.</td>
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<tr>
<td>26. When I need to convince myself of the correctness of a geometric</td>
<td>☐</td>
<td>☐</td>
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<tr>
<td>result, I tend to use geometry software to test it out.</td>
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</tbody>
</table>
If you have further comments, please provide them below:

___________________________________________________________________________
___________________________________________________________________________
___________________________________________________________________________
___________________________________________________________________________

Please answer the following questions concerning you and your background:
1. Age_________
2. Gender_________
3. The year I finished high school_________
4. The program that I am in at the University_________________________________________
5. Did you complete high school in Canada?  ❏ YES  ❏ NO
   If NO,
   Country _________________________________________________________________
   Course/Program ___________________________________________________________

6. Do you agree to be interviewed?  ❏ YES  ❏ NO
   If YES,
   Your name: _______________________________________________________________
   E-mail address/telephone number: ___________________________________________

Thank you for your contribution.
## Appendix B

### Classroom Observation Checklist

<table>
<thead>
<tr>
<th>Course code</th>
<th>Topic</th>
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<tbody>
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<table>
<thead>
<tr>
<th>Date</th>
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<table>
<thead>
<tr>
<th></th>
<th>Yes</th>
<th>No</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>Print materials</td>
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<tr>
<td>Visual aids</td>
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<tr>
<td>Group work/discussion</td>
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<tr>
<td>Proof(s) completed by instructor</td>
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</tr>
<tr>
<td>Proof(s) completed by students</td>
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<tr>
<td>Students’ work presented</td>
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<tr>
<td>Key idea(s) addressed by instructor</td>
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<tr>
<td>Key idea(s) discussed by students</td>
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</tr>
<tr>
<td>Key idea(s) identified by students</td>
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</tbody>
</table>
Appendix C

Student Interview Questions

1. Perceptions of proof
   (1) What kind of proof have you encountered this year? What do you enjoy about doing a proof? What do you struggle with?
   (2) Is there a difference between mathematical proof and scientific proof? What is the difference?
   (3) How do you know whether a proof is valid or not?
   (4) How do axioms and theorems differ? What is the role of axioms and theorems?

2. Learning experiences and practices of proving
   (1) How do you begin when you encounter a proving task? How do you know when you are done?
   (2) After constructing a proof, have you or your group reflected on how you tackled it? If so, what have you learned from the reflections?
   (3) Have you had opportunities to discuss with your peers about a key idea used in a proof? Have you used the same key idea in one proof to construct another? How did key ideas help you with proof reconstruction?

3. Challenges in proof construction and reconstruction
   (1) What challenges have you faced in constructing a proof? Could you describe what a challenge may look like?
   (2) Do you think the challenges you have encountered are different from other students? How are your challenges different from others?
   (3) What challenges have you faced in reconstructing a proof?

4. Supports needed in proof construction and reconstruction
   (1) How would you describe a supportive learning environment in which promotes the learning of proof and mathematical understanding?
   (2) What supports would you expect from course instructors to advance your skills and techniques in proof construction and reconstruction?
Appendix D

Followed-up Questions For Class B and C

Class B

1. You were asked to summarize the key idea of this proof in the class. Do you consciously use or identify key ideas in a proof when you study on your own?
☐ Always     ☐ Sometimes     ☐ Never
Please explain.

2. Do you think the identification of key ideas is an important practice that would help you reproduce a proof that you have seen before? Please provide a few sentences to support your answer.

Class C

1a. Do you remember the carpet proof of the irrationality of $\sqrt{2}$?  ☐ Yes    ☐ No

1b. You were asked to summarize the key idea of this proof in the class. Do you consciously use or identify key ideas in a proof when you study on your own?
☐ Always     ☐ Sometimes     ☐ Never
Please explain.

2. Do you think the identification of key ideas is an important practice that would help you reproduce a proof that you have seen before? Please provide a few sentences to support your answer.
Appendix E

Research Consent Form

Dear student,

I am investigating the factors that promote undergraduate students’ proof construction and reconstruction in the classrooms. The purpose of the research is to contribute to the knowledge base regarding the role that key ideas might play in fostering students’ conceptual understanding and procedural fluency, particularly in geometric proof reconstruction.

I would like you to participate in this research by allowing me to administer a survey questionnaire, to conduct an interview with you, as well as to observe the class you are attending. The survey will take approximately 15 minutes to complete. The interview will take approximately 45 minutes and will be tape-recorded. I will administer the survey and conduct the interview during the school day and in your university.

You are free to change your mind at any time, and to withdraw even after you have consented to participate. You may decline to answer any specific questions or participate in any parts of the procedures without any negative consequences. There are no incentives and known risks to you for assisting in the research. To ensure confidentiality, both hardcopy and electronic data will be securely stored in a locet filing cabinet and a secure server environment until destroyed. All identifiable electronic information outside of a secure server environment will be encrypted at all times. The identifiable or confidential data will only be taken offsite if absolutely necessary and permitted by REB approvals and research agreements. Confidentiality will be provided to the fullest extent possibly by law.

The research has been reviewed and approved for compliance to research ethics protocols by the Research Oversight and Compliance Office – Human Research Ethics Program of University of Toronto, and the Review Subcommittee of your university. If you have any questions or concerns, please feel free to contact the Human Research Ethics Program, University of Toronto, at ethics.review@utoronto.ca or 416-946-3273, and the Office of Research Ethics at your University.

The research study you are participating in may be reviewed for quality assurance to make sure that the required laws and guidelines are followed. If chosen, (a) representative(s) of the Human Research Ethics Program (HREP) of University of Toronto may access study-related data and/or consent materials as part of the review. All information accessed by the HREP will be upheld to the same level of confidentiality that has been stated by the research investigator.

If you agree to participate in the research, please sign the attached form. The second copy is for your records. Thank you for your time and support.

Yours sincerely,

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Proof Construction and Reconstruction: An Investigation in the Undergraduate Mathematics Classrooms

I acknowledge that the topic and purposes of this research have been explained to me and that any question that I have asked has been answered to my satisfaction. I understand that I can withdraw at any time without penalty.

I have read the letter provided to me by Xiaoheng (Kitty) Yan and agree to participate in the research for the purpose described.

Signature of participant: ____________________________________________

Name (printed): ________________________ Date: ______________

Signature of researcher: ________________________

Name (printed): ________________________ Date: __________
Consider this problem and solution from Krantz, Techniques of Problem Solving.

PROBLEM 3.2.2 A ten foot pole is dropped into a milling saw and randomly cut into three shorter poles. What is the probability that these three pieces will form a triangle?

Solution: What is the characterizing property of three lengths, call them A; B; C, that form a triangle? It is that the triangle inequality will hold: the sum of any two of the lengths will be at least as great as the third. For instance, lengths 1, 2, 4 could not form a triangle.

If any of the pieces has length 5 feet then the triangle will be at and trivial; so we rule out this situation (the probability that any piece will have length five feet, or any particular pre-specified length, is zero). So we assume that each of the three pieces has length either greater than 5 feet or less than 5 feet.

But if any piece, say A, has length exceeding 5 feet then the triangle inequality A ≤ B + C must fail. Thus, to form a triangle, all three pieces must have length less than 5 feet. But when this is the case, then all three-triangle inequalities will hold (since the sum of any two lengths will exceed 5 feet) and hence a triangle can indeed be formed.

So the problem posed is equivalent to the question: “What is the probability that each piece will have length less than five feet?”

It is convenient, for the purposes of calculating our probabilities, to translate this last condition into a question of where the saw cuts will fall. First the saw cuts must lie on opposite sides of the midpoint of the pole (otherwise the largest piece will have length exceeding 5 feet). Second, the distance between the two saw cuts must not be greater than or equal to 5 feet (otherwise the middle piece cut from the pole will have length greater than or equal to 5 feet). These two conditions taken together will guarantee that all three pieces have length less than 5 feet.

A moment’s thought (there are four possibilities) reveals that the probability that the cuts lie on opposite sides of the midpoint is .5. Now the distance d between the cuts (if we rule out distance exactly 5, as discussed above) is either 0 < d < 5 or 5 < d < 10. These eventualities are equally likely. Thus the probability that the distance between the cuts is less than 5 feet is .5.

The probability that these two eventualities - (i) cuts on opposite sides of the mid-point and (ii) cuts with distance less than 5 feet between them - will both occur is the product of the two probabilities, that is p = .5 × .5 = .25.

The probability is .25 that the three pieces will form a triangle. □
Appendix G

The Irrationality of for $\sqrt{k}$ Non-square $k$

The irrationality of $\sqrt{k}$ for non-square $k$:

In this proof, we interpret $\sqrt{k}$ as the slope of a line through the origin, as illustrated below.

\[ y = \sqrt{k}x = \frac{m}{n}x \]

**Theorem:** If $k$ is not the square of an integer, then $\sqrt{k}$ is irrational.

**Proof.** Assume $\sqrt{k} = \frac{m}{n}$ in lowest terms. Then the point on the line $y = \sqrt{k}x = \frac{m}{n}x$ closest to the origin with integer coordinates is $(n, m)$. However, if we let $p$ be the greatest integer less than $\sqrt{k}$ so that $p < \sqrt{k} < p + 1$, then the point with integer coordinates $(m - pn, kn - pm)$ lies on the line and is closer to the origin since $\frac{m}{n}(m - pn) = \frac{m^2}{n} - pm = kn - pm$, and $p < \frac{m}{n} < p + 1$ implies $0 < m - pn < n$ and $0 < kn - pm < m$. Thus we have a contradiction and $\sqrt{k}$ is irrational.

1. Let $\alpha$ be rational. Prove that the line $y = \alpha x$ has a point both of whose coordinates are integral.
2. In the proof, the line $y = \frac{m}{n}x$ has a point with positive integral coordinates closest to the origin. Why is that point $(n, m)$?
3. Why does the point $(m - pn, kn - pm)$ lie on the line $y = \sqrt{k}x$? Provide an appropriate calculation.
4. Explain how to use $p < \frac{m}{n} < p + 1$ to prove that $0 < m - np < n$ and $0 < kn - pm < m$.
   What is the relevance of these inequalities?
5. Verify that $n(\sqrt{k} - p) = m - pn$ and $m(\sqrt{k} - p) = kn - pm$. This explains the choice of the point $(m - pn, kn - pm)$.
6. What is the method of the proof?
7. What is the idea of the proof? Limit yourself to three or four sentences.
Appendix H

The Carpet Proof of the Irrationality of $\sqrt{2}$

Proving that $\sqrt{2}$ is irrational invariably involves proof by contradiction. Consider the “carpet proof” as follows.

![Diagram of carpet proof]

Proof:

Assume that there exist positive integers $a$ and $b$ such that $a/b = \sqrt{2}$ or equivalently, $a^2 = 2b^2$. Let $a$ be minimal positive integer with the property that there exists $b$ such that $a^2 = 2b^2$.

Begin with one square of side $a$ and two squares of side $b$. Since $a^2 = 2b^2$, the area of the large square is equal to the sum of the areas of the two smaller squares.

The area of the overlap equals the sum of the areas of the two unshaded squares. But the overlap is a square with side of integer length less than $a$ and the two unshaded squares have sides of positive integer length as well. This contradicts the choice of $a$ as the smallest integer with the property that there exists an integer $b$ with $a^2 = 2b^2$.

Prompt questions:

1. The second diagram shows that the two $b \times b$ squares overlap. Prove that this must occur.
2. Explain why the area of the central square (overlap) equals the sum of the areas of the two shaded squares.
3. Explain why the central square and the two shaded squares each have sides of integer length.
4. The Well Ordering Property for the positive integers states that a nonempty subset of the positive integers has a smallest element. Where is this used in the proof?
5. What is the key idea (essence) of the proof? Please be concise. Your answer should fit in the space provided.