Nonlinear Subgrade Reaction Solution for Circular Tunnel Lining Design Based on Mobilized Strength of Undrained Clay
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Abstract:
This paper presents a nonlinear solution of radial subgrade reaction–displacement ($p_k-u_r$) curve for circular tunnel lining design in undrained clay. With the concept of soil shear strength nonlinearly mobilized with shear strain, an analytical solution of $p_k$ is obtained using the mobilized strength design (MSD) method. Two typical deformation modes are considered, namely oval and uniform modes. A total of 197 orthogonally designed cases are used to calibrate the proposed nonlinear solution of $p_k$ using the finite element method (FEM) with the Hardening soil (HS) model. The calibration results are summarized using a correction factor $\eta$, which is defined as the ratio of $p_{k,FEM}$ over $p_{k,MSD}$. It is shown that $\eta$ is correlated to some input parameters. If this correlation is removed by a regression equation $f$, the modified solution $f \times p_{k,MSD}$ agrees very well with $p_{k,FEM}$. Although the mobilized soil strength varies with principal stress direction in reality, it is found that a simple average of plane strain compression and extension results is sufficient to produce the above agreement. The proposed nonlinear $p_k-u_r$ curve is applied to an actual tunnel lining design example. The predicted tunnel deformations agree very well with the measured data. In contrast, a linear $p_k$ model would produce an underestimation of tunnel convergence and internal forces by 2-4 times due to the overestimation of $p_k$ at large strain level.

Keywords:
Radial subgrade reaction, Tunnel, Mobilized strength design (MSD), Nonlinear, Undrained clay.
Introduction

The embedded beam spring model is the most frequently used design model among all the models recommended by design codes for the structural design of tunnel linings (ITA 2004; JSCE 2007; GB50157 2013). Some key aspects in lining design, e.g., soil load, lining structure and soil-structural interactions, are considered explicitly in this model. Specifically, the tunnel lining is represented by beams supported on soil springs around tunnel perimeter to simulate the soil-structural interaction caused by initial earth pressure. However, the soil spring constant, also named as the radial subgrade modulus $k_r$, is difficult to evaluate and it is often selected from empirical correlations with some engineering judgment (Mair 2008). For example, the standard penetration test (SPT) blow count $N$ has been related to a range of $k_r$ in some local design regulations (JSCE 2007; DGJ08-10 2010). It is a matter of judgment to pick a specific value of $k_r$ from this broad guideline, which would translate to significant uncertainty in the calculated lining thrust and moment (Lee et al. 2001; Gong et al. 2015). Hence, some analytical solutions of $k_r$ have been proposed as a more rational method to estimate the magnitude of this parameter for design (Arnau and Molins 2011).

These analytical solutions of $k_r$ are usually derived with a basic assumption of elastic soil in infinite or semi-infinite space. The value of $k_r$ is defined as the ratio of calculated subgrade reaction $p_k$ over soil radial displacement $u_r$ around the tunnel perimeter given a prescribed deformation mode of soil-lining interaction. Wood (1975) proposed a solution of $k_r$ under the assumption of oval deformation mode shape in infinite space. Sagaseta (1987) presented a solution of $k_r$ under a uniform deformation mode shape in semi-infinite space. Later on, more general ground deformation modes are developed by combining these two basic mode shapes (Verruijt and Booker 1996; Park 2005; Pinto and Whittle 2014). Based on this observation, Zhang et al. (2014) proposed a general elastic solution of $k_r$ that can be
decomposed into a series of basic $k_r$, each produced by a simple soil deformation mode shape.

However, it has been observed from centrifuge test results that tunneling may produce non-linear soil behavior around the tunnel (Mair 1979; Taylor 1998). In addition, in-situ measurements of tunnels in operation have shown that large deformations far beyond the elastic range can happen (Shen et al. 2014). Huang and Zhang (2016) presented a detailed field case where the tunnel deformation caused by extreme surface surcharge is almost seven times larger than the design value. The large tunnel deformation inevitably will drive the soil around the tunnel to an elasto-plastic state (Osman et al. 2006b; Klar et al. 2007). Thus, the elastic solution of $k_r$ might not be sufficient to describe the full range of behavior of this key input parameter in a tunnel lining design. Hence, there is a practical motivation to develop an elasto-plastic solution of $k_r$ to model tunnel lining behavior in non-linear soils.

The task at hand is to obtain a non-linear solution of $k_r$, which is a non-linear curve of subgrade reaction $p_k$ versus the radial displacement $u_r$. This is conceptually similar to the well-known non-linear $p-y$ curve for laterally loaded piles (Nogami et al. 1992; Bransby 1999; McGann et al. 2011). However, unlike the $p-y$ curve for laterally loaded pile design, limited solutions have been proposed for the non-linear $p_k-u_r$ curve for tunnel lining design. A non-linear curve with the assumption of a hyperbolic function for subgrade reaction has been proposed and utilized for the design of rock tunnel supports (Oreste 2007; Do et al. 2013; Do et al. 2014). The rationality of using the hyperbolic shape function is not well described and its relation to rock properties is unclear.

The mobilized strength design (MSD) method can provide a more rational method to derive the non-linear $p_k-u_r$ curve and can provide an explicit linkage to relevant soil properties. The MSD method is built around an assumed plastic deformation mechanism of the soil movements due to soil-lining interaction and the concept of “mobilized soil strength”
that is founded on measured soil behavior (Bolton and Powrie 1988; Bolton et al. 1990).

For the problem with tunneling induced ground movement, Osman et al. (2006a) has developed a non-linear curve of initial support pressure $p$ varying with the ground surface settlement $s_m$ by using the MSD method. The empirical Gaussian function (Peck 1969; Osman et al. 2006b) is adopted to describe the ground movement in the model. However, the soil-lining interactions that cause the common oval deformation mode (Klar et al. 2007; Pinto et al. 2014) are ignored in their model. The MSD method is useful, because it provides an analytical solution to a fairly complex soil-structure interaction problem. Although the finite element method (FEM) can solve the same problem, it is arguably less physically insightful than the MSD method and it may require a more detailed characterization of soil behavior (in terms of parameters for constitutive model) beyond what is produced by a routine site investigation. The cost of achieving analytical simplicity in the MSD method is that it is potentially less accurate than the FEM (assuming all information required in the FEM method is fully available). Nonetheless, Zhang et al. (2015) showed that the MSD method and the FEM could produce comparable solutions if the former is corrected by a simple factor. A key contribution of this paper is the characterization of this correction factor for the tunnel lining problem.

The objective of this paper is to develop a non-linear curve of subgrade reaction $p_k$ varying with radial displacement $u_r$ for a circular lined tunnel in elasto-plastic undrained soil using the MSD method. The non-linear solution of radial subgrade modulus $k_r$ is derived as the derivative of this $p_k-u_r$ curve. The mobilized soil strain $\varepsilon_{mob}$ is expressed as a function of radial displacement $u_r$ using a typical uniform or oval deformation mode for soil-lining interactions in semi-infinite space (Verruijt and Booker 1996). The subgrade reaction $p_k$ is then calculated with the concept of mobilized soil strength $s_{u,mob}$ under the minimum energy principle. The finite element method (FEM) with a widely used nonlinear constitutive
model for undrained soil, namely the Hardening Soil (HS) model, is adopted to validate the applicability of the proposed nonlinear solution of $p_k$. The MSD solution ($p_{k,MSD}$) can be corrected to agree with the FEM solution ($p_{k,FEM}$) using a simple approach proposed in Zhang et al. (2015). Finally, the performance of the corrected MSD method is demonstrated using a tunnel lining design example based on a field case reported by Huang and Zhang (2016).

**Deformation modes subjected to subgrade reaction**

**Mode shape for tunnel lining**

A tunnel lining interacts with the surrounding ground after it is installed. It deforms under the initial earth pressure coming from the ground. The ground responds to the lining deformation by a subgrade reaction. The lining deformation would adjust to the subgrade reaction and subsequently induce a new but small subgrade reaction. This iteration would finally converge and the lining would form a specific distribution of deformation that is consistent to the distribution of subgrade reaction. Although this distribution is complicated and different from case to case, it could be reasonably characterized by basic mode shapes such as uniform and oval shapes or more general shapes that can be viewed as combinations of these basic mode shapes (Pinto and Whittle 2014; Pinto et al. 2014; Zhang et al. 2014). It should be noted that all deformation mechanisms discussed in this paper take place under the plane strain condition since a tunnel is a long linear structure.

In the case of a uniform mode shape (see in Fig. 1a), the lining deforms uniformly around the tunnel perimeter at a constant value, saying $\Delta$ (Sagasetta 1987):

$$u_{r1} = \Delta \quad (1a)$$

where $u_{r1}$ means the uniform radial convergence at the perimeter of the tunnel lining. In the case of an oval mode shape (see in Fig. 1b), the lining deforms following a cosine function of the sectional angle $\theta$ with vertical axis, as shown below (Wood 1975):

$$u_{r2} = \delta \cos 2\theta \quad (1b)$$
where \( u_{r2} \) means the oval radial convergence at tunnel perimeter. The parameter \( \delta \) is the value of deformation at the tunnel crown (i.e., \( \theta = 0^\circ \)). More general deformation shapes can be formed by combining these two basic mode shapes. It is clear that the parameters \( \Delta \) and \( \delta \) are the maximum magnitude of the lining convergence around the tunnel perimeter. It is more convenient to use these parameters as key performance indicators of the tunnel lining under earth pressure including subgrade reaction, rather than to consider the entire shape.

It has been reported that these deformation indicators (i.e., \( \Delta \) and \( \delta \)) are closely related to the internal forces of lining and structural defects, e.g., crack, leakage, concrete spalling or joint opening for segmental lining of shield tunnel (Yuan et al. 2012; Huang and Zhang 2016). Large internal forces or severe structural defects could affect the serviceability and safety of tunnels. Hence, it is not surprising these deformation indicators appear in the definition of ultimate and serviceability limit states in tunneling codes and standards. For example, the British Tunnel Standard (BTS 2004) has set an ultimate limit for the maximum convergence normalized by tunnel radius (\( \Delta/R \) or \( \delta/R \), \( R \) is the tunnel radius) at 2\%. As for serviceability, the Chinese code (GB50157 2013) has set a serviceability limit of \( \Delta/R \) (or \( \delta/R \)) at 0.3\%-0.4\%, while the Shanghai local regulation (DGJ08-10 2010) relaxes the serviceability limit to 0.5\%.

**Deformation modes for soils around the tunnel**

Assuming that soil and tunnel lining deforms in a compatible way, one expects the soil around the deformed tunnel to follow the two main deformation modes, namely uniform and oval modes. Equations characterizing these two modes in infinite space and semi-infinite space are available (Timoshenko and Goodier 1970; Wood 1975; Pinto and Whittle 2014). Although these equations are derived analytically with a deformation field based on elasticity, measured data reported in the literature have justified this assumption, namely the observed deformation field for mobilized soils in a nonlinear state is proportional to those elastic deformation fields (Mair 1979; Klar et al. 2007; Pinto et al. 2014). Since the mode in
semi-infinite space is suitable for both the shallow and deep tunnels, it is applied to describe
the elasto-plastic deformation field of mobilized soils in this paper. Based on the solutions
proposed by Verruijt and Booker (1996), the uniform and oval deformation modes for
undrained soil with a Poisson’s ratio \( \nu \) of 0.5 in semi-infinite space are briefly introduced
below.

\( \textit{a). Uniform mode} \)

A typical uniform deformation field is plotted in Fig. 2a. The tunnel with a radius of \( R \) is
buried in a depth of \( h \) below the ground surface, as shown in Fig. 2a. The horizontal ground
movement \( u_x \) could be represented by Eq. 2a, while the vertical ground movement \( u_z \) could be
represented by Eq. 2b, as shown below:

\[
\begin{align*}
\text{a). Uniform mode} \\
\text{A typical uniform deformation field is plotted in Fig. 2a. The tunnel with a radius of } R \text{ is}
buried in a depth of } h \text{ below the ground surface, as shown in Fig. 2a. The horizontal ground}
movement } u_x \text{ could be represented by Eq. 2a, while the vertical ground movement } u_z \text{ could be}
represented by Eq. 2b, as shown below:}
\end{align*}
\]

\[
\begin{align*}
u_x &= -\Delta \cdot R \left\{ \frac{x}{x^2 + (z-h)^2} + \frac{x}{x^2 + (z+h)^2} - \frac{4xz(z+h)}{\left[ x^2 + (z+h)^2 \right]^2} \right\} \\
u_z &= -\Delta \cdot R \left\{ \frac{z-h}{x^2 + (z-h)^2} + \frac{z+h}{x^2 + (z+h)^2} - \frac{2(z+h)}{x^2 + (z+h)^2} + \frac{2z}{{\left[ x^2 + (z+h)^2 \right]^2}} \right\}
\end{align*}
\]

where \( x \) is the horizontal distance of calculated soil element from tunnel axis, and \( z \) is the
depth of the element. The parameter \( \Delta \) is the radial convergence of the deformed tunnel
lining in a uniform shape mentioned previously. Although the tunnel deforms uniformly, it
is clear from Fig. 2a that the surrounding soils do not deform uniformly due to the effect of
the image part of singularity solution in a semi-infinite space (Sagaseta 1987).

In a semi-infinite space, the boundary of the deformation field is located at the ground
surface above the tunnel. There is no boundary at the two sides of the tunnel and below the
tunnel. But it should be noted from Eq. 2 that, as the radial distance \( r \) between soil and the
center of tunnel increases, the soil displacement vanishes quickly following a decay function
of \( 1/r \). That is to say, when the distance \( r \) is about 100 times of radius \( R \), the corresponding
soil element would have a displacement less than 1% of \( \Delta \), which exert a negligible effect on
the behaviors of soil around tunnel lining (Mair et al. 1993).

In this deformation mode, the calculation of soil strain by a first derivation of Eq. 2 have revealed that the direction of principal strain is horizontal for soils locating above tunnel crown and along its vertical axis, while the direction of principal strain is vertical for soils locating at the two sides of the springline. It has been validated by ground movement in 1g model test (Seneviratne 1979; Osman et al. 2006a). Hence, the soil behaviors along the vertical axis are more likely to be modelled under the condition of plane strain extension (PSE), while the soils at two sides of springline are to be modelled under the condition of plane strain compression (PSC), as shown in Fig. 2b.

b). Oval mode

The typical oval deformation field is plotted in Fig. 3a. Similar to the uniform mode, the tunnel has a radius of \( R \) and a cover depth of \( h \). The horizontal ground movement \( u_x \) is expressed by Eq. 3a, and the vertical ground movement \( u_z \) is represented by Eq. 3b, as shown below:

\[
\begin{align*}
\text{Eq. 3a:} & \quad u_x = -\delta \cdot R \left( \frac{x(x^2 - z^2) + x^2 - (2h - z)^2}{(x^2 + z^2)^2} - 4xh \frac{(h - z)[x^2 - 3(2h - z)^2]}{[x^2 + (2h - z)^2]^2} \right) \\
\text{Eq. 3b:} & \quad u_z = \delta \cdot R \left( \frac{-z(x^2 - z^2)}{(x^2 + z^2)^2} + \frac{2(h - z)[x^2 - (2h - z)^2]}{[x^2 + (2h - z)^2]^2} - 2h \frac{x^2 - (2h - z)^2}{[x^2 + (2h - z)^2]^2} \right) \\
\end{align*}
\]

where \( x \) is the horizontal distance between the calculated soil element and the tunnel axis, and \( z \) is the depth of the element. The parameter \( \delta \) is the magnitude of lining displacement at crown as the tunnel deforms into an oval shape mentioned previously. Similar to the case of uniform mode, the boundary of the oval mode is located at the ground surface above tunnel, and the soil displacement will decrease with the increase of radial distance \( r \) following a
In the oval deformation mode, similar to the case of uniform mode, the calculated principal strain of soil along the vertical axis from Eq. 3 is in the horizontal direction. The soil at both sides of the springline experiences principal strains in the horizontal direction due to the extension of soil at the springline location as part of an oval shape. This extension ground movement mode at the springline location can be seen from measured data in field cases (Hashimoto et al. 1996; Standing and Selemetas 2013). Hence, the plane strain extension (PSE) condition is more suitable for modeling mobilized soil behavior in an oval mode, as shown in Fig. 3b.

**Energy Conservation in Mobilized Soils**

Following the MSD concept that the soil is in continuous yielding corresponding to a mobilized strength, the potential energy loss caused by deformed soil induced by soil-lining interaction is equal to the work done by corresponding subgrade reaction load \( p_k \) around the tunnel perimeter. This energy conservation formulation is the same as those general formulation used for elastic solutions of subgrade reaction (Verruijt and Booker 1996; Zhang et al. 2014). It should be emphasized that this energy conservation formulation only accounts for soil-lining interaction at a distance behind the tunnel face. It does not include the work done by the face support pressure during tunnel excavation. The MSD framework for this tunneling procedure has been presented elsewhere by Osman et al. (2006a) and Klar and Klein (2014). With the above consideration in mind, the formulation of energy conservation for subgrade reaction can be expressed as below:

\[
\int_{\pi D} p_k u_r \, ds = \int_{\Delta \epsilon_{\text{mob}}} s_{u,\text{mob}} \epsilon_s \, dA
\]

(4)

where the \( u_r \) is soil radial displacement around the tunnel perimeter calculated by using the \( u_x \) and \( u_z \) mentioned previously (\( u = u_r \cos \theta - u_z \sin \theta \)). The one-dimensional integral length for left side of Eq. 4 is essentially along the tunnel perimeter i.e., \( \pi D \) in Eq. 4, while the
two-dimensional integral area for the right side, i.e., Area in Eq. 4, is the semi-infinite space described in previous section (see Fig. 1). The parameters \( s_{u,mob} \) and \( \varepsilon_s \) are the mobilized soil undrained shear strength and the corresponding mobilized engineering shear strain, respectively. Adopting the rule-of-thumb that the engineering shear strain \( \varepsilon_s \) is 1.5 times of the deviatoric strain \( \varepsilon_q \) leads to:

\[
\varepsilon_s = 1.5 \varepsilon_q = \sqrt{\frac{3}{2} \varepsilon_y \varepsilon_y}, \quad (i,j \text{ principal directions})
\]

where \( \varepsilon_q \) are the shear strain in the principal directions. All these strains could be calculated by Eqs. 2 and 3 in closed form. The mobilized undrained shear strength \( s_{u,mob} \) is obtained by a suitable nonlinear interpolation of the stress-strain curve given a calculated mobilized engineering shear strain \( \varepsilon_s \) for each soil element within the semi-infinite integral area. The soil elements around tunnel are in a PSC, PSE, or an intermediate state, depending on the rotation of the principal strain. Hence, soils within the integral area in the right side of Eq. 4 theoretically should be discretized into separate soil element that follows a stress-strain curve compatible to the rotation of the principal strain calculated by Eqs. 2 and 3. In other words, for this tunnel problem, the stress-strain curve varies with spatial location, because the rotation of the principle strain varies with spatial location.

**Practical Approximation of Energy Conservation**

It is evident that the exact solution of Eq. 4 can be tedious if each soil element is assigned a stress strain curve as a function of the principal strain rotation at each location. A practical solution to Eq. 4 is to examine the possibility of replacing \( s_{u,mob} \) which varies according to principal strain rotation by an equivalent \( s^*_{u,mob} \) that does not depend on principal strain rotation. Since both PSC and PSE states are present in the soil elements around the deformed tunnel (as see in Fig. 2b), a simple way to approximate the exact mobilized shear strength \( s_{u,mob} \) is to average the results of \( s_{u,mob} \) from plane strain compression tests and the results of \( s_{u,mob} \) from plane strain extension tests, as recommended by Koutsofias and Ladd.
Note that both the $s^*_{u,mob|PSE}$ from plane strain extension test and $s^*_{u,mob|PSC}$ from plane strain compression tests are evaluated at the depth of the tunnel axis following the assumption adopted by Osman and Bolton (2006a) for a tunneling-induced ground settlement problem. The mobilized shear strength $s_{u,mob}$ in Eq. 4 could be replaced by the above approximate $s^*_{u,mob}$ in Eq. 6. It is worth pointing out here that the fundamental value of MSD is related to its ability to produce an answer sufficiently accurate for design at a significantly lower cost than FEM. If MSD is only marginally less costly than FEM, there is no reason to adopt MSD. Eq. 6 is adopted with this pragmatic consideration in mind. The focus of this study is to clarify if the error associated with this approximation as benchmarked against the finite element solution is acceptable for design purposes.

In addition, when Eq. 4 is integrated numerically, the semi-infinite space is approximated by a large circular domain with a radius about 100 times the tunnel radius $R$, which is reasonably large enough to minimize the effect of ignoring soil displacements outside this domain on the integral results (Klar et al. 2007). Hence, equation 4 is re-written based on the above simplification as follows:

$$
\int_{z_D} p_k u_k ds = \int_{\text{Area}} s^*_{u,mob|PSC} \varepsilon_z dA \quad (\text{Area} \in \{z \geq 0, r \leq 100R\})
$$

(7)

As the tunnel lining deforms into a prescribed mode shape (uniform or oval), the distribution of subgrade reaction would match a similar shape to the deformation shape following the basic Winkler assumption. Hence, two distribution shapes for subgrade reaction $p_k$ is assumed based on the previously assumed deformation shapes, namely the uniform and oval shapes. Equation 7 is further developed based on each specific deformation mode below.

In the case of the uniform mode, subgrade reaction $p_k$ is distributed with a constant value...
$p_{k0}$ at any position along lining as below:

$$p_k = p_{k0}$$  \hspace{1cm} (8)

By substituting Eq. 8 and Eq. 1a into Eq. 7, the $p_{k0}$ can be derived as following equation:

$$p_{k0} = \frac{\int_{\text{area}} s^*_{u,mob} \times \varepsilon_\delta(\Delta) \, dA}{\Delta \int_{\text{c}D} \, ds}$$  \hspace{1cm} (9)

where $\Delta$ is the radial displacement of lining under uniform shape (see in Eq. 1a). When given a specific value of $\Delta$, the engineering shear strain $\varepsilon_\delta$ could be derived by Eq. 5. The approximate mobilized shear strength $s^*_{u,mob}$ can then be read off a representative shear stress-strain curve from Eq. 6 corresponding to the calculated $\varepsilon_\delta$. It is worth pointing out that the stress-strain curve is an input obtained from an actual laboratory test. Subsequently, subgrade reaction is derived by using Eq. 9. Hence, subgrade reaction $p_k$ is a nonlinear function of the parameter $\Delta$ given a specific soil stress-strain curve from the average of the PSC test and PSE test. The subgrade modulus $k_r$ is just the first-order derivative of Eq. 9 with parameter $\Delta$.

In the case of oval shape, subgrade reaction $p_k$ is distributed in an oval shape represented by following equation:

$$p_k = p_{k0} \cos 2\theta$$  \hspace{1cm} (10)

where $p_{k0}$ is the subgrade reaction at tunnel crown and the angle $\theta$ is equal to zero. By substituting Eq. 1b and Eq. 10 into Eq. 7, the $p_{k0}$ is further derived as below:

$$p_{k0} = \frac{\int_{\text{area}} s^*_{u,mob} \cdot \varepsilon_\delta(\delta) \, dA}{\delta \int_{\text{c}D} \cos^2 2\theta \, ds}$$  \hspace{1cm} (11)

where $\delta$ is the tunnel radial displacement at tunnel crown (see in Eq. 1b). Similar to the uniform case, when the parameter $\delta$ is obtained, the subgrade reaction $p_{k0}$ is derived as a function of radial displacement $\delta$ given a specific soil stress-strain curve from the average of the PSE and PSC tests. Subgrade modulus $k_r$ under oval mode is also obtained from the
first-order derivative of Eq. 11.

**Validation of Proposed Nonlinear Solution of** $p_k$

The proposed nonlinear subgrade reaction $p_k$ is derived analytically using the mobilized strength concept and an approximate mobilized shear strength $s^*_{u,mob}$ that is independent of principal strain rotation. In this section, this nonlinear subgrade reaction $p_k$ is validated by comparing with the numerical result of $p_k$ from a two-dimensional finite element method (FEM) analysis. A homogeneous soil condition is assumed in FEM and MSD to simplify the validation. The FEM analysis requires a soil constitutive model, such as the Hardening soil (HS) model. The MSD does not require a soil constitutive model. In practice, it uses the measured laboratory test data from the average of PSE and PSC test results. In this section, the PSE and PSC stress-strain curves are calculated numerically using a commercial FEM code PLAXIS2D (Brinkgreve et al. 2006) on an element extracted at the depth of tunnel axis with an element length, width and height equal to $1\text{m} \times 1\text{m} \times 1\text{m}$. The same Hardening soil (HS) model is used in this numerical element soil test. Hence, given a prescribed displacement $u_r$, the analytical $p_{kh}$ is calculated by using Eqs. 9 and 11 with a calculated stress-strain curve from the average of numerical PSC and PSE test results.

The PLAXIS2D code is also used for FEM analysis. Figure 4 is a representative finite element mesh used in this paper. A tunnel with radius $R$ of 3m is buried 22m below the ground surface. The cover depth $h$ is thus equal to 25m. Due to the symmetrical feature of this problem, only half of the model is established as shown in Fig. 4a. The half-width and depth of the mesh is equal to 200m and 120m, respectively, which are sufficiently large to minimize boundary effects. The horizontal displacement along the vertical boundaries at two sides of the mesh are constrained, while the vertical displacement along the horizontal boundary at the bottom of mesh is constrained. An example of stress boundary conditions following Eqs. 8 and 10 is shown in Fig. 4b and 4c.
The input stress boundary around the tunnel perimeter is the subgrade reaction $p_k$. The response obtained from PLAXIS2D with a specific constitutive soil model (Hardening soil model) is the soil radial displacement $u_r$ at the tunnel crown. Note that the radial subgrade modulus $k_r$ varies around the tunnel perimeter – it is smaller at the tunnel crown than at the invert (Verruijt 1997). In other words, a uniform subgrade reaction around the tunnel perimeter produces a larger soil radial displacement $u_r$ and consequently a smaller subgrade modulus $k_r$ (i.e., $k_r = p_k / u_r$) at the tunnel crown compared to the results at other positions around the tunnel perimeter. Hence, a design based on the smallest subgrade modulus $k_r$ at the tunnel crown would be most conservative, i.e. the calculated internal lining forces are the highest. This study focuses on the nonlinear $p_k$ versus $u_r$ curve at the tunnel crown for this reason.

The Hardening soil (HS) model that is widely used in tunnel designs is selected for this validation study. The HS model is essentially a hyperbolic nonlinear model that considers both shear hardening and compression hardening. Details of this soil model are given elsewhere (Möller and Vermeer 2008). Typical stress strain curves for HS model are illustrated in Fig. 5, i.e., solid line for the plane strain compression condition and dash line for the plane strain extension condition. The average mobilized strength obtained by averaging the results from both PSC and PSE tests is also plotted against shear strain in Fig. 5 as the dot-dash line. A set of input soil parameters required for HS model are provided in Table 1 for typical undrained Shanghai soft clay following the suggestion by Zhang et al. (2015). Note that the soil unit weight input for FEM analysis is set to zero because the work done by soil weight is not included in the energy conservation equation (see in Eq. 4). Undrained analysis with undrained strength parameter ($s_u$) (method B in Plaxis) is carried out in this validation study. For method B in PLAXIS, note that the stiffness modulus is no longer stress level dependent, because the effective friction angle is equal to zero. Hence, the HS...
model adopted in this paper exhibits no compression hardening. As a result, the soil used in
FEM model exhibits homogeneous soil stiffness. But the model retains its
unloading-reloading modulus and shear hardening characteristics. The nonlinear shear
hardening in the HS model contributes to the nonlinear mobilization of the soil undrained
shear strength.

**Comparison with FEM Analysis**

Figure 6 compares the calculated nonlinear curve of $p_k$, between MSD and FEM under
the uniform deformed shape condition. The horizontal axis refers to the ratio of the tunnel
convergence $\Delta$ over the tunnel radius $R$, and the vertical axis refers to the representative
subgrade reaction $p_{k0}$ at tunnel crown. The solid square dots in Fig. 6 are the FEM results
$p_{k0,FEM}$ from PLAXIS, and the solid line denotes the MSD results $p_{k0,MSD}$ with $s_{u,mob}^*$
determined from the average of $s_{u,mob}$ from PSC and PSE test condition. It is clear from Fig.
6 that the MSD results $p_{k0,MSD}$ using an average stress strain curve is almost equal to the
numerical results of $p_{k0,FEM}$. Thus, the approximation $s_{u,mob}^*$ composed by the average of
PSC and PSE test results (Eq. 6) appears to be reasonable for this example. Similar findings
have also been presented for the tunneling-induced ground movement calculation where the
same approximation of mobilized strength is adopted (Osman et al. 2006a).

In addition, the linear elastic solution $p_{k0,linear}$ presented by Verruijt and Booker (1996) and
Zhang et al. (2013) is also plotted against the convergence ratio ($\Delta/R$) in Fig. 6 (denoted by
dotted lines). When the $\Delta/R$ is relatively small, say within a value of 0.1-0.2%, the
nonlinear $p_{k0,MSD}$ is almost equal to the linear solution, which shows that the nonlinear
solution reduces correctly to the linear solution at small strain level. However, the
nonlinearity of $p_{k0,MSD}$ could become significant as the tunnel convergence increases. The
shape of nonlinear variation of $p_k$ with $\Delta$ is similar to the nonlinear shape of stress strain
curve. Hence, these figures indicate that linear $p_{k0,linear}$ is only appropriate for the design of
tunnel lining where the convergence ratio ($\Delta/R$) is smaller than 0.2%. The convergence ratios recommended in various codes are indicated by arrows in Fig. 6, where 0.2% is still below the lower bound of the ratio for serviceability limit state (SLS). Otherwise, the tunnel structural design, i.e., deformation and internal forces, could be significantly underestimated since the subgrade reaction $p_k$ is overestimated by the linear solution. This design impact will be further discussed by a design example presented later in this paper.

Figure 7 compares the calculated $p_{k0}$ between MSD and FEM for the oval deformed shape condition. The solid line in Fig. 7 is the MSD results $p_{k0,MSD}$ with the average stress-strain curve from PSE and PSC condition, while the solid dots denote the $p_{k0,FEM}$ calculated from FEM analysis. The variation of MSD results ($p_{k0,MSD}$) with convergence ratio ($\delta/R$) matches the same trend from FEM results reasonably well. The linear solution of $p_k$ varying with the convergence ratio is also illustrated in Fig. 7 (denoted by dotted line). The nonlinear solution of $p_{k0,MSD}$ at small strain range also agree well with the linear solution presented in Fig. 7. It should be specifically noted that the magnitude of $p_{k0}$ under the oval mode is almost 1.5-2 times of the magnitude of $p_{k0}$ under the uniform mode. It is consistent with results from the elastic solution presented by Zhang et al. (2013). One possible explanation is that an oval deformation shape absorbs more soil energy at a given strain level compared to the uniform deformation shape. Overall, the proposed nonlinear solution of $p_k$ produced by the MSD method agrees reasonably well with the FEM solution for both uniform and oval shapes for one example. More extensive validation is discussed in the following section.

**Correction factor $\eta$ for proposed nonlinear solution of $p_k$**

One baseline example with input parameters given in Table 1 has been studied above. From an application perspective, engineers would be interested to know the detailed difference between this analytical MSD solution and the actual field measurement over a wider range of conditions encountered in practice. This difference is typically characterized
by a model factor (ratio of measured \( p_k \) over the calculated value) in geotechnical reliability based design (Phoon and Kulhawy 2005). Strictly speaking, the numerical subgrade reaction \( p_k \) from FEM analysis is not equal to the measured \( p_k \), but comparisons between FEM solutions and field data for a number of geotechnical structures indicate that this difference (i.e., model factor \( \varepsilon_{\text{FEM}} \)) is small. Several calibrations of finite element analysis including finite element limit analysis (FELA) with field cases or physical model tests show that mean of \( \varepsilon_{\text{FEM}} \) is between 0.95 and 1.01 and COV is between 0.06 and 0.18, as shown in Table 2 (Phoon and Tang 2015a, 2015b; Tang and Phoon 2016a, 2016b, 2017). It is reasonable to say that mean and COV of \( \varepsilon_{\text{FEM}} \) should be around 1 and around 0.1, respectively. Nonetheless, to maintain a strict distinction between FEM solutions and field measurements, Zhang et al. (2015) defined a ratio of FEM solution over calculated MSD value as a correction factor. The model factor can be constituted as the product of the correction factor and the model factor for FEM (typically unbiased, i.e. mean close to 1 and precise, i.e. coefficient of variation around 10%). The purpose of this section is to characterize the correction factor for the nonlinear analytical solution of \( p_k \) using the method proposed in Zhang et al. (2015). The characterization of the correction factor for the oval mode shape is presented in detail below.

It is clear from Eqs. 2, 3, 9 and 11 that the calculated nonlinear solution \( p_k \) depends on the input parameters such as cover depth \( h \), tunnel diameter \( D \) (or radius \( R \)), soil undrained shear strength \( s_u \) and convergence deformation \( \delta \). Besides, the mobilized stress-strain curve could also contribute to the evaluation of \( p_k \) in the calculation. Because the Hardening Soil model is applied, the soil unload-reloading modulus \( E_{ur} \) is also included as an input parameter. In summary, a total of four dimensionless parameters from the above five parameters are considered in the orthogonal design of numerical cases required for characterization of the correction factor, i.e., cover depth ratio \( h/D \), undrained strength ratio \( s_u/\sigma_v' \), soil
unload-reloading modulus ratio $E_{ur}/S_u$, and the tunnel convergence ratio $\delta/R$.

Table 3 shows the typical ranges of these four parameters for tunnels in soft undrained clay condition. The 197 numerical cases are designed in two stages. In the first stage, only the parameters $h/D$,  $s_u/\sigma_v'$, $E_{ur}/S_u$ are included to generate 25 orthogonal FEM models. Each parameter has five different levels. In the second stage, for each one of the generated 25 models, different levels of convergence ratio ($\delta/D$) are generated within the range of 0.001% – 3%, i.e., typical range of tunnel deformation on site (Huang et al. 2016). Similar to the procedure carried out for the baseline example, the $p_k$ is prescribed as the stress boundary condition for FEM model in PLAXIS2D, while the calculated displacement $\delta$ from PLAXIS2D is applied as an input parameter for MSD calculation to obtain an analytical solution of $p_k$. In total, there are 197 cases generated using these two-stage orthogonal design principle.

Figure 8 plots the calculated $p_k$ from MSD (denoted as $p_{k, MSD}$ hereon) against the prescribed $p_k$ in FEM model (denoted as $p_{k, FEM}$ hereon) for the 197 cases in a logarithm form of the axes, denoted by grey square dots. It is observed from Fig. 8 that the $p_{k, MSD}$ distributed relatively closely around the 45 degree line (i.e., the equality line $p_{k, MSD} = p_{k, FEM}$). However, it is evident from Fig. 8 that the discrepancy between grey dots and 45 degree line becomes larger as the absolute value of $p_k$ increases. To characterize this discrepancy statistically, the correction factor $\eta$ is defined as the ratio of the magnitude of $p_{k, FEM}$ over the magnitude of $p_{k, MSD}$.

$$\eta = \frac{p_{k, FEM}}{p_{k, MSD}}$$  (12)

The model factor for $p_{k, MSD}$ is related the correction factor $\eta$ as follows:

$$\varepsilon_{MSD} = \frac{P_{k, m}}{P_{k, MSD}} = \frac{P_{k, FEM}}{P_{k, MSD}} \times \frac{P_{k, m}}{P_{k, FEM}} = \eta \times \varepsilon_{FEM}$$  (13)

where $p_{k,m}$ is the measured subgrade reaction. Table 2 shows that the mean and coefficient...
of variation (COV) of $\varepsilon_{\text{FEM}}$ is close to 1 (unbiased) and relatively small (less than 20%) for a number of different geotechnical systems studied thus far. The mean of $\eta$ for the 197 cases is equal to 1.02, which means the calculated $p_{k,MSD}$ is almost equal to the $p_{k,FEM}$ on the average. The minimum and maximum value of $\eta$ are 0.66 and 1.28, respectively, as shown by the grey dashed lines in Fig. 8. The COV of the correction factor $\eta$ is 0.15, which is quite small compared to the MSD for other geotechnical structures, e.g., 0.47 for cantilever deflection as shown in Table 2 (Zhang et al. 2015). The mean and COV of the correction factor $\eta$ for a variety of geotechnical problems such as foundation systems in different type of soils (Table 2) appear to be around 1 and 0.10, respectively.

It is worthwhile to examine if the correction factor $\eta$ is dependent on input parameters, because it is clearly dependent on the magnitude of $p_k$ as shown in Fig. 8. The customary practice for engineers to correct for model bias using the average model factor (deterministic method) or as a random variable (reliability method) is only correct when the model factor is random. To clarify this critical dependency issue, the calculated correction factor $\eta$ is plotted against the four input parameters $h/D$, $s_u/\sigma_v'$, $E_{ur}/s_u$ and $\delta/R$ as shown in Fig. 9a, 9b, 9c and 9d, respectively. Note that the vertical axis in Fig. 9a – 9c is represented by averaged correction factor $\eta_{\text{ave},i}$ for 25 orthogonally designed FEM models in the first stage, which is calculated as below:

$$\eta_{\text{ave},i} = \frac{\sum_j \eta_{i,j}}{m}$$

(14)

where $\eta_{i,j}$ is the calculated correction factor for the total of 197 designed FEM cases, $i$ is equal to 1, 2, ..., 25 which stands for the number of 25 orthogonal designed case in the first stage, and $j$ is equal to 1, 2, ..., $m$ which stands for the number of $m$ cases in the second stage given the parameter set of $h/D$, $s_u/\sigma_v'$, $E_{ur}/s_u$ have been selected in the first stage. For example, the averaged $\eta_{\text{ave},1}$ is obtained by taking the arithmetic average of $\eta_{1,j}$ values from 6
simulations (i.e., \( m = 6 \)) with the following input parameters:

1. \( h/D = 2, \ s_u/\sigma_v' = 0.5, \ E_{ur}/s_u = 200, \ \delta/R = 0.025\%, \ \eta_{1,1} = 0.91; \)

2. \( h/D = 2, \ s_u/\sigma_v' = 0.5, \ E_{ur}/s_u = 200, \ \delta/R = 0.061\%, \ \eta_{1,2} = 0.94; \)

3. \( h/D = 2, \ s_u/\sigma_v' = 0.5, \ E_{ur}/s_u = 200, \ \delta/R = 0.122\%, \ \eta_{1,3} = 0.97; \)

4. \( h/D = 2, \ s_u/\sigma_v' = 0.5, \ E_{ur}/s_u = 200, \ \delta/R = 0.242\%, \ \eta_{1,4} = 1.02; \)

5. \( h/D = 2, \ s_u/\sigma_v' = 0.5, \ E_{ur}/s_u = 200, \ \delta/R = 0.605\%, \ \eta_{1,5} = 1.15; \)

6. \( h/D = 2, \ s_u/\sigma_v' = 0.5, \ E_{ur}/s_u = 200, \ \delta/R = 1.419\%, \ \eta_{1,6} = 1.19; \)

Thus, the calculated averaged \( \eta_{\text{ave},1} \) is equal to 1.04, i.e., \((0.91+0.94+0.97+1.02+1.15+1.19)/6.\)

By doing so, the averaged correction factor \( \eta_{\text{ave}} \) is plotted against the input parameters in Fig. 9a to 9c. It is clear that \( \eta_{\text{ave}} \) is strongly dependent on the cover depth ratio \( h/D \), but is relatively independent of strength ratio \( s_u/\sigma_v' \) and modulus ratio \( E_{ur}/s_u \). The dependency of \( \eta_{\text{ave}} \) on the parameter \( h/D \) seems to follow a sigmoid function. In Fig. 9d, the calculated correction factor \( \eta \) for all the 197 cases is plotted directly against \( \delta/D \) on a logarithm scale. A linear correlation between \( \eta \) and \( \ln(\delta/R) \) is obtained as shown in Fig. 9d. By conducting Spearman correlation tests for all these four parameters, the \( p \)-value associated with a null hypothesis of zero-rank (Spearman) correlation is much less than the strict 1\% level of significance for parameter \( h/D \) and \( \delta/R \). Hence, the null hypothesis that the correction factor \( \eta \) is independent of the input parameter \( h/D \) and \( \delta/R \) can be rejected at a level of significance of 1\%.

Because the correction factor is correlated to two input parameters, it cannot be modeled as a random variable (Phoon and Kulhawy 2005). Figure 9 indicates that the correction factor \( \eta \) varies with the input parameter \( h/D \) as a sigmoid function and varies with the input parameter \( \ln(\delta/R) \) as a linear function. This systematic variation should be removed by regression. Based on the above observed trends, the proposed regression equation is:
where \( a_i \) is the coefficients in Eq. 15. A multiple regression analysis is carried out to determine the regression coefficients \( a_i \). The values of the regression coefficients \( a_i \) are shown in Table 4. The coefficient of determination \( (R^2) \) of Eq. 15 associated with the coefficients given in Table 4 is about 0.9. The \( R^2 \) is relative high, which means Eq. 15 describes the systematic part of correction factor \( \eta \) very well.

The cover depth ratio \( h/D \) affects the displacement field (Eq. 3) and the tunnel convergence ratio \( \delta/R \) is considered by the energy conservation equation (Eq. 11). Equation 15 describes the systematic part of the ratio between the subgrade reaction computed by FEM versus MSD. The existence of this regression equation as a function of \( h/D \) and \( \delta/R \) implies that these input parameters produce secondary effects beyond the primary effects already covered by the displacement field and the energy conservation equation. Figure 10 compares the tunnel deflection at the springline \( s \) produced by FEM and MSD against the parameters \( h/D \) and \( \delta/R \) shown in Eq. 15. The tunnel deflection \( s \) (an output) is distinct from the imposed displacement at the tunnel boundary \( \delta \) (a “loading” input). The data points refer to the results produced by the 197 orthogonal scenarios involving a wide range of input parameters described previously. Figure 10a shows that the discrepancy between FEM and MSD becomes significant as the tunnel cover depth decreases (say \( h/D \) less than 1 or 2). The average ratio of \( s_{\text{FEM}}/s_{\text{MSD}} \) converges to 0.94 for deep tunnels. For shallow tunnels, the average ratio can be as low as 0.64 for \( h/D = 1 \). It is possible that the elastic displacement field assumed in MSD model departs from the elasto-plastic field used in FEM for shallow tunnels. Figure 10b shows the ratio of \( s_{\text{FEM}}/s_{\text{MSD}} \) versus the convergence ratio \( \delta/R \). With the exception of results from \( h/D = 1 \) (closed markers), the ratio generally increases with \( \delta/R \). One expects the FEM displacement field to be close to elastic (assumed in MSD) and
therefore $s_{\text{FEM}}/s_{\text{MSD}} \approx 1$ for small “loading” $\delta/R$. However, this is not the trend shown in Fig. 10b. This may imply that there are other reasons besides the elastic displacement field assumption in MSD, such as the lack of a soil unload-reload modulus in the stress-strain curve for MSD and the averaging strength assumption in Eq. 6. The physical reasons underlying Eq. 15 were not systematically studied in this paper.

The residual of the regression equation $f_i$, denoted as $\eta_i^*$, should be the random part of the correction factor $\eta$.

$$\eta = f \times \eta^*$$

The residual $\eta^*$ is also plotted against the four parameters in Fig. 9 (denoted by hollow dots in each figure). It is clear that the residual $\eta^*$ is not correlated to any input parameter, particularly to $h/D$ and $\delta/R$, and it is appropriate to model it as a random variable. The histogram of calculated residual $\eta^*$ is plotted in Fig. 11 (grey bars). The residual $\eta^*$ is observed to be lognormally distributed. The mean of residual $\eta^*$ is equal to 1.00 and the COV is about 0.05, which suggest the precision of the proposed nonlinear solution of $p_{k,\text{MSD}}$ could be further improved if Eq. 16 is adopted. In summary, the analytical MSD solution of $p_k$ under the oval deformation mode can be as good as the corresponding FEM solution when it is corrected as follows:

$$p_{k,0} = \int_{\Delta x} \frac{s_{\text{mob}}^* \cdot \varepsilon_x (\delta) dA}{\int_{xD} \cos^2 2\theta ds} \times f \times \eta^*$$

The only marginal cost is the model uncertainty $\eta^*$, the associated COV of 0.05 is negligible for all practical purposes within the geotechnical context. The proposal in this paper is to modify the original MSD subgrade reaction ($p_{k,\text{MSD}}$) by the regression function $f$. Figure 8 also plots the modified MSD subgrade reaction ($f \times p_{k,\text{MSD}}$) against the $p_{k,\text{FEM}}$, denoted by the solid circular dots. It is clear from Fig. 8 that the discrepancy between the $p_{k,\text{MSD}}$ and the $p_{k,\text{FEM}}$ is reduced in particular at a large magnitude of $p_k$. The range between upper and
lower bound limits for $\eta^*$ has narrowed down to a maximum of 1.11 and a minimum of 0.85.

The same characterization procedure is carried out for the uniform deformation mode. The regression equation $f$ is derived by using 146 orthogonally designed numerical cases. The analytical MSD solution of $p_k$ for the uniform deformation mode can be as good as the corresponding FEM solution when it is corrected as follows:

$$p_{k,0} = \frac{\int_{s_{\text{mob}}} \cdot \mathbb{E}_s(\Delta) dA}{\Delta \int_{s_D} ds} \cdot \exp \left[ b_0 + b_1 \frac{h}{D} + b_2 \left( \frac{\Delta}{R} \right)^{-1} \right] \cdot \eta^*$$

(18)

The coefficients $b_i$ in the correlation function $f$ are also given in Table 4. The original correction factor $\eta$ has a mean of 1.08 with a COV at 0.15. However, by removing the dependency on input parameters using the regression equation $f$, the residual $\eta^*$ can be modeled as a lognormally distributed random variable with a mean of 1.00 and a COV of 0.07.

**Application Example**

The proposed nonlinear solution of subgrade reaction $p_k$ for tunnel lining design is applied to a field case reported by Huang and Zhang (2016). The site of this case located in southeast part of Shanghai in China. A running metro tunnel with a diameter of 6.2m and a wall thickness of 0.35m was buried 16.4m below the initial ground surface. A sectional profile is shown in Fig. 12. Unfortunately, 360 lining rings of this tunnel has been subjected to an unexpected ground surcharge due to soil dumping during daily metro operation. The dumped soil has a height $H$ ranging from 1.7m to 7m (shaded area in Fig. 12) causing extreme surcharge loads from 30kPa to 120kPa. The design level for surcharge is 20kPa, which is much smaller than this accidental surcharge. Hence, performance of the segmental lining experienced a severe disruption as many of structural defects could be observed on site (Huang and Zhang 2016). In this circumstance, the structural loads in the segmental lining can be re-calculated to better understand the structural behavior and performance robustness.
of this tunnel subjected to such an unexpected surcharge.

Since the tunnel lining ring of this case is composed of six concrete segments jointed by steel bolts, the embedded beam model incorporating discontinuous joint are adopted in the design. This structural model is proposed by Hashimoto et al. (1994). A typical layout of the model is shown in Fig. 13a. Segmental joint is simulated by a three dimensional joint spring representing shear, axial and rotational discontinuous between segments. Thus, spring constants in these three dimensions are the input parameters for modeling of the joint. These parameters are extracted from the literature (Ding et al. 2004). The loads acting on the lining, e.g., vertical total pressure \( p_1, p_2 \), lateral total pressure \( q_1, q_2 \) and dead load \( p_g \), are determined by following the ITA guideline (ITA 2004). Note that the extreme surcharge is assumed to be fully added on the calculation of vertical load \( p_1 \) ignoring the effect of load spreading along depth. It is acceptable due to the relative shallow depth of the tunnel in Shanghai soft clay (DGJ08-10 2010). The properties of soil and segmental linings used for the calculation are provided in Table 5. The mobilized stress-strain curve is built based on the HS model from the element soil tests in PLAXIS2D using the soil properties given in Table 5. The measured sectional radial displacements of this case indicate a significant oval deformation mode (Huang et al. 2016). Hence, the nonlinear solution of \( p_k \) under oval mode is applied in this example. The nonlinear \( p_k-u_r \) curve is first calculated by using the proposed MSD method both for the corrected solution (Eq. 17) and the uncorrected solution (Eq. 11). The calculated nonlinear \( p_k-u_r \) curves are plotted in Fig. 13b (\( p_k \) without correction represented by solid dot line and \( p_k \) with correction represented by hollow dot line). For comparison, the embedded beam model with linear soil spring is also included in the design. The linear \( p_k-u_r \) curve with a slope \( (k_r) \) identically equal to 6870kN/m\(^3\) is also plotted in Fig. 13b (represented by dash line). The discrepancy between linear and nonlinear \( p_k-u_r \) curve is quite significant especially when convergence is large.
The structural responses of tunnel to the extreme surcharge are also analyzed by finite element method using the PLAXIS2D code. The finite element mesh corresponding to this case is illustrated in Fig. 14. Only half of the model is simulated assuming that the surcharge is also distributed symmetrically with the vertical axis of metro tunnel. This assumption generally agrees well with the field condition, as shown in Fig. 12. The mesh has a width of 200m and a depth of 120m. The HS model is used to simulate the soil behavior, and the elastic model is used to simulate the lining behavior. Input parameters for these two models are shown in Table 5. The joint between two segment linings is simulated by a continuous rotation spring, as shown in the insert in Fig. 14, with a rotation stiffness equal to that of the embedded beam spring model given in Table 5. Compared to the embedded beam spring model, it should be noted that the shear and axial discontinuity have not been simulated in this PLAXIS2D model. The stress reduction method, also named as $\beta$-method, is adopted to simulate the installation of tunnel lining in numerical analysis (Möller and Vermeer 2008). A reduction factor $\beta$ equal to 0.25 is selected to match the typical volume loss at 0.5% commonly encountered in tunneling practice in Shanghai (Ding et al. 2004). The extreme surcharge above the ground surface is simulated by line load with a same distribution range as that in the field. The load levels for surcharge are set equivalent to the height of dumped soils from 2m to 8m. Hence, for this design example, the tunnel convergence, bending moment and axial forces are calculated by four design models, namely embedded beam with the corrected nonlinear soil spring model (nonlinear corrected in short), embedded beam with the uncorrected nonlinear soil spring model (nonlinear uncorrected in short), the embedded beam with linear soil spring model (linear model) and the FEM model.

Figure 15 plots the calculated tunnel convergence $\delta$ at crown by using four design models mentioned previously. The in-situ tunnel convergence at the crown for 360 rings also has
been measured and is plotted in Fig. 15 (solid circular dots). To minimize the effect of initial cover depth on the measured convergence, the horizontal axis only covers the relative surcharge $H$ over initial cover depth to tunnel crown $C$. It is clear from Fig. 15 that both the predicted and measured convergence increase as the surcharge level increases. Among the results from the four types of design models, the convergence predicted by nonlinear corrected model and FEM model could capture both the trend and magnitude of measured convergence more accurately. The prediction by using linear model is approximately half of the magnitude of measured convergence. The prediction by using nonlinear uncorrected model is slightly larger than the measured data. Corresponding to the measured convergence ratio $\delta/R$ ranging from 1.5% to 3.5%, the magnitude of subgrade reaction $p_k$ for linear spring model could be 2-4 times of the value for nonlinear spring model (see the shaded area in Fig. 13b). The overestimation of $p_k$ by linear spring model could be the main reason that produces an unsafe prediction of tunnel convergence in Fig. 15.

The prediction of internal forces from the above four models are plotted in Fig. 16 (Fig. 16a for bending moments and Fig. 16b for axial forces). The predictions both from nonlinear corrected and nonlinear uncorrected models are quite comparable to the results from FEM model, but they are much larger than the results from linear model. It is not surprising to obtain these results due to the overestimation of $p_k$ in the linear model. An overestimated subgrade reaction would over-confine the lining deformation and thus result in a small bending moment. Similar results can be observed in Fig. 16b for axial forces. The axial force from linear spring model is much smaller than those from nonlinear corrected model, nonlinear uncorrected model and FEM model. But it is also interesting to find that the axial forces from FEM model is larger than those from nonlinear models. The discrepancy of results will increase as the surcharge increases. This observation could be explained by the fact that the discontinuity of segmental lining are not simulated in FEM.
A joint model with shear and axial discontinuity would absorb some of the shear and axial forces. Hence, the axial forces in the segmental lining would be smaller compared to those results produced by a joint model with rotational discontinuity only. Similar numerical results could also be observed in the paper elsewhere (Ding et al. 2004). By comparing to the measured axial forces, the discontinued model for segmental lining could predict the results much better than the conventional continuous FEM model does. To sum up from Fig. 16, it is clear that a nonlinear subgrade reaction can be important when the tunnel convergence is large.

Conclusion

In the practical application of embedded beam model for structural design of tunnels, the soil spring is always regarded to be linear elastic. This could hardly match the reality of significant nonlinear properties of soils on site. To solve this problem, this paper has presented an analytical nonlinear solutions of subgrade reaction-displacement ($p_k-u_r$) curve by using the mobilized strength design (MSD) method. Two typical deformation modes for soil-lining interaction, i.e., uniform and oval shape functions, are used to calculate the energy loss of ground movement induced by subgrade reaction. With the concept of nonlinearly mobilized shear strength with engineering shear strain by averaging the curves both from compression test and extension test, the subgrade reaction $p_k$ is thus obtained via the energy conservation in the mobilized undrained clays.

The proposed nonlinear solution of $p_k$ varying with tunnel convergence ($\Delta$ or $\delta$) is successfully validated by a rigorous 2D FEM analysis with the Hardening soil (HS) model. From the statistical point of view, the subgrade reaction $p_k$ from FEM analysis could be predicted by the $p_k$ from MSD analysis multiplying a correction factor $\eta$ of 1.02. However, the $\eta$ is correlated with the input parameters. If the systematic correlation characterized by a regressed function $f$ is removed in the correction factor ($\eta^* = \eta/f$), the bias of proposed...
nonlinear model of $p_{k,\text{MSD}}$ compared to FEM result $p_{k,\text{FEM}}$ could be further reduced to a residual $\eta^*$. The residual $\eta^*$ could be a lognormally distributed random variable with a mean of 1.00 and a COV of 0.05 for oval mode and with a mean of 1.00 and a COV of 0.07 for uniform mode, which are negligible for all practical purposes within the geotechnical context.

As shown in the application example at the end of this paper, the merit of nonlinear solution of $p_k$ should be greatly appreciated when tunnel has a large deformation. The prediction of convergence from nonlinear solution of $p_k$ has a very well agreement with measured data and FEM model in this example. However, the magnitude of $p_k$ from linear solution could be overestimated by 2-4 times the value from nonlinear solution, resulting in a significant underestimation of the predicted tunnel convergence and inner forces by 2-4 times. Hence, the proposed nonlinear solution of $p_k$ should be helpful in the structural design of tunnel lining especially for the case that soil nonlinear behavior is mobilized significantly.

However, it should be noted that the proposed nonlinear solution of $p_k$ is largely based on either the uniform mode or the oval mode. It also has been found in this paper that the value of $p_k$ for oval shape could be 1.5-2 times the value for uniform shape. In other words, a pre-judgement might be made before the application of proposed method in that the dominant deformation mode should be selected. Based on the application example, the oval deformation mode appears to agree with the measured data. This is the cost of analytical solution by using MSD compared to the purely numerical analysis. Furthermore, soil shearing may occur before the installation of the tunnel lining, resulting in a possible reduction in soil strength among others. This aspect and other construction aspects that could affect the determination of internal forces in the lining are not considered in this paper.

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Captions for tables and figures

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Figure 15 Comparison of predicted tunnel deformation from different design model with measured data; Note: measured data after (Huang and Zhang 2016).

Figure 16 Comparison of the predicted internal forces from different design models between linear subgrade reaction and nonlinear subgrade reaction model: a) bending moment; and b) axial force.
Table 1 Input parameters for HS model used in the validation (modified after data from Zhang et al. (2015))

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter (Unit)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>General</td>
<td>Reference vertical earth pressure $p_{ref}^{*}$ (kPa)</td>
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</tr>
<tr>
<td></td>
<td>Soil unit weight $\gamma$ (kN/m$^3$)</td>
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<tr>
<td></td>
<td>Soil initial void ratio $e_0$</td>
<td>1</td>
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<tr>
<td>HS</td>
<td>Soil secant stiffness in standard triaxial test $E_{s0}$ (kPa)</td>
<td>9800</td>
</tr>
<tr>
<td></td>
<td>Soil tangent stiffness for primary oedometer loading $E_{oed}$ (kPa)</td>
<td>7840</td>
</tr>
<tr>
<td></td>
<td>Soil unloading-reloading stiffness $E_{ur}$ (kPa)</td>
<td>29400</td>
</tr>
<tr>
<td></td>
<td>Soil undrained shear strength $s_u$ (kPa)</td>
<td>80</td>
</tr>
</tbody>
</table>
Table 2 Summary of statistics for correction factors and FEM (or FELA) model factor for different geotechnical systems

<table>
<thead>
<tr>
<th>Problem</th>
<th>$\eta^{(a)}$</th>
<th>$\eta^{(b)}$</th>
<th>$\varepsilon^{(c)}_{\text{FEM or FELA}}$</th>
<th>$N^{(d)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lateral deflection at top of cantilever wall (MSD method) (Zhang et al., 2015)</td>
<td>1.01</td>
<td>0.47</td>
<td>1.01</td>
<td>59</td>
</tr>
<tr>
<td>Strip footings on sand under positive combined loading (Phoon and Tang 2015a)</td>
<td>1.65</td>
<td>0.22</td>
<td>1.03</td>
<td>120</td>
</tr>
<tr>
<td>Strip footings on sand under negative combined loading (Phoon and Tang 2015b)</td>
<td>2.00</td>
<td>0.24</td>
<td>1.06</td>
<td>72</td>
</tr>
<tr>
<td>Strip footings on sand under general combined loading (Phoon and Tang 2015b)</td>
<td>2.00</td>
<td>0.25</td>
<td>1.04</td>
<td>192</td>
</tr>
<tr>
<td>Helical anchors in clay under tension loading (Tang and Phoon 2017)</td>
<td>1.37</td>
<td>0.26</td>
<td>0.98</td>
<td>78</td>
</tr>
<tr>
<td>Circular footings on dense sand (Tang &amp; Phoon 2016a)</td>
<td>1.18</td>
<td>0.32</td>
<td>1.02</td>
<td>26</td>
</tr>
<tr>
<td>Bearing capacity of dense sand overlying clay (punching shear method) (Tang and Phoon 2016b)</td>
<td>2.63</td>
<td>0.31</td>
<td>1.01</td>
<td>62</td>
</tr>
<tr>
<td>Bearing capacity of dense sand overlying clay (load spread method tan$\alpha_p$=1/3) (Tang and Phoon 2016b)</td>
<td>1.83</td>
<td>0.30</td>
<td>1.01</td>
<td>62</td>
</tr>
<tr>
<td>Bearing capacity of dense sand overlying clay (load spread method tan$\alpha_p$=1/5) (Tang and Phoon 2016b)</td>
<td>2.32</td>
<td>0.33</td>
<td>1.02</td>
<td>62</td>
</tr>
</tbody>
</table>

Note:

a) $\eta =$ correction factor
b) $\eta^* =$ residual part of the correction factor $\eta$, namely, $\eta^* = \eta / f$.
c) $\varepsilon_{\text{FEM}}$ (or $\varepsilon_{\text{FELA}}$) = model factor of finite element method analysis or finite element limit analysis, defined as $\varepsilon_{\text{FEM}} = Q_m / Q_{C,FEM}$ (or $\varepsilon_{\text{FELA}} = Q_m / Q_{C,FELA}$), where $Q_m =$ measured data.
d) $N =$ number of measured data for field cases and model tests.
Table 3 Parameters and Their Ranges of Values used in the Numerical Simulations

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Ranges of Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h/D$</td>
<td>1.00 – 5.00</td>
</tr>
<tr>
<td>$s_u/\sigma_v$</td>
<td>0.30 – 0.70</td>
</tr>
<tr>
<td>$E_{ur}/s_u$</td>
<td>200.00 – 900.00</td>
</tr>
<tr>
<td>$\delta/R \ (A/R) \ (%)$</td>
<td>0.001 – 3.000</td>
</tr>
</tbody>
</table>
Table 4 Best regressed coefficient $a_i$ for systematic part $f$ in correction factor $\eta$

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Oval mode Value</th>
<th>Coefficient</th>
<th>Uniform mode Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>0.172</td>
<td>$b_0$</td>
<td>-0.109</td>
</tr>
<tr>
<td>$a_1$</td>
<td>-0.361</td>
<td>$b_1$</td>
<td>0.0404</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.0591</td>
<td>$b_2$</td>
<td>0.00520</td>
</tr>
<tr>
<td>$a_3$</td>
<td>1.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.90</td>
<td>$R^2$</td>
<td>0.91</td>
</tr>
</tbody>
</table>
Table 5 Input parameters for the design example (modified after data from Ding et al. 2004; Huang and Zhang 2016).

<table>
<thead>
<tr>
<th>Design objective</th>
<th>Design input parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tunnel Lining</td>
<td>Outer-diameter $D$ (m)</td>
<td>6.2</td>
</tr>
<tr>
<td></td>
<td>Wall thickness $t$ (m)</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>Elastic modulus $E$ (kPa)</td>
<td>$3.45 \times 10^6$</td>
</tr>
<tr>
<td></td>
<td>Joint compressive stiffness $k_{o_{\text{joint}}}$ (kN/m)</td>
<td>$1 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Joint shear stiffness $k_{s_{\text{joint}}}$ (kN/m)</td>
<td>$8 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Joint rotate stiffness $k_{r_{\text{joint}}}$ (kN/rad)</td>
<td>$3.5 \times 10^4$</td>
</tr>
<tr>
<td>Soil</td>
<td>Cover depth to tunnel crown C (m)</td>
<td>16.4</td>
</tr>
<tr>
<td></td>
<td>Soil unit weight $\gamma$ (kN/m$^3$)</td>
<td>16.7</td>
</tr>
<tr>
<td></td>
<td>Coefficient of earth pressure at-rest $K_0$</td>
<td>0.54</td>
</tr>
<tr>
<td></td>
<td>Surcharge at ground surface $H$ (m)</td>
<td>1.7 – 7</td>
</tr>
<tr>
<td></td>
<td>Soil undrained shear strength $s_u$ (MPa)</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td>Soil unloading-reloading modulus $E_{ur}$ (MPa)</td>
<td>21.3</td>
</tr>
</tbody>
</table>
Figure 1 Deformation modes for tunnel lining: a) uniform; and b) oval.
Figure 2 Uniform mode for soils in semi-infinite space: a) contour lines for ground movement and b) directions of principle strain.
Figure 3 Oval mode for soils in semi-infinite space: a) contour lines for ground movement and b) directions of principle strain
Figure 4 Finite element model used for verification of Eqs. 9 and 11: a) a typical mesh of half model; b) Stress boundary condition under uniform mode; and c) stress boundary condition under oval mode.
Figure 5 Typical stress-strain curves of HS model used for the validation under two different test conditions: 1) dash line for PSE; 2) solid line for PSC; and 3) dot-dash line for average of PSE and PSC.
Figure 6 Comparison of $p_{k0}$ between MSD and FEM under uniform mode shape
Figure 7 Comparison of $p_{k0}$ between MSD and FEM under oval mode shape
Figure 8 Comparison of $p_{k0}$ calculated by MSD with the results from FEM for the orthogonally designed 197 validation cases under oval mode shape: a) grey square dots for original MSD using Eq. 11; b) black circular dots for corrected MSD results.
Figure 9 Correlation of calculated correction factor with input parameters in MSD under oval mode shape: a) cover depth ratio $h/D$; b) shear strength ratio $s_u/\sigma_v'$; c) unload-reload modulus ratio $E_{ur}/s_u$; d) convergence ratio $\delta/R$. 
Figure 10 Comparison of tunnel deflection at the springline between FEM and MSD

$\left( \frac{s_{FEM}}{s_{MSD}} \right)$ with: a) cover depth ratio $h/D$; b) convergence ratio $\delta/R$
Figure 11 Histogram for correction factor from 197 numerical cases under oval mode shape.
Figure 12 Sectional profile of the design example
Figure 13 Nonlinear embedded beam spring with joint model and FEM analysis with PLAXIS for design example: a) spring model layout; and b) nonlinear $p_k-u_r$ curve for the design.
Figure. 14 Element mesh in FEM analysis for the design example
Figure 15 Comparison of predicted tunnel deformation from different design model with measured data; Note: measured data after (Huang and Zhang 2016)
Figure 16 Comparison of the predicted internal forces from different design models between linear subgrade reaction and nonlinear subgrade reaction model: a) bending moment; and b) axial force