**Effective Action of $\lambda \phi^3$ in Krein Space Quantization**

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Effective Action of $\lambda \phi^3$ in Krein Space Quantization

B. Forghan*, S. Razavi S†

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Department of Physics, Parand Branch, Islamic Azad University, Tehran, Iran
Department of Physics, Robatkarim Branch, Islamic Azad University, Tehran, Iran

Abstract

The appearance of divergence creates computational issues in the process of calculating the one-loop effective action of $\lambda \phi^3$ in quantum field theory. In this paper, it is demonstrated that using Krein space quantization with Ford's method of fluctuated metrics, divergence can be removed and that without using any traditional regularization method, it is possible to arrive at a finite solution for the effective action.

Keywords: Krein regularization; Krein Space Quantization; Quantum metric fluctuation

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1 Introduction

In quantum field theory (QFT), an issue that often leads to computational break down and significant hindrance is the appearance of divergences, which necessitates regularization and renormalization procedures. The methodology introduced by Takook et al and termed Krein Space Quantization (KSQ) provides a robust framework for facilitating the calculation of values pertinent to interacting fields and QFT in general[1, 2, 3].

The unique properties of KSQ arise from the optimal use of states with negative norms, which are maintained as regulators to preserve causality. If a direct-sum decomposition of a vector space with an indefinite scalar product exists into a Hilbert space and an anti-Hilbert space, then that vector field is a Krein space. The Hilbert space includes positive norm states and the anti-Hilbert includes negative norm states. It has been proved that using these positive and negative norm states allows for the preservation of causality and covariance [1, 3, 4, 5].

In QFT, at short relative distances, ultraviolet divergence appears, while at large relative distances, infrared divergence emerges in the Green function. The divergent terms manifest in

*e-mail: b.forghan@piau.ac.ir
†e-mail: srazavis.phys@gmail.com

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the following forms in the Green function $G(x, x')$, in the limit $x \to x'$ or $\sigma \to 0$:

$$\frac{1}{\sigma}, \ln \sigma, \text{ and } \delta(\sigma), \text{ where } \sigma = \frac{1}{2}(x - x')^2.$$

Quantum metric fluctuation may smear out the singularity on the light cone, i.e. $\delta(\sigma)$, but the ultraviolet divergences of QFT persevere[5, 6]. Takook et al proved that quantization in Krein space removes all ultraviolet divergences of QFT except the light cone singularity and KSQ in tandem with the quantum metric fluctuation yields a QFT without divergences[3, 7, 8]. This combined approach is called "Krein regularization" [9].

In the one-loop approximation the propagator of the scalar field in KSQ is

$$G_{Kr} = PP \frac{m^2}{p^2(p^2 - m^2)}.$$

(1.1)

This is precisely the same as the Green function which appears in the s-channel contribution of the transition amplitude and interestingly, this propagator also appears in the supersymmetry theory [7, 10].

In KSQ, the auxiliary negative norm states do not interact with the physical states or the real physical world; these states disappear entirely and the theory is unitary under the following conditions [9]:

i) The negative norm states do not appear in the external legs of the Feynmann diagram. This "reality condition" guarantees that the negative norm states only appear in the internal legs and in the disconnected parts of the diagram.

ii) The S matrix elements must be renormalized in the following form:

$$S'_{tf} \equiv \text{probability amplitude} = \frac{<\text{physical states, in}|\text{physical states, out}>}{<0, \text{in}|0, \text{out}>}.$$

the negative norm states are eliminated by this condition in the disconnected parts.

The Casimir effect, one-loop approximation of Moller scattering, $\lambda \phi^4$, QED and QCD have been determined successfully by using Krein regularization [9, 11, 12, 13, 14]. This method has also removed infrared divergence in linear gravity in de Sitter space and has provided an efficient solution to the problem of non-renormalizability of linear quantum gravity [5]. The effective action for $\lambda \phi^4$ and QED has been calculated using KSQ combined with the quantum metric fluctuation [15, 16].

Krein regularization has certain advantages over the alternative Pauli-Villars method, as the former gives a finite value for the vacuum expectation of $T_{\mu\nu}$ without using the normal ordering procedure and without considering any contour terms in writing the Lagrangian; in other words there is no need for renormalization and the calculation procedure is greatly shortened in $\lambda \phi^4$, $\lambda \phi^3$, QED and QCD theories [9, 13, 14, 17].

Replacing the Feynman propagator with the Krein propagator removes all ultraviolet and infrared divergence terms in the effective action of the Lagrangian of $\lambda \phi^3$, owing to the existence of positive and negative modes and the quantum metric fluctuation.

The next section of this paper presents the Krein regularization method which is used to calculate the effective action of $\lambda \phi^3$. 

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2 Effective Action of $\lambda \phi^3$ in Krein Space

The effective action of $\lambda \phi^3$ in the one-loop approximation is defined by [18]

$$\Gamma[\phi] = I(\phi) + \frac{i}{2} \hbar Tr \ln[1 - G_K V''(\phi)] + O(\hbar^2), \quad (2.1)$$

Applying the Fourier transformation and using (1.1), we have

$$Tr \ln[1 - G_K V''(\phi)] = \int d^4x \int \frac{d^4p}{(2\pi)^4} \ln[1 - V''(\varphi)PP \frac{m^2}{p^2(p^2 - m^2)}]. \quad (2.2)$$

The effective potential of $\lambda \phi^3$ is as follows:

$$V_{eff} = V_{eff}^{(0)} + \lambda \phi \frac{V_{eff}^{(1)}}{m^2} + \lambda^2 \phi^3 + \ldots \quad (2.3)$$

$$V_{eff}^{(0)} = \frac{m^2}{2} \phi^2 + \frac{\lambda \phi^3}{3!}$$

$$V_{eff}^{(1)} = -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \ln[1 - V''(\varphi)PP \frac{m^2}{p^2(p^2 - m^2)}]. \quad (2.4)$$

Contrast (2.4) with its counterpart in Hilbert space:

$$V_{eff}^{(1)} = -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \ln[1 - V''(\varphi)PP \frac{m^2}{p^2(p^2 - m^2)}]. \quad (2.5)$$

In (2.5), the divergent terms $\Gamma(0)$ and $\Gamma(-1)$ have been removed by applying regularization; whereas in Krein regularization the appearance of $PP \frac{m^2}{p^2(p^2 - m^2)}$ makes the relation independent of traditional regularization methods and the answer is automatically finite. Here, $V''(\varphi)$ is $\lambda \phi$, and $V_{eff}^{(1)}$ can be written as [15, 16]

$$-\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \ln[1 - \frac{\lambda \phi m^2}{p^2(p^2 - m^2 + i\epsilon)}] - \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \ln[1 - \frac{\lambda \phi m^2}{p^2(p^2 - m^2 - i\epsilon)}] - \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \ln[1 - \frac{\lambda \phi m^2}{(p^2 - m^2 + i\epsilon)}(1 - \frac{\lambda \phi m^2}{p^2(p^2 - m^2 - i\epsilon)})]. \quad (2.6)$$

By applying Wick rotation such that $p_0$ changes to $ik_0$, the corresponding Euclidean four momentum $k$, the first and second integrals disappear and the third integral converts to two integrals as below:

$$\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln[1 - \frac{\lambda \phi m^2}{k^2(k^2 + m^2)}] - \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln[1 + \frac{\lambda \phi m^2}{k^2(k^2 + m^2)}]. \quad (2.7)$$
These integrals can be solved as follows:

\[
\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^{\frac{\lambda m^2 \phi}{2}} du \int_0^{\frac{m^2 \lambda \phi}{2} - u} \frac{du}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4}} \left( \frac{m^2 \lambda \phi}{2} - u \right) + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^{\frac{\lambda m^2 \phi}{2}} du \int_0^{\frac{m^2 \lambda \phi}{2} + u} \frac{du}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4}} \left( \frac{m^2 \lambda \phi}{2} + u \right),
\]

(2.8)

and replacing \( k^2 + \frac{m^2}{2} \) with \( q \) we have

\[
V_{eff}^{(1)} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \int_0^{\frac{\lambda m^2 \phi}{2}} du \int_0^{\frac{m^2 \lambda \phi}{2} - u} \frac{du}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4}} \left( \frac{m^2 \lambda \phi}{2} - u \right) e^{-t\left(q^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi}{2} - u\right)}
\]

\[+ \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \int_0^{\frac{\lambda m^2 \phi}{2}} du \int_0^{\frac{m^2 \lambda \phi}{2} + u} \frac{du}{(k^2 + \frac{m^2}{2})^2 - \frac{m^4}{4}} \left( \frac{m^2 \lambda \phi}{2} + u \right) e^{-t\left(q^2 - \frac{m^4}{4} - \frac{m^2 \lambda \phi}{2} + u\right)}.
\]

(2.9)

A necessary condition is \( q^2 > \frac{m^4}{4} + m^2 \lambda \phi \) and after applying cutoff momentum calculations[15], we have \( k_c = \sqrt{\lambda \phi} \) and \( k_c < k < \infty \). Equation (2.9) consequently becomes

\[V_{eff}^{(1)} = -\frac{1}{32\pi^2} \int_0^{\infty} dt \int_0^{\infty} dq \left( q - \frac{m^2}{2} \right) e^{-tq^2} \int_0^{\frac{m^2 \lambda \phi}{2} - u} \frac{du}{\left( k^2 + \frac{m^2}{2} \right)^2 - \frac{m^4}{4}} e^{-tu + t(u + \frac{m^2 \lambda \phi}{2})}
\]

\[+ \frac{1}{32\pi^2} \int dt \int_0^{\infty} dq \left( q - \frac{m^2}{2} \right) e^{-tq^2} \int_0^{\frac{m^2 \lambda \phi}{2} + u} \frac{du}{\left( k^2 + \frac{m^2}{2} \right)^2 - \frac{m^4}{4}} e^{-tu + t(u - \frac{m^2 \lambda \phi}{2})}
\]

\[= -\frac{1}{32\pi^2} \int dt \left( \frac{e^{-tq^2}}{2t} - \frac{m^2}{4} \sqrt{\frac{\pi}{t}} e^{-tq^2} \left( \frac{e^{\frac{m^2 \lambda \phi}{2} - t - 1}}{t} \right) e^{t\left(\frac{\frac{m^4}{4} + \frac{m^2 \lambda \phi}{2}}{2}\right)}
\]

\[+ \frac{1}{32\pi^2} \int dt \left( \frac{e^{-tq^2}}{2t} - \frac{m^2}{4} \sqrt{\frac{\pi}{t}} e^{-tq^2} \left( \frac{e^{\frac{m^2 \lambda \phi}{2} - t - 1}}{t} \right) e^{t\left(\frac{\frac{m^4}{4} + \frac{m^2 \lambda \phi}{2}}{2}\right)}
\]

\[= -\frac{1}{32\pi^2} \int dt \frac{e^{-t\left(\lambda^2 \phi^2 + \frac{m^2 \lambda \phi}{2}\right)}}{t^2} \left( \cosh \frac{m^2 \lambda \phi}{2} t - 1 \right)
\]

\[+ \frac{m^2 \phi}{64\pi^2} \int dt \frac{e^{-t\left(\lambda^2 \phi^2 + \frac{m^2 \lambda \phi}{2}\right)}}{t^2} \left( \cosh \frac{m^2 \lambda \phi}{2} t - 1 \right).
\]

(2.10)

After replacing the expansion of the hyperbolic cosine, we have

\[V_{eff}^{(1)} = -\frac{1}{32\pi^2} \sum_{n=1}^{\infty} \int dt \frac{(2\lambda \phi m^2)^{2n} t^{2n-2} e^{-t\frac{\lambda \phi m^2}{2} (1 + \frac{2 \lambda \phi}{m^2})}}{4^{2n} n!} + \frac{m^2}{64\pi^2} \sum_{n=1}^{\infty} \int dt \frac{(2\lambda \phi m^2)^{2n} t^{2n-2} e^{-t\frac{\lambda \phi m^2}{2} (1 + \frac{2 \lambda \phi}{m^2})}}{4^{2n} n!}
\]

\[-\frac{2m^2 \lambda \phi}{(16\pi)^2} \left( \frac{1}{1 + \frac{2 \lambda \phi}{m^2}} \right)^2 \left[ \ln(1 + \frac{2 \lambda \phi}{m^2}) + \ln(1 + \frac{\lambda \phi}{m^2}) + \ln \frac{\lambda \phi}{m^2} + 2 \ln 2 + \ln(1 + \frac{\lambda \phi}{m^2}) - \ln \frac{\lambda \phi}{m^2} \right]
\]

\[-\frac{m^2 (2m^2 \lambda \phi)^{2}}{64\pi^2} \left[ 1 + \frac{m^2}{2 \lambda \phi} \right] - 2 \left( 1 + \frac{m^2}{2 \lambda \phi} \right)^{\frac{3}{2}}.
\]

(2.11)

This result is achieved without employing any conventional regularization methods, and is finite.
3 Conclusion

In this paper, KSQ and quantum metric fluctuation (Krein Regularization) has been used to analyse the effective action of $\lambda \phi^3$ theory in the one-loop approximation. For $\lambda \phi^3$, this quantization eliminates the singularity in the theory without changing the physical content. Conversely, in Hilbert space, after using Pauli-Villars or dimensional regularization, the infinite terms remain and renormalization is needed. KSQ has been used without applying any traditional regularization methods and renormalization to compute the magnetic anomaly, Lamb shift and running coupling constant of QED, $\lambda \phi^3$, $\lambda \phi^4$ and QCD; remarkably, it has yielded the same results as conventional techniques and experimental observations \cite{9,13,14,17,19}. KSQ can be generalized to linear gravity \cite{1,2,20,21}. It is noteworthy that in all of these solutions, the reality condition ensures that the unphysical states do not contribute to the S matrix elements, so unitary is preserved.

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