Symplectic Reduction and Convexity of Moment Maps

by

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Abstract
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This thesis is was written as part of a contribution to a book on symplectic geometry. The first section is corresponds to a chapter on symplectic reduction and the second section to a chapter on convexity of moment maps for torus actions. While the results are all well known I hope that the presentation is unique.

I am very grateful to Lisa Jeffrey for her introducing me to symplectic geometry and for giving me the opportunity to contribute to a book on the subject.
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Chapter 1

Symplectic Reduction

1.1 Introduction

In the presence of a Hamiltonian action on a symplectic manifold we may reduce the size of the symplectic structure by quotienting out by the symmetries. When and how we can do this is the primary subject of this chapter and sufficient conditions are described by the theorem of Marsden-Weinstein and Meyer. The rest of the chapter is devoted to studying various equivalent definitions of symplectic quotients, the structure of symplectic quotients by subgroups, and finally constructing symplectic cuts which will be a useful tool in later chapters. Many of the results and proofs found in this chapter are borrowed from Cannas da Silva’s book. [4]

Let us start by recalling some standard definitions and results on smooth manifolds which are covered for instance in Lee’s book on smooth manifolds [5].

Definition 1.1. A smooth map \( f : M \to N \) is said to be transverse to a sub manifold \( S \subset N \) when for every \( x \in f^{-1}(S) \) the tangent space \( T_{f(x)}N \) is spanned by the subspaces \( T_{f(x)}S \) and \( df_x(T_xM) \).

Theorem 1.2. If \( f : M \to N \) is a smooth map which is transverse to an embedded sub manifold \( S \subset N \) then \( f^{-1}(S) \) is an embedded sub manifold of \( M \) with codimension in \( M \) equal to that of \( S \) in \( N \). In particular if \( S = \{y\} \) for a regular value \( y \) of \( f \) then \( f^{-1}(S) \) is an embedded sub manifold of \( M \) with codimension equal to the dimension of \( N \).

Regular level sets provide a wealth of examples of embedded sub manifolds and we might hope that by considering certain functions on a symplectic manifold \( (M,\omega) \) the regular levels with inherit a natural symplectic structure. If \( (M,\omega) \) comes with a Hamiltonian \( G \)-action then the moment map is a good candidate to study the regular levels of. However because the moment map is constant on the orbits it follows that the restriction of \( \omega \) to a regular level will be degenerate along any direction which is also tangent to the orbit,

\[
i_{X^\xi}i^*\omega = -i^*(d\mu_X) \equiv 0 \quad \forall X^\xi \in T_xG : x \bigcap T_x\mu^{-1}(\xi).
\]

On each tangent space \( T_x\mu^{-1}(\xi) \) we can remove the degeneracies of the linear form \( i^*\omega_x \) by quotienting out the infinitesimal symmetries. Then we hope that \( i^*\omega_x \) will push forward to a non degenerate linear form on this quotient space. As the infinitesimal symmetries are the tangent vectors to the \( G \)-orbits we can hope that the action of \( G \) will extend this fibrewise notion to a global. To pursue this approach we must first recall some definitions and results surrounding principal bundles.[9]

Definition 1.3. An action \( G \xrightarrow{\psi} \text{Diff}(M) \) by a Lie group \( G \) on a smooth manifold \( M \) is said to be free if the stabilizer of every point is trivial and is said to be locally free if the stabilizer of every point is finite. The action is said to be proper when the map \( G \times M \to M \times M \) by \( (g,x) \mapsto (\psi_g(x),x) \) is a proper map. Note that if \( G \) is assumed to be compact then the action is automatically proper.
\textbf{Definition 1.4} (Fibre Bundles). A fibre bundle is a map \( p : M \to B \) of smooth manifolds which is locally a projection, that is there is an open covering of \( B \) by sets \( U_i \) and diffeomorphisms \( \phi_i : p^{-1}(U_i) \to U_i \times E \) so that \( p : p^{-1}(U_i) \to U_i \) is the composition of \( \phi_i \) with the projection onto the first factor. The spaces \( M \), \( B \), and \( E \) are called the total space, base space, and typical fibre of the fibre bundle \( (p : M \to B) \) respectively and we refer to the preimages of points \( p^{-1}(y) \) as a the fibres. On any fibre bundle there is a natural short exact sequence of vector bundles obtained by differentiating the map \( p \), the kernel of \(Tp\) is the canonically defined vertical bundle which we denote by \( VM = \ker Tp \),

\[ 0 \longrightarrow VM \longrightarrow TM \xrightarrow{Tp} TB \longrightarrow 0. \]

\textbf{Definition 1.5} (Principle Bundles). A principle \( G \)-bundle with structure group \( G \) is a fibre bundle \( p : M \to B \) with a right action of \( G \) on the total space which is also free and transitive on the fibres, that is

\[ \forall x \in M, g \in G \quad g \cdot x \in p^{-1}(p(x)) \]
\[ \forall x \in M \quad g \cdot x = x \implies g = 1 \]
\[ \forall x, y \in M \quad p(x) = p(y) \implies \exists g \in G \quad g \cdot x = y. \]

The fibres of the bundle are therefore diffeomorphic to \( G \) and so the base \( B \) may be identified with the orbit space \( M/G \).

\textbf{Proposition 1.6.} If a Lie group \( G \) acts freely and properly on a manifold \( M \) then the orbit space \( M/G \) is a smooth manifold and the orbit map \( p : M \to M/G \) is a principle \( G \)-bundle.

\textbf{Definition 1.7} (Horizontal and Basic forms). A horizontal form on a fibre bundle is a differential form on the total space such that the contraction with any vertical vector field vanishes. A basic form on a principle bundle is an invariant horizontal form.

\[ \Omega^k_{\text{hor}}(p : M \to B) = \left\{ \alpha \in \Omega^k(M) \big| \iota_X \alpha \equiv 0 \quad \forall X \in VM \right\} \]
\[ \Omega^k_{\text{bas}}(p : M \to B, G) = \left\{ \alpha \in \Omega^k_{\text{hor}}(p : M \to B) \big| \psi^*_g \alpha = \alpha \quad \forall g \in G \right\}. \]

The reason for considering basic forms on principal bundles is that they are precisely those forms on the total space that can be pushed forward to the base.

\textbf{Proposition 1.8.} Every basic \( k \)-form \( \alpha \in \Omega^k_{\text{bas}}(P) \) on a principal \( G \)-bundle \( p : M \to B \) determines a unique \( k \)-form \( p_* \alpha \in \Omega^k(B) \) which is a push forward of \( \alpha \) in the sense that \( p^* p_* \alpha = \alpha \). Moreover if \( \alpha \) is closed then so is the push forward \( p_* \alpha \).

We now have all the tools to make precise our vague description of quotienting regular levels to obtain a symplectic structure. Before doing so in generality we exhibit the basic idea through the example of the usual \( U(1) \) action on \( \mathbb{C}^{n+1} \).

\textbf{Example 1.9} (Fubini Study structure on Complex Projective Space). Consider the complex vector space \( \mathbb{C}^{n+1} \) with coordinates \( (z_0, \ldots, z_n) \) along with the standard symplectic structure and the usual Hamiltonian action of \( U(1) \) with associated vector field \( \partial_\theta \) and moment map \( \mu \).

\[ \omega = \frac{1}{2i} \sum_{j=0}^n dz_j \wedge d\bar{z}_j \quad \partial_\theta = \sum_{j=0}^n i(z_j \partial_{\bar{z}_j} - \bar{z}_j \partial_{z_j}) \quad \mu(z) = \frac{1}{2} \|z\|^2. \]

Any non-zero level of \( \mu \) is a codimension 1 sphere and we denote by \( i : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1} \) the inclusion of the unit sphere \( \mu^{-1}(1/2) \). We can represent the restriction of \( \omega \) to \( S^{2n+1} \) explicitly as a 2-form which vanishes on the normal bundle \( NS^{2n+1} \) and agrees with \( \omega \) on vectors tangent to \( S^{2n+1} \); to this end consider the 2-form \( \omega + d\mu \wedge d\theta \). Showing that
this 2-form vanishes on the normal bundle amounts to checking that it vanishes on global Euler vector field \( \partial_e \) which generates \( NS^{2n+1} \),

\[
\iota_{\partial_e}d\mu \equiv 1 \quad \Rightarrow \quad \partial_e \omega = \sum_{j=0}^{n} (z_j \partial_{z_j} + \bar{z}_j \partial_{\bar{z}_j}) \quad \iota_{\partial_e} \omega = \sum_{j=0}^{n} \frac{z_j \partial \bar{z}_j - \bar{z}_j \partial z_j}{2i} = -\|z\|^2 d\theta = -d\theta.
\]

Since \( d\theta \) vanishes on \( \partial_e \) it follows that \( \iota_{\partial_e}(d\mu \wedge d\theta) = -d\theta = -\iota_{\partial_e} \omega \) and so \( \iota_{\partial_e}(\omega + d\mu \wedge d\theta) = 0 \) as desired. On the other hand \( \omega + d\mu \wedge d\theta \) agrees with \( \omega \) whenever contracted with a vector field tangent to \( S^{2n+1} \). Indeed \( \iota_{\partial_e} \omega = \iota_{X^*} \omega = -d\mu X = -Xd\mu = -d\theta(\partial_e) d\mu = d\mu \wedge d\theta \). Locally complete \( \partial_e, \partial_\theta \) to a basis so that the \( d\mu \wedge d\theta \) vanishes for tangent vectors orthogonal to the orbits.

The 2-form \( \omega + d\mu \wedge d\theta \) is certainly closed but fails to be symplectic when restricted to \( S^{2n+1} \) since \( \partial_\theta \) is tangent to \( S^{2n+1} \) but as we have already seen \( \iota_{\partial_\theta}(\omega + d\mu \wedge d\theta) = 2d\mu \) and \( d\mu \) vanishes on \( TS^{2n+1} \) by definition. This makes \( \omega + d\mu \wedge d\theta \) a horizontal form for the principle \( U(1) \)-bundle \( p : S^{2n+1} \to \mathbb{C}P^n \), and since both \( \omega \) and \( d\mu \wedge d\theta \) are \( U(1) \)-invariant \( \omega + d\mu \wedge d\theta \) is in fact basic. We therefore have a well defined push forward of \( \omega + d\mu \wedge d\theta \) to the base \( \mathbb{C}P^n \) which we denote by \( \omega^{FS} \).

The closed 2-form \( \omega^{FS} \) is in fact symplectic since by definition we have \( \iota_{p_*X} \omega^{FS} = \iota_X \omega + d\mu \wedge d\theta \) and the latter vanishes if and only if \( X \) is a multiple of \( \partial_\theta \) in which case \( p_*X \) is zero. The resulting symplectic structure \( (\mathbb{C}P^n, \omega^{FS}) \) is called the Fubini-Study structure on complex projective space and realizing it as above allows us to express it in coordinates as below

\[
p^* \omega^{FS} = \frac{1}{2 \|z\|^2} (\omega + d\mu \wedge d\theta) = \frac{1}{2 \|z\|^2} \left( \sum_{j=0}^{n} d z_j \wedge d \bar{z}_j \right) - \frac{1}{4 \|z\|^4} \left( \sum_{j=0}^{n} z_j \partial \bar{z}_j + \bar{z}_j \partial z_j \right) \wedge \left( \sum_{k=0}^{n} \bar{z}_k d z_k - z_k d \bar{z}_k \right) \wedge \left( \sum_{k=0}^{n} \bar{z}_k d z_k - z_k d \bar{z}_k \right) = \frac{1}{\|z\|^4} \sum_{k=0}^{n} \sum_{j \neq k} \left( z_j \bar{z}_j d z_k \wedge d \bar{z}_k - z_j \bar{z}_k d z_j \wedge d \bar{z}_k \right).
\]

### 1.2 Symplectic Quotients

For this section let us fix a connected symplectic manifold \( (M, \omega) \) equipped with a Hamiltonian action of a compact and connected Lie group \( G \) having moment map \( \mu \). As in our example in the previous section we will want to consider the restriction of \( \omega \) to a regular level of \( \mu \) and then remove the degeneracies by pushing forward to the space of \( G\) orbits.

The latter step is essential a fibrewise notion and it will be useful to isolate the linear version of this process after recalling some basic definitions from linear symplectic spaces.

**Definition 1.10.** Let \( (V, \Omega) \) be a symplectic vector space and \( F \subset V \) a subspace. The *symplectic orthogonal* of \( F \) is the subspace

\[
F^\Omega = \{ v \in V : \Omega(v, u) = 0 \ \forall u \in F \}.
\]

The subspace \( F \) is said to be *isotropic* if \( F \subset F^\Omega \) and *coisotropic* if \( F^\Omega \subset F \). Lastly we say that \( F \) is a *Lagrangian* subspace if it is both isotropic and coisotropic, ie if it is equal to its symplectic orthogonal \( F = F^\Omega \).

**Lemma 1.11 (Linear Reduction).** Suppose \( F \) is a subspace of a symplectic vector space \( (V, \Omega) \) with inclusion \( I : F \to V \) and denote by \( P \) the projection of \( F \) onto the quotient \( F/ (F \cap F^\Omega) \). Then there exists a unique linear symplectic form \( \Omega^F \) on \( F \) satisfying \( P^* \Omega^F = I^* \Omega \).

**Proof.** The pullback relation forces us to define \( \Omega^F \) as follows and in particular uniqueness is immediate,

\[
\Omega^F([u], [v]) = \Omega(u, v) \quad u, v \in F.
\]
We can check that $\Omega^F$ is well defined one argument at a time in which case this follows from the universal property of the kernel since $\Omega^F$ is defined on equivalence classes modulo $\ker i^\ast \Omega = F^\Omega \cap F$. It is clear that $\Omega^F$ is bilinear and that $\Omega^F$ is symplectic is again a result of the universal property of the kernel but we can also see this explicitly; for any $[u] \in F$ we have
\[
\Omega^F([u], F) = 0 \Rightarrow \Omega(u, F) = 0 \Rightarrow u \in F^\Omega \Rightarrow [u] = 0.
\]

Unlike in our example it possible that the action of $G$ may not preserve the levels of $\mu$ so to obtain our principle bundle we therefore consider the action of subgroups which do preserve $\mu^{-1}(\xi)$. Equivariance of $\mu$ implies that those elements of $G$ which do preserve $\mu^{-1}(\xi)$ are precisely those that fix $\xi$ under the coadjoint action. So the largest possible subgroup which acts on the level $\mu^{-1}(\xi)$ is the coadjoint stabilizer $G_\xi$ of $\xi$. This means that we can at best remove the degenerate directions which are tangent to the orbit of $G_\xi$. Thankfully the following lemma ensures that this will be enough to recover a non-degenerate form from $i^\ast \omega$.

**Lemma 1.12.** For any point $x \in \mu^{-1}(\xi)$ the symplectic orthogonal of the tangent space to $\mu^{-1}(\xi)$ at any $x \in \mu^{-1}(\xi)$ is the tangent space to the entire $G$-orbit through $x$. Moreover the intersection of these subspaces is $T_x \mu^{-1}(\xi)$ is precisely the tangent space to the $G_\xi$ orbit through $x$

\[
(T_x \mu^{-1}(\xi))^\omega = T_x G \cdot x \quad T_x \mu^{-1}(\xi) \cap T_x G \cdot x = T_x G_\xi \cdot x.
\]

It follows from Lemma 11 that there is a family of uniquely determined (linear) symplectic forms $\omega^\xi_x$ on the quotients $T_x \mu^{-1}(\xi)/T_x (G_\xi \cdot x)$.

**Proof.** The statement of the lemma amounts to the following equalities;
\[
\ker(i^\ast \omega_x) = T_x \mu^{-1}(\xi) \cap (T_x \mu^{-1}(\xi))^\omega_x
\]
\[
= T_x \mu^{-1}(\xi) \cap T_x (G \cdot x)
\]
\[
= T_x (G_\xi \cdot x).
\]

The first is straight from the definition and the second can be verified by showing that $\ker T_x \mu = (T_x G \cdot x)^\omega_x$. Indeed since $\mu$ is a moment map we have for any $Y_x \in T_x M$

\[
\langle (T_x \mu) Y_x, X \rangle = \iota_Y d\mu_x^X = \omega_x (X^\xi_x, Y_x) \quad \forall X \in \mathfrak{g}.
\]

So $(T_x \mu) Y_x$ vanishes if and only if $Y_x$ is $\omega_x$-orthogonal to the span of the $X^\xi_x$, which is to say that $T_x \mu^{-1}(\xi) = (T_x G \cdot x)^\omega_x$ and taking the symplectic orthogonal of both sides yields the equality. The last equality can be checked by writing out the intersection in terms of the infinitesimal action and using equivariance of $T_x \mu$,

\[
T_x \mu^{-1}(\xi) \cap T_x (G \cdot x) = \{ X^\xi_x \mid 0 = T_x \mu (X^\xi_x) = \text{ad}_{X}^\ast \xi \}
\]
\[
= \{ X^\xi_x \mid \text{ad}_{\exp X}^\ast \xi = \xi \}
\]
\[
= \{ X^\xi_x \mid X \in \mathfrak{g}_\xi \}.
\]

Now we are ready to present the main result of this chapter.

**Theorem 1.13 (Symplectic Reduction at a regular level).** Let $(M, \omega)$ be a connected symplectic manifold endowed with a Hamiltonian action of a connected Lie group $G$ having moment map $\mu$. For a point $\xi \in \mathfrak{g}^\ast$ let $i : \mu^{-1}(\xi) \rightarrow M$ denote the inclusion, $M^\xi$ the orbit space $\mu^{-1}(\xi)/G_\xi$ with corresponding projection $p : \mu^{-1}(\xi) \rightarrow M^\xi$. If $\xi$ is a regular
value of $\mu$ and $G_\xi$ acts freely and properly on $\mu^{-1}(\xi)$ then there is a unique symplectic structure $\omega^\xi$ on $M^\xi$ satisfying $p^*\omega^\xi = i^*\omega$.

Proof. Since $\xi$ is assumed to be a regular value the level $\mu^{-1}(\xi)$ is a smooth sub manifold of $M$ with codimension $k = \dim G$ and requiring the action of $G_\xi$ on $\mu^{-1}(\xi)$ to be free and proper ensures the orbit mapping is a principle $G_\xi$-bundle. Then for any $x \in \mu^{-1}(\xi)$ we have short exact sequences induced by the inclusion and projection

$$0 \longrightarrow T_x(G_\xi \cdot x) \longrightarrow T_x\mu^{-1}(\xi) \xrightarrow{T_x\mu} T_{\mu(x)}M^\xi \longrightarrow 0$$

$$0 \longrightarrow T_x\mu^{-1}(\xi) \xrightarrow{T_xi} T_xM \xrightarrow{T_x\mu^*} T_{\mu(x)}\mathfrak{g}^* \longrightarrow 0.$$

Now the pull back $i^*\omega$ is a smooth closed 2-form on $\mu^{-1}(\xi)$ and by our lemma on linear reduction $(i^*\omega)_x$ vanishes precisely on $T_xG_\xi \cdot x$. So $i^*\omega$ is a closed basic 2-form on $\mu^{-1}(\xi)$ since it vanishes on the vertical bundle and $G_\xi$-equivariance follows from $G$-equivariance of $\omega$ and $i$. Therefore there exists a unique push forward to a closed 2-form $\omega^\xi$ on the base,

$$\omega^\xi \in \Omega^2(M^\xi) \quad p^*\omega^\xi = i^*\omega.$$

This proves the uniqueness claim since any symplectic form on $M^\xi$ satisfying the pullback relation is necessarily the push forward of $i^*\omega$. For existence it remains only to verify that $\omega^\xi$ is in fact symplectic which amounts to verifying non-degeneracy in the fibres. This too follows from the linear reduction lemma since $(\omega^\xi)_{p(x)}$ is push forward of the linear form $(i^*\omega)_x$ to the quotient of $T_x\mu^{-1}(\xi)$ by its kernel. 

\[\square\]

1.3 Reduction at coadjoint orbits and the shifting trick.

It was the equivariance of the moment map led us to consider the action of the coadjoint stabilizer on a regular level instead of the action of the entire group $G$. On the other hand it follows from the equivariance of $\mu$ that we can consider an action of the entire group $G$ if we are willing to enlarge the level $\xi$ to the preimage of its entire coadjoint orbit $O_\xi$. The following lemma shows us that the sufficient conditions to form the symplectic quotient $(M^\xi, \omega^\xi)$ as in the previous section are enough to ensure the action of $G$ on $\mu^{-1}(O_\xi)$ defines a principle $G$-bundle $p_{O_\xi} : \mu^{-1}(O_\xi) \rightarrow M^{O_\xi}$.

Lemma 1.14. If the coadjoint orbit $O_\xi$ contains a single regular value of $\mu$ then every point in $O_\xi$ must be a regular value of $\mu$. In this case the moment map is transverse to the coadjoint orbit and hence $\mu^{-1}(O_\xi)$ is a smooth sub manifold of $M$ of codimension equal to the codimension of $G_\xi$ in $G$.

Proof. Assume a point $\xi \in O_\xi$ is a regular value of $\mu$ so that $T_x\mu$ is surjective for all $x \in \mu^{-1}(\xi)$. Recall that the image of $T_x\mu$ is given by the annihilator of the Lie algebra of the stabilizer of $x$ so that for $x \in \mu^{-1}(\xi)$ the stabilizer must be discrete

$$\text{Ann}(\mathfrak{g}_x) = \text{Im}T_x\mu = \mathfrak{g}^* \quad \Leftrightarrow \quad \mathfrak{g}_x = \{0\}.$$ 

Now for any $\eta \in O_\xi$ we have some $g \in G$ with $\eta = \text{Ad}_g^*\xi$. Then the stabilizer for any point $y \in \mu^{-1}(\eta)$ is conjugate to the stabilizer of $g^{-1} \cdot y \in \mu^{-1}(\xi)$ which we know to be discrete. Therefore the stabilizers $\text{Stab}(y)$ are discrete for every $y \in \mu^{-1}(\eta)$ and so their Lie algebras are trivial in $\mathfrak{g}$. It follows that $T_y\mu$ is surjective for ever $y \in \mu^{-1}(\eta)$ and hence $\eta$ is also a regular value. Since $\eta$ was arbitrary we conclude that the entire coadjoint orbit must consist of regular values of the moment map. 

\[\square\]

Lemma 1.15. The action of $G_\xi$ on $\mu^{-1}(\xi)$ is free if and only if the action of $G$ on $\mu^{-1}(O_\xi)$ is free.

Proof. Given $x \in \mu^{-1}(O_\xi)$ let $\eta = \mu(x)$ and take any $g \in G$ so that $\eta = \text{Ad}_g^*\xi$. If some $h \in G$ fixes $x$ then by applying
we see right away that \( h \) must belong to the coadjoint stabilizer of \( \eta \) which is necessarily conjugate to that of \( \xi \)

\[
\eta = \mu(x) = \mu(h \cdot x) = \text{Ad}_h^* \eta \quad h \in G \ni gG\xi g^{-1}.
\]

So we may write \( h = gh_0g^{-1} \) for some \( h_0 \in G\xi \) which must fix \( g^{-1} \cdot x \). Now from the equivariance of \( \mu \) we see that \( g^{-1} \cdot x \in \mu^{-1}(\xi) \) and because the action of \( G\xi \) was assume to be free it follows that \( h_0 \) and therefore also \( h \) must be the identity element in \( G \).

So under the assumptions of theorem 13 the action of \( G \) on the preimage of the coadjoint orbit defines a principle \( G \)-bundle which we denote along with the inclusion of the preimage as below

\[
i_{O_\xi} : \mu^{-1}(O_\xi) \to M \quad p_{O_\xi} : \mu^{-1}(O_\xi) \to M^{O_\xi}.
\]

We might hope that \( i_{O_\xi}^* \omega \) is a basic form so that we may push it forward to a symplectic form \( M^{O_\xi} \) however this is not the case since it fails to vanish on the vertical bundle. Indeed for any point \( x \in \mu^{-1}(O_\xi) \) and vertical vector \( w \in \ker T_xp_{O_\xi} \) we want to consider \( \omega_x(w,v) \) for an arbitrary \( v \) tangent to \( \mu^{-1}(O_\xi) \) at \( x \). The tangent vectors \( v \) and \( w \) are described as follows

\[
T_x\mu^{-1}(O_\xi) = (T_x\mu)^{-1}(T_x(O_\xi))
\]

\[
= (T_x\mu)^{-1} \left\{ X^{\text{Ad}^*} = \text{ad}_{X}^* \eta \quad X \in \mathfrak{g} \right\}
\]

\[
= \{ v \in T_xM \mid \exists X_v \in \mathfrak{g} \quad T_x\mu(v) = \text{ad}_{X_v}^* \mu(x) \}
\]

\[
\ker T_xp_{O_\xi} = T_x(G \cdot x)
\]

\[
= \{ w \in T_xM \mid \exists X_w \in \mathfrak{g} \quad w = (X_w)^2 \}.
\]

We see then that while \( i_w \omega_x \) need not vanish on \( T_x\mu^{-1}(O_\xi) \) it is controlled by the canonical 2-form on \( \omega^{KKS} \in \Omega^2(\mathfrak{g}^*) \),

\[
\omega_x(w,v) = -\langle T_x\mu(v) \mid X_w \rangle
\]

\[
= -\langle \text{ad}_{X_v}^* \eta \mid X_w \rangle
\]

\[
= \langle \eta \mid \text{ad}_{X_v}X_w \rangle
\]

\[
= \langle \eta \mid [X_v,X_w] \rangle
\]

\[
= \omega^{KKS}_\eta(\text{ad}_{X_v}^* \eta, \text{ad}_{X_w}^* \eta)
\]

\[
= \langle \mu^*\omega^{KKS} \rangle_x(v,w).
\]

Now \( \mu^*\omega^{KKS} \) is \( G \)-invariant as \( \mu \) is equivariant and \( \omega^{KKS} \) is invariant under the coadjoint action. Therefore the difference \( \omega - \mu^*\omega^{KKS} \) is a closed invariant 2-form and we have shown that by restricting to \( \mu^{-1}(O_\xi) \) we obtain a basic 2-form \( i_{O_\xi}^*(\omega - \mu^*\omega^{KKS}) \). Define \( \omega^{O_\xi} \) to be the push forward of this basic 2-form to \( M^{O_\xi} \) so that

\[
p_{O_\xi}^*\omega^{O_\xi} = i_{O_\xi}^*(\omega - \mu^*\omega^{KKS})
\]

The push forward \( \omega^{O_\xi} \) vanishes only on \( (p_{O_\xi})_*X \) for which \( i_X \omega = i_X \mu^*\omega^{KKS} \) and this holds only for those \( X \) tangent to the \( G \)-orbits, that is, when \( X \) is a vertical vector field and hence \( (p_{O_\xi})_*X = 0 \). Thus \( \omega^{O_\xi} \) is non-degenerate and therefore defines a symplectic structure \( (M^{O_\xi},\omega^{O_\xi}) \) which we could also call a symplectic quotient. There is a natural diffeomorphism from \( M^{O_\xi} \) onto \( M^\xi \) and we should hope that this is also an isomorphism of the symplectic structures. Before investigating this we construct a third possible candidate for the symplectic quotient.

Consider \( M \times O_\xi \) with the skewed-product symplectic structure \( \nu = \omega \oplus (-\omega^{KKS}) \) so that the component wise action of \( G \) is Hamiltonian with moment map \( \Phi(x,\eta) = \mu(x) - \eta \). Then \( \Phi^{-1}(0) = \{ \mu^{-1}(\eta) \mid \eta \in O_\xi \} \) is equivariantly
diffeomorphic to $\mu^{-1}(O_\xi)$; we refer to $\Phi$ as the shifted moment map since it is effectively shifting $\mu$ so as to vanish on $\mu^{-1}(O_\xi)$. The conditions to form the symplectic quotient $(M^\xi,\omega^\xi)$ as in theorem 13 are sufficient to form the symplectic quotient at the zero level of $\Phi$ as the following lemma shows.

**Lemma 1.16.** If $\xi$ is a regular value for $\mu$ and $G_\xi$ acts freely and properly on $\mu^{-1}(\xi)$ then zero is a regular value for $\Phi$ and $G$ acts freely and properly on $\Phi^{-1}(0)$.

**Proof.** If $G_\xi$ acts freely on $\mu^{-1}(\xi)$ then $G_\eta$ must act freely on the level $\mu^{-1}(\eta)$ for any $\eta \in O_\xi$ since the stabilizers are conjugate. Now for any point $p \in \Phi^{-1}(0)$ we may write $p = (x, \mu(x))$ for $\eta = \mu(x) \in O_\xi$. Suppose some $g \in G$ fixes $p$ so that

$$(g \cdot x, \text{Ad}_g^* \eta) = g \cdot p = p = (x, \eta).$$

It follows from equality in the second component that $g \in G_\eta$ but then since the action of $G_\eta$ on $\mu^{-1}(\eta)$ is free the equality in the second component must force $g$ to be the identity element. We conclude that $G$ acts freely on $\Phi^{-1}(0)$ and zero is therefore also a regular value of $\Phi$.

So under the hypothesis of theorem 13 we have three candidates for the symplectic quotient to consider;

$$(M^\xi,\omega^\xi) \quad (M^{O_\xi},\omega^{O_\xi}) \quad ((M \times O_\xi)^0,\nu^0).$$

and the following proposition ensures that these are all naturally isomorphic to one another.

**Proposition 1.17.** There are canonical symplectomorphisms between the three symplectic quotients constructed above.

**Proof.** There natural inclusion of $\mu^{-1}(\xi)$ into $\mu^{-1}(O_\xi)$ descends to a smooth map $\phi : M^\xi \to M^{O_\xi}$. The graph the moment map provides an equivariant diffeomorphism of $\mu^{-1}(O_\xi)$ onto $\Phi^{-1}(0)$. Consider the graph of $\mu$ when restricted to $\mu^{-1}(O_\xi)$ and denote it by $\text{Gr}(i_{O_\xi}^* \mu) = i_{O_\xi}^* (\text{Id}_M \times \mu)$. This is necessarily an injection since it is injective in the first component and equivariant since both the inclusion and moment map are. Therefore $\text{Gr}(i_{O_\xi}^* \mu)$ factors through an equivariant diffeomorphism $\gamma_{O_\xi}$ onto its image which we see below is the zero level of the shifted moment map $\Phi$,

$$\text{Im} \left( \text{Gr}(i_{O_\xi}^* \mu) \right) = \{(x, \mu(x)) \in M \times O_\xi \mid x \in \mu^{-1}(O_\xi) \} = \{(x, \eta) \in M \times O_\xi \mid \mu(x) = \eta \in O_\xi \} = \Phi^{-1}(0) \quad \Rightarrow \quad \text{Gr}(i_{O_\xi}^* \mu) = i_0 \circ \gamma_{O_\xi}$$

As an equivariant diffeomorphism between principle $G$-bundles $\gamma_{O_\xi}$ descends to a diffeomorphism $\psi$ on the orbit spaces. Similarly the natural inclusion on $\mu^{-1}(\xi)$ into $\mu^{-1}(O_\xi)$ descends to an injective immersion $\phi$ on the orbit spaces. All of this is summarized in the following commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{i_{O_\xi}} & M \times O_\xi \\
\downarrow{\mu^{-1}(\xi)} & \searrow{\gamma_{O_\xi}} & \nearrow{\Phi^{-1}(0)} \\
M^\xi & \xrightarrow{\phi} & M^{O_\xi} & \xrightarrow{\psi} & (M \times O_\xi)^0, \\
\end{array}$$

It is straight forward to verify that $\phi$ is surjective and hence a diffeomorphism. To see that $\phi$ is a symplectomorphism...
we establish the following
\[ p_\xi^* \phi^* \omega_O = j_\xi^* p_O^* \omega \]
\[ = j_\xi^* (\omega - \mu^* \omega_{KKS}) \]
\[ = i_\xi^* \omega - (\mu \circ i_\xi)^* \omega_{KKS} \]

The first equality follows from commutativity of the diagram, the second uses the push forward property defining \( \omega_O \) and the third is just a composition of inclusions. Moreover the pull back \((\mu \circ i_\xi)^* \omega_{KKS}\) must vanish because \( \mu \) is constant on \( \mu^{-1}(\xi) \). Hence the pull back \( \phi^* \omega_O \) satisfies the universal property defining \( \omega^\xi \) and so by uniqueness they must coincide
\[ p_\xi^* \phi^* \omega_O = i_\xi^* \omega \Rightarrow \phi^* \omega_O = \omega^\xi. \]

We similarly verify that \( \psi \) is a symplectomorphism by showing that \( \psi^* \nu^0 \) satisfies the universal property defining \( \omega_O \). This is established by the following string of equalities using commutativity of the diagram, the factorization \( Gr(i_O^* \mu) = i_0 \circ \gamma_\xi \) and the definition of \( \nu \),
\[ p_{O\xi}^* \psi^* \nu^0 = \gamma_\xi^* p_0^* \nu^0 \]
\[ = \gamma_\xi^* i_0 \nu \]
\[ = (i_O \oplus \mu \circ i_O)^* (\omega \oplus -\omega_{KKS}) \]
\[ = i_{O\xi}^* \omega - (\mu \circ i_{O\xi})^* \omega_{KKS} \]
\[ = i_{O\xi}^* (\omega - \mu^* \omega_{KKS}). \]

**Corollary 1.18.** For any two points \( \xi \) and \( \eta \) in the same coadjoint orbit there is a natural isomorphism of the symplectic quotients \( (M^{O\xi}, \omega^\xi) \cong (M^{O\eta}, \omega^\eta) \).

**Proof.** Both are isomorphic to \( (M^{O\xi}=O_\eta, \omega^{O\xi}=\omega_\eta). \)

**Remark.** Whenever we can form the symplectic quotient \( \mu^{-1}(\xi)/G_\xi \) we may choose instead to work with the symplectic quotient at the zero level of the shifted moment map \( \Phi \). We may always assume that our symplectic quotients are formed at the zero level of the moment map and in particular are \( G \)-orbit spaces. Unless specified otherwise the symplectic quotients in the rest of the section are assumed to be taken at the zero level of the given moment map.

### 1.4 Reduction In Stages

Assume that in addition to the Hamiltonian action of a connected compact Lie group \( G \) on the connected symplectic manifold \((M, \omega)\) with moment map \( \mu_G \) we have a Lie subgroup \( H \) of \( G \). Then there is an induced Hamiltonian action of \( H \) on \( M \) with moment map given by projecting \( \mu_G \) onto \( \mathfrak{h}^* \). If we form the symplectic quotient at a regular level of this projected moment map it may carry some residual symmetry from the original \( G \)-action.

Consider the natural short exact sequence associated with the inclusion \( j : H \to G \) of a normal Lie subgroup \( H \) of \( G \). Differentiating and then dualizing this gives two more short exact sequences as below,
The action of $H$ on $M$ has moment map $\mu_H = J^* \circ \mu_G$ and if we assume that zero is a regular level for $\mu_H$ on which $H$ acts freely then we may form the symplectic quotient which we will denote by $(M_H = \mu_H^{-1}(0))/H, \omega^H)$. The kernel of $J^*$ consist of those $\xi \in g^*$ such that for every $X \in \mathfrak{h}$ the pairing $\langle J^*\xi, X \rangle = \langle \xi, JX \rangle$ vanishes. Identifying $\mathfrak{h}$ as a subspace of $g$ via the linear map $J$ this means that $\ker J^* = \text{Ann}(\mathfrak{h})$. Now conjugation by $G$ preserves the normal subgroup $H$ and therefore the adjoint action of $G$ preserves the Lie sub algebra $\mathfrak{h} \subset \mathfrak{g}$. If follows that $G$ preserves the annihilator of $\mathfrak{h}$ and therefore also the zero level of $\mu_H$

$$\mu_{H}^{-1}(0) = \mu_G^{-1}(\ker J^*) = \mu_G^{-1}(\text{Ann}\mathfrak{h}).$$

Normality of $H$ implies furthermore that $G$ preserves the $H$-orbits and therefore acts on the orbit space $M_H$ in such a way that $\rho_H$ is equivariant. Finally this action on $M_H$ clearly reduces to an action of the quotient group $G/H$ which we will see is Hamiltonian.

**Proposition 1.19.** Let $M_H$ be the symplectic quotient with principle $H$-bundle denoted by $p_H : \mu_H^{-1}(0) \to M_H$ and the inclusion of the zero level by $i_H : \mu_H^{-1}(0) \to M$. There is a natural Hamiltonian action of the quotient group $G/H$ on $M_H$ with moment map $\mu_{G/H}$ satisfying $Q^* \circ (p_H^*\mu_{G/H}) = i_H^*\mu_G$.

**Remark 1.20.** If we use $Q^*$ to identify $(\mathfrak{g}/\mathfrak{h})^*$ with its image in $\mathfrak{g}^*$ then the relation between the moment maps becomes the same as the relation between the symplectic form on $M$ and the reduced form $\omega^0$,

$$p_H^*\mu_{G/H} = i_H^*\mu_G \leftrightarrow p_H^*\omega^\xi = i_H^*\omega.$$

**Proof.** Having already seen that the quotient $G/H$ acts on the orbit space $M_H$ is remains to verify that this action is Hamiltonian and the moment map verifies the given relation. Because $H$ acts trivially on $\text{Ann}\mathfrak{h}$ the equivariance of $\mu_G$ makes the restriction of $\mu_G$ to $\mu_H^{-1}(0)$ constant on $H$-orbits and it therefore descends to a smooth map from the orbit space $M_H \to \text{Ann}\mathfrak{h}$. Furthermore since $\text{Ann} \mathfrak{h} = \ker J^* = \text{Im} Q^*$ this map factors uniquely through $Q^*$. We summarize this in the diagram below,

$$\begin{array}{ccc}
0 \longrightarrow (\mathfrak{g}/\mathfrak{h})^* & \xrightarrow{Q^*} & \mathfrak{g}^* \\
& \searrow & \downarrow i_H^*\mu_G \\
& & \mu_{H}^{-1}(0) \\
\exists \mu_{G/H} & \xrightarrow{\exists \mu_{G/H}} & \mathfrak{g}/\mathfrak{h} \\
M_H & \xleftarrow{p_H} & \mu_{H}^{-1}(0) \\
\end{array}$$

It follows from equivariance of $\mu_G$ that $\mu_{G/H}$ is equivariant with respect to the action of $G/H$. Indeed let $gH \in G/H$ and $H \cdot x \in M_H$, to show $\mu_{G/H}(gH \cdot (H \cdot x)) = \text{Ad}^*_H\mu_{G/H}(H \cdot x)$ it suffices to check this equality holds when paired with any element of $\mathfrak{g}/\mathfrak{h}$ which may be written as $QX$ for some $X \in \mathfrak{g}$. This is verified by the following string
equalities where we use the commutativity of the diagram above and the equivariance of both $\mu_G$ and $Q$

$$\langle \mu_{G/H}(gH \cdot (H \cdot x)), QX \rangle = \langle Q^* \mu_{G/H}(p_H(g \cdot x)), X \rangle$$

$$= \langle \mu_G(g \cdot x), X \rangle$$

$$= \langle \text{Ad}_g^* \mu_G(x), X \rangle$$

$$= \langle \text{Ad}_g^* Q^* \mu_{G/H}(H \cdot x), X \rangle$$

$$= \langle \text{Ad}_g^* \mu_{G/H}(H \cdot x), QX \rangle.$$ 

This establishes equivariance of $\mu_{G/H}$ and it remains only to verify the moment mapping property. We must check that at every point $H \cdot x \in M^H$ and for every $v \in \mathfrak{g}/\mathfrak{h}$ and $w \in T_{H \cdot x} M^H$ we have

$$t_{v \cdot w}^H_{\mathfrak{g}/\mathfrak{h}}(w) = -\langle T_{H \cdot x} \mu_{G/H}(w), v \rangle.$$ 

Choose $X \in \mathfrak{g}$ and $Y \in T_x \mu_{G/H}^{-1}(0)$ so that $QX = v$ and $T_x p_H(Y) = w$. Using the fact that the projection $p_H$ pushes forward the fundamental vector fields $(QX)_x^H = T_x p_H(X_x^H)$, the commutativity of the diagram, and the moment mapping property for $\mu_G$ we arrive at the desired equality

$$t_{v \cdot w}^H_{\mathfrak{g}/\mathfrak{h}}(w) = \omega_{p(x)}^H(T_x p_H(X_x^H), T_x p_H(Y))$$

$$= (p_H^* \omega_H)_x(X_x^H, Y)$$

$$= (i_H^* \omega)_x(X_x^H, Y)$$

$$= -\langle T_x \mu_G(Y), X \rangle$$

$$= -\langle Q^* T_{p_H(x)} \mu_{G/H} T_x p_H Y, X \rangle$$

$$= -\langle T_{p_H(x)} \mu_{G/H}(w), v \rangle.$$ 

\[\square\]

Reduction by a normal subgroup is simplified in the case of a product group $G = G_1 \times G_2$ with $H = G_1$. In this situation the reduced moment map is essential the moment map for the action of the $G_2$ component.

**Corollary 1.21.** Suppose $(M, \omega)$ carries a Hamiltonian action of $G = G_1 \times G_2$ with moment map $\mu$ and $M^1$ is the symplectic quotient at the zero level of the moment map $\mu_1$ for $G_1$. Then the moment map $\mu_{G/G_1}$ for the action of $G/G_1$ on $M^1$ satisfies $p_{G_1}^* \mu_{G/G_1} = i_{G_1}^* \mu_2$ where $p_1$ and $i_1$ are the $G_1$-bundle and $\mu_1^{-1}(0)$ inclusion maps respectively.

**Proof.** The splitting $G = G_1 \times G_2$ induces the identifications $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$ so that $Q$ is the projection onto the second factor and $Q^*$ is the inclusion $\xi \mapsto (0, \xi)$. Then the composition $QQ^*$ is just the identity on $\mathfrak{g}_2^*$ and $Q\mu = \mu_2$ where $\mu = \mu_1 \oplus \mu_2$. Applying $Q$ to the relation defining the moment map $\mu_{G/G_1}$ for the $G_2 = G_1$ action on $M^1$ produces the result

$$QQ^*(p_{G_1}^* \mu_{G/G_1}) = Q i_{G_1}^* \mu \ \Rightarrow \ p_{G_1}^* \mu_{G/G_1} = i_{G_1}^* \mu_2.$$ 

\[\square\]

If $G$ acts freely on $\mu_2^{-1}(0)$ and $H$ acts freely on $\mu_H^{-1}(0)$ then $G/H$ acts freely on $\mu_{G/H}^{-1}(0)$ and we can form the symplectic quotient $(M^H)^{G/H} = \mu_{G/H}^{-1}(0)/(G/H)$. This space is clearly diffeomorphic to $M^G = \mu_{G}^{-1}(0)/G$ and we should hope that the symplectic structures $(\omega^H)^{G/H}$ and $\omega^G$ are also isomorphic. The following theorem verifies this and effectively allows us to perform symplectic reduction in multiple stages given a tower of normal subgroups.

**Theorem 1.22** (Reduction in stages). Whenever the iterated symplectic quotient $((M^H)^{G/H}, (\omega^H)^{G/H})$ can be formed as above there is a natural symplectomorphism with the symplectic quotient $(M^G, \omega^G)$. 


Proof. From the definition of $\mu_{G/H}$ we see that

$$\mu_{G}^{-1}(0) = (Q^* \circ \mu_{G/H} \circ p_H)^{-1}(0) = p_H^{-1}(\mu_{G/H}^{-1}(0))$$

and so it follows that $\mu_{G/H}^{-1}(0) = p_H(\mu_{G}^{-1}(0)) = \mu_{G}^{-1}(0)/H$. Now consider that the sub-bundle map $p_{G,G/H} : \mu_{G}^{-1}(0) \to \mu_{G/H}^{-1}(0)$ is equivariant with respect to the $G$ action in the sense that $p_{G,G/H}(g \cdot x) = gH \cdot p_{G,G/H}(x)$ and so induces a smooth map $\psi$ on the base spaces. It is straightforward to check that $\psi$ is in fact a symplectomorphism consider the following commutative diagram incorporating the various bundle projections and inclusions,

![Diagram](image)

Using the universal properties defining $\omega^{G/H}$ and $\omega^H$ as well as the commutativity of the above diagram shows that $p_G^*(\psi^*\omega^{G/H}) = i_G^*\omega$ and so uniqueness for the reduced symplectic form $\omega^G$ on $M^G$ implies that $\psi^*\omega^{G/H} = \omega^G$ as desired. 

\[\square\]

1.5 Symplectic Cutting

In this section we introduce a construction due to Eugene Lerman called symplectic cutting.\[6\] This will not only provide some new (and old) examples of symplectic structures but also prove a useful theoretical tool in the following chapters. In particular we will use symplectic cuts in the proof of the Delzant correspondence for toric manifolds. Symplectic cuts are also used to prove a non-abelian version of the convexity theorem appearing in the next chapter.\[7\]

Suppose we have a connected symplectic manifold $(M, \omega)$ is endowed with a Hamiltonian $U(1)$ action having moment map $H : M \to \mathbb{R}$. If we give $M \times \mathbb{C}$ the twisted product symplectic structure then the component wise $U(1)$ action will also be Hamiltonian with moment map given by the difference of $H$ and half the norm squared. Any level set of this map is therefore a disjoint union of two $U(1)$ invariant subsets as described below,

$$\Phi : (x, w) \mapsto H(x) - \frac{1}{2}|w|^2$$

$$\Phi^{-1}(\xi) = \left(\Phi^{-1}(\xi) \times \{0\} \sqcup \bigcup_{r \geq 0} H^{-1}(\xi + r) \times \left\{\frac{1}{2}|w|^2 = r\right\}\right) \cong (H \geq \xi) \times S^1.$$

Any regular value $\xi$ of $H$ is also a regular value for $\Phi$ and whenever $U(1)$ acts freely on $H^{-1}(\xi)$ so to is the action on $\Phi^{-1}(\xi)$. Indeed the action is always free on the interior since it is free on the second component and the action is free on the boundary since it is assumed to be free on the first component. Under these conditions we may therefore form the symplectic quotient at the level $\Phi^{-1}(\xi)$ which we denote by $(M_{\geq \xi}, \omega_{\geq \xi})$. Since the boundary and interior are
invariant the reduced space is diffeomorphic to the disjoint union of the orbit spaces as below

\[
\overline{M}_{\geq \xi} \cong \frac{H^{-1}(\xi)}{U(1)} \bigsqcup \frac{\{H > \xi\} \times S^1}{U(1)} \cong M^\xi \bigsqcup \{H > \xi\}.
\]

**Definition 1.23 (Symplectic Cuts).** We call \((\overline{M}_{\geq \xi}, \omega_{\geq \xi})\) the *symplectic cut* of \(M\) (above \(\xi\) and with respect to \(H\)) since it is obtained by cutting \(M\) along the level \(H^{-1}(\xi)\) and then performing symplectic reduction along the boundary to recover a symplectic manifold. An analogous construction produces the symplectic cut of \(M\) below \(\xi\) and with respect to \(H\) which we will denote by \((\overline{M}_{\leq \xi}, \omega_{\leq \xi})\). These two cut spaces can be glued together along the sub manifolds equivalent to symplectic quotient \(M^\xi\) to recover the original symplectic manifold \((M, \omega)\). [3]

**Proposition 1.24.** Assume now that \((M, \omega)\) carries a Hamiltonian action of some Lie group \(G\) with moment map \(\mu : M \to g^*\) in addition to a \(U(1)\) action with moment map \(H\). If the action of \(G\) commutes with that of \(U(1)\) then the cut spaces will also carry a Hamiltonian \(G\) action.

**Proof.** Extend the action of \(G\) to the product space \(M \times \mathbb{C}\) by letting it act trivially on the second component. This commutes with the \(U(1)\) action on the product and so there is a well defined Hamiltonian action of \(G \times U(1)\) having moment map

\[
\phi : (x, w) \mapsto \mu(x) \oplus \Phi(x, w).
\]

Then our corollary on reduction by a subgroup for product groups implies the natural \(G\) action on \(\overline{M}_{\geq \xi}\) is Hamiltonian with moment map \(\overline{\mu}_{\geq \xi}\) satisfying

\[
p^*\overline{\mu}_{\geq \xi} = i^*\mu.
\]

Under the identification of \(\{H > \xi\}\) with an open subset of \(M_{\geq \xi}\) the moment map \(\overline{\mu}_{\geq \xi}\) agrees with the restriction of \(\mu\) to \(\{H > \xi\}\). On the subset isomorphic to \(M^\xi\) agrees with \(\mu\) (\(\mu\) descends to \(M^\xi\) since it is necessarily constant on \(U(1)\) orbits).
Chapter 2

Convexity

2.1 Introduction

In this chapter we consider the Hamiltonian action of torus action on a compact symplectic manifold. The image of a moment map for such a Hamiltonian system has some very nice properties, in particular it is convex. The main goal of this chapter is to establish the classical convexity result for the moment map. Then we establish some relations between the structure of the image of the moment map and the Hamiltonian action on $M$. We have borrowed many of the results and techniques in the chapter from the books by Audin [2] and McDuff-Salamon [3].

Let us begin with a familiar example.

Example 2.1. Using reduction in stages we obtained a Hamiltonian action of an $n$-torus $T = U(1)^n/S^1$ (where $S^1$ was the diagonal circle group) on the complex projective space $\mathbb{C}P^n$ and the resulting moment map is given in homogeneous coordinates as

$$
\mu : [z_0 : \cdots : z_n] \mapsto \frac{1}{2} \left( \frac{|z_0|^2, \cdots, |z_n|^2}{\sum_{j=0}^{n} |z_j|^2} \right). 
$$

The image of $\mu$ is the intersection of an affine hyperplane with the positive orthant,

$$
\mu(\mathbb{C}P^n) = H \cap \mathbb{R}_{\geq 0}^{n+1} \quad H = \left\{ \xi \in \mathbb{R}^{n+1} \mid \langle \xi, (1, \cdots, 1) \rangle = \frac{1}{2} \right\}.
$$

Observe that the image of $\mu$ is the intersection of an affine hyperplane with the positive orthant,

$$
\mu(\mathbb{C}P^n) = H \cap \mathbb{R}_{\geq 0}^{n+1} \quad H = \left\{ \xi \in \mathbb{R}^{n+1} \mid \langle \xi, (1, \cdots, 1) \rangle = \frac{1}{2} \right\}.
$$

This intersection is an $n$-simplex the vertices of which correspond precisely to the images of the fixed points of the action

$$
\mu(\mathbb{C}P^n) = \Delta^n \quad \mu : [0 : \cdots : z_j : \cdots : 0] \mapsto \frac{1}{2}e_j.
$$

Similarly the edges of the simplex are obtained as the image under $\mu$ of the 2-dimensional sub manifolds determined by setting all but two homogeneous coordinates to zero. So on it goes for higher dimensional sub manifolds fixed by a sub-torus mapping to the faces of $\Delta^n$.

These properties occur more generally and the relationship between convexity, connectedness of levels, fixed points and extremal points is the subject of this chapter. The main result we present is the classical convexity theorem for Hamiltonian torus actions due to Atiyah and Guillemin-Sternberg. [10, 1] We state the theorem below and postpone the proof until we establish some necessary preliminary results.

Theorem 2.2 (Convexity Theorem). Let $(M, \omega)$ be a compact and connected $2d$-dimensional symplectic manifold endowed with the Hamiltonian action of an $n$-dimensional torus $T$ along with a moment map $\mu : M \to t^*$. Then the image of $\mu$ is convex in $t^*$ and the non-empty levels $\mu^{-1}(\xi)$ are connected. Moreover the fixed points form a finite union
of connected symplectic sub manifolds \( C_1, \ldots, C_N \) on each of which the moment map is constant \( \mu(C_j) = c_j \) and \( \mu(M) \) is the convex hull of the points \( c_1, \ldots, c_N \).

**Definition 2.3.** Recall that the convex hull of a subset \( A \subset \mathbb{R}^n \) is the intersection of all convex subsets of \( \mathbb{R}^n \) which contain \( A \) and is denoted by \( \text{Conv}(A) \). A convex hull of finitely many points is a convex polytope and any convex polytope is an intersection of finitely many half spaces. We will now refer to the image of \( \mu \) as the moment polytope associated to the action and denote it by \( \Delta = \mu(M) \). This language will be justified once the convexity theorem is established.

### 2.2 Digression on Morse Theory

In this section we cover some basic definitions and results from Morse Theory and we refer to Milnor’s book on the subject for the omitted proofs.[8] Consider a compact Riemannian manifold \((M,g)\) and a smooth function \( f : M \to \mathbb{R} \) and recall that a critical point of \( f \) is one where \( df \) vanishes and denote the collection of all critical points of \( f \) by \( \text{Crit}(f) \). Assume furthermore that \( \text{Crit}(f) \) is a smooth sub-manifold of \( M \). Then we can define the second derivative of \( f \) at any critical point as follows.

**Definition 2.4 (Hessian).** The Hessian of \( f \) at a critical point \( p \) is a symmetric bilinear form \( D_p^2 f \) on the tangent space \( T_p M \) defined for any two tangent vectors \( X \) and \( Y \) at \( p \) by

\[
D_p^2 f(X,Y) = X(Y(f)) \bigg|_p - Y(X(f)) \bigg|_p - df([X,Y]) \bigg|_p.
\]

In a local coordinate system \((x_1, \ldots, x_n)\) about a critical point \( p \) of \( f \) the Hessian of \( f \) at \( p \) is the matrix with \( j,k\)-entry

\[
\frac{\partial^2 f}{\partial x_j \partial x_k} \bigg|_p
\]

and so \( D_p^2 f \) agrees with the familiar notion for functions on \( \mathbb{R}^d \). The family of symmetric bilinear forms \( p \mapsto D_p^2 f \) depend smoothly on \( p \in C \) and we refer to this family as the Hessian of \( f \) denoted \( D^2 f \). We are forced to restrict the definition of the Hessian to the critical sub-manifold of \( f \) to ensure that each \( D_p^2 f \) it is in fact symmetric.

At any critical point \( p \) the Hessian defines a linear map \( D_p^2 f : T_p M \to T_p^* M \) by contraction \( D_p^2 f(v, -) : w \mapsto D_p^2 f(v, w) \). The kernel of this map must contain any vector tangent to \( \text{Crit}(f) \) at \( p \) since \( f \) is constant on the connected component containing \( p \). We are interested in those functions with are minimally degenerate; those for which the kernel is precisely the tangent space to \( \text{Crit}(f) \) at every critical point.

**Definition 2.5 (Morse-Bott functions).** If the critical set of a smooth function \( f : M \to \mathbb{R} \) is a closed sub-manifold of \( M \) and if the Hessian \( D^2 f \) of is non-degenerate in directions normal to \( \text{Crit}(f) \) then \( f \) is said to be a Morse-Bott function on \( M \).

**Remark.** Note that this definition differs from that of a Morse function on \( M \) which requires that the Hessian be entirely non-degenerate at all critical points. This requirement forces the critical set of \( f \) to be discrete. The Morse functions we are interested in, those coming from moment maps, may well have non-discrete critical components and so must consider the more general Morse-Bott functions.

It follows that for a Morse-Bott function the Hessian induces a non-degenerate bilinear form \( Q_f \) on the normal bundle \( N \) to \( \text{Crit}(f) \). Non-degeneracy of \( Q_f \) implies that the normal is a direct sum of two vector bundles \( \nu^+ \) and \( \nu^- \) on which \( Q_f \) is positive and negative definite respectively,

\[
Q_{f,p}([X_p],[Y_p]) = D_p^2 f(X,Y) \quad NZ = \nu^+ \oplus \nu^-.
\]
Definition 2.6 (Index). The index of a Morse-Bott function \( f \) at a critical point \( p \) is defined to be the dimension of the negative normal bundle at \( p \) and we denote this by \( \lambda_f(p) = \dim(\nu^-_p) \). The coindex of \( f \) is the dimension of the positive normal bundle at \( p \). Equivalently, the index of \( f \) at \( p \) is the number of negative eigenvalues of the associated quadratic form \( Q_{f,p} \).

Proposition 2.7. The index is constant on connected components of the critical set.

Proof. Let \( p \) be a critical point of \( f \) with \( C_0 \) the connected component of \( \text{Crit}(f) \) which contains it and consider any vector field \( X \) which is non-vanishing near \( p \). Then \( q \rightarrow D^2_q f(X,X) \) is a continuous function on \( C_0 \) and must be non-vanishing if the Hessian is to be non-degenerate on the normal bundle \( N_0 \) to \( C_0 \). Therefore the sign of \( D^2_q f(X,X) \) is preserved near \( p \) as must be the dimension of \( \nu^-_p \). The index is therefore locally constant and on the connected component \( C_0 \) must be constant. \( \square \)

Proposition 2.8. Let \( f \) be a Morse-Bott function on \( M \) and \( p \) an arbitrary critical point contained in the connected component \( C_0 \) of \( \text{Crit}(f) \). Then there are local coordinates \( (x,y) = (x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \) about \( p \) such that

a) the critical sub-manifold \( C_0 \) is described by \( y = 0 \)

b) there is a quadratic form \( q(x,y) \) which is non-degenerate in the \( y \)-variables so that \( f(x,y) = f(p) + q(x,y) \).

c) There are finitely many connected components of \( \text{Crit}(f) \).

The major result we will need to borrow from Morse theory describes the homotopy types of levels of a Morse-Bott function. In the following assume that \( f \) is a Morse-Bott function on \( M \) and for any value \( c \in \mathbb{R} \) denote the sublevels and superlevels by \( M^+_c = f^{-1}([c,\infty)) \quad M^-_c = f^{-1}((-\infty,c]). \)

Theorem 2.9. a) If \( f^{-1}(a,b) \) contains no critical points of \( f \) then there are homotopy equivalences \( f^{-1}(a) \simeq f^{-1}(b) \) and \( M_a \simeq M_b \).

b) If \( f^{-1}(a,b) \) contains one critical component \( C_0 \) then there is a homotopy equivalence

\[ M^+_b \simeq M^+_a \cup D(\nu_-(C_0)) \]

where \( D(\nu_-(C_0)) \) is the disk bundle of the negative normal bundle of \( C_0 \). Moreover, if \( C_0 \) is an isolated point then as a topological space \( M^+_b \) is obtained by adding to \( M^+_a \) a cell of dimension equal to the index of \( C_0 \).

Corollary 2.10. Suppose \( f : M \rightarrow \mathbb{R} \) is a Morse function for which there is no critical manifold of index 1 or \( n-1 \). Then

a) \( f \) has a unique local maximum and minimum and

b) every level of \( f \) is either empty or connected.

The Morse-Bott functions we will consider in the rest of the chapter will always have even index and since the dimension of \( M \) will be even as well the above corollary will always apply.

2.3 Almost Periodic Hamiltonians

Return now to the setting of a Hamiltonian action of an \( n \)-torus \( \mathbb{T} \) on a symplectic manifold \( (M, \omega) \) of dimension \( 2d \) with moment map \( \mu \). We are in search of Morse-Bott functions on \( M \) which are also related to the Hamiltonian action. Since the moment map generally does not take values in \( \mathbb{R} \) we must improvise and take a projection onto some 1-dimensional subspace by pairing with some \( X \in \mathfrak{t} \). The hope is that by choosing an appropriate subspace the projected moment map \( \mu^X = \langle \mu, X \rangle \) will be a Morse-Bott function that retains sufficient information about the action.
Lemma 2.11. For $X \in \mathfrak{t}$ consider denote subgroup generated by $X$ by $T_X = \overline{\text{Fix}(\exp(\mathbb{R} \cdot X))}$ ($\mathbb{T}$ is also a torus as a closed subgroup of $\mathbb{T}$). The fixed point set of $T_X$ coincides with the critical sub-manifold of the almost periodic Hamiltonian $\mu^X$

$$\text{Fix}(T_X) = \text{Crit}(\mu^X).$$

Proof. A point $p \in M$ is a critical point of $\mu^X$ if and only if $X^t$ vanishes at $p$ since for a moment map $d\mu^X = -\iota_{X^t} \omega$. By linearity this extends to the entire subspace generated by $X$,

$$d\mu^X_p = 0 \iff (\mathbb{R} \cdot X)^t_p = 0.$$ 

It follows that the critical points $p$ of $\mu^X$ correspond to those points where the $X^t$ vanish

$$\text{Crit}(\mu^X) = \text{Fix}(\exp(\mathbb{R} \cdot X)).$$

\[\square\]

If the 1-parameter subgroup generated by $X$ is dense in $\mathbb{T}$ the previous Lemma says that the critical sub-manifold of $\mu^X$ coincides with the fixed points for the entire $\mathbb{T}$ action. This is the case whenever $X$ is chosen to have rationally independent coefficients. Such functions $\mu^X : M \to \mathbb{R}$ make good candidates to study the entire moment map $\mu$ since they retain the information of the fixed points and, as a result of the convexity theorem, the fixed points completely determine the moment polytope $\Delta$ as the convex hull of their images under $\mu$. It will be useful to describe the behavior of the functions $\mu^X$ without reference to an existing Hamiltonian torus action and so we make the following definition.

Definition 2.12. A smooth function $H \in C^\infty(M, \mathbb{R})$ is said to be an almost periodic Hamiltonian if the associated Hamiltonian vector field $X_H$ generates a one parameter group of diffeomorphisms $\{\exp(tX_H) \mid t \in \mathbb{R}\}$ the closure of which is a torus.

Since we are considering compact manifold $M$ we may choose a $T$-invariant Riemannian metric $g$ on $M$ and almost complex structure $J$ so that $\omega(X, JY) = g(X, Y)$. Let us fix a choice of such a compatible triple for the remainder of this section.

Lemma 2.13. For any subgroup $G$ of $T$ the fixed point set $\text{Fix}(G)$ is a symplectic sub-manifold.

Proof. Let $\psi_g$ denote the diffeomorphism associated to any $g \in G$ and consider for any fixed point $p \in \text{Fix}(G)$ the differential $(d\psi_g)_p$ which necessarily preserves the $T$-invariant almost complex structure $J$,

$$d\psi_g(p) : T_pM \to T_pM \quad d\psi_g(p) \circ J_p = J_p \circ d\psi_g(p).$$

Now let $\exp_p : T_pM \to M$ be the exponential mapping taken with respect to the chosen invariant metric $g$ on $M$ and suppose $\gamma : [0, 1] \to M$ is a geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ for some $v \in T_pM$. Then $c = \psi_g \circ \gamma$ is also a geodesic with $c(0) = \psi_g(p) = p$ and $\dot{c}(0) = (D\psi_g \cdot \dot{\gamma})(0) = D\psi_g \cdot v$ so that

$$\exp_p(D\psi_g \cdot v) = c(1) = (\psi_g \circ \gamma)(1) = \psi_g(\exp_p v).$$

The exponential therefore provides a correspondence between fixed points of $\psi_g$ in a neighborhood of $p$ with the fixed points of $(d\psi_g)_p$. Thus the fixed point set of $G$ is the intersection of the eigenspaces with eigenvalue 1 of each $(d\psi_g)_p$ as $g$ ranges over $G$

$$T_p \text{Fix}(G) = \bigcap_{g \in G} \ker (\text{Id} - d\psi_g(p)).$$

Each eigenspace is invariant under $J_p$ since each $(d\psi_g)_p$ is a unitary transformation and therefore so is the intersection. We conclude that $T_p \text{Fix}(G)$ is $J_p$-invariant for the $\omega$-calibrated almost complex structure $J$ and so $\text{Fix}(G)$ is a symplectic sub-manifold as claimed. \[\square\]
**Proposition 2.14.** For all $X \in \mathfrak{t}$ the almost periodic Hamiltonian $\mu^X = \langle X, \mu \rangle$ is a Morse-Bott function with even dimensional critical sub manifolds of even index. Moreover Crit($\mu^X$) is a symplectic sub-manifold of $M$.

**Proof.** By possibly considering the action of some sub torus we can assume that $X$ generates all of $\mathbb{T}$ and so the vanishing locus of $X^i$ is the fixed point set Fix($\mathbb{T}$). For any fixed point $p$ of $\mathbb{T}$ there is a lift of the action to a linear representation on the tangent space $T_pM$ which preserves the hermitian metric $h$. So the torus acts on $T_pM$ as a subgroup of $U(d)$ and so there is a choice of basis for $T_pM$ in which all elements of $\mathbb{T}$ are diagonal. We can assume this basis is induced by a choice of local coordinates $(x_1, y_1, \ldots, x_d, y_d)$ on $M$ near $p$. With these coordinates $T_pM$ decomposes as a direct sum of complex lines $V_1 \oplus \cdots \oplus V_d$ and $\exp(X)$ is an element of the diagonal torus $U(1)^d \subset U(d)$ acting by multiplication by $e^{i\lambda_j}$ on $V_j$. Furthermore we can assume that $T_p(\text{Fix}(\mathbb{T}))$ is given by the last $d-k$-components $V_j$ so that $\lambda_j = 0$ for $j > k$ and because $X$ generates the action of the entire torus we must have $\lambda_j \neq 0$ whenever $j \leq k$. Now we can express the associated vector field $X^i$ in these coordinates according to the scalars $\lambda_j$,

$$X^i = \sum_{j=1}^k \lambda_j \left( x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right),$$

$$d\mu^X = -i_{X^i} \omega = \sum_{j=1}^k \lambda_j \underbrace{(x_j dx_j + y_j dy_j)}_{\text{pairs}}.$$

In these coordinates the Hessian of $\mu^X$ is given by the matrix of second derivatives and it follows that $\mu^X$ is a Morse-Bott function since the $\lambda_j$ are non-zero for those $j$ with $V_j$ normal to Crit($\mu^X$). Because the $\lambda_j$ occur in pairs the index of $\mu^X$ will necessarily be even. \qed

### 2.4 Proof of Convexity theorem

At last we are ready to give a proof of the convexity theorem following that given by Atiyah. We proceed by induction on $n = \dim(\mathbb{T})$ and for each $n$ let us separate the statements of the theorem into the following three parts,

$A_n$: $\mu^{-1}(\xi)$ is either empty or connected for every $\xi \in \mathfrak{t}^*$

$B_n$: $\mu(M)$ is convex

$C_n$: There are finitely many connected components $C_j$ of the fixed point set, $\mu(C_j) = c_j$ is a point and $\mu(M) = \text{Conv}(c_1, \ldots, c_N)$.

To prove the connectedness of the levels of $\mu$ its will be convenient to work with regular values. So before carrying on we verify that there are enough regular values of $\mu$.

**Proposition 2.15.** The regular values of $\mu$ are dense in $\Delta$.

**Proof.** Let $C$ denote the union of all critical manifolds for $\mu^X$ as $X$ ranges over $\mathfrak{g}$ and by lemma 11 each critical manifold is the fixed point set of the action of a corresponding sub-torus $\mathbb{T}_X$ and so we have established the middle inequality below

$$M \setminus C = \left( \bigcup_{X \in \mathfrak{t}} \text{Crit}(\mu^X) \right)^c = \bigcap_{X \in \mathfrak{t}} \text{Fix}(\mathbb{T}_X)^c = \bigcap_{X \in \mathbb{Z}^{n+1}} \text{Fix}(\mathbb{T}_X)^c.$$

The rightmost equality above can be seen as follows: each $\text{Fix}(\mathbb{T}_X)$ will be the intersection of the fixed point sets for the action of circle subgroups whose product is $\mathbb{T}_X$. We may therefore consider only the intersection over $X \in \mathfrak{t}$ which generate circle subgroups $\mathbb{T}_X$, that is the $X \in \mathfrak{t}$ which have rational components. Moreover each such circle subgroup $\mathbb{T}_X$ can be obtained by rescaling $X$ to lie on the integer lattice $\mathbb{Z}^{n+1} \subset \mathfrak{t}$. 

Since each Fix($T\times$) is a proper closed sub-manifold of $M$ and so has open and dense complement. Thus the complement of $C$ in $M$ is a countable intersection of open dense sets and the Baire category theorem tells us that $M\setminus C$ must also be open and dense. An arbitrary $\xi \in \Delta$ can therefore be approximated by $\{\mu(x_j)\}$ for a sequence of points $x_j \in M\setminus C$. The image $\Delta = \mu(M)$ necessarily contains a neighborhood of each $\mu(x_j)$ and by Sard’s theorem we may find a sequence of regular values $\{\xi_{j,k}\}$ converging to $\mu(x_j)$ for every $x_j$. The diagonal sequence $\{\xi_{j,j}\}$ will then converge to $\xi$ and so we conclude that regular values of $\mu$ are in fact dense in $\Delta$. 

}\hspace{1cm} \square

$A_n$ holds for all $n$. The statement $A_1$ is an immediate consequence of corollary 10 and proposition 14 since for $n=1$ the moment map $\mu : M \to \mathbb{R}$ is itself an almost periodic Hamiltonian. Now suppose that $A_k$ holds for all Hamiltonian actions by a torus of dimension $k \leq n$ and let $T$ be a torus of dimension $n+1$ acting in a Hamiltonian fashion on $(M, \omega)$ with moment map $\mu$. We may assume that the action is effective since otherwise we may reduce to the action of some quotient of $T$ and apply the induction hypothesis to conclude the result for regular levels of the reduced moment map $\tilde{\mu}$.

To apply the induction hypothesis we need to work with the action of a sub-torus of dimension not more than $n$. Take then the action of the sub-torus of the first $n$-components and let $\tilde{\mu}$ be the reduced moment map. With respect to the associated basis on $t^* = \mathbb{R}^{n+1}$ we may then decompose $\mu$ into $n+1$ component functions and $\mu = (\tilde{\mu}, \mu_{n+1})$. Similar we have $\xi = (\tilde{\xi}, \xi)$ for all $\xi \in t^*$. Now consider the restoration of $\mu_{n+1}$ to the level $Q = \tilde{\mu}^{-1}(\tilde{\xi})$,

$$Q = \tilde{\mu}^{-1}(\tilde{\xi}) = \left\{ j = 1 \right\}_{j=1}^{n} \mu_{j}^{-1}(\xi_{j}) \quad \mu_{Q} = \mu_{n+1} \right|_{Q} \quad \mu^{-1}(\xi) = \mu^{-1}_{Q}(\xi_{n+1}).$$

It will suffice then to check that $\mu^{-1}_{Q}(\xi_{n+1})$ is connected and we do this by showing $\mu_{Q}$ is a Morse-Bott function with only even index and coindex. A similar argument to that in the proof of proposition 15 shows that the set of $\xi \in t^*$ for which $\tilde{\xi}$ is a regular value for $\tilde{\mu}$ is also dense in the image and so we will assume that this is the case.

Now a point $p \in Q$ is a critical point for $\mu_{Q}$ if and only if $\sum_{j=1}^{n} a_{j} d\mu_{j} + d\mu_{n+1}$ vanishes at $p$ for some choice of constants $a_{1}, \ldots, a_{n}$. Then $p$ is also a critical point for the almost periodic Hamiltonian $\mu_{A}$ where $A = (a_{1}, \ldots, a_{n}, 1) \in t$. Then if $C_{0}$ is the critical component of $\text{Crit}(\mu_{A})$ containing $p$ we know from corollary 10 that $C_{0}$ is a critical component on which $\mu_{A}$ has even index and coindex.

The Hamiltonian vector field $X_{j}$ of each component $\mu_{j}$ preserves the sub-manifold $C_{0}$ since it must commute with that of $\mu_{A}$ and therefore each $X_{j}$ is tangent to $C_{0}$. Moreover the Hamiltonian vector fields $X_{j}$ are linearly independent since the torus action was assumed to be effective and so every linear combination $B_{X} = \sum_{j=1}^{n} b_{j} X_{j} \in t$ is non-vanishing at $p$ and since $C_{0}$ is symplectic there must be a non-zero $v \in T_{p} C_{0}$ so that $\iota_{B_{X} \omega} \xi = \sum_{j=1}^{n} b_{j} d\mu_{j}$ is non-zero at $p$. This says that the $d\mu_{j}(p)$ are linearly independent and thus $C$ is transverse to $Q$.

Transversality implies that $i_{Q}^{*} \mu_{A}$ is a Morse-Bott function and $C_{0} \cap Q$ a critical component of $i_{Q}^{*} \mu_{A}$ with even index and coindex. The same must hold for $\mu_{Q}$ since $\mu_{Q} - i_{Q}^{*} \mu_{A} = \sum_{j=1}^{n} a_{j} \xi_{j}$ is a constant. Since $p$ was arbitrary we conclude that $\mu_{Q}$ satisfies the hypothesis of corollary 10 and so the level $\mu_{Q}^{-1}(\xi_{n+1}) = \mu^{-1}(\xi)$ is connected. 

Note that the statement $B_{n}$ for $n = 1$ is simply that $\mu(M)$ is connected and this is immediate since $M$ is connected by assumption and $\mu$ is continuous. Now we show that the statement holds for all $n \geq 2$.

**Proof that $A_{n}$ implies $B_{n+1}$**. To prove that the image is convex it is enough to verify that the intersection of $\mu(M)$ with any straight line in $\mathbb{R}^{n+1}$ is either empty for connected. Any such straight line is an affine linear subspace and can be written as $\pi^{-1}(\eta)$ where $\pi$ is the projection onto some codimension 1 subspace $V \subset \mathbb{R}^{n+1}$ and $\eta$ is any element of $V$. We want to check the following set is empty or connected

$$\mu(M) \cap \pi^{-1}(\eta) = \mu((\pi \circ \mu)^{-1}(\eta))$$

The composition $\pi \circ \mu$ describes the moment map for the action of the sub-torus $S \subset \mathbb{T}$ corresponding to the subspace $V \subset \mathbb{R}^{n+1} = t^*$. The property $A_{n}$ for this action of the sub-torus then says that $(\pi \circ \mu)^{-1}(\eta)$ is either empty or
connected and so too must be \( \mu((\pi \circ \mu)^{-1}(\eta)) \) since \( \mu \) is continuous. Since \( \pi \) and \( \eta \) were arbitrary this completes the proof.

\[\]

**Proof that \( B_n \) implies \( C_n \).** We have seen that the fixed point set of \( T \) coincides with the critical set for any almost periodic Hamiltonian \( \mu^X \) satisfying \( T_X = T \) and so proposition 8 implies that this set has only finitely many connected components \( C_1, \ldots, C_N \). The fixed point set \( \text{Fix}(T) \) is contained in the critical set \( \text{Crit}(\mu^X) \) for any \( X \in t \) and therefore \( \mu^X \) must be constant on any connected component \( C_j \) of \( C \). Since this holds for all \( X \in t \) it follows that \( \mu \) is constant on each \( C_j \) and \( \mu(C_j) = c_j \) is indeed a single point.

The image of \( \mu \) is convex by assumption and certainly contains all of the \( c_j \)'s and so also their convex hull; \( \Delta = \mu(M) \supset \text{Conv}(c_1, \ldots, c_N) \). To obtain the reverse inclusion let \( \xi \in t^* \) be any point not contained within the convex hull of the \( c_j \)'s. As a compact convex set there is a hyperplane separating the convex hull and the point \( \xi \); that is there is some \( X \in t \) such that \( \langle \eta, X \rangle < \langle \xi, X \rangle \) for all \( \eta \in \text{Conv}(c_1, \ldots, c_N) \). Moreover we may choose \( X \) to have rationally independent coordinates since the distance between these compact sets must be positive and such \( X \) are dense in \( t \).

By doing this we ensure that \( \text{Crit}(\mu^X) \) coincides with \( \text{Fix}(T) \) and so the maximum of \( \mu^X \) must occur at some \( p \in C_j \).

This proves the reverse inclusion since if \( \xi \) were to lie in the image of \( \mu \) it would violate the following inequality

\[ \sup_{x \in M} \langle \mu(x), X \rangle = \langle \mu(p), X \rangle < \langle \xi, X \rangle. \]

\[\]

## 2.5 Applications and Examples

**Theorem 2.16 (Schur-Horn).** Let \( A \in \mathcal{H} \) be a hermitian with spectrum \( \lambda = \{\lambda_1, \ldots, \lambda_n\} \) and diagonal elements \( a_{11}, \ldots, a_{nn} \). Then \( (a_{11}, \ldots, a_{nn}) \) is contained in the convex hull generated by the permutations of the eigenvalues \( \text{Conv}(\lambda_{\sigma \in S_n}) \). Conversely any element of \( \text{Conv}(\lambda_{\sigma \in S_n}) \) is the diagonal for some hermitian matrix with spectrum equal to \( \lambda \).

**Proof.** The coadjoint action of \( U(n) \) can be identified with conjugation on the space of skew hermitian matrices \( i\mathcal{H} \). Then the orbit \( \mathcal{O}_\lambda \) containing the diagonal matrix \( \text{diag}(i\lambda_1, \ldots, i\lambda_n) \) is precisely the skew hermitian matrices with spectrum \( i\lambda \). The torus subgroup \( T = U(1)^n \) of diagonal matrices in \( U(n) \) acts on \( \mathcal{O}_\lambda \) and has a moment map \( \mu \) given by projecting onto the diagonal

\[ \mu : A \mapsto (a_{11}, \ldots, a_{nn}). \]

We now know from the convexity theorem that the image of \( \mu \) is the convex hull of \( \{c_j = \mu(C_j)\} \). Now \( A \in i\mathcal{H} \) is fixed by \( T \) if and only if it is diagonal so the fixed points are precisely the elements \( \text{diag}(i\lambda_{\sigma(1)}, \ldots, i\lambda_{\sigma(n)}) \).

The moment polytope for the above torus action must lie in a hyperplane since the trace is constant on any \( \mathcal{O}_\lambda \)

\[ \langle (a_{11}, \ldots, a_{nn}), (1, \ldots, 1) \rangle = \sum_{j=1}^{n} a_{jj} = \text{tr}(A) = \sum_{j=1}^{n} \lambda_j. \]

**Example 2.17.** Consider the case for three distinct eigenvalues having zero trace, that is

\[ \lambda_1 < \lambda_2 < \lambda_3 \quad \lambda_1 + \lambda_2 + \lambda_3 = 0. \]

Then \( S_3 \) acts freely on the set \( \{\lambda_1, \lambda_2, \lambda_3\} \) and so \( \text{diag}(\mathcal{O}_\lambda) \) is the convex hull of six distinct points. So \( \Delta \) is a hexagon which lies in the hyperplane orthogonal to \( (1,1,1) \).

Recall that we chose a compatible triple \( (\omega, J, g) \) on \( M \) so that at a fixed point \( p \) the torus acts on the tangent space as a subgroup of \( U(d) \). We also made a choice of basis for \( T_p M = V_1 \oplus \cdots \oplus V_d \) which simultaneously diagonalizes
Then the rank of Corollary 2.20. manifolds are the subject of the next chapter but we observe some of the properties of their moment polytopes now. has dimension exactly half that of \( \text{dim}(\mathbb{T}) \).

**Proof.** First recall that cone generated by the \( \alpha_k \)'s at \( p \) is the image of the moment map for the linear action of \( \mathbb{T} \) with respect to the usual symplectic form

\[
\mu_0 = p + \frac{1}{2} \left( \sum_{j=1}^{n} \alpha_k^j |v_j|^2, \ldots \right) \quad \omega_0 = \sum_{j=1}^{n} dv_j \wedge d\bar{v}_j \quad \mu_0(T_z M) = \text{Cone}_p(\alpha_1, \ldots, \alpha_n)
\]

Using the equivariant Darboux theorem we can describe the moment map \( \mu \) in a neighborhood of \( z \) in terms of \( \mu_0 \). Since \( \omega_z \) and \( \omega_0 \) agree at the origin there is a neighborhood \( U_0 \) of zero in \( T_z M \) and an equivariant map \( \psi : U_0 \to T_z M \) preserving the origin and satisfying \( \psi^* \omega_z = \omega_0 \). We may assume that \( U_0 \) has been taken small enough so that the exponential corresponding to our invariant Riemannian metric provides a diffeomorphism with a neighborhood \( U_z \) of \( z \) in \( M \),

\[
\exp_z|U_0 : U_0 \cong U_z.
\]

Since the exponential is equivariant the composition \( \mu' = \mu \circ \exp_z \) provides a moment map for the linearized action of \( \mathbb{T} \) on \( (T_z M, \omega_z) \). It follows that \( \psi^* \mu' \) is a moment map for the action of \( \mathbb{T} \) with respect to \( \omega_0 \) and therefore can only differ from \( \mu_0 \) by a constant. We conclude that these maps are in fact equal since the agree at zero

\[
(\psi^* \mu')(0) = \mu'(\psi(0)) = \mu'(0) = \mu(\exp_z(0)) = \mu(z) = p = \mu_0(0).
\]

Therefore \( \mu_z \) takes \( U_z \) to the image of \( \mu_0 \) which we know to be contained in the given cone at \( p \)

\[
\mu(U_z) = \mu(\exp_z U_0) = \mu'(U_0) = \mu_0(U_0).
\]

\( \square \)

**Corollary 2.19.** The hyperplanes defining the moment polytope have integral equations.

Recall that an action is said to be effective if the intersection of all stabilizers is trivial. For a Hamiltonian torus action to be effective on a symplectic manifold of dimension \( 2d \) we know we must have \( \dim(\mathbb{T}) \leq d \). When the torus has dimension exactly half that of \( M \) we say that \( M \) along with the Hamiltonian action is a *toric* manifold. Toric manifolds are the subject of the next chapter but we observe some of the properties of their moment polytopes now.

**Corollary 2.20.** Then the rank of \( \mu \) at any point \( x \) in \( M \) is equal to the dimension of the face of \( \Delta \) that contains \( p = \mu(x) \).

**Proof.** Let \( k \) be the dimension of the stabilizer \( T_x = \text{Stab}(x) \) and consider the action of the subgroup \( T_x \) on \( M \) with moment map \( \pi_x \circ \mu \) where \( \pi_x \) is the appropriate projection. We can choose a basis so that \( \pi_x \) is given by

\[
\pi_x : (\xi_1, \ldots, \xi_d) \mapsto (\xi_1, \ldots, \xi_k)
\]

Since \( x \) is a fixed point of this action by design we can apply proposition 18 to obtain a neighborhood \( U \) of \( x \) and \( V \) of \( p \) for which

\[
(\pi_x \circ \mu)(U) = V \cap \text{Cone}_p(\alpha_1, \ldots, \alpha_k).
\]
Bibliography


