Pricing and Hedging of Variable Annuities on Mixed Funds under Lévy Processes

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Statistical Sciences
University of Toronto

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Abstract

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2015

Variable annuities (VAs) are deferred annuities whose future benefits are linked to the performance of a portfolio of securities during the deferment period. The VA policyholders can benefit from favorable movements in financial markets and are simultaneously protected against adverse events. As a result, variable annuities have options and option-like features embedded in their contracts. In this thesis, we consider pricing and hedging of such embedded options when the underlying reference portfolio is a mixed fund consisting of a bond index and a stock index. The bond index and the stock index are both assumed to follow exponential Lévy processes while the risk-free interest rate is modeled by an Ornstein-Uhlenbeck process. As a key mathematical result, we solve the valuation partial-integro differential equation in semi-analytic closed-form using Fourier transform method. In our numerical illustrations, variance gamma processes are considered and calibrated to financial data and their parameters are estimated under the real world and risk neutral probability measures. The formulae are then used to value the minimal rate of return guarantee when the underlying Lévy drivers are variance gamma processes. Our numerical results indicate that the method is fast, accurate and relatively straight-forward to implement. The sensitivities of the fair management fee with respect to different model parameters are also examined.

In addition to determining the fair management fee, effective hedging strategies are crucial for insurance companies to prevent potentially large losses. Here, since jump risk
and mortality risk result in an incomplete market, the usual dynamic hedging strategies are not applicable. We propose instead to resort to mean variance and local risk minimization hedging. By applying this technique, we are able to obtain semi-analytical closed-form hedging positions when using the mixed fund and an option on the mixed fund to hedge. Our numerical results show the advantage of local risk minimization over delta and delta-gamma hedging methods. Finally three common types of unit-linked life insurance: (i) pure endowment; (ii) term insurance; and (iii) endowment insurance; are used to demonstrate the efficiency of local risk minimization trading strategies.
Acknowledgements

I want to express my deep and sincere gratitude to my supervisors, Professor Sheldon Lin and Professor Sebastian Jaimungal. Their guidance illuminated my way towards my doctoral program. Their wide knowledge and logical way of thinking have been of great value to me. This work would hardly have been possible without their encouragement and support.

I would also like to thank Professor Andrei Badescu who is my committee member. His advice and patience are greatly appreciated. I would also like to thank Professor Samuel Broverman, Professor Nancy Reid, Professor Radford Neal, Professor Fang Yao, Professor Philip McDunnough and Professor Radu Craiu for their interesting courses that prepared me well for the challenges in my research. I also wish to thank the external examiner, Professor Mathieu Boudreault of the Université du Québec à Montréal, for his insightful comments and suggestions which improve the final form of the thesis.

The staff members in my department, especially Andrea Carter, Angela Fleury, Christine Bulguryemez, Annette Courtemanche and Dermot Whelan, are always willing to help. Their professionalism is highly regarded.

Finally I dedicate this thesis to my family and my parents, for being always understanding, supporting and loving me.
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$B_t$ Money market account ........................................... 23
$r_t$ Risk-free short rate ............................................... 23
$p(t,T)$ Price of risk free zero-coupon bond at time $t$ with maturity $T$ . 25
$S_t$ Stock index .......................................................... 26
$J^P$ $\mathbb{P}$-jump measure for the bond index ......................... 28
$\nu^P$ $\mathbb{P}$-Lévy density for the bond index .......................... 28
$\Phi^S(dt \times dy)$ Compensated $\mathbb{P}$-jump measure for stock index .......................... 29
$\Phi^P(dt \times dy)$ Compensated $\mathbb{P}$-jump measure for the bond index .......................... 29
$\hat{W}^S$ Standard Brownian motion driving $S$ under $\mathbb{Q}$ ................. 29
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Chapter 1

Introduction

1.1 Variable annuities and existing literature

Traditionally, insurance companies offer fixed annuities which guarantee a stream of fixed payments over the life of the contract. Fixed annuities were attractive to investors in the context of high interest rate market. However, bullish market and low interest rate environment motivate the investors to look for higher returns than those provided by the fixed annuities.

1.1.1 Variable annuities and unit-linked life insurance

Variable annuities were introduced in the 1970s in the United States (see Sloane (1970)). Their future benefits are based on the performance of a portfolio of securities including equities. Companies design VAs so that the policyholders can benefit from favorable movements in the markets but are protected against the downside movements in the market. Variable annuities in the U.S. are similar to the unit-linked annuities in the U.K. and the segregated funds in Canada.

Variable annuities are very attractive to investors because they are tax-deferred and they offer different types of guarantees. There are two common categories of guarantees including:

- The Guaranteed Minimum Death Benefits (GMDB)-where the investor’s beneficiaries receive a guaranteed minimum payment upon the investor’s death.
- The Guaranteed Minimum Living Benefits (GMLB)- where the investor receives a guaranteed stream of income while alive. The GMLB options can be categorized
in three main groups.

- Guaranteed Minimum Accumulation Benefits (GMAB) provide a guaranteed minimum survival benefit at some specified point in the future to protect policyholders against decreasing stock markets.

- Guaranteed Minimum Income Benefits (GMIB) provide a similar guaranteed value at some point in time. However, the guarantee only applies if this guaranteed value is converted into an annuity using given annuitization rates.

- The third kind of guaranteed minimum living benefits are so-called Guaranteed Minimum Withdrawal Benefits (GMWB). Here, a specified amount is guaranteed for withdrawals during the life of the contract as long as both the amount that is withdrawn within each policy year and the total amount that is withdrawn over the term of the policy stay within certain limits. Figure 1.1 gives a few sample paths of the fund value for the GMWB product. In this example, the maturity is 6 years and withdrawals happen during the first 3 years.

![Sample paths of the fund value for GMWB](image)

Figure 1.1: Sample paths of the fund value for GMWB with 1 million paid back over 3 years, semi-annual payments of 166K

Due to the innovative guarantee features, VAs have rapidly grown in popularity around the world in recent years. The 2014 IRI Fact Book (Insured Retirement Institute) reports the total sales for the variable annuities were $142.8 billion in 2013 - which contrasts with fixed annuities sales of $78.1 billion for the same year. The Figure
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1.2 plots the total sales for the VAs in the U.S. market from 2002 to 2013. From the plot, we can see the total sales increased from 2002 to 2008. In 2009 there was a drop in the total sales due to the subprime mortgage crisis in 2008. As the economy of U.S. gradually recovered, the surge in variable annuity assets began in 2010.

![Annual sale of VA in the US market](image)

Figure 1.2: Total sales of VAs in the U.S. market, Source: Morningstar.

U.K., Canada, Japan, and South Korea also experienced a similar rapid growth in VAs sales. As a result, almost all the major insurance companies in these countries are managing very large VAs portfolios. This success exists because of the opportunity VAs offer to manage long-term savings and to possibly provide post-retirement income. Their tax advantages also give incentives to invest in these products. The fact that retirees and near-retirees hold such a substantial portion of their assets in VAs motivates the need of valuation, hedging and risk management of these products. Considering the aging population in the world, and the contribution to the financing of post-retirement income that these contracts can bring, there should be a continuing expansion of these markets in the coming years.

Another product similar to variable annuities is the unit-linked life insurance. It appeared for the first time around 1950 in the Netherlands and around 1954 in the United States of America. Later, in 1957, they were introduced to the United Kingdom. Unit-linked life insurance is also known as equity-linked life insurance. Unlike traditional life insurance the benefits and possibly the premiums of a unit-linked life insurance are random and linked to the development of some specified reference portfolio. Such contracts are usually equipped with a minimal rate of return guarantee and allow the policyholder to decide how his premiums are to be invested. Hence, they transfer parts of the investment risk to the policyholder while, simultaneously, providing a limited exposure to the
downside risk and a high level of investment transparency. Moreover, the policyholder benefits directly from the upside potential of his investment and the flexibility of the contract allowing to adjust the riskiness of the investment depending on the remaining time of the policy. In recent years these specific life insurance products, especially those with capital guarantees, have become very popular in most countries with a fully developed insurance industry.

These kinds of insurance products combine financial and insurance risks and require techniques from both financial and actuarial mathematics for their proper treatments. The first researchers who analyzed unit-linked insurance products using modern financial mathematics were Brennan and Schwartz (1976) and Boyle and Schwartz (1977) who recognized that the payoff of a unit-linked with guarantee life insurance equaled the payoff of a European call option plus the guaranteed amount. Furthermore, referring to a law of large numbers argument they replaced the uncertainty of insured lives by their expected values and did not pay attention to mortality risk. For this reason they could treat unit-linked with guarantee life insurance contracts as contingent claims with financial uncertainty in a complete financial market setting and apply the standard valuation and hedging techniques that had been developed by Black and Scholes (1973). The idea of replacing the uncertain course of mortality by its expected development was later also used by Delbaen (1986), Bacinello and Ortu (1993) and Aase and Persson (1994) to price unit-linked life insurance contracts involving the martingale based financial techniques of Harrison and Pliska (1981) and Harrison and Kreps (1979).

In the literature, there are many papers dealing with valuation, hedging, and risk management of these guarantees. Milevsky and Posner (2001) used risk neutral valuation technique to price GMDB in a VA contract. Hardy (2001) considered valuation of the maturity guarantee under regime-switching lognormal model. Lin et al. (2009) used Esscher transform to determine an equivalent martingale measure for the fair valuation of a VA contract under a regime-switching model. Jiang and Chang (2010) used the Black-Scholes model to derive an analytical solution of the cost of the GMAB rider. In recent years, variable annuities with guaranteed minimum withdrawal benefits (GMWB) have attracted significant attention and sales. Milevsky and Salisbury (2006) propose the pricing formulations of GMWB with static and dynamic withdrawals under a constant interest rate. In fact, there are many papers - most of them in the insurance literature - written on the topic. For a selected bibliography and books on the topic, we refer to Hardy (2003).
In order to reflect real world products, the underlying reference portfolio will be composed of different assets such as equity, fixed-income and real estate. Such an investment fund is also known as a mix fund or mixed fund. Vellekoop et al. (2006) proposed a valuation and hedging strategy for a guaranteed minimal rate of return on a mix fund when bond index and stock index followed geometric Brownian motion and bond market was modeled by the Hull-White extension of the Vasiček short rate model. Also they derived delta-hedging positions to hedge the guaranteed returns on a mix fund. The mix fund provides the flexibility of choosing how much to invest in the stock index and the bond index for the policyholder, which makes this product very appealing. Risk-loving investors can choose more aggressive fund mixtures (e.g., 70% in the stock index and 30% in the bond index) and risk-averse investors can choose more conservative fund mixtures (e.g., 30% in the stock index and 70% in the bond index).

1.1.2 Modeling with jump processes

Almost all of the articles mentioned above assume a Black-Scholes type of financial market, in which asset prices are driven by a Brownian motion featuring a normal distribution of asset returns and continuous sample paths. Recent research, however, favors more general Lévy-process financial market models, in which the Brownian motion is replaced by a more general Lévy process resulting in models with more flexible asset return distributions and discontinuous sample paths. In the last decade, also the research departments of major banks started to accept Lévy processes as a valuable tool in their day-to-day modeling. This increasing interest to jump models in finance is mainly due to the following reasons.

1. Asset prices do jump, and some risks simply cannot be handled within continuous-path models.

2. Implied volatility smile in option markets indicates that the risk-neutral returns are non-Gaussian. While the smile itself can be explained within a model with continuous paths, the fact that it becomes much more pronounced for short maturities is a clear indication of the presence of jumps. In continuous-path models, the law of returns for shorter maturities becomes closer to the Gaussian law, whereas in reality and in models with jumps returns actually become less Gaussian as the horizon becomes shorter.
3. Jump processes correspond to genuinely incomplete markets, whereas all continuous-path models are complete with a small number of additional assets. This fundamental incompleteness makes it possible to carry out a rigorous analysis of the hedging error and find ways to improve the hedging performance using additional instruments such as liquid European options.

Figure 1.3 gives continuous compounded daily returns of S&P 500 and Dow Jones Corporate Bond Index from 2003-05-22 to 2013-05-22. In the plots, daily returns are widely dispersed in their amplitude and manifest frequent large peaks corresponding to jumps in the price. This high variability is a constantly observed feature of financial asset returns. In statistical terms this results in heavy tails in the empirical distribution of returns. This well-known fact leads to a poor representation of the distribution of returns by a normal distribution. For a normal random variable the probability of occurrence of a value three times the standard deviation is 0.0027, which implies that in a Gaussian model a daily return of such magnitude occurs less than once in 1.5 years. In Figure 1.3 we can see many points (red circle points) are outside the range of $[\mu - 3\sigma, \mu + 3\sigma]$. Furthermore, Eraker (2004) found empirical evidence for jumps in stock prices and jumps in the volatility whereas German (2002) and Ornthanalai (2014) have found out evidence of Lévy dynamics in stock prices.

Figure 1.3: S&P 500 daily returns vs Dow Jones bond index daily returns

A great advantage of exponential Lévy models is their mathematical tractability, which makes it possible to perform many computations explicitly and to present deep results of modern mathematical finance in a simple manner. This has led to an explosion of the literature on option pricing and hedging in exponential Lévy models in the late
1990s and early 2000s. As a consequence, research in general Lévy-process financial markets has gained a major role in modern financial mathematics over the last decade. Chan (1999), Gallucio (2001) and Lewis (2001) used Lévy processes to price financial derivatives without insurance risk exposures. Jaimungal (2004) studied the pricing problem for various equity-indexed annuity products with jumps in the underlying risky asset. In that work, the asset dynamics was assumed to follow an exponential Lévy process directly under the risk-neutral measure and mortality risk was treated via the actuarial present value principle. Within this framework, he obtained closed form solutions for the premium and the analogs of the Black-Scholes Greek hedging parameters. Jaimungal and Young (2005) discussed the pricing of equity-linked pure endowments in a finite variation Lévy-process financial market using indifference pricing, more precisely, the principle of equivalent utility. Gerber et al. (2013) proposed a method based on discounted joint density function to price the GMDB rider under jump diffusion model.

1.1.3 Stochastic interest rate

All authors mentioned above used constant interest rate when pricing and hedging the unit-linked life insurance. While these assumptions might be adequate for most options offered by the exchanges, it is sometime undesirable to extrapolate that these assumptions are also applicable to the guarantees embedded in unit-linked insurance products. Most of the options offered by exchanges typically are short-dated with maturity less than one year, and hence a Black-Scholes framework would provide a reasonable approximation for pricing purposes. In contrast, the embedded guarantees associated with unit-linked insurance products have maturities from 1 to 10 years. It is therefore unreasonable to assume that the interest rate would remain level for such a long duration. More reliable pricing models should allow for stochastic interest rates. Bacinello and Ortu (1994), Nielsen and Sandmann (1995) and Bacinello and Persson (2002) focused on the pricing of guarantees under stochastic interest rates. Lin and Tan (2003) and Kijima and Wong (2007) considered the pricing of equity-indexed annuities under stochastic interest rates. Donnelly, Jaimungal and Rubisov (2014) considered valuing GWB (guaranteed withdrawal benefits) under stochastic interest rates and volatility when the reference portfolio was a mixed fund composed of both equity and fixed-income securities.

1.1.4 Hedging under the incomplete market

In a complete market, a contingent claim can be replicated perfectly by an investment portfolio. In an incomplete market, however, the usual perfect hedging technique does
not work anymore. We have to choose other approaches to hedge derivatives. The risk minimization concept was first discussed in Föllmer and Sondermann (1986) when the asset price process was a martingale under the empirical measure. Schweizer (1991) introduced the concept of local risk minimization for price processes which were semimartingales and this criterion was similar to performing risk minimization using the minimal martingale measure. Møller (1998, 2001a, 2001b) considered a model describing the uncertainty of the financial market and a portfolio of insured individuals simultaneously, the risk minimizing trading strategies and the associated intrinsic risk processes were determined for different types of unit-linked life insurance contracts. Chan (1999) found out a locally risk minimizing strategy when the price process was driven by a general Lévy process. Coleman et al. (2006) used local risk minimization to study the discrete hedging of the guarantees embedded in a VA contract with both equity risk and interest rate risk and concluded that hedging with standard options is superior to hedging with the underlying. Riesner (2006) extended the Møller’s model. He supposed that the risky asset price process was discontinuous as it followed a geometric Lévy process, and obtained the risk minimizing hedging strategy of unit-linked life insurance contracts in a Lévy process financial market. However, Vandaele and Vanmaele (2008) pointed out that the result of Riesner (2006) was not correct, and found that the risk minimizing hedging strategy was not the locally risk minimizing hedging strategy under the original measure.

1.2 Summary of the thesis

In this thesis, we consider pricing and hedging of embedded guarantees in the variable annuities when the underlying reference portfolio is a mixed fund consisting of a bond index and a stock index. It is assumed that the bond index and the stock index follow exponential Lévy processes and the risk-free interest rate is modeled by the Vasicek model. We develop a pricing methodology via Fourier transform. In our numerical illustrations, variance gamma processes are considered and calibrated to financial data and their parameters are estimated under the real world and risk neutral probability measures. The pricing formulae are then used to value the minimal rate of return guarantee when the underlying Lévy drivers are variance gamma processes. Our numerical results indicate that the method is fast, accurate and relatively straightforward to implement. The sensitivities of the fair management fee with respect to different model parameters are also examined. Since the jump risk and mortality risk lead to an incomplete market framework, the usual perfect hedging technique does not work anymore.
Effective hedging strategies for variable annuities are crucial for insurance companies in preventing potentially large losses. We propose the mean variance hedging and local risk minimization hedging in this situation. By applying this technique, we are able to obtain semi-analytical closed-form hedging positions when using the mixed fund and an option on the mixed fund to hedge. Our numerical results show the advantage of local risk minimization over delta and delta-gamma hedging methods. Finally three types of unit-linked contracts, i.e. pure endowment, term insurance and endowment insurance are used to demonstrate the efficiency of local risk minimization trading strategies.

The thesis is structured as follows. In chapter 2, we give an introduction to the Lévy process including definitions, path properties, and common examples such as jump diffusion and variance gamma processes. Since the interest rate is assumed to be stochastic, we also review interest rate models and bond valuations. At the end of this chapter, we introduce general Lévy-process financial market together with measure change from real world to risk neutral measure. In chapter 3, pricing model under Lévy process and constant interest rate is derived. This simple case can serve as an example of our pricing methodology. We first review the pricing methodology under the Black-Scholes model. Then we derive the pricing PIDE for the option price under Lévy model. After applying Fourier transform, the PIDE is transformed to an ODE which has an explicit solution. The results via Fourier transform give general semi-analytic closed-form solutions that can be used for a large class of Lévy processes. In chapter 4, we deal with calibrations of model parameters. Since the hedging performance is very sensitive to the differences of Lévy densities between real world measure and risk neutral measure, choosing parameters becomes a very important task. We calibrate real world parameters of variance gamma processes using historical index values and risk neutral parameters of variance gamma processes using historical option prices. In chapter 5, we propose mean-variance hedging and local risk minimization hedging strategies in an incomplete market framework. Numerical examples show the advantage of local risk minimization over other hedging methods.

In chapter 6, the pricing methodology is extended to the general case of mix funds and the stochastic interest rate. To avoid calculations of joint density of the mix fund and the interest rate, we introduce forward measure using the bond price as a numéraire. We also derive the pricing PIDE for the option price. Then we use the Fourier transform and finite difference method to solve the PIDE. Comparisons among Fourier transform method, finite difference method and Monte-Carlo simulations indicate the method via
Chapter 1. Introduction

Fourier transform is fast, accurate and easy to implement. After prices of embedded options can be obtained, calculations of fair management fees are also given. Fair management fees are very important to the insurance company as they provide compensations for the risks they take. Accurate and efficient calculations of fair management fees are very crucial for them. At the end of chapter 6, sensitivities of the fair management fee with respect to different model parameters are also examined. We conclude that the jump size can affect the fair management fees and jump direction does not matter. In chapter 7, local risk minimization hedging for the guaranteed minimal rate of return is derived under the case of mix funds and the stochastic interest rate. Numerical examples indicate the advantage of local risk minimization hedging over delta and delta-gamma hedging.

Chapter 8 considers pricing and hedging of unit-linked life insurance contracts under the constant interest rate. As Møller (1998, 2001a), we assume stochastic independence between the financial market and the insurance model and consider them combined in a common product space. This picks up the idea to model the uncertain development of the stock index and the insured lives simultaneously not averaging away mortality. Then we derive the prices and local risk minimization hedging strategies for three common types of unit-linked life insurance: pure endowment, term insurance and endowment insurance. Numerical examples are used to demonstrate the efficiency of local risk minimization trading strategies. In chapter 9 we extend pricing and hedging of unit-linked life insurance contracts to the case of stochastic interest rate. Efficiency of local risk minimization in the presence of interest rate risk is demonstrated by numerical examples. Chapter 10 concludes the thesis and also describes the future works that can be done.
Chapter 2

Model setup

In this chapter we provide the modeling background required in this thesis. In doing so we begin with an introduction to the theory of Lévy processes and review those properties of a Lévy process which are beneficial for a Lévy-process financial market. Moreover, we briefly discuss specific Lévy processes that are in common use for financial market modeling, such as jump-diffusions and variance gamma processes. Since the interest rate is assumed to be stochastic in this thesis, we also review some interest rate models and bond valuations. At the end of this chapter, we introduce general Lévy-process financial model together with measure change from real world to risk neutral measure.

2.1 Preliminaries

This short section at the beginning provides assumptions made throughout the thesis and states frequently used and common notation. For a detailed treatment of basic tools such as martingales, semimartingales and stochastic integrations please refer to Protter (1990). Throughout this thesis we only consider insurance contracts and related financial markets, so we only treat the case of a finite time horizon \( T \in [0, \infty) \).

Definition 2.1.1. Suppose the probability space is \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) satisfies the usual conditions if \((\Omega, \mathcal{F}, \mathbb{P})\) is complete, all the null sets of \( \mathcal{F} \) are contained in \( \mathcal{F}_0 \), and \((\mathcal{F}_t)_{0 \leq t \leq T}\) is a right-continuous filtration, that is, \( \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \) are \( \sigma \)-algebras for \( 0 \leq s \leq t \leq T \), and

\[
\mathcal{F}_s = \bigcap_{t > s} \mathcal{F}_t, \quad \forall \ 0 \leq s \leq T.
\]

In particular, we assume \( \mathcal{F} = \mathcal{F}_T \).

For a filtration, \( \mathcal{F}_t \) is interpreted as the information known in the model at time \( t \)
which increases over time, i.e. a filtration describes how information is revealed in the model. All filtered probability spaces mentioned in this thesis are assumed to satisfy the usual conditions.

A stochastic process \( X = (X_t)_{0 \leq t \leq T} \) on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) is said to be càdlàg if its paths are right-continuous admitting left limits, i.e.

\[
\lim_{s \to t, s > t} X_s = X_t \quad \text{and} \quad \exists \lim_{s \to t, s < t} X_s =: X_{t-}.
\]

For such a stochastic process we define its jump at time \(0 \leq t \leq T\) by

\[
\Delta X_t =: X_t - X_{t-}, \quad \text{where we set} \quad X_{0-} = X_0.
\]

We always denote the expectation operator with respect to the canonical measure \(\mathbb{P}\) by \(\mathbb{E}[\cdot]\), whereas the expectation with respect to any other measure \(\mathbb{Q}\) is written as \(\mathbb{E}^Q[\cdot]\). In the context of this thesis we generally use \(\mathbb{P}\) for the so called real world or historical measure and respectively \(\mathbb{Q}\) for the so called risk neutral measure.

**Definition 2.1.2.** Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be some filtered probability space and let \(\mathbb{Q}\) be a second probability measure on it. Then \(\mathbb{P}\) is said to be locally equivalent with respect to \(\mathbb{Q}\) \((\mathbb{P} \operatorname{loc} \sim \mathbb{Q})\) if

\[
\mathbb{P}_t(A) = 0 \iff \mathbb{Q}_t(A) = 0, \quad \forall a \in \mathcal{F}_t, \quad 0 \leq t \leq T.
\]

\(\mathbb{P}_t\) and \(\mathbb{Q}_t\) denote restrictions of \(\mathbb{P}\) and \(\mathbb{Q}\) to the \(\sigma\)-algebra \(\mathcal{F}_t\), respectively.

### 2.2 Definitions and properties of Lévy processes

Lévy processes are named after the French mathematician Paul Lévy (1886-1971). Together with Andrei N. Kolmogorov (1903-1987) Lévy is considered as one of the founding fathers of the modern theory of stochastic processes.

Lévy processes are a class of stochastic processes that allow to have discontinuous paths, which are at the same time simple enough for applications, or at least to be used as building blocks of more realistic models.

**Definition 2.2.1.** A stochastic process \(X\) is a Lévy process if it is càdlàg, satisfies \(X_0 = 0\) and possesses the following properties:
Chapter 2. Model setup

- Independent increments: for every increasing sequence of times \( t_0, \cdots, t_n \), the random variables \( X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}} \) are independent.

- Stationary increments: the law of \( X_{t+h} - X_t \) does not depend on \( t \).

- Stochastic continuity: \( \forall \varepsilon, \lim_{s \to 0} P[|X_{t+s} - X_t| > \varepsilon] = 0 \).

The third condition does not imply in any way that sample paths are continuous. It is verified by the Poisson process. It serves to exclude processes with jumps at fixed (non-random) times, which can be regarded as "calendar effects" and are not interesting for our purpose. It means that for a given time \( t \), the probability of seeing a jump at \( t \) is zero: discontinuities occur at random times.

The notion of the jump measure is central for the theory of Lévy processes. We give its definition first. Jump measure is a kind of random measures and the definition of the random measure is as follows.

**Definition 2.2.2** (Random measure). Let \((\Omega, \mathbb{P}, \mathcal{F})\) be a probability space and \((E, \mathcal{E})\) a measurable space. Then \(M : \Omega \times \mathcal{E} \to \mathbb{R}\) is a random measure if

- For every \( \omega \in \Omega \), \( M(\omega, \cdot) \) is a measure on \( \mathcal{E} \).

- For every \( A \in \mathcal{E} \), \( M(\cdot, A) \) is measurable.

**Definition 2.2.3** (Poisson random measure). Let \((\Omega, \mathbb{P}, \mathcal{F})\) be a probability space, \((E, \mathcal{E})\) a measurable space and \( \nu \) a measure on \((E, \mathcal{E})\). Then \(M : \Omega \times \mathcal{E} \to \mathbb{R}\) is a Poisson random measure with intensity \( \mu \) if

- For all \( A \in \mathcal{E} \) with \( \mu(A) < \infty \), \( M(A) \) follows the Poisson law with parameter \( \mathbb{E}[M(A)] = \mu(A) \).

- For any disjoint sets \( A_1, \cdots, A_n, M(A_1), \cdots, M(A_n) \) are independent.

In particular, the Poisson random measure is a positive integer-valued random measure. It can be constructed as the counting measure of randomly scattered points, as shown by the following theorem.

**Theorem 2.2.4.** Let \( \mu \) be a \( \sigma \)-finite measure on a measurable subset \( E \) of \( \mathbb{R}^d \). Then there exists a Poisson random measure on \( E \) with intensity \( \mu \).

**Proof.** Please refer to Proposition 2.14 in Cont and Tankov (2004). \( \square \)
Corollary (Exponential formula). Let $M$ be a Poisson random measure on $(E, \mathcal{E})$ with intensity $\mu$, $B \in \mathcal{E}$ and let $f$ be a measurable function with $\int_B |e^{f(x)} - 1| \mu(dx) < \infty$. Then

$$E \left[ e^{\int_B f(x) M(dx)} \right] = \exp \left[ \int_B (e^{f(x)} - 1) \mu(dx) \right]$$

Definition 2.2.5 (Jump measure). Let $X$ be a $\mathbb{R}^d$-valued càdlàg process. The jump measure of $X$ is a random measure on $\mathcal{B}([0,\infty) \times \mathbb{R}^d)$ defined by

$$J(B) = \# \{ t : \Delta X_t \neq 0 \text{ and } (t, \Delta X_t) \in B \}.$$ 

The jump measure of a set of the form $[s,t] \times A$ counts the number of jumps of $X$ between $s$ and $t$ such that their sizes fall in $A$. For a counting process, since the jump size is always equal to 1, the jump measure can be seen as a random measure on $[0,\infty)$.

Definition 2.2.6 (Lévy measure). Let $X$ be an $\mathbb{R}^d$-valued Lévy process. The measure $\nu$ defined by

$$\nu(A) = E \left[ \# \{ t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A \} \right], \quad A \in \mathcal{B}(\mathbb{R}^d)$$ 

is called the Lévy measure of $X$.

The following is the decomposition theorem about Lévy process.

Theorem 2.2.7 (Lévy-Itô-decomposition). Let $X$ be an $\mathbb{R}^d$-valued Lévy process with Lévy measure $\nu$. Then

1. The jump measure $J$ of $X$ is a Poisson random measure on $[0,\infty) \times \mathbb{R}^d$ with intensity $dt \times \nu$.

2. The Lévy measure $\nu$ satisfies $\int_{\mathbb{R}^d} (||x||^2 \wedge 1) \nu(dx) < \infty$.

3. There exists $\gamma \in \mathbb{R}^d$ and a $d$-dimensional Brownian motion $W$ with covariance matrix $A$ such that

$$X_t = \gamma t + W_t + M^J_t + \lim_{\varepsilon \downarrow 0} M^\varepsilon_t,$$

where

$$M^J_t = \int_{|x| \geq 1, s \in [0,t]} x J(ds \times dx),$$

$$M^\varepsilon_t = \int_{\varepsilon \leq |x| < 1, s \in [0,t]} x \{ J(ds \times dx) - \nu(dx)ds \} - \int_{\varepsilon \leq |x| < 1, s \in [0,t]} x \Phi(ds \times dx)$$
where $\Phi(ds \times dx) = J(ds \times dx) - \nu(dx)ds$ is defined as compensated measure. The three terms are independent and the convergence in the last term is almost sure and uniform in $t$ on $[0, T]$.

**Proof.** The Lévy-Itô-decomposition was originally found by Lévy using a direct analysis of the paths of Lévy processes and completed by Itô. There are many proofs available in the literature. A probabilistic approach close to the original proof of Lévy is given in Gikhman and Skorokhod (1996).

The Lévy-Itô-decomposition entails that for every Lévy process there exist a vector $\gamma$, a positive definite matrix $A$ that uniquely determine its distribution. The triple $(A, \nu, \gamma)$ is called the characteristic triplet of the process $X_t$.

Another immediate consequence of the Lévy-Itô-decomposition is the Lévy-Khinchin formula for the characteristic function of a Lévy process.

**Theorem 2.2.8** (Lévy-Khinchin representation). Let $X$ be a Lévy process with characteristic triplet $(A, \nu, \gamma)$. Its characteristic function is given by

$$E[e^{iuX_t}] = \exp \left\{ t \left( i\gamma u - \frac{Au^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx) \right) \right\} \quad (2.1)$$

According to the Lévy-Khinchin formula the characteristic triplet of a Lévy process uniquely determine its characteristic function and hence its distribution.

**Theorem 2.2.9** (Path properties). Let $L$ be a Lévy process and $(A, \nu, \gamma)$ its characteristic triplet. Then it holds:

1. $\nu \equiv 0$ if and only if almost all paths of $L$ are continuous.
2. (a) (Finite activity) If $\nu(\mathbb{R}) < \infty$, then almost all paths of $L$ have only finitely many jumps on any compact interval.
   (b) (Infinite activity) If $\nu(\mathbb{R}) = \infty$, then almost all paths of $L$ have infinitely many jumps on any compact interval.
3. (a) (Finite variation) If $A = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, then almost all paths of $L$ are of finite variation on any compact interval.
   (b) (Infinite variation) If $A > 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, then almost all paths of $L$ are of infinite variation on any compact interval.

**Proof.** Please refer to the Appendix A.
2.3 Common examples

Our aim in this section is to discuss some examples of Lévy processes frequently used in the financial market modeling. Classified by their jump activities (cf. Theorem 2.2.9), we distinguish between two main categories: finite and infinite activity Lévy-processes. For a related discussion we refer to Chapter 4 in Cont and Tankov (2004).

2.3.1 Finite activity Lévy-processes: jump-diffusions

The most simple examples and at the same time the most well-known finite activity Lévy processes are clearly the Brownian motion possessing almost sure continuous sample paths and suitable normally distributed increments, and the Poisson process.

We recall that in the Black-Scholes model the log-asset-price is modeled using a diffusion, that is,

\[ X_t = \beta t + cW_t, \quad 0 \leq t \leq T, \]

for \( \beta \in \mathbb{R}, c > 0 \) and \( \{W_t\}_{0 \leq t \leq T} \) a standard Brownian motion. Another important example is the compound Poisson process which we write as

\[ X_t = \sum_{i=1}^{N_t} Y_i, \quad 0 \leq t \leq T \]

where the jump sizes \( Y_i \) are i.i.d. with distribution having a density \( f(x) \) with respect to the Lebesgue measure and where \( \{N_t\}_{0 \leq t \leq T} \) is a Poisson process with intensity \( \lambda t \), independent from \( Y_i \), for all \( i \). Compound Poisson processes are completely characterized as Lévy processes with almost sure piecewise constant sample paths. For a compound Poisson process the Lévy measure is particularly simple.

**Theorem 2.3.1.** Let \( X \) be a compound Poisson process. Then its Lévy measure is given by

\[ \nu(dx) = \lambda f(x)dx \]

**Proof.** Let \( B \in \mathcal{B} \). Then, conditionally on the trajectory of the Poisson process \( N_t \), the measure \( J([0,1] \times B) \) is a sum of \( N_1 \) i.i.d. Bernoulli random variables taking value 1 with
probability \( \int_B f(x)dx \). Hence,

\[
\nu(B) = \mathbb{E}[J([0, 1] \times B)] = \mathbb{E}(\mathbb{E}[J([0, 1] \times B)|\mathcal{F}_1]) \\
= \mathbb{E} \left[ N_1 \int_B f(x)dx \right] = \lambda \int_B f(x)dx
\]

In particular, this shows once again that each compound Poisson process is a finite activity Lévy process. Together with the Brownian motion compound Poisson processes constitute so called jump-diffusions. A Lévy process of jump-diffusion type has the following form

\[
X_t = \beta t + cW_t + \sum_{i=1}^{N_t} Y_i, \quad 0 \leq t \leq T
\]

where \( \beta \in \mathbb{R}, c > 0, (W_t)_{0 \leq t \leq T} \) a standard Brownian motion and \( \sum_{i=1}^{N_t} Y_i, 0 \leq t \leq T, \) is a compound Poisson process. Opposed to the Black-Scholes model, using a jump-diffusion to describe log-prices in a financial market implies that the “normal” evolution of log-prices is given by a diffusion process interrupted by jumps at random occurrence. These jumps might represent rare events due to unexpected new market information resulting in, for example, large drawdowns or large quotation gains. To make the definition of a jump-diffusion model complete one needs to specify the jump-size distribution. The challenge here is to find the “right” tail behavior to reproduce extremal events, since the tail behavior of the process depends significantly on the tail behavior of the jump-size distribution. There are essentially two jump-size distributions considered in the context of financial markets. They are especially attractive since at least for European style options both lead to an almost closed option pricing formula in the sense of being a quickly converging series. First, the Merton model (cf. Merton (1976)) assumed that the jump-sizes \( Y_i \) followed a normal distribution: \( Y_i \sim \mathcal{N}(\mu, \delta^2) \). This implies that the probability density of \( X_t \) satisfies

\[
p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{\lambda t}{k!} \frac{\exp \left( -\frac{(x-\beta t-\mu)^2}{2(\sigma^2 t + k\delta^2)} \right)}{\sqrt{2\pi(\sigma^2 t + k\delta^2)}}
\]

and its Lévy measure is given by

\[
\nu(dx) = \frac{\lambda}{\delta \sqrt{2\pi}} \exp \left( -\frac{(x-\mu)^2}{2\delta^2} \right) dx
\]
The second common jump-diffusion model is known as the Kou model (cf. Kou (2002)). Here the jump size distribution is assumed to follow an asymmetric double exponential distribution with Lebesgue density

\[ f(x) = p\eta_1 e^{-\eta_1 x} 1_{[0,\infty)}(x) + q\eta_2 e^{\eta_2 x} 1_{(-\infty,0)}(x), \quad \eta_1 > 0, \eta_2 > 0, \]

where \( p, q \geq 0 \) and \( p + q = 1 \). The jumps may be interpreted as rare events due to unexpected new market information. According to Kou (2002) this model featured several advantages compared to the Black-Scholes financial market. First, from empirical investigations it is widely known that asset return distributions incorporate asymmetric leptokurtic features. This means they may be skewed to the left and may have a higher peak and two asymmetric heavier tails than those of a normal distribution. This follows by the fact that markets tend to have both overreaction and underreaction to various good or bad news. Further, in an option pricing framework one usually observes a volatility smile effect. Both the leptokurtic features and the volatility smile effect can be explained by the asymmetric double exponential jump diffusion model. Second, under a specific change of measure, it is possible to derive in this model closed-form solutions to a number of option pricing problems such as European call and put options as well as path dependent options like American or barrier options (cf. Kou (2002), Kou and Wang (2003) and Kou and Wang (2004)).

### 2.3.2 Infinite activity Lévy-processes

The most frequently used infinite activity Lévy processes are generated through Brownian subordination with an independent increasing Lévy process. This means a Brownian motion with a possible drift is evaluated at a different, new and stochastic time scale which is given by an independent subordinator. This time change is interpreted in financial terms as business time. Those models stay in the class of Lévy processes, that is, if \( \beta t + cW_t \) is a diffusion and \( \{\tau_t\}_{0 \leq t \leq T} \) is an independent subordinator, then the process \( \beta \tau_t + cW_{\tau_t} \) is again a Lévy process (cf. Cont and Tankov (2004) Theorem 4.2). Although generated through a Brownian motion, the Lévy-Itô-decomposition of this kind of Lévy processes does not necessarily contain a Brownian motion leading to purely discontinuous Lévy processes. Modeling asset prices with purely discontinuous but infinite activity Lévy processes is justified in the literature (e.g. Carr, Geman, Madan and Yor (2002)) by the argument that the jump structure with an infinite number of arbitrarily small jumps is already rich enough to give a nontrivial small time behavior.
Chapter 2. Model setup

2.3.3 Variance gamma processes

In this thesis, the Lévy density used in numerical examples is a variance gamma process. A variance gamma process is a Lévy process of infinite activity but of finite variation. It was first studied in Madan and Seneta (1990) for the symmetric case (where $\theta = 0$ in (2.2) below). The general case of an asymmetric variance gamma processes was then developed in Madan and Milne (1991) and Madan, Carr and Chang (1998). Besides the volatility, those processes feature only two additional parameters allowing to control skewness and kurtosis of the distribution. The additional parameters also correct for pricing biases of the Black Scholes model. In Madan, Carr and Chang (1998), they showed that the superior performance of the VG model was reflected in orthogonality tests conducted on the pricing errors. Option pricing errors from the Black Scholes model and the symmetric special case of the VG model were observed to be correlated with the degree of moneyness and the maturity of the options. Orthogonality tests showed that the VG model was relatively free of these biases. This is also the main reason that we use variance gamma process in this thesis.

The VG process is obtained by evaluating Brownian motion with drift at a random time given by a gamma process. The gamma process $\gamma(t; \mu, \nu)$ with mean rate $\mu$ and variance rate $\nu$ is the process of independent gamma increments over non-overlapping intervals of time $(t, t+h)$. The density $p_h(x)$, of the increments $x = \gamma(t+h; \mu, \nu) - \gamma(t; \mu, \nu)$ is given by the gamma density function with mean $\mu h$ and variance $\nu h$. Specifically,

$$p_h(x) = \left(\frac{\mu^2}{\nu}\right)^{\frac{x^2}{\nu}} \frac{x^{\mu^2 h - 1} e^{-\frac{x^2}{\nu}}}{\Gamma\left(\frac{\mu^2 h}{\nu}\right)}, \quad x > 0$$

where $\Gamma(x)$ denotes the gamma function. The gamma density has a characteristic function, $\phi_{\gamma(t)}(u) = \mathbb{E}[\exp(iu\gamma(t; \mu, \nu))]$, given by,

$$\phi_{\gamma(t)}(u) = \left(\frac{1}{1 - i u \frac{\mu}{\nu}}\right)\frac{\mu^2}{\nu}$$

Given now a gamma process, a standard Brownian motion $(W_t)_{0 \leq t \leq T}$, some $\sigma > 0$ and $\theta \in \mathbb{R}$, the variance gamma process is defined by

$$X_t = \theta \gamma(t; 1, \nu) + \sigma \gamma(t; 1, \nu), \quad 0 \leq t \leq T,$$  (2.2)
which is a Brownian motion with drift evaluated at a time given by the gamma process. The VG process has three parameters: (i) $\sigma$ the volatility of the Brownian motion with drift, (ii) $\nu$ the variance rate of the gamma time change and (iii) $\theta$ the drift in Brownian motion with drift. The process therefore provides two dimensions of control on the distribution over and above that of the volatility. We observe below that control is attained over the skew via $\theta$ and over kurtosis with $\nu$. The Lévy measure of a variance gamma process has the following form

$$\nu(dy) = a \left( \frac{e^{-b^+y}}{y} 1_{\{y>0\}} + \frac{e^{-b^-|y|}}{|y|} 1_{\{y<0\}} \right) dy.$$  \hspace{1cm} (2.3)

where

$$a = \frac{1}{v},$$

$$b^+ = \frac{\sqrt{\theta^2 + 2\sigma^2/v} - \theta}{\sigma^2},$$

$$b^- = \frac{\sqrt{\theta^2 + 2\sigma^2/v} + \theta}{\sigma^2}.$$

The form of its Lévy measure also implies that the VG process has paths of finite variation, since $\int_{|x|\leq 1} |x| \nu(dx) < \infty$. Further the asymptotic behavior $\nu(dx) \sim \frac{1}{|x|} dx, \,(x \to 0)$, yields the infinite activity of the process. Another remarkable fact is that the variance gamma process may be written as a difference of two independent gamma process implying as well the almost sure finite variation of its sample paths. From the moments of the variance gamma process,

$$E[X_t] = \theta t$$

$$E[(X_t - E[X_t])^2] = (\theta^2 v + \sigma^2) t$$

$$E[(X_t - E[X_t])^3] = (2\theta^3 v^2 + 3\sigma^2 \theta v) t$$

$$E[(X_t - E[X_t])^4] = (3\sigma^4 v + 12\sigma^2 \theta^2 v^2 + 6\theta^4 v^3) t + (3\sigma^4 + 6\sigma^2 \theta^2 v + 3\theta^4 v^2) t^2$$

one derives that $\theta$ is the parameter generating skewness while kurtosis is primarily influenced by $v$.

Contrary to much of the literature on option pricing, the proposed VG process for log stock prices has no continuous martingale component. In contrast, it is a pure jump process that accounts for high activity (as in Brownian motion) by having an infinite number of jumps in any interval of time. Poisson type jump components in jump dif-
fusion models are designed to address these concerns. For the VG process, however, as the Black Scholes model is a parametric special case already, and high activity is already accounted for, it is not necessary to introduce a diffusion component in addition; hence the absence of a continuous martingale component.

Unlike Brownian motion, the sum of the absolute log-price changes is finite for the VG process (Brownian motion is a process of infinite variation but finite quadratic variation and the log-price changes must be squared before they are summed, to get a finite result). Since the VG process is one of finite variation, it can be written as the difference of two increasing processes, the first of which accounts for the price increases, while the second explains the price decreases. In the case of VG process, the two increasing processes that are differenced to obtain the VG process are themselves gamma processes.

The variance gamma distribution is a subclass of the so-called CGMY distributions. CGMY distributions are infinitely divisible and were introduced in Carr, Geman, Madan and Yor (2002) as an extension of the variance gamma distribution to model log returns in financial markets. The Lévy density of CGMY process is

$$\nu(dx) = \left( \frac{c_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} 1_{x>0} + \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_- |x|} 1_{x>0} \right) dx$$

with $\alpha_+ < 2$ and $\alpha_- < 2$.

### 2.4 Interest rate models and the bond valuation

One of the most crucial economic factors in the pricing and hedging of variable annuities is the term structure of interest rates. In this section, we will provide an overview of some aspects of interest rate models and bond valuations.

**Definition 2.4.1.** A zero coupon bond with maturity $T$, also called a $T$-bond, is a contract which guarantees the holder $\$1$ to be paid on the date $T$. The price at time $t$ of a bond with maturity $T$ is denoted by $p(t, T)$.

We now make an assumption to guarantee the existence of a sufficiently rich bond market.

**Assumption.** We assume that

1. There exists a (frictionless) market for $T$-bonds for every $T > 0$. 

2. For every fixed $T$, the process $\{p(t, T); 0 \leq t \leq T\}$ is an optional stochastic process with $p(t, t) = 1$ for all $t$.

3. For every fixed $t$, $p(t, T)$ is a $\mathbb{P}$-a.s. continuously differentiable in the $T$ variable.

Given the above bond market, we can define a number of riskless interest rates.

**Definition 2.4.2.**

1. The simple **forward rate for** $[S, T]$ **contracted at** $t$, henceforth referred to as the LIBOR forward rate, is defined as
   $$L(t; S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)}.$$  

2. The simple **spot rate for** $[S, T]$, henceforth referred to as the LIBOR spot rate, is defined as
   $$L(S, T) = -\frac{p(S, T) - 1}{(T - S)p(S, T)}.$$  

3. The continuously compounded **forward rate for** $[S, T]$ **contracted at** $t$ is defined as
   $$R(t; S, T) = -\frac{\log(p(t, T)) - \log(p(t, S))}{T - S}.$$  

4. The **continuously compounded spot rate**, $R(S, T)$, for the period $[S, T]$ is defined as
   $$R(S, T) = -\frac{\log(p(S, T))}{T - S}.$$  

5. The **instantaneous forward rate with maturity** $T$, **contracted at** $t$, is defined by
   $$f(t, T) = -\frac{\partial \log(p(t, T))}{\partial T}.$$  

6. The instantaneous **short rate at time** $t$ is defined by
   $$r(t) = f(t, t).$$

We note that spot rates are forward rates where the time of contracting coincides with the start of the interval over which the interest rate is effective, i.e. $t=S$. The instantaneous forward rate, which will be of great importance below, is the limit of the continuously compounded forward rate when $S \to T$. It can thus be interpreted as the
riskless interest rate, contracted at \( t \), over the infinitesimal interval \([T, T+dt]\).

We now go on to define the money market account process \( B \).

**Definition 2.4.3.** The money market account process \( B_t \) is defined by

\[
B_t = e^{\int_0^t r(s) ds}
\]

i.e.

\[
dB(t) = r(t)B(t)dt, \quad B(0) = 1.
\]

The interpretation of the money market account is that you may think of it as describing a bank with the stochastic short rate \( r \). It can also be shown that investing in the money market account is equivalent to investing in a self-financing “rolling over” trading strategy, which at each time \( t \) consists entirely of “just maturing” bonds, i.e. bonds which will mature at \( t + dt \).

We recall the following fundamental result in the arbitrage theory.

**Theorem 2.4.4.** Let \( X \in \mathcal{F}_T \) be a \( T \)-claim, i.e., a contingent claim paid out at time \( T \), and let \( Q \) be the “risk neutral” martingale measure with \( B \) as the numeraire. Then the arbitrage free price is given by

\[
\Pi(t; X) = \mathbb{E}^Q \left[ e^{-\int_t^T r_s ds} X \middle| \mathcal{F}_t \right]. \tag{2.4}
\]

In particular we have

\[
p(t, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]. \tag{2.5}
\]

In the literature there are a large number of proposals on how to specify the \( Q \)-dynamics for the risk-free short rate \( r_t \). We present a (far from complete) list of the most popular models. If a parameter is time dependent this is written out explicitly. Otherwise all parameters are constant and positive.

1. Vasiček

\[
dr_t = \kappa(\theta - r_t)dt + \sigma R d\hat{W}_t^R \tag{2.6}
\]

2. Cox-Ingersoll-Ross(CIR)

\[
dr_t = \kappa(\theta - r_t)dt + \sigma \sqrt{r_t} d\hat{W}_t^R
\]
3. Dothan

\[ dr_t = ar_t dt + \sigma^R r_t \hat{d}W^R_t \]

4. Black-Derman-Toy (BDT)

\[ dr_t = \Theta(t)r_t dt + \sigma^R(t)r_t \hat{d}W^R_t \]

5. Ho-Lee

\[ dr_t = \Theta(t)dt + \sigma^R \hat{d}W^R_t \]

6. Hull-White (extended Vasiček)

\[ dr_t = \kappa(t)\{\theta(t) - r_t\} dt + \sigma^R(t) \hat{d}W^R_t \]

7. Hull-White (extended CIR)

\[ dr_t = \kappa(t)\{\theta(t) - r_t\} dt + \sigma^R(t) \sqrt{r_t} \hat{d}W^R_t \]

where \( \hat{W}^R_t \) is standard Brownian motion under risk neutral measure \( \mathbb{Q} \).

We now briefly comment on the models listed above. The models of Vasiček, Ho-Lee and Hull-White all describe the short rate using a linear SDE. Such SDEs are easy to solve and the corresponding \( r \)-processes can be shown to be normally distributed. In contrast with the linear models above, in the Dothan model the short rate will be log-normally distributed, which means that in order to compute bond prices we are faced with determining the distribution of an integral of log-normal random variables. This is analytically intractable. We also note that all of the models above, except Dothan, BDT and Ho-Lee exhibit the phenomenon of mean reversion, i.e., the short rate has a tendency to revert to a (possibly time dependent) mean value. Mean reversion is a typical requirement of a short rate model (as opposed to stock price models). The reason for this is basically political/institutional: if the short rate becomes very high one expects that the government and/or central bank will intervene to lower it.

In this thesis we will use Vasiček model for illustration purposes. Vasiček model has good computational tractability and mean reversion property. In the future, our methodology can be extended to other interest rate models without difficulty.
It can be shown under the dynamics of (2.6) a closed-form solution for \( r_t \) exists and is given by

\[
r_t = r_0 e^{-\kappa t} + \theta \int_0^t \kappa e^{-\kappa(t-u)} du + \sigma R \int_0^t e^{-\kappa(t-u)} d\tilde{W}_R^R. \tag{2.7}
\]

The stationary distribution of \( r_t \) is a normal distribution with mean \( \theta \) and standard deviation \( \sigma R / \sqrt{2\kappa} \). Hence \( \theta \) is called long term mean level and \( \kappa \) is called speed of reversion.

It should be pointed out that the Vasićek model could generate negative short rate since \( r_t \) is a normal random variable for each \( t \). However, in most practical applications the probability of having a negative interest rate is very small and hence it still serves as a reasonable model due to its tractability. For further information on term structure, interest rate models and their parameter estimation issues, see James and Webber (2000).

**Theorem 2.4.5.** The price of risk free zero-coupon bond \( p(t,T) = \mathbb{E}^Q \left( e^{-\int_0^T r_s^t ds} \mid \mathcal{F}_t \right) \) will satisfy the following PDE:

\[
\begin{align*}
p_t + \kappa (\theta - r) p_r + \frac{1}{2} \sigma^R p_{rr} &= rp \\
p(T, T) &= 1
\end{align*} \tag{2.8} \tag{2.9}
\]

**Proof.** Since \( e^{-\int_0^t r_s^t ds} p(t, T) = \mathbb{E}^Q \left( e^{-\int_0^T r_s^t ds} \mid \mathcal{F}_t \right) \) is a Doob-martingale under \( \mathbb{Q} \), applying Itô’s lemma to the LHS leads to

\[
d \left( e^{-\int_0^t r_s^t ds} p(t, T) \right) = e^{-\int_0^t r_s^t ds} \left[ \left( -rp + p_t + \kappa (\theta - r) p_r + \frac{1}{2} \sigma^R p_{rr} \right) dt + p_r \sigma R d\tilde{W}_R^R \right]
\]

Applying zero drift condition to the above equation, we obtain (2.8).

**Theorem 2.4.6.** The solution to the above PDE is

\[
p(t, T) = e^{A_t(T) - C_t(T)r_t}
\]

where

\[
C_t(T) = \frac{1}{\kappa} \left( 1 - e^{-\kappa(T-t)} \right)
\]

\[
A_t(T) = \left[ \theta - \frac{1}{2} \left( \frac{\sigma^R}{\kappa} \right)^2 \right] \left( C_t(T) - (T-t) \right) - \frac{[\sigma^R C_t(T)]^2}{4\kappa}
\]

**Proof.** Since the coefficients of equation (2.8) are linear, the solution should have affine
Chapter 2. Model setup

structure. Suppose it has the following form

\[ p(t, T) = e^{A_t(T) - C_t(T) r_t} \]

Plug it into the equation (2.8), then the ODEs satisfied by \( A_t(T) \) and \( C_t(T) \) are

\[
\begin{align*}
-C' + \kappa C - 1 &= 0 \\
A' - \kappa \theta C + \frac{1}{2}(\sigma^R)^2 C^2 &= 0
\end{align*}
\]

with boundary condition \( A_T(T) = 0 \) and \( C_T(T) = 0 \). The functions \( A \) and \( C \) can be obtained by solving ODEs.

The dynamics of \( p(t, T) \) under \( \mathbb{Q} \) is obtained by applying Itô’s Lemma

\[
\frac{d p(t, T)}{p(t, T)} = r_t dt - C_t(T) \sigma^R d\hat{W}^R_t. \tag{2.10}
\]

2.5 Lévy models for the stock index and the bond index

Now that we introduced the most significant properties of a Lévy process and after having seen many examples of Lévy processes, we consider the Lévy process financial market that will be used in this thesis.

Suppose the stock index \( S_t \) follows an exponential Lévy process with the canonical decomposition into a drift, a pure diffusion and a pure jump process; that is,

\[
S_t = S_0 \exp\{\gamma^S t + \sigma^S W^S_t + L^S_t\} = S_0 \exp\left\{\gamma^S t + \sigma^S W^S_t + \int_0^t \int_{-\infty}^{+\infty} y J^S(ds \times dy)\right\}
\]

where \( \{W^S_t\}_{0 \leq t \leq T} \) is a \( \mathbb{P} \)-standard Brownian motion and \( L^S \) is a pure \( \mathbb{P} \)-Lévy jump process with jump measure \( J^S \).

The predictable compensator \( A^S_t \) of the jump component is defined as the \( \mathcal{F}_t \)-adapted process which makes \( L^S_t - A^S_t \) a \( \mathbb{P} \)-martingale. This process is expressed in terms of the Lévy density \( \nu^S \), which is independent of \( t \) because of the stationarity of Lévy process, as follows:

\[
A^S_t = t \int_{-\infty}^{\infty} y \nu^S(dy).
\]
We assume that the jump process has finite variation, that is,
\[ \int_{-\infty}^{\infty} |y| \nu^S(dy) < +\infty \]

The Lévy-Khintchine representation of the log-stock process is \((\sigma^S)^2, \nu^S, \gamma^S)\), and the characteristic function of \(Y_t \equiv \log(S_t)\) is given by
\[
\Psi_{Y_t}(z) = \mathbb{E}[\exp\{izY_t\}] = \exp\{t\psi_{Y_t}(z)\}
\]
where the cumulant \(\psi_{Y_t}(z)\) is provided by the Lévy-Khintchine theorem resulting in
\[
\psi_{Y_t}(z) = i\gamma^S z - \frac{1}{2}(\sigma^S)^2 z^2 + \int_{-\infty}^{\infty} (e^{zy} - 1) \nu^S(dy)
\]

The drift parameter \(\gamma^S\) is chosen so that the price process \(S_t\) has an observed drift \(\mu^S\). This can be achieved by setting
\[
\mu^S = \Psi_{Y_t}(-i) \Rightarrow \gamma^S = \mu^S - \frac{1}{2}(\sigma^S)^2 - \int_{-\infty}^{\infty} (e^{y} - 1) \nu^S(dy)
\]
where we have further assumed the integrability condition
\[ \int_{-\infty}^{\infty} (e^{y} - 1) \nu^S(dy) < +\infty \]
on the Lévy density.

The dynamics of the stock index \(S_t\) can be obtained by applying Itô’s Lemma for jump process:
\[
dS_t = S_{t-} \left\{ \left( \gamma^S + \frac{1}{2}(\sigma^S)^2 \right) dt + \sigma^S dW_t^S + \int_{-\infty}^{\infty} [e^{y} - 1] J^S(dt \times dy) \right\} \tag{2.11}
\]
The subscript \(t-\) represents the process just prior to any jump at time \(t\).

Next we will focus on the bond index. The bond fund portfolio invests in a basket of bonds with laddered maturities. It is computed from the prices of selected bonds (typically a weighted average). It can be categorized based on their broad characteristics, such as whether they are composed of government bonds, municipal bonds, corporate bonds, high-yield bonds, mortgage-backed securities, etc. They can also be classified based on
their credit rating or maturity. Most bond indices are weighted by market capitalization. This results in the bums problem, in which less creditworthy issuers with a lot of outstanding debt constitute a larger part of the index than more creditworthy ones. In Vellekoop et al. (2006), they supposed the bond fund portfolio was continuously rebalanced in a self-financing way to match certain desire values of its interest rate sensitivity, or “Duration”. In fact, we do not need to care about what the weight for each bond is and how the weight changes over time. We can focus on the behavior of the bond fund portfolio just like modeling the stock index.

Suppose the bond index $\tilde{P}_t$ follows exponential Lévy process, i.e.

$$\tilde{P}_t = \tilde{P}_0 \exp\left\{\gamma^P t + \sigma^P W^P_t + L^P_t\right\} = \tilde{P}_0 \exp\left\{\gamma^P t + \sigma^P W^P_t + \int_0^t \int_{-\infty}^{+\infty} y J^P(ds \times dy)\right\}$$

where $\{W^S_t\}_{0 \leq t \leq T}$ is a $\mathbb{P}$-standard Brownian motion and the correlation with $W^S_t$ is $\rho_{PS}$ and $L^P$ is a pure $\mathbb{P}$-Lévy jump process with jump measure $J^P$. We also suppose $J^S$ and $J^P$ are independent, i.e. they will never jump together.

The predictable compensator $A^P_t$ of the jump component is defined as the $\mathcal{F}_t$-adapted process which makes $J^P_t - A^P_t$ a $\mathbb{P}$-martingale. This process is expressed in terms of the Lévy density $\nu^P$, as follows:

$$A^P_t = t \int_{-\infty}^{\infty} y \nu^P(dy).$$

We assume that the jump process has finite variation, that is,

$$\int_{-\infty}^{\infty} |y| \nu^P(dy) < +\infty$$

Similar to the case of stock index, the drift parameter $\gamma^P$ is chosen so that the price process $\tilde{P}_t$ has an observed drift $\mu^P$. This can be achieved by setting

$$\gamma^P = \mu^P - \frac{1}{2}(\sigma^P)^2 - \int_{\mathbb{R}} (e^y - 1) \nu^P(dy)$$

where we have further assumed the integrability condition

$$\int_{-\infty}^{\infty} (e^y - 1) \nu^P(dy) < +\infty.$$
Applying Itô’s lemma,

\[ d\tilde{P}_t = \tilde{P}_t \left\{ \left( \hat{\gamma}^P + \frac{1}{2}(\sigma^P)^2 \right) dt + \sigma^P d\tilde{W}^P_t + \int_{-\infty}^{\infty} [e^y - 1] J^P(dt \times dy) \right\} \]  

(2.12)

We will define compensated \( \mathbb{P} \)-jump measures for stock index and bond index as \( \Phi^S(dt \times dy) = J^S(dt \times dy) - \nu^S(dy)dt \) and \( \Phi^P(dt \times dy) = J^P(dt \times dy) - \nu^P(dy)dt \).

According to the Fundamental Theorem of Asset Pricing: there exists at least one risk-neutral measure \( \mathbb{Q} \) in an arbitrage-free market. Applying measure change from \( \mathbb{P} \) to \( \mathbb{Q} \), the dynamics of \( S_t, \tilde{P}_t, r_t \) under \( \mathbb{Q} \) become

\[ dS_t = S_t \left( \hat{\gamma}^S + \frac{1}{2}(\sigma^S)^2 \right) dt + \sigma^S d\tilde{W}^S_t + \int_{\mathbb{R}} (e^y - 1) \hat{J}^S(dt \times dy) \]  

(2.13)

\[ d\tilde{P}_t = \tilde{P}_t \left( \hat{\gamma}^P + \frac{1}{2}(\sigma^P)^2 \right) dt + \sigma^P d\tilde{W}^P_t + \int_{\mathbb{R}} (e^y - 1) \hat{J}^P(dt \times dy) \]  

(2.14)

\[ dr_t = \kappa (\theta - r_t) dt + \sigma^R d\tilde{W}^R_t \]  

(2.15)

where \( \tilde{W}^S_t, \tilde{W}^P_t, \tilde{W}^R_t \) are correlated standard \( \mathbb{Q} \)-Brownian motions with correlations \( \rho_{RS}, \rho_{PR}, \rho_{PS} \). \( \hat{J}^S(dt \times dy) \) and \( \hat{J}^P(dt \times dy) \) are \( \mathbb{Q} \)-jump measures and they are independent. The compensators are \( \hat{\nu}^S(dy)dt \) and \( \hat{\nu}^P(dy)dt \). \( \hat{\Phi}^S(dt \times dy) = \hat{J}^S(dt \times dy) - \hat{\nu}^S(dy)dt \) and \( \hat{\Phi}^P(dt \times dy) = \hat{J}^P(dt \times dy) - \hat{\nu}^P(dy)dt \) are compensated \( \mathbb{Q} \)-jump measures for the stock index and bond index.

In order for \( e^{-\int_0^t r_s ds} S_t \) and \( e^{-\int_0^t r_s ds} \tilde{P}_t \) to be \( \mathbb{Q} \)-martingales, \( \hat{\gamma}^S \) and \( \hat{\gamma}^P \) need to satisfy the following conditions,

\[ \hat{\gamma}^S = r_t - \frac{1}{2}(\sigma^S)^2 - \int_{-\infty}^{\infty} (e^y - 1) \hat{\nu}^S(dy), \quad \hat{\gamma}^P = r_t - \frac{1}{2}(\sigma^P)^2 - \int_{-\infty}^{\infty} (e^y - 1) \hat{\nu}^P(dy) \]

After having dynamics of stock index and bond index, we now describe the mix fund process. Suppose the mix fund is continuously rebalanced in such a way that a determined percentage \( \pi_t \) of capital is invested in the bond index, and the rest is in the stock index. These weights are maintained on a daily basis (approximated as a continuous rebalancing) and may change continuously or, as is more typical, on an annual basis. This allows the investor to have a more aggressive portfolio when the contract is new and a more conservative portfolio when nearing the contract’s maturity. Then the dynamics of
the mix fund $M_t$ is
\[ \frac{dM_t}{M_{t-}} = \pi_t \frac{d\hat{P}_t}{\hat{P}_{t-}} + (1 - \pi_t) \frac{dS_t}{S_{t-}}. \]

The dynamics of mix fund under $Q$ is
\[
\frac{dM_t}{M_{t-}} = \left[ \pi_t \left( \tilde{\gamma}^P + \frac{1}{2}(\sigma^P)^2 \right) + (1 - \pi_t) \left( \tilde{\gamma}^S + \frac{1}{2}(\sigma^S)^2 \right) \right] dt + \pi_t \sigma^P d\hat{W}_t^P + (1 - \pi_t) \sigma^S d\hat{W}_t^S \\
+ \pi_t \int_{-\infty}^{\infty} (e^y - 1) \hat{J}^P(dt \times dy) + (1 - \pi_t) \int_{-\infty}^{\infty} (e^y - 1) \hat{J}^S(dt \times dy).
\]

The reference portfolio $\hat{M}_t$ will be charged the proportional management fees $\alpha$. Hence relationship between $M_t$ and $\hat{M}_t$ is $\hat{M}_t = M_t e^{-\alpha t}$.

## 2.6 Measure change

Risk-neutral valuation requires a locally $\mathbb{P}$-equivalent measure such that the discounted tradable asset is a martingale under this measure. We call such a measure equivalent martingale measure. In Chan (1999) it is shown that there are arbitrary many equivalent martingale measures in a general Lévy-process financial market. Such markets are well-known to be free of arbitrage, but riskless hedging is not possible, that is, those markets are by far not complete. For the theory of arbitrage-free and complete markets we refer to the classical paper of Harrison and Pliska (1981) or to the fundamental paper of Delbaen and Schachermayer (1998). A review of this theory is also contained in Cont and Tankov (2004).

In the Black-Scholes model, the unique equivalent martingale measure could be obtained by changing the drift of the Brownian motion. In models with jumps, if the Gaussian component is absent, this is no longer possible, but a much greater variety of equivalent measures can be obtained by altering the distribution of jumps. The following theorem describes the possible measure changes under which a Lévy process remains a Lévy process.

**Theorem 2.6.1.** Let $(X, \mathbb{P})$ be a Lévy process on $\mathbb{R}^d$ with characteristic triple $(A, \nu, \gamma)$; choose $\eta \in \mathbb{R}^d$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ with
\[
\int_{\mathbb{R}^d} (e^{\phi(x)/2} - 1)^2 \nu(dx) < \infty \tag{2.16}
\]
and define
\[ U_t := \eta \cdot X^c + \int_0^t \int_{\mathbb{R}^d} (e^{\phi(x)} - 1) \Phi(ds \times dx), \]
where $X^c$ denotes the continuous martingale (Brownian motion) part of $X$ and $\Phi$ is the compensated jump measure of $X$.

Then $\mathcal{E}(U)_t$ is a positive martingale such that the probability measure $Q$ defined by

$$
\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = \mathcal{E}(U)_t
$$

(2.17)

is equivalent to $P$ and under $Q$, $X$ is a Lévy process with characteristic triplet $(A, \hat{\nu}, \hat{\gamma})$ where $\hat{\nu} = e^{\phi} \nu$

$$
\hat{\gamma} = \gamma + \int_{|x| \leq 1} x(\hat{\nu} - \nu)(dx) + A\eta.
$$

where $\mathcal{E}(U)$ is the stochastic exponential or the Doléans-Dade exponential of $U$, i.e. if $dZ_t = Z_{t-}dU_t$ and $Z_0 = 1$, then $Z_t$ is denoted as $\mathcal{E}(U)$.

Proof. Please refer to Sato (1999), Theorems 33.1 and 33.2. $\square$

A useful example that will be the basis of our construction of an equivalent martingale measure is the Esscher measure. Let $X$ be a Lévy process on $\mathbb{R}^d$ with characteristic triplet $(A, \nu, \gamma)$, and let $\theta \in \mathbb{R}^d$ be such that $\int_{|x| > 1} e^{\theta \cdot x} \nu(dx) < \infty$. Applying a measure transformation of the above theorem with $\eta = \theta$ and $\phi(x) = \theta \cdot x$, we obtain an equivalent probability under which $X$ is a Lévy process with Lévy measure $\hat{\nu}(dx) = e^{\theta \cdot x} \nu(dx)$ and the third component of the characteristic triplet $\hat{\gamma} = \gamma + A\theta + \int_{|x| \leq 1} x(e^{\theta \cdot x} - 1) \nu(dx)$.

Using the above theorem, the Radon-Nikodym derivative corresponding to this measure change is found to be

$$
\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = \frac{e^{\theta \cdot X_t}}{E(e^{\theta \cdot X_1})} = \exp(\theta \cdot X_t - \kappa(\theta) t),
$$

where $\kappa(\theta) := \log E(e^{\theta \cdot X_1}) = \psi(-i\theta)$.

Let us consider the measure change for the case of variance gamma processes. Suppose the $P$-Lévy density is

$$
\nu(x) = a \left( \frac{e^{-b^+x}}{x} 1_{x > 0} + \frac{e^{-b^-|x|}}{|x|} 1_{x < 0} \right)
$$

Applying Esscher transform, the relationship between real world and risk neutral param-
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The model setup parameters is as follows.

\[ \hat{\nu}(x) = e^{\phi(x)} \nu(x) \]

\[ \hat{\gamma}^S = \gamma^S + \int_\mathbb{R} x(e^{\phi(x)} - 1) \nu(dx) \]

Suppose \( \phi(x) = \alpha + \beta^+ x_{x>0} + \beta^- |x|1_{x<0} \), \( \hat{\nu}(x) \) will be

\[ \hat{\nu}(x) = e^{\phi(x)} \nu(x) = e^{\alpha} a \left( \frac{e^{-(b^+-\beta^+)x}}{x} 1_{x>0} + \frac{e^{-(b^-\beta^-)|x|}}{|x|} 1_{x<0} \right) \]

The relationships between real world parameters and risk neutral parameters are

\[ \hat{a} = e^{\alpha} a, \quad \hat{b}^+ = b^+-\beta^+, \quad \hat{b}^- = b^- - \beta^- \]

Next, we will check when \( \int_\mathbb{R} (e^{\phi(x)/2} - 1)^2 \nu(dx) < \infty \) is satisfied. The left-hand side of condition (2.16) on the positive half-axis writes

\[ \int_0^\infty (e^{\phi(x)/2} - 1)^2 \nu(dx) = a \int_0^\infty \left( e^{\alpha} e^\beta x - 1 \right)^2 \frac{e^{-b^+x}}{x} dx \]

when \( \alpha \neq 0 \), the integrand is equivalent to \( \frac{1}{x} \) near zero and, hence is not integrable. However, when \( \alpha = 0 \) the integrand is equivalent to \( x \) near zero and is always integrable. This means \( a \) will be the same under real world measure and risk neutral measure. Only \( b^+ \) and \( b^- \) can be changed under risk neutral measure. From this one can change freely the distribution of large jumps (as long as the new Lévy measure is absolutely continuous with respect to the old one) but one should be very careful with the distribution of small jumps (which is determined by the behavior near zero). This is a good property since large jumps are the ones which are important from the point of view of options pricing: they affect the tail of the return distribution and option prices in an important way.

From the above calculations, we can see the problem happens at the small jump size. One way of getting rid of this constraint is to truncate small jumps. For example, when the jump size \( |y| \leq \varepsilon \), we will use Brownian motion to replace the VG process. But in this thesis, we will still use VG process and parameters of real world and risk neutral need to satisfy the above constraint.
Chapter 3

Pricing under Lévy model and constant interest rate

Before considering pricing of a guaranteed minimal rate of return under mix funds and the stochastic interest rate, we first consider a simple model in which all the money is invested in the stock index and the interest rate is constant. The simple model can help us understand the approaches and methodologies better. In this chapter, we first give a review of the Black-Scholes pricing model and then derive the pricing partial integro differential equation (PIDE) under Lévy model. Finally we solve the PIDE using the Fourier transform method.

Now we want to price a guaranteed minimal rate of return $R_{\text{min}}$ on a nominal value $S_0$ over the next $T$ years, i.e. a contingent claim which pays at a certain time of maturity $T > 0$ the amount

$$\max(K, \hat{S}_T) = \max(K - \hat{S}_T, 0) + \hat{S}_T,$$

where $K = S_0 e^{R_{\text{min}} T}$ and reference portfolio is $\hat{S}_t = S_t e^{-\alpha t}$.

In fact the payoff of a guaranteed minimal rate of return is equal to the sum of a European put option and the underlying reference portfolio. Hence pricing and hedging of a guaranteed minimal rate of return are equivalent to pricing and hedging of a European put option.
Chapter 3. Pricing under Lévy model and constant interest rate

3.1 Review of the Black-Scholes model

A fundamental idea of pricing a contingent payoff in finance using the risk-neutral probability measure is that one can perfectly replicate the payoff of the contingent claim by rebalancing a portfolio of the underlying risky assets and the money market account with a self-financing strategy. Under the Black-Scholes model, the stock index follows geometric Brownian motion and the interest rate is constant. Under these assumptions, the market is complete and perfect hedging exists. The price of the put option will be equal to the value of the replicating portfolio.

The price of this European put option with payoff \( G(S_T) = \max(K - \hat{S}_T, 0) \) is different from the Black-Scholes formula in which the underlying asset pays dividends. Dividends are paid to the holder of the underlying asset and proportional to positions of underlying asset while the management fee is paid to the seller of the option and proportional to the number of options. In this section, we will re-derive its pricing formula using dynamic hedging method.

Under the Black-Scholes model, the stock index \( S_t \) and money market account \( B_t \) have the following dynamics under real measure \( \mathbb{P} \),

\[
\frac{dS_t}{S_t} = \mu^S dt + \sigma^S dW^S_t
\]

\[
\frac{dB_t}{B_t} = r dt.
\]

where \( W^S_t \) is a standard Brownian motion under \( \mathbb{P} \). The constant \( \sigma^S > 0 \) is assumed to be fixed and known while the fixed value \( \mu^S \in \mathbb{R} \) does not need to be known explicitly.

Suppose the price of the put option with maturity \( T \) at time \( t \) is \( H(t, S_t, T) \). In order to hedge the sold option we buy \( \xi_t \) units of \( S_t \) and \( \beta_t \) units of \( B_t \). Our portfolio is

\[
V_t = \xi_t S_t + \beta_t B_t - H_t.
\]

According to the self-financing condition, after the next short period of time \( dt \), the change in the portfolio comes from the changes in each asset and receiving management fees \( \alpha \hat{S}_t dt \) from the contract buyer, i.e.,

\[
dV_t = \xi_t dS_t + \beta_t dB_t - dH_t + \alpha \hat{S}_t dt
\]
Applying Ito’s Lemma,
\[
dV_t = \left( \xi_t \mu^S_t S_t + \beta_t r B_t - \partial_t H + \mu^S_t \partial_S H + \frac{1}{2} (\sigma^S_t)^2 \partial_{SS} H + \alpha \hat{S}_t \right) dt + (\xi_t - \partial_S H) \sigma^S_t S_t dW_t^S.
\]

The completeness of the market leads to perfect hedging of the option. Hence the coefficient of \(dW_t^S\) must be zero to eliminate the uncertainty. Then we will have
\[
\xi_t = \partial_S H.
\]

Therefore
\[
dV_t = \left( \beta_t r B_t - \partial_t H + \frac{1}{2} (\sigma^S_t)^2 \partial_{SS} H + \alpha \hat{S}_t \right) dt
\]

According to the no-arbitrage argument, \(dV_t\) must be 0. Since the initial value \(V_0 = 0\), \(V_t\) will be always 0 for any \(t\). Then
\[
\beta_t = \frac{H_t - \xi_t S_t}{B_t} = \frac{H_t - \partial_S H S_t}{B_t}
\]

and the function \(H\) satisfies the following PDE
\[
\left( \partial_t + r S_t \partial_S + \frac{1}{2} (\sigma^S_t)^2 \partial_{SS} \right) H = r H_t + \alpha \hat{S}_t \quad (3.1)
\]

\[
H(T, S_T, T) = G(S_T) \quad (3.2)
\]

Comparing to the Black-Scholes PDE, there is an extra term \(\alpha \hat{S}_t\) on the RHS of (3.1).

The following theorem will give the explicit solution to the above PDE.

**Theorem 3.1.1.** The solution to the PDE (3.1) together with terminal condition (3.2) can be written as the sum of two expectations under risk neutral measure \(Q\), i.e.
\[
H(t, S_t, T) = \mathbb{E}^Q \left[ e^{-r(T-t)} \cdot G(S_T) | \mathcal{F}_t \right] + \mathbb{E}^Q \left[ -\alpha \int_t^T e^{-r(u-t)} \hat{S}_u du | \mathcal{F}_t \right] \quad (3.3)
\]

The price of the option is equal to the sum of risk neutral expectation of discounted terminal payoff and expectation of discounted future cash flows.

**Proof.** The first expectation is denoted as \(H^E_t\) and the second is denoted as \(H^F_t\). Then \(e^{-rt} H^E_t = \mathbb{E}^Q[e^{-rT} \cdot G(S_T)|\mathcal{F}_t]\) is a Doob-martingale under \(Q\). Applying Itô’s
lemma,
\[ d(e^{-rt}H_t^E) = -re^{-rt}H_t^E dt + e^{-rt}dH_t^E = e^{-rt} \left[ (-rH_t^E + (\partial_t + \mathcal{L}^r)H_t^E) dt + \partial_S H_t^E \sigma S d\tilde{W}_S^t \right], \]
where $\mathcal{L}^r$ is the infinitesimal generator with drift $r$ and $\tilde{W}_S^t$ is a standard $Q$-Brownian motion. The martingale property leads to zero drift. Hence $H_t^E$ will satisfy the following PDE,
\[ (\partial_t + \mathcal{L}^r)H_t^E = rH_t^E. \]

Since $H_t^E e^{-rt} - \alpha \int_0^t e^{-ru} \hat{S}_u du = -\alpha \mathbb{E}^Q \left[ \int_0^T e^{-ru} \hat{S}_u du | \mathcal{F}_t \right]$, it is a Doob-martingale under $Q$. Making the drift term become zero will lead to
\[ (\partial_t + \mathcal{L}^r)H_t^E = rH_t^E + \alpha \hat{S}_t. \]
Hence
\[ (\partial_t + \mathcal{L}^r)(H_t^E + H_t^F) = r(H_t^E + H_t^F) + \alpha \hat{S}_t. \]
Hence $H_t^E + H_t^F$ satisfies equation (3.1) and the terminal condition (3.2).

The first expectation can be obtained using Black-Scholes formula. Since $e^{-rt}S_t$ is a martingale under $Q$, the second expectation can be simplified as
\[ \mathbb{E}^Q \left[ -\alpha \int_t^T e^{-r(u-t)} \hat{S}_u du | \mathcal{F}_t \right] = -\alpha \left( \int_t^T S_t e^{-au} du \right) = S_t(e^{-\alpha T} - e^{-\alpha t}). \]
The price of this put option at time $t$ is
\[ H(t, S_t, T) = Ke^{-r(T-t)}N(-d_2) - S_t e^{-\alpha(T-t)}N(-d_1) - S_t(e^{-at} - e^{-\alpha T}), \quad (3.4) \]
where
\[ d_1 = \frac{\ln(S_t/K) + (r - \alpha + (\sigma^S)^2/2)(T-t)}{\sigma^S \sqrt{T-t}}, \quad d_2 = d_1 - \sigma^S \sqrt{T-t}. \]

### 3.2 Pricing under the Lévy model

The pricing formula (3.3) in the Black-Scholes model can be extended to the case of exponential Lévy process. The first expectation $\mathbb{E}^Q [ e^{-r(T-t)} \cdot G(S_T) | \mathcal{F}_t ]$ can not be calculated explicitly under Lévy model. According to the Markov property, $\mathbb{E}^Q[G(S_T) | \mathcal{F}_t]$ only
depends on time $t$ and stock index $S_t$. We will denote it as $g(t, x_t)$, where $x_t = \ln(S_t/K)$. $x_t$ is introduced to simplify the expression of PIDE. In this section, we will derive the PIDE for the function $g$.

**Theorem 3.2.1.** The function $g(t, x_t)$ satisfies the following PIDE

$$
\partial_t g + \hat{\gamma}^S \partial_x g + \frac{1}{2}(\sigma^S)^2 \partial_{xx} g + \int_{-\infty}^{\infty} [g(t, x_{t-} + y) - g(t, x_{t-})] \hat{\nu}^S(dy) = 0 \quad (3.5)
$$

$$
g(T, x_T) = \max(K\left[1 - e^{-\alpha T}e^{x_T}\right], 0) \quad (3.6)
$$

**Proof.** Applying Itô’s Lemma to the function $g$ leads to

$$
dg = \left[\partial_t g + \hat{\gamma}^S \partial_x g + \frac{1}{2}(\sigma^S)^2 \partial_{xx} g\right] dt + \partial_x g \sigma^S d\hat{W}^S_t + \int_{\mathbb{R}} (g(x_{t-} + y) - g(x_{t-})) \hat{J}^S(dy \times dt).
$$

Since $g$ is a Doob-martingale, $\mathbb{E}^Q(dg) = 0$. We can obtain (3.5).

In the above PIDE, a non-local integral term must be solved. The difficulties in working with Lévy processes led to the development of transform-based methods, such as Carr and Madan (1999), Raible (2000) and Lewis (2001). In this thesis we will use Fourier Space Time-stepping (FST) algorithm. This method, in its first appearance together with extensive numerical experiments can be found in Jackson, Jaimungal and Surkov (2007).

The FST scheme is related to the method employed by Carr and Madan (1999) for valuing European options. In their work, the authors performed a transform in the log-strike price to obtain prices of options of all strikes in “one shot” using a Fast Fourier Transform (FFT) operation. Here, we employ the FFT to obtain prices for all log-spot levels at intermediate decision dates. Another difference is that the adjustment factor which pushed the pole singularity for calls and puts off the real axis is absent. This issue can be treated in two ways: (i) shift the integration path appropriately; or (ii) truncate the payoff for sufficiently large spot values. We will use truncating to avoid the singularity problems and we find that the results are insensitive to the truncation point for sufficiently large truncation. Furthermore, FST scheme can be efficiently applied to pricing of multi-dimensional options with path dependency.

Before applying FST scheme to solve PIDEs, we will first give the definitions of the Fourier transform and inverse Fourier transform.
A function in the space domain \( f(x) \) can be transformed to a function in the frequency domain \( \mathcal{F}[f](\omega) \) (where \( \omega \) is given in radians per second) and vice-versa using the Fourier transform:
\[
\mathcal{F}[f](\omega) := \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx, \quad \mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega.
\]

The Fourier transform is a linear operator that maps spatial derivatives \( \partial_x^n \) into multiplications in the frequency domain:
\[
\mathcal{F}[\partial_x^n f](t,\omega) = (i\omega)^n \mathcal{F}[f](t,\omega).
\]

**Theorem 3.2.2.** The solution to the above PIDE can be written as
\[
g(t, x) = \mathcal{F}^{-1} \left[ \mathcal{F}[g](T, \omega) e^{\psi(\omega)(T-t)} \right]
\]
where \( \psi(\omega) = i\gamma S \omega - \frac{1}{2}(\sigma S)^2 \omega^2 + \int_{-\infty}^{\infty} (e^{i\omega y} - 1) \nu^S(dy) \). In fact, \( \psi(\omega) \) is the characteristic function of \( x_1 \).

**Proof.** Applying Fourier transform to both sides of the PIDE (3.5) and boundary condition (3.6), we will have the following ODE together with terminal boundary condition.
\[
\partial_t \mathcal{F}[g] + \left[ i\gamma S \omega - \frac{1}{2}(\sigma S)^2 \omega^2 + \int_{-\infty}^{\infty} (e^{i\omega y} - 1) \nu^S(dy) \right] \mathcal{F}[g] = 0.
\]
\[
\mathcal{F}[g](T, \omega) = e^{-rT} \int_{-\infty}^{\infty} \max \left[ K \left( 1 - e^{x-\alpha T} \right), 0 \right] \cdot e^{-i\omega x} \, dx.
\]

The solution to the above ODE can be obtained explicitly.
\[
\mathcal{F}[g](t, \omega) = \mathcal{F}[g](T, \omega) \exp \left( \left[ i\gamma S \omega - \frac{1}{2}(\sigma S)^2 \omega^2 + \int_{-\infty}^{\infty} (e^{i\omega y} - 1) \nu^S(dy) \right] (T-t) \right)
\]
Taking the inverse transform leads to the final result. □

Alternatively, it is possible to derive (3.7) directly from the expectation representation of prices rather than going through the PIDE. Recall that \( g(t, x_t) \) is a \( \mathbb{Q} \)-martingale.
Consequently,

\[ g(t, x_t) = \mathbb{E}_Q^Q[g(T, x_T)] = \int_{-\infty}^{\infty} g(T, x_t + y)f_{x_T}(y)dy = \int_{-\infty}^{\infty} g(T, x_t + y)f_{x_T}(y)dy. \]

Here, \( f_{x_t}(y) \) denotes the p.d.f. of the process \( x_t \) and third line follows from the independent and stationary properties of the process \( x_t \). Furthermore, \( \mathcal{F}[f_{x_t}](\omega) = e^{i\psi(-\omega)} \) and since a convolution in real space corresponds to multiplication in Fourier space, we have

\[ \mathcal{F}[g](t, \omega) = \mathcal{F}[g](T, \omega)e^{\psi(\omega)(T-t)}. \]

After we solve the function \( g \), the price of the put option is

\[ H(t, S_t, T) = h(t, S_t, T) - S_t(e^{-\alpha t} - e^{-\alpha T}) = e^{-r(T-t)}g(t, x_t) - S_t(e^{-\alpha t} - e^{-\alpha T}). \] (3.10)

**Theorem 3.2.3.** The derivatives of the function \( h \) can be obtained using inverse Fourier transform.

\[ \frac{\partial h}{\partial S} = \frac{e^{-r(T-t)}}{S_t} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x_t} \cdot i\omega \mathcal{F}[g](T, \omega)e^{\psi(\omega)(T-t)}d\omega \]

\[ \frac{\partial^2 h}{\partial \Delta^2} = \frac{e^{-r(T-t)}}{S_t^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x_t} \cdot (-\omega^2 - i\omega) \mathcal{F}[g](T, \omega)e^{\psi(\omega)(T-t)}d\omega \]

**Proof.** The results can be obtained by using the chain rule and properties of Fourier transform. \( \square \)
Chapter 4

Calibration of variance gamma processes

In this thesis, we choose variance gamma process as our Lévy density. In order to have reasonable parameters in the numerical examples, we will consider calibrations of the variance gamma process under real world measure using historical index data. Since we derived the option price under Lévy model in the last chapter, we can also calibrate the variance gamma process under risk neutral measure using historical option data.

4.1 Calibrations of real world parameters

In this section, we will use market data of stock index and bond index from Datastream to calibrate variance gamma processes under real world measure. The S&P 500, or the Standard & Poor’s 500, is a stock market index based on the market capitalizations of 500 leading companies publicly traded in the U.S. stock market, as determined by Standard & Poor’s. It differs from other U.S. stock market indices such as the Dow Jones Industrial Average and the Nasdaq due to its diverse constituency and weighting methodology. It is one of the most commonly followed equity indices. The Dow Jones Corporate Bond Index is an equally weighted basket of 96 recently issued investment-grade corporate bonds with laddered maturities. The index intends to measure the return of readily tradable, high-grade U.S. corporate bonds.

We will use the Maximum Likelihood Estimation method to obtain estimates of variance gamma processes. Suppose \( S_t \) follows the process of equation (2.2). The density for the log-price relatives \( z_t = \ln(S_t/S_0) \) is derived in Madan, Carr and Chang (1998) by
making use of the mixture representation in (2.2). It has the following form

\[ f_t(z) = \frac{2 \exp(\theta x/\sigma^2)}{\nu^{\nu/2} \sqrt{2\pi \sigma \Gamma(\nu/2)}} \left( \frac{x^2}{2\sigma^2/\nu + \theta^2} \right)^{-\nu/2} K_{\nu/2 - 1} \left( \frac{1}{\sigma^2 \sqrt{x^2(2\sigma^2/\nu + \theta^2)}} \right) \]

where \( K \) is the modified bessel function of second kind,

\[ x = z - \mu t - \frac{t}{\nu} \ln \left( 1 - \theta \nu - \sigma^2 \nu / 2 \right). \]

In fact we have a series of continuously compounded daily returns \( z_i = \ln(S_{i+1}/S_i) \). We want to choose \( \{\theta, \sigma, \nu\} \) to minimize the negative log-likelihood function

\[ - \sum_{i=1}^{N} \ln(f_{\Delta t}(z_i)). \]

After we have estimates of \( \{\theta, \sigma, \nu\} \), we can compute estimates of \( \{a, b^+, b^-\} \). The estimates and standard deviations are given in Table 4.1. After we get the maximum likelihood estimates by minimizing the negative log-likelihood, we also check the first derivatives and second derivatives at the optimal points. We find out the first derivatives of negative log-likelihood are very close to zero and the matrix of second derivatives is positive definite, which confirms our estimates are optimal.

<table>
<thead>
<tr>
<th>Parameter Estimates</th>
<th>( \sigma )</th>
<th>( \nu )</th>
<th>( \theta )</th>
<th>( \mu )</th>
<th>( a )</th>
<th>( b^+ )</th>
<th>( b^- )</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P500 composite</td>
<td>0.1917 ( (0.0044) )</td>
<td>0.0053 ( (0.0003) )</td>
<td>-0.1284 ( (0.0629) )</td>
<td>0.1733 ( (0.0537) )</td>
<td>187.72</td>
<td>104.63</td>
<td>97.64</td>
</tr>
<tr>
<td>Dow Jones bond index</td>
<td>0.0593 ( (0.0011) )</td>
<td>0.0019 ( (0.0002) )</td>
<td>-0.1028 ( (0.0486) )</td>
<td>0.008 ( (0.0138) )</td>
<td>534.87</td>
<td>581.59</td>
<td>523.14</td>
</tr>
</tbody>
</table>

Table 4.1: Maximum likelihood estimates of variance gamma processes for daily returns of the stock index and the bond index based on: May 22, 2003-May 22, 2013

The following figures plot the negative log likelihood surface when the other two parameters are the maximum likelihood estimates. From the plots we can see the negative log likelihood is more sensitive to \( \sigma \) and \( \nu \). Hence the estimates of \( \theta \) and \( \nu \) have smaller standard deviations.

The comparisons between histograms and fitted distribution are given in Figure 4.2. From this figure, we can see the fitted distributions match the shapes of histograms.

In order to see how the variance gamma fits improve the fits in the tails, we also draw q-q plots to compare the empirical quantile and theoretical quantile in Figure 4.3. The
Figure 4.1: Negative log-likelihood surface

Figure 4.2: Comparing fitted distribution with 10 years’ data

q-q plot displays the sample quantiles $X_{(1)}, \cdots, X_{(n)}$ against the distribution quantiles $F^{-1}(p_1), \cdots, F^{-1}(p_n)$, where $p_i = i - 1/2 \cdot n$. From Figure 4.3, we can see the tails of the fitted distribution do not match those of the empirical distribution.

In fact the distribution for a 10-year period is not completely stationary. To calibrate 10 years’ data properly we may consider adding stochastic volatility to the model. In order to calibrate our model, we will consider using data from a short period. Table 4.2 and Table 4.3 give estimates of variance gamma densities for S&P 500 and Dow
Chapter 4. Calibration of variance gamma processes

Comparing theoretical quantile with empirical quantile

(a) q-q plot for the stock index

(b) q-q plot for the bond index

Figure 4.3: q-q plots for 10 years’ data

Jones corporate bond index using one-year data. The estimates for different years are quite different, which confirms the non-stationary property for the 10 years’ data. The comparisons between histograms and fitted distribution and q-q plots are given in Figure 4.4 and Figure 4.5. For the annual data of 2010, we can see the tails of the fitted distributions can match those of the empirical distributions.

Figure 4.4: Comparing fitted distribution with annual data of 2010
Chapter 4. Calibration of variance gamma processes

4.2 Calibration of risk neutral parameters

In this section, we will consider calibration of risk neutral variance gamma parameters using prices of options on S&P 500. Our aim is to minimize the sum of the square errors between implied volatilities using market price and implied volatilities using variance gamma model across different strikes.

First we will define implied volatility using market price. The solution to (4.1) is

![Comparing theoretical quantile with empirical quantile](a) q-q plot for stock index

![Comparing theoretical quantile with empirical quantile](b) q-q plot for bond index

Figure 4.5: q-q plots for annual data of 2010

<table>
<thead>
<tr>
<th>Parameter Estimates</th>
<th>$\sigma$</th>
<th>$\nu$</th>
<th>$\theta$</th>
<th>$\mu$</th>
<th>$a$</th>
<th>$b^+$</th>
<th>$b^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 2009</td>
<td>0.2767</td>
<td>0.0045</td>
<td>-0.4352</td>
<td>0.2491</td>
<td>224.52</td>
<td>82.48</td>
<td>71.11</td>
</tr>
<tr>
<td>Year 2010</td>
<td>0.1868</td>
<td>0.0055</td>
<td>-0.0389</td>
<td>0.1899</td>
<td>185.23</td>
<td>104.83</td>
<td>102.62</td>
</tr>
<tr>
<td>Year 2011</td>
<td>0.2304</td>
<td>0.0042</td>
<td>-0.6021</td>
<td>0.0159</td>
<td>238.05</td>
<td>106.74</td>
<td>84.05</td>
</tr>
</tbody>
</table>

Table 4.2: Maximum likelihood estimates of variance gamma process for daily returns of S&P 500 based on annual data

<table>
<thead>
<tr>
<th>Parameter Estimates</th>
<th>$\sigma$</th>
<th>$\nu$</th>
<th>$\theta$</th>
<th>$\mu$</th>
<th>$a$</th>
<th>$b^+$</th>
<th>$b^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 2009</td>
<td>0.076</td>
<td>0.0005</td>
<td>0.7321</td>
<td>0.1051</td>
<td>1966.5</td>
<td>708.4</td>
<td>962.1</td>
</tr>
<tr>
<td>Year 2010</td>
<td>0.0584</td>
<td>0.0003</td>
<td>-1.0975</td>
<td>0.0344</td>
<td>3123.7</td>
<td>1713.2</td>
<td>1069.5</td>
</tr>
<tr>
<td>Year 2011</td>
<td>0.0602</td>
<td>0.0002</td>
<td>-0.5786</td>
<td>0.0351</td>
<td>1537.4</td>
<td>1094.1</td>
<td>775</td>
</tr>
</tbody>
</table>

Table 4.3: Maximum likelihood estimates of variance gamma process for daily returns of Dow Jones corporate bond index based on annual data
Chapter 4. Calibration of variance gamma processes

denoted as $\sigma_{imp}(K, T)$. The LHS of (4.1) is Black-Scholes price and RHS is market price.

$$Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1) = P_{market} \quad (4.1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Next we will define the implied volatility using variance gamma process. The solution to (4.2) is denoted as $\sigma_{VG}(K, T)$. The LHS of (4.2) is Black-Scholes price and RHS is price of the option under variance gamma process,

$$Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1) = h(0, S, T) \quad (4.2)$$

where $h(0, S, T)$ is the option price calculated in chapter 3 using variance gamma process as Lévy density.

We want to choose the parameters of variance gamma process to minimize the difference between $\sigma_{imp}(K, T)$ and $\sigma_{VG}(K, T)$ across all the strikes which are 25% below or above the current stock index, i.e.

$$\varepsilon^2 = \frac{\sum\limits_{i \mid \frac{K_i}{St} - 1 < 0.25} \left[ 1 - \frac{\sigma_{VG}(K_i, T)}{\sigma_{imp}(K_i, T)} \right]^2}{\# \ of \ points}$$

Hence $\varepsilon$ is called Root of Mean Errors (RME).

Each of the option pricing models requires a continuously-compounded interest rate as input. This interest rate is calculated from a collection of continuously-compounded zero-coupon interest rates at various maturities, collectively referred to as the zero curve. The zero curve used in this thesis is derived from BBA LIBOR rates.

For a given option, the appropriate interest rate input corresponds to the zero-coupon rate that has a maturity equal to the option’s expiration, and is obtained by linearly interpolating between the two closest zero-coupon rates on the zero curve. The zero curve is calculated as follows.

Step 1. The BBA LIBOR rates for maturities of 1 week and 1-12 months are converted
to discount factors (DF) using the formula:

\[ DF = (1 + r \times d/360)^{-1} \]

where \( r \) is the BBA LIBOR rate and \( d \) is the actual number of days to maturity.

Step 2. The LIBOR discount factors are converted to continuous LIBOR zero rates using the Actual/365 day-count convention:

\[ L = -365/d \times \ln(DF) \]

where \( L \) is the continuously-compounded LIBOR zero rate.

Step 3. The zero rate on the nearest futures contract date (greater than one week) is obtained by linear interpolation between the two closest LIBOR zero rates computed in Step 2.

We will use option prices on May 2, 2013 and February 8, 2010 to calibrate the risk neutral variance gamma parameters. Figure 4.6 plots the zero curves on these two days.

![Figure 4.6: Zero curve obtained from LIBOR USD](image-url)

When the underlying equity or index pays dividends, we will also require an estimate of dividends to be paid up until the option’s expiration. For dividend-paying indices, we assume that the security pays dividends continuously, according to a continuously-compounded dividend yield. A put-call parity relationship is assumed, and the implied
index dividend is calculated from the following linear regression model:

\[ C - P = b_0 + b_1 S + b_2 ST + b_3 K + b_4 KT + b_5 D_{BA} \]

In this model, \( C - P \) is the difference between prices of a call option and a put option with the same expiration and the same strike. When calculating this difference, the bid price of the call is used with the offer price of the put, and vice versa. \( D_{BA} \) is a dummy variable set equal to 1 if the call option’s bid price is used. \( S \) is the underlying security’s (index’s) closing price, \( K \) is the strike price of the call and put options, and \( T \) is the time to expiration in years. The regression is calculated using three months of option data across all strikes and expirations with an exception of contracts expiring in less than 15 days. For a single underlying index the dividend rate will be approximately equal to the negative of the estimated parameter \( b_2 \).

For option price on May 2, 2013, the estimate dividend rate is 1.55% with 95% confidence interval [0.94%, 2.17%]. Similarly, for the option prices on Feb 8, 2010, the estimate dividend rate is 1.93% with 95% confidence interval [1.78%, 2.09%].

It is well-known that prices of in-the-money options are dominated by the payoff of options. In order to get the right implied volatility, we will use implied volatility of out of the money options. Suppose current stock price is \( S \). When \( K < S \), implied volatilities are obtained from out of the money (OTM) calls and when \( K > S \), implied volatilities are obtained from out of the money (OTM) puts. The calibration results for options prices on May 2, 2013 are shown in Table 4.4. Another calibration results for option prices on Feb 8, 2010 are shown in Table 4.5.

<table>
<thead>
<tr>
<th>( T ) (days)</th>
<th>( \hat{a} )</th>
<th>( b^+ )</th>
<th>( b^- )</th>
<th>( \min \varepsilon )</th>
<th>( b^+/b^- )</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>4.62</td>
<td>46.17</td>
<td>16.2</td>
<td>0.0375</td>
<td>2.85</td>
<td>0.141</td>
</tr>
<tr>
<td>79</td>
<td>3.3</td>
<td>39.84</td>
<td>12.84</td>
<td>0.021</td>
<td>3.10</td>
<td>0.149</td>
</tr>
<tr>
<td>107</td>
<td>2.35</td>
<td>32.66</td>
<td>10.28</td>
<td>0.0197</td>
<td>3.18</td>
<td>0.156</td>
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<tr>
<td>233</td>
<td>1.18</td>
<td>22.69</td>
<td>6.42</td>
<td>0.0176</td>
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<tr>
<td>261</td>
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<td>20.92</td>
<td>6.03</td>
<td>0.0163</td>
<td>3.47</td>
<td>0.177</td>
</tr>
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<td>324</td>
<td>0.795</td>
<td>17.63</td>
<td>4.9</td>
<td>0.0211</td>
<td>3.6</td>
<td>0.189</td>
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</tbody>
</table>

Table 4.4: Risk neutral variance gamma parameters for option prices on May 2, 2013

From Table 4.4 and Table 4.5 we can see that the variance of calibrated variance gamma model in unit time increases as the maturity of the option increases. That is because the decay rate of the implied volatility for the variance gamma model is quicker
than the decay of implied volatility of the market. These can also be seen from Figure 4.7 and Figure 4.8. The implied volatility plot of variance gamma model with parameters of the shortest maturity (\(T = 51\) days in Figure 4.7 and \(T = 40\) days in Figure 4.8) is flatter than that of the market. In order to compensate this effect, the calibrated variance gamma model needs to have increased variance as the maturity increases. From these two plots, we also note the calibrated volatility plots are very close to those of the market, which is a sign of good calibrations. We also note that there is a negative skewness to the risk neutral process associated with risk aversion while the real world process is almost symmetric. The enhancement of skewness in the risk neutral process relative to the real world process is an expected consequence of risk aversion in equilibrium.
Chapter 4. Calibration of variance gamma processes

In Madan, Carr and Chang (1998), they also calibrated real world parameters and risk neutral parameters under variance gamma process. They never mentioned the relationship between real world parameters and risk neutral parameters since they mainly worked on the option pricing under variance gamma process. In this thesis, we will also deal with hedging problem and the performances of hedging strategies are sensitive to the difference between real world parameters and risk neutral parameters. Hence we will let variance gamma parameters satisfy the requirements of measure change from $\mathbb{P}$ to $\mathbb{Q}$, i.e, $\hat{a}$ should keep the same as its value under $\mathbb{P}$. Hence we will redo the calibration with only two free parameters $\hat{b}^+$ and $\hat{b}^-$.

For the option prices on May 2, 2013, we will use the values of index from May 2, 2013 to March 22, 2014 to calibrate real world parameters. Hence $\hat{a} = a = 902.3$. For the option prices on February 8, 2010, we will use the values of index from February 8, 2010 to December 31, 2010 to calibrate real world parameters. Hence $\hat{a} = a = 185.23$. Table 4.6 and Table 4.7 give estimates of risk neutral variance gamma parameters for option prices on May 2, 2013 and February 8, 2010 when the risk neutral parameters satisfy the measure change constraint.

According to Figure 4.9 and Figure 4.10, we can see the fits of keeping $a$ fixed perform
Table 4.6: Risk neutral variance gamma parameters for option prices on May 2, 2013 when satisfying constraints

<table>
<thead>
<tr>
<th>T (days)</th>
<th>( \hat{a} )</th>
<th>( b^+ )</th>
<th>( b^- )</th>
<th>min ( \varepsilon )</th>
<th>( b^+/b^- )</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>902.3</td>
<td>1204.9</td>
<td>214.6</td>
<td>0.308</td>
<td>5.61</td>
<td>0.142</td>
</tr>
<tr>
<td>79</td>
<td>902.3</td>
<td>10937</td>
<td>232</td>
<td>0.287</td>
<td>47.2</td>
<td>0.13</td>
</tr>
<tr>
<td>107</td>
<td>902.3</td>
<td>10524</td>
<td>220</td>
<td>0.272</td>
<td>47.9</td>
<td>0.137</td>
</tr>
<tr>
<td>233</td>
<td>902.3</td>
<td>10654</td>
<td>217</td>
<td>0.2221</td>
<td>49.1</td>
<td>0.139</td>
</tr>
<tr>
<td>261</td>
<td>902.3</td>
<td>10656</td>
<td>215</td>
<td>0.2331</td>
<td>49.7</td>
<td>0.14</td>
</tr>
<tr>
<td>324</td>
<td>902.3</td>
<td>10573</td>
<td>208</td>
<td>0.2266</td>
<td>50.9</td>
<td>0.145</td>
</tr>
</tbody>
</table>

Table 4.7: Risk neutral variance gamma parameters for option prices on Feb 8, 2010 when satisfying constraints

<table>
<thead>
<tr>
<th>T (days)</th>
<th>( \hat{a} )</th>
<th>( b^+ )</th>
<th>( b^- )</th>
<th>min ( \varepsilon )</th>
<th>( b^+/b^- )</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>185.23</td>
<td>2054.1</td>
<td>73.1</td>
<td>0.1674</td>
<td>28.1</td>
<td>0.186</td>
</tr>
<tr>
<td>68</td>
<td>185.23</td>
<td>2268.27</td>
<td>81.3</td>
<td>0.164</td>
<td>27.9</td>
<td>0.168</td>
</tr>
<tr>
<td>103</td>
<td>185.23</td>
<td>2351.9</td>
<td>83.4</td>
<td>0.173</td>
<td>28.2</td>
<td>0.163</td>
</tr>
<tr>
<td>142</td>
<td>185.23</td>
<td>2382.9</td>
<td>84.8</td>
<td>0.171</td>
<td>28.1</td>
<td>0.161</td>
</tr>
<tr>
<td>234</td>
<td>185.23</td>
<td>2419.4</td>
<td>84.3</td>
<td>0.164</td>
<td>28.7</td>
<td>0.162</td>
</tr>
<tr>
<td>326</td>
<td>185.23</td>
<td>2430.2</td>
<td>83.5</td>
<td>0.164</td>
<td>29.1</td>
<td>0.163</td>
</tr>
</tbody>
</table>

Figure 4.9: Implied vol plots on May 2, 2013 when satisfying constraints

worse than the previous fits. This is obvious since previous fits have one more free parameter than current fits. Although they are not the best fits, we will still use them in
testing the hedging performance since they are the best fits when risk neutral parameters satisfy the measure change constraint.
Chapter 5

Hedging under Lévy models and constant interest rate

Under Lévy models, jump risk leads to incomplete market and perfect hedging does not exist. Effective hedging strategies are crucial for insurance companies in preventing potentially large losses. Thus we are forced to reconsider hedging in the more realistic sense of approximating a target payoff with a trading strategy: one has to recognize that option hedging is a risky affair, specify a way to measure this risk and then try to minimize it. Different ways to measure risk thus lead to different approaches to hedging. Quadratic hedging can be defined as the choice of a hedging strategy which minimizes the hedging errors in a mean square sense. This corresponds to the market practice measuring risk in terms of “variance” and in some cases leads to explicitly computable hedging strategies. The criterion to be minimized in a least squares sense can be either the hedging error at maturity or the one step ahead hedging error, i.e., the hedging errors measured locally in time. The first notion leads to mean-variance hedging and the second notion leads to local risk minimization hedging. In order to take expectations, a probability measure has to be specified. Mean variance hedging and local risk minimization hedging are equivalent when the discounted price is a martingale measure but different otherwise.

At the beginning of this chapter, we will first introduce hedging methods commonly used in the complete market setting. The next two sections will derive hedging positions in the mean variance hedging and local risk minimization hedging. Then numerical examples are used to compare the hedging performance of these three hedging methods. Finally, we extend pricing and hedging methodologies to the product with annual ratchet guarantee.
CHAPTER 5. HEDGING UNDER LÉVY MODELS AND CONSTANT INTEREST RATE

5.1 Delta hedging and Delta Gamma hedging

Suppose the portfolio we have is

\[ V_t = \xi_t S_t + \beta_t B_t - H_t. \]

The idea of Delta hedging is to make the portfolio delta neutral, i.e. \( \partial_S V = 0 \). Then we can get the optimal hedging position in the Delta hedging method is

\[ \hat{\xi}_t(T) = \partial_S H = \partial_S h - (e^{-\alpha t} - e^{-\alpha T}). \]

When we introduce another risky asset as second hedging instrument, the portfolio is

\[ V_t = \xi_t^{(1)} S_t + \beta_t B_t + \xi_t^{(2)} f_t - H_t. \]

The idea of Delta Gamma hedging is to make the portfolio delta neutral and gamma neutral, i.e. \( \partial_S V = 0 \) and \( \partial_{SS} V = 0 \). Then the optimal hedging positions in the Delta Gamma hedging method are

\[ \hat{\xi}_t^{(1)}(T) = \frac{\partial_S h \partial_{SS} f - \partial_S f \partial_{SSH} h}{\partial_{SS} f} - (e^{-\alpha t} - e^{-\alpha T}) \]

\[ \hat{\xi}_t^{(2)}(T) = \frac{\partial_{SSH} h}{\partial_{SS} f}. \]

The derivatives \( \partial_S h \) and \( \partial_{SSH} h \) can be obtained by Theorem 3.2.3.

5.2 Mean-variance hedging

The Black-Scholes model and generalization of it where the dynamics of prices \( X_t = \{X_t^1, \ldots, X_t^m\} \) of several assets are described by a diffusion process driven by Brownian motion have strongly influenced risk management practices in derivatives markets since the 1970s. In such models, the question of hedging a given contingent claim with payoff \( Y \) paid at a future date \( T \) can be theoretically tackled via a representation theorem for Brownian martingale: by switching to a unique equivalent martingale measure \( Q \), we obtain a unique self-financing strategy \( \xi_t \) such that

\[ Y = E^Q[Y | \mathcal{F}_0] + \int_0^T \xi_t dX_t \quad Q \text{ a.s.} \]  

(5.1)
This representation holds almost surely under any measure equivalent to $\mathbb{Q}$, thus yielding a strategy $\xi_t$ with initial capital $c = E^\mathbb{Q}[Y|\mathcal{F}_0]$ which “replicates” the terminal payoff $Y$ almost surely. On the computational side, $\xi_t$ can be computed by differentiating the option price w.r.t. the underlying asset(s) $X_t$. These ideas are central to the use of diffusion models in option pricing and hedging.

Stochastic processes with discontinuous trajectories are being increasingly considered, both in the research literature and in practice. A natural question is therefore to examine what becomes of the above assertions in presence of discontinuities in asset prices. It is known that, except in very special cases, martingales with respect to the filtration of a discontinuous process cannot be represented in the form (5.1), leading to market incompleteness. Far from being a shortcoming of models with jumps, this property corresponds to a genuine feature of real markets: the impossibility of replicating an option by trading in the underlying asset.

A natural extension, due to Föllmer and Sondermann (1986), has been to approximate the target payoff $Y$ by optimally choosing the initial capital $c$ and a self-financing trading strategy $\xi = \left\{\xi_t^{(1)}, \cdots, \xi_t^{(m)}\right\}$ in the assets $X^{(1)}, \cdots, X^{(m)}$ in order to minimize the quadratic hedging error:

$$\text{minimize } E\left[\left(c + \sum_{i=1}^m \int_0^T \xi_t^{(i)} dX_t^{(i)} - Y\right)^2\right] = E[\epsilon_T^2(c, \xi)] \quad (5.2)$$

where $\epsilon_T(c, \xi) = c - Y + \int_0^T \xi_t dX_t$ is the residual hedging error at time $T$.

The expectation in (5.2) can be understood either as being computed under an “objective” measure meant as a statistical model of price fluctuation or as being computed under a martingale (“risk-adjusted”) measure. There are practical and theoretical motivations for using a risk-adjusted (martingale) measure fitted to market prices of options for computing the hedging performance.

- When $X$ is a martingale, problem (5.2) is related to the Kunita-Watanabe decomposition of $Y$, which has well-known properties guaranteeing the existence of a solution under mild conditions. By contrast, quadratic hedging with discontinuous processes under an arbitrary measure may lead to negative prices or not having a solution in general.
Ideally, the probability measure used to compute expectations in (5.2) should reflect future uncertainty over the lifetime of the option. When using the “statistical” measure as estimated from historical data, this only holds if increments are stationary. On the other hand, the risk-adjusted measure retrieved from quoted option prices using a “calibration” procedure is naturally interpreted as encapsulating the market anticipation of future scenarios.

In this section, we will study the problem (5.2) when underlying asset prices are modeled by an exponential Lévy process. In accordance with the above remarks, we will assume that the expectation in (5.2) is computed using a martingale measure estimated from observed prices of options.

We consider a market consisting of \( m \) traded assets \( X^{(i)}, i = 1, \ldots, m \) that can be used for hedging a contingent claim \( Y \in \mathcal{F}_T \) with \( E[Y^2] < \infty \). We suppose that the prices of traded assets are expressed using the money market account as numeraire. We assume that, using market prices of options, we have identified a pricing measure under which the prices of traded assets \( X^{(1)}, \ldots, X^{(m)} \) are martingales. The evolution of prices under this probability measure will be described by the following stochastic integrals:

\[
X_t = X_0 + \int_0^t \delta_s d\hat{W}_s + \int_{[0,t] \times \mathbb{R}} \eta_s(y) \hat{\Phi}(ds \times dy) \quad (5.3)
\]

where \( \hat{\Phi}(ds \times dy) \) is compensated \( \mathbb{Q} \)-jump measure.

We denote \( Y_t = E^\mathbb{Q}[Y|\mathcal{F}_t] \) the value of the option and assume that \( Y_t \) can be represented by a stochastic integral:

\[
Y_t = Y_0 + \int_0^t \delta^0_s d\hat{W}_s + \int_{[0,t] \times \mathbb{R}} \eta^0_s(y) \hat{\Phi}(ds \times dy) \quad (5.4)
\]

The initial values \( X_0 \) and \( Y_0 \) are deterministic. Such a representation can be formally obtained by applying Itô formula to the function of the option price \( Y_t = h(t, X_t, T) \).

We assume the coefficients satisfy the following conditions:
(i) \( \delta : \Omega \times [0, \infty) \rightarrow \mathbb{R}^m \times \mathbb{R}^d \) and \( \delta^0 : \Omega \times [0, \infty) \rightarrow \mathbb{R}^d \) are càdlàg \( \mathcal{F}_t \)-adapted processes.
(ii) \( \eta : \Omega \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^m \) and \( \eta^0 : \Omega \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) are càdlàg predictable random functions such that
∀t ∈ [0, T], ∀z ∈ R, ||ηₜ(z)||² ≤ τ(z)Cₛ and |ηₜ(z)ηₜ₀(z)| ≤ τ(z)Cₛ

hold almost surely for some finite-valued adapted process C and some deterministic function τ satisfying \(\int_{\mathbb{R}} \tau(z) \nu(dz) < \infty\).

(iii) We fix a time horizon T and assume

\[ E \int_0^T (||\delta_s||^2 + C_s)ds < \infty \]

These assumptions imply in particular that the stochastic integrals (5.3) and (5.4) exist and define square-integrable martingales. Below we give an example of stock price model satisfying (5.3) and assumptions (i)-(iii).

**Example 1 (Exponential Lévy models).** Let \(L\) be a Lévy process with characteristic triplet \((σ, ν, γ)\). For \(e^L\) to be a martingale, the characteristic triplet must satisfy

\[ \int |y| > 1 e^y \nu(dy) < \infty, \quad \text{and} \quad \gamma + \frac{\sigma^2}{2} + \int (e^y - 1 - y 1_{|y| \leq 1}) \nu(dy) = 0. \]

In this case, \(X_t = X_0 e^{L_t}\) satisfies the following:

\[ X_t = X_0 + \int_0^t \sigma X_s dW_s + \int_{[0,t] \times \mathbb{R}} X_{s-}(e^z - 1)\Phi(ds \times dz) \]

where \(\Phi\) is the compensated jump measure of \(L\). From the Lévy-Khinchin formula,

\[ E[X_t^2] = X_0^2 \exp \left\{ t\sigma^2 + t \int (e^z - 1)^2 \nu(dx) \right\}, \]

hence, \(X_t\) is square integrable if \(\int_{|x| \geq 1} e^{2x} \nu(dx) < \infty\).

Let us consider hedging under the exponential Lévy model. Consider an agent who has sold at time \(t = 0\) the contingent claim with terminal payoff \(Y\) for the price \(c\) and wants to hedge the associated risk by trading in assets \((X^{(1)}, \ldots, X^{(m)}) = X\). We call an admissible hedging strategy a predictable process \(ξ : \Omega \times [0, T] \rightarrow \mathbb{R}^m\) such that \(\int_0^T \xi_t dX_t\) is a square integrable martingale. Denote by \(\mathcal{A}\) the set of such strategies.

**Theorem 5.2.1.** Let \(X\) and \(Y\) as be in (5.3) and(5.4) satisfying the hypotheses (i)-(iii)
and suppose in addition that the matrix
\[ D_t = \delta_t \delta_t^* + \int_{\mathbb{R}} \dot{\nu}(dz) \eta_t(z) \eta_t(z)^* \] (5.5)
is almost surely nonsingular for all \( t \in [0, T] \), where the star denotes the matrix transposition. Then the mean-variance hedge \((\hat{c}, \hat{\xi})\), solution of
\[ E^Q[\epsilon_T^2(\hat{c}, \hat{\xi})] = \inf_{(c, \xi) \in \mathbb{R} \times A} E^Q[\epsilon_T^2(c, \xi)] \]
is given by
\[ \hat{c} = E^Q[Y] = Y_0 \] (5.6)
\[ \hat{\xi}_t = D_t^{-1} \left( \delta_t^0 \delta_t^* + \int_{\mathbb{R}} \dot{\nu}(dz) \eta_t^0(z) \eta_t(z)^* \right) \] (5.7)

This theorem is very general and can be applied to derive hedging positions when using \( n \) risky assets.

**Proof.** First, for every admissible strategy \( \xi \),
\[ E^Q[\epsilon_T^2(c, \xi)] = (c - E^Q[Y])^2 + E^Q \left[ \left( E^Q[Y] - Y + \int_0^T \xi_t dX_t \right)^2 \right] \]
This shows that the initial capital is given by \( \hat{c} = E^Q[Y] \). Substituting \( c = \hat{c} \) yields
\[ E^Q[\epsilon_T^2(c, \xi)] = \int_0^T E^Q \|\xi_t \delta_t - \delta_t^0\|^2 dt + \int_0^T dt \int_{\mathbb{R}} \dot{\nu}(dz) E^Q \left[ (\xi_t \eta_t(z) - \eta_t^0(z))^2 \right] \]
This expression is clearly minimized by the strategy \( \hat{\xi} \). Moreover, under the assumption of this proposition, almost surely, \( (\xi_t)_{0 \leq t \leq T} \) is càglàd and therefore admissible. \( \Box \)

Suppose we have two risky assets, one is the underlying \( S_t \), and the other is the liquid options \( f_t \). When using money market accounts as numeraire, their dynamics are (we will use symbols \( \tilde{S}_t \) and \( \tilde{f}_t \) to represent the discounted process):
\[
\tilde{S}_t = \tilde{S}_0 + \int_0^t \sigma \tilde{S}_u \tilde{d}W_u + \int_0^t \int_{\mathbb{R}} \tilde{S}_u \left( e^y - 1 \right) \tilde{\Phi}^S(du \times dy)
\]
\[
\tilde{f}_t = \tilde{f}_0 + \int_0^t \sigma \tilde{S}_u \tilde{\partial}_S \tilde{f} \tilde{d}W_u + \int_0^t \int_{\mathbb{R}} \left[ \tilde{f}(u, \tilde{S}_u - e^y) - \tilde{f}(u, \tilde{S}_u) \right] \tilde{\Phi}^S(du \times dy)
\]
The dynamics of function $\hat{h}_t$ is

$$\hat{h}_t = \hat{h}_0 + \int_0^t \sigma^S S_u \partial h \, dW_u^S + \int_0^t \int_\mathbb{R} \left[ h(u, S_u e^y, T) - h(u, S_u, T) \right] \hat{\Phi}^S(du \times dy)$$

Then we can get the following two theorems by applying the Theorem 5.2.1. In the following two theorem, we suppose there is no management fees, i.e. $\alpha = 0$.

**Theorem 5.2.2.** The mean-variance hedging position of underlying asset to hedge the option with maturity $T$ is

$$\hat{\xi}_t(T) = \frac{(\sigma^S)^2 \partial h + \frac{1}{S_t} \int_\mathbb{R} (e^y - 1) \Delta h \hat{\nu}^S(dy)}{(\sigma^S)^2 + \int_\mathbb{R} (e^y - 1)^2 \hat{\nu}^S(dy)}$$

where $\Delta h = h(t, S_t e^y, T) - h(t, S_t, T)$.

**Proof.** When we consider hedging with only the underlying asset, there is one risky asset. In Theorem 5.2.1,

$$\delta_t = \sigma^S \partial S_t, \quad \eta_t(y) = \hat{S}_t(e^y - 1)$$

$$\delta_0^0 = \sigma^S \partial \hat{h}_t, \quad \eta_0^0(y) = \hat{h}(t, S_t e^y, T) - \hat{h}(t, S_t, T)$$

Applying the formula (5.7), we can finish the proof. \qed

In incomplete markets, options are not redundant assets; therefore, if options are available as hedging instruments they can and should be used to improve hedging performance. Forty years after opening of the first organized option markets, options have become liquidly traded securities in their own right and “vanilla” call and put options on indices, exchange rates and major stocks are traded in a way similar to their underlying stocks. Thus, liquidly traded options are available as instruments for hedging more complex, exotic or illiquid options and are commonly used by risk managers.

**Theorem 5.2.3.** The mean-variance hedging positions $\hat{\xi}_t = \left( \hat{\xi}^{(1)}_t, \hat{\xi}^{(2)}_t \right)$ of underlying and liquid options to hedge the option with maturity $T$ are

$$\hat{\xi}^{(1)}_t(T) = \frac{BC - DE}{AB - E^2}, \quad \hat{\xi}^{(2)}_t(T) = \frac{AD - CE}{AB - E^2}$$
where

\[
A = S_t^2 \left( (\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1)^2 \hat{\nu}^S(dy) \right)
\]

\[
B = S_t^2 \left( \partial_s f (\sigma^S)^2 + \int_{\mathbb{R}} (\Delta f / S_t)^2 \hat{\nu}^S(dy) \right)
\]

\[
C = S_t^2 \left( \partial_s h (\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1) \Delta h / S_t \hat{\nu}^S(dy) \right)
\]

\[
D = S_t^2 \left( \partial_s f \partial_s h (\sigma^S)^2 + \int_{\mathbb{R}} \Delta f \Delta h / S_t \hat{\nu}^S(dy) \right)
\]

\[
E = S_t^2 \left( \partial_s f (\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1) \Delta f / S_t \hat{\nu}^S(dy) \right).
\]

**Proof.** When we hedge with both the underlying asset and liquid option with maturity \( T_s \), there are two risky assets. In Theorem 5.2.1,

\[
\delta_t = \left( \begin{array}{c}
\sigma^S \hat{S}_t - \sigma^S \hat{S}_t \partial \hat{f} \\
\sigma^S \hat{S}_t - \partial \hat{f} 
\end{array} \right), \quad \eta_t(y) = \left( \begin{array}{c}
\hat{S}_t - (e^y - 1) \\
\Delta \hat{f}
\end{array} \right)
\]

\[
\delta^0_t = \sigma^S \hat{S}_t \partial \hat{h}, \quad \eta^0_t(y) = \tilde{h}(t, S_t - e^y, T) - \tilde{h}(t, S_t, T)
\]

Applying formula (5.5),

\[
D_t = \left( \begin{array}{cc}
A & E \\
E & B
\end{array} \right)
\]

and

\[
\delta^0_t \delta^*_t + \int_{\mathbb{R}} \hat{\nu}^S(dy) \eta^0_t(y) \eta_t(y)^* = \left( \begin{array}{c}
B \\
C
\end{array} \right)
\]

Hence the optimal hedging positions are

\[
\hat{\xi}_t(T) = \left( \begin{array}{c}
\hat{\xi}^{(1)}_t(T) \\
\hat{\xi}^{(2)}_t(T)
\end{array} \right) = \left( \begin{array}{cc}
A & E \\
E & B
\end{array} \right)^{-1} \left( \begin{array}{c}
B \\
C
\end{array} \right) = \left( \begin{array}{c}
\frac{BC - DE}{AB - E^2} \\
\frac{AD - CE}{AB - E^2}
\end{array} \right).
\]

**5.3 Local risk minimization hedging**

In the previous section, minimizing the \( \mathbb{Q} \)-variance of hedging error as a criterion for the measuring risk is not very natural: \( \mathbb{Q} \) represents a pricing rule and not a statistical description of market events, so the profit and loss (PnL) of a portfolio may have a large variance while its “risk neutral” variance can be small. A natural generalization is therefore to try to repeat the same analysis under a statistical model \( \mathbb{P} \). Unfortunately the analysis is not as easy as in the martingale case and explicit solutions are difficult to obtain for models with jumps. Instead we will consider local risk minimization hedging,
Theorem 5.3.1. The local risk minimization hedging position \( \hat{\xi}_t \) of underlying to hedge the option with maturity \( T \) is

\[
\hat{\xi}_t(T) = \frac{(\sigma^S)^2 \partial_S h + \frac{1}{S_t} \int_{\mathbb{R}} (e^y - 1) \Delta h \nu^S(dy)}{(\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu^S(dy)} - e^{-\alpha t} + e^{-\alpha T}
\]

(5.9)

where \( \Delta h = h(t, S_t, e^y, T) - h(t, S_t, T) \).

Proof. Applying Itô’s lemma under real measure \( \mathbb{P} \) and letting \( \epsilon_t = \xi_t + e^{-\alpha t} - e^{-\alpha T} \) we will have

\[
dV_t = \epsilon_t S_t \left( \gamma^S + \frac{1}{2} (\sigma^S)^2 \right) dt + \epsilon_t S_t \sigma^S dW^S_t + \epsilon_t \int_{\mathbb{R}} S_t \nu^S(dy) dt + \beta_t B_t dt
\]

The variance will be

\[
Var(dV_t) = Var\left( \int_{\mathbb{R}} [\epsilon_t S_t - (e^y - 1) - \Delta h] J^S(dy) dt \right)
\]

\[
= (\epsilon_t - \partial_S h)^2 S_t^2 (\sigma^S)^2 dt + \int_{\mathbb{R}} [\epsilon_t S_t - (e^y - 1) - \Delta h]^2 \nu^S(dy) dt
\]

\[
- \left( \int_{\mathbb{R}} [\epsilon_t S_t - (e^y - 1) - \Delta h] \nu^S(dy) dt \right)^2.
\]

The hedging position should not depend on \( dt \). We will let \( dt \) tend to 0 to eliminate the \( dt \) term,

\[
\lim_{dt \to 0} \frac{Var(dV_t)}{dt} = (\epsilon_t - \partial_S h)^2 S_t^2 (\sigma^S)^2 + \int_{\mathbb{R}} [\epsilon_t S_t - (e^y - 1) - \Delta h]^2 \nu^S(dy).
\]

The optimal \( \hat{\epsilon}_t \) is obtained as

\[
\hat{\epsilon}_t = \frac{(\sigma^S)^2 \partial_S h + \frac{1}{S_t} \int_{\mathbb{R}} (e^y - 1) \Delta h \nu^S(dy)}{(\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu^S(dy)}.
\]

(5.10)
Then we can obtain the expression of $\hat{\xi}_t$.

Note that the above equation makes sense in a much more general setting than for instance the delta hedging strategy which requires that the option price be differentiable, a property which can fail in pure jump models.

From the theorem, we can see local risk minimization hedging positions have the same structure as the mean-variance hedging positions with only replacing the $Q$-Lévy density with $P$-Lévy density. Hence the mean variance hedging and local risk minimization hedging are equivalent when the discounted price is a martingale but different otherwise.

Next we will interpret the formulas of local risk minimization hedging positions using a simple case. Suppose the stock index follows an exponential Lévy process with a non-zero diffusion component and a two possible jump size. For simplicity, we also suppose there is no management fees, i.e., $\alpha = 0$.

$$\frac{dS_t}{S_t} = \left(\gamma^S + \frac{1}{2}(\sigma^S)^2\right) dt + \sigma^S dW_t^S + \left[q(e^{\eta_1} - 1) + (1 - q)(e^{\eta_2} - 1)\right] dN_t$$

where $N$ is a Poisson process with intensity $\lambda$ and $q > 0$ is the probability of jump with size $\eta_1$.

The optimal $\hat{\xi}_t$ is

$$\hat{\xi}_t(T) = \frac{\partial_S h (\sigma^S)^2 + \lambda \left[(e^{\eta_1} - 1)\Delta h(\eta_1) \ q + (e^{\eta_2} - 1)\Delta h(\eta_2) \ (1 - q)\right]}{\left((\sigma^S)^2 + \lambda \left[(e^{\eta_1} - 1)^2q + (e^{\eta_2} - 1)^2(1 - q)\right]\right)}$$

where $\Delta h(\eta) = h(t, S_t e^{\eta}, T) - h(t, S_t, T)$.

If the jump sizes are small, $\Delta h(\eta) \approx \partial_S h \cdot S_t (e^\eta - 1)$. The optimal $\hat{\xi}_t(T) = \partial_S h$ which is just the same as delta hedging.

The money market account is determined according to the self-financing condition. The following theorem will give the hedging positions on each hedging date.

**Theorem 5.3.2.** Suppose we have a series of hedging dates $0 = T_0 < T_1 < \cdots < T_N = T$ with equal time step $\Delta t = T_{j+1} - T_j$. The stock index at time $T_n$ is denoted as $S_n$. The
hedging positions on the date $T_n$ are denoted by $\{\xi_n, \beta_n\}$, where

$$\xi_n = \xi_{T_n}(T), \quad n \geq 0$$

$$\beta_n = \beta_{n-1}e^{r\Delta t} - (\xi_n - \xi_{n-1})S_n + \alpha \hat{S}_{n-1}\Delta t, \quad n \geq 1$$

$$\beta_0 = H_0 - \xi_0S_0.$$

**Proof.** At $T_0 = 0$, we have $V_0 = 0$. Then we will hold $\xi_0$ units of $S_0$, and the money in the bank account is $\beta_0 = H_0 - \xi_0S_0$.

At $t = T_1$, value of hedging portfolio is $\xi_0S_1 + \beta_0e^{r\Delta t} + \alpha \hat{S}_0\Delta t$. Now units of stock index have changed to $\xi_1$. $\beta_1$ needs to satisfy the following self-financing condition,

$$\xi_1S_1 + \beta_1 = \xi_0S_1 + \beta_0e^{r\Delta t} + \alpha \hat{S}_0\Delta t.$$

Hence $\beta_1 = \beta_0 e^{r\Delta t} - (\xi_1 - \xi_0)S_1 + \alpha \hat{S}_0\Delta t$.

Repeat, $\beta_n = \beta_{n-1}e^{r\Delta t} - (\xi_n - \xi_{n-1})S_n + \alpha \hat{S}_{n-1}\Delta t$.

Define the book value at time $T$ as

$$PnL(T) = \xi_{N-1}S_N + \beta_{N-1}e^{r\Delta t} + \alpha \hat{S}_{N-1}\Delta t - G(S_N).$$  \hfill (5.11)

This book value is also known as the Profit and Loss (PnL).

In the following we will derive local risk minimization hedging position when using both underlying and liquid options. The portfolio we have is

$$V_t = \xi_t^{(1)}S_t + \xi_t^{(2)}f(t, S_t, T_s) + \beta_tB_t - H(t, S_t, T).$$

The one step ahead hedging error is

$$dV_t = \xi_t^{(1)}dS_t + \xi_t^{(2)}df_t + \beta_t dB_t - [dh_t - (e^{-\alpha t} - e^{-\alpha T})dS_t] + \alpha dt\hat{S}_t.$$

**Theorem 5.3.3.** The local risk minimization hedging positions $\{\hat{\xi}_t^{(1)}, \hat{\xi}_t^{(2)}\}$ of the underlying and liquid options to hedge the option with maturity $T$ are

$$\hat{\xi}_t^{(1)}(T) = \frac{B^pC^p - D^pE^p}{A^pB^p - (E^p)^2} - e^{-\alpha t} + e^{-\alpha T},$$  \hfill (5.12)

$$\hat{\xi}_t^{(2)}(T) = \frac{A^pD^p - C^pE^p}{A^pB^p - (E^p)^2}.$$  \hfill (5.13)
where

\[
A^\mathbb{P} = S_t^2 \left[ (\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu^S(dy) \right]
\]
\[
B^\mathbb{P} = S_t^2 \left[ \partial_S f (\sigma^S)^2 + \int_{\mathbb{R}} (\Delta f / S_t - \sigma^S)^2 \nu^S(dy) \right]
\]
\[
C^\mathbb{P} = S_t^2 \left[ \partial_S h (\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1) \Delta h / S_t - \nu^S(dy) \right]
\]
\[
D^\mathbb{P} = S_t^2 \left[ \partial_S f \partial_S h (\sigma^S)^2 + \int_{\mathbb{R}} \Delta f \Delta h / S_t^2 - \nu^S(dy) \right]
\]
\[
E^\mathbb{P} = S_t^2 \left[ \partial_S f (\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1) \Delta f / S_t - \nu^S(dy) \right]
\]
\[
\Delta f = f(t, S_t e^{y}, T_s) - f(t, S_t, T_s).
\]

Proof. Applying Itô’s lemma under \( \mathbb{P} \) and letting \( \varepsilon_t = \xi_t + e^{-\alpha t} - e^{-\alpha T} \) we will have

\[
dV_t = \varepsilon_t S_t \left( \gamma^S + \frac{1}{2} (\sigma^S)^2 \right) dt + \varepsilon_t S_t \sigma^S dW_t^S + \varepsilon_t \int_{\mathbb{R}} S_t \left( (e^y - 1) \right) J^S(dt \times dy) + \beta_t B_t r dt
\]
\[
+ \xi_t^{(2)} \left[ [\partial_t f + \mathcal{L}^\mu f] dt + \partial_S f S_t \sigma^S dW_t^S + \int_{\mathbb{R}} \Delta f J^S(dt \times dy) \right]
\]
\[- \left[ \partial_t h + \mathcal{L}^\nu h \right] dt - \partial_S h S_t \sigma^S dW_t^S - \int_{\mathbb{R}} \Delta h J^S(dt \times dy).
\]

The variance is

\[
\text{Var}(dV_t) = \text{Var} \left[ \varepsilon_t \varepsilon_t S_t \left( \gamma^S + \frac{1}{2} (\sigma^S)^2 \right) dt + \varepsilon_t S_t \sigma^S dW_t^S + \varepsilon_t \int_{\mathbb{R}} S_t \left( (e^y - 1) \right) J^S(dt \times dy) \right]
\]
\[
= \left( \varepsilon_t + \xi_t^{(2)} \partial_S f - \partial_S h \right) S_t \sigma^S dW_t^S + \int_{\mathbb{R}} \left[ \varepsilon_t \varepsilon_t S_t \left( (e^y - 1) \right) + \xi_t^{(2)} \Delta f - \Delta h \right] J^S(dt \times dy)
\]
\[- \left( \int_{\mathbb{R}} \left[ \varepsilon_t \varepsilon_t S_t \left( (e^y - 1) \right) + \xi_t^{(2)} \Delta f - \Delta h \right] J^S(dt \times dy) \right)^2.
\]

We will let \( dt \) tend to 0 so that hedging positions will not depend on \( dt \).

\[
\lim_{dt \to 0} \frac{\text{Var}(dV_t)}{dt} = \left( \varepsilon_t + \xi_t^{(2)} \partial_S f - \partial_S h \right)^2 S_t^2 (\sigma^S)^2 + \int_{\mathbb{R}} \left[ \varepsilon_t \varepsilon_t S_t \left( (e^y - 1) \right) + \xi_t^{(2)} \Delta f - \Delta h \right]^2 \nu^S(dy).
\]

We will choose \( \varepsilon_t \) and \( \xi_t^{(2)} \) such that (5.14) is minimized. (5.14) is a bivariate quadratic
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function and its minimum is obtained at

\[ \hat{\xi}_t(T) = \frac{B^p C^p - D^p E^p}{A^p B^p - (E^p)^2}, \quad \hat{\xi}^{(2)}_t(T) = \frac{A^p D^p - C^p E^p}{A^p B^p - (E^p)^2}. \]

In order to interpret the local risk minimization hedging positions of underlying and liquid options, we will use the case of a single jump size as an example. Suppose the stock index follows an exponential Levy process with a non-zero diffusion component and a single possible jump size:

\[ \frac{dS_t}{S_{t-}} = \left( \gamma^S + \frac{1}{2}(\sigma^S)^2 \right) dt + \sigma^S dW_t^S + (e^{\eta} - 1)dN_t \]

where \( N \) is a Poisson process with intensity \( \lambda \).

Under this simple case, the expressions for all the coefficients can be simplified.

\[ A^p = S_{t-}^2 \left[ (\sigma^S)^2 + \lambda(e^m - 1)^2 \right] \]
\[ B^p = S_{t-}^2 \left[ (\sigma^S \partial_S f)^2 + \lambda \left( \frac{\Delta f(\eta_1)}{S_{t-}} \right)^2 \right] \]
\[ C^p = -2S_{t-}^2 \left[ (\sigma^S)^2 \partial_S h + \lambda(e^m - 1) \frac{\Delta h(\eta_1)}{S_{t-}} \right] \]
\[ D^p = -2S_{t-}^2 \left[ (\sigma^S)^2 \partial_S f \partial_S h + \lambda \frac{\Delta f(\eta_1) \Delta h(\eta_1)}{S_{t-}^2} \right] \]
\[ E^p = 2S_{t-}^2 \left[ (\sigma^S)^2 \partial_S f + \lambda(e^m - 1) \frac{\Delta f(\eta_1)}{S_{t-}} \right] \]

where \( \Delta h(\eta) = h(t, S_t e^\eta, T) - h(t, S_t, T) \) and \( \Delta h(\eta) = h(t, S_t e^\eta, T_s) - h(t, S_t, T_s) \).

Plug them into the expressions of \( \hat{\xi}^{(1)}_t \) and \( \hat{\xi}^{(2)}_t \) and we will have

\[ \hat{\xi}^{(1)}_t(T) = \frac{\partial_S h(\Delta f(\eta_1) - \partial_S f \Delta h(\eta_1))}{\Delta f(\eta_1) - \partial_S f (e^m - 1)S_t} - e^{-\alpha t} + e^{-\alpha T} \]
\[ \hat{\xi}^{(2)}_t(T) = \frac{\Delta h(\eta_1) - \partial_S h(e^m - 1)S_t}{\Delta f(\eta_1) - \partial_S f (e^m - 1)S_t}. \]

When the jump size \( \eta_1 \) is small, the optimal hedge is approximated by delta gamma
hedge ratios.

\[
\begin{align*}
\dot{\xi}_t^{(1)}(T) &\approx \frac{\partial_s h \partial_{ss} f - \partial_s f \partial_{ss} h}{\partial_{ss} f} - e^{-\alpha t} + e^{-\alpha T} \\
\dot{\xi}_t^{(2)}(T) &\approx \frac{\partial_{ss} h}{\partial_{ss} f}.
\end{align*}
\]

The positions in the money market account are determined by the self-financing condition.

**Theorem 5.3.4.** Suppose we have a series of hedging dates \(0 = T_0 < T_1 < \cdots < T_N = T\) with equal time step \(\Delta t = T_{j+1} - T_j\). The stock index at time \(T_n\) is denoted as \(S_n\). The hedging positions on the date \(T_n\) are \(\{\xi_n^{(1)}, \xi_n^{(2)}, \beta_n\}\), where

\[
\begin{align*}
\xi_n^{(1)} &= \xi_{T_n}^{(1)}(T), \quad n \geq 0 \\
\xi_n^{(2)} &= \xi_{T_n}^{(2)}(T), \quad n \geq 0 \\
\beta_n &= S_n \left( \xi_{n-1}^{(1)} - \xi_n^{(1)} \right) + \xi_{n-1}^{(2)} f(\Delta t, S_n, T_s, K_{n-1}) - \xi_n^{(2)} f(0, S_n, T_s, K_n) + \beta_{n-1} e^{r \Delta t} + \alpha \hat{S}_{n-1} \Delta t, n \geq 1 \\
\beta_0 &= H_0 - \xi_0^{(1)} S_0 - \xi_0^{(2)} f(0, S_0, T_s, K_0).
\end{align*}
\]

**Proof.** At \(T_0 = 0\), we have \(V_0 = 0\). Then we will hold \(\xi_0^{(1)}\) units of \(S_0\), \(\xi_0^{(2)}\) units of at-the-money options \(f(0, S_0, T_s, K_0)\) maturing at \(T_s\) with current stock index \(S_0\) and strike \(K_0\). Hence the money in the bank account is given by \(\beta_0 = H_0 - \xi_0^{(1)} S_0 - \xi_0^{(2)} f(0, S_0, T_s, K_0)\).

At \(t = T_1\), value of hedging portfolio is \(\xi_1^{(1)} S_1 + \xi_1^{(2)} f(0, S_1, T_s, K_1) + \beta_1 = \xi_0^{(1)} M_1 + \xi_0^{(2)} f(\Delta t, S_1, T_s, K_0) + \beta_0 e^{r \Delta t} + \alpha \hat{S}_0 \Delta t\). We will liquidate all the positions in the options and buy new at-the-money options with maturity \(T_s\). Now units of underlying and options have updated as \(\xi_1^{(1)}\) and \(\xi_1^{(2)}\). \(\beta_1\) needs to satisfy the following self-financing condition.

\[
\begin{align*}
\xi_1^{(1)} S_1 + \xi_1^{(2)} f(0, S_1, T_s, K_1) + \beta_1 &= \xi_0^{(1)} M_1 + \xi_0^{(2)} f(\Delta t, S_1, T_s, K_0) + \beta_0 e^{r \Delta t} + \alpha \hat{S}_0 \Delta t \\
\beta_1 &= S_1 \left( \xi_0^{(1)} - \xi_1^{(1)} \right) + \xi_0^{(2)} f(\Delta t, S_1, T_s, K_0) - \xi_1^{(2)} f(0, S_1, T_s, K_1) + \beta_0 e^{r \Delta t} + \alpha \hat{S}_0 \Delta t.
\end{align*}
\]

Hence

\[
\begin{align*}
\beta_1 &= S_1 \left( \xi_0^{(1)} - \xi_1^{(1)} \right) + \xi_0^{(2)} f(\Delta t, S_1, T_s, K_0) - \xi_1^{(2)} f(0, S_1, T_s, K_1) + \beta_0 e^{r \Delta t} + \alpha \hat{S}_0 \Delta t.
\end{align*}
\]

Similarly,

\[
\begin{align*}
\beta_n &= S_n \left( \xi_{n-1}^{(1)} - \xi_n^{(1)} \right) + \xi_{n-1}^{(2)} f(\Delta t, S_n, T_s, K_{n-1}) - \xi_n^{(2)} f(0, S_n, T_s, K_n) + \beta_{n-1} e^{r \Delta t} + \alpha \hat{S}_{n-1} \Delta t.
\end{align*}
\]

\(\Box\)
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After we have the values of money market account we can define the book value (Profit and Loss) at time $T$ as

$$
PnL(T) = \xi_{N-1} S_{N} + \xi_{N-1}^{(2)} f(\Delta t, S_{N}, T_{s}, K_{N-1}) + \beta_{N-1} e^{\Delta t} + \alpha \hat{S}_{N-1} \Delta t - G(S_{N}).$$

(5.15)

5.4 Numerical examples

In this section, we will compare hedging performances of different hedging strategies under variance gamma process. This Lévy process can, as every semimartingale, be written as a Brownian motion evaluated at a random time. In particular the variance gamma process can be obtained by replacing the time in the Brownian motion with a gamma process. The variance gamma model, in its general version, has two additional parameters compared with the Black-Scholes model. These parameters allow to control the skewness and kurtosis of the process followed by the underlying returns. In the following we will directly use the calibration results from chapter 4. The parameters of variance gamma process under $P$ and $Q$ are as follows.

$$\nu^{S}(dy) = a \left( \frac{e^{-b^{+}y}}{y} 1_{\{y>0\}} + \frac{e^{-b^{-}|y|}}{|y|} 1_{\{y<0\}} \right)$$

$$\hat{\nu}^{S}(dy) = \hat{a} \left( \frac{e^{-\hat{b}^{+}y}}{y} 1_{\{y>0\}} + \frac{e^{-\hat{b}^{-}|y|}}{|y|} 1_{\{y<0\}} \right)$$

where $a = 185.23, b^{+} = 104.83, b^{-} = 102.62$ and $\hat{a} = 185.23, \hat{b}^{+} = 2430.2, \hat{b}^{-} = 83.5$. Other parameters are $\alpha = 0, S_{0} = 100, R_{\text{min}} = 0, r = 0.02, \mu^{S} = 0.015, \sigma^{S} = 0.001, N_{\text{sim}} = 2000$. The hedging frequency is daily, i.e. $dt = 1/252$. The option used as hedging instrument has a shorter maturity $T_{s} = 1/12$ and is liquidated on each hedging date, i.e., we will liquidate all the positions of options with shorter maturity and buy new at-the-money options with maturity $T_{s}$. This is because at-the-money options have the best liquidity. For simplicity we suppose there are no transaction costs.

Table 5.1 illustrates the hedging performance for the three hedging methods described above. The numerical results show means and standard deviations of PnL at maturity for different hedging strategies and different maturities. From this table we can see when using one risky asset, i.e. underlying asset to hedge, local risk minimization performs the best and mean variance hedging performs the worst. The performance of delta hedging is similar to local risk minimization. In Figure 5.1, we use two sample
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<table>
<thead>
<tr>
<th>Hedging method</th>
<th>$T = 1$</th>
<th>$T = 3$</th>
<th>$T = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean variance with 1 asset</td>
<td>-0.9</td>
<td>-1.55</td>
<td>-2</td>
</tr>
<tr>
<td>Local risk min with 1 asset</td>
<td>-0.9</td>
<td>-1.56</td>
<td>-2.03</td>
</tr>
<tr>
<td>Delta hedging</td>
<td>-0.9</td>
<td>-1.56</td>
<td>-2.03</td>
</tr>
<tr>
<td>Mean variance with 2 assets</td>
<td>-0.037</td>
<td>-0.076</td>
<td>0.32</td>
</tr>
<tr>
<td>Local risk min with 2 assets</td>
<td>-0.057</td>
<td>-0.083</td>
<td>0.19</td>
</tr>
<tr>
<td>Delta-Gamma hedging</td>
<td>-0.09</td>
<td>-0.15</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Table 5.1: Means and standard deviations of PnL when risk neutral parameters satisfy constraints.

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>$70%$</th>
<th>$90%$</th>
<th>$95%$</th>
<th>$70%$</th>
<th>$90%$</th>
<th>$95%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean variance with 1 asset</td>
<td>1.9882</td>
<td>2.8187</td>
<td>3.5292</td>
<td>2.9869</td>
<td>3.6808</td>
<td>4.1378</td>
</tr>
<tr>
<td>Local risk min with 1 asset</td>
<td>1.976</td>
<td>2.8436</td>
<td>3.3244</td>
<td>2.7382</td>
<td>3.5015</td>
<td>3.9264</td>
</tr>
<tr>
<td>Delta hedging</td>
<td>1.9809</td>
<td>2.8589</td>
<td>3.3814</td>
<td>2.7554</td>
<td>3.5499</td>
<td>3.9781</td>
</tr>
<tr>
<td>Mean variance with 2 assets</td>
<td>0.2215</td>
<td>0.4645</td>
<td>0.6214</td>
<td>0.4645</td>
<td>0.7421</td>
<td>0.9574</td>
</tr>
<tr>
<td>Local risk min with 2 assets</td>
<td>0.1648</td>
<td>0.323</td>
<td>0.4109</td>
<td>0.3073</td>
<td>0.4629</td>
<td>0.5687</td>
</tr>
<tr>
<td>Delta-Gamma hedging</td>
<td>0.2132</td>
<td>0.3885</td>
<td>0.5142</td>
<td>0.3933</td>
<td>0.5933</td>
<td>0.7472</td>
</tr>
</tbody>
</table>

Table 5.2: VaR and CVaR for different hedging methods in the case of $T = 3$.

Figure 5.1: Results of Kolmogorov-Smirnov test in the case of $T = 3$

Kolmogorov-Smirnov test to compare the distributions of PnL between hedging methods. In statistics, the Kolmogorov-Smirnov test (K-S test) is a nonparametric test for
Figure 5.2: Comparing difference of PnL between hedging methods in the case of $T = 3$

the equality of continuous, one-dimensional probability distributions that can be used to compare a sample with a reference probability distribution, or to compare two samples. The two-sample K-S statistic quantifies a distance between the empirical distribution functions of two samples. The two-sample K-S test is one of the most useful and general nonparametric methods for comparing two samples, as it is sensitive to differences in both location and shape of the empirical cumulative distribution functions of the two samples.

When we perform the two-sample K-S test to compare the distributions of PnLs between two hedging methods, we get only the difference between local risk minimization hedging using one risky asset and delta hedging is not statistically significant at significance level 5%. All the other differences are statistically significant.

In Figure 5.2, we plot the histograms of difference of PnL between two hedging methods. We also perform the t-test with

$$H_0 : \text{mean} = 0$$
$$H_a : \text{mean} \neq 0$$

From the test results, we conclude the difference of means between local risk minimization and mean-variance hedging is not statistically significant. But mean-variance
hedging has bigger standard deviation than local risk minimization which confirms the advantage of local risk minimization hedging strategy. Since the PnLs of local risk minimization using underlying asset and delta hedging come from the same distribution from K-S test, hence the difference between their means is not significant. The difference of means between local risk minimization using two risky assets and delta-gamma hedging is significant, which confirms the advantage of local risk minimization using two assets over delta-gamma hedging.

Table 5.2 shows the Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) for 70%, 90% and 95%. VaR(\(\alpha\)) corresponds to the maximum amount of money the portfolio is short of meeting the liability with \(\alpha\) probability. CVaR(\(\alpha\)) gives information about the right tail of the total risk, it is the expected amount of money the portfolio is short of meeting the liability, conditional on the total risk being larger than VaR(\(\alpha\)). In table 5.1, the values of VaR and CVaR for local risk minimization using 1 assets very close to the corresponding values for delta hedging and mean variance hedging using 1 asset. When using 2 assets, local risk minimization hedging has smaller VaR and C-VaR compared to delta-gamma hedging and mean variance hedging. We also note that hedging with underlying and options leads to larger reduction in terms of VaR and CVaR.

Figure 5.3: Hedging positions for hedging methods using one asset

Figure 5.3(a) plots the hedging position of underlying asset for three hedging methods. The hedging positions of underlying asset are very close to each other. In order to see clearly, we also plot the differences between hedging positions in Figure 5.3(b). We can see the difference between mean-variance hedging and local risk minimization is
We compare hedging positions of underlying among three hedging methods using two assets in Figure 5.4(a) and Figure 5.4(b). The difference of positions of underlying between mean-variance hedging and local risk minimization is bigger than that between delta gamma hedging and local risk minimization hedging. We also compare hedging positions of liquid options in Figure 5.4(c) and Figure 5.4(d). The difference of positions of options between mean-variance hedging and local risk minimization is much bigger than that between delta gamma hedging and local risk minimization hedging. This makes advantage of local risk minimization over delta-gamma hedging become more significant comparing to the case using one risky asset.

Figure 5.5 plots one sample path of stock index and its corresponding daily log return.
The sample path reflects jump behaviors of the stock index. Figure 5.6 plots histograms of PnL for different hedging methods. From the histogram we can know the shape of the distribution. The histograms of PnLs using one risky asset are more left-skewed compared to those of hedging methods using two risky assets. We also note that histogram of PnL of mean variance hedging with two assets is far away from normal distribution which confirms its bad performance. Figure 5.7 compares the portfolio value at maturity with terminal payoff of the option. From this figure, we can see adding liquid option as second risky asset largely reduces mean and the variance of PnL.

When using one risky asset, local risk minimization and delta hedging are very similar and they perform better than the mean variance hedging. When using liquid options as the second risky asset, we can see both the mean and standard deviation become much smaller, which shows that options are not redundant in the incomplete market. Among these three hedging methods using two risky assets, local risk minimization performs the best, which has the the smallest standard deviation.

In the following we will compare the performances of different hedging methods when real world parameters and risk neutral parameters for variance gamma do not satisfy the measure change constraint, i.e. $\hat{a} \neq a$. We will use risk neutral parameters $\hat{a} = 1.82$, $\hat{b}^+ = 37.12, \hat{b}^- = 4.9$ to compare the hedging performances. Table 5.3 gives the means and standard deviations of PnL for different hedging methods.

According to Table 5.3, if we only compare the standard deviation of the PnLs, the local risk minimization hedging with two risky assets performs the best among all the hedging methods. Figure 5.9 can confirm this. Now if we also consider both the mean
and standard deviation, we will find the difference between local risk minimization using one risky asset and delta hedging are not statistically significant and they perform better than mean-variance hedging. When using two assets, mean-variance hedging has the biggest positive mean and biggest variance with biggest Sharpe ratio (defined as the ratio
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<table>
<thead>
<tr>
<th>Hedging method</th>
<th>$T = 1$ mean</th>
<th>$T = 1$ std</th>
<th>$T = 3$ mean</th>
<th>$T = 3$ std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean variance with 1 asset</td>
<td>-0.55</td>
<td>4.05</td>
<td>-0.753</td>
<td>4.53</td>
</tr>
<tr>
<td>Local risk min with 1 asset</td>
<td>-0.648</td>
<td>1.75</td>
<td>-0.743</td>
<td>1.9</td>
</tr>
<tr>
<td>Delta hedging</td>
<td>-0.653</td>
<td>1.83</td>
<td>-0.742</td>
<td>1.94</td>
</tr>
<tr>
<td>Mean variance with 2 assets</td>
<td>4.43</td>
<td>3.68</td>
<td>9.34</td>
<td>5.1</td>
</tr>
<tr>
<td>Local risk min with 2 assets</td>
<td>0.614</td>
<td>1.18</td>
<td>1.38</td>
<td>1.27</td>
</tr>
<tr>
<td>Delta-Gamma hedging</td>
<td>2.5</td>
<td>2.67</td>
<td>4.41</td>
<td>2.42</td>
</tr>
</tbody>
</table>

Table 5.3: Means and standard deviations of PnL when risk neutral parameters do not satisfy constraints

Figure 5.8: Hedging positions when risk neutral parameters do not satisfy constraints

of rate of return and standard deviation). According to the personal preference, someone may think mean-variance hedging performs the best. From Figure 5.9, we can see choosing risk neutral parameters not satisfying the constraint reduces hedging performances of all hedging methods. From this numerical example, we can see performances of all hedging strategies are sensitive to the difference of Lévy densities between the real world measure and the risk neutral measure. Choosing parameters becomes a very important task.

5.5 The annual ratchet guarantee

In this section, we consider the pricing and hedging of a $n$-year guaranteed rate of return with annual ratchet feature. Under the ratchet contract design, the participation in the stock index is evaluated year by year. Each year the guaranteed payoff is stepped up by the greater of the guaranteed rate of return $R_{min}$ and the return of the sub-account.
Suppose the sub-account starts at $S_0$, then at time 1, the wealth of the policyholder is guaranteed to be the maximum of $S_0 e^{R_{\text{min}}}$ and $S_1$, denoted by $S^*_1$. The policyholder reinvests any extra amount to the sub-account so that the number of shares he holds for the next period is $\frac{S^*_1}{S_1}$. At time 2, the account value is the maximum of $S^*_1 e^{R_{\text{min}}}$ and $\frac{S^*_1}{S_1} S_2$, denoted by $S^*_2$, \ldots. This process of annual ratcheting continues to the end of year $n$, resulting in a total discounted cost for the insurer

$$G_{\text{ratchet}}(n) = e^{-r} \left[ S_0 e^{R_{\text{min}}} - S_1 \right] + e^{-2r} \left[ S^*_1 e^{R_{\text{min}}} - \frac{S^*_1}{S_1} S_2 \right] + \cdots + e^{-nr} \left[ S^*_{n-1} e^{R_{\text{min}}} - \frac{S^*_{n-1}}{S^*_{n-1}} S_n \right]$$

where

$$S^*_i = \max \left\{ S^*_{i-1} e^{R_{\text{min}}}, \frac{S^*_{i-1}}{S^*_{i-1}} S_i \right\} = S^*_{i-1} \max \left( e^{R_{\text{min}}}, \frac{S_i}{S^*_{i-1}} \right), \quad S^*_0 = S_0 \quad (5.17)$$

In this section, we are looking at the cost to the insurer, instead of the payoff to the policyholder. Indeed, the annual ratcheting at the end of each year brings an instant cost to the insurer, as it has to boost the sub-account with money from its own pocket. However, the increase in the sub-account is not cashed out by the policyholder until the
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maturity of the contract. So from the policyholder’s perspective, the annual increase is accumulated to the maturity, resulting in a payoff of a form similar to that in Hardy (2004). But for the insurer, the cost is the sum of each annual boosting, the present value of which is given by (5.16)

Next we will focus on the pricing of this type of contract. First we will rewrite $G_{\text{ratchet}}(n)$ as

$$G_{\text{ratchet}}(n) = e^{-r}S_0 \left[ e^{R_{\min}} - \frac{S_1}{S_0} \right]_+ + e^{-2r}S_1^* \left[ e^{R_{\min}} - \frac{S_2}{S_1} \right]_+ + \cdots + e^{-nr}S_{n-1}^* \left[ e^{R_{\min}} - \frac{S_n}{S_{n-1}} \right]_+ \tag{5.18}$$

For the $i$-th term on the RHS, we have

$$E^Q \left\{ e^{-ir}S_{i-1}^* \left[ e^{R_{\min}} - \frac{S_i}{S_{i-1}} \right]_+ \right\} = e^{-ir}E^Q(S_{i-1}^*)E^Q \left\{ \left( e^{R_{\min}} - \frac{S_i}{S_{i-1}} \right)_+ \right\}$$

where we use the fact that $S_{i-1}^*$ is independent of $\frac{S_i}{S_{i-1}}$.

According to the property of Lévy process, $\frac{S_i}{S_{i-1}}$ will have the same distribution as $S_1$ and with initial condition 1. The following expectation is just the put option with maturity 1 and strike $e^{R_{\min}}$ and its value can be obtained using the methodology from chapter 3.

$$E^Q \left\{ \left( e^{R_{\min}} - \frac{S_i}{S_{i-1}} \right)_+ \right\} = E^Q \left\{ \left( e^{R_{\min}} - S_1 \right)_+ \right\}$$

By definition of $S_i^*$, we have

$$E^Q(S_i^*) = E^Q(S_{i-1}^*)E^Q \left[ \max \left( e^{R_{\min}}, \frac{S_i}{S_{i-1}} \right) \right] = E^Q(S_{i-1}^*)E^Q \left[ \left( e^{R_{\min}} - \frac{S_i}{S_{i-1}} \right)_+ + \frac{S_i}{S_{i-1}} \right]$$

Suppose

$$A = E^Q \left[ \left( e^{R_{\min}} - \frac{S_i}{S_{i-1}} \right)_+ + \frac{S_i}{S_{i-1}} \right] = E^Q \left[ \left( e^{R_{\min}} - S_1 \right)_+ \right] + e^{r}$$

Note that $A$ is a constant. Thus,

$$E^Q(S_i^*) = S_0 A^i, \quad i \in \{1, 2, \cdots, n\}$$
In summary, the price of the \( n \)-year annual ratchet guarantee is

\[
h_{\text{ratchet}}(n) = E^Q [G_{\text{ratchet}}(n)] = S_0 E^Q \left[ (e^{R_{\min}} - S_1)_+ \right] \sum_{i=1}^n e^{-ir} A^{i-1}.
\]

The hedging strategy for the annual ratchet guarantee can be described as follows. Suppose we sell a \( n \)-year annual ratchet guarantee at time 0 with payoff structure given by \( G_{\text{ratchet}}(n) \). In the period \((0, 1]\), we hedge \( S_0^* = S_0 \) units of a put option with payoff \( \left[ e^{R_{\min}} - \frac{S_i}{S_0} \right]_+ \) at time 1; in period \((1, 2]\), we hedge \( S_1^* \) (note that, this strategy is feasible since \( S_1^* \) is known at time 1) units of a put option with payoff \( \left[ e^{R_{\min}} - \frac{S_2}{S_1} \right]_+ \) at time 2; ... In general, in period \((i-1, i] \) \( (i \in \{1, 2, \cdots, n\}) \), we hedge \( S_{i-1}^* \) units of a put option with payoff \( \left[ e^{R_{\min}} - \frac{S_i}{S_{i-1}} \right]_+ \) at time \( i \).

It is worthwhile to point out that in the strategy we just described, the option we hedge in period \((i-1, i] \) does not depend on \( i \), only the units of such an option differ from period to period. In fact, the option we hedge for each period is simply a vanilla put option with strike \( K = e^{R_{\min}} \), time to maturity \( T = 1 \) and initial sub-account value 1. Hence all the hedging methods we described in the previous sections can be applied to hedge the \( n \)-year annual ratchet guarantee.
Chapter 6

Pricing under Lévy models and stochastic interest rate

In this chapter we will extend the pricing methodology for the simple model to the case of mix funds and the stochastic interest rate. In order to avoid calculations of the joint distribution of the mix fund and the interest rate we will introduce the forward measure. In this chapter we first derive the PIDE for the price of the guarantee. Then we use two methods to solve the PIDE. One is to use Fourier transform method, the other is to use finite difference method. We will also compare their performance using Monte-Carlo simulations. After having the price of the guarantee, we focus on calculations of the fair management fee which plays an important role in the insurance company. Finally we examine sensitivities of the fair management fee with respect to different model parameters.

6.1 Pricing under Lévy models

If we recall from the Chapter 2, the mix fund and the interest rate have the following dynamics under \( Q \),

\[
\frac{dM_t}{M_{t-}} = \left[ \pi_t \left( \tilde{\gamma}^P + \frac{1}{2} \left( \sigma^P \right)^2 \right) + (1 - \pi_t) \left( \tilde{\gamma}^S + \frac{1}{2} \left( \sigma^S \right)^2 \right) \right] dt + \pi_t \sigma^P d\tilde{W}^P_t + (1 - \pi_t) \sigma^S d\tilde{W}^S_t
\]

\[
+ \pi_t \int_{-\infty}^{\infty} (e^y - 1) \tilde{j}^P (dt \times dy) + (1 - \pi_t) \int_{-\infty}^{\infty} (e^y - 1) \tilde{j}^S (dt \times dy)
\]

\[dr_t = \kappa(\theta - r_t)dt + \sigma^R d\tilde{W}^R_t\]
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Under this model, the price of the European put option can be written as

\[ H(t, Mt, rt, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r_s \, ds} \cdot G(M_T) | \mathcal{F}_t \right] + \mathbb{E}^Q \left[ -\alpha \int_t^T e^{-\int_t^s r_u \, du} \hat{M}_u \, du | \mathcal{F}_t \right] 
\]

\[ = \mathbb{E}^Q \left[ e^{-\int_t^T r_s \, ds} \cdot G(M_T) | \mathcal{F}_t \right] - M_t (e^{-\alpha t} - e^{-\alpha T}) 
\]

\[ = h(t, Mt, rt, T) - M_t (e^{-\alpha t} - e^{-\alpha T}). \]

Since both interest rate and mix fund are stochastic processes, we need their joint distribution to calculate the function \( h \). To avoid it, we will introduce the forward measure \( Q_T \) which is equivalent to \( Q \). The forward measure uses the bond price \( p(t, T) \) as a numeraire where the bond has the same maturity as the guarantee. Under the forward measure \( Q_T \), for any tradable asset \( M_t \), \( \frac{M_t}{p(t, T)} \) is a martingale. The measure change is defined as

\[ \left( \frac{dQ_T}{dQ} \right)_t = \frac{p(t, T)/p(0, T)}{e^{\int_0^t r_s \, ds}}. \]

If we recall from the Equation (2.10) in chapter 2, the bond price \( p(t, T) \) has the following dynamics

\[ \frac{d}{dt} p(t, T) = r_t dt - C_t(T) \sigma^R d\tilde{W}_t^R \]

where \( C_t(T) = \frac{1}{T} (1 - e^{-\kappa(T-t)}) \).

Hence the function \( h \) will become

\[ h(t, Mt, rt, T) = \mathbb{E}^Q \left[ e^{-\int_t^T r_s \, ds} \cdot G(M_T) | \mathcal{F}_t \right] 
\]

\[ = p(t, T) \mathbb{E}^T \left( \frac{G(M_T)}{p(T, T)} | \mathcal{F}_t \right) 
\]

\[ = p(t, T) \mathbb{E}^T \left[ \max \left( K - \frac{M_T e^{-\alpha T}}{p(T, T)}, 0 \right) | \mathcal{F}_t \right]. \]

If we define \( \tilde{M}_t = \frac{M_t}{p(t, T)} \), the process \( \tilde{M}_t \) is a martingale under the forward measure \( Q_T \).

We can conclude \( \tilde{M}_t \) has the following dynamics under \( Q_T \),

\[ \frac{d\tilde{M}_t}{\tilde{M}_t} = \left[ \pi_t \hat{\tau}^P + (1 - \pi_t) \hat{\tau}^S \right] dt + \pi_t \sigma_t^P d\tilde{W}_t^P + (1 - \pi_t) \sigma_t^S d\tilde{W}_t^S + C_t(T) \sigma_t^R d\tilde{W}_t^R 
\]

\[ + \pi_t \int_{\mathbb{R}} (e^{y} - 1) \hat{J}^P (dt \times dy) + (1 - \pi_t) \int_{\mathbb{R}} (e^{y} - 1) \hat{J}^S (dt \times dy) \]
where $\tilde{W}^P, \tilde{W}^S$ and $\tilde{W}^R$ are standard Brownian motions with correlations $\rho_{SP}, \rho_{RP}, \rho_{RS}$ under forward measure $\mathbb{Q}^T$. $\hat{\tau}^S$ and $\hat{\tau}^P$ are determined by the martingale condition.

\[
\hat{\tau}^S = \int_{-\infty}^{\infty} (1 - e^y) \hat{\nu}^S(dy), \quad \hat{\tau}^P = \int_{-\infty}^{\infty} (1 - e^y) \hat{\nu}^P(dy).
\]

The measure change from $\mathbb{Q}$ to $\mathbb{Q}^T$ only affects the Brownian motion and does not affect the jump process.

In order to simplify the expressions, we define

\[
\hat{\tau}^M = \pi_t \hat{\tau}^P + (1 - \pi_t) \hat{\tau}^S
\]

\[
(\sigma^M_t)^2 = (\pi_t \sigma^P_t)^2 + ((1 - \pi_t) \sigma^S_t)^2 + (C_t(T) \sigma^R_t)^2 + 2\pi_t (1 - \pi_t) \sigma^R_t \sigma^S_t \rho_{SP} + 2\pi_t C_t(T) \sigma^P_t \sigma^R_t \rho_{RP} + 2(1 - \pi_t) C_t(T) \sigma^S_t \sigma^R_t \rho_{RS}
\]

Then the dynamics of $\tilde{M}_t$ can be written as

\[
\frac{d\tilde{M}_t}{\tilde{M}_t} = \hat{\tau}^M dt + \sigma^M_t d\tilde{W}^M_t + \pi_t \int_{\mathbb{R}} (e^y - 1) \hat{J}^P(dy \times dy) + (1 - \pi_t) \int_{\mathbb{R}} (e^y - 1) \hat{J}^S(dy \times dy)
\]

where $\tilde{W}^M_t$ is a standard Brownian motion under $\mathbb{Q}^T$.

Define

\[
\mathbb{E}^T \left[ \max \left( K - \tilde{M}_T e^{-\alpha T}, 0 \right) \right] \mid \mathcal{F}_t \equiv g(t, x_t),
\]

where $x_t = \log \left( \frac{\tilde{M}_t}{K_{P(t,T)}} \right) = \log \left( \frac{M_t}{K_{P(t,T)}} \right)$. Then we will have the following theorem about the function $g$.

**Theorem 6.1.1.** The function $g(t, x)$ will satisfy the following PIDE.

\[
\partial_t g + \left( \hat{\tau}^M - \frac{1}{2} (\sigma^M_t)^2 \right) \partial_x g + \frac{1}{2} (\sigma^M_t)^2 \partial_{xx} g + \int_{\mathbb{R}} [g(t, x + \log(1 + \pi[e^y - 1])) - g(t, x)] \hat{\nu}^P(dy) + \int_{\mathbb{R}} [g(t, x + \log(1 + (1 - \pi)[e^y - 1])) - g(t, x)] \hat{\nu}^S(dy) = 0
\]

\[
g(T, x) = \max \left( K \left[ 1 - e^{x - \alpha T} \right], 0 \right)
\]

**Proof.** The function $g(t, x)$ is a Doob-martingale under $\mathbb{Q}^T$. Applying zero drift condition together with boundary condition, we can get the above PIDE. \qed
6.1.1 Using the Fourier transform method to solve PIDE

Just like the simple case in Chapter 3, we first consider using the Fourier transform method to solve the above PIDE based on many advantages of Fourier transform method over finite difference method.

Theorem 6.1.2. The solution to the above PIDE is given by

\[ g(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\omega} \mathcal{F}[g](T, \omega) \exp \left( \int_{t}^{T} \psi(s, \omega) ds \right) d\omega \]  

(6.2)

where

\[ \psi(t, \omega) = \left( \hat{\tau}^M - \frac{1}{2}(\sigma_t^M)^2 \right) i\omega - \frac{1}{2}(\sigma_t^M)^2 \omega^2 + \int_{\mathbb{R}} \left[ (1 + \pi(e^y - 1))^{i\omega} - 1 \right] \hat{\nu}^P(dy) \]

\[ + \int_{\mathbb{R}} \left[ (1 + (1 - \pi)(e^y - 1))^{i\omega} - 1 \right] \hat{\nu}^S(dy). \]

**Proof.** Applying Fourier transform to the above PIDE leads to the following ODE,

\[ \partial_t \mathcal{F}[g] + \left[ \left( \hat{\tau}^M - \frac{1}{2}(\sigma_t^M)^2 \right) i\omega - \frac{1}{2}\omega^2(\sigma_t^M)^2 \right] \mathcal{F}[g] + \int_{\mathbb{R}} \left[ (1 + \pi(e^y - 1))^{i\omega} - 1 \right] \hat{\nu}^P(dy) + \int_{\mathbb{R}} \left[ (1 + (1 - \pi)(e^y - 1))^{i\omega} - 1 \right] \hat{\nu}^S(dy) \mathcal{F}[g] = 0. \]

The ODE is just

\[ \partial_t \mathcal{F}[g] + \psi(t, \omega) \mathcal{F}[g] = 0 \]

The solution to the above ODE is

\[ \mathcal{F}[g](t, \omega) = \mathcal{F}[g](T, \omega) \exp \left( \int_{t}^{T} \psi(s, \omega) ds \right) \]  

(6.3)

Then \( g \) is the inverse Fourier transform of \( \mathcal{F}[g] \).

The price of the put option at time \( t \) is

\[ H(t, M_t, r_t, T) = h(t, M_t, r_t, T) - M_t \left( e^{-at} - e^{-aT} \right) = p(t, T)g(t, x) - M_t \left( e^{-at} - e^{-aT} \right). \]

Theorem 6.1.3. The derivatives of the function \( h \) can be obtained using inverse Fourier
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\[
\frac{\partial h}{\partial M} = \frac{p(t, T)}{M_t} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x_t} \cdot i\omega \mathcal{F}[g](T, \omega) \exp \left( \int_t^T \psi(s, \omega)ds \right) d\omega
\]

\[
\frac{\partial^2 h}{\partial M^2} = \frac{p(t, T)}{M_t^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x_t} \cdot (-\omega^2 - i\omega) \mathcal{F}[g](T, \omega) \exp \left( \int_t^T \psi(s, \omega)ds \right) d\omega
\]

\[
\frac{\partial h}{\partial r} = \frac{p(t, T)}{\kappa} \left(1 - e^{-\kappa(T-t)}\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x_t} \cdot (i\omega - 1) \mathcal{F}[g](T, \omega) \exp \left( \int_t^T \psi(s, \omega)ds \right) d\omega
\]

Proof. The results can be obtained by using the chain rule and properties of Fourier transform.

6.1.2 Using the finite difference method to solve PIDE

An assortment of finite difference method for solving these PIDEs has been proposed in the literature, see e.g. Andersen and Andreasen (2000), Cont and Tankov (2004), and d’Halluin, Forsyth, and Vetzal (2005). Although the methods are diverse, they all treat the integral and diffusion terms of the PIDE separately. Invariably, the integral term is evaluated explicitly in order to avoid solving a dense system of linear equations. Unfortunately, these methods require several approximations such as:

- in infinite activity processes, small jumps are approximated by a diffusion and incorporated into the diffusion term.

- the integral term must be localized to the bounded domain of the diffusion term, i.e. large jumps are truncated.

- the option price behavior outside the solution domain must be assumed.

- the separate treatment of diffusion and integral components requires that function values are interpolated and extrapolated between the diffusion and integral grids in order to compute the convolution term.

These factors together make finite difference methods for option pricing under Lévy models quite complex, and potentially prone to accuracy and stability problems. As a consequence, many methods are only applicable to a specific class of Lévy model. Moreover, for infinite activity Lévy processes, finite difference methods typically suffer from slow convergence.
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In this section we will consider solving PIDE using finite difference method to provide a comparison with the Fourier transform method. Given, a grid of spatial points \( \{x_{\text{min}} = x_1 < x_2 < \cdots < x_M = x_{\text{max}}\} \) and time points \( \{0 = t_1 < t_2 < \cdots < t_N = T\} \), where \( x_i = x_{\text{min}} + (i - 1)\Delta x \) and \( t_n = (n - 1)\Delta t \). Using \( g_i^n \) to denote \( g(t_n, x_i) \) and \( \sigma_{M,n+1}^+ \) to denote \( \sigma^+_{M,n+1} \). We will use Crank-Nicolson method for the space derivatives and forward finite Euler for the time derivative.

\[
\begin{align*}
\frac{g_i^{n+1} - g_i^n}{\Delta t} + \left( \frac{\Delta t}{2} \right) \left[ \frac{1}{2} \left( \frac{g_{i+1}^{n+1} - g_{i-1}^{n+1}}{2\Delta x} + \frac{g_{i+1}^{n+1} - g_{i-1}^{n+1}}{2\Delta x} \right) \right] \\
+ \frac{(\sigma_{M,n}^+)^2}{4(\Delta x)^2} \left[ (g_{i+1}^{n+1} - 2g_i^{n+1} + g_{i-1}^{n+1}) + (g_{i+1}^n - 2g_i^n + g_{i-1}^n) \right] \\
+ \int_{\mathbb{R}} \left[ g(t_{n+1}, x_i + \log[1 + \pi(e^y - 1)]) - g(t_{n+1}, x_i) \right] \hat{\nu}^P(dy) \\
+ \int_{\mathbb{R}} \left[ g(t_{n+1}, x_i + \log[1 + (1 - \pi)(e^y - 1)]) - g(t_{n+1}, x_i) \right] \hat{\nu}^S(dy) = 0
\end{align*}
\]

When calculating \( g(t_{n+1}, x_i + \log[1 + \pi(e^y - 1)]) \) and \( g(t_{n+1}, x_i + \log[1 + (1 - \pi)(e^y - 1)]) \), linear extrapolation is needed. The terminal condition is

\[
g_i^N = \max \left( K(1 - e^{-\sigma^+T}), 0 \right)
\]

we assume (as usual) that the second derivatives vanish along the minimum and maximum spatial directions for each \( n \), i.e.,

\[
\begin{align*}
g_0^n - 2g_1^n + g_2^n &= 0 \\
g_{M-1}^n - 2g_M^n + g_{M+1}^n &= 0
\end{align*}
\]

These are the boundary conditions.

If we use vectors to represent all the function values at time \( t_n \), i.e. \( Z^n = \{g_0^n, g_1^n, \ldots, g_M^n\}^* \), the above equations can be written as

\[
A^n Z^n + B^n = C^n Z^{n+1} \Rightarrow Z^n = (A^n)^{-1}(C^n Z^{n+1} - B^n) \tag{6.4}
\]

where \( A^n \) and \( C^n \) are tri-diagonal matrices, \( B^n \) is a known vector and * represents the transposition of the vector. Then we can start with the terminal condition and use Equation (6.4) to obtain all the function values at all time points and all spatial points.
6.1.3 Numerical results

In this section we will compare the numerical results among using Fourier transform method, finite difference method and Monte-Carlo simulations. We will use variance gamma processes for both stock index and bond index. Parameters used in Figure 6.1 are $\alpha = 0, \pi = 0.4, M_0 = 100, T = 3, R_{\min} = 0.04, r_0 = 0.04, \kappa = 0.2, \theta = 0.05, \sigma^R = 0.01, \sigma^S = 0, \sigma^P = 0, \rho_{SP} = 0, \rho_{RP} = 0, \rho_{RS} = 0$. The parameters for $\hat{\nu}^P$ are $\hat{a}^P = 25, \hat{b}^{P+} = 65, \hat{b}^{P-} = 61$. The parameters for $\hat{\nu}^S$ are $\hat{a} = 12, \hat{b}^+ = 15, \hat{b}^- = 14$. The number of simulations is $N_{\text{sim}} = 2000$.

![Comparisons among different numerical methods](image1)

(a) Comparing $g$ among Fourier transform, finite difference and Monte Carlo simulation

![Value of function $g$ at different times](image2)

(b) Comparing $g$ at different times

Figure 6.1: Comparing the performance of numerical methods

From Figure 6.1(a), we can see the results using Fourier transform method are within the 95% confidence intervals of Monte Carlo simulation results. But results using finite difference method are not always within the confidence interval due to many approximations used in the finite difference method. From this chapter on, we will only use Fourier transform method to solve the PIDE because of its accuracy and efficiency in the computations. Figure 6.1(b) compares the values of function $g$ at time 0, 3, 5, 7 using Fourier transform method. From this figure, we can see as the time to maturity increases, the time value of the option also increases. At maturity the time value will become zero. The change of the time value is more significant for the out-of-the-money range.

6.2 Sensitivities of fair Management fees

Now that we have the price of the guarantee, we can calculate another important value for the insurance company: the fair management fee.
Before we introduce the definition of the fair management fee, let us see the Figure 6.2 first. In this figure the value of the contract at time 0 is shown as a function of the management fee $\alpha$. As expected, the value of the embedded option decreases as the management fee increases. The fair management fee is defined such that the value of the contract at time 0 equals the initial investment amount $M_0$. Hence the fair management fee $\alpha_m$ should satisfy the following equation.

$$
M_0 = \mathbb{E}^Q \left( e^{-\int_0^T r_s ds} \max(K, \hat{M}_T) \right)
= \mathbb{E}^Q \left( e^{-\int_0^T r_s ds} \max(K - \hat{M}_T, 0) \right) + \mathbb{E}^Q \left( e^{-\int_0^T r_s ds} \hat{M}_T \right)
= h(0, M_0, r_0, T) + e^{-\alpha T} M_0
$$

where the function $h$ also depends on $\alpha$. This definition can be interpreted from the point of the policyholder. The LHS is his initial investment and RHS is the present value of the payoff at time $T$. The fair management fee renders the value of the cash-flows received by the investor equal to the initial investment. The above equation does not have explicit solution. We will use numerical methods to solve it.

In Figure 6.2, the contract parameters are $M_0 = 100, T = 5, R_{\min} = 0$. Interest rate parameters are $r_0 = 0.04, \kappa = 0.2, \theta = 0.05, \sigma_R = 0.005$. Percentage of the bond index is $\pi = 0.2$. Stock index parameters are $\sigma_S = 0$ and $\hat{a} = 100, \hat{b}^+ = 65, \hat{b}^- = 60$ for the variance gamma process. Bond index parameters are $\sigma_P = 0$ and $\hat{a}^P = 381, \hat{b}^{P+} = 400, \hat{b}^{P-} = 380$ for the variance gamma process. Since there are no diffusion terms in the
Chapter 6. Pricing under Lévy models and stochastic interest rate

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>302</td>
<td>193</td>
<td>104</td>
<td>42</td>
<td>14</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 6.1: Fair management fees (in basis points) for different percentage of bond index variance gamma processes, the correlation coefficients should be zero. For the contract in Figure 6.2, the fair management fee is 0.0193 or 193 bps.

6.2.1 Sensitivity to the percentage of bond index

It is instructive to investigate the sensitivities of fair management fee with respect to different model parameters. First we will consider the sensitivity of fair management fee with respect to the percentage of the bond index. Mix fund gives the shareholder the flexibility of choosing between stock index and bond index. We will vary the percentage of bond index $\pi$ and keep other parameters the same as in Figure 6.2.

From Table 6.1, we can see fair management fee decreases as the percentage in the bond index increases as we expect. The variance of variance gamma process for the bond index is $381(1/400^2 + 1/380^2) = 0.005$ which is less than that of stock index $100(1/65^2 + 1/60^2) = 0.0514$. This means the stock index is more risky than the bond index. As the percentage in the bond index decreases, the value of the underlying option increases, management fees must increase to compensate for this increased option value.

6.2.2 Sensitivity to jump parameters

Next we will discuss effects of jump parameters on the fair management fee. We will vary variance gamma parameters for the stock index and keep other parameters the same as Figure 6.2. Table 6.2 and Table 6.3 show the results of fair management fees for different jump parameters.

<table>
<thead>
<tr>
<th>$\hat{a}$</th>
<th>140</th>
<th>120</th>
<th>100</th>
<th>80</th>
<th>60</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>272</td>
<td>234</td>
<td>193</td>
<td>152</td>
<td>108</td>
<td>64</td>
</tr>
</tbody>
</table>

Table 6.2: Fair management fees (in basis points) for different $\hat{a}$

From Table 6.2, increasing $\hat{a}$ will raise the fair management fees as a result of increased variance. From Table 6.3, increasing $\hat{b}^+$ or $\hat{b}^-$ will lower the fair management fees as a result of decreased variance. We also notice that the matrix in the Table 6.3 is almost symmetric, i.e. when we switch $\hat{b}^+$ and $\hat{b}^-$, the fair management fees are almost equal.
This means $\hat{b}^+$ and $\hat{b}^-$ have symmetric effects on the fair management fees. We conclude only the jump size will affect the fair management fees and jump direction does not matter.
Chapter 7

Local risk minimization hedging

In a complete market, the terminal payoff of a European option can be replicated by a self-financing portfolio consisting of the underlying asset and money market account. Thus, one can hedge the risk of a short position in the option perfectly by buying the replicating portfolio. However, modeling jumps with stochastic jump size leads to incompleteness of the market model. Thus a self-financing replicating portfolio no longer exists: perfectly hedging the option is no longer possible. In this chapter, we particularly focus on the local risk minimization hedging strategy. Föllmer and Sondermann (1986) pioneered risk minimization hedging in a special case where the discounted actual stock price followed a martingale. At each point in time they required that the risk, defined as the expected quadratic hedging error, was minimized. However, in semimartingale models, a risk minimization hedging strategy does not always exist. Therefore Schweizer (1991) introduced a locally risk minimizing strategy and showed that - under certain assumptions - a strategy was a local risk minimization hedging if the cost process was a martingale which was orthogonal to the martingale part of the stock price process. Colwell and Elliott (1993) presented a general formula for the minimal equivalent martingale measure and the local risk minimization strategy in jump diffusion models. However, their assumptions only allowed for bounded jumps.

In this chapter we will derive a local risk minimization hedging strategy in the case of mix funds and the stochastic interest rate. The hedging strategy is chosen by minimizing the \( \mathbb{P} \)-variance of “one step ahead hedging error” over instantaneous time period. Our hedge ratios can be applied to any Lévy density. In the first section we consider hedging using one risky asset: the underlying. In the second section we consider hedging using two risky assets: the underlying and liquid options. At the end of this chapter, we compare the performance of different hedging strategies using numerical examples.
7.1 Hedging with the underlying

At first, we will consider hedging using the underlying only. Under real world measure $\mathbb{P}$, the bank account $B_t$, mix fund index $M_t$ and the short rate $r_t$ have the following dynamics

$$
\frac{dB_t}{B_t} = r_t \, dt \\
\frac{dM_t}{M_t} = \tau^A_t \, dt + (1 - \pi_t)\sigma^S dW^S_t + \pi_t \sigma^P dW^P_t + \pi_t \int_{\mathbb{R}} (e^y - 1) J^P(\,dt \times dy) \\
+ (1 - \pi_t) \int_{\mathbb{R}} (e^y - 1) J^S(\,dt \times dy) \\
= \tau^A_t \, dt + \sigma^A_t dW^A_t + \pi_t \int_{\mathbb{R}} (e^y - 1) J^P(\,dt \times dy) + (1 - \pi_t) \int_{\mathbb{R}} (e^y - 1) J^S(\,dt \times dy) \\
dr_t = \kappa (\hat{\theta} - r_t) + \sigma^R dW^R_t
$$

where $W^A_t$ is a standard Brownian motion under $\mathbb{P}$ and

$$
\tau^A_t = \pi_t \left( \gamma^P + \frac{1}{2} (\sigma^P)^2 \right) + (1 - \pi_t) \left( \gamma^S + \frac{1}{2} (\sigma^S)^2 \right) \\
(\sigma^A_t)^2 = \left[ (1 - \pi_t) \sigma^S \right]^2 + \left[ \pi_t \sigma^P \right]^2 + 2 \pi_t (1 - \pi_t) \sigma^S \sigma^P \rho_{SP}
$$

**Theorem 7.1.1.** The local risk minimization hedging position of the underlying asset to hedge the option with maturity $T$ is $\hat{\xi}_t(T) = \hat{\xi}(T) - e^{-\alpha t} + e^{-\alpha T}$, where

$$
\hat{\xi}(T) = \frac{\partial_M h \left( (\sigma^A)^2 \right) + \frac{\sigma^P}{M_t} \partial_r h + \frac{\pi_t}{M_t} \cdot \int_{\mathbb{R}} (e^y - 1) \Delta_P h \nu^P(dy) + \frac{\pi_t}{M_t} \cdot \int_{\mathbb{R}} (e^y - 1) \Delta_S h \nu^S(dy)}{(\sigma^A)^2 + \pi_t^2 \int_{\mathbb{R}} (e^y - 1)^2 \nu^P(dy) + (1 - \pi_t)^2 \int_{\mathbb{R}} (e^y - 1)^2 \nu^S(dy)}
$$

(7.1)

where

$$
\rho = \pi_t \sigma^P \rho_{RP} + (1 - \pi_t) \sigma^S \rho_{RS} \\
\Delta_P h = h(t, M_t[1 + \pi_t(e^y - 1)], r_t, T) - h(t, M_t, r_t, T) \\
\Delta_S h = h(t, M_t[1 + (1 - \pi_t)(e^y - 1)], r_t, T) - h(t, M_t, r_t, T).
$$

**Proof.** The portfolio we have is $V_t = \xi_t M_t + \beta_t B_t - H_t = \xi_t M_t + \beta_t B_t - h_t + (e^{-\alpha t} - e^{-\alpha T}) M_t$. According to the self-financing condition, the change in $V_t$ comes from the change in the underlying asset and the money market account and receiving management fees from...
Let \( \varepsilon_t = \xi_t + e^{-\alpha t} - e^{-\alpha T} \). Applying Ito’s lemma under \( \mathbb{P} \),

\[
dV_t = \xi_t dM_t + \beta_t dB_t - dh_t + (e^{-\alpha t} - e^{-\alpha T})dM_t + \alpha dt \hat{M}_t
\]

The coefficient of \( \varepsilon \) is obtained.
The money market account is determined according to the self-financing condition. The following theorem will give the hedging positions on each hedging date.

**Theorem 7.1.2.** Suppose we have a series of hedging dates \(0 = T_0 < T_1 < \cdots < T_N = T\) with equal time step \(\Delta t = T_{j+1} - T_j\). The short rate and mix fund at time \(T_n\) are denoted as \(r_n\) and \(M_n\). The hedging positions on hedging date \(T_n\) are denoted by \(\{\xi_n, \beta_n\}\), where

\[
\xi_n = \xi_{T_n}(T), \quad n \geq 0 \\
\beta_n = \beta_{n-1}e^{r_{n-1}\Delta t} - (\xi_n - \xi_{n-1})M_n + \alpha \hat{M}_{n-1}\Delta t, \quad n \geq 1 \\
\beta_0 = H_0 - \xi_0M_0.
\]

**Proof.** At \(T_0 = 0\), we have \(V_0 = 0\). Then we will hold \(\xi_0\) units of \(M_0\), and the money in the bank account is \(\beta_0 = H_0 - \xi_0M_0\).

At \(t = T_1\), value of hedging portfolio is \(\xi_0M_1 + \beta_0e^{r_0\Delta t} + \alpha \hat{M}_0\Delta t\). Now units of mix fund have changed to \(\xi_1\). \(\beta_1\) needs to satisfy the following self-financing condition.

\[
\xi_1M_1 + \beta_1 = \xi_0M_1 + \beta_0e^{r_0\Delta t} + \alpha \hat{M}_0\Delta t
\]

Hence \(\beta_1 = \beta_0e^{r_0\Delta t} - (\xi_1 - \xi_0)M_1 + \alpha \hat{M}_0\Delta t\).

Repeat, \(\beta_n = \beta_{n-1}e^{r_{n-1}\Delta t} - (\xi_n - \xi_{n-1})M_n + \alpha \hat{M}_{n-1}\Delta t\).

At maturity \(t = T_N\), we do not need to rebalance. (If transaction cost is considered, there will be difference between rebalancing or not rebalancing at maturity). At maturity, we have \(\xi_{N-1}\) units of underlying and \(\beta_{N-1}e^{r_{N-1}\Delta t}\) in the bank account and receive management fee from policy holders for the period of \([T_{N-1}, T_N]\). The value of our hedging portfolio is \(\xi_{N-1}M_N + \beta_{N-1}e^{r_{N-1}\Delta t} + \alpha \hat{M}_{N-1}\Delta t\) and we owe the policyholders \(G(M_N) = \max(K - \hat{M}_N, 0)\). Define the book value at time \(T\) as

\[
PnL(T) = \xi_{N-1}M_N + \beta_{N-1}e^{r_{N-1}\Delta t} + \alpha \hat{M}_{N-1}\Delta t - G(M_N).
\]

This book value is also called the Profit and Loss (PnL). We will focus on the mean and standard deviation of PnL at maturity and they will serve as our criterion of comparing hedging strategies. Later in this chapter, we will use numerical examples to compare the performances of different hedging strategies.
7.2 Hedging with both the underlying and options

To improve the hedging performance, we will introduce another risky asset: liquid option with shorter maturity. We will use $f(t, M_t, r_t, T_s)$ to represent its value at time $t$ with maturity $T_s$. The option used to hedge is just a financial option, no management fee is involved, i.e. the terminal payoff is just $\max(K - M_{T_s}, 0)$. $T_s$ is usually less than the maturity $T$ of the option we want to hedge. The following theorem gives hedging positions when using two risky assets.

**Theorem 7.2.1.** The local risk minimization hedging positions $\left(\hat{\zeta}_t^{(1)}, \hat{\zeta}_t^{(2)}\right)$ to hedge the option with maturity $T$ are

$$\hat{\zeta}_t^{(1)}(T) = -\frac{2BC - DE}{4AB - E^2} - e^{-\alpha t} + e^{-\alpha T}$$

$$\hat{\zeta}_t^{(2)}(T) = -\frac{2AD - CE}{4AB - E^2}$$

where

$$A = M_t^2 \left[ (\sigma_t^A)^2 + \pi_t^2 \int_{\mathbb{R}} (e^y - 1)^2 \nu^P(dy) + (1 - \pi_t)^2 \int_{\mathbb{R}} (e^y - 1)^2 \nu^S(dy) \right]$$

$$B = M_t^2 \left[ (\sigma_t^A \partial_M f)^2 + \left(\frac{\sigma_t^R}{M_t}\right)^2 + 2 \frac{\sigma_t^R \partial_M f \partial_r f}{M_t} + \int_{\mathbb{R}} \left( \frac{\Delta_p f}{M_t} \partial_r f \right)^2 \nu^P(dy) + \int_{\mathbb{R}} \left( \frac{\Delta_s f}{M_t} \right)^2 \nu^S(dy) \right]$$

$$C = -2M_t^2 \left[ (\sigma_t^A)^2 \partial_M h + \pi_t \int_{\mathbb{R}} (e^y - 1) \frac{\Delta_p h}{M_t} \nu^P(dy) + (1 - \pi_t) \int_{\mathbb{R}} (e^y - 1) \frac{\Delta_s h}{M_t} \nu^S(dy) + \sigma_t^R \partial_r h \right]$$

$$D = -2M_t^2 \left[ (\sigma_t^A)^2 \partial_M f \partial_M h + \left(\frac{\sigma_t^R}{M_t}\right)^2 \partial_r f \partial_r h + \sigma_t^R \rho (\partial_M f \partial_r h + \partial_M h \partial_r f) \right] - 2 \int_{\mathbb{R}} \Delta_p f \Delta_s h \nu^P(dy)$$

$$E = 2M_t^2 \left[ (\sigma_t^A)^2 \partial_M f + \frac{\sigma_t^R \rho}{M_t} \partial_r f + \pi_t \int_{\mathbb{R}} (e^y - 1) \frac{\Delta_p f}{M_t} \nu^P(dy) + (1 - \pi_t) \int_{\mathbb{R}} (e^y - 1) \frac{\Delta_s f}{M_t} \nu^S(dy) \right]$$

and

$$\Delta_p h = h(t, M_{t-}[1 + \pi_t(e^y - 1)], r_t, T) - h(t, M_{t-}, r_t, T)$$

$$\Delta_s h = h(t, M_{t-}[1 + (1 - \pi_t)(e^y - 1)], r_t, T) - h(t, M_{t-}, r_t, T)$$

$$\Delta_p f = f(t, M_{t-}[1 + \pi_t(e^y - 1)], r_t, T) - f(t, M_{t-}, r_t, T)$$

$$\Delta_s f = f(t, M_{t-}[1 + (1 - \pi_t)(e^y - 1)], r_t, T) - f(t, M_{t-}, r_t, T)$$
Chapter 7. Local risk minimization hedging

Theorem 7.2.2. Suppose we have a series of hedging dates \( 0 = T_0 < T_1 < \cdots < T_N = T \) with equal time step \( \Delta t = T_{j+1} - T_j \). The short rate and mix fund at time \( T_n \) are denoted...
Chapter 7. Local risk minimization hedging

as \( r_n \) and \( M_n \). The hedging positions on date \( T_n \) are \( \{\xi_n^{(1)}, \xi_n^{(2)}, \beta_n\} \), where

\[
\begin{align*}
\xi_n^{(1)} &= \xi_{T_n}^{(1)}(T), \quad n \geq 0 \\
\xi_n^{(2)} &= \xi_{T_n}^{(2)}(T), \quad n \geq 0 \\
\beta_n &= M_n \left( \xi_{n-1}^{(1)} - \xi_n^{(1)} \right) + \xi_{n-1}^{(2)} f(\Delta t, M_n, r_n, T_s, K_{n-1}) - \xi_n^{(2)} f(0, M_n, r_n, T_s, K_n) \\
&\quad + \beta_{n-1} e^{r_{n-1}\Delta t} + \alpha \hat{M}_{n-1}\Delta t, \quad n \geq 1 \\
\beta_0 &= H_0 - \xi_0^{(1)} M_0 - \xi_0^{(2)} f(0, M_0, r_0, T_s, K_0)
\end{align*}
\]

Proof. At \( T_0 = 0 \), we have \( V_0 = 0 \). Then we will hold \( \xi_0^{(1)} \) units of \( M_0 \), \( \xi_0^{(2)} \) units of at-the-money options \( f(0, M_0, r_0, T_s, K_0) \) maturing at \( T_s \) with current mix fund value \( M_0 \), interest rate \( r_0 \) and strike \( K_0 \). Hence the money in the bank account is given by \( \beta_0 = H_0 - \xi_0^{(1)} M_0 - \xi_0^{(2)} f(0, M_0, r_0, T_s, K_0) \).

At \( t = T_1 \), value of hedging portfolio is \( \xi_0^{(1)} M_1 + \xi_0^{(2)} f(\Delta t, M_1, r_1, T_s, K_0) + \beta_0 e^{r_0\Delta t} + \alpha \hat{M}_0\Delta t \). We will liquidate all the positions in the options and buy new at-the-money options with maturity \( T_s \). Now units of underlying and options have updated as \( \xi_1^{(1)} \) and \( \xi_1^{(2)} \) and \( \beta_1 \) needs to satisfy the following self-financing condition.

\[
\xi_1^{(1)} M_1 + \xi_1^{(2)} f(0, M_1, r_1, T_s, K_1)+ \beta_1 = \xi_0^{(1)} M_1 + \xi_0^{(2)} f(\Delta t, M_1, r_1, T_s, K_0) + \beta_0 e^{r_0\Delta t} + \alpha \hat{M}_0\Delta t
\]

Hence

\[
\beta_1 = M_1 \left( \xi_0^{(1)} - \xi_1^{(1)} \right) + \xi_0^{(2)} f(\Delta t, M_1, r_1, T_s, K_0) - \xi_1^{(2)} f(0, M_1, r_1, T_s, K_1) + \beta_0 e^{r_0\Delta t} + \alpha \hat{M}_0\Delta t.
\]

Similarly,

\[
\beta_n = M_n \left( \xi_{n-1}^{(1)} - \xi_n^{(1)} \right) + \xi_{n-1}^{(2)} f(\Delta t, M_n, r_n, T_s, K_{n-1}) - \xi_n^{(2)} f(0, M_n, r_n, T_s, K_n) + \beta_{n-1} e^{r_{n-1}\Delta t} + \alpha \hat{M}_{n-1}\Delta t.
\]

Similar to hedging using one risky asset, we will define the book value or PnL at maturity

\[
PnL(T) = \xi_{N-1}^{(1)} M_N + \xi_{N-1}^{(2)} f(\Delta t, M_N, r_N, T_N, K_{N-1}) + \beta_{N-1} e^{r_{N-1}\Delta t} + \alpha \hat{M}_{N-1}\Delta t - G(M_N).
\]

(7.3)
7.3 Numerical examples

In this section, we will use numerical examples to compare performances of different hedging strategies. First we will generate sample paths of mix fund and interest rate. Then for each simulation, we calculate the PnLs of different hedging methods. After we have PnLs for all the sample paths, we can calculate their means and standard deviations.

In the numerical examples we use the following parameter set. The hedging frequency is daily, i.e. $\Delta t = 1/252$. The option used as hedging instrument has a shorter maturity $T_s = 1/12$ and is liquidated on each hedging date, i.e., we liquidate all the positions of options with shorter maturity and buy new at-the-money options with maturity $T_s$. The minimal rate of return $R_{\text{min}}$ is supposed to be zero. The parameters for interest rate are $r_0 = 3\%$, $\kappa = 0.2$, $\theta = 3.5\%$, $\sigma^R = 0.005$. The percentage of bond index is $\pi = 0.3$ and initial mix fund value is $M_0 = 100$. We suppose the stock index and bond index follow variance gamma process. The parameters for the variance gamma process are obtained from calibrations in chapter 4. The parameters for the stock index are $a^S = 185.23$, $b^{S+} = 104.83$, $b^{S-} = 102.62$, $\tilde{a}^S = 185.23$, $\tilde{b}^{S+} = 2430.2$, $\tilde{b}^{S-} = 83.5$. The parameters for the bond index are $a^P = 382$, $b^{P+} = 484$, $b^{P-} = 462$, $\tilde{a}^P = 382$, $\tilde{b}^{P+} = 3500$, $\tilde{b}^{P-} = 420$. Since there are no diffusion terms in the stock index and bond index, we have $\sigma^S = 0$ and $\sigma^P = 0$. The number of simulations is $N_{\text{sim}} = 2000$. For simplicity, we suppose there is no management fee, i.e. $\alpha = 0$. The real returns of stock index and bond index are supposed to be $\mu^S = 0.15$, $\mu^P = 0.04$. Table 7.1 gives means and standard deviations of PnL of different hedging methods and different maturities.

<table>
<thead>
<tr>
<th>Method</th>
<th>Mix fund</th>
<th>Stock index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 1$</td>
<td>$T = 3$</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
</tr>
<tr>
<td>LRM with 1 asset</td>
<td>-0.54</td>
<td>0.548</td>
</tr>
<tr>
<td>Delta</td>
<td>-0.537</td>
<td>0.556</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>-0.0346</td>
<td>0.149</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>-0.0559</td>
<td>0.171</td>
</tr>
</tbody>
</table>

Table 7.1: Means and standard deviations of PnL in the case of $\sigma^R = 0.005$

The difference between local risk-minimization using underlying asset only and delta hedging is not statistically significant when using two-sample Kolmogorov-Smirnov test at significance level 5%. The difference between local risk-minimization using two risky assets and delta-gamma hedging is significantly different. These can be seen from Figure 7.1. Then we compare the differences of PnLs between local risk minimization and delta
Chapter 7. Local risk minimization hedging

Figure 7.1: Results of Kolmogorov-Smirnov test in the case of mix funds and $T = 3$

Figure 7.2: Comparing differences of PnL in the case of mix funds and $T = 3$

hedging or delta-gamma hedging in Figure 7.2. After performing t-test with

$$H_0 : \text{mean} = 0$$

$$H_a : \text{mean} > 0$$

we conclude local risk minimization using one asset has the same mean as delta hedging while local risk minimization using two assets has bigger mean than delta-gamma hedging.

The reason for the above results can be explained as follows. When using one risky asset, the positions of underlying asset for local risk minimization is very close to those of delta hedging which can be seen from Figure 7.3(a) and 7.3(b). The difference is of order $10^{-4}$. When using two risky assets, difference of positions for underlying asset between
local risk minimization and delta-gamma hedging is of order $10^{-3}$. This can be seen from Figure 7.4(a) and 7.4(b). Besides the difference of positions for underlying asset, the difference of positions for liquid options also leads to significant difference between local risk minimization using two risky assets and delta-gamma hedging. This can be seen from Figure 7.4(c) and 7.4(d). Hence we can conclude that when using one risky asset, the advantage of local risk minimization over delta-hedging is not obvious. Considering the simplicity of delta hedging, we suggest using delta hedging if we only use one risky asset. If we use two risky assets, the advantage of local risk minimization over delta-gamma hedging is very obvious, i.e., it has bigger mean and smaller standard deviation. It is worthwhile to use local risk minimization.

Figure 7.5 plots the sample paths of mix fund and interest rate. Sample path of mix fund shows jump behaviors and sample path of interest rate shows mean-reverting property. The histograms of PnL for the mix fund case can be found in Figure 7.6. When using one risky asset, histograms are left-skewed. When using two risky assets, histograms are almost symmetric. Comparing the case of mix fund and stock index, we conclude adding bond index to the portfolio can reduce the standard deviation of PnLs since the bond index is less volatile than the stock index. The policyholders can choose different bond index percentage according to their risk preferences. Furthermore, as the maturity increases, the difference between local risk minimization and delta-gamma hedging becomes smaller which is due to the property of Lévy density. Figure 7.7 compares terminal payoffs with hedging portfolio. When using one risky asset, the hedging
(a) Comparing positions of the underlying between LRM 2 and Delta-Gamma hedging

(b) Comparing difference between the LRM 2 and Delta-Gamma hedging

(c) Comparing positions of options between LRM 2 and Delta-Gamma hedging

(d) Comparing difference between LRM 2 and Delta-Gamma hedging

Figure 7.4: Hedging positions for hedging methods using two assets in the case of mix funds and $T = 3$

(a) Sample path of mix fund $M_t$

(b) Sample path of interest rate $r_t$

Figure 7.5: Sample paths of mix fund and interest rate
errors mostly happen when the underlying asset price is near the strike. Adding liquid option as second hedging instrument can greatly reduce the hedging errors near the strike.

In order to evaluate the effect of interest rate risk on the hedging performance, we will use another interest rate parameters with big $\sigma^R$. The new interest rate parameters are $\kappa = 0.2, \theta = 0.08, \sigma^R = 0.02, r_0 = 0.07$. The means and standard deviations are in table 7.2. Comparing with table 7.1, the standard deviations increases as the volatility of interest rate increases. This is because the bigger volatility leads to bigger interest rate risk. Since we do not use any bond as hedging instrument, the interest rate risk can not be hedged. The VaR and CVaR are in table 7.3. As we expect, the values of VaR and CVaR for local risk minimization with 1 asset are very close to those for delta hedging. The VaR and CVaR for local risk minimization with 2 assets are smaller than those of delta-gamma hedging. We conclude PnL of local risk minimization using two assets has thinner left tail compared to delta-gamma hedging.
Figure 7.7: Terminal payoffs in the case of mix funds and $T = 3$

<table>
<thead>
<tr>
<th>Method</th>
<th>Mix fund $T = 1$</th>
<th>Mix fund $T = 3$</th>
<th>Stock index $T = 1$</th>
<th>Stock index $T = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>Mean</td>
<td>Std</td>
</tr>
<tr>
<td>LRM with 1 asset</td>
<td>-0.755</td>
<td>0.8542</td>
<td>-0.4384</td>
<td>1.0405</td>
</tr>
<tr>
<td>Delta</td>
<td>-0.71</td>
<td>0.8678</td>
<td>-0.4358</td>
<td>1.0427</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>-0.0893</td>
<td>0.4242</td>
<td>0.0137</td>
<td>0.8576</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>-0.1186</td>
<td>0.4402</td>
<td>-0.0066</td>
<td>0.8627</td>
</tr>
</tbody>
</table>

Table 7.2: Means and standard deviations of PnL in the case of $\sigma^R = 0.02$
### Table 7.3: VaR and CVaR for different hedging methods in the case of mix funds, $T = 3$ and $\sigma^R = 0.02$

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>$VaR_\alpha$</th>
<th></th>
<th>$CVaR_\alpha$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>70%</td>
<td>90%</td>
<td>95%</td>
<td>70%</td>
</tr>
<tr>
<td>Local risk min with 1 asset</td>
<td>0.6684</td>
<td>1.6752</td>
<td>2.3848</td>
<td>1.6164</td>
</tr>
<tr>
<td>Delta hedging</td>
<td>0.6639</td>
<td>1.6733</td>
<td>2.3904</td>
<td>1.6163</td>
</tr>
<tr>
<td>Local risk min with 2 assets</td>
<td>0.2548</td>
<td>0.9059</td>
<td>1.4056</td>
<td>0.8912</td>
</tr>
<tr>
<td>Delta-Gamma hedging</td>
<td>0.2742</td>
<td>0.9389</td>
<td>1.4431</td>
<td>0.9196</td>
</tr>
</tbody>
</table>
Chapter 8

Pricing and hedging of unit-linked life insurance

Mortality is a major risk factor for life insurance companies and pension funds. For calculating premium payments of life insurance contracts and deriving the market reserves for general life insurance liabilities mortality risk is a crucial input parameter.

In this chapter we consider pricing and hedging of three common types of unit-linked life insurance contracts when the interest rate is constant. As in Møller (1998, 2001a), we assume stochastic independence between the financial market and the insurance model and consider them combined in a common product space. This picks up the idea to model the uncertain development of the stock index and the insured lives simultaneously not averaging away mortality.

We assume that the lifetimes in a group of individuals are mutually independent and identically distributed. The i.i.d. assumption implies that the individuals are selected from a cohort of equal age $x$ and we denote by $l_x$ the number of persons in the group. Mathematically, this is described by representing the individual remaining lifetimes as a sequence $T_1, \cdots, T_{l_x}$ of i.i.d. non-negative random variables. Assuming that the distribution of $T_i$ is absolutely continuous with hazard rate function $\mu_{x,t}^H$, the survival function is

$$p_x = P(T_i > t) = \exp \left( - \int_0^t \mu_{x+s}^H ds \right)$$

Now define a univariate process $N = (N_t)_{0 \leq t \leq T}$ counting the number of deaths in the
group,
\[ N_t = \sum_{i=1}^{l_x} 1(T_i \leq t) \]
and denote by \( H = (H_t)_{0 \leq t \leq T} \) the natural filtration generated by \( N \), i.e. \( H_t = \sigma\{N_u, u \leq t\} \). By definition \( N \) is càdlàg (right-continuous with left limits) and, since the lifetimes \( T_i \) are i.i.d., the counting process \( N \) is a \( H \)-Markov process. The stochastic intensity process \( \lambda \) of the counting process \( N \) can be informally defined by
\[
E[dN_t | H_t] = (l_x - N_t - 1)\mu_{x+t}dt = \lambda_t dt,
\]
the hazard rate function \( \mu_{x+t} \) times the number of individuals under exposure just before time \( t \). The compensated counting process \( \Phi \) defined by
\[
\Phi_t = N_t - \int_0^t \lambda_udu
\]
is a \( H \)-martingale.

### 8.1 Combined model

Now introduce the filtration \( (G_t)_{0 \leq t \leq T} \) generated by the economy and the insurance portfolio, that is
\[
G_t = F_t \vee H_t.
\]
We assume throughout that \( F_T \) and \( H_T \) are independent and take
\[
G_T = F_T \vee \sigma\{I(T_i \leq u), 0 \leq u \leq T, i = 1, \ldots, l_x\}.
\]

Next we will discuss the choice of martingale measure in the combined model. For any \( H \)-predictable process \( l \), such that \( l > -1 \), define a likelihood process \( L \) by
\[
dL_t = L_t d\hat{N}_t
\]
and the initial condition \( L_0 = 1 \). Provided that \( E[L_T] = 1 \), a new probability measure \( \mathbb{Q}^* \) can be defined by
\[
\frac{d\mathbb{Q}^*}{d\mathbb{P}^*} = \mathcal{E}(U)_T \cdot L_T,
\]
where \( \mathcal{E}(U)_T \) is given by (2.17). Using the definition of the measure \( \mathbb{Q}^* \) and the independence between \( N \) and \( S \) under \( \mathbb{P}^* \) we see that \( e^{-rt}S_t \) is also a \( \mathbb{Q}^* \) martingale: for \( u < t \)
we have

$$
\mathbb{E}^{Q^*}[e^{-rt}S_t|\mathcal{G}_u] = \mathbb{E}^{P^*}[e^{-rt}S_t\mathcal{E}(U)_T \cdot L_T|\mathcal{G}_u] = \mathbb{E}^{P^*}[e^{-rt}S_t\mathcal{E}(U)_T|\mathcal{G}_u]\mathbb{E}^{P^*}[L_T|\mathcal{G}_u] = \mathbb{E}^{Q}[e^{-ru}S_u]
$$

using that $e^{-rt}S_t$ is a $Q$-martingale. So each $Q^*$ is an equivalent martingale measure. Due to this non-uniqueness of the equivalent martingale measure, contracts can not in general be priced uniquely by no-arbitrage.

At time zero, the insurance company issues an insurance contract for each of the $l_x$ individuals. These contracts specify payments of benefits and premiums that are contingent on the remaining lifetime of the policyholder, and are linked to the development of the financial market.

In the following sections, we will consider pricing and hedging unit-linked life insurance contracts. Since in this section we mainly focus on the effect of mortality risk on the hedging strategies, we will use the model in which the reference portfolio is stock index and interest rate is constant. We also suppose there is no management fees. As briefly mentioned earlier, unit-linked life insurance contracts link the insurance benefit and possibly the premiums to the market value of some specified reference portfolio. They have various structures. Some only provide death benefit, some only maturity benefit. Next we will deal with three common types of unit-linked payoffs: pure endowment, term insurance and endowment insurance.

### 8.2 Unit-linked pure endowment

With a “pure unit-linked” insurance contract the amount to be paid is given by the value of $S$ at expiration, that is, $S_T$. In this case all financial risk transfers to the policyholders. Opposed to this, a “unit-linked with guarantee” contract is equipped with a guarantee that assures the policyholder a minimum amount if the reference portfolio falls below a certain level at due date. For a prefixed guarantee $K > 0$ the payoff has the form

$$
\max(S_T, K) = S_T + G(S_T), \text{ where } G(S_T) = \max(K - S_T, 0).
$$

Recall that with a pure endowment the sum insured is to be paid at the end of the
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If the insured person is still alive. Therefore for each insured person the present value of obligation of the insurance company is given by

\[ 1_{(T_i > T)} e^{-rT} G(S_T), \]

which is \( G_T \)-measurable.

The entire portfolio generates the claim

\[ G^N(S_T, N_T) = e^{-rT} G(S_T) \sum_{i=1}^{l_x} 1_{(T_i > T)} = e^{-rT} G(S_T)(l_x - N_T). \]

where \((l_x - N_T)\) is the number of survivors at the end of the insurance period.

The value of the contract at time \( t \) is

\[ H^N(t, S_t, N_t, T) = E^{Q^*}[G^N(S_T, N_T)|G_t] \]

\[ = E^{Q^*}[l_x - N_T|G_t] e^{-rt} E^{Q^*}[e^{-r(T-t)} G(S_T)|G_t] \]

The second equality is due to the stochastic independence between \( N \) and \( S \) under \( Q^* \). The first expectation is easily determined as

\[ E^{Q^*}[l_x - N_T|G_t] = E^{Q^*}\left[\sum_{i=1}^{l_x} 1_{\{T_i > T\}}|G_t\right] = \sum_{i:T_i > t} E^{Q^*}[1_{\{T_i > T\}}|T_i > t] \]

\[ = \sum_{i:T_i > t} T - tp_{x+t} = (l_x - N_t)T - tp_{x+t} \]

that is, at any time \( t \) the expected number of individuals alive at the time of maturity \( T \) is simply the number of survivors at time \( t \) multiplied by the probability \( T - tp_{x+t} \) of survival to \( T \) for an individual, conditional on his/her survival to \( t \). In the present model, the insured lives are included in the filtration \( \mathcal{H} \). However, as \( N \) and \( S \) are stochastically independent, the conditional distribution of \( S \) given \( \mathcal{F}_t \) does not depend on information concerning the insured lives \( \mathcal{H}_t \) and thus

\[ E^{Q^*}[e^{-r(T-t)} G(S_T)|G_t] = E^{Q}[e^{-r(T-t)} G(S_T)|\mathcal{F}_t] = h(t, S_t, T) \]

where \( h(t, S_t, T) \) is the function defined in chapter 3.
Hence $H^N(t, S_t, N_t, T)$ will be

$$H^N(t, S_t, N_t, T) = (l_x - N_t)_{T-t} p_{x+t} e^{-rt} h(t, S_t, T).$$

Next we will consider hedging unit-linked pure endowment. The following theorem gives the expression of the hedging position for the unit-linked pure endowment and its relationship with local risk-minimization hedging positions without mortality risk.

**Theorem 8.2.1.** The local risk minimization hedging position $\tilde{\xi}_t(T)$ of the underlying to hedge the unit-linked pure endowment with maturity $T$ is

$$\tilde{\xi}_t(T) = (l_x - N_{t-})_{T-t} p_{x+t} e^{-rt} \hat{\xi}_t(T)$$  \hspace{1cm} (8.2)

where $\hat{\xi}_t(T)$ is the hedging position of underlying asset in chapter 5 when the reference portfolio is stock index and interest rate is constant.

**Proof.** Applying Itô’s lemma under real world measure $\mathbb{P}^*$,

$$dV_t = \xi_t dS_t + \beta_t dB_t - dH^N_t$$

$$= \xi_t dS_t + \beta_t B_t dt - (l_x - N_{t-})_{T-t} p_{x+t} e^{-rt} dh_t - e^{-rt} h_t (l_x - N_{t-})_{T-t} p_{x+t} \mu_{x+t} dt$$
$$+ e^{-rt} h_t (l_x - N_{t-})_{T-t} p_{x+t} dN_t$$

The variance will be

$$\text{Var}^*(dV_t)$$

$$= \text{Var}^*[\xi_t dS_t(e^y - 1) + (l_x - N_{t-}) T-t p_{x+t} e^{-rt} \Delta h_S)_{S_t} dW^S_t] + (h_t e^{-rt} T-t p_{x+t})^2 \text{Var}^*(dN_t)$$

$$+ \text{Var}^*\left( \int_{\mathbb{R}} [\xi_t S_t(e^y - 1) + (l_x - N_{t-}) T-t p_{x+t} e^{-rt} \Delta h_S)_{S_t} dW^S_t] J^S(dt \times dy) \right)$$

$$= \left[ (\xi_t - (l_x - N_{t-}) T-t p_{x+t} e^{-rt} \Delta h_S)_{S_t} dW^S_t \right]^2 dt + (h_t e^{-rt} T-t p_{x+t})^2 (l_x - N_{t-}) \mu_{x+t} dt$$

$$+ \int_{\mathbb{R}} [\xi_t S_t(e^y - 1) + (l_x - N_{t-}) T-t p_{x+t} e^{-rt} \Delta h_S)_{S_t} dW^S_t] J^S(dt \times dy)$$

$$- \left( \int_{\mathbb{R}} [\xi_t S_t(e^y - 1) + (l_x - N_{t-}) T-t p_{x+t} e^{-rt} \Delta h_S)_{S_t} dW^S_t] \nu^S(dy) dt \right)^2$$
The hedging position should not depend on \( dt \). We will let \( dt \) tend to 0.

\[
\lim_{dt \to 0} \frac{Var^*(dV_t)}{dt} = \left[ \xi_t - (l_x - N_{t-}) T_{-t} p_{x+t} e^{-rt} \partial_S h \right] S_t \sigma^S_t \]

\[
+ \int_{\mathbb{R}} \left[ \xi_t S_t (e^y - 1) - (l_x - N_{t-}) T_{-t} p_{x+t} e^{-rt} \Delta h \right]^2 \nu^S(dy) + (h_t e^{-rt} T_{-t} p_{x+t})^2 (l_x - N_{t-}) \mu^H_{x+t}
\]

The optimal \( \tilde{\xi}_t(T) \) that minimizes the variance is given by (8.2).

**Theorem 8.2.2.** The local risk minimization hedging positions \( \{ \tilde{\xi}_t^{(1)}(T), \tilde{\xi}_t^{(2)}(T) \} \) of underlying assets and liquid options to hedge the unit-linked pure endowment with maturity \( T \) are

\[
\tilde{\xi}_t^{(1)}(T) = (l_x - N_{t-}) T_{-t} p_{x+t} e^{-rt} \tilde{\xi}_t^{(1)}(T)
\]

\[
\tilde{\xi}_t^{(2)}(T) = (l_x - N_{t-}) T_{-t} p_{x+t} e^{-rt} \tilde{\xi}_t^{(2)}(T)
\]

where \( \tilde{\xi}_t^{(1,2)}(T) \) are the hedging positions defined in chapter 5.

**Proof.** Applying Itô’s lemma under \( \mathbb{P}^* \),

\[
dV_t = \xi_t^{(1)} dS_t + \beta_t B_t rd t + \xi_t^{(2)} df_t - (l_x - N_{t-}) e^{-rt} T_{-t} p_{x+t} dh_t - h_t (l_x - N_{t-}) e^{-rt} T_{-t} p_{x+t} \mu_{x+t} dt \\
+ h_t e^{-rt} T_{-t} p_{x+t} dN_t
\]

The variance is

\[
Var^*(dV_t) = Var^* \left[ (\xi_t^{(1)} + \xi_t^{(2)} \partial_S f - (l_x - N_{t-}) T_{-t} p_{x+t} \partial_S h) S_t \sigma^S_t dW_t^S \right] + (h_t e^{-rt} T_{-t} p_{x+t})^2 Var^*(dN_t)
\]

\[
+ \int_{\mathbb{R}} \left[ (\xi_t^{(1)} + \xi_t^{(2)} \partial_S f - (l_x - N_{t-}) e^{-rt} T_{-t} p_{x+t} \partial_S h) S_t \sigma^S_t \right]^2 dt + (h_t e^{-rt} T_{-t} p_{x+t})^2 (l_x - N_{t-}) \mu^H_{x+t} dt
\]

\[
+ \int_{\mathbb{R}} \left[ (\xi_t^{(1)} + \xi_t^{(2)} \partial_S f - (l_x - N_{t-}) e^{-rt} T_{-t} p_{x+t} \partial_S h) S_t \sigma^S_t \right]^2 \nu^S(dy) dt
\]

\[
- \left( \int_{\mathbb{R}} [\xi_t^{(1)} S_t (e^y - 1) + \xi_t^{(2)} \Delta f - (l_x - N_{t-}) e^{-rt} T_{-t} p_{x+t} \Delta h] \nu^S(dy) dt \right)^2
\]
The hedging position should not depend on \( dt \). We will let \( dt \) go to 0.

\[
\lim_{dt \to 0} \frac{\text{Var}^*(dV_t)}{dt} = (\xi^{(1)}_t + \xi^{(2)}_t \partial_S f - (l_x - N_{t-})e^{-rt} T-t p_{x+t} \partial_S h)^2 S_t^2 (\sigma^S)^2 + (h_t e^{-rt} T-t p_{x+t})^2 (l_x - N_{t-}) \mu^H_{x+t}
\]

\[
+ \int_{\mathbb{R}} \left[ \xi^{(1)}_t S_t (e^y - 1) + \xi^{(2)}_t \Delta f - (l_x - N_{t-})e^{-rt} T-t p_{x+t} \Delta h \right]^2 \nu^S(dy)
\]

The above expression is a bivariate quadratic function and the optimal \( \{\tilde{\xi}^{(1)}_t, \tilde{\xi}^{(2)}_t\} \) are given by the theorem.

### 8.3 Unit-linked term insurance

A term insurance is in some sense the opposite of a pure endowment. Its benefit is due immediately upon death before time \( T \). Hence, payments can occur at any time during \([0, T]\). For simplicity, we assume that death benefits are paid at the end of year of death.

The present value of the payoff \( G^L \) can be defined as

\[
G^L = \sum_{i=1}^{l_x} \sum_{j=0}^{T-1} e^{-r(j+1)} G(S_{j+1})1_{\{j< T_i \leq j+1\}}
\]

For \( t = 0, 1, \cdots, T \), the value of the contract at time \( t \) is

\[
H^L(t, S_t, N_t, T) = E^{Q^*}(G^L|\mathcal{F}_t)
\]

\[
= \sum_{i=1}^{l_x} \sum_{j=0}^{t-1} e^{-r(j+1)} G(S_{j+1})1_{\{j< T_i \leq j+1\}} + (l_x - N_t)e^{-rt} \sum_{j=t}^{T-1} h(t, S_t, j + 1) j-t p_{x+t} q_{x+j}
\]

where \( h(t, S_t, T) \) is defined in chapter 3 and the last parameter in the function represents the maturity.

For the special case \( t = 0 \), the above expression can be simplified as

\[
H^L(0, S_0, N_0, T) = l_x \sum_{j=0}^{T-1} h(0, S_0, j + 1) j p_x q_{x+j}
\]
If the time \( t \) is between two payment dates, i.e., there is an integer \( k \) such that \( k < t < k + 1 \), \( H_t^L \) is obtained by

\[
H_t^L(t, S_t, N_t, T) = \sum_{i=1}^{l_x} \sum_{j=0}^{k-1} e^{-r(j+1)} G(S_{j+1}) 1_{\{j \leq T_i \leq j+1\}} + \sum_{i=1}^{l_x} e^{-rt} h(t, S_t, k + 1) 1_{\{k < T_i \leq t\}} + (l_x - N_t) e^{-rt} \left[ h(t, S_t, k + 1) k_{-t+1} q_{x+t} + \sum_{j=k+1}^{T-1} h(t, S_t, j + 1) j_{-t} p_{x+t} q_{x+j} \right]
\]

**Theorem 8.3.1.** The local risk minimization hedging position \( \tilde{\xi}_t(T) \) of the underlying to hedge the unit-linked term insurance is as follows.

\[
\tilde{\xi}_t(T) = (l_x - N_t-) e^{-rt} \sum_{j=t}^{T-1} j_{-t} p_{x+t} q_{x+j} \hat{\xi}_t(j + 1), \quad t = 0, 1, \ldots, T
\]

\[
\tilde{\xi}_t(T) = e^{-rt} \hat{\xi}_t(k + 1) \left( \sum_{i=1}^{l_x} 1_{\{k < T_i \leq t\}} + (l_x - N_t-) k_{-t+1} q_{x+t} \right) + (l_x - N_t-) e^{-rt} \sum_{j=k+1}^{T-1} j_{-t} p_{x+t} q_{x+j} \hat{\xi}_t(j + 1), \quad k < t < k + 1.
\]

where \( \hat{\xi}_t(T) \) is the hedging position of underlying asset in chapter 5 when the reference portfolio is stock index and interest rate is constant.

**Proof.** The proof is similar to the case of pure endowment. In the term insurance, we are hedging multiple options with different maturities. The portfolio we have is \( V_t = \xi_t S_t + \beta_t B_t - H_t^L \). When \( k < t < k + 1 \), we apply Itô’s Lemma under real world measure \( \mathbb{P}^* \),

\[
dV_t = \xi_t dS_t + \beta_t dB_t - dH_t^L
\]

\[
= \xi_t dS_t + \beta_t dB_t + (\cdot) dt - \left( \sum_{i=1}^{l_x} 1_{\{k < T_i \leq t\}} + (l_x - N_t) k_{-t+1} q_{x+t} \right) e^{-rt} dh^{k+1}
\]

\[
- (l_x - N_t) e^{-rt} \sum_{j=k+1}^{T-1} j_{-t} p_{x+t} q_{x+j} dh^{j+1} + e^{-rt} \left[ h^{k+1} k_{-t+1} q_{x+t} + \sum_{j=k+1}^{T-1} h^{j+1} j_{-t} p_{x+t} q_{x+j} \right] dN_t
\]

\[
= \xi_t dS_t + \beta_t r B_t dt + (\cdot) dt - \sum_{j=k}^{T-1} w_j dh^{j+1} + e^{-rt} \left[ h^{k+1} k_{-t+1} q_{x+t} + \sum_{j=k+1}^{T-1} h^{j+1} j_{-t} p_{x+t} q_{x+j} \right] dN_t
\]
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The variance will be

\[
Var^*(dV_t) = Var^* \left[ \left( \xi_t - \sum_{j=k}^{T-1} w_j \partial_S h^{j+1} \right) S_t \sigma^S dW^S_t \right] \\
+ Var^* \left( \int_{\mathbb{R}} \left[ \xi_t S_t (e^y - 1) - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right] J^S(dt \times dy) \right) \\
+ \left[ e^{-rt} \left( h^{k+1}_{k-t+1} q_{x+t} + \sum_{j=k+1}^{T-1} h^{j+1}_{j-t} p_{x+t} q_{x+j} \right) \right]^2 Var^*(N_t) \\
= \left( \xi_t - \sum_{j=k}^{T-1} w_j \partial_S h^{j+1} \right)^2 S_t^2 (\sigma^S)^2 dt + \int_{\mathbb{R}} \left[ \xi_t S_t (e^y - 1) - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right] ^2 \nu^S(dy) dt \\
- \left( \int_{\mathbb{R}} \xi_t S_t (e^y - 1) - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right) ^2 \nu^S(dy) dt \\
+ \left[ e^{-rt} \left( h^{k+1}_{k-t+1} q_{x+t} + \sum_{j=k+1}^{T-1} h^{j+1}_{j-t} p_{x+t} q_{x+j} \right) \right]^2 (l_x - N_t) \mu^H_x dt
\]

The hedging position should not depend on \( dt \). We will let \( dt \) go to 0. The optimal \( \xi \) to minimize \( \lim_{dt \to 0} \frac{Var(dV_t)}{dt} \) is

\[
\tilde{\xi}_t(T) = \frac{\sum_{j=k}^{T-1} w_j \left( (\sigma^S)^2 \partial_S h^{j+1} + \frac{1}{S_t} \int_{\mathbb{R}} (e^y - 1) \Delta h^{j+1} \nu^S(dy) \right)}{(\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu^S(dy)} = \sum_{j=k}^{T-1} w_j \hat{\xi}_t(j + 1)
\]

The proof for the case of \( t = 0, 1, \cdots T \) can be obtained similarly.

Theorem 8.3.2. The local risk minimization hedging positions \( \{\tilde{\xi}_t^{(1)}, \tilde{\xi}_t^{(2)}\} \) of the underlying and liquid option to hedge the unit-linked term insurance are as follows.
For $t = 0, 1, \cdots, T$,

\[
\tilde{\xi}_t^{(1)}(T) = (l_x - N_{t-})e^{-rt} \sum_{j=t}^{T-1} \hat{\xi}_t^{(1)}(j + 1)_{j-t}q_{x+t}q_{x+j}
\]

\[
\tilde{\xi}_t^{(2)}(T) = (l_x - N_{t-})e^{-rt} \sum_{j=t}^{T-1} \hat{\xi}_t^{(2)}(j + 1)_{j-t}q_{x+t}q_{x+j}
\]

Similarly for $k < t < k + 1$

\[
\tilde{\xi}_t^{(1)}(T) = e^{-rt} \xi_t^{(1)}(k + 1) \left( \sum_{i=1}^{l_x} 1(k < T_i \leq t) + (l_x - N_{t-})_{k-t+1}q_{x+t} \right)
\]

\[
+ (l_x - N_{t-})e^{-rt} \sum_{j=k+1}^{T-1} \hat{\xi}_t^{(1)}(j + 1)_{j-t}q_{x+t}q_{x+j}
\]

\[
\tilde{\xi}_t^{(2)}(T) = e^{-rt} \xi_t^{(2)}(k + 1) \left( \sum_{i=1}^{l_x} 1(k < T_i \leq t) + (l_x - N_{t-})_{k-t+1}q_{x+t+1} \right)
\]

\[
+ (l_x - N_{t-})e^{-rt} \sum_{j=k+1}^{T-1} \hat{\xi}_t^{(2)}(j + 1)_{j-t}q_{x+t}q_{x+j}
\]

where $\xi_t^{(1),(2)}(T)$ are the hedging positions of underlying asset and liquid options in chapter 5 when the reference portfolio is stock index and interest rate is constant.

**Proof.** The portfolio we have is $V_t = \xi_t^{(1)} S_t + \xi_t^{(2)} f_t + \beta_t B_t - H_t^L$.

When $k < t < k + 1$, we apply Itô’s Lemma under real world measure $\mathbb{P}^*$,

\[
dV_t = \xi_t^{(1)} dS_t + \xi_t^{(2)} df_t + \beta_t dB_t - dH_t^L
\]

\[
= \xi_t^{(1)} dS_t + \xi_t^{(2)} df_t + \beta_t dB_t + (\cdot)dt - \left( \sum_{i=1}^{l_x} 1(k < T_i \leq t) + (l_x - N_{t-})_{k-t+1}q_{x+t} \right) e^{-rt} dh_t^{k+1}
\]

\[
- (l_x - N_{t-})e^{-rt} \sum_{j=k+1}^{T-1} \eta_{j-t}q_{x+t}q_{x+j} dh_t^{j+1} + e^{-rt} \left[ h_t^{k+1}_{k-t+1}q_{x+t} + \sum_{j=k+1}^{T-1} h_t^{j+1}_{j-t}q_{x+t}q_{x+j} \right] dN_t
\]

\[
= \xi_t^{(1)} dS_t + \xi_t^{(2)} df_t + \beta_t dB_t dt + (\cdot)dt - \sum_{j=k}^{T-1} w_j dh_t^{j+1}
\]

\[
+ e^{-rt} \left[ h_t^{k+1}_{k-t+1}q_{x+t} + \sum_{j=k+1}^{T-1} h_t^{j+1}_{j-t}q_{x+t}q_{x+j} \right] dN_t
\]
The variance will be
\[ Var(dV_t) = Var \left( \left( \xi_t^{(1)} + \xi_t^{(2)} \frac{\partial S}{\partial Sf} - \sum_{j=k}^{T-1} w_j \frac{\partial S_h}{\partial S} \right) S_t \sigma^S dW_t^S \right) \]
\[ + Var \left( \int_{\mathbb{R}} \left[ \xi_t^{(1)} S_t (e^y - 1) + \xi_t^{(2)} \Delta f - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right] J^S (dt \times dy) \right) \]
\[ + Var^* \left( \int_{\mathbb{R}} \left[ \xi_t S_t (e^y - 1) - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right] J^S (dt \times dy) \right) \]
\[ = \left( \xi_t^{(1)} + \xi_t^{(2)} \frac{\partial S}{\partial Sf} - \sum_{j=k}^{T-1} w_j \frac{\partial S_h}{\partial S} \right)^2 S_t^2 (\sigma^S)^2 dt \]
\[ + \int_{\mathbb{R}} \left[ \xi_t^{(1)} S_t (e^y - 1) + \xi_t^{(2)} \Delta f - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right]^2 \nu^S (dy) dt \]
\[ - \left( \int_{\mathbb{R}} \left[ \xi_t^{(1)} S_t (e^y - 1) + \xi_t^{(2)} \Delta f - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right] \nu^S (dy) \right)^2 \]
\[ + \left[ e^{-rt} \left( h^{k+1} + \sum_{j=k+1}^{T-1} h^{j+1} - \sum_{j=k+1}^{T-1} q_{x+j} \right) \right] (l_x - N_t) \mu_{x+t}^H dt \]

The hedging positions should not depend on \( dt \). We will let \( dt \) go to 0.

\[ \lim_{dt \to 0} \frac{Var(dV_t)}{dt} \]
\[ = \left( \xi_t^{(1)} + \xi_t^{(2)} \frac{\partial S}{\partial Sf} - \sum_{j=k}^{T-1} w_j \frac{\partial S_h}{\partial S} \right)^2 S_t^2 (\sigma^S)^2 \]
\[ + \int_{\mathbb{R}} \left[ \xi_t^{(1)} S_t (e^y - 1) + \xi_t^{(2)} \Delta f - \sum_{j=k}^{T-1} w_j \Delta h^{j+1} \right]^2 \nu^S (dy) \]
\[ + \left[ e^{-rt} \left( h^{k+1} + \sum_{j=k+1}^{T-1} h^{j+1} - \sum_{j=k+1}^{T-1} q_{x+j} \right) \right] (l_x - N_t) \mu_{x+t}^H \]

we will choose \( \xi_t^{(1)} \) and \( \xi_t^{(2)} \) such that the \( \mathbb{P} \)-variance of \( \lim_{dt \to 0} \frac{Var(dV_t)}{dt} \) is minimized.

The coefficient of \( \left( \xi_t^{(1)} \right)^2 \) is

\[ A = S_t^2 \left( (\sigma^S)^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu^S (dy) \right) \]
The coefficient of $\left(\xi_t^{(2)}\right)^2$ is

$$B = S_t^2 \left( \partial_S (\sigma^S)^2 + \int_R (\Delta f / S_t)^2 \nu^S(dy) \right).$$

The coefficient of $\xi_t^{(1)}$ is

$$C = \sum_{j=k}^{T-1} w_i \left[-2S_t^2 \left( \partial_S h^{j+1} (\sigma^S)^2 + \int_R (e^y - 1) \Delta h^{j+1} / S_t \nu^S(dy) \right)\right].$$

The coefficient of $\xi_t^{(2)}$ is

$$D = \sum_{j=k}^{T-1} w_i \left[-2S_t^2 \left( \partial_S \partial_S h^{j+1} (\sigma^S)^2 + \int_R \Delta f \Delta h^{j+1} / S_t^2 \nu^S(dy) \right)\right].$$

The coefficient of $\xi_t^{(1)} \xi_t^{(2)}$ is

$$E = 2S_t^2 \left[ \partial_S f (\sigma^S)^2 + \int_R (e^y - 1) \Delta f \nu^S(dy) \right].$$

If $4AB - E^2 > 0$, the function has a minimum if $A > 0$. The minimum is obtained at

$$\tilde{\xi}_t^{(1)} = -\frac{2BC - DE}{4AB - E^2} = \sum_{j=k}^{T-1} w_j \xi_t^{(1)} (j + 1)$$

and

$$\tilde{\xi}_t^{(2)} = -\frac{2AD - CE}{4AB - E^2} = \sum_{j=k}^{T-1} w_j \xi_t^{(2)} (j + 1).$$

The proof for the case of $t = 0, 1, \cdots T$ can be obtained similarly. \hfill \Box

### 8.4 Unit-linked endowment insurance

A common unit-linked life insurance contract in practice is the endowment insurance, which is a combination of a pure endowment and a term insurance policy. Suppose an endowment insurance with $T$ years maturity is sold to a policyholder aged exactly $x$. We assume further that $T$ is an integer, and if death occurs before the maturity, then the death benefit is payable at the end of the year of death; otherwise, a contingent claim is payable at $T$. In other word, the payoff of endowment insurance in the presence of
mortality risk can be represented as follows:

\[ G(S_T), \quad \text{if } T(x) > T, \]
\[ G(S_k), \quad \text{if } k - 1 < T(x) \leq k, \text{for } k = 1, 2, \ldots, T. \]

The present value of the payoff can be defined as

\[ G^l = \sum_{i=1}^{l_x} \sum_{k=0}^{T-1} e^{-r(k+1)} G(S_{k+1}) 1_{\{k<T_i \leq k+1\}} + \sum_{i=1}^{l_x} e^{-rT} G(S_T) 1_{\{T_i > T\}} \]

Hence the price of the endowment insurance is the sum of the prices of pure endowment and term insurance. For \( t = 0, 1, \cdots, T \), the value of the contract at time \( t \) is

\[ H^l(t, S_t, N_t, T) = E^Q(G^l | F_t) \]
\[ = \sum_{i=1}^{l_x} \sum_{j=0}^{t-1} e^{-r(j+1)} G(S_{j+1}) 1_{\{j<T_i \leq j+1\}} + (l_x - N_t) e^{-rt} \sum_{j=t}^{T-1} h(t, S_t, j + 1) p_{x+t} q_{x+j} + (l_x - N_t) e^{-rt} h(t, S_t, T) p_{x+t} \]

where \( h(t, S_t, T) \) is defined in chapter 3.

For the special case \( t = 0 \), the above expression can be simplified as

\[ H^l(0, S_0, N_0, T) = l_x \sum_{j=0}^{T-1} h(0, S_0, j + 1) p_{x+j} q_{x+j} + l_x h(0, S_0, T) p_{x} \]

If the time \( t \) is between two payment dates, i.e., there is an integer \( k \) such that \( k < t < k + 1 \), \( H^l \) is obtained by

\[ H^l(t, S_t, N_t, T) = \sum_{i=1}^{l_x} \sum_{j=0}^{k-1} e^{-r(j+1)} G(S_{j+1}) 1_{\{j<T_i \leq j+1\}} + \sum_{i=1}^{l_x} e^{-rt} h(t, S_t, k + 1) 1_{\{k<T_i \leq t\}} \]
\[ + (l_x - N_t) e^{-rt} \left[ h(t, S_t, k + 1) p_{x+t} q_{x+j} + \sum_{j=k+1}^{T-1} h(t, S_t, j + 1) p_{x+t} q_{x+j} \right] \]
\[ + (l_x - N_t) e^{-rt} h(t, S_t, T) p_{x+t} \]
Since each individual is independent, hedging positions should be the sum of hedging positions for pure endowment and term insurance. The expressions of hedging positions will be as follows.

For $t = 0, 1, \cdots, T$,

\[
\tilde{\xi}_t^{(1)}(T) = (l_x - N_{t-})e^{-rt} \left( \tilde{\xi}_t^{(1)}(T) T - t p_{x+t} + \sum_{j=t}^{T-1} \tilde{\xi}_t^{(1)}(j+1) j - t p_{x+t} q_{x+j} \right)
\]

\[
\tilde{\xi}_t^{(2)}(T) = (l_x - N_{t-})e^{-rt} \left( \tilde{\xi}_t^{(2)}(T) T - t p_{x+t} + \sum_{j=t}^{T-1} \tilde{\xi}_t^{(2)}(j+1) j - t p_{x+t} q_{x+j} \right)
\]

Similarly for $k < t < k + 1$

\[
\tilde{\xi}_t^{(1)}(T) = e^{-rt} \tilde{\xi}_t^{(1)}(k+1) \left( \sum_{i=1}^{l_x} 1_{\{T_x \leq t\}} + (l_x - N_{t-})k - t + 1 q_{x+t} \right)
\]

\[
+ (l_x - N_{t-})e^{-rt} \left( \sum_{j=k+1}^{T-1} \tilde{\xi}_t^{(1)}(j+1) j - t p_{x+t} q_{x+j} + \tilde{\xi}_t^{(1)}(T) T - t p_{x+t} \right)
\]

\[
\tilde{\xi}_t^{(2)}(T) = e^{-rt} \tilde{\xi}_t^{(2)}(k+1) \left( \sum_{i=1}^{l_x} 1_{\{T_x \leq t\}} + (l_x - N_{t-})k - t + 1 q_{x+t} \right)
\]

\[
+ (l_x - N_{t-})e^{-rt} \left( \sum_{j=k+1}^{T-1} \tilde{\xi}_t^{(2)}(j+1) j - t p_{x+t} q_{x+j} + \tilde{\xi}_t^{(2)}(T) T - t p_{x+t} \right)
\]

8.5 Numerical examples

In this section we will use numerical examples to compare performances of different hedging methods when hedging the above three types of unit-linked life insurance.

We will use mortality table downloaded from mortality.org. Using mortality table, $n p_x = \frac{i_x + n}{l_x}, \ n q_x = \frac{l_x - i_x + n}{l_x}$. We also suppose the $n p_x$ and $n q_x$ are linear between integer ages, i.e. $[n, n+1]$, for $n = 0, 1, \cdots, 109$. To simulate $N_t = \sum_{i=1}^{l_x} 1_{\{T_x \leq t\}}$, we first generate the times until death $T_x$ for all individuals at age $x$ and count the number of people dying before $t$ to obtain $N_t$. The c.d.f. of $T_x$ is

\[P(T_x \leq t) = t q_x = F(t)\]
We will generate a uniform random variable $u$ and one sample value of $T_x$ is solved by $F^{-1}(u)$.

The parameters used in this section are $S_0 = 100, R_{min} = 0, T = 3, r = 0.01, \alpha = 0$. The real world and risk neutral parameters for the variance gamma processes are $a = 185.23, b^+ = 104.83, b^- = 102.62$ and $\hat{a} = 185.23, \hat{b}^+ = 2430.2, \hat{b}^- = 83.5$. The volatility for the stock index is $\sigma^S = 0$. The real return of stock index is $\mu^S = 0.15$. The number of simulations is $N_{sim} = 10000$. The hedging frequency is $dt = 1/63$. The option used as hedging instrument has a shorter maturity $T_s = 1/12$ and is liquidated on each hedging date, i.e., we will liquidate all the positions of options with shorter maturity and buy new at-the-money options with maturity $T_s$. Since we have $l_x$ people, we will divide the total PnL by $l_x$ in order to obtain the PnL for a single person. Table 8.1, table 8.3 and table 8.5 report means and standard deviations of PnL for a single person for different numbers of people when we hedge the unit-linked pure endowment, the term insurance and the endowment insurance.

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>$l_x = 10$ Mean</th>
<th>$l_x = 100$ Mean</th>
<th>$l_x = 1000$ Mean</th>
<th>$l_x = 10$ Std</th>
<th>$l_x = 100$ Std</th>
<th>$l_x = 1000$ Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM with 1 asset</td>
<td>-1.2525</td>
<td>-1.2461</td>
<td>-1.2474</td>
<td>1.1331</td>
<td>1.0377</td>
<td>1.0302</td>
</tr>
<tr>
<td>Delta</td>
<td>-1.2422</td>
<td>-1.2358</td>
<td>-1.2371</td>
<td>1.14</td>
<td>1.0452</td>
<td>1.0375</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>-0.0257</td>
<td>-0.0204</td>
<td>-0.0225</td>
<td>0.464</td>
<td>0.2230</td>
<td>0.1841</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>-0.0951</td>
<td>-0.0897</td>
<td>-0.0914</td>
<td>0.4756</td>
<td>0.2413</td>
<td>0.2043</td>
</tr>
</tbody>
</table>

Table 8.1: Means and standard deviations of PnL for the pure endowment of a single person

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>VaR $\alpha$ 70%</th>
<th>VaR $\alpha$ 90%</th>
<th>VaR $\alpha$ 95%</th>
<th>CVaR $\alpha$ 70%</th>
<th>CVaR $\alpha$ 90%</th>
<th>CVaR $\alpha$ 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM with 1 asset</td>
<td>1.5841</td>
<td>2.5765</td>
<td>3.2132</td>
<td>2.4842</td>
<td>3.4952</td>
<td>4.1274</td>
</tr>
<tr>
<td>Delta</td>
<td>1.5353</td>
<td>2.5769</td>
<td>3.2236</td>
<td>2.4817</td>
<td>3.5059</td>
<td>4.1473</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>0.0555</td>
<td>0.1976</td>
<td>0.3106</td>
<td>0.2017</td>
<td>0.3857</td>
<td>0.5244</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>0.1195</td>
<td>0.2896</td>
<td>0.4125</td>
<td>0.2928</td>
<td>0.5085</td>
<td>0.6728</td>
</tr>
</tbody>
</table>

Table 8.2: VaR and CVaR for the pure endowment in the case of $l_x = 1000$

In the table 8.1, the difference between LRM with 1 asset and the delta hedging is not statistically significant while the difference between LRM with 2 assets and the delta-gamma hedging is statistically significant after we perform two-sample K-S test at 5%
### Chapter 8. Pricing and hedging of unit-linked life insurance

#### Age = 60

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM with 1 asset</td>
<td>-0.0945</td>
<td>0.5671</td>
<td>-0.1001</td>
<td>0.2238</td>
<td>-0.0991</td>
<td>0.1349</td>
</tr>
<tr>
<td>Delta</td>
<td>-0.094</td>
<td>0.567</td>
<td>-0.0997</td>
<td>0.224</td>
<td>-0.0987</td>
<td>0.1352</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>-0.0524</td>
<td>0.5319</td>
<td>-0.0568</td>
<td>0.2068</td>
<td>-0.0559</td>
<td>0.1197</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>-0.0547</td>
<td>0.5339</td>
<td>-0.0592</td>
<td>0.2078</td>
<td>-0.0583</td>
<td>0.1206</td>
</tr>
</tbody>
</table>

Table 8.3: Means and standard deviations of PnL for the term insurance of a single person

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM with 1 asset</td>
<td>1.667</td>
<td>2.8049</td>
<td>3.4652</td>
<td>2.6741</td>
<td>3.7517</td>
<td>4.4047</td>
</tr>
<tr>
<td>Delta</td>
<td>1.6523</td>
<td>2.8079</td>
<td>3.4727</td>
<td>2.6718</td>
<td>3.7637</td>
<td>4.4271</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>0.1065</td>
<td>0.3208</td>
<td>0.4715</td>
<td>0.3114</td>
<td>0.5576</td>
<td>0.7231</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>0.1808</td>
<td>0.4193</td>
<td>0.5872</td>
<td>0.4141</td>
<td>0.6894</td>
<td>0.8829</td>
</tr>
</tbody>
</table>

Table 8.4: VaR and CVaR for the term insurance in the case of $l_x = 1000$

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM with 1 asset</td>
<td>-1.347</td>
<td>1.1198</td>
<td>-1.3462</td>
<td>1.1014</td>
<td>-1.3468</td>
<td>1.0981</td>
</tr>
<tr>
<td>Delta</td>
<td>-1.3363</td>
<td>1.1275</td>
<td>-1.3355</td>
<td>1.1094</td>
<td>-1.3361</td>
<td>1.1061</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>-0.0791</td>
<td>0.3221</td>
<td>-0.0782</td>
<td>0.236</td>
<td>-0.0787</td>
<td>0.2231</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>-0.1503</td>
<td>0.3398</td>
<td>-0.1494</td>
<td>0.2612</td>
<td>-0.15</td>
<td>0.2498</td>
</tr>
</tbody>
</table>

Table 8.5: Means and standard deviations of PnL for the endowment insurance of a single person

<table>
<thead>
<tr>
<th>Hedging method</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM with 1 asset</td>
<td>1.667</td>
<td>2.8049</td>
<td>3.4652</td>
<td>2.6741</td>
<td>3.7517</td>
<td>4.4047</td>
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<tr>
<td>Delta</td>
<td>1.6523</td>
<td>2.8079</td>
<td>3.4727</td>
<td>2.6718</td>
<td>3.7637</td>
<td>4.4271</td>
</tr>
<tr>
<td>LRM with 2 assets</td>
<td>0.1065</td>
<td>0.3208</td>
<td>0.4715</td>
<td>0.3114</td>
<td>0.5576</td>
<td>0.7231</td>
</tr>
<tr>
<td>Delta-Gamma</td>
<td>0.1808</td>
<td>0.4193</td>
<td>0.5872</td>
<td>0.4141</td>
<td>0.6894</td>
<td>0.8829</td>
</tr>
</tbody>
</table>

Table 8.6: VaR and CVaR for the endowment insurance in the case of $l_x = 1000$

significance level. As the number of people $l_x$ increases, the standard deviation of PnL decreases. This is because the mortality risk can be averaged away by selling policies to enough people.
The values of VaR and CVaR for the pure endowment are reported in the table 8.2. The values of VaR and CVaR for LRM with 1 asset are close to those for the delta hedging as the difference between the difference between these two hedging methods is not statistically significant. Values of VaR and CVaR for LRM with 2 assets are smaller than those for the delta-gamma hedging as a result of thinner left tail of PnL for LRM with 2 assets. This can be seen from the figure 8.1 comparing histograms of PnL for different hedging methods. This also confirms the advantage of LRM with 2 assets over other hedging methods. We also note that distributions of PnL for all hedging methods are left-skewed. Figure 8.2 compares terminal payoffs with hedging portfolios. The worst hedging happens around the strike and adding liquid option as the second instrument can greatly improve the hedging performance.

In the table 8.3, the hedging performance for the unit-linked term insurance is given. The difference between LRM with 1 asset and delta hedging is not statistically significant while the difference between LRM with 2 assets and delta-gamma hedging is statistically significant. We also find that means and standard deviations of the term insurance are smaller than those of the pure endowment. This is because the number of options need to be hedged are smaller than the case of pure endowment. After looking at histograms of the PnL in the figure 8.3, we conclude distributions of PnL are all left-skewed. The values of VaR and CVaR for the term insurance are reported in the table 8.4. The values of VaR and CVaR for hedging methods with 1 asset are close to each other. LRM with 2 assets has smaller VaR and CVaR than the delta-gamma hedging.

At the end, we consider hedging the endowment insurance. In the table 8.5, the difference between LRM with 1 asset and delta hedging is not statistically significant while the difference between LRM with 2 assets and delta-gamma hedging is statistically significant. This can be seen from the figure 8.7 comparing the empirical c.d.f of two distributions. The histograms of difference in the PnL between LRM with 1 asset and the delta hedging and the difference between LRM with 2 assets and the delta-gamma hedging can be found in the figure 8.8. After performing the t-test, LRM with 2 assets has bigger mean than the delta-gamma hedging. The figure 8.5 compares histograms of PnL for different hedging methods. The distributions of PnL for all hedging methods are all left-skewed. The values of VaR and CVaR for the term insurance are reported in the table 8.6. For the endowment insurance, we have the same conclusion. The values of VaR and CVaR for hedging methods with 1 asset are close to each other while LRM
with 2 assets has smaller VaR and CVaR than delta-gamma hedging. Terminal payoffs can be found in the figure 8.6. Adding liquid options can greatly reduce hedging errors near the strike.

Numerical results in this section demonstrate the efficiency of local risk minimization hedging method over other hedging methods in the presence of mortality risk for different types of unit-linked life insurance contracts.

Figure 8.1: Histograms of PnL for the pure endowment of a single person in the case of $l_x = 1000$ and $age = 60$. 
Figure 8.2: Terminal payoff for the pure endowment of a single person in the case of $l_x = 1000$ and age = 60
Figure 8.3: Histograms of PnL for the term insurance of a single person in the case of $l_x = 1000$ and $age = 60$. 
Figure 8.4: Terminal payoffs for the term insurance of a single person in the case of $l_x = 1000$ and age = 60.
Chapter 8. Pricing and hedging of unit-linked life insurance

Figure 8.5: Histograms of PnL for the endowment insurance of a single person in the case of $l_x = 1000$ and age $= 60$. 

(a) LRM with 1 asset

(b) delta hedging

(c) LRM with 2 assets

(d) delta gamma hedging
Figure 8.6: Terminal payoffs for the endowment insurance of a single person in the case of $l_x = 1000$ and $age = 60$

Figure 8.7: Results of the Kolmogorov-Smirnov test for the endowment insurance
Figure 8.8: Comparing the difference of PnL for the endowment insurance
Chapter 9

Pricing and hedging under stochastic interest rate

In this chapter, we consider pricing and hedging of unit-linked life insurance contracts under mix funds and the stochastic interest rate. In the first section, we give the main results for three common types of unit-linked life insurance. The proofs are similar to chapter 8 with replacing the constant interest rate with the stochastic interest rate. We will skip proofs in this chapter. In the second section, we use numerical examples to compare the performance of different hedging methods.

9.1 Main results

Recall that with a pure endowment the sum insured is to be paid at the end of the term $T$ if the insured person is still alive. Therefore for each insured person the present value of obligations of the insurance company is given by

$$1_{(T_i > T)} e^{-\int_0^T r_s ds} G(M_T),$$

where $G(M_T) = \max(K - M_T, 0)$.

The entire portfolio generates the claim

$$G^N(M_T, N_T) = e^{-\int_0^T r_s ds} G(M_T) \sum_{i=1}^{l_x} 1_{(T_i > T)} = e^{-\int_0^T r_s ds} G(M_T)(l_x - N_T).$$

where $(l_x - N_T)$ is the number of survivors at the end of the insurance period.
Chapter 9. Pricing and hedging under stochastic interest rate

**Theorem 9.1.1.** The value of pure endowment contract at time $t$ is

$$H^N(t, M_t, r_t, N_t, T) = E^{Q^*}[G^N(M_T, N_T)|G_t]$$

$$= (l_x - N_t)T - tp_{x+t} e^{-\int_0^t r_s ds} h(t, M_t, r_t, T).$$

where $h(t, M_t, r_t, T)$ is computed in chapter 6.

Next we consider hedging unit-linked pure endowment. The following theorem gives the expression of hedging positions for the unit-linked pure endowment.

**Theorem 9.1.2.** The local risk minimization hedging position $\tilde{\xi}_t(T)$ of the underlying to hedge the unit-linked pure endowment with maturity $T$ is

$$\tilde{\xi}_t(T) = (l_x - N_t - t - tp_{x+t} e^{-\int_0^t r_s ds} \hat{\xi}_t(T)$$

(9.1)

where $\hat{\xi}_t(T)$ is the hedging position of mix funds in the chapter 7 when the reference portfolio is mix funds and the interest rate is stochastic.

**Theorem 9.1.3.** The local risk minimization hedging positions $\{\tilde{\xi}_t^{(1)}(T), \tilde{\xi}_t^{(2)}(T)\}$ of mix funds and liquid options to hedge the unit-linked pure endowment with maturity $T$ are

$$\tilde{\xi}_t^{(1)}(T) = (l_x - N_t - t - tp_{x+t} e^{-\int_0^t r_s ds} \hat{\xi}_t^{(1)}(T)$$

(9.2)

$$\tilde{\xi}_t^{(2)}(T) = (l_x - N_t - t - tp_{x+t} e^{-\int_0^t r_s ds} \hat{\xi}_t^{(2)}(T)$$

(9.3)

where $\hat{\xi}_t^{(1),(2)}(T)$ are hedging positions of mix funds and liquid options computed in the chapter 7.

A term insurance is in some sense the opposite of a pure endowment. Its benefit is due immediately upon death before time $T$. Hence, payments can occur at any time during $[0, T]$. For simplicity, we assume that death benefits are paid at the end of year of death. The present value of the payoff $G^L$ can be defined as

$$G^L = \sum_{i=1}^{l_x} \sum_{j=0}^{T-1} e^{-\int_0^{j+1} r_s ds} G(M_{j+1})1_{\{j < T, T \leq j+1\}}$$

**Theorem 9.1.4.** The value of the term insurance contract at time $t$ is
For $t = 0, 1, \cdots, T$, 
\[
H^L(t, M_t, r_t, N_t, T) = E^{Q^*}(G^L|\mathcal{F}_t)
\]
\[
= \sum_{i=1}^{l_x} \sum_{j=0}^{t-1} e^{-\int_0^{j+1} r_s ds} G(M_{j+1}) 1_{\{j < T_i \leq j+1\}}
\]
\[
+ (l_x - N_t) e^{-\int_0^{T-1} r_s ds} \sum_{j=t}^{T-1} h(t, M_t, r_t, j+1) (j-p_{x+t-q_{x+j}})
\]

If the time $t$ is between two payment dates, i.e., there is an integer $k$ such that $k < t < k+1$, the value is 
\[
H^L(t, M_t, r_t, N_t, T) = \sum_{i=1}^{l_x} \sum_{j=0}^{k-1} e^{-\int_0^{j+1} r_s ds} G(M_{j+1}) 1_{\{j < T_i \leq j+1\}} + \sum_{i=1}^{l_x} e^{-\int_0^{k} r_s ds} h(t, M_t, r_t, k+1) 1_{\{k < T_i \leq t\}}
\]
\[
+ (l_x - N_t) e^{-\int_0^{T-1} r_s ds} \left[ h(t, M_t, r_t, k+1) k-p_{x+t-q_{x+j}} + \sum_{j=k+1}^{T-1} h(t, M_t, r_t, j+1) (j-p_{x+t-q_{x+j}}) \right]
\]
where $h(t, M_t, r_t, j)$ is computed in the chapter 6.

**Theorem 9.1.5.** The local risk minimization hedging positions of the underlying asset for the term insurance are as follows.

\[
\tilde{\xi}_t(T) = (l_x - N_{t-}) e^{-\int_0^t r_s ds} \sum_{j=t}^{T-1} \hat{\xi}_t(j+1) (j-p_{x+t-q_{x+j}}), \quad t = 0, 1, \cdots, T,
\]

Similarly for $k < t < k+1$

\[
\tilde{\xi}_t(T) = e^{-\int_0^{k} r_s ds} \hat{\xi}_t(k+1) \left( \sum_{i=1}^{l_x} 1_{\{k < T_i \leq t\}} + (l_x - N_{t-}) k-p_{x+t-q_{x+j}} \right)
\]
\[
+ (l_x - N_{t-}) e^{-\int_0^{T-1} r_s ds} \sum_{j=k+1}^{T-1} \hat{\xi}_t(j+1) (j-p_{x+t-q_{x+j}})
\]
where $\hat{\xi}_t(j+1)$ represents the local risk minimization hedging position of the underlying
asset at time $t$ for the guarantee with maturity $j+1$ in the chapter 7.

Next we consider hedging unit-linked term insurance contracts using both underlying assets and liquid options.

**Theorem 9.1.6.** The local risk minimization hedging positions of underlying assets and liquid options for the unit-linked term insurance are as follows. For $t = 0, 1, \ldots, T$,

\[
\tilde{\xi}_t^{(1)}(T) = (l_x - N_{t-}) e^{-\int_0^t r_s ds} \sum_{j=t}^{T-1} \hat{\xi}_t^{(1)}(j+1) \cdot j - t \cdot p_{x+t} \cdot q_{x+j}
\]

\[
\tilde{\xi}_t^{(2)}(T) = (l_x - N_{t-}) e^{-\int_0^t r_s ds} \sum_{j=t}^{T-1} \hat{\xi}_t^{(2)}(j+1) \cdot j - t \cdot p_{x+t} \cdot q_{x+j}
\]

Similarly for $k < t < k + 1$

\[
\tilde{\xi}_t^{(1)}(T) = e^{-\int_0^t r_s ds} \sum_{i=1}^{l_x} 1_{\{k < T_i \leq t\}} + (l_x - N_{t-}) \cdot k - t + 1 \cdot q_{x+t}
\]

\[
\tilde{\xi}_t^{(2)}(T) = e^{-\int_0^t r_s ds} \sum_{i=1}^{l_x} 1_{\{k < T_i \leq t\}} + (l_x - N_{t-}) \cdot k - t + 1 \cdot q_{x+t}
\]

where $\hat{\xi}_t^{(1),(2)}(j+1)$ represent local risk minimization hedging positions of underlying assets and liquid options at time $t$ for the guarantee with maturity $j+1$ in the chapter 7.

A common unit-linked life insurance contract in practice is the endowment insurance, which is a combination of a pure endowment and a term insurance policy. In other words, the payoff of the endowment insurance in the presence of mortality risk can be represented as follows:

\[
G(M_T), \text{ if } T(x) > T,
\]

\[
G(M_k), \text{ if } k - 1 < T(x) \leq k \text{ for } k = 1, 2, \ldots, T.
\]
The present value of the payoff can be defined as

\[ G^I = \sum_{i=1}^{l_x} \sum_{k=0}^{T-1} e^{-\int_{k+1}^{k+1} r_s ds} G(M_{k+1})1_{\{k<T_i\leq k+1\}} + \sum_{i=1}^{l_x} e^{-\int_0^T r_s ds} G(M_T)1_{\{T_i>T\}} \]

i.e., the sum of payoffs from a pure endowment and a term insurance. Hence the price of the endowment insurance is the sum of the prices of the pure endowment and the term insurance.

**Theorem 9.1.7.** The value of the endowment insurance contract at time \( t \) is as follows. For \( t = 0, 1, \cdots, T \),

\[
H^I(t, M_t, r_t, N_t, T) = E^{Q^*}(G^I|\mathcal{F}_t)
\]

\[
= \sum_{i=1}^{l_x} \sum_{j=0}^{T-1} e^{-\int_{j+1}^{j+1} r_s ds} G(M_{j+1})1_{\{j<T_i\leq j+1\}} + (l_x - N_t)e^{-\int_0^T r_s ds} \sum_{j=t}^{T-1} h(t, M_t, r_t, j + 1) j - \alpha x_t q_j + (l_x - N_t)e^{-\int_0^T r_s ds} h(t, M_t, r_t, T) T - \alpha x_t
\]

If the time \( t \) is between two payment dates, i.e., there is an integer \( k \) such that \( k < t < k + 1 \), \( H^I \) is obtained by

\[
H^I(t, M_t, r_t, N_t, T) = \sum_{i=1}^{l_x} \sum_{j=0}^{k-1} e^{-\int_{j+1}^{j+1} r_s ds} G(M_{j+1})1_{\{j<T_i\leq j+1\}} + \sum_{i=1}^{l_x} e^{-\int_0^T r_s ds} h(t, M_t, r_t, k + 1)1_{\{k<T_i\leq t\}}
\]

\[
+ (l_x - N_t)e^{-\int_0^T r_s ds} \left[ h(t, M_t, r_t, k + 1)k - \alpha x_t q_{k+1} + \sum_{j=k+1}^{T-1} h(t, M_t, r_t, j + 1) j - \alpha x_t q_j \right]
\]

\[
+ (l_x - N_t)e^{-\int_0^T r_s ds} h(t, M_t, r_t, T) T - \alpha x_t
\]

where \( h(t, M_t, r_t, T) \) is defined in the chapter 6.

Since each individual is independent, hedging positions for the endowment insurance should be the sum of hedging positions for the pure endowment and the term insurance.

**Theorem 9.1.8.** The local risk minimization hedging positions of underlying assets for the unit-linked endowment insurance are as follows.
For $t = 0, 1, \cdots T$,

$$
\tilde{\xi}_t(T) = (l_x - N_{t-}) e^{-\int_0^t r_s ds} \left( \frac{T-1}{j=t} \tilde{\xi}_t(j+1) \right)_{j-t} q_{x+j} + \tilde{\xi}_t(T)_{T-t} q_{x+t}
$$

Similarly for $k < t < k + 1$

$$
\tilde{\xi}_t(T) = e^{-\int_0^t r_s ds} \tilde{\xi}_t(k+1) \left( \sum_{i=1}^{l_x} 1_{\{k < T_i \leq t\}} + (l_x - N_{t-})_{k-t+1} q_{x+j} \right)
$$

where $\tilde{\xi}_t(j+1)$ represents the local risk minimization hedging position of the underlying asset at time $t$ for the guarantee with maturity $j+1$ in the chapter 7.

**Theorem 9.1.9.** The local risk minimization hedging positions of underlying and liquid options for the unit-linked endowment insurance are as follows.

For $t = 0, 1, \cdots T$,

$$
\tilde{\xi}_t^{(1)}(T) = (l_x - N_{t-}) e^{-\int_0^t r_s ds} \left( \tilde{\xi}_t^{(1)}(T)_{T-t} q_{x+j} + \sum_{j=t}^{T-1} \tilde{\xi}_t^{(1)}(j+1)_{j-t} q_{x+j} \right)
$$

Similarly for $k < t < k + 1$

$$
\tilde{\xi}_t^{(1)}(T) = e^{-\int_0^t r_s ds} \tilde{\xi}_t^{(1)}(k+1) \left( \sum_{i=1}^{l_x} 1_{\{k < T_i \leq t\}} + (l_x - N_{t-})_{k-t+1} q_{x+j} \right)
$$

where $\tilde{\xi}_t^{(1),(2)}(j+1)$ represent local risk minimization hedging positions of underlying assets and liquid options at time $t$ for the guarantee with maturity $j+1$ in the chapter 7.
9.2 Numerical examples

In this section we use numerical examples to compare the performance of different hedging methods for hedging three common types of unit-linked life insurance contracts.

We use the following parameter set. The hedging frequency is $\Delta t = 1/63$. The option used as hedging instrument has a shorter maturity $T_s = 1/12$ and is liquidated on each hedging date, i.e., we will liquidate all the positions of options with shorter maturity and buy new at-the-money options with maturity $T_s$. Contract parameters are $R_{\text{min}} = 0$ and $T = 3$. Parameters for the interest rate are $r_0 = 3\%$, $\kappa = 0.2$, $\theta = 3.5\%$, $\sigma^R = 0.005$. The mix fund parameters are $\pi = 0.3$, $M_0 = 100$. We suppose the stock index and bond index follow variance gamma processes. There are no diffusion terms in the stock index and the bond index, i.e., $\sigma^S = 0$ and $\sigma^P = 0$. Parameters for variance gamma processes are obtained from calibrations in the chapter 4. The parameters for the stock index are $a = 185.23$, $b^+ = 104.83$, $b^- = 102.62$, $\hat{a} = 185.23$, $\hat{b}^+ = 2430.2$, $\hat{b}^- = 83.5$. The parameters for the bond index are $a^P = 382$, $b^{P+} = 484$, $b^{P-} = 462$, $\hat{a}^P = 382$, $\hat{b}^{P+} = 3500$, $\hat{b}^{P-} = 420$. The number of simulations is $N_{\text{sim}} = 10000$. For simplicity, we suppose there is no management fee, i.e. $\alpha = 0$. The number of people is $l_x = 1000$ and the age is $x = 60$. The real returns of the stock index and the bond index are $\mu^S = 0.15$, $\mu^P = 0.04$.

<table>
<thead>
<tr>
<th>Hedge</th>
<th>Stock index Mean</th>
<th>Std</th>
<th>Mix fund Mean</th>
<th>Std</th>
<th>VaR$_\alpha$ 70%</th>
<th>90%</th>
<th>95%</th>
<th>CVaR$_\alpha$ 70%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
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<tr>
<td>LRM 1</td>
<td>-0.7455</td>
<td>0.9919</td>
<td>-0.4836</td>
<td>0.6673</td>
<td>0.6686</td>
<td>1.3299</td>
<td>1.7904</td>
<td>1.2714</td>
<td>1.9332</td>
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<td>Delta</td>
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<td>-0.4796</td>
<td>0.6727</td>
<td>0.6581</td>
<td>1.3305</td>
<td>1.79</td>
<td>1.2728</td>
<td>1.9473</td>
<td>2.3539</td>
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<tr>
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<td>0.5648</td>
<td>0.0334</td>
<td>0.2022</td>
<td>0.0306</td>
<td>0.1525</td>
<td>0.239</td>
<td>0.159</td>
<td>0.3228</td>
<td>0.4538</td>
</tr>
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<td>DG</td>
<td>0.2452</td>
<td>0.5663</td>
<td>0.0105</td>
<td>0.2146</td>
<td>0.0493</td>
<td>0.1782</td>
<td>0.2668</td>
<td>0.1901</td>
<td>0.3744</td>
<td>0.5276</td>
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</table>

Table 9.1: Pure endowment of a single person when $\sigma^R = 0.005$

<table>
<thead>
<tr>
<th>Hedge</th>
<th>Stock index Mean</th>
<th>Std</th>
<th>Mix fund Mean</th>
<th>Std</th>
<th>VaR$_\alpha$ 70%</th>
<th>90%</th>
<th>95%</th>
<th>CVaR$_\alpha$ 70%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM 1</td>
<td>-0.0719</td>
<td>0.1098</td>
<td>-0.0436</td>
<td>0.0673</td>
<td>0.0545</td>
<td>0.1251</td>
<td>0.1743</td>
<td>0.1188</td>
<td>0.1898</td>
<td>0.2325</td>
</tr>
<tr>
<td>Delta</td>
<td>-0.0715</td>
<td>0.1101</td>
<td>-0.0434</td>
<td>0.0675</td>
<td>0.0542</td>
<td>0.1249</td>
<td>0.1745</td>
<td>0.1188</td>
<td>0.1901</td>
<td>0.2328</td>
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<tr>
<td>LRM 2</td>
<td>-0.0308</td>
<td>0.0898</td>
<td>-0.018</td>
<td>0.0542</td>
<td>0.0239</td>
<td>0.0712</td>
<td>0.1115</td>
<td>0.0708</td>
<td>0.1255</td>
<td>0.1639</td>
</tr>
<tr>
<td>DG</td>
<td>-0.0332</td>
<td>0.0961</td>
<td>-0.0194</td>
<td>0.058</td>
<td>0.0253</td>
<td>0.0765</td>
<td>0.122</td>
<td>0.076</td>
<td>0.1349</td>
<td>0.1752</td>
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</tbody>
</table>

Table 9.2: Term insurance of a single person when $\sigma^R = 0.005$
Chapter 9. Pricing and hedging under stochastic interest rate

<table>
<thead>
<tr>
<th>Hedge</th>
<th>Stock index</th>
<th>Mix fund</th>
<th>VaR</th>
<th>CVaR</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM 1</td>
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<td>0.7343</td>
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<td>1.8674</td>
<td>1.3577</td>
<td>2.0358</td>
<td>2.4564</td>
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</tr>
<tr>
<td>Delta</td>
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<td>1.051</td>
<td>-0.521</td>
<td>0.7194</td>
<td>0.7282</td>
<td>1.4074</td>
<td>1.8659</td>
<td>1.3558</td>
<td>2.0402</td>
<td>2.4655</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LRM 2</td>
<td>0.2715</td>
<td>0.5776</td>
<td>0.1437</td>
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<td>0.2942</td>
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<td>0.2742</td>
<td>0.5266</td>
<td>0.7022</td>
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<td>0.3236</td>
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<td>0.5719</td>
<td>0.7542</td>
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</tbody>
</table>

Table 9.3: Endowment insurance of a single person when \( \sigma^R = 0.005 \)

In the following, we will choose another set of interest rate parameters to test the effect of interest rate risk on the hedging performances. The new interest rate parameters are \( r_0 = 7\%, \kappa = 0.2, \theta = 8\%, \sigma^R = 0.02 \).

<table>
<thead>
<tr>
<th>Hedge</th>
<th>Stock index</th>
<th>Mix fund</th>
<th>VaR</th>
<th>CVaR</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
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</thead>
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<tr>
<td>LRM 1</td>
<td>-0.2807</td>
<td>1.5074</td>
<td>-0.1821</td>
<td>0.984</td>
<td>0.4505</td>
<td>1.2756</td>
<td>1.8303</td>
<td>1.2243</td>
<td>2.1162</td>
<td>2.7124</td>
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<tr>
<td>Delta</td>
<td>-0.2748</td>
<td>1.5084</td>
<td>-0.1798</td>
<td>0.9847</td>
<td>0.4464</td>
<td>1.2756</td>
<td>1.8284</td>
<td>1.2231</td>
<td>2.1185</td>
<td>2.7181</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>LRM 2</td>
<td>0.4945</td>
<td>1.3548</td>
<td>0.2195</td>
<td>0.8791</td>
<td>0.1165</td>
<td>0.6356</td>
<td>1.0183</td>
<td>0.6182</td>
<td>1.2062</td>
<td>1.6177</td>
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<tr>
<td>DG</td>
<td>0.4566</td>
<td>1.3586</td>
<td>0.2018</td>
<td>0.8820</td>
<td>0.1312</td>
<td>0.6569</td>
<td>1.0431</td>
<td>0.6403</td>
<td>1.2386</td>
<td>1.6558</td>
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</tbody>
</table>

Table 9.4: Pure endowment of a single person when \( \sigma^R = 0.02 \)

<table>
<thead>
<tr>
<th>Hedge</th>
<th>Stock index</th>
<th>Mix fund</th>
<th>VaR</th>
<th>CVaR</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRM 1</td>
<td>-0.0394</td>
<td>0.1055</td>
<td>-0.0203</td>
<td>0.0654</td>
<td>0.0301</td>
<td>0.0841</td>
<td>0.1213</td>
<td>0.0809</td>
<td>0.1417</td>
<td>0.1831</td>
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<td></td>
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</tr>
<tr>
<td>Delta</td>
<td>-0.0392</td>
<td>0.1056</td>
<td>-0.0202</td>
<td>0.0655</td>
<td>0.03</td>
<td>0.0841</td>
<td>0.1218</td>
<td>0.0811</td>
<td>0.1424</td>
<td>0.1835</td>
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<tr>
<td>LRM 2</td>
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<td>0.0873</td>
<td>-0.0017</td>
<td>0.0574</td>
<td>0.0121</td>
<td>0.044</td>
<td>0.0698</td>
<td>0.0448</td>
<td>0.0881</td>
<td>0.1223</td>
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<tr>
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<td>0.0981</td>
<td>-0.003</td>
<td>0.0613</td>
<td>0.0137</td>
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<td>0.0785</td>
<td>0.0498</td>
<td>0.0976</td>
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</tbody>
</table>

Table 9.5: Term insurance of a single person when \( \sigma^R = 0.02 \)

<table>
<thead>
<tr>
<th>Hedge</th>
<th>Stock index</th>
<th>Mix fund</th>
<th>VaR</th>
<th>CVaR</th>
<th>Mean</th>
<th>Std</th>
<th>Mean</th>
<th>Std</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
<th>70%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
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<tr>
<td>LRM 1</td>
<td>-0.3201</td>
<td>1.5647</td>
<td>-0.2024</td>
<td>1.0227</td>
<td>0.4799</td>
<td>1.3303</td>
<td>1.9253</td>
<td>1.2854</td>
<td>2.2182</td>
<td>2.8366</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>Delta</td>
<td>-0.3141</td>
<td>1.566</td>
<td>-0.2001</td>
<td>1.0236</td>
<td>0.4759</td>
<td>1.3299</td>
<td>1.923</td>
<td>1.2842</td>
<td>2.2212</td>
<td>2.8431</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>LRM 2</td>
<td>0.4919</td>
<td>1.3988</td>
<td>0.2177</td>
<td>0.9102</td>
<td>0.1264</td>
<td>0.6614</td>
<td>1.0508</td>
<td>0.648</td>
<td>1.2595</td>
<td>1.6886</td>
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</tr>
<tr>
<td>DG</td>
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<td>0.1988</td>
<td>0.9153</td>
<td>0.1437</td>
<td>0.6857</td>
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<td>0.6729</td>
<td>1.2964</td>
<td>1.7332</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9.6: Endowment insurance of a single person when \( \sigma^R = 0.02 \)

From above tables we can see values of standard deviation, VaR and CVaR increase as a result of the presence of the interest rate risk compared to corresponding values in the
chapter 8. In the case of $\sigma^R = 0.005$, the differences between LRM with 1 asset and the delta hedging are not statistically significant for all three types unit-linked life insurance contracts after we perform two-sample K-S test at 5% significance level. The differences between LRM with 2 assets and the delta-gamma hedging are statistically significant for all three types of products. In the case of $\sigma^R = 0.02$, the differences between LRM with 1 asset and the delta hedging and are not statistically significant for all three types unit-linked life insurance contracts. The difference between LRM with 2 assets and the delta-gamma hedging is statistically significant for the unit-linked term insurance while differences for the pure endowment and the endowment insurance are not statistically significant. In the case of big volatility of the interest rate, the advantage of local risk minimization in hedging the jump risk is not obvious. In the future, we will consider introducing other bonds as hedging instruments to eliminate the interest rate risk.

Histograms of PnL are give in figures 9.1, 9.3 and 9.5. When comparing values of VaR and CVaR, we conclude hedging methods using 1 asset are close to each other while LRM with 2 assets has smaller VaR and CVaR than the delta-gamma hedging methods. Figures 9.2, 9.4 and 9.6 plot histograms of difference between hedging method using 1 asset and difference between hedging method using 2 assets. After performing t-test, we get all differences are statistically significant. Figures 9.7, 9.8 and 9.9 compare terminal payoffs with hedging portfolios. From these plots we can see introducing liquid options as the second hedging instrument is useful for hedging the jump risk, but the interest rate risk is not negligible when the $\sigma^R$ is bigger. Hence the reductions in terms of the standard deviation are smaller than those in the case of constant interest rate (results from Chapter 8). In order to hedge the interest rate risk, we need to introduce bonds as hedging instrument in the future.
Figure 9.1: Histogram for the pure endowment of a single person when $\sigma^R = 0.02$

Figure 9.2: Comparing differences of PnL for the pure endowment when $\sigma^R = 0.02$
Chapter 9. Pricing and hedging under stochastic interest rate

Figure 9.3: Histogram for the term insurance of a single person when $\sigma^R = 0.02$.  

(a) LRM with 1 asset  
(b) Delta hedging  
(c) LRM with 2 assets  
(d) Delta gamma hedging

Figure 9.4: Comparing differences of PnL for the term insurance when $\sigma^R = 0.02$.

(a) Reject Null, non-zero mean  
(b) Reject Null, non-zero mean
Figure 9.5: Histograms for the endowment insurance of a single person when $\sigma^R = 0.02$.

Figure 9.6: Comparing the difference of PnL for the endowment insurance when $\sigma^R = 0.02$. 
Figure 9.7: Terminal payoffs for the pure endowment of a single person when $\sigma^R = 0.02$. 
Figure 9.8: Terminal payoffs for the term insurance of a single person when $\sigma^R = 0.02$. 
Figure 9.9: Terminal payoffs for the endowment insurance of a single person when $\sigma^R = 0.02$. 
Chapter 10
Conclusions and future works

10.1 Conclusions

In this thesis, we considered pricing and hedging of the guaranteed rate of return and the unit-linked life insurance. We derived the pricing methodology under Lévy model and stochastic interest rate and then used Fourier transform method to solve the pricing PIDE numerically. After comparing Fourier transform method with finite difference method and Monte-Carlo method, we concluded that method via Fourier transform was fast, accurate and easy to implement. Sensitivities of the fair management fee to different model parameters were also examined. The fair management fee decreased as the percentage in the bond index increased. Only the jump size would affect the fair management fees and jump direction did not matter. Hedging in an incomplete market is a challenging task. We proposed mean variance hedging using multiple risky assets and local risk minimization hedging strategy using one and two risky assets. Since hedging performance was very sensitive to the difference between real world and risk neutral Lévy density. Before using numerical examples to compare hedging performances, we calibrated real world parameters using historical data on indices and risk neutral parameters using historical option prices. We calculated mean, standard deviation, VaR and CVaR of Profit and Loss at maturity for different hedging methods and concluded local risk minimization hedging using two risky assets performed the best over other hedging methods. Finally three types of unit-linked contracts, i.e. pure endowment, term insurance and endowment insurance were used to demonstrate the efficiency of local risk minimization trading strategies under the constant interest rate and stochastic interest rate.
10.2 Future works

In this thesis, we are considering local risk minimization using at most two risky assets as hedging instruments. In order to improve the hedging performance, local risk minimization using multiple risky assets can also be derived in the future. Since GMWB (guaranteed minimum withdrawal benefit) products are becoming more and more popular nowadays, pricing and hedging of GMWB on mixed funds under Lévy model and stochastic interest rate can also be considered. Since GMWB are path-dependent payoffs, the pricing and hedging may become more complicated. Similar to Jaimungal and Chong (2014), we can also model the clustering in jumps of the stock index and bond index. As a major risk in long term insurance products, mortality risk has material impact on liabilities calculation. Stochastic mortality rate can be incorporated later. Another extension is to calculate optimal percentage of bond index by maximizing the expected utility.
Appendices
Appendix A

Proof of Theorem 2.2.9

Proof. 1. $B \in \mathbb{R}$, $\nu(B) = 0$ implies $J([0, t] \times B) = 0$ almost surely for all $0 \leq t \leq T$ and for all Borel sets $B$. Hence by the Lévy-Itô-decomposition almost all paths are continuous. If almost all paths of $L$ are continuous, then $\Delta L_t = 0$ for all $t$, which implies $\nu(dx) = 0$.

2. Let $\nu(\mathbb{R}) < \infty$, then for any $0 \leq t_1 < t_2 \leq T$ one has that

$$J([t_1, t_2] \times \mathbb{R}) = \sum_{t_1 \leq s \leq t_2} 1_{\{\Delta L_s \neq 0\}} 1_{\{\Delta L_s \in \mathbb{R}\}} < \infty \quad a.s.$$ 

This shows that $L$ has almost surely only finitely many jumps on $[t_1, t_2]$.

Let $\nu(\mathbb{R}) = \infty$ and consider the sequence $\varepsilon_n$ given by

$$\varepsilon_1 = \sup \left\{ r : \int_{\{ |x| \geq r \}} \nu(dx) \geq 1 \right\}$$

$$\varepsilon_n = \sup \left\{ r : r < \varepsilon_{n-1}, \int_{\{ |x| \geq r \}} \nu(dx) \geq n \right\}, \quad n \geq 2.$$ 

We define for $0 \leq t_1 < t_2 \leq T$ the following sequence of independent random variables

$$Y_n = \int_{t_1}^{t_2} \int_{\{ \varepsilon_{n+1} \leq |x| < \varepsilon_n \}} J(dt \times dx), \quad n \geq 1.$$ 

The variables $Y_n$ are Poisson distributed with intensity $\lambda_n = (t_2 - t_1)$ for all $n$ and the total number of jumps of $L$ in the interval $[t_1, t_2]$ is equal to $\sum_{n=1}^{\infty} Y_n$. Now it is
not very hard to show that
\[ \mathbb{P}(Y_n \geq 1) = e^{-\lambda_n} \sum_{k=1}^{\infty} \frac{\lambda_n^k}{k!} = 1 - \frac{1}{e^{\lambda_n}} \geq 1 - \frac{1}{e^{t_2 - t_1}} \quad \text{all } n. \]

Hence \( \sum_{n=1}^{\infty} \mathbb{P}(Y_n \geq 1) = \infty \), using the Borel-Cantelli Lemma,
\[ \mathbb{P}(Y_n \geq 1, \text{i.o.}) = 1. \]

Therefore, \( \sum_{n=1}^{\infty} Y_n = \infty \) almost surely.

3. Let \( c = 0 \) and \( \int_{\{|x| \leq 1\}} |x| \nu(dx) < \infty \), then the Lévy-Itô-decomposition implies that
\[ L_t = \gamma t + L_t^I + M_t = \left[ \gamma - \int_{|x| \leq 1} x \nu(dx) \right] t + L_t^I + \int_0^t \int_{\{|x| \leq 1\}} x J(ds \times dx). \]

The first two terms are clearly of finite variation. It remains to consider the third term having total variation over \( [0, T] \) given by
\[ TV_{[0, T]} \left( \int_0^T \int_{|x| \leq 1} x J(ds \times dx) \right) = \int_0^T \int_{|x| \leq 1} |x| J(ds \times dx). \]

We have equality here, since the total variation of any càdlàg function is greater or equal to the sum of its jumps. Therefore,
\[ \mathbb{E} \left[ TV_{[0, T]} \left( \int_0^T \int_{\{|x| \leq 1\}} x J(ds \times dx) \right) \right] = T \int_{\{|x| \leq 1\}} |x| \nu(dx), \]
which implies that the total variation of \( L \) is finite almost surely.

For the converse statement we observe that for every \( n \geq 1 \) and every \( 0 \leq t_1 < t_2 \leq T \) it holds,
\[ TV_{[t_1, t_2]}(L) \geq \int_{t_1}^{t_2} \int_{\{|x| \leq 1\}} |x| J(ds \times dx) \]
\[ = (t_2 - t_1) \int_{\{|x| \leq 1\}} |x| \nu(dx) + \int_{t_1}^{t_2} \int_{\{|x| \leq 1\}} |x| \Phi(ds \times dx). \]

Let now \( Y_n := \int_{t_1}^{t_2} \int_{\{|x| \leq 1\}} |x| \Phi(ds \times dx) \) and \( X_n := Y_{n+1} - Y_n \). Then one has that
\[ \mathbb{E}[X_n] = 0 \text{ all } n \text{ and} \]
\[ \sum_{n=1}^{\infty} \text{Var}[X_n] = \sum_{n=1}^{\infty} \mathbb{E}[X_n^2] = (t_2 - t_1) \sum_{n=1}^{\infty} \int_{\{1/n+1 \leq |x| < 1/n \}} x^2 \nu(dx) < \infty \]

Kolmogorov’s three series theorem implies then that \( \sum_{n=1}^{\infty} X_n \) converges to a finite limit almost surely. Therefore
\[ \int_{\{|x| \leq 1 \}} |x| \nu(dx) = \infty \Rightarrow TV_{[t_1, t_2]}(L) = \infty \text{ a.s.} \]

If \( A > 0 \) then \( L \) contains a Brownian motion component which has almost surely paths of infinite variation.
Appendix B

Simulations of variance gamma processes

The variance gamma process can be represented either as a Brownian motion subordinated by a gamma process or as the difference of two gamma processes. So there are two ways of simulating it.

Suppose we have the variance gamma process \( X(t) \) with parameters \( \{\theta, \sigma, \nu\} \). Then \( X(t) \) can be represented as follows.

\[
X(t) = \theta \gamma(t; 1, \nu) + \sigma W_{\gamma(t; 1, \nu)} \tag{B.1}
\]

where \( W_t \) is a standard Brownian motion and \( \gamma(t; \mu, \nu) \) is a gamma process with mean rate \( \mu \) and variance rate \( \nu \).

The first way of simulating the variance gamma process is as follows.

- simulate \( n \) independent gamma variables \( \Delta J_1, \cdots, \Delta J_n \) with parameters \( \frac{\mu_i}{\nu}, \frac{t_i - t_{i-1}}{\nu}, \cdots, \frac{t_n - t_{n-1}}{\nu} \).
  
  Set \( \Delta J_i = \nu \Delta J_i \).

- Simulate \( n \) i.i.d \( N(0, 1) \) random variables \( N_1, \cdots, N_n \). Set \( \Delta X_i = \sigma N_i \sqrt{\Delta J_i} + \theta \Delta J_i \) for all \( i \).

The discretized trajectory is \( X(t_i) = \sum_{k=1}^{i} \Delta X_k \).

The other way of simulating \( \{X(t_1), \cdots, X(t_n)\} \) is to use the difference of two Gamma processes.

- Set \( X_0 = 0 \). From \( i = 1 \) to \( n \), generate \( \Delta J_i^+ \sim \Gamma \left( a(t_i - t_{i-1}), \frac{1}{\nu} \right) \) and \( \Delta J_i^- \sim \Gamma \left( a(t_i - t_{i-1}), \frac{1}{\nu} \right) \) independently and independent of past r.v.s.
Appendix B. Simulations of variance gamma processes

• Set $\Delta X_i = \Delta J_i^+ - \Delta J_i^-$ for all $i$.

The discretized trajectory is $X(t_i) = \sum_{k=1}^{i} \Delta X_k$. 

Bibliography


