NON-EQUILIBRIUM DYNAMICS OF STRONGLY INTERACTING FERMIONS

by

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Abstract

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This thesis summarizes experimental work investigating strongly interacting Fermi gases in two and three dimensions. Radio-frequency spin manipulation is used to initialize and probe strongly interacting Fermi gases dynamically to measure relaxation to a steady-state. Time-resolved spectroscopy is used to discover the $p$-wave contacts by observing how correlations in the system develop after quenching the atoms into an interacting state. By combining quasi-steady state measurements with new contact relations, an attractive $p$-wave interaction energy as large as half the Fermi energy is inferred. Spin echo measurements of the magnetization dynamics are used to observe spin transport in two and three dimensions in the presence of strong interactions. The bare spin diffusivity is distinguished from the Leggett-Rice effect, in which demagnetization is slowed by precession of a spin current around the local magnetization. Observations of the demagnetization dynamics support the conjecture that a general quantum bound on spin transport is obeyed where the diffusivity $D \gtrsim \hbar/m$, where $m$ is the particle mass.
Acknowledgements

When I started graduate school I really didn’t know what I wanted to accomplish or do with a postgraduate degree. At the end of graduate school I still don’t think I’ve learned anything more about what I can do with a post-graduate degree but I have learned a lot about myself, my relationships, and an “ultra”-cool little corner of science.

When I decided to come to Toronto and join Joseph’s group I have to admit that it wasn’t really because of the science they were pursuing. Instead, I was drawn into the group because of the people. Alma, (big) Scott, and Stefan were incredibly welcoming and made my first two years of study rewarding scientifically and personally. There is no one I would have rather had teaching me why ultracold atoms are interesting or how to manage the complexity of the apparatus. As I moved into the senior position in the lab (barely out of my qualifying exam) I was grateful for Stefan’s careful analysis and (little) Scott’s diligence in running the experiment. But more importantly, I’m grateful that we could go to Sichuan Garden upwards of three times a week without judgement. As the group has changed over the years I have enjoyed working alongside a number of fantastic scientists. Thank you to the “Chip Labbers” over the years: Simon, Will, Nico, Matt, Fabian, Haille, and Ben. Even though they were a world (door) away, the folks in the “Lattice Lab” have also been an amazing source of procrastination, help, and inspiration. Thank you Dylan, Graham, Ryan, Rhys, Vijin, Peihang, and Fudong. While I may have eventually fallen in love with the research we are pursuing, it has always been the people that have kept me coming back for more.

Outside of the lab I’ve been fortunate to have found a couple of good sources of stress relief. From soccer games with the Skinny Arms and DST to water polo with the Triggerfish, there was always a ball (or person) to take out some of my frustration when the experiment mysteriously stopped working again. In particular, my soccer husband Josh has always helped me forget whatever was going wrong in the lab with some combination of hot tub, BBQ, and/or beer.

I don’t think I could have made it to the end of my PhD without the support and guidance of my supervisor Joseph. We didn’t always agree but I think every hiccup along the way made us a better and more productive team. I’m excited to watch how the lab continues to evolve and see the discoveries I’m sure that your new team of researchers will uncover. Throughout my formal education I have been fortunate to have been taught by some excellent teachers whose passion for their subject motivated me to continuing studying and I’m happy that Joseph will cap off that list.

My family has been a constant and never-ending source of love and support. Thank you for limiting the number of times you’ve asked me “what are you going to do with that” and for encouraging me to describe my research as “it’s lasers: pew pew”. Thank you Adam for the stream of webcomics that could always cheer me up. Thank you Dad for being the one who actually (sort of?) understands my research. Thank you Mum for always being there to listen even when it was just me grumping about how much
everything sucks. My family has encouraged me to explore and be a scientist from as early as I can remember. Whether it was Adam and I mixing random ingredients from the cabinet and seeing if they would become cake when put in the microwave, curing cancer in our make-believe lab in the basement, or going to as many science camps as sports camps, I always felt that I should explore and push the boundaries of what I understood. If a PhD is really just standing on the shoulders of the people who came before me and trying to carve out a new piece of knowledge then I think my family deserves most of the credit for my research; they’ve always put me on their shoulders and made me feel like I can do anything.

At the end of the day my success and well-being boils down to the two most important men in my life. What started in the first few months of acting in the play-that-must-not-be-named, to our burgeoning friendship, and now relationship Emre has been a constant source of happiness and personal growth. I’m excited to continue to grow and learn from you. Of course, I could not have finished this journey without my partner Alex. You’ve been with me through the (mostly) thick and thin. From nights where I could only complain about how I didn’t know what to do make it through another day at the lab to nights where I didn’t even make it home from the lab, you have been there for me and supported me through it all. You are my most handsome cheerleader. I love you and can’t wait for my turn to support you while you do your PhD!
Preface

The work culminating in this thesis resulted in a number of papers which I list here:


This thesis serves as an extended version of the results presented in these papers supplemented with additional measurements and calculations. As such I would like to acknowledge my co-authors S. Trotzky and S. Smale who shared in the responsibilities of data taking, analysis, and continued improvements to the experiment throughout my PhD. I would also like to acknowledge A. Bardon and S. Beattie who helped develop some of the measurement techniques used in this thesis. More recently, H. Sharum, F. Böetcher, and B. Olsen contributed to taking data. I must also acknowledge my theory collaborators who contributed greatly to my understanding of the physics presented in this thesis and for their enumerable calculations that helped explain the measurements presented here. E. Taylor and S. Zhang contributed to the understanding of the meaning of the Leggett-Rice effect in terms of the effective interaction. T. Enss has been the driving force in providing quantitative results with which to compare the spin transport data. S. Zhang and Z. Yu were instrumental in developing the theory of the $p$-wave contacts.

Lastly, I would like to acknowledge J. Thywissen for his guidance, mentorship, and careful attention to detail.

During the first year of my PhD I contributed to taking and analyzing data as well as developing experimental tools and techniques which resulted in the following paper,


Many of the experimental techniques first used in this work have been subsequently developed and extended and as such I include a brief discussion in this thesis where appropriate.
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strongly interacting Fermi systems, or strongly correlated electron systems, underly some of the biggest open questions in condensed matter research, for instance, what mechanism underlies high-temperature superconductivity [1]? Fermi gases are ensembles of fermions, which are particles that obey Fermi-Dirac statistics. At zero temperature fermions fill every available energy level up to an energy level known as the Fermi energy. In contrast, a gas of bosons undergoes a phase transition at low temperature to form a Bose-Einstein Condensate (BEC) where all the bosons aggregate in the lowest energy state.

Problems involving strongly interacting fermions have remained at the forefront of active research because they underpin many exotic systems in nature and yet a complete description of the physics is often analytically intractable or computationally impractical in a reasonable timescale. Experimentally, these problems are also notoriously difficult to study because they require extremely low temperatures, very high densities, large magnetic fields, or that the samples be very pure. Connecting experiments and theoretical treatments of these problems can be complex as the physical systems have complicating features like lattice structures, impurities, and thermal excitations that must be considered.

The field of ultracold atomic gases provides an alternative perspective from which to study highly correlated systems. Ultracold atomic gases allow researchers to challenge the traditional “top-down” approach taken by condensed matter research. In a top-down approach, a material is probed and its complexity is simplified to fit into a theoretical framework. Instead, an ultracold atom experiment lets researchers take a “bottom-up” approach. A simplified toy model is nearly replicated by the conditions of the experiment allowing for direct observation of the emergent behaviour of the system as relevant parameters in the model are varied. The tunability of ultracold atomic systems can be used to access previously inaccessible phases of matter that have yet to be fully studied.

The state of the field of ultracold atomic gases has been rapidly advancing since the observation of the first BEC in 1995 [2–4]. As laser cooling of atoms was further developed, the first degenerate Fermi gases were produced [5, 6]. Subsequently, more exotic atomic species have been cooled and trapped expanding the field to include atoms with dipolar interactions [7, 8]. Despite the varied realizations of ultracold systems, they are all often characterized by two essential length scales. They are extremely dilute. The density $n$ is typically less than $10^{15}$ particles per cubic centimetre. This implies that the scattering between atoms happens infrequently and the interparticle spacing $n^{-1/3} \sim k_F^{-1}$, where $k_F$ is the Fermi wave vector, is much larger than the range of the interaction. Ultracold atoms are also extremely cold, with typical temperatures on the order of $10^{-6} – 10^{-9}$ K. At these temperatures, the
thermal deBroglie wavelength is larger than the typical interparticle distance and quantum statistics become important.

The physics of the simplest ultracold atomic systems are well understood which allows researchers to add increasing layers of complexity. For instance, optical lattices can be used to vary the dimensionality [9], disorder can be added to the underlying potentials [10], impurities can be studied [11], and interactions between atoms can be varied [12]. Tailoring simple atomic systems makes them resemble complex condensed matter systems with a complete bottom-up understanding.

Of particular relevance to this thesis is the ability to tune the interactions between atoms. Arbitrary tuning of the strength of the interactions between repulsive and attractive is accomplished by using a Fano-Feshbach resonance [13]. When the interactions are engineered to be strong enough, atomic Fermi systems resemble their strongly correlated electron counterparts.

On one side of the Feshbach resonance the ground-state of the system is formed from Bardeen-Cooper-Schrieffer (BCS) pairs. Here BCS-like superfluidity can be studied at low temperature. On the other side of the Feshbach resonance a two-body bound state exists and pairs of fermions (a composite boson) can condense to form a BEC at low temperature. The regime between these two limits is called the BCS-BEC crossover. The ability to arbitrarily tune interactions makes ultracold Fermi gases ideal to study this crossover. Numerous studies have taken place to understand the normal-state and superfluid properties of such a system [12, 14].

A Feshbach resonance also allows degenerate Fermi gases to access the regime of “unitary” interactions. An ultracold atomic unitary Fermi gas is characterized by a separation of length scales. As the thermal deBroglie wavelength and the interparticle spacing are large (and much greater than the range of the scattering potential), the two-body interaction is solely characterized by the scattering length. At unitarity the scattering length diverges leaving no small parameter with which to perturbatively treat the problem. The unitary regime is also encountered in exotic matter; for instance, neutron stars or the quark-gluon plasma [15].

Ultracold unitary Fermi gases have a number of universal properties. The properties are universal in that they do not depend on the underlying interaction as all length scales associated with the microscopic interaction have dropped out of the problem. The universal properties relate both to the thermodynamics of the gas [16] and to some microscopic properties of the gas [17]. Surprisingly, a direct connection between the microscopic and macroscopic properties exists and universal relations have been formulated that connect the two [18].

Many of the universal properties of strongly interacting degenerate Fermi gases have been studied in equilibrium. However, advances in dynamical studies of these systems have raised the question of whether there is any universality in transport and dynamics of strongly interacting Fermi gases. Ultracold atomic gases provide a unique opportunity to study non-equilibrium physics because they possess convenient timescales associated with preparation and measurement of the system. The characteristic frequencies of many-body systems of ultracold atoms are typically much slower than their condensed matter counterparts. Ultracold systems need to be manipulated and probed on kilohertz timescales, in sharp contrast to the giga or terahertz frequencies required in condensed matter systems. Ultracold atoms are also isolated from their environments implying that any dynamics are decoupled from environmental sources of dissipation. The evolution of atomic systems can then follow coherent quantum dynamics over long time scales.

A variety of transport problems are being investigated in the context of ultracold atomic systems.
Similar to condensed matter systems, particle currents are often studied. In cold atom experiments, particle currents have been observed and studied in the superfluid flow of a BEC in a ring trap [19–21] and in the transport of atoms through a narrow optical channel from one reservoir to another [22]. The transport of ultracold atoms in optical lattices has also been extensively studied [23]. Recently, the advent of artificial gauge fields and synthetic spin-orbit coupling in these systems has allowed for the realization of the quantum Hall effect [24, 25]. In other systems, transport properties are inferred from the collective modes or dynamics. From the dynamics of ultracold gases in the presence of strong interactions, the viscosity [26] and spin dynamics [27, 28] have been observed to obey limiting universal behaviour.

Outline

This thesis presents non-equilibrium measurements of strongly interacting fermions confined in two and three dimensions. All of the results are based off of the same simple question: what happens if the atoms are rapidly manipulated, or quenched, to be strongly interacting? In some cases, the quench will be directly into an interacting state. In another case the quench creates a superposition of interacting states, the dynamics of which is observed. In all cases the evolution of the system to a quasi-equilibrium is studied in the presence of strong interparticle scattering.

Chapter 2 reviews the essentials of trapped Fermi gases and scattering theory in dilute gases. I detail the differences between s-wave scattering and p-wave scattering in two and three dimensions. I also provide an overview of Feshbach resonances and their use in ultracold atom experiments. Finally, I introduce the concept of the contact and highlight its utility in a set of universal relations, connecting microscopic and macroscopic properties of the system.

Chapter 3 summarizes the experimental uses of radio-frequency spectroscopy to probe strongly interacting Fermi gases. I discuss the characteristics of a typical spectrum at unitarity and in the BCS-BEC crossover. Spectroscopy is used to show the decay of the metastable upper-branch in three dimensions and the prospects for measuring a “repulsive contact”. Spectroscopy is also used to probe a two dimensional Fermi gas with strong interactions. Lastly, the technique of dynamical spectroscopy is used to measure the dynamics of the contact in two dimensions.

Chapter 4 presents the first observation of the p-wave contacts and evidence supporting a set of universal relations to describe Fermi gases with p-wave interactions. The p-wave contacts are measured via the high-energy tails of the momentum distribution and radio-frequency spectrum. The contacts are used to infer the free energy of a strongly interacting gas with p-wave interactions via recently derived universal relations. The dynamics of the contact is also observed near resonance.

Chapter 5 presents measurements of spin transport in strongly interacting Fermi gases. I introduce the model used to describe the transverse demagnetization dynamics and I discuss the experimental technique used to extract various spin transport coefficients. I then present measurements of the spin transport in strongly interacting three-dimensional and two-dimensional Fermi gases showing that universal bounds on transport appear to be respected by varied systems.

Chapter 6 summarizes the measurements made finding connections between the local dynamics of the contact and the global magnetization dynamics. I present prospects for future investigations of the non-equilibrium properties of ultracold Fermi gases.
Chapter 2

Dilute Fermi Gases

The dilute nature of ultracold atomic gases implies that many of the system properties are determined by two-body collisions. In this chapter I review some basic information about trapped Fermi gases and the two-body scattering problem. The ability to tune the two-body scattering parameters arises from the use of magnetic Feshbach resonances and optical potentials that allow one to vary the dimensionality of the system. Together, these tools allow experiments with ultracold atoms to tune between strong and weak attractive or repulsive interactions in one, two, and three dimensions.

In this chapter I will discuss the ways in which the two-body scattering parameters can impact the many-body properties of the gas. In particular, I will introduce the contact, which quantifies the strength of interactions and relates microscopic properties of the gas to macroscopic properties. Throughout, emphasis will be placed on the differences between two and three dimensions as well as the description of \( p \)-wave interactions in addition to \( s \)-wave interactions.
Chapter 2. Dilute Fermi Gases

Figure 2.1: (a) At zero temperature fermions fill each energy level up to the Fermi energy. (b) As the temperature is increased some states above the Fermi energy begin to be occupied.

2.1 Fundamentals of Degenerate Fermi Gases

Quantum degeneracy of an ensemble of atoms obeying Fermi-Dirac statistics was achieved in 1999 [5, 6]. Cooling atoms of mass \( m \), leads to an increase in the deBroglie wavelength \( \lambda_T = \sqrt{\frac{2\pi \hbar^2}{mk_B T}} \), where \( k_B \) is the Boltzmann constant and \( T \) is the temperature. When the deBroglie wavelength becomes comparable to the interparticle spacing \( n^{-1/3} \), where \( n \) is the density, the wave functions begin to overlap. At this point the phase-space density \( n\lambda^3_T \sim 1 \) which signifies the onset of quantum degeneracy.

In a Fermi gas, the characteristic temperature at which quantum statistics become important is called the Fermi temperature, \( T_F \). The thermodynamics of non-interacting Fermi gases have been extensively discussed [9, 12, 29] and so I only provide a brief overview of the theory to introduce relevant quantities discussed throughout this thesis.

Fermions are particles that obey Fermi-Dirac statistics. This means that two identical fermions cannot occupy the same quantum state, and hence they fill up all available states starting from the lowest energy level (see Fig. 2.1). At zero temperature, the highest occupied energy level is called the Fermi energy, \( E_F \). For a gas at finite temperature, a small fraction of atoms occupy states above this energy level as in Fig. 2.1b.

For a gas of non-interacting fermions, the average occupation of state \( i \) with energies \( E_i \) is given by the Fermi-Dirac distribution [30]

\[
\langle n_i \rangle = \frac{1}{e^{(E_i - \mu)/k_B T} + 1}.
\] (2.1)

The chemical potential \( \mu \) for a fixed number of particles \( N \) is chosen such that \( N = \langle N \rangle = \sum_i \langle n_i \rangle \). The total energy is \( U = \sum_i E_i \langle n_i \rangle \).

Atoms are typically confined in a harmonic trap \( V(r) = \frac{1}{2} m \sum_j \omega_j^2 x_j^2 \) in two or three dimensions where \( \omega_j \) is the trapping frequency in the \( x_j \)-direction. In a harmonic potential, the density of states in two and three dimensions are listed in Tab. 2.1. The Fermi energy can be determined by integrating the


\[
\begin{array}{|c|c|c|}
\hline
& 2D & 3D \\
\hline \rho(E) & \frac{E}{(\hbar^2\omega_y\omega_z)} & \frac{E^2}{(2\hbar^6\omega_y\omega_z)} \\
E_F & \frac{h(\omega_y\omega_z)^{1/2}}{(2^N)^{1/2}} & \frac{h(\omega_y\omega_z)^{1/3}}{(6^N)^{1/3}} \\
k_F & \frac{(4\pi n_{pk})^{1/2}}{2} & \frac{(6\pi n_{pk})^{1/3}}{2} \\
\hline
\end{array}
\]

Table 2.1: A summary of thermodynamic quantities in harmonically trapped Fermi gases in two and three dimensions. Here \(n_{pk}\) is the peak density.

\[
\begin{array}{|c|c|c|}
\hline
& 2D & 3D \\
\hline \rho(E) & \frac{V m/(2\pi h^2)}{h^24\pi n/(2m)} & \frac{V(2m)^{3/2}\sqrt{E}/((2\pi)^2\hbar^3)}{h^2(6\pi n)^{2/3}/(2m)} \\
E_F & \frac{1/2}{(4\pi n)^{1/2}} & \frac{1/3}{(6\pi n)^{1/3}} \\
k_F & \frac{(4\pi n)^{1/2}}{2} & \frac{(6\pi n)^{1/3}}{2} \\
\hline
\end{array}
\]

Table 2.2: A summary of thermodynamic quantities in homogeneous Fermi gases in two- and three-dimensions. \(V\) is the depth of the homogeneous potential.

density of states at \(T = 0\)

\[
N = \int_0^{E_F} \rho(E)dE \tag{2.2}
\]

and is related to the Fermi wave vector \(k_F = \sqrt{2mE_F/\hbar^2}\). The reduced temperature, which quantifies the degeneracy of the gas, is defined as \(T/T_F = Tk_B/E_F\). I summarize some useful quantities for harmonically trapped Fermi gases in Tab. 2.1 and for homogeneous Fermi gases in Tab. 2.2 in two and three dimensions. More explicit discussion of the thermodynamics and thermometry of trapped Fermi gases is discussed in Appendix A.

## 2.2 Scattering Theory

Typically, the interaction between two colliding atoms of equal mass is described by a short ranged central potential \(V(r)\) where \(r\) is the vector between the positions of the atoms. At short range this potential is strongly repulsive. At larger distances the potential resembles an attractive van der Waals potential and scales as \(-C_6/r^6\). If the interaction occurs with nonzero relative angular momentum \(\ell\) of the atoms the radial potential includes a centrifugal barrier \(\hbar^2\ell(\ell + 1)/(mr^2)\). The centrifugal barrier suppresses higher order partial wave scattering at low temperatures. To fully treat the scattering problem, details of the short range potential are needed to determine all of the scattering phases. However, in some cases the phase and its dependence on the relative momentum \(k\) can be parametrized simply. In ultracold atomic systems the interparticle spacing is \(\sim 5000a_0\) which is much larger than the range of the potential \(r_0 \sim (mC_6/(2\hbar^2))^{1/4} \sim 60a_0\), where \(a_0\) is the Bohr radius.

The Schrödinger equation for two particles colliding in the centre-of-mass frame via a short ranged potential is

\[
\left[\frac{k^2}{m} + V(r)\right] \psi(r) = E\psi(r) \tag{2.3}
\]

where \(k\) is the relative momentum. Far from the scattering potential the wave function \(\psi\) is the sum of
an incident plane wave and an outgoing scattered wave. In three dimensions \[30\]
\[
\psi = e^{ik \cdot r} + f(k, k') \frac{e^{ikr}}{r},
\]
(2.4)
or in two dimensions \[31\]
\[
\psi = e^{ik \cdot r} - f(k, k') \sqrt{\frac{\ell}{8\pi kr}} e^{ikr}.
\]
(2.5)
In both expressions, \(f(k, k')\) is the scattering amplitude for an incoming plane wave to scatter into the
direction \(k' = k \hat{r}\). I summarize some useful properties of the scattering amplitude in two and three
dimensions for both \(s\)-wave scattering (\(\ell = 0\)) and \(p\)-wave scattering (\(\ell = 1\)).

The scattering amplitude in three dimensions can be expanded in terms of spherical harmonics \(Y_{\ell \nu}\)
\[32\]
\[
f(k, k') = 4\pi \sum_{\ell = 0}^{\infty} \sum_{\nu = -\ell}^{\ell} f_{\ell \nu}(k) Y_{\ell \nu}^*(\hat{k}) Y_{\ell \nu}(\hat{k}'),
\]
(2.6)
where \(\ell = 0, 1, 2, \ldots\) is the relative angular momentum of the atoms and \(\nu\) is the projection of the angular
momentum onto the quantization axis. To determine the total cross section one sums up contributions
in all directions

\[
\sigma_k = \int d\Omega_{k'} |f(k, k')|^2.
\]
(2.7)
Using the optical theorem \[33\], \(\text{Im}\{f(k, k')\} = \frac{k}{4\pi} \sigma_k\), in combination with knowledge of the scattering
cross section gives

\[
|f_{\ell \nu}(k)|^2 = \frac{\text{Im}\{f_{\ell \nu}(k)\}}{k}.
\]
(2.8)
Equation (2.8) can be rewritten \(|1 + 2ikf_{\ell \nu}(k)|^2 = 1\), which implies that \(1 + 2ikf_{\ell \nu}(k)\) is a phase factor,
of unity magnitude. One can therefore define the scattering phase \(\delta_{\ell \nu}(k)\) via \(1 + 2ikf_{\ell \nu}(k) = e^{2i\delta_{\ell \nu}(k)}\),
which is related to the scattering amplitude

\[
f_{\ell \nu}(k) = \frac{1}{k \cot \delta_{\ell \nu}(k) - ik}.
\]
(2.9)

The scattering phase can be related to various scattering parameters using the effective range expan-
sion which in three dimensions is \[34\]
\[
k^{2\ell+1} \cot \delta_{\ell \nu}(k) = -\frac{1}{a_{\ell \nu}^{2\ell+1}} + \frac{1}{2} r_{\ell \nu}^{2\ell+1} k^2.
\]
(2.10)
For \(s\)-wave interactions the effective range expansion is

\[
k \cot \delta_s = -\frac{1}{a} + \frac{1}{2} r_{\text{eff},s} k^2
\]
(2.11)
where \(a\) is the \(s\)-wave scattering length and \(r_{\text{eff},s}\) is the effective range. For \(p\)-wave interactions the
effective range expansion is

\[
k^3 \cot \delta_p = -\frac{1}{v} + \frac{1}{2r_{\text{eff},p} k^2}
\]
(2.12)
where \(v\) is the \(p\)-wave scattering volume and \(r_{\text{eff},p}\) is the effective range.
In two dimensions the scattering amplitude can be expanded in partial waves

$$f(k, k') = f_0(k) + \sum_{\ell=1}^{\infty} 2\cos(\ell \theta) f_{\ell}(k).$$

(2.13)

Using the two-dimensional optical theorem,

$$\text{Im}\{f(k, k')\} = -k \sigma_k \frac{d}{dk} \ln \left( \frac{1}{k a_{2D}} \right)$$

[31], the scattering phase can be related to the scattering amplitude

$$f_{\ell}(k) = -\frac{4}{\cot \delta_{\ell}(k) - i}.$$  

(2.14)

The effective range expansion for s-wave scattering in two dimensions is [31]

$$\cot \delta_s = -\frac{2}{\pi} \ln \left( \frac{1}{k a_{2D}} \right) + \frac{r_{\text{eff},s} k^2}{2\pi}$$

(2.15)

which defines the two-dimensional s-wave scattering length $a_{2D}$ and the effective range $r_{\text{eff},s}$. The effective range expansion for p-wave scattering in two dimensions is [35]

$$k^2 \cot \delta_p = -\frac{1}{A} + \frac{2k^2}{\pi} \ln \left( r_{\text{eff},p} k \right)$$

(2.16)

which defines the p-wave scattering area $A$ and the effective range $r_{\text{eff},p}$.

2.3 Tuning Interactions in Ultracold Atomic Gases

The scattering parameters can be tuned in ultracold gases by use of a Feshbach resonance [13]. In general, a Feshbach resonance occurs when there is resonant coupling between a free scattering state and a bound molecular state. The coupling between two molecular potentials is illustrated conceptually in Fig. 3.3a. Two molecular potential curves $V_o(R)$ and $V_c(R)$ are shown. The potential $V_o(R)$ represents the open scattering channel that asymptotically connects two free atoms. $V_c(R)$ is called the closed-channel and supports a series of bound molecular states near the threshold of the open-channel. If the magnetic moment of the unbound pair differs from that of the bound state by $\delta \mu$ the energy of the bound molecular state can be tuned into resonance with the energy of free atoms using a magnetic field (see Fig. 3.3b). Alternatively, resonant coupling can be achieved with optical coupling between scattering atoms and an excited molecular state resulting in similar features [36, 37].

Magnetic Feshbach resonances have been discussed at length (c.f. [13, 30]) and so I summarize the key result. The s-wave scattering length in three dimensions as a function of the magnetic field $B$ is given by a simple expression,

$$a(B) = a_{bg} \left( 1 - \frac{\Delta B}{B - B_0} \right)$$

(2.17)

where $a_{bg}$ is the background scattering length, $\Delta B$ characterizes the width of the resonance, and $B_0$ is the magnetic field at which the scattering length diverges.

2.3.1 Narrow Versus Broad Resonances

When the interaction can be described by a single parameter, the scattering length, the Feshbach resonance is typically called universal or broad [12, 13]. To understand what is meant by a broad resonance it is instructive to construct a hierarchy of length scales. I start by considering the three-dimensional
Figure 2.2: A cartoon illustration of a Feshbach resonance. (a) The open- and closed-channel are brought into resonance with each other by tuning a magnetic field. (b) In the dressed state picture, the avoided crossing shows resonant behaviour as the magnetic field tunes the energy of the open- and closed-channel.

$s$-wave scattering amplitude

$$f_s(k) = \frac{1}{-\frac{1}{a} + \frac{1}{2}r_{\text{eff},s}k^2 - ik}.$$  \hfill (2.18)

A single scattering parameter can be used to describe three-dimensional $s$-wave scattering if $r_{\text{eff},s} = 0$. As an approximation, a broad resonance requires that $r_{\text{eff},s}k \ll 1$ for all relevant momenta in the system [12] or for a degenerate Fermi gas $r_{\text{eff},s}k_F \ll 1$. If this criteria is satisfied, the resonance is also considered universal in that the scattering only depends on the scattering length and not on the details of the short-range potential. It is considered broad in the sense that when $ka \to \infty$ all collisions in the gas are resonant, or all collisions have the unitary limited value for the scattering amplitude $f \to i/k$.

Another way to approach constructing this hierarchy is to define an energetic width associated with the resonance and compare that to the relevant energies available for scattering, i.e., the Fermi energy. I start by considering the cross section for resonant ($\delta_s = \pi/2$) $s$-wave scattering in three dimensions

$$\sigma = \frac{4\pi}{k^2 + (Rk^2 - Rk_0^2)^2}$$  \hfill (2.19)

where $R = -r_{\text{eff},s}/2$. Identifying the scattering energy $E = \hbar^2k^2/m$, the resonant energy $E_0 = -\hbar^2k_0^2/m$ where $k_0^2 = -(aR)^{-1}$, and $E_r = \hbar^2/mR^2$ one can write the scattering cross section in a more useful form

$$\sigma = \frac{4\pi\hbar^2}{m} \frac{E_r}{(E - E_0)^2 + EE_r}.$$  \hfill (2.20)

Within this Lorentzian approximation one can assign a width $\gamma = 2\sqrt{EE_r}$. The form of the scattering cross section can be compared to that for resonant $p$-wave scattering in three dimensions

$$\sigma = \frac{4\pi\hbar^2}{m} \frac{E^2/E_r}{E^3/E_r + (E - E_p)^2}$$  \hfill (2.21)

where $E_p = -\hbar^2/mvR$ (note $v < 0$), which defines a width in the Lorentzian approximation $\gamma = 2\sqrt{E^3/E_r}$ (see Chapter 4 for more discussion).
A narrow resonance is defined as occurring when the width is much smaller than the resonant scattering energy. In terms of scattering parameters this requires the relative width $\gamma/E_0 \ll 1$, or $a \ll R$ for s-wave scattering. For $p$-wave resonances, $\gamma/E_p$ is identically zero as $v^{-1} \to 0$ and so all $p$-wave resonances are intrinsically narrow at resonance. Similarly, one can compare the width to the many-body properties of the gas, namely the Fermi energy. A narrow resonance is characterized by $\gamma \ll E_F$. For s-wave interactions, where the scattering energy is approximately the Fermi energy, a narrow resonance requires $k_F R \gg 1$. Conversely, for a $p$-wave resonance a narrow resonance requires $k_F R \ll 1$.

### 2.4 Bound States

Near the Feshbach resonance in three dimensions, where the two channels are strongly coupled and the scattering length is large and positive, a molecular state exists with binding energy given by

$$E_b = \frac{\hbar^2}{2ma^2}. \quad (2.22)$$

In this region, the bound state has universal properties. The state can be described in terms of a single effective molecular potential having scattering length $a$. Away from the resonance, the energy varies linearly as a function of $B$ and with a slope given by the differential magnetic moment of the open- and closed-channels.

The energy spectrum of states is shown in Fig. 2.3 for two particles in a harmonic trap in two and three dimensions. The energy in units of $\hbar\omega_j$ is plotted versus the interaction strength $-a^{-1}$ in three dimensions and $\ln(a)$ in two dimensions where $a$ is in units of the harmonic oscillator length $a_H = \sqrt{\hbar/m\omega_j}$. The interaction strength expressed in these units is the two-body equivalent to the coupling constant used to parametrize the interaction strength throughout this thesis.

The ground state of the system is referred to as the “lower-branch”. Figure 2.3a shows the energy spectrum of states in three dimensions. For a large and positive scattering length, the ground state of the system is a molecular state. The energy of this state approaches the universal form of the binding energy $-1/a^2$ as $-a$ decreases. The energy of the lower-branch is shifted to larger energy due to the trapping potential. An “upper-branch” also exists which is the higher lying energy state of the system. In this two particle system the upper-branch exists for all values of the scattering length, for $-a^{-1} < 0$ the atoms interact via a repulsive interaction, while for $-a^{-1} > 0$ the interaction is attractive.

The two-dimensional energy spectrum is shown in Fig 2.3b. Of note is the decrease of the ground state energy by $\hbar\omega_j/2$ relative to the three-dimensional case. The shift in energy can be understood as resulting from the presence of a bound state for all interaction strengths in two dimensions, discussed below.

When adding a third particle to the calculation, the energy spectrum becomes much more complicated [38] which suggests that the notion of an upper-branch in the many body system may be ill-defined.

Lastly, the underlying two-body interaction between atoms is always attractive, or in other words, the ground state is always a gas of dimers. Therefore, a gas prepared in the upper-branch suffers from atom loss as scattering atoms combine to form dimers when a bound state is present. This motivates non-equilibrium studies of a potentially metastable upper-branch as discussed in Chapters 3, 4, and 5.
Chapter 2. Dilute Fermi Gases

2.4.1 Bound States in Two Dimensions

This thesis details simple dynamical experiments in the presence of strong scattering. One open question addressed in this thesis is how two-body scattering impacts the dynamics that are observed. In these experiments the two-body interaction can be tuned by changing the dimensionality and/or by changing the scattering parameters (by changing the magnetic field). The effects of varying the scattering parameters has been explored by utilizing Feshbach resonances to experimentally access Fermi gases in the BCS-BEC crossover. However, the study of dimensionality in degenerate Fermi gases has been limited experimentally.

Two-dimensional gases possess a number of interesting properties. One such property is the fact that a bound state exists for arbitrarily weak attractive interactions, in contrast to the three-dimensional case where a threshold exists for the formation of a bound state. The presence of the bound state for arbitrary interactions in two dimensions can be understood in a number of ways. Simply, the scattering length in two dimensions must always be positive for elastic scattering and therefore the condition typically imposed for the existence of a bound state (i.e., $a > 0$) is always satisfied, even for weak attractive interactions. An alternative perspective is that the pole in the scattering amplitude associated with the bound state is always in the upper half of the complex plane. Conversely, in three dimensions, elastic scattering allows for both $a > 0$ and $a < 0$. When $a < 0$ the pole in the scattering amplitude exists in the lower half of the complex plane which corresponds to a virtual state [33].

Alternatively, the existence of the bound state for arbitrarily weak interactions in two dimensions can be demonstrated by considering the density of states in a homogeneous potential. The equation for the bound state energy $E$ can be formulated in terms of the density of states $\rho_D$ in $D$ dimensions (see Tab. 2.2) as [30]

$$\frac{1}{V_0} = \frac{1}{\Omega} \int_0^{E_r} d\epsilon \frac{\rho_D(\epsilon)}{2\epsilon + |E|} \quad (2.23)$$

where $\Omega$ is the system volume and $E_r = 2\hbar^2 / mR^2$ is an energy cutoff associated with the range $R$ of the potential. $V_0$ is the approximately constant size of the potential well over the range $R$, $V_0 \approx RV^D$ where
$V$ is the potential well depth. For a bound state to be supported by this potential with arbitrarily weak interactions $V_0 \to 0$, the right hand side of Eq. (2.23) must diverge for $|E| \to 0$.

In two dimensions the solution to the integral in Eq. (2.23) is

$$\frac{m}{4\pi \hbar^2} \ln \left( \frac{2E_r + |E|}{|E|} \right)$$

which diverges logarithmically as $|E| \to 0$. The logarithmic divergence implies that for arbitrarily weak interactions a bound state is supported by this potential in two dimensions. However, in three dimensions the solution to the integral in Eq. (2.23) is

$$\frac{1}{2\pi^2} \frac{m^{3/2}}{\hbar^3} \left( \sqrt{2E_r} - \frac{\pi}{2} \sqrt{|E|} \right)$$

which remains finite as $|E| \to 0$. The solution as $|E| \to 0$ implies that there is threshold behaviour for the potential to support a bound state. The threshold for the potential is $V_{0c} = \sqrt{2\pi^2 E_r R^3}$.

A consequence of the finite bound state pair size is that for the attractive two-dimensional Fermi gas, scale invariance is broken. More precisely, a harmonically trapped two-dimensional Fermi gas possesses a classical dynamical $SO(2,1)$ scaling symmetry [39]. While true at the classical level, this scaling symmetry is broken by the procedure of renormalization, the so-called quantum anomaly [40]. The broken scaling symmetry has a number of consequences for the collective modes [41–44] and dynamics of harmonically confined strongly interacting two dimensional Fermi gases.

For a classical scale invariant system, the breathing mode frequencies should be equally spaced at two times the trapping frequency [39]. Despite the breaking of the classical symmetry by the quantum anomaly, experiments measuring the breathing mode have shown no appreciable shift from the classical expectation [41]. It has been suggested that thermal fluctuations at finite temperature may have effectively washed out the effect of the quantum anomaly [42]. Interestingly, the operator product that quantifies the breaking of scaling symmetry is the contact, introduced below. Alternatively, it has been suggested that corrections to the scattering cross section depend logarithmically on density, leading to small corrections, which make the system appear scale invariant despite the finite bound state pair size [43]. Both explanations coupled with the experimental investigation suggest that the effect of interactions in two dimensions is much weaker than is theoretically expected.

Conversely, a dynamical measurement of the spin diffusivity in a two-dimensional Fermi gas [45] has observed a surprisingly small value of the spin diffusion constant that violates the theoretically conjectured quantum bound [46–49] suggesting that the effect of interactions is, in fact, important in two dimensions. The contradictory observations of collective modes and transport dynamics remain to be understood and are investigated in Chapter 5.

### 2.4.2 Confinement Induced Resonance

In realistic experimental conditions, true two-dimensional scattering is not attainable as the extent of the gas perpendicular to the plane is necessarily finite. Instead, experiments are in the quasi-two-dimensional regime where the harmonic oscillator length is much smaller than the interparticle spacing and the thermal wavelength, such that the transverse degrees of freedom are frozen out. However, the length scale associated with the confinement is bigger than the range of the van der Waals interaction and thus at short distances the two-body interactions are unaffected by the confinement.
Atoms are trapped in a one-dimensional optical lattice with approximately harmonic confinement in the $x$-direction, $V(x) = \frac{1}{2} m \omega_x^2 x^2$, perpendicular to the two-dimensional plane (see Appendix A for more details). For sufficiently deep lattices and low energy scattering only the ground band is populated and collisions are not impacted by excitations to higher states. The harmonic oscillator length $l_x = \sqrt{\hbar/m \omega_x}$ and the three-dimensional scattering length determine the two-dimensional scattering length via the scattering amplitude for two particles near the Fermi surface

$$a_{2D} = l_x \sqrt{\frac{\pi}{B}} \exp \left( -\sqrt{\frac{\pi}{2}} \frac{l_x}{a} \right)$$

(2.26)

where $B = 0.905$. Equation (2.26) is valid for all values of $l_x/a$ as long as the scattering energy is sufficiently small with respect to the confinement (i.e., $E_F \ll \hbar \omega_x$) [31]. In true two-dimensional systems one can define a binding energy in analogy with the three-dimensional case, $E_{2D}^b = \hbar^2/(ma_{2D}^2)$. In general, $E_{2D}^b$ is different from the binding energy of the confinement induced dimer, $E_c^b$. The binding energy of the confinement induced dimer can be determined from solutions to the transcendental equation [31]

$$\frac{l_x}{a} = F(\Omega) = \int_0^\infty \frac{du}{\sqrt{4\pi u^3}} \left( 1 - \frac{e^{-\Omega u}}{\sqrt{(1-e^{-2u})/(2u)}} \right)$$

(2.27)

where $F$ appears in the scattering amplitude and captures the effect of the finite length in the tightly confined direction and $\Omega = E_c^b/\hbar \omega_x$. In contrast to the situation in three dimensions, where a bound state exists only for positive scattering length, in the quasi-two-dimensional situation a bound state exists for a zero-range interaction of arbitrary strength. Therefore, the dimer at negative three dimensional scattering length is called confinement induced. The presence of the two-dimensional bound state results from an increase of the continuum by $\hbar \omega_x/2$ as illustrated in Fig. 2.4.

When $l_x/a \ll -1$ the expression for $E_c^b$ (black line in Fig. 2.4) matches the two-dimensional binding energy $E_{2D}^b$ (red line). Here the dimer spreads out in the two-dimensional plane and the binding energy follows from Eq. (2.26). On the other hand, when the binding energy is large, the molecular size is much less than the confinement and the molecules are essentially three-dimensional (blue line in Fig. 2.4). Near resonance, however, the dimer binding energy becomes strongly modified as shown by the black line in Fig. 2.4.

The dimer binding energy has been measured using radio-frequency spectroscopy in experiments with tight optical confinement [50, 51]. The effect of confinement on a Feshbach resonance can be inferred by observing the location of maximum atom loss after 5 ms hold time as a function of the confinement expressed in units of the recoil energy $E_R = \hbar^2/(2m \lambda^2)$, where $\lambda$ is the wavelength of the light used to generate the confinement. Measurements of the fraction of atoms remaining as a function of the magnetic field is shown in Fig. 2.5a. As the confinement strength is increased, the location of maximum atom loss shifts to larger magnetic fields, shown in Fig. 2.5b. The location of maximum atom loss can be compared to a calculation of the magnetic field at which $\ln (k_F a_{2D}) = 0$ (for a fixed atom number) plotted in the solid line which shows qualitatively similar behaviour; the location of maximum scattering is shifted to higher magnetic fields as the confinement is increased.
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Figure 2.4: Various binding energies plotted versus $l_x/a$. The red line shows $E_b^{2D}$ compared to the solution from the transcendental equation $E_b^c$ (black line). For large binding energy $E_b^c$ approaches the three dimensional binding energy $\hbar^2/ma^2$ shown in blue.

Figure 2.5: (a) Measurements of atom loss after 5 ms hold time as a function of magnetic field in three dimensions. A fit using a Lorentzian (solid line) determines the centre of the loss feature which is shifted slightly from the Feshbach resonance at 202.1 G (dashed line). (b) As the lattice depth is increased the centre of the loss feature shifts to larger magnetic field. The shift in the centre of the loss feature roughly matches the theoretical prediction shown by the solid line.
2.5 The Contact

The contact quantifies the strength of two-body correlations at short range. It is a quantity central to a number of universal relations that have been derived to describe the behaviour of systems with strong short-ranged interactions. The relations are universal in that they hold for any state of the system: normal state or superfluid, few-body or many-body, ground state or excited state, homogeneous or in a trapping potential. Operationally, the contact can be thought of as a measure of the number of opposite spin pairs in a small volume. More conceptually, the contact appears as the normalization of the many-body wave function at short range. The powerful nature of the contact comes from the connection it provides between microscopic quantities and macroscopic or thermodynamic quantities. Once the contact has been measured or calculated, the universal relations it is central to allow one to determine a multitude of properties about the system. In this section I summarize the current theoretical and experimental status of these universal relations and the contact.

The relations centering around the contact were first derived by Shina Tan and are often referred to as Tan’s relations [18, 52, 53]. These relations were formulated to apply to systems consisting of fermions in two spin states whose scattering length $a$ is large compared to the range $r_0$ of their interactions. Additionally, the density and temperature must be small such that $n^{-1/3} \gg r_0$ and $\lambda T r_0$. For a trapped system the length scale associated with the potential (i.e., the harmonic oscillator length) should be large compared to the range of the interaction. In Tan’s formulation, the universal relations involve an extensive quantity called the contact $C$ which is the integral over space of a local quantity, the contact density $C$

$$C = \int d^3R C(R). \quad (2.28)$$

I follow the derivation of Zhang and Leggett [54] and summarize the important conceptual points here and more technical details in Appendix B. Consider two atoms interacting via $V(r)$ with $\ell = 0$. When atoms interact via this short-ranged potential the many-body wave function can be simplified to have the form of the two-body wave function because the interparticle spacing is much greater than the range of the potential $n^{-1/3} \sim k_F^{-1} \gg r_0$, i.e., the gas is ultra dilute such that interactions involve only two particles. This argument suggests that the form of the short range part ($r \ll k_F^{-1}$) of the many-body wave function is determined by the two-body interaction. The normalization of the short-ranged part of the many-body wave function is determined by the many-body physics and is the quantity defined as the contact.

Another common formulation of the contact is to have it defined as the coefficient of the large-momentum tail of the momentum distribution for atoms in spin state $\sigma$ [52]:

$$n_\sigma(k) \to C/k^4. \quad (2.29)$$

It is important to note that the momentum distribution has been normalized such that $\int d^3k/(2\pi^3)n_\sigma(k) = N_\sigma$ where $N_\sigma$ is the total atom number in state $\sigma$. The large-momentum behaviour of the momentum distribution is closely related to the high-frequency behaviour of the spectral intensity [55]

$$I_\sigma(\omega) = \sum_k \int_{-\infty}^{\infty} d\Omega A_\sigma(k, \Omega)n_F(\Omega)\delta(\Omega - \epsilon_k - \mu_\sigma + \hbar\omega) \quad (2.30)$$
where \( A(k, \Omega) \) is the spectral function for energy \( \Omega \), \( n_F(\Omega) \) is the Fermi function, \( \epsilon_k = \hbar^2 k^2 / (2m) \) and \( \delta \) is the Kronecker delta function. In the large frequency and low energy scattering limit the delta function enforces both \( |\Omega| \) and \( \epsilon_k \) large and comparable in magnitude. The quantity \( I_\sigma \) can then be calculated as \([55, 56]\)
\[
I_\sigma \approx \hbar \sum_k n_\sigma(k) \delta(\hbar \omega - 2\epsilon_k). \tag{2.31}
\]

In the thermodynamic limit where \( \sum_k \to \int d^Dk/(2\pi)^D \) in \( D \) dimensions (\( D = 2, 3 \)) Eq. (2.31) becomes
\[
I_\sigma(\omega) \approx \hbar \pi^{1-D/2} \int dk C(\zeta) k^{D-1-\zeta} \delta(\hbar \omega - 2\epsilon_k) \tag{2.32}
\]

where I have expressed \( n_\sigma \) in terms of the power law with exponent \( \zeta \) appearing in the momentum distribution. For the \( s \)-wave case, \( \zeta = 4 \) corresponding to \( 1/k^4 \) behaviour. In Chapter 4 the power law in the momentum distribution will be seen to be different. Substituting \( dk = d\epsilon_k \sqrt{m/(2\hbar \epsilon_k)} \) gives
\[
I_\sigma(\omega) \approx \pi^{1-D/2} (D-6-\zeta)/2m(D-\zeta)/2 \hbar^{\zeta+1-D} \int d\epsilon_k C(\zeta) \epsilon_k^{(D-2-\zeta)/2} \delta(\hbar \omega/2 - \epsilon_k). \tag{2.33}
\]

Performing the integral over the delta function gives
\[
I_\sigma(\omega) \to \frac{\pi^{1-D}}{4} C(\zeta) \left( \frac{\hbar}{m} \right)^{(\zeta-D)/2} \omega^{(D-2-\zeta)/2} \tag{2.34}
\]

which is valid for any leading order power law in the momentum distribution and in two or three dimensions. It has been shown that the form of the spectral intensity is modified when considering interactions in the final states \([57]\), which will be discussed further in Chapter 3.

The contact appears in a number of other quantities related to the microscopic behaviour of the system. For instance, the pair correlation function at short distance in three dimensions is \([58, 59]\)
\[
g^{(2)}(r) \sim \frac{1}{(4\pi)^2} \left( \frac{1}{r^2} - \frac{2}{ar} \right) C. \tag{2.35}
\]

The contact also appears in photoassociation measurements \([60, 61]\) as a measure of the number of closed-channel molecules. In addition, the contact density appears in the static structure factor in three dimensions \([58, 62]\)
\[
S(q) \to \frac{1}{8} \left( \frac{1}{q} - \frac{4}{\pi aq^2} \right) C. \tag{2.36}
\]

Many of Tan’s relations relate to the thermodynamics of the system. One relation that showed the central nature of the contact to the properties of the system is called the adiabatic relation. In three dimensions the adiabatic relation states that the rate of change of the free energy of the system (with the temperature held fixed) as the scattering length is changed is proportional to the contact \([52]\)
\[
\left( \frac{dF}{da^{-1}} \right)_T = -\frac{\hbar^2}{4\pi m} C. \tag{2.37}
\]

In other words, the contact is the thermodynamic “force” conjugate to the scattering length (just as pressure is the conjugate force to volume). The adiabatic theorem directly shows that the contact determines the thermodynamics of the system. After the contact is measured or calculated as a function
of scattering length, the free energy can be determined by integrating Eq. (2.37) with respect to the
scattering length. The adiabatic theorem and associated contact relations have now been derived in a
variety of theoretical approaches [18, 52–54, 61, 63–65].

I note a few other relations related to the energy of the system. The generalized virial theorem for
a three-dimensional gas confined in a harmonic potential reads [58]

\[ T + U - V = -\frac{\hbar^2}{8\pi ma} C, \tag{2.38} \]

where \( T \) is the kinetic energy, \( U \) is the interaction energy, and \( V \) is the energy associated with the
external potential. A relation between the pressure density \( P \) and energy density \( E \) in a homogeneous
system has also been derived [58]

\[ P = \frac{2}{3} E + \frac{\hbar^2}{12\pi ma} C. \tag{2.39} \]

Measurements of the photoassociation rate in a gas of \(^6\)Li [60] were later interpreted [61] as the first
experimental observation of the contact. Measurements of the contact were not readdressed experimentally
to 2010 when the Jin group tackled the problem [17]. The contact was measured in a number of
different ways microscopically: using radio-frequency spectroscopy, measuring the momentum distribution,
and using atom photoemission spectroscopy [17]. Additionally, the release energy and interaction
energy were directly measured to verify the adiabatic and virial theorem [17]. Quickly on their heels, the
Vale group measured the contact via the static structure factor allowing for an additional comparison
between measurement techniques (and atomic species) [66]. More recent work from these groups has
been focused on measuring the homogeneous contact [67–69]. Measurements of the homogeneous contact
have been used to explore the sensitive temperature dependence of the contact across the superfluid
transition temperature [67, 68, 70]. Other work has shown that dynamical measurements of the contact
agree well with equilibrium measurements [28].

More recently the formalism surrounding the contact has been extended to bosons [71–75]. The two-
body contact for bosons has been observed using radio-frequency spectroscopy [71, 75] and by observing
the momentum distribution [73, 76]. An experiment has also measured the three-body contact [75],
a novel feature that results from the symmetric bosonic wave function. Contact relations have also
been extended to treat one- [77–80] and two-dimensional [59, 81–83] Fermi gases. The contact has also
been experimentally studied in a two-dimensional Fermi gas finding good agreement with theoretical
predictions [81].

The contact formalism has also been extended to systems outside of ultracold atoms. Coulomb gases
have been described in this formalism [84]. In addition, neutron-proton scattering experiments are being
closely compared to measurements of the contact for strongly interacting ultracold gases [85–87].

Theoretically, an interesting extension of the contact formalism has been to detail the impact of
higher order terms appearing in the momentum distribution. Werner and Castin [59] have shown that
including effects of the effective range in the scattering problem introduces a \( 1/k^6 \) term in the momentum
distribution. Including the effective range also introduces a new adiabatic theorem detailing how the
free energy changes as the effective range is changed. The coefficient to this term has been dubbed the
“derivative contact” [88]. Additionally, new terms arise of order \( 1/k^6 \) in the momentum distribution
related to the centre-of-mass motion of the interacting pairs [89–92]. It is important to note that no
previous experimental work has sought to treat these higher order terms.

Lastly, the contact formalism has been extended to higher partial wave scattering [93–97]; \( p \)-wave
scattering and the $p$-wave contacts will be the focus of Chapter 4.
Chapter 3

Measuring the Contact with Radio-Frequency Spectroscopy

The primary tool used to manipulate and probe atoms in the experiments detailed in this thesis is radio-frequency (rf) radiation. Rf state manipulation has been useful for exploring degenerate Fermi gases. It has been used to measure energy shifts [98–101], to associate and dissociate molecules [50, 51, 102–106], to construct atom interferometers [45, 107–109], to explore pairing and energy gaps [110–113], and to infer thermodynamic properties [17, 28, 81, 95]. Rf state manipulation of atomic Fermi gases has become a workhorse tool for probing one-body, few-body, and many-body physics.

Rf radiation is primarily used to manipulate the internal spin degree of freedom of atoms. Typical rf frequencies are in the 1 kHz -100 MHz range. Rf radiation changes the internal state of the atom while leaving the kinetic energy and momentum of the system mostly unchanged. In other words, by observing the momentum distribution and energy spectrum of atoms following state-manipulation via rf, one can determine the spectral response $A(\omega, k)$ of atoms in an energy- and momentum-resolved way [111, 113]. Measuring $A(\omega, k)$ is the atom analog to photoemission spectroscopy which has been a valuable tool to understand condensed matter systems. The atom analog to photoemission spectroscopy has helped to develop the understanding of the pairing gap in strongly interacting Fermi gases [69, 81, 113, 114].

In this chapter, momentum-integrated rf spectroscopy is used in a time-resolved way to explore the evolution of correlations in strongly interacting Fermi gases [28, 95, 115]. Exploiting the universal relations discussed in Chapter 2, the spectral response of the gas can be used to characterize the emergence of local pairing correlations as well as to infer information about the energetics of the gas. Special attention is paid to the time evolution of the spectral response as the interaction strength is varied. In addition, the interplay between weakly bound molecules, called dimers, and free scattering atoms is studied.

This chapter begins with measurements of rf spectra at unitarity. I first discuss how to understand and analyze the high-frequency behaviour. I then present measurements of dimers and their binding energy in three dimensions. The absence of dimers is used to infer a region of interaction strength where a metastable upper-branch exists. The upper-branch is linked to the prospect of measuring the contact for a three-dimensional gas with repulsive interactions. Next, the contact for a two-dimensional gas is measured. In two dimensions, a weakly bound dimer exists for all interaction strengths, as opposed to only for repulsive interactions in three dimensions. However, the dimer state is not observed to be populated and the magnitude of the contact is substantially smaller than theoretical expectations. Possible hypotheses for the magnitude of the two-dimensional contact are presented. Lastly, the dynamics
of the contact in two dimensions is explored.

### 3.1 The s-Wave Contact in Three Dimensions

One important characteristic of the rf spectrum of strongly interacting Fermi gases is the frequency scaling related to the quantity known as the contact (see Chapter 2). The contact in three dimensions has been measured extensively \[17, 66, 67, 69, 70\] and has been shown to vary monotonically across the Feshbach resonance. Further, at unitarity, the homogeneous contact has been measured as a function of temperature \[68, 70\] and dynamical measurements have shown good agreement with equilibrium measurements of the contact \[116\].

The contact appears in spectroscopy as the prefactor to the high-frequency tail of the transfer rate to a spin state \(\sigma\)

\[
I_\sigma(\omega) \rightarrow \frac{1}{4\pi^2} C^{3D}_s \left(\frac{\hbar}{m}\right)^{1/2} \omega^{-3/2} \tag{3.1}
\]

where \(C^{3D}_s\) is the s-wave contact in three dimensions and \(\omega\) is the detuning from resonance. The transfer rate obeys the normalization \(\int I_\sigma(\omega)d\omega = N_\sigma\) where \(N_\sigma\) is the total atom number in state \(\sigma\). In the experiments discussed in this thesis \(\Gamma(\omega) = N_p(\omega)/t_{rf}\) is measured by applying a rf pulse with resonant Rabi frequency \(\Omega\) and duration \(t_{rf}\) chosen such that the pulse is a weak perturbation on the system (i.e., linear regime). The rate \(\Gamma(\omega)\) obeys the sum rule \(\int \Gamma(\omega)d\omega = \Omega^2\pi N_\sigma/2\) which is equivalent to the sum rule for an ensemble of \(N_\sigma\) coupled two-level systems calculated via Fermi’s golden rule using the coupling matrix element \(\hbar\Omega/2\). Combining the identities listed above, one can identify \(\Gamma(\omega) = \Omega^2\pi I_\sigma(\omega)/2\).

Adopting the normalization convention detailed in \[28, 95, 115\] one can introduce the normalized (dimensionless) detuning \(\Delta = \hbar\omega/E_F\) and define the normalized transfer rate

\[
\tilde{\Gamma}(\Delta) = \frac{E_F}{\hbar\Omega^2\pi N_\sigma} \Gamma(\Delta E_F/\hbar) = \frac{E_F}{2\hbar N_\sigma} I_\sigma(\Delta E_F/\hbar), \tag{3.2}
\]

which obeys \(\int \tilde{\Gamma}(\Delta)d\Delta = 1/2\). Combining the normalization with Eq. (3.1) gives

\[
\tilde{\Gamma}(\Delta) \rightarrow \frac{1}{2^{5/2}\pi^2} \frac{C^{3D}_s}{N_\sigma k_F} \Delta^{-3/2}. \tag{3.3}
\]

Of note is recent work that has shown a second term proportional to \(\Delta^{-5/2}\) arises in the spectrum when including the effect of the effective range \[59, 88, 94, 96\]. A term proportional to \(\Delta^{-5/2}\) has yet to be considered experimentally with application to systems with s-wave interactions. A separate modification of the high-frequency scaling of the spectrum results from interactions between atoms in the final states. Including final state interactions and ignoring the second, higher-order term, the normalized transfer rate takes the form \[57\]

\[
\tilde{\Gamma}(\Delta) \rightarrow \frac{1}{2^{5/2}\pi^2} \frac{C^{3D}_s}{N_\sigma k_F} \Delta^{-3/2} \times \frac{((k_F a)_{\downarrow\downarrow} - (k_F a_{\downarrow\uparrow})^{-1})^2}{(k_F a_{\downarrow\uparrow})^{-2} + \Delta/2}, \tag{3.4}
\]

where \(a_{\downarrow\uparrow}\) is the scattering length between the final states \(|\downarrow\rangle\) and \(|p\rangle\) introduced below. The correction
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Figure 3.1: The time sequence of rf pulses to probe an interacting Fermi gas spectroscopically. A $\pi/2$ pulse initializes a superposition of interacting states $|\downarrow\rangle$ and $|\uparrow\rangle$. After a hold time $t$ the sample is probed with a rf pulse whose amplitude is modulated with a Blackman envelope. An optional $\pi$ pulse can be applied at time $t/2$ such that the spins rephase at time $t$.

due to final state interactions is only a few percent and has been discussed at length in [117].

Previous work measured the contact in an equilibrium mixture of two spin states. This work differentiates itself as the sample is prepared in a superposition of two spin states that evolves into a mixture as the superposition decoheres. The global relaxation and related timescale for this process is discussed at length in Chapter 5.

Spectroscopic measurements described in this thesis utilize the three lowest hyperfine states of the $F = 9/2$ electronic ground state. The high-field $m_F = -9/2, -7/2, -5/2$ states are labelled $|\downarrow\rangle$, $|\uparrow\rangle$, and $|p\rangle$ respectively. The interaction strength between $|\downarrow\rangle$ and $|\uparrow\rangle$ atoms is tuned by the Feshbach resonance near 202.1 G [12] while the state $|p\rangle$ remains weakly interacting with the other two when the respective scattering lengths are small. The experimental sequence is shown in Fig. 3.1 and begins with atoms in an equal superposition of states $|\downarrow\rangle$ and $|\uparrow\rangle$. After a hold time $t$ a spectrum is taken by varying the frequency of a rf pulse coupling state $|\uparrow\rangle$ to state $|p\rangle$. In the final states, the nearby Feshbach resonance at 224 G determines the scattering length and binding energy for states $|\downarrow\rangle$ and $|p\rangle$ [118].

The open circles in Fig. 3.2 show measurements of a spectrum immediately after creating the superposition where $s$-wave scattering is forbidden as all atoms are in identical spin states. The closed circles in Fig. 3.2 show a measurement of the spectrum after the superposition has decohered where maximal $s$-wave scattering occurs. At large detuning there is significant spectral weight in the spectrum from the resonantly interacting mixture as compared to the non-interacting initial state. From a fit using Eq. (3.3) the contact is measured to be $C_{3D} = 1.53(4)NkF$ at unitarity. Previous work showed that as the superposition evolves in time, Eq (3.3) correctly describes the high-frequency tail of the spectrum allowing for a dynamical determination of the contact [116]. The inset of Fig. 3.2 shows the spectra multiplied by $\Delta^{3/2}$ such that the power law in the high-frequency tail appears as a plateau at large detuning. The value at which the spectrum plateaus is the normalized contact per particle. By rescaling the spectra in this way an empirical value for a high-frequency cutoff, above which the fit is performed, can be chosen. The detuning at which the plateau first appears is chosen as the high-frequency cutoff.

3.2 Molecular Dissociation in Three Dimensions

As discussed in Chapter 2 the molecular potential supports a series of bound states which can be probed by rf spectroscopy. These bound states may be probed by directly associating dimers using rf [119] or by first forming a sizeable fraction of dimers and dissociating them [12]. Creating a large fraction of
Figure 3.2: A sample spectrum at unitarity. Immediately after creating the superposition the gas is non-interacting (open circles). After 2.5 ms the gas is strongly interacting (closed circles) and spectral weight shifts to the high-frequency tail. From the fit to the high-frequency tail the contact is determined to be $C^{3D}_s = 1.53(4)Nk_F$ at unitarity. The inset shows the spectrum multiplied by $\Delta^{3/2}$ such that the power law in the high-frequency tail appears as a plateau at large detuning.

dimers is often accomplished by sweeping the magnetic field across the Feshbach resonance, utilizing the avoided crossing to remain in the ground state of the system, adiabatically converting the gas of BCS pairs into a gas of dimers [12]. Alternatively, dimers can be formed as the system relaxes to the ground state. Conservation of energy requires that the binding energy of the dimer be given to a third atom present during scattering or by the Fermi sea if the binding energy is less than the maximum energy that can be absorbed, $2E_F$ [120]. Figure 3.3 illustrates the various processes that contribute to the formation of dimers. Feshbach dimers are bound by an energy $E_b = \hbar^2/(ma^2)$. Universality is broken by the finite range of the interaction $r_0 \approx 60a_0$ which modifies the binding energy, giving $E_b = \hbar^2/(m(a - r_0)^2)$.

When probing weakly bound dimers with rf radiation there are two transitions possible. The first is called the bound-to-bound transition where a weakly bound dimer is converted into a more deeply bound dimer with a different internal state. The second transition is called the bound-to-free transition and involves disassociating the weakly bound dimer into free atoms. A spin-flip transition in scattering states is also possible and called the free-to-free transition. The bound-to-bound spectral feature contributes spectral weight to the spectrum and needs to be accounted for. In particular, as the Franck-Condon factor (or overlap between the bound wave functions) increases, more spectral weight will shift to the bound-to-bound feature of the spectrum. The spectral weight $S_b$ in the bound-to-bound feature is [121]

$$S_b = 1 - \left(\frac{a - a'}{a + a'}\right)^2$$  \hspace{1cm} (3.5)

where $a$ is the scattering length in the initial scattering channel and $a'$ is the scattering length in the scattering channel that supports the more deeply bound dimer. Equation (3.5) only holds for $a' > 0$ when the more deeply bound state exists. The corresponding spectral weight $S_f$ in the bound-to-free transition obeys $S_b + S_f = 1$. The line shape of the bound-to-free transition is more complicated [121]

$$F_f(\omega) = \frac{2}{\pi} \left(1 - \frac{a'}{a}\right)^2 \frac{(\omega - E_b)^{1/2}E_b^{1/2}}{\omega^2(\omega - E_b + E_b')^2}$$  \hspace{1cm} (3.6)
Figure 3.3: A cartoon showing the upper- and lower-branch of a Feshbach resonance and the routes to form dimers. On the BCS side of resonance, only the lower-branch is energetically well-defined and atoms in opposite spin states (red and blue) form BCS-like pairs. As the magnetic field is ramped through resonance atoms remain in the lower-branch and form weakly bound dimers. On the BEC side of resonance, a gas prepared in the upper-branch can spontaneously decay to the lower-branch. This process can occur during a collision if the Fermi sea (or a third particle) provides the binding energy of the dimer.

Figure 3.4: The calculated spectrum for three binding energies: $\hbar \times 62$ kHz, $\hbar \times 84$ kHz, and $\hbar \times 134$ kHz. There are three features present in the spectra (from left to right): the bound-to-bound transition, the free-to-free transition, and the bound-to-free transition. As the binding energy increases spectral weight shifts from the bound-to-free transition to the bound-to-bound transition. For a constant dimer fraction the free-to-free transition is unchanged.
where $E_b$ is the binding energy of the weakly bound dimer in the initial scattering channel and $E_b'$ is the binding energy of the more deeply bound molecule.

Figure 3.4 shows examples of spectra for various magnetic fields. The bound-to-bound transition is a narrow feature with energy $|E_b - E_b'|$. The free-to-free feature is shown as a narrow transition at the spin-flip frequency. The bound-to-free feature has a sharp onset at $E_b$ and extends out to large detuning. The integrated spectral weight in the bound-to-free feature decreases as the binding energy of the dimer increases. The spectral weight of the bound-to-free transition trades off to the bound-to-bound transition. The relative strength of the bound transitions as compared to the free-to-free transition is determined by the relative fraction of atoms in bound dimer states as compared to free scattering states. The resulting products of the bound-to-free transition are two free atoms with total kinetic energy $K = \omega - E_b$. Atoms with $K$ larger than the trap depth will not remain trapped.

The experimental protocol to study dimers is shown in Fig. 3.1. After the spectroscopy pulse a mixture of free atoms and dimers are held in the trap. The nearby Feshbach resonance at 224 G for states $|\downarrow\rangle$ and $|p\rangle$ determines the scattering length and binding energy for these two states [118]. An illustration of the transitions that are probed is shown in Fig. 3.5.

The goal of measuring dimers is to determine the region near the Feshbach resonance where a gas prepared in free scattering states spontaneously decays to a gas of dimers. To better understand and calibrate the rf signal, a gas of almost entirely dimers is created by adiabatically ramping the magnetic field across the Feshbach resonance. The end value of the ramp is chosen to be 200 G to ensure that there is large separation between the free-to-free feature and the bound-to-free feature. At this magnetic field the system is observed to be robust to various loss processes. Unfortunately, the bound-to-bound feature is not observed, although at this field it should contribute $\sim 70\%$ to the total spectral weight.
Figure 3.6: An experimentally obtained spectrum after preparing a mixture of spin states above the Feshbach resonance at 204 G and sweeping the magnetic field through resonance to associate dimers. The final magnetic field value is chosen to be 200 G where the dimerized gas is stable to losses to deeply bound states. Two features are present: the free-to-free (f-f) feature and the bound-to-free (b-f) feature.

Figure 3.6: An experimentally obtained spectrum after preparing a mixture of spin states above the Feshbach resonance at 204 G and sweeping the magnetic field through resonance to associate dimers. The final magnetic field value is chosen to be 200 G where the dimerized gas is stable to losses to deeply bound states. Two features are present: the free-to-free (f-f) feature and the bound-to-free (b-f) feature.

A typical spectrum measured after ramping the magnetic field across the Feshbach resonance is shown in Fig. 3.6. The spectrum has two notable features: the free-to-free resonance and the bound-to-free feature. The free-to-free resonance is located at the spin-flip frequency and the Fourier width is determined by the duration of the spectroscopy pulse. The bound-to-free feature is observed with a sharp onset at roughly the binding energy. The initial line shape matches the theoretical prediction with some broadening due to the power and duration of the rf pulse. The large-detuning behaviour goes to zero quickly around 600 kHz above the onset of the feature which is different from the theoretical prediction. The difference is due to the finite trap depth the atoms are confined in (see Appendix A). When the kinetic energy of the atoms is greater than the trap depth they are lost from the trap and are not counted in the final state.

From the measurement shown in Fig. 3.6 the fraction of dimers can be extracted. Extracting the dimer fraction involves normalizing the spectrum to account for the total atom number and Rabi frequency of the spectroscopy pulse and integrating the region of the spectrum corresponding to the bound-to-free contribution. Comparing the fraction of the total spectral weight in the bound-to-free feature plus the inferred spectral weight in the bound-to-bound transition with respect to the free-to-free feature gives an estimate of the fraction of dimers in the gas. Previously, the dimer fraction was determined by ramping the magnetic field to create deeply bound dimers that are dark to the imaging light used and inferring the dimer fraction from the number of atoms lost [12].

The biggest systematic issues with the approach presented here are the kinetic energy cut-off imposed by the trap depth and the inference of the spectral weight in the bound-to-bound transition. The cut-off due to the finite trap depth is shown in Fig. 3.7 for two trap depths. As the trap depth increases the number of atoms at large kinetic energy counted after dissociation is larger.

From the data shown in Fig. 3.7 the number of atoms lost from the trap can be calibrated. Alternatively, Eq. (3.6) can be used to fit the small kinetic energy behaviour of the data. The best-fit values can then be used to determine the spectral weight in the bound-to-free feature. The data in Fig. 3.7 is obtained by looking at the number of atoms in the final state after dissociation. The complementary
signal is the number of atoms lost from the initial state, which is not subject to the same constraints from the trap depth (atoms lost from the trap still contribute to the negative signal measured). However, this method is extremely noisy due to atom number fluctuations washing out small features of atom loss at large detuning.

Accurately measuring the dimer fraction is difficult because of the systematic uncertainties outlined above. The first measurements attempted involved initializing a superposition of $|\downarrow\rangle$ and $|\uparrow\rangle$ atoms at the desired magnetic field and after a 2 ms hold time, during which the superposition decoheres and collisions populate the dimer state, measuring a spectrum. By numerically integrating the area under the free-to-free feature and the bound-to-free feature the fraction of dimers can be estimated. The estimation of the fraction of dimers as a function of interaction strength $-(k_F a)^{-1}$ is shown in Fig. 3.8a (closed circles) for values of $a$ such that the bound-to-free feature is distinct from the free-to-free feature. In this data, the bound-to-free feature is not corrected with the Franck-Condon factor accounting for the bound-to-bound transition and so is likely an underestimate of the dimer fraction. Two- and three-body interactions that can convert dimers into more deeply bound molecules [122] are ignored which contributes to an underestimation of the dimer fraction. As such, the errors bars in Fig. 3.8a are suppressed. Despite the numerous estimates that enter into this data, it does show a qualitative decrease in the dimer fraction as $a$ decreases. The same trend is reproduced in Fig. 3.8b (and the open circles in Fig. 3.8a). The data is obtained in a slightly modified way: after 2 ms of hold time at the desired magnetic field the magnetic field is ramped to 200 G in 5 ms and then the spectroscopy pulse is applied. Using this method benefits from the fact that the bound-to-free feature will be located over the same range of frequencies for direct comparison between initial magnetic fields. The additional 5 ms of time in the sequence allows for loss processes that can decrease the measured dimer fraction. Regardless, the spectral weight at large detuning is consistent with noise for $-(k_F a)^{-1} \lesssim -1.3$.

Figure 3.8 shows that dimers are formed in the range $-1.3 \lesssim -(k_F a)^{-1} \leq 0$. The instability towards the formation of pairs has been discussed at length with respect to a strongly repulsive Fermi gas [120, 123–125]. Theoretical predictions suggest that the onset of the instability to form pairs in two-body processes occurs at $-(k_F a)^{-1} \approx -1$. The discrepancy between the measurement and the
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Figure 3.8: (a) The estimated molecular fraction determined with (open markers) and without (closed markers) a field sweep to 200 G as described in the text. In both cases, the estimated fraction reaches the noise floor of the experiment at $-(k_Fa)^{-1} \sim -1.3$. (b) Sample spectra taken with a field sweep to 200 G. The molecular feature decreases in amplitude as $-(k_Fa)^{-1}$ decreases.

predicted value could arise from the formation of dimers due to a three-body process for values of $-(k_Fa)^{-1} < -1$. The absence of significant dimer fraction in the measurements shown in Fig. 3.8 allows for the determination of a range of interaction strengths where a superposition of these two spin states remains in the upper-branch of the energy spectrum near the Feshbach resonance as it decoheres.

3.2.1 Can the Repulsive Contact be Measured?

Having delineated a regime where the upper- and lower-branch can be distinctly populated, an open question was whether or not one could observe a “repulsive contact” in the upper-branch. To measure the repulsive contact the usual spectroscopic sequence is used, as illustrated in Fig. 3.1. A sample spectrum, after the superposition has decohered, is shown taken at 200.5 G in Fig. 3.9. At large detuning $\Delta \gg 1$, there is excess spectral weight.

Fitting the spectrum using Eq. (3.3) allows one to measure the contact if the spectrum only contains a free-to-free feature with a high-frequency tail. However, when a bound-to-free feature is present in the spectrum it complicates the analysis. Figure 3.9 shows that only a narrow window in detuning exists over which a value of the contact can be determined before the high-frequency tail is contaminated by the bound-to-free feature.

The spectrum must be probed with $\Delta \gg 1$, which is determined experimentally to correspond to values of $\Delta \gtrsim 3$ from data at unitarity. Empirically, a frequency cut-off is chosen by plotting $\Delta^{3/2} \tilde{\Gamma}$ versus $\Delta$. The power law in the tail of the spectrum appears as a plateau when plotted this way. By observing the value of $\Delta$ for which $\tilde{\Gamma} \Delta^{3/2}$ begins to saturate at a constant value the frequency cut-off can be determined. This imposes a narrow range of interaction strengths over which the contact can be faithfully measured from the high-frequency tail before the signal is contaminated by the bound-to-free feature.

To measure the contact the binding energy must be $E_b/E_F \gtrsim 3$ which implies that $-(k_Fa)^{-1} \lesssim -1.2$. Decreasing the value of the interaction strength increases the binding energy of the
Figure 3.9: A sample spectrum taken at 200.5 G for an interacting (filled circles) and non-interacting (open circles) gas. The inset shows three regions of the spectrum: the spin-flip resonance (broadened by the rf pulse), the high-frequency tail where the contact can be measured, and the bound-to-free feature.

dimer and therefore moves the bound-to-bound feature to larger detuning. The spectrum then has more data points above the frequency cut-off and below the bound-to-free feature which makes fitting the spectrum with Eq. (3.3) more robust.

In my analysis, I attempted to determine the power law describing the high-frequency tail of the spectrum. Simply performing a nonlinear fit using \( f \propto \Delta^\eta \) to determine the exponent \( \eta \) led to poorly constrained fits. Instead, a one-dimensional chi-squared map was constructed as a function of \( \eta \) for half-integer values of \( \eta \). The best-fit exponent was determined by the minimum in chi-squared to be \( \eta = -1.5 \) (see Chapter 4 for more details). This method worked well when the bound-to-free feature was separated from the free-to-free feature but often failed when the number of data points included in the fit decreased. The value of \( \eta \) was also sensitive to the choice of frequency cut-off. In the data that follow, a single detuning \( \Delta = 3 \) is chosen from which a value of the contact is extracted [28].

Measurements of the trap-averaged contact per particle are shown in Fig. 3.10 across the Feshbach resonance. For \(-(k_F a)^{-1} > 0\) (attractive interactions) the lower-branch of the Feshbach resonance is populated by scattering atoms forming BCS pairs. The BCS prediction for the contact \(4(k_F a)^2/3\) (red dashed line) shows that for weak interactions the dynamical measurement presented here agrees well with theory. As the interaction strength is increased the measurement begins to disagree with simple BCS theory but matches other experiments that found \( C/(N k_F) \approx 1.5 \) at \((k_F a)^{-1} = 0\) [17, 66, 69]. For \(-(k_F a)^{-1} < 0\) (repulsive interactions) the value of the contact depends on the branch of the energy spectrum that is populated. Measurements using momentum-resolved rf spectroscopy have measured the homogeneous contact for a gas in equilibrium in the lower-branch [69]. These measurements tend towards the BEC prediction for the contact \(4\pi(k_F a)^{-1}\) (blue dotted line) for \(-(k_F a)^{-1} < -1\). Most notably, they find a value for the homogeneous contact of \( C/(N k_F) \approx 10 \) at \((k_F a)^{-1} \approx -0.75\) [69]. Conversely, in this experiment the contact for a gas of atoms with repulsive interactions is measured in the upper-branch of the energy spectrum. The magnitude of the contact is two orders of magnitude smaller than in the lower-branch, remaining relatively constant at \( C^{\text{3D}}/(N k_F) \approx 0.1 \) for \(-(k_F a)^{-1} < -1\). The sharp decrease in the magnitude of the contact across the Feshbach resonance is suggestive of a switching of energetic branch (see discussion below and Chapter 4).
Figure 3.10: A summary of spectroscopic measurements of the contact in the upper- and lower-branch. Measurements of the contact are plotted with various predictions; the BCS prediction (red), the BEC prediction (blue), and the Fermi-Liquid theory prediction (black). The contact shows relatively constant behaviour for \( -(k_F a)^{-1} < 0 \).

From the perspective of the adiabatic relation, a constant value of \( C^{3D}_{s} \) makes little sense as the energy in the system increases as one moves towards the Feshbach resonance. However, when interpreting the contact as a measure of the number of pairs at small volume it is not surprising that increasing the strength of repulsive interactions does not increase the number of these pairs. Perhaps there is an interplay between these two contributions that results in the relatively constant value that is observed. More rigorous measurements are need to draw any conclusions from the data shown in Fig. 3.10 for \( -(k_F a)^{-1} < 0 \).

3.3 The \( s \)-Wave Contact in Two Dimensions

The contact in two-dimensional Fermi gases with \( s \)-wave interactions has been largely unstudied. One experiment has measured the contact in a two-dimensional Fermi gas with weak attractive interactions [81] showing good agreement with theoretical predictions from the zero temperature equation of state [83, 126–130].

The scattering problem in two dimensions is different from three dimensions as a bound state exists for arbitrarily weak attractive interactions [31]. The binding energy of the two-dimensional bound state has been studied in the dimensional crossover [50, 51] and recently pair condensation has been observed in a two-dimensional Fermi gas [131]. The presence of a bound state for weakly attractive interactions complicates a spectroscopic approach to measuring the contact as discussed in the previous section; the high-frequency tail of the free-to-free feature can be difficult to observe if the binding energy is small. Ideally, one would like to remove the contribution of the dimers before measuring the spectrum to see the contribution to the contact from free scattering atoms.

At this point in the discussion, a question arises: how does one interpret the dimer contribution to the contact? Dimers are atom pairs in a small volume and therefore contribute sizeably to the magnitude of the contact. An intuitive understanding comes from the equation of state literature where
the contribution to the contact from the dimers is explicitly separate. Notably, the ground-state energy is parametrized as \( E/N - E_b/2 \) where the dimer contribution is subtracted from the energy per particle. In fact, the ground state energy is largely determined by the two-body physics. Utilizing the adiabatic theorem one can define the “many-body” contact \( C_{2D}^s - C_0 \) where \( C_{2D}^s \) is the two-dimensional \( s \)-wave contact and

\[
C_0 = \frac{\pi m}{\hbar^2} \frac{dE_b}{d(\ln (k_F a_{2D}))}
\]

encapsulates the contribution from dimers. The many-body contact describes the contribution to the energy of the lower-branch from atom-dimer and dimer-dimer interactions. In two dimensions the dimer binding energy can be greater than \( E_F \) so that an attractive upper-branch is energetically well defined (see Chapter 2). In the dynamical approach to measuring the contact, either the upper-branch, the lower-branch, or both (as the upper-branch decays to the lower-branch) may be populated.

When performing spectroscopy of two-dimensional Fermi gases a spin-polarized gas of \( |\downarrow\rangle \) atoms is loaded into a one-dimensional optical lattice to create an ensemble of two-dimensional gases (see Appendix A). The same spectroscopic measurement as in three dimensions is carried out, following the timing sequence shown in Fig. 3.1. At high-frequency the transfer rate from an interacting state to a non-interacting probe state obeys

\[
I_\sigma(\omega) \rightarrow \frac{1}{4\pi} C_{2D}^s \left( \frac{\hbar}{m} \right) \omega^{-2}.
\]

When combined with the normalization in Eq. (3.2) the transfer rate at high-frequency becomes

\[
\bar{\Gamma}(\Delta) \rightarrow \frac{1}{4\pi} \frac{C_{2D}^s}{N_{\sigma} k_F^2} \Delta^{-2}.
\]

In two dimensions, the interactions in the final state also influence the shape of the spectrum. Correcting for these final state interactions in the spectrum gives [81, 132, 133]

\[
\bar{\Gamma} \rightarrow \frac{1}{4\pi} \frac{C_{2D}^s}{N_{\sigma} k_F^2} \Delta^{-2} \times \frac{\ln^2 (E'_F/E_b)}{\ln^2 (\Delta E_F/E'_b) + \pi^2}.
\]

A sample spectrum is shown in Fig. 3.11 with and without final state corrections. From the high-frequency tail the contact can be measured. As discussed for the case in three dimensions, a high frequency cut-off is chosen above which the tail of the spectrum is fit. The spectrum obeys the sum rule with \( \int \bar{\Gamma}(\Delta)d\Delta \approx 0.5 \) measured when the transition is probed in the linear regime.

At \( \ln (k_F a) \approx 1 \) the contact is measured to be \( C_{2D}^s/(Nk_F^2) \approx 0.15 \) which is in sharp contrast to the value determined in [81], \( C_{2D}^s/(Nk_F^2) \approx 1 \) at a similar interaction strength. The difference between the measurement presented here and the previous measurement [81] is further exaggerated as the interaction strength is increased. One hypothesis for the discrepancy arose from considering the many-body contact. The magnitude of the contribution to the contact from dimers is \( \sim 0.85 \) at \( \ln (k_F a) \approx 1 \) which would correctly rescale the measured value to match previous observations and to approximately match the predictions from the zero temperature equation of state. The magnitude of the contact observed suggests that a contribution from dimers is not observed despite the presence of a weakly bound state for attractive interactions. The hypothesis that these measurements do not probe the contribution from dimers is lent more weight by measurements of the temperature increase as the gas relaxes from a superposition to a
mixture (see Chapter 5). The heating is approximately three times smaller than that which would be expected from a calculation matching initial energy and atom number to the equilibrium two-dimensional equation of state [83, 126–130, 134, 135]. These observations suggest that equilibration in the lower-branch to a dimerized state does not occur. However, the data is insufficient to distinguish between two possibilities: that dimers are lost from the trap before they equilibrate, or that the system remains in an attractive upper-branch of the Feshbach resonance.

Figure 3.12b shows measurements of the two-dimensional contact across the Feshbach resonance. The data are peaked near $\ln (k_F a_{2D}) = 0$ and decrease quickly as the interaction strength is varied. The shape of the data is asymmetric about $\ln (k_F a_{2D}) = 0$, almost immediately tending to zero for $\ln (k_F a_{2D}) < 0$. Figure 3.12a shows measurements of atom loss over a similar range of interaction strength. The measurement protocol recovers weakly bound dimers but more deeply bound dimers are lost from the trap. The loss feature in Fig. 3.12a gives an estimate of the region of increased dimer formation as dimers are a precursor to loss. The total fraction lost is $\sim 20\%$. The location of the loss feature is shifted from the three-dimensional case (see Fig. 2.5a) such that the maximum occurs near $\ln (k_F a) \approx 1$. The onset of the loss and shape of the feature is in general qualitative agreement with a calculation of the instability of the upper-branch in two dimensions [136]. Combining the understanding of this loss feature with the magnitude of the contact and the heating induced by demagnetization (and measurements of the sign of the effective interactions, see Chapter 5) suggests that equilibration in the lower-branch does not occur.

### 3.4 Contact Dynamics in Two Dimensions

I extend previous research measuring contact dynamics in three dimensions [28, 116] to two dimensions. To measure the time-evolution of the contact the experimental protocol follows the same time sequence introduced in Fig. 3.1. At $\ln (k_F a_{2D}) = 0$ a resonant $\pi/2$ pulse is applied to create a superposition of $|\downarrow\rangle$ and $|\uparrow\rangle$. The superposition is probed spectroscopically as a function of time $t$ as it decoheres. Figure 3.13 shows the time evolution of the spectrum as the superposition goes from being non-interacting to resonantly interacting. The spectral weight at large detuning increases from zero at short hold times to a maximal value $C_s^{2D} \approx 0.2 N k_F^2$. Using Eq. (3.9) at each hold time suggests that, as in three
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Figure 3.12: (a) Measurements of the fraction of atoms remaining after 3 ms. (b) Measurements of $C_{2D}^s$ across the Feshbach resonance.

dimensions [28, 116], the dynamical measurements of the contact follow the same universal relations as their equilibrium counterparts.

Measuring the contact dynamically provides a connection to the magnetization dynamics discussed in Chapter 5. For pairs of fermions to interact via an $s$-wave interaction they must have a spin singlet wave function. The singlet fraction can be no larger that $1 - |M|$ and would be $(1 - |M|^2)/4$ for uncorrelated spins, where $M$ is the magnetization [28, 80, 137]. The contact can be measured in the spectroscopic sequence with a $\pi$ pulse applied at time $t/2$ to match the protocol used for the magnetization measurement. Figure 3.14a shows the dynamics of the contact with (open circles) and without (closed circles) a $\pi$ pulse. The evolution of $C_{2D}^s$ with a $\pi$ pulse is a factor of $4^{-1/3}$ slower than without and otherwise identical (see fit lines of Fig. 3.14a). Figure 3.14b directly compares the contact and the magnetization for a two-dimensional Fermi gas. Fitting the data using $C_{2D}^s \propto 1 - |M|^2$ finds $\eta = 1.4$, between the fully correlated and uncorrelated limits. The relationship between $C_{2D}^s$ and $M$ has been discussed as an entanglement witness [137] which would be interesting to study as a function of temperature. The relationship between the contact and the magnetization is exploited in Chapter 5 to verify the understanding of the observed magnetization dynamics.

3.5 Discussion

Understanding rf spectroscopy of strongly interacting Fermi gases has been a theme of my research. Rf spectroscopy has been shown to help elucidate both the microscopic and (through universal relations) the macroscopic properties of these systems. The data I present push beyond an equilibrium understanding of rf spectra. Dynamical measurements have allowed for the observation of the evolution of pairing correlations across the BCS-BEC crossover in two and three dimensions.

In three dimensions, a regime where dimers are largely not formed has been observed and the signal
Figure 3.13: Time-resolved spectroscopy in two dimensions is carried out as a superposition decoheres. After a few milliseconds the gas is strongly interacting and spectral weight has shifted to the high-frequency tail. The inset shows $C_{2D}^D(t)$ determined by fitting the high-frequency tail of each spectrum.

Figure 3.14: (a) The dynamical evolution of the contact with (open circles) and without (closed circles) a $\pi$ pulse. The two data sets are identical except that the data with a $\pi$ pulse is $4^{-1/3}$ times slower than without. (b) A comparison of the contact and the magnetization. The dashed line shows the best fit $\eta = 1.4$ assuming a functional form $C \propto 1 - |M|^\eta$. 
from upper-branch scattering atoms as compared to lower-branch dimers can be distinguished. Delin-
eating upper-branch and lower-branch physics may allow for measurements of the contact in the upper-
branch of the Feshbach resonance which can inform understandings of the energetics near the pairing
instability [120, 123, 125]. Measurements of the contact in a two-dimensional Fermi gas suggest that the
method employed here does not measure the contribution from dimers. However, the experiment cannot
distinguish between having probed the contact in the attractive upper-branch of the Feshbach resonance
or merely measuring the contact in the lower-branch with the contribution from dimers missing.

The data presented in this chapter open a few avenues of research that should be explored. Firstly,
the measurements of the repulsive contact should be more rigorously understood. The contact could be a
useful proxy for measuring the energy in the upper-branch. Next, future research should investigate the
dynamics of dimer formation. Perhaps utilizing the dynamical technique discussed in this chapter would
allow for the observation and determination of the timescale on which dimers form. Lastly, this chapter
opens a large question about the value of the contact measured in two dimensions. More study is needed
to determine whether the dynamical technique used populates the upper-branch or if the contribution
from dimers is simply missing due to a technical limitation of the apparatus.
Chapter 4

The \( p \)-Wave Contacts

As discussed in Chapter 2 and Chapter 3, in dilute gases with short-ranged \( s \)-wave interactions, a set of universal relations are known to connect microscopic properties of the gas with macroscopic thermodynamics. At centre stage in these relations is a quantity known as the contact (see Chapter 2). The contact has been a unique lens with which to bring together theory and experiment over the past ten years in ultracold atoms. In particular, the measured value of the contact has been a benchmark against which various theories of unitary Fermi gases can be compared \([67, 68, 70]\). As these measurements have become more and more precise, theoretical treatments of the unitary Fermi gas are more and more constrained which helps identify the most successful models.

The formalism surrounding the contact has been so successful that it recently has been extended to a diverse collection of systems. The contact has been discussed in the context of bosons, in one, two, and three dimensions, for Coulomb gases, and for neutron-proton scattering. In all of these contexts the contact and its universal relations have been limited to short-ranged \( s \)-wave interactions.

In this chapter I present the first experimental observation of the \( p \)-wave contacts and extensions of the contact formalism to \( p \)-wave scattering. This chapter serves as a detailed treatment of the work published in \([95]\).
4.1 Ultracold Atoms with $p$-Wave Interactions

The observation of the first degenerate Fermi gas occurred a few years after the observation of the first Bose-Einstein condensate. The delay can be partially attributed to the quantum statistics associated with fermions as compared to bosons. In the boson case, evaporative cooling is extremely successful as a spin-polarized bosonic gas can thermalize via $s$-wave collisions. Conversely, a spin-polarized fermionic gas can only thermalize via $p$-wave collisions as $s$-wave collisions between identical fermions are forbidden. As discussed in Chapter 2, $p$-wave collisions are suppressed at low temperature due to the presence of the centrifugal barrier in the radial potential. The suppression of $p$-wave scattering was observed in a spin-polarized gas of potassium atoms where as the temperature decreased, the scattering cross section decreased according to the Wigner threshold law [138]. The first measurements of evaporative cooling of spin-polarized Fermi gases showed that evaporative cooling was only possible at temperatures $T \gtrsim 20\mu K$. Therefore, experiments with fermionic atoms began using spin mixtures (whose scattering properties could be tuned with Feshbach resonances) or mixtures of atomic species to perform evaporative cooling.

$p$-wave physics in ultracold atoms resurfaced following the observation of Feshbach resonances that enhanced $p$-wave scattering [139–141]. Similar to $s$-wave Feshbach resonances, $p$-wave Feshbach resonances allow one to tune the scattering volume $v$ to resonantly enhance the $p$-wave scattering cross section by tuning the magnetic field

\begin{equation}
    v = v_{bg} \left(1 - \frac{\Delta B}{B_0 - B}\right) \tag{4.1}
\end{equation}

where $v_{bg}$ is the background scattering volume, $\Delta B$ is the width of the resonance, and $B_0$ is the magnetic field at which the scattering volume diverges. $p$-wave resonances possess the property that the various projections of the angular momentum of the pair of scattering atoms onto the magnetic field axis causes there to be a unique resonance for each value $\nu \in \{x, y, z\}$ of the projection, $v \equiv v_\nu$. The location of these resonances may be different, split by the magnetic dipole-dipole interaction [142, 143] or potentially modified by vector light shifts on molecular transitions [144]. Experimental work in an optical lattice has shown that the projection of angular momentum can be deterministically controlled by varying the direction of the external confinement [145, 146].

Interest in studying $p$-wave interactions in ultracold atoms was also motivated by the observation of pair condensation and superfluidity at $s$-wave Feshbach resonances. It was thought that the same techniques used to observe $s$-wave superfluidity could be applied to studying $p$-wave superfluidity. As was done for $s$-wave interactions, the study of $p$-wave Feshbach resonances began to focus on the collisional properties and lifetime of $p$-wave Feshbach molecules.

For $v_\nu > 0$ there is a Feshbach dimer state, which is a superposition of the open- and closed-channels. The energy of this dimer is

\begin{equation}
    E_{d,\nu} = -\frac{\hbar^2 R_\nu}{mv_\nu} \approx \frac{\hbar^2 R_\nu (B - B_{0\nu})}{mv_\nu^2 \Delta B_\nu} \tag{4.2}
\end{equation}

where $m$ is the atomic mass and $R_\nu$ is related to the effective range. Close to resonance these molecules are closed-channel dominated [147]. For $v_\nu < 0$ the dimer state rises above threshold ($E_{d,\nu} > 0$) and decays at a rate

\begin{equation}
    \gamma_\nu = \frac{2\sqrt{m} R_\nu E_{d,\nu}^{3/2}}{\hbar^2} \tag{4.3}
\end{equation}
At low energy, all \(p\)-wave resonances are narrow, \(\hbar\gamma_{\nu} \ll E_F\) (see Chapter 2) because \(\gamma_{\nu}/E_{d,\nu}\) increases with energy.  

\(p\)-wave Feshbach dimers have been created and studied in experiments with both \(^{40}\text{K}\) and \(^{6}\text{Li}\) [105, 106, 146–149]. The binding energy has been measured showing the linear dependence on magnetic field detuning predicted by Eq. (4.2). For both atomic species, the lifetime near the Feshbach resonance is short due to the closed-channel nature of the resonance. In \(^{40}\text{K}\) the lifetime of a gas of purely dimers was determined to be on the order of a few milliseconds [117], limited by spin-relaxation as the Feshbach resonance does not occur between atoms in their absolute ground state. In lithium the lifetime was found to be tens of milliseconds [148] as the Feshbach resonance occurs between atoms in their ground state, approaching the thermalization timescale required for the formation of a condensate of dimers.  

Due to the limited lifetime near \(p\)-wave Feshbach resonances many groups stopped pursuing this work as thermalization times were not fast compared to loss rates which is typically required for (metastable) ultracold atom experiments. New proposals for observing \(p\)-wave pairing were proposed that circumvented the large atom loss associated with \(p\)-wave Feshbach resonances [150–152].

### 4.1.1 Dynamical Spectroscopy

The conventional wisdom in the lab was that “choosing magnetic field values near the \(p\)-wave Feshbach resonance is bad because the atoms are lost faster than you can do anything meaningful”. The high density gases used in these experiments makes this warning especially true as the loss rates appear as a product with density squared (two-body loss) or density cubed (three-body loss). Without knowing about this common lab warning and being in charge of the lab while I was on vacation, Scott Smale inadvertently tuned the magnetic field directly on top of the \(p\)-wave Feshbach resonance expecting to observe the slow increase in the \(s\)-wave contact as determined by a nearby \(s\)-wave Feshbach resonance (see Chapter 3 for example). Opposite to his expectation, the weight at high-frequency was almost immediately present and decreased as a function of time! This seemingly bizarre feature went unexplained until I returned to the lab. My immediate hypothesis was that this was something associated with the \(p\)-wave Feshbach resonance and the decrease in signal was a result of the fast loss associated with the narrow resonance. However, it would take many months and a few theory papers before the signal observed was fully understood.

The measurement technique used follows directly from the dynamical scheme presented in Chapter 3. The atom loss is circumvented by using a short \(40\mu\text{s}\) \(\pi\) pulse to quickly transfer all of the atoms into the interacting state and subsequently probe the sample as a function of time spectroscopically before too many of the atoms are lost.

A typical spectrum observed near resonance is shown in Fig. 4.1 after \(100\mu\text{s}\) of time in the interacting state. At first glance, the spectrum looks similar to what is observed near the \(s\)-wave Feshbach resonance. However, I soon discovered an important difference. The tail of the spectrum at high-frequency for \(s\)-wave interactions (in three dimensions) is expected to scale as \(\omega^{-3/2}\), where \(\omega\) is the detuning from resonance. I developed a quick statistical test to determine the “best fit” power law to describe the tail of the spectrum. From my previous work, I had seen that doing a simple nonlinear fit with a free exponent could be highly dependent on the range of values of detuning used in the fit and typically the value determined for the power law would fall ambiguously between two half-integer values.  

Instead, I constructed one-dimensional maps of the reduced chi-squared \(\chi^2\) as a function of the exponent to determine the best fit value with the assumption that a half-integer power law described the
Figure 4.1: The first spectrum taken at 198.4 G near the $p$-wave Feshbach resonance. The fraction of atoms transferred from the $p$-wave interacting state to a probe state is measured and significant spectral weight at large detuning is observed.

high-frequency tail (see Chapter 2). Using a function $f = A\omega^\eta$ I perform a nonlinear fit to determine $A$ for each exponent $\eta \in \{-5, -4.5, ..., 5\}$. Then for each exponent and the corresponding best-fit value of $A$ I calculate the reduced chi-squared for various ranges of detuning in the high-frequency tail, expecting that for large detuning, $\hbar \omega/E_F \gg 1$, the power law would be representative of some short-ranged physics. Representative chi-squared maps plotted as a function of the exponent $\eta$ are shown in Fig. 4.2 for various detuning ranges in the high-frequency tail. The minimum in the reduced chi-squared is then determined for each range of detuning. For large detuning (but when there are still many points included in the fit) the best-fit power law determined this way is $\omega^{-1/2}$ as shown in Fig. 4.2.

The power law observed was not consistent with what was expected from $s$-wave scattering. However, I could calculate that such a frequency scaling would arise in three dimensions if the momentum distribution scaled as $1/k^2$ at large-momentum. A hypothesis existed that the momentum distribution near a $p$-wave Feshbach resonance would exhibit exactly this scaling [153].

To test the hypothesis that the frequency scaling resulted from $p$-wave interactions, additional measurements of the spectra near the $p$-wave Feshbach resonances (recall, there are two, the $\nu = z$ resonance is distinct from the degenerate $\nu = x$ and $\nu = y$ resonances) were taken. These measurements revealed another mystery: the power law in the high-frequency tail changed from $\omega^{-1/2}$ near resonance to be $\omega^{-3/2}$ further from resonance and in between these limiting cases the best-fit exponent was highly variable. Using an empirically determined function $g(\omega) = A\omega^{-1/2} + B\omega^{-3/2}$ to fit the data showed a smooth crossover between the two power laws. Figure 4.3 shows the ratio of the $A$ and $B$ coefficients determined from such a fit. In closed circles, $A/(A+B)$ is plotted while in open circles $B/(A+B)$ is plotted (with sigmoid functions to guide the eye). The spectra appear to change from being dominated by the $\omega^{-1/2}$ frequency scaling to being dominated by the $\omega^{-3/2}$ frequency scaling. The magnetic field values where a single frequency scaling describes the data will be referred to as “pure points”. The pure points could be treated using the analysis techniques introduced in Chapter 3 but understanding the crossover behaviour between the two regimes left me stumped. To fully understand what was happening a theoretical treatment of the problem was needed.
Figure 4.2: One dimensional $\chi^2$ maps showing the best-fit exponent describing the power law at high-frequency in the spectrum. For all ranges of detuning used in the fit, the minimum in $\chi^2$ occurs for an exponent of -0.5.

Figure 4.3: The ratio of $A$ and $B$ coefficients as a function of detuning from the Feshbach resonance. Closed circles show $A/(A+B)$ while open circles show $B/(A+B)$. The dashed lines are sigmoid functions to guide the eye.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{bg}^x$</td>
<td>$(101.6 \ a_0)^3$</td>
<td>[142]</td>
</tr>
<tr>
<td>$v_{bg}^{xy}$</td>
<td>$(96.74 \ a_0)^3$</td>
<td>[142]</td>
</tr>
<tr>
<td>$\Delta z$</td>
<td>21.95 G</td>
<td>[142]</td>
</tr>
<tr>
<td>$\Delta_{xy}$</td>
<td>24.99 G</td>
<td>[142]</td>
</tr>
<tr>
<td>$B_{0,z}$</td>
<td>198.792(13) G</td>
<td>this work</td>
</tr>
<tr>
<td>$B_{0,xy}$</td>
<td>198.301(11) G</td>
<td>this work</td>
</tr>
<tr>
<td>$R_{bg}^z$</td>
<td>47.19 $a_0$</td>
<td>[142]</td>
</tr>
<tr>
<td>$R_{bg}^{xy}$</td>
<td>46.22 $a_0$</td>
<td>[142]</td>
</tr>
<tr>
<td>$\Delta R_z$</td>
<td>-18.71 G</td>
<td>[142]</td>
</tr>
<tr>
<td>$\Delta_{R,xy}$</td>
<td>-22.46 G</td>
<td>[142]</td>
</tr>
</tbody>
</table>

Table 4.1: Scattering parameters describing the $p$-wave resonances near 198.5 G in $^{40}$K. $a_0 = 0.529 \times 10^{-10}$ m is the Bohr radius.

4.2 Universal Relations for Short-Ranged $p$-Wave Scattering

As discussed in Chapter 2, tuning of the scattering volume $v_\nu$ is accomplished with a Feshbach resonance, where the relative energy of a closed-channel molecular bound state is tuned near energetic resonance with an open-channel scattering state of free atoms. Tuning is accomplished with a magnetic field due to the differential magnetic moment $\delta \mu$ between the open- and closed-channels. The effective range $R_\nu \equiv -(2r_{eff,p})^{-1}$ (see Chapter 2 for a definition of $r_{eff,p}$) does not depend strongly on the magnetic field near resonance. Far from resonance, $v_\nu$ and $R_\nu$ take on background values $v_{bg}^\nu$ and $R_{bg}^\nu$, but near resonance $v_\nu$ is resonantly enhanced according to Eq. (4.1). The $\nu = z$ resonance is split from the degenerate $\nu = x$ and $\nu = y$ resonances such that both the strength and the anisotropy of $p$-wave interactions can be controlled with the magnetic field, $B$.

The parametrization of the scattering resonance used in this thesis is shown in Fig. 4.4. The parameters $v_\nu$ and $R_\nu$ are parametrized as a function of the magnitude of the magnetic field using

$$\frac{1}{v_\nu} \approx \frac{1}{v_{bg}^\nu} \left[ -\frac{\delta B_\nu}{\Delta_\nu} + \left( \frac{\delta B_\nu}{\Delta_\nu} \right)^2 \right]$$

$$\frac{1}{R_\nu} = \frac{1}{R_{bg}^\nu} \left[ 1 + \frac{\delta B_\nu}{\Delta_{R,\nu}} \right]$$

(4.4)

where $\delta B_\nu = B - B_{0,\nu}$. The values for the scattering parameters are summarized in Tab. 4.1. The next order contributions to $v_\nu$ and $R_\nu$ are small as measurements take place close to resonance.

That $p$-wave interactions might be described with an $\ell = 1$ analogue of the contact was conjectured in [153], and has more recently been followed with a full theory [93, 94]. The most significant structural difference from the $s$-wave contact theory is that each scattering channel has two contacts, $C_{v,\nu}$ and $C_{R,\nu}$, which are conjugate to $v_\nu$ and $R_\nu$ in the respective adiabatic relation. I summarize here what can be learned from the contacts once they are measured or calculated.

The adiabatic relation can be framed in terms of the free energy of the system. Keeping the temperature, number, and volume constant the free energy changes as [93, 94]

$$\frac{dF}{dv_\nu^{-1}} = -\frac{\hbar^2}{2m} \sum_\nu C_{v,\nu}$$

(4.5)
Figure 4.4: The nature of interactions can be controlled by selecting internal states and tuning the magnetic field. Near the $p$-wave resonances studied here, spin-triplet collisions are resonant for atoms in the $|\uparrow\rangle$ state. This can be contrasted with the $s$-wave interactions near a Feshbach resonance at 202.1 G, where spin-singlet collisions in a mixture of $|\downarrow\rangle$ and $|\uparrow\rangle$ atoms are enhanced. Unlike $s$-waves, $p$-waves are normally suppressed at low energy due to a centrifugal energy barrier. The scattering volume $v_\nu$ diverges at distinct $\nu = z$ and $\nu = xy$ Feshbach resonances, while $R_\nu$ is only weakly dependent on $B$ and nearly isotropic.
as the scattering volume is changed. Equation (4.5) is the $\ell = 1$ analogue to the $s$-wave adiabatic relation Eq. (2.37). The factor of two in the denominator arises due to the fact that the $p$-wave Feshbach resonance enhances spin-triplet scattering between identical fermions, as opposed to between fermions in two spin states. Notice, there is also a factor of $4\pi$ difference in the definition of the $p$-wave contacts as compared to Tan’s definition of the $s$-wave contact. An additional adiabatic relation must exist to include $C_{R,\nu}$ in the theoretical framework. The second adiabatic relation relates changes in the free energy to changes in the effective range

$$\frac{dF}{dR_{\nu}} = -\frac{\hbar^2}{2m} \sum_{\nu} C_{R,\nu}.$$ (4.6)

At short range, $r \ll k_F^{-1}$, the many-body wave function has a form that is controlled by the two-body physics, but a normalization that is controlled by the contacts [54]. The pair correlation function is

$$g^{(2)}(r) \rightarrow \frac{6\pi^2}{N} \sum_{\nu} Y_{1\nu}(\hat{r}) \left[ \frac{C_{v,\nu} k_F}{(k_F r)^4} + \frac{C_{R,\nu}/k_F}{(k_F r)^2} \right].$$ (4.7)

in the regime $r_0 \ll r \ll k_F^{-1}$, where $r = r_1 - r_2$ is the relative coordinate, $\hat{r} = r/r$, $Y_{1,\nu}$ are the spherical harmonics for $\ell = 1$, and $N$ is the atom number.

The spectral weight of excitations that probe the high-energy or short-range sector of the many-body wave function are also controlled by the contacts [56, 57, 154]. For rf transfer to a non-interacting probe state, the high-frequency tail of the spectral density is

$$I(\omega) \rightarrow \frac{\pi}{\pi} \sum_{\nu} \left[ C_{v,\nu} \sqrt{\frac{m}{\hbar}} \omega^{-1/2} + \frac{3}{2} C_{R,\nu} \sqrt{\frac{\hbar}{m}} \omega^{-3/2} \right].$$ (4.8)

Notice that the numerical coefficient leading the $C_R$ term does not match the expectation from the calculation in Chapter 2 using $n(k)$ defined below. The coefficient in the sub-leading term must be determined in a full calculation of the spectral response of the system. The spectral response at high-frequency is linked to the momentum distribution $n_k$ at large-momentum. For $p$-waves in the asymptotic regime $k_F \ll k \ll r_0^{-1}$, $n_k$ has contributions with $k^{-2}$ and $k^{-4}$ [89, 92, 94, 155]

$$n_k \rightarrow \sum_{\nu} |Y_{1\nu}(\hat{k})|^2 \left[ \frac{16\pi^2 C_{v,\nu}}{V k^2} + \frac{32\pi^2 C_{R,\nu}}{V k^4} - \frac{8\pi^2 C_{cm,\nu}}{3V k^4} \right] + \pi C_{cm,\nu}$$ (4.9)

where $V$ is the volume of the system. The momentum distribution has normalization $\sum_k n(k) = N$ and $C_{cm,\nu}$ is a term arising from the centre-of-mass of the pairs. It is important to note that $C_{cm,\nu}$ is not a “contact” in the sense that it has no related adiabatic theorem, but it will contribute to the measured momentum distribution.

The fraction of closed-channel molecules $f_{c,\nu}$ is also related to the contact [93, 94]. Close to the Feshbach resonance, where $v_{\nu} \gg v_{bg,\nu}$, $f_{c,\nu}$ is proportional to $C_{v,\nu}$

$$f_{c,\nu} = \ell_{c,\nu}^{-1} C_{v,\nu}/2N$$ (4.10)

where $\ell_{c,\nu} = m\delta \mu \nu^{bg}_F \Delta_{\nu}/\hbar^2$. In this aspect $C_{v,\nu}$ is similar to the $s$-wave contact [54, 61]. In contrast, $C_{R,\nu}$ is an energy-weighted quantity that, in the two-channel model, also involves atom-dimer interactions.
With these universal relations in mind the data became more tractable. The theoretical framework also presented an opportunity to show that the universality expected from previous studies of the s-wave contact \[17\] applied to the p-wave contacts despite their increased structural complexity.

### 4.3 Momentum Distribution

While the momentum distribution was not the first signal I measured and investigated, it quickly became obvious that the contacts would be observable through a measurement of the large-momentum tail. In addition, the momentum distribution could give direct access to the anisotropy of the interaction.

The greatest signal to noise in this experiment is obtained when imaging the cloud along the z spatial direction, which is also the direction of the magnetic field used to access the Feshbach resonance. The momentum distribution is obtained from the column density measured by imaging the cloud after sufficiently long time-of-flight (TOF) expansion once released from the trapping potential. While this two dimensional image can be used to determine properties of the gas (i.e., the temperature or high momentum tail), an inverse Abel inversion is generally used to determine the three-dimensional momentum distribution from the two-dimensional projection. While the inverse Abel transform is generally well-suited to this task it does require that the three-dimensional momentum distribution possess cylindrical symmetry. For the anisotropic p-wave interactions probed, combined with the imaging direction that maximizes the signal-to-noise (and the asymmetric trap), the inverse Abel transform cannot be used to reconstruct the momentum distribution. Therefore, the contacts are measured from the column density.

A typical TOF experiment measures the column-integrated atom density $\rho_\perp(x, y, t_{\text{TOF}}) = \int \rho(x, y, z, t_{\text{TOF}}) dz$. For large TOF the initial cloud size is rendered unimportant and $\rho(\mathbf{r}) \propto n(\mathbf{k})$. When normalizing $\rho$ by its pixel-sum one obtains a normalized momentum distribution $\tilde{n}(k)$ with $\int \tilde{n}(k) d^3 k = 1$. With this normalization the form of the momentum distribution at large-momentum is

$$\tilde{n}_k \rightarrow \sum_\nu |Y_\nu(\hat{k})|^2 \left[ \frac{16\pi^2 C_{v,\nu}}{NVk^2} + \frac{32\pi^2 C_{R,\nu}}{NVk^4} - \frac{8\pi^2 C_{cm,\nu}}{3NVk^4} \right] + \frac{\pi C_{cm,\nu}}{NVk^4}. \quad (4.11)$$

The column density measured experimentally, $\tilde{n}(k_\perp) = \int n(k_\perp, k_z) dk_z$ where $k_\perp^2 = k_x^2 + k_y^2$, is the integral of the density along the $z$ spatial direction. I perform the column integration using Eq. (4.11) assuming that the $\nu = x$ and $\nu = y$ resonances are degenerate and separate from the $\nu = z$ resonance and $2C_{xy} = C_x + C_y$ for each contact. I calculate the momentum distribution at the $\nu = z$ and $\nu = xy$ resonance separately. At the $\nu = z$ resonance

$$\tilde{n}_z(k) \rightarrow \frac{3}{4\pi} \frac{C_{v,z}}{Nk_\perp^2} + \frac{3}{8\pi} \frac{C_{R,z}}{Nk_\perp^4} + \frac{1}{32\pi} \frac{C_{cm,z}}{Nk_\perp^4}, \quad (4.12)$$

while at the $\nu = xy$ resonance

$$\tilde{n}_{xy}(k) \rightarrow \frac{3}{4\pi} \frac{C_{v,xy}}{Nk_\perp^2} + \frac{9}{8\pi} \frac{C_{R,xy}}{Nk_\perp^4} + \frac{1}{32\pi} \frac{C_{cm,xy}}{Nk_\perp^4}. \quad (4.13)$$

Notice that in this measurement the anisotropy of the interaction has been integrated out.

The work published in \[95\] neglected to include the centre-of-mass terms in the momentum distribution. As the centre-of-mass terms appear at the same order in $k_\perp$ as the $C_R$ term this resulted in an overestimation of the magnitude of $C_R$. For a low temperature Fermi gas one can estimate the relative
magnitude of the centre-of-mass terms. Following [92, 94] I define the contacts

\[
C_{v,\nu} = \sum_{P,j} n_{P,j,\nu} \int dk \int dk' b_{jk} b_{jk'}
\]

\[
C_{R,\nu} = \frac{1}{2} \sum_{P,j} n_{P,j,\nu} \int dk \int dk' b_{jk} b_{jk'} (k^2 + k'^2)
\]

\[
C_{cm,\nu} = \sum_{P,j} n_{P,j,\nu} \int dk \int dk' b_{jk} b_{jk'} P^2
\]

(4.14)

where \( P \) is the centre-of-mass momentum of pairs, the \( b_{jk} \) are the real expansion coefficients of the radial wave function and the integration is over all scattering states (see Appendix B).

For two particles each of momentum \( k_1 \) and \( k_2 \) with centre-of-mass momentum \( k_1 + k_2 \) and relative momentum \( k = (k_1 - k_2)/2 \) one can write, assuming \( \langle k_1 k_2 \rangle = 0 \),

\[
\langle P^2 \rangle = 2 \langle k_1^2 \rangle
\]

\[
\langle (k^2 + k'^2) \rangle = \langle k_1^2 \rangle.
\]

(4.15)

Using these identities and Eqs. (4.14) allows one to write \( C_{cm,\nu} = 4C_{R,\nu} \).

Utilizing the relation between \( C_R \) and \( C_{cm} \) allows one to estimate the magnitude of the correction from the centre-of-mass terms. I rewrite the momentum distribution at each resonance as

\[
\tilde{n}_z(k) \rightarrow \frac{3}{4\pi N k_\perp} C_{v,z} + \frac{3}{8\pi N k_\perp^3} \left[ 1 + \frac{1}{3} \right]
\]

\[
\tilde{n}_{xy}(k) \rightarrow \frac{3}{4\pi N k_\perp} C_{v,xy} + \frac{9}{8\pi N k_\perp^3} \left[ 1 + \frac{1}{9} \right]
\]

(4.16)

where the terms in brackets show the relative correction to \( C_R \): 1/3 at the \( \nu = z \) resonance and 1/9 at the \( \nu = xy \) resonance. The result presented in [95] therefore overestimated the value of \( C_{R,z} \) by 25% and the value of \( C_{R,xy} \) by 10%. I use the form of the momentum distribution including the centre-of-mass corrections in the remainder of this chapter.

Figure 4.5 shows the time sequence used to measure the momentum distribution. Initially, the gas is spin-polarized in a non-interacting state (labelled \( |\downarrow\rangle \)). The magnetic field is ramped to the desired Feshbach resonance and a 40 \( \mu \)s rf \( \pi \) pulse coherently transfers all atoms into the interacting state \( |\uparrow\rangle \) at time \( t = 0 \). After a hold time \( t \) the magnetic field is jumped away from the Feshbach resonance, preserving the interacting momentum distribution which determines the ballistic flight after release from the trap. As the atoms are released from the trap they are transferred back into the Feshbach resonance, preserving the interacting momentum distribution which determines the ballistic flight after release from the trap. The distribution \( \rho_\perp \) is normalized by its pixel sum to obtain \( \tilde{n}(k_\perp) \).

To improve the signal to noise 30 to 40 images of the momentum distribution are averaged. To do so the centre of the two-dimensional distribution is identified by performing a two-dimensional Gaussian fit and then these centres are overlaid to construct an average image. An additional step of azimuthal averaging further improves the signal to noise. Using the centre that was previously identified, the radial distance to each pixel is determined \( \kappa = \sqrt{k_x^2 + k_y^2/k_F} \) and the average of all pixels at the same radius
Figure 4.5: Interactions are initiated by a resonant $40 \mu s \pi$ pulse that transfers all atoms into the interacting state $|\uparrow\rangle$. After a hold time $t$ the magnetic field is jumped in $20 \mu s$ to $209$ G where all states are noninteracting. The trapping potential is simultaneously turned off and the atoms are allowed to expand ballistically in TOF for $5.5$ ms. Before imaging the atoms are transferred to the $|\downarrow\rangle$ state.
Figure 4.6: The rescaled momentum distribution $\tilde{n}(\kappa)$ measured at $\delta B_{xy} = 0.1 \text{ G}$ and $t = 160 \mu s$. There is increased weight at large momentum, that is not present for a non-interacting gas (red line in inset). The dashed line shows the best-fit asymptote $C_v k_F/N = 0.028(6)$ and $C_R = 0$.

is calculated. The averaged momentum distribution is fit with

$$
\tilde{n}_z(\kappa) \rightarrow \frac{3}{4\pi} \frac{C_{v,z} k_F}{N} \kappa^{-1} + \frac{1}{2\pi} \frac{C_{R,z}}{N k_F} \kappa^{-3}
$$

$$
\tilde{n}_{xy}(\kappa) \rightarrow \frac{3}{4\pi} \frac{C_{v,xy} k_F}{N} \kappa^{-1} + \frac{5}{4\pi} \frac{C_{R,xy}}{N k_F} \kappa^{-3}
$$

(4.17)

to determine the contacts $C_v$ and $C_R$ at the respective resonance.

A normalized and averaged momentum distribution is shown in Fig. 4.6 at $\delta B_{xy} = 0.1 \text{ G}$ taken after a $160 \mu s$ hold time. This magnetic field detuning corresponds to a pure point where $C_R$ is zero and so the large-$\kappa$ behaviour can be revealed as a plateau by plotting $\kappa \tilde{n}(\kappa)$. The plateau is shown in the inset to Fig. 4.6. Notably, the signal at large-$\kappa$ is comparable to the noise. The small signal is caused by a competition of effects: where the contact signal is largest, the loss rates are greatest. The competition between signal and loss restricts the range of $\delta B$ over which the contacts can be observed in the momentum distribution. Equations (4.17) are used to fit data such as shown in Fig. 4.6 to measure $C_v$ and $C_R$. The fit is sensitive to the momentum range chosen to fit. The form in Eqs. (4.17) is valid for $\kappa \gg 1$ and so a high-momentum cutoff, $\kappa \geq \kappa_c$, must be chosen before fitting. For each momentum distribution the cutoff is chosen empirically by carrying out the fit for a range of values of the cutoff. The values of $C_v$ and $C_R$ typically vary for different values of $\kappa_c$ but a region near $\kappa \sim 1.2$ shows a range of values over which the contacts do not depend strongly on the cutoff. The behaviour of the contacts as a function of the cutoff is shown in Fig. 4.7. The mean value of $\kappa_c$ in the region where the contacts do not change is chosen as the momentum cutoff for the data set. As the atom number changes for each value of $\delta B$, the determination of the high-momentum cutoff is repeated for each data set.

Qualitatively, the power law behaviour is visible in a log-log plot of the momentum distribution. Figure 4.8a shows $\ln \tilde{n}(\kappa)$ versus $\ln (\kappa)$ at a pure point where the momentum distribution is dominated by $C_v$. The expected slope at large detuning is $-1$, shown in the dashed line at large $\kappa$. This can be compared to Fig. 4.8b which shows the same quantities measured at a pure point where $C_R$ dominates the momentum distribution. In this case the expected slope is $-3$ shown by the dashed line. The differences between Fig. 4.8a and Fig. 4.8b are subtle. From linear fits to the respective log-log plots
Figure 4.7: The contacts $C_v$ (circles) and $C_R$ (squares) determined from fits of the momentum distribution for various values of the large-momentum cutoff $\kappa_c$. For a range of values near $\kappa_c \sim 1.2$ the contacts are insensitive to the choice of cutoff.

The expected power laws are recovered. However, given the magnitude of the systematic and statistical uncertainties an alternative probe of the contacts is needed to verify the measurements.

There is an additional systematic error associated with the rate at which the magnetic field is jumped away from the Feshbach resonance. Previous work at a broad $s$-wave Feshbach resonance showed that the value of the contact asymptotically approaches its true value as the jump rate is increased [17]. The decrease in the value of the contact can be thought of as a time average over all the intermediate momentum distributions during the jump. The longer the time spent at weaker interactions the larger the contribution from these distributions to the measured momentum distribution. While technical limitations made it difficult to directly measure the dependence of the contact on the jump rate this effect seems to be minimal as independent measures of the contact agree well (see discussion below). The effect of the jump rate is likely minimized due to the small width of the $p$-wave Feshbach resonance with respect to the $s$-wave Feshbach resonance that was previously studied.

4.4 Spectroscopy

In order to corroborate measurements of the $p$-wave contacts via the momentum distribution and to verify the recently derived universal relations, rf spectroscopy is used as another probe of the high-energy behaviour of the system. Starting from Eq. (4.8) I derive the normalized transfer rate measured from the large-detuning tail of a spectrum. As with the momentum distribution, the normalization is fixed to be $\int I_x(\omega)d\omega = N$. The transfer rate to a non-interacting probe state $\Gamma = N_p(\omega)/t_{\text{rf}}$ is measured by applying a rf pulse with Rabi frequency $\Omega_R$ for duration $t_{\text{rf}}$. The power of the rf pulse is chosen such that the response of the system is in the linear regime. The transfer rate obeys the sum rule $\int \Gamma(\omega)d\omega = \Omega_R^2\pi N/2$. I introduce the normalized detuning $\bar{\omega} \equiv \hbar \omega/E_F$ and the normalized transfer rate

$$\bar{\Gamma}(\bar{\omega}) = \frac{E_F}{\hbar \Omega_R^2 \pi N} \Gamma(\bar{\omega}E_F/\hbar)d\bar{\omega}$$ (4.18)
Figure 4.8: A demonstration of the power law in the high-$\kappa$ tail of the momentum distribution. (a) A log-log plot of the high-momentum tail for a momentum distribution taken at $\delta B_{xy} = 0.1$ G. A fit to the high-momentum tail determines a power law $-0.4 \pm 1.2$ (shaded region) which agrees with the expected slope of $-1$ (dashed line). (b) A log-log plot of the high-momentum tail for a momentum distribution taken at $\delta B_{xy} = 0.3$ G. The best fit power law $-2.2 \pm 1.5$ (shaded region) agrees with the expected slope of $-3$ (dashed line).

which obeys the normalization $\int \Gamma(\tilde{\omega})d\tilde{\omega} = 1/2$. These definitions allow one to write the high-frequency tail of the spectrum as

$$\Gamma(\tilde{\omega}) \to \frac{1}{2^{3/2}\pi} \frac{C_{v,\nu}k_F}{N} \tilde{\omega}^{-1/2} + \frac{3}{2^{3/2}\pi} \frac{C_{R,\nu}k_F}{N} \tilde{\omega}^{-3/2}.$$  \hspace{1cm} (4.19)

Notice that the centre-of-mass terms do not enter into the expression for the high-frequency tail of the spectrum.

To measure the transfer rate to a probe state all atoms are initially spin-polarized in state $|\downarrow\rangle$. At time $t = 0$ all atoms are transferred into state $|\uparrow\rangle$ using a 40 $\mu$s rf $\pi$ pulse. After a hold time $t$ a fraction of the atoms are transferred to a non-interacting probe state $|p\rangle$ using an 80–120 $\mu$s spectroscopic pulse whose amplitude is modulated with a Blackman envelope. The Rabi frequency of this pulse is chosen to be $\Omega_R = 2\pi \times 30(5)$ kHz when probing the spectrum with $\tilde{\omega} \gg 1$. The Rabi frequency is decreased for $\tilde{\omega} \leq 1$ to remain in the regime of linear response. A diagram of the timing is shown in the inset of Fig. 4.9.

Figure 4.9 shows a sample spectrum taken at $\delta B_{xy} = 0.1$ G after a 160 $\mu$s hold time. The spectrum is reminiscent of data presented in Chapter 3, however, the high-frequency tail looks qualitatively different. This magnetic field corresponds to a pure point where $C_v$ dominates the spectrum. The scaling with detuning is found to be $\tilde{\omega}^{-1/2}$, as opposed to $\tilde{\omega}^{-3/2}$ familiar from the contact relations for $s$-wave interactions. The power law behaviour can be determined quantitatively at magnetic fields corresponding to pure points where either only $C_v$ or $C_R$ contribute to the spectrum. Figure 4.10b shows a log-log plot of a spectrum taken at $\delta B_z = 0.05$ G where only $C_v$ contributes to the high-frequency tail. Fitting the high-frequency tail of this spectrum with a function $f(\tilde{\omega}) = A\tilde{\omega}^\eta$, finds an exponent $\eta = -0.43 \pm 0.06$. This power law is plotted as the shaded region in Fig. 4.10b compared to the theoretical prediction $\eta = -0.5$ shown by the dashed line. The power law can be compared to the power law determined by fitting $f(\tilde{\omega})$ to the high-frequency tail of a spectrum measured at $\delta B_{xy} = 0.3$ G. Figure 4.10d shows a log-log plot of the high-frequency tail with the fitted power law $\eta = -1.35 \pm 0.21$ shown as the shaded region compared to the theoretical expectation $\eta = -1.5$ shown by the dashed line. The scatter in this data is much smaller than that presented for the momentum distribution providing a more convincing
determination of the power law expected from the $p$-wave contact relations. A qualitative change in the spectra from being $C_v$ dominated to being $C_R$ dominated can be seen when comparing the shape of the high-frequency tails in Fig. 4.10a and Fig. 4.10c. The weight at high frequency in Fig. 4.10c goes to zero much quicker as $\tilde{\omega}$ is increased than in Fig. 4.10a.

In the intermediate regime of $\delta B$ where both $C_v$ and $C_R$ contribute to the spectrum the values of $C_v$ and $C_R$ determined from the fits using Eq. (4.19) are sensitive to the value chosen for the high-frequency cutoff, $\tilde{\omega} \geq \tilde{\omega}_c$, above which the fit takes place (recall, Eq. (4.19) is only valid for $\tilde{\omega} \gg 1$). Throughout this chapter the value for the cutoff is chosen to be $\tilde{\omega}_c = 3.5$. Figure 4.11 explores the impact of the cutoff on the values of $C_v$ and $C_R$ for a range of magnetic fields $\delta B$ and shows that the error bar for the values determined by the fits is similar in magnitude to the scatter from varying the high-frequency cutoff. The error bar (especially in the determination of $C_R$) could be minimized by increasing the data sampling at high frequency.

Significant work was invested in determining the power law in the regime where the spectrum is dominated by $C_v$ and the regime where the spectrum is dominated by $C_R$. However, the $C_R$ dominated behaviour was initially unexpected by theory [93] and the origin of this term was poorly understood. One hypothesis for the $\tilde{\omega}^{-3/2}$ scaling was residual $s$-wave interactions between $|\downarrow\rangle$ and $|\uparrow\rangle$ atoms as a result of an imperfect initialization pulse. The strength of $s$-wave interactions between these two states is controlled by the nearby Feshbach resonance at 202.1 G discussed at length in Chapter 3. The results presented in Chapter 3 show that the $s$-wave contact at magnetic fields near the $p$-wave Feshbach resonance is small. Further, the $\pi$ pulse initializing the interactions transfers roughly 98% of the atoms into the $|\uparrow\rangle$ state (leaving only 2% of the atoms available to interact via $s$-wave collisions) which further reduces any signal from the $s$-wave contact. It is unlikely that the $\tilde{\omega}^{-3/2}$ behaviour arises from residual $s$-wave interactions. An alternative hypothesis that could change the high-frequency scaling in the spectrum is motivated by the literature for $s$-wave interactions; interactions between atoms in the final states could impact the shape of the spectrum (see Chapter 3). The $s$-wave interactions between atoms in the $|\uparrow\rangle$ and $|p\rangle$ states are tuned by nearby Feshbach resonances [118] but at the relevant magnetic fields near the $p$-wave Feshbach resonance the scattering length is approximately given by the background scattering length $\sim 160a_0$. While the magnitude of the final state interactions cannot be determined
Figure 4.10: (a) At $\delta B_z = 0.05$ G the spectrum is dominated by $C_v$. (b) A log-log plot of the high frequency tail shows the expected slope of $-0.5$ (dashed line). A fit to the high-frequency tail is shown in the shaded region finding a best-fit exponent $-0.43 \pm 0.06$. (c) At $\delta B_{xy} = 0.3$ G the spectrum is dominated by $C_R$. (d) When plotted on a log-log plot the fitted power law with slope $-1.35 \pm 0.21$ (shaded region) is consistent with the prediction of $-1.5$ (dashed line).
Figure 4.11: The $p$-wave contacts $C_v$ and $C_R$ determined for various values of the high-frequency cutoff $\tilde{\omega}_c$. The values plotted here use $\tilde{\omega}_c = 3.0, 3.5, 4.0$. 
Figure 4.12: The normalized rf transfer rate $\tilde{\Gamma}$ versus frequency $\tilde{\omega}$ at $\delta B_z = 0.1$ G and $t = 260 \mu s$ taken using $| \downarrow \rangle$ as the probe state. A pronounced tail is present at $\tilde{\omega} < -1$ that can be fit to determine $C_v$ and $C_R$.

exactly without a full calculation it is expected that the contribution is only a few percent as seen when analyzing s-wave contact data [156]. Performing spectroscopy using a different final state, $| \downarrow \rangle$ instead of $| p \rangle$, can be used to check this hypothesis. A spectrum using $| \downarrow \rangle$ as the final state is shown in Fig. 4.12 taken at $\delta B_z = 0.1$ G and at $t = 260 \mu s$. Qualitatively the spectrum looks identical to that presented in Fig. 4.9, although the high-frequency behaviour now appears for $\tilde{\omega} \ll -1$. Fitting the data shown in Fig. 4.12 with Eq. (4.19) finds $C_v k_F / N = 0.19(2)$ which is a small deviation from the fits to data obtained using the state $| p \rangle$ as the final state. The evidence collected here, combined with theory, provides strong evidence that the $\tilde{\omega}^{-3/2}$ scaling is due to $p$-wave interactions.

4.5 Comparing Spectra and Momentum Distributions

The measurements of $\tilde{n}(\kappa)$ and $\tilde{\Gamma}(\tilde{\omega})$ described above are repeated for various values of the magnetic field $\delta B$ near both the $\nu = z$ and $\nu = xy$ resonances. For each magnetic field value, a high-frequency or high-momentum cutoff is chosen as outlined in the previous sections.

A fitting routine was used to determine the best fit values for $C_v$ and $C_R$. The fits are restricted to $C_v \geq 0$ and $C_R \geq 0$. Initially, the fitting routine used was the nonlinear fitting routine built in to Mathematica where $C_v$ and $C_R$ were both forced to be greater than zero. For both $\tilde{n}(\kappa)$ and $\tilde{\Gamma}(\tilde{\omega})$ these fits would often fail or return error bars associated with the fitted parameters that were unconstrained. As such, I adopted a routine that constructed maps of the apparent reduced chi-squared as a function of $C_v$ and $C_R$. I call this an apparent chi-squared as the restriction to non-negative values for both parameters means they cannot be chosen from a true normal distribution. I use the minimum in the apparent chi-squared as an estimator for the values of best fit. The error bars are determined by the value at which the apparent chi-squared doubles from its minimum value. I choose to double the effective reduced chi-squared instead of finding where it increases by one as the minimum value is sometimes (especially when both $C_v$ and $C_R$ contribute to the spectrum) greater than one. A sample contour showing the $1 \sigma$ error bars from a fit using $\tilde{\Gamma}$ is shown at $\delta B_{xy} = 0.2$ G in Fig. 4.13a while a contour from a fit using $\tilde{n}$ at the same field is shown in Fig. 4.13b. The contours are generally the same shape. The value of $C_v$ is
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Figure 4.13: Effective chi-squared maps from (a) spectroscopy and (b) the momentum distribution at \( \delta B_{xy} = 0.2 \) G where both \( C_v \) and \( C_R \) contribute to the measurement. I plot the 1\( \sigma \) contours and the best fit values of the contacts. In both cases \( C_R \) is less well determined than \( C_v \).

Figure 4.14: Overlaying the reduced chi-squared maps from both the momentum distribution and the spectroscopy. The overlapping region provides a more precise determination of the contacts.

always more tightly constrained than the value of \( C_R \). When comparing Fig. 4.13a and Fig. 4.13b it is also clear that the values of the contacts are much more precisely determined when fitting \( \tilde{\Gamma} \) as opposed to \( \tilde{n} \).

The contours from the two independent measurements can be overlaid to improve the statistical determination of the contacts shown in Fig. 4.14. This method of analysis can be used to fully utilize the universal relations in order to minimize the uncertainty in the final value measured for \( C_v \) and \( C_R \). However, it does not independently verify that the two measurements are compatible with each other, which is discussed at length below.

Using this fitting routine I attempt to check the assumption that \( C_v > 0 \) and \( C_R > 0 \). \( C_v \) is expected to be positive definite [94]. A calculation based on a virial expansion at \( T = 2T_F \) [94] shows that at this temperature \( C_R \) is also positive for all \( \delta B > 0 \) but analysis allowing for the possibility that \( C_R < 0 \) is included below. A Quantum Monte Carlo calculation at zero temperature including the subleading analogous \( C_R \) term for \( s \)-wave interactions found a negative value of this parameter but only for \( \delta B < 0 \) [88]. In general, the fits are sensitive to noise, atom loss at high energy, and small offsets due to imperfect background subtraction. Constraining the values of \( C_R > 0 \) is motivated in part by the observation that
Figure 4.15: A comparison of the contact values measured via momentum distribution and spectroscopy with (a) $C_v > 0$ and $C_R > 0$ compared to (b) $C_v > 0$ and $C_R$ unconstrained. A linear correlation is found between the measurements. Fitting the data gives the shaded regions which are compared to the ideal prediction of unity (dashed line). Circles show values of $C_v$ while squares show values of $C_R$. Red points are taken at the $\nu = z$ resonance while blue points are taken at the $\nu = xy$ resonance.

a small positive offset can change the value of $C_R$ determined in the fit. I demonstrate this by simulating high-frequency tails with $C_v \sim 0.03$, $C_R = 0$ and a small positive offset. I then add Gaussian noise to the data on the order of the scatter observed and use the fitting routine outlined above to determine the best fit values of $C_v$ and $C_R$. I find that statistically, the fitting routine is unable to distinguish between $C_R = 0$ with a small positive offset and $C_R < 0$. In the spectroscopic data, a background is subtracted that is on the order of 50 atoms which is a large enough offset to make $C_R$ apparently negative as determined by this analysis.

Figure 4.15a shows a comparison of the values of $C_v$ and $C_R$ measured from the momentum distribution and spectroscopy where both contacts are forced to be positive while Fig. 4.15b shows the same comparison with $C_v > 0$ and $C_R$ unconstrained. The correlation between the observables in Fig. 4.15a is $1.00(9)$ shown by the shaded region and compared to a slope of unity shown by the dashed line. Similarly, in Fig. 4.15b the correlation is $0.88(9)$. The agreement between measures of $C_v$ and $C_R$ along with the asymptotic scaling of Eqs. (4.19) and (4.17) is strong evidence that the $p$-wave contact relations are valid.

4.6 Magnetic Field Dependence of the $p$-Wave Contacts

The dependence of $C_v$ and $C_R$ on magnetic field at both the $\nu = z$ and $\nu = xy$ resonances is shown in Fig. 4.16. In this plot both $C_v$ and $C_R$ are restricted to be greater than zero. The data in Fig. 4.16 is obtained from $\tilde{\Gamma}$ at $t = 160 \mu s$, from $\tilde{n}$ at $t = 160 \mu s$, and from the asymptotic values from $\tilde{\Gamma}$ versus $t$ (discussed in more detail below). The variable time data also identifies a loss-dominated regime for $0.00 \text{ G} \leq \delta B \leq 0.04 \text{ G}$, outside of which the contacts reach a steady-state value despite loss of up to 20%, indicated by the grey region in the plot. There is a pronounced asymmetry about each resonance: significant contacts are only observed for $\delta B > 0$. $C_v$ is largest near resonance and decreases with $\delta B$
Figure 4.16: $C_v k_F/N$ and $C_R/Nk_F$ versus magnetic field $\delta B_\nu$ (lower axes) or dimer energy $E_{d\nu}/E_F$ (upper axes). Data are from both the $\nu = xy$ (blue) and $\nu = z$ (red) resonances, from rf spectra (filled) and momentum distributions (open). Most data is taken at $t = 160 \mu s$, however long-time asymptotes from dynamical data are also shown as smaller filled points. Solid lines show the result of a toy model assuming equilibration between fermions and dimers in a harmonic trap.

until it vanishes at $\delta B \approx 0.3$ G. $C_R$ instead peaks at $\delta B \approx 0.3$ G before abruptly falling to zero for larger fields. Figure 4.17 shows the same dependence of $C_v$ and $C_R$ on magnetic field but allowing for negative values of $C_R$. The same qualitative behaviour in $C_v$ is seen. The behaviour of $C_R$ is similar but when $\delta B < 0.2$ G, $C_R$ takes a small negative value.

The qualitative features of the data shown in Fig. 4.16 can be explained by a simple model in which $N_d = \sum \rho_{d,\nu}$, non-interacting closed-channel dimers are in equilibrium with $N_f$ free fermions. The connection between the number of closed-channel dimers and the contacts is detailed in Appendix B. Each dimer has $C_{v,\nu} = 2R_\nu$ and $C_{R,\nu} = -2R_\nu^2/\nu$, but free fermions make no contribution to the contacts. The $\nu = z$ and $\nu = xy$ resonances are well separated which allows one to write $C_v k_F/N \approx 2k_F R_\nu(N_d/N)$ and $C_R/(Nk_F) \approx 2k_F R_\nu(E_{d\nu}/E_F)(N_d/N)$. Assuming equilibrium between the free fermions and dimers in a harmonic trap at zero temperature gives $N_d = N(1 - (E_d/2E_F)^3)/2$ [157] which allows the contacts
Figure 4.17: $C_v k_F / N$ and $C_R / N k_F$ versus magnetic field with $C_R$ unconstrained in the fitting routine. Data is otherwise the same as in Fig. 4.16.
to be estimated as
\[
\frac{C_\nu k_F}{N} \approx k_F R_\nu \left[ 1 - \left( \frac{E_d}{2E_F} \right)^3 \right]
\]
\[
\frac{C_R}{N k_F} \approx k_F R_\nu \frac{E_d}{E_F} \left[ 1 - \left( \frac{E_d}{2E_F} \right)^3 \right]
\]
(4.20)

shown in the solid lines of Fig. 4.16. The toy model suggests that \(C_\nu\) and \(C_R\) are the same near the \(\nu = xy\) and \(\nu = z\) resonances. Further, it suggests that \(C_\nu\) and \(C_R\) both go to zero as \(E_d\) approaches \(2E_F\). When the gas is fully dimerized, i.e., \(N_d \to N/2\), this model predicts \(C_\nu k_F/N \approx 0.04\) as \(k_F R_\nu \approx 0.04\) in typical experimental conditions. At resonance, the model predicts \(C_R = 0\) and a peak value \(C_R/(N k_F) \approx 0.03\) at \(E_d/E_F \approx 1.6\).

The simple toy model seems to explain the peak value of \(C_\nu\) as well as the general behaviour of the contacts as a function of \(\delta B\). However, it does not explain the location of the peak value of \(C_R\) or the apparent factor of two difference between \(C_{R,z}\) and \(C_{R,xy}\). To capture the behaviour of \(C_R\) a more realistic model including finite temperature and interactions between dimers, between atoms, and/or between atoms and dimers is needed. For instance, resonant enhancement of atom-dimer interactions have been seen in a three-body calculation [158, 159].

### 4.7 Adiabatic Relation

Independent of any particular model, but assuming adiabaticity, the thermodynamic implications of the observed contacts can be inferred using Eq. (4.5). The adiabatic theorem Eq. (4.5) can be rewritten in terms of the the experimentally varied \(\delta B\) as
\[
\frac{\partial F}{\partial B} = \frac{\partial F}{\partial \nu^{-1}} \frac{\partial \nu^{-1}}{\partial B} \approx \frac{-\hbar^2 C_{\nu,\nu}}{2m b_\nu^{bg} \Delta_\nu}.
\]
(4.21)

Similarly, the adiabatic theorem for \(C_R\) can be rewritten
\[
\frac{\partial F}{\partial B} \approx \frac{-\hbar^2 C_{R,\nu}}{2m R_\nu^{bg} \Delta_{R,\nu}}.
\]
(4.22)

In Fig. 4.16 the same peak values of \(C_\nu\) and \(C_R\) are observed, however, when comparing the coefficients in Eq. (4.21) and Eq. (4.22) the contribution to the free energy from \(C_R\) is negligible as \(k_F^2 v_\nu^{bg} / R_\nu^{bg}\) (the ratio of the coefficients) is roughly \(10^{-2}\) for typical experimental parameters. As such the contribution from \(C_\nu\) is the focus of the discussion that follows.

To determine the change in free energy between two magnetic field values \(B_1\) and \(B_2\) one must evaluate \(F(B_2) - F(B_1) = \int_{B_1}^{B_2} (\partial F/\partial B) dB\). In terms of the measured values of \(C_\nu\) this amounts to
calculating the numerical integrals

\[
F(B) \approx F(B_{\text{min}}) + \int_{B_{\text{min}}}^{B} \frac{C_{v,\nu}(B')}{2m_{\nu}b_{\nu}} dB' , B < B_0
\]

(4.23)

\[
F(B) \approx F(B_{\text{max}}) - \int_{B}^{B_{\text{max}}} \frac{C_{v,\nu}(B')}{2m_{\nu}b_{\nu}} dB' , B > B_0
\]

(4.24)

where \(B_{\text{min}}\) is the smallest magnetic field value at which there is data with \(B < B_0\), while \(B_{\text{max}}\) is the largest magnetic field value at which there is data with \(B > B_0\). In other words, for \(B < B_0\) the integration is carried out from the left while for \(B > B_0\) the integration is carried out from the right.

The numerical integration is carried out using the trapezoidal rule,

\[
\int_{a}^{b} f(x) dx \approx \frac{(b-a)}{2} (f(a) + f(b)).
\]

The resulting free energy estimated from the \(C_v\) data is shown in Fig. 4.18. At each resonance, the data from \(\tilde{n}\) and \(\tilde{\Gamma}\) are averaged together before doing the numerical integration, while the asymptotic values from \(\tilde{\Gamma}\) versus \(t\) are excluded. The error bars are determined statistically by repeating the numerical integration with different values of \(C_v\) chosen from a normal distribution determined by the error bars for the measured \(C_v\). After repeating the numerical integration 10,000 times to generate a distribution of values of \(F(B)\) the standard deviation of this distribution is assigned to be the error reported in Fig. 4.18. In addition to statistical error, a number of systematic errors contribute to the uncertainty in the inferred free energy. First, some of the other variables that determine \(F\) are varied as \(\delta B_{\nu}\) is varied. For instance, \(N\) decreases owing to loss and \(T\) increases by \(\sim 0.05 E_F/k_B\) near resonance. A second source of error may lie in the calibration of atom number. Finally, equilibrium is likely to be only local and not trap-wide. However, the toy model introduced previously still captures the essential behaviour observed (which is unsurprising as it seemed to capture the behaviour of \(C_v\)) shown in the solid line of Fig. 4.18. The integrated data is also sufficient to demonstrate several qualitative regimes.

In regime (i), below resonance with \(E_d < 0\), the gas is weakly repulsive with \(0 \leq \Delta F \ll E_F\). Here, resonant scattering is inaccessible to free particles and the gas remains in the upper-branch \([160, 161]\). Few or no dimers are formed because energy conserving two-body collisions cannot produce a dimer with a finite binding energy. Instead, the gas has weakly repulsive \(p\)-wave interactions.

In regime (ii), at resonance, a value for \(F\) is not extracted because a steady-state in \(C_v k_F/N\) is not achieved as discussed in the next section. However, the discontinuity in \(F\) between regime (i) and regime (iii) implies that the system shifts from the upper-branch to the lower-branch in region (ii).

In regime (iii), above resonance in the range \(0.25 \lesssim E_d/E_F \lesssim 2\), a reduction in \(F\) per particle approaching half the Fermi energy near resonance is inferred. The significant reduction of free energy could be partially explained by the formation of dimers, whose binding energy could contribute up to \(\Delta F = -3E_F/4\) in a harmonic trap \([157]\). As seen in Chapter 3, the ground state energy has additional contributions which include dimer-dimer and atom-dimer interactions. Accompanying the large \(\Delta F\) in this regime are the largest observed contacts, and therefore the strongest \(p\)-wave correlations as described by Eq. (4.7).

In regime (iv), farther above resonance with \(E_d > 2E_F\), the scattering resonance at \(E_d\) exceeds the maximum collision energy \(2E_F\) in a zero temperature Fermi sea, leaving primarily non-resonant interactions between atoms. In this regime \(p\)-wave interactions are weakly attractive.
Figure 4.18: Numerical integration of $C_v$ gives the shift of free energy $\Delta F = F - F_{bg}$ due to near-resonant interactions. Data is referenced to $F_{bg} = F(B_{\text{max}})$ for $\delta B_\nu > 0$ and to $F_{bg} = F(B_{\text{min}})$ for $\delta B_\nu < 0$. Blue points are from the $\nu = xy$ resonance while red points are from the $\nu = z$ resonance. Illustrations depict the dimer energy compared to the range of collision energies available in the Fermi sea. The solid line shows the result of a toy model assuming equilibration between fermions and dimers in a harmonic trap.
Chapter 4. The $p$-Wave Contacts

4.8 Dynamics Near Resonance

So far in this chapter I have assumed that the values of the contacts measured at $t = 160 \mu s$ represent a steady-state measurement. The validity of such a hypothesis was supported by measurements near an $s$-wave Feshbach resonance which had demonstrated that local equilibration appeared to occur on a timescale set by the Fermi energy, approximately tens of microseconds [116, 162]. I was somewhat astonished to observe that the dynamics of the $p$-wave contact could be resolved when the gas was probed spectroscopically near resonance.

Evolution of the many-body wave function is required to adjust to the $p$-wave interactions initiated by transferring atoms from $|\downarrow\rangle$ to $|\uparrow\rangle$. The dynamics of the contact is observed in the range $0 \leq \delta B_{\nu} \leq 0.1$ G by varying the time $t$ at which $\tilde{\Gamma}$ is measured. This range of magnetic field values corresponds to pure points where only $C_v$ contributes to the spectrum (see Fig. 4.16) and so the dynamics are measured using a single frequency $\tilde{\omega} \sim 5$ to determine $C_v(k_F)/N$. The atom number $N$ at each time is also measured. At a single frequency, the contact is given by

$$\frac{C_v k_F}{N} = \sqrt{\frac{2}{\Omega_R^2 \hbar}} \frac{\hbar}{k_F} \sqrt{\tilde{\omega} N_p(\tilde{\omega})}.$$  

Figure 4.19 shows $N(t)$ and $C_v(t)k_F/N$ measured at various values of $\delta B_\nu$. The value of $C_v k_F/N$ rises to an apparent steady-state value whereas $N$ decays relatively slowly. The initial growth rate of $C_v k_F/N$ increases with $\delta B_\nu$ until it can no longer be resolved at $\delta B_\nu \gtrsim 0.1$ G.

Near resonance, the wave function of each Feshbach dimer is closed-channel dominated [147, 163] (see Appendix B), allowing one to interpret the contact dynamics through a multichannel model and Eq. (4.10). Three closed-channels of $p$-wave dimers are initially empty but come to equilibrium with the initially populated open-channel of atoms, but all contribute to the observed atom number $N = N_f + 2 \sum_{\nu} N_{d,\nu}$. The time evolution of the number of fermions $N_f$ and the number of dimers $N_{d,\nu}$ is calculated with rate equations (see Appendix B) that omit coherence between the channels but include
Figure 4.20: (a) A typical $\chi^2_{\text{red}}$ map at $\delta B_z = 0.08$ G showing two local minima corresponding to the association-dominated (i) and the loss-dominated solutions (ii). The dashed line shows the condition $\tau_z L_{fd} = 1$. (b) The best fit values for the association rate at both resonances and for both loss models. The discarded local solutions for the $\nu = z$ resonance (small points), which clearly violate the predicted proportionality to $\gamma_\nu$, are also included. Solid lines show a fit to the association rates while dashed lines are the theoretical prediction $A_\nu \to 8 \gamma_\nu$. Blue circles are data taken at the $\nu = xy$ resonance while red squares are taken at the $\nu = z$ resonance.

$$\dot{N}_f = -\sum_\nu (2A_\nu N_f - 2\gamma_\nu N_{d,\nu} + L_{fd,\nu} N_f N_{d,\nu})$$

$$\dot{N}_{d,\nu} = A_\nu N_f - \gamma_\nu N_{d,\nu} - L_{fd,\nu} N_f N_{d,\nu} - \sum_{\nu'} L_{dd,\nu,\nu'} N_{d,\nu} N_{d,\nu'}$$

(4.26)

where $A_\nu$ is the dimer association rate and $L_{fd,\nu}$ is the fermion-dimer loss coefficient and $L_{dd,\nu}$ is the dimer-dimer loss coefficient. Pure fermion-dimer loss ($L_{dd,\nu} = 0$) or pure dimer-dimer loss ($L_{fd,\nu} = 0$) give compatible results but including both loss coefficients leads to poorly constrained fits. Further, the model is constrained to have $N_{d,x} = N_{d,y} = 0$ near the $\nu = z$ resonance and $N_{d,z} = 0$ near the $\nu = xy$ resonance. The degenerate modes at the $\nu = xy$ resonance are combined to $N_{d,xy} = N_{d,x} + N_{d,y}$, $A_{xy} = A_x + A_y$, $\gamma_{xy} = \gamma_{x,y}$, $L_{fd,xy} = L_{fd,x,y}$, and $L_{dd,xy} = L_{dd,x,y}$. Therefore the same set of two coupled equations can be used for the $\nu = z$ and the $\nu = xy$ resonances.

The data in Fig. 4.19 is fit using the solutions of Eqs. (4.26) with pure dimer-dimer (dashed line) and pure fermion-dimer (solid line) loss. Previously, the growth rate of dimers $N_{d,\nu}$ was observed for $\delta B_\nu > 0$ [106, 148]. In each iteration of the fit a resonance position is assumed, which is used to calculate $\gamma_\nu$. This leaves four fit parameters: the prefactor $\alpha = C_v / N_d$, the phenomenological association rate $A_\nu$, the respective loss rate $L$ and the initial atom number $N_0 = N(t = 0)$ which needs to be included due to an unknown calibration factor in the state-selective imaging scheme. For a fixed grid of pairs $(A_\nu, L)$, the reduced chi-squared value $\chi^2_{\text{red}}(A_\nu, L)$ is calculated for the optimal values of $\alpha$ and $N_0$.

A typical $\chi^2_{\text{red}}(A_\nu, L)$ distribution is shown in Fig. 4.20a for $\delta B_z = 0.08$ G. There are typically two local minima with similar values of $\chi^2_{\text{red}}$ for each set of dynamical data. One of the minima, where $L \gg A$,
is identified as being loss-dominated. The other minimum is associated with the system reaching a steady-state with $L \ll A$. Figure 4.20b shows the association rates determined with either pure fermion-dimer loss or pure dimer-dimer loss. In both loss models, the association rates are compatible. The small points correspond to the “unphysical” second minimum from the $\chi^2_{red}$ distribution which is determined by using the result of a Fermi’s golden rule calculation that predicts $A_\nu \propto \gamma_\nu$. The remaining best-fit values for $A_\nu$ outside of the loss dominated regime are then extrapolated to $A_\nu = 0$ to determine a new resonance position $B_{0,\nu}$. This new field value is assumed for the next iteration until the fit routine has converged.

With $L_{fd,\nu} = L_{dd,\nu} = 0$, Eq. (4.26) would lead to a dimer population that asymptotically tends towards

$$N_{dx,y} = \frac{A_{xy}}{\gamma_{xy}}N_f$$

and

$$N_{dz} = \frac{A_z}{\gamma_z}N_f.$$  

The associated equilibration rates are

$$\frac{1}{\tau_{xy,z}} = \gamma_{xy,z} + 2A_{xy,z}. \quad (4.27)$$

As noted above, with loss present the fitting routine can differentiate between regime (ii) and regime (iii) identified in Fig. 4.18. If $\tau_\nu L_\nu \ll 1$, the system reaches a quasi-steady-state from which it decays slowly. This scenario corresponds to regime (iii) in Fig. 4.18. If however, $\tau_\nu L_\nu \gg 1$, the loss rate of the system is faster than the association rate and equilibrium between $N_d$ and $N_f$ is inhibited. As a consequence $N_d$ reaches a significantly reduced steady-state value on a timescale that is dominated by the loss rate. In turn, atom loss occurs on the timescale of dimer association. This case corresponds to regime (ii) in Fig. 4.18.

From the analysis of the dynamical data an asymptotic value of $C_\nu k_F/N$ can be determined. These asymptotic values are included in Fig. 4.16 as the small points. They are within the uncertainty of $C_\nu$ at $t = 160 \mu s$ for $\delta B \geq 0.1$ G but provide an upward correction for $\delta B \sim 0.05$ G. With the inclusion of this correction, $C_\nu$ decreases monotonically with $\delta B$ throughout regime (iii).

Figure 4.21 shows the measured association rates $A_\nu$ assuming pure fermion-dimer loss where loss-dominated data is shown in the open points. Using only data in which $C_\nu$ reaches a quasi-steady-state $A_\nu$ is fit assuming proportionality to $\gamma_\nu$. The best-fit ratios $A_{xy}/2\gamma_{xy} = 8.1(1.8)$ and $A_z/\gamma_z = 9.9(2.5)$ are shown by the solid lines in Fig. 4.21. These ratios are consistent with a perturbative treatment of resonant closed-channel molecular formation in a $T = 0$ Fermi cloud (see Appendix B) that predicts $A_\nu \rightarrow 8\gamma_\nu$ as $\nu^{-1}_\nu \rightarrow 0^{-}$ shown by the dashed lines. The consistency between the dynamical response of $C_\nu$ and a model of dimer population supports the validity of Eq. (4.10).

### 4.9 Discussion

The evidence presented in this chapter are first steps towards verifying the universality of the $p$-wave contact relations. The verification of the asymptotic scaling in $\tilde{n}$ and $\tilde{\Gamma}$, the congruence between measurements of the contacts using various observables and the consistency between the dynamical response of $C_\nu$ and a model of dimer population all provide evidence that the $p$-wave contact relations are valid. Although the experiments in this chapter were restricted to a Fermi gas in a metallic state, the $p$-wave contact relations are expected to hold for any type of particle, whether boson or fermion, in any dimensionality, and in any state (superfluid or normal), so long as the interactions are short-ranged and $p$-wave.

To fully verify the universality of the $p$-wave contact relations future studies should directly measure the energy, structure factor, and dimer number as has been done for the $s$-wave contact [17, 60, 66].
Figure 4.21: The dimer association rates $A_{xy}$ (blue circles) and $A_z$ (red squares) versus magnetic field. The open squares indicate a loss-dominated solution. Solid lines are fits to $A_{xy}$ and $A_z$ with assumed proportionality to $\gamma$, finding $A_{xy}/\gamma_{xy} = 16.3(3.7)$ and $A_z/\gamma_z = 9.9(2.5)$. Dotted lines represent the theoretical prediction of $16\gamma_{xy}$ (blue) and $8\gamma_z$ (red).

In this work, a regime where the atom-dimer equilibration is much faster than loss is identified. After reaching a quasi-steady-state, strong $p$-wave correlations persist for at least half a millisecond, eventually limited by dipolar and three-body loss rates. This opens a new dynamical window within which to study systems with strong $p$-wave interactions. The advent of such a dynamical method has recently invigorated the study of unitary Bose gases which suffer from similar (albeit faster) three-body loss rates [71–75, 164]. I am optimistic that the dynamical method exploited here will continue to be used to explore the many-body physics of strongly interacting Fermi gases with $p$-wave interactions. In particular, searching for the onset of pair condensation in this dynamical window at low temperature would be of great interest.

Another problem to address would be to study three-body physics at a $p$-wave Feshbach resonance. While a three-body contact [72, 75] may be observed in a three-dimensional Fermi gas with $p$-wave interactions, a three-body effect analogous to the Efimov effect [165] is expected to exist when the gas is confined to two dimensions. This so-called “super Efimov effect” [166–169] has attracted considerable theoretical attention and could be observed in spectroscopy [170]. Whatever the direction, I hope that this work will bring new life to the study of Fermi gases with $p$-wave interactions.
Chapter 5

Spin Transport in Strongly Interacting Fermi Gases with $s$-Wave Interactions

Experiments with ultracold atoms have begun to explore non-equilibrium dynamics to probe transport in the relaxation of samples or in their collective behaviour [23]. While the dynamics of condensed matter systems must be probed at giga or terahertz frequencies, cold atoms possess much more convenient timescales associated with preparation and measurement of the system (typically kilohertz). Thus, measurements of transport in cold atom systems can follow coherent quantum dynamics over long timescales.

For instance, the conductance has been measured across a narrow channel connecting two reservoirs of fermions [22]. These experiments make strides towards performing successful quantum simulations of mesoscopic conductors [171–173]. Other experiments probe more “exotic” transport properties like shear viscosity by measuring collective modes or observing the expansion of the gas when released from a trapping potential [26, 41, 44, 174–176].

Many experiments employ probes of cold atoms that focus on coupling electromagnetic radiation to the internal state of the atom to measure non-equilibrium properties of the system. This method exploits the relatively slow rates of relaxation of cold atom systems when compared to the timescale of the initialization and probes. Using simple rf manipulation of the spin degree of freedom, the transport of spin in ultracold atoms has been extensively studied [27, 28, 45, 107–109, 115, 177, 178].

Combining the favourable timescales for initialization and probes with the tunability of the interactions between atoms, these examples highlight the fact that ultracold atoms present a novel opportunity to probe transport in strongly correlated systems where a theoretical understanding is incomplete. Despite the difficult theoretical treatment, many of these transport parameters are expected to obey quantum bounds in a wide range of systems, from the quark-gluon plasma [179–182] to bad metals [183–186] to unitary Fermi gases [26–28, 46, 47, 49, 109, 174–176, 187–194]. In this chapter I will present measurements of the spin transport in strongly interacting two- and three-dimensional Fermi gases. The measurement technique probes the transverse demagnetization dynamics and characterizes the spin currents in the sample. These results support the conjecture that the spin diffusivity obeys a quantum bound $D \gtrsim \hbar/m$ in two and three dimensions.
5.1 Spin Currents and Magnetization

In a kinetic theory \[195, 196\], the local magnetization density \( \mathcal{M} \) obeys a continuity equation

\[
\partial_t \mathcal{M} + \mathbf{\Omega}_0 \times \mathcal{M} = -\nabla_j \mathcal{J}_j \tag{5.1}
\]

where \( \mathbf{\Omega}_0 = (0, 0, \Omega_0) \) with \( \Omega_0 \) the Larmour frequency due to an external field, and \( \mathcal{J}_j \) is the spin current density. Here the subscript \( j \in \{1, 2, 3\} \) indicates spatial direction, \( \partial_t \) is a time derivative, and bold quantities denote vectors in Bloch space. The magnetization density is a vector whose magnitude is \( |\mathcal{M}| = (n_\uparrow - n_\downarrow)/2 \) where \( n_\sigma \) is the density in the \( |\uparrow\rangle \) and \( |\downarrow\rangle \) spin states. While \( \mathcal{M} \) is conserved in the frame rotating with the external field, the spin current is in general not conserved. In the hydrodynamic limit, the local steady-state spin current obeys \[195, 196\]

\[
\mathcal{J}_j + \mathcal{J}_j \times \mu \mathcal{M} + D_0 \nabla_j \mathcal{M} = 0 \tag{5.2}
\]

where \( \nabla_j \) denotes a spatial derivative in the \( j \)th direction, \( D_0 \) is the bare spin diffusivity, and \( \mu \) is the Leggett-Rice parameter. The solution of Eq. (5.2) is

\[
\mathcal{J}_j = -\frac{D_0}{1 + \mu^2 |\mathcal{M}|^2} \left[ \nabla_j \mathcal{M} - \mu \mathcal{M} \times \nabla_j \mathcal{M} - \mu^2 (\mathcal{M} \cdot \nabla_j \mathcal{M}) \right] \tag{5.3}
\]

When \( \mu = 0 \) Eq. (5.3) reduces to Fick’s law, \( \mathcal{J}_j = -D_0 \nabla_j \mathcal{M} \).

The magnetization density can be written \( \mathcal{M} = \mathcal{M} \mathbf{m} \) with \( \mathcal{M} = |\mathcal{M}| \) and \( \mathbf{m} \) is a unit vector pointing in the direction of the magnetization. Expressing the magnetization density in this form leads to a decomposition of the spatial gradient

\[
\nabla_j \mathcal{M} = (\nabla_j \mathcal{M}) \mathbf{m} + (\nabla_j \mathbf{m}) \mathcal{M}. \tag{5.4}
\]

The first term is parallel to \( \mathcal{M} \) and the second term is perpendicular to \( \mathcal{M} \).

A gradient \( \nabla_j \mathcal{M} \) that is parallel to \( \mathcal{M} \) induces a longitudinal current. The gradient is in the magnitude of the magnetization density. Assuming a purely longitudinal current, the second term in Eq. (5.3) is zero which gives

\[
\mathcal{J}^\parallel_j = -D_0^\parallel \nabla_j \mathcal{M} \tag{5.5}
\]

where the dependence on the Leggett-Rice parameter has dropped out, and only a bare longitudinal diffusivity, \( D_0^\parallel \), remains.

A gradient that is perpendicular to \( \mathcal{M} \) generates a transverse spin current. For a purely transverse spin current the gradient is in the direction of the magnetization density. Simplifying Eq. (5.3) gives

\[
\mathcal{J}^\perp_j = -D_0^\perp \left[ \nabla_j \mathcal{M} + \mu \mathcal{M} \times \nabla_j \mathcal{M} \right] \tag{5.6}
\]

where \( D_0^\perp = D_0^\perp/(1 + \mu^2 |\mathcal{M}|^2) \) where \( D_0^\perp \) is the bare transverse spin diffusivity.

At low temperatures the diffusivity can be different between the longitudinal and transverse modes \[197, 198\]. The difference can be understood conceptually as illustrated in Fig. 5.1. At temperatures well below the degeneracy temperature the phase space for collisions responsible for spin diffusion is different between longitudinal and transverse modes. For longitudinal spin diffusion the Fermi sphere
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Figure 5.1: (a) An illustration of longitudinal spin transport. The gradient in magnetization is in the magnitude of the magnetization. Scattering occurs in a thin region near the Fermi surface. (b) An illustration of transverse spin transport where the gradient in magnetization is in the direction of the magnetization. Scattering can occur throughout the Fermi surface.

at position $x$ and position $x + dx$ is shown in Fig. 5.1a. Consider an up spin moving from position $x$ to position $x + dx$. If that spin is in the thin annular region that represents the difference between the two Fermi spheres it is out of equilibrium when it reaches $x + dx$ and must scatter to become equilibrated. Up spins deep in the Fermi sphere may not be able to scatter as their momentum states at $x + dx$ are already occupied. Scattering is therefore restricted to the Fermi surface and the spin diffusion coefficient will have the characteristic $T^{-2}$ scaling. The phase space for longitudinal spin diffusion is restricted at low temperatures, scattering decreases, and the spin diffusivity increases.

Conversely, for transverse spin diffusion the Fermi spheres at $x$ and $x + dx$ are the same size as illustrated in Fig. 5.1b. An up spin that moves from $x$ to $x + dx$ in any momentum state between the up and down Fermi sphere is out of equilibrium and must scatter to become equilibrated. Therefore, the scattering does not follow the $T^{-2}$ scaling and instead saturates to a constant value as $T \to 0$. As a result the transverse spin diffusivity can be much lower than the longitudinal spin diffusivity at low temperatures. The so-called “anisotropy temperature” below which these two modes are different has been measured in liquid helium experiments [199] and is predicted to be $T \sim 0.14T_F$ in a three-dimensional unitary Fermi gas [49]. The phase space argument also highlights the magnetization dependence of the transverse spin diffusivity. A larger magnetization means there is more phase space available for scattering and therefore spin diffusivity will be suppressed.
5.1.1 Transverse Spin Diffusion

In this chapter the magnetization is measured through the density-weighted trap average of dimensionless polarization $M = 2\mathcal{M}/n$ with $|M| \leq 1$. Alongside $M$ is a polarization current $J_j = 2\mathcal{J}_j/n$. Using these definitions and treating only the transverse component of Eq. (5.3) gives

$$J_{\perp j} = -D_{\text{eff}}^\perp \nabla_j M - D_{\text{eff}}^\perp \gamma M \times \nabla_j M$$

(5.7)

where a homogeneous density is assumed and $\gamma \equiv \mu n/2$ such that $\gamma M = \mu \mathcal{M}$. The continuity equation can now be written in terms of the polarization as

$$\partial_t M + \Omega_0 \times M = -\nabla_j J_{\perp j}$$

(5.8)

where repeated indices are summed.

The dynamics of the transverse magnetization $M_{xy} = M_x + iM_y$ in a linear magnetic field gradient $B'$ are described by

$$\partial_t M_{xy} = -i\alpha x_3 M_{xy} + D_{\text{eff}}^\perp (1 + i\gamma M_z) \nabla_x^2 M_{xy}$$

(5.9)

where $\alpha = B'\Delta\mu/\hbar$, and $\Delta\mu$ is the differential magnetic moment between spin states. Here, $M_z$ is conserved globally and even locally in a uniform system.

Following [196] I introduce the ansatz solution

$$M_{xy}(t) = A(t) \exp [i\phi(t) - i\xi(t)]$$

(5.10)

where $A = |M_{xy}|$ describes the amplitude of the transverse magnetization and $\phi$ characterizes the phase lag that results from the precession of the spin current around the local magnetization known as the Leggett-Rice effect. The parameter $\xi$ encapsulates the phase accumulated as a result of the applied magnetic field gradient and any other uniform phase increments applied experimentally. Here, $A$, $\phi$ and $\xi$ are real-valued. In a spin-echo experiment [200–202] in the presence of a linear magnetic field gradient $\xi(t) = \alpha x_3 t$. Substituting the ansatz into Eq. (5.9) gives coupled differential equations for the amplitude and the phase

$$\frac{\partial \phi}{\partial t} = -\gamma M_z D_{\text{eff}}^\perp \alpha^2 t^2$$

$$\frac{\partial A}{\partial t} = -AD_{\text{eff}}^\perp \alpha^2 t^2.$$ 

(5.11)

Integrating these equations gives

$$\phi(t) - \phi(t_0) = \gamma M_z \ln \left(\frac{A(t)}{A(t_0)}\right)$$

$$\left(1 + \gamma^2 M_z^2\right) \ln \left(\frac{A(t)}{A(t_0)}\right) = -\frac{1}{2} \gamma^2 \left(A(t_0)^2 - A(t)^2\right) = -\frac{D_{\text{eff}}^\perp \alpha^2 t^3}{12}$$

(5.12)

at the echo time $t$ for an initial amplitude $A(t_0)$ and initial phase $\phi(t_0)$.

For $\gamma = 0$ the amplitude of the transverse magnetization decays as $A'(t) = A(t_0) \exp \left(-D_{\text{eff}}^\perp \alpha^2 t^3/12\right)$ [200–202]. The solution $A'(t)$ has been used to analyze the magnetization dynamics in a number of cold atom experiments [27, 28, 45]. Moreover, $A'(t)$ is a solution to Eq. (5.8) with purely longitudinal
Figure 5.2: The (a) amplitude and (b) phase of the transverse magnetization as a function of dimensionless time \((D_0^\perp \alpha^2)^{1/3}t\) for various values of \(|\gamma| = 0, ..., 1\) (light to dark). The Leggett-Rice effect modifies the amplitude to extend the coherence time. An unambiguous signature of the Leggett-Rice effect is in the phase evolution of the transverse magnetization.

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5.2 Measuring Magnetization

The magnetization is measured using a \(\theta - \pi - \pi/2\) spin-echo sequence [196]. As introduced previously, the two spin states \(|\downarrow\rangle\) and \(|\uparrow\rangle\) represent the high-field \(m_F = -9/2, -7/2\) states of the \(F = 9/2\) hyperfine manifold of the electronic ground state. The spin-echo experiment begins with a spin-polarized gas of \(|\downarrow\rangle\) atoms. The first rf pulse has a pulse area \(\theta\) that creates a superposition of \(|\downarrow\rangle\) and \(|\uparrow\rangle\) with a transverse magnetization \(M_{xy} = i \sin (\theta)\) and \(M_z = - \cos (\theta)\). The initial pulse area is calibrated by
driving resonant Rabi oscillations between the two spin states to optimize a $\pi$ pulse which inverts the two populations. The pulse area is then decreased fractionally to obtain the desired value of $\theta$.

An applied linear magnetic field gradient $B'$ twists the initial magnetization into a spiral texture which drives irreversible spin currents. These spin currents precess around the local magnetization in the sample; a phenomenon called the Leggett-Rice effect. The spin currents also erode the magnetization. After a hold time $t_\pi$ a spin refocussing $\pi$ pulse is applied which reverses the phases of the spins. After an additional precession time $t_\pi$ the spins have rephased and the transverse magnetization is measured with a final $\pi/2$ pulse. The phase of the final pulse is varied with respect to the phase of the first two pulses so that the contrast in the population in the two spin states measures $A$ and $\phi$. By varying the total hold time $t = 2t_\pi$ the dynamics of the transverse magnetization can be measured.

The measurement protocol is shown in Fig. 5.3a. The relative fraction of atoms in each spin state as a function of the phase of the final rf pulse forms a fringe. For hold times greater than $\sim 1.5$ ms the magnetic field stability is insufficient to preserve a reproducible relative phase between the pulses resulting in a randomized phase for long hold times. To accurately measure $\phi$ the demagnetization dynamics need to occur on a timescale shorter than 1.5 ms. The desired demagnetization timescale is engineered by working with relatively large magnetic field gradients ($\sim 20$ G/cm). From each fringe the oscillation in the relative population is fit using a sinusoidal function to measure $A(t)$, $\phi(t)$, and an offset. By hand a factor accounting for the imaging efficiency of the two spin states is manipulated to keep the offset near 1/2. A time series of these fringes is shown in Fig. 5.3b.

To ensure that the amplitude $A$ is not underestimated in the fit, the fitted value is compared to two alternative estimates of the amplitude: the standard deviation of the points (normalized to the standard deviation from a sinusoid), and the visibility $(\text{max-min})/(\text{max+min})$. All three measures agree within their statistical uncertainties. The agreement between the estimates of the amplitude was further demonstrated by simulations of fringes with increasing phase noise which show that for less than 1 rad
of phase noise the three methods produce the same amplitude within their statistical uncertainties [116].

5.3 Measuring $D_0^\perp$ and $\gamma$

From a measurement of $A(t)$ and $\phi(t)$ the bare transverse spin diffusivity and the Leggett-Rice parameter can be determined using Eqs. (5.13). In general, all other parameters (i.e., $B'$, $A(t_0)$, $M_z$) can be calibrated independently. The time evolution of the amplitude and phase are shown in Fig. 5.4a. To determine $\gamma$, $\phi$ is plotted as a function of $M_z \ln |A(t)/A(t_0)|$ as in Fig. 5.4b. The slope of this data is $\gamma$. Varying the sign of $M_z$ reverses the sign of $\phi$ but retains the same slope further demonstrating the relationship in Eqs. (5.13). Figure 5.4b shows two values of $M_z$ used to extract a single value of $\gamma$.

Having determined the Leggett-Rice parameter, $D_0^\perp$ can be determined from the amplitude of the transverse magnetization dynamics. After fixing $\gamma$, the amplitude $A(t)$ is fit using the analytic solution in Eqs. (5.13) to determine $D_0^\perp$. Again, multiple values of $M_z$ can be used to find the best-fit value of the bare transverse spin diffusivity as shown in Fig. 5.4a.

A number of systematic effects impact the analysis of the time evolution of the transverse magnetization. Firstly, the initial amplitude is often different from the calibrated value determined by the initialization pulse and there can be a transient-like oscillation in the amplitude at short times. Combined, these effects contribute a large uncertainty when determining the value of $M_z$ from the initial measured magnetization which in turn impacts the fitted value of $\gamma$ as $\gamma$ and $M_z$ appear as a product in Eq. (5.13). The hypothesis that the deviations in the initial amplitude were caused by a magnetic field transient was thoroughly investigated and the behaviour was minimized after finding that a small detuning of each rf pulse could produce the transient behaviour observed. A slight deviation from the expected amplitude still occurred for short hold times which has been hypothesized to result from the finite duration of the rf pulses (see Appendix C). A numerical simulation of Eq. (5.9) using finite difference methods which included finite pulses showed similar transient-like behaviour in the short-time amplitude. Simulations of spins on a lattice in the presence of a magnetic field gradient with finite
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pulses also showed that the effect of the pulse duration is most pronounced for short hold times. The investigation of deviations in the initial magnetization occurred over the course of my entire graduate work and as such the three-dimensional data uses a fitting algorithm to reconstruct $M_z$ from the initial amplitude while the two-dimensional data uses $M_z$ as determined from the calibration of the rf pulses. The difference of method does not greatly impact the measured transport parameters.

Another systematic effect in the analysis arises from the increased phase noise for long hold times and small $|M_{xy}|$. To avoid this regime I apply a low-amplitude cut-off, where any phase data with a corresponding value of $A < 0.1$ is excluded from the linear fit to determine $\gamma$. I still include this data in the fit of $A(t)$ as the fractional scatter is smaller.

The last systematic effect investigated was the simultaneous fitting of $\gamma$ and $D_{\perp}^0$ using $A(t)$ data. In a typical dataset, the reduced chi-squared is on the order of 50 when simultaneously fitting both parameters. In addition, the values are substantially correlated and the value of $\gamma$ is poorly constrained. Fits of the amplitude often fail to converge. Therefore, the fitting procedure where $\gamma$ is determined first is used when the phase data is available.

During demagnetization the initially non-interacting spin-polarized gas relaxes to become a strongly interacting spin-mixture. The dynamical transformation has a number of consequences. The relaxation of the Fermi surfaces results in an increase in the temperature [28]. For an equal superposition and a non-interacting gas the temperature rise can be calculated as shown in Fig. 5.5 for a three-dimensional (green) and two-dimensional (blue) gas. For an interacting gas, the release of interaction energy also contributes to the temperature rise and can be calculated given knowledge of the finite temperature equation of state. A calculation of the total temperature rise for a unitary three-dimensional Fermi gas was carried out in [116]. The polarization of the sample also changes throughout the dynamics. A magnetization dependence of the transport parameters could influence the measurement as the dynamics occur over a range of polarizations. Finally, the rms size of the cloud changes during demagnetization. The change in size of the cloud can be understood from the generalized virial theorem for a three-dimensional [18] or two-dimensional [82] Fermi gas. The change in rms size of the cloud in turn causes a decrease in the density of the gas. It is important to note that for a three-dimensional unitary Fermi gas the rms size of the cloud remains unchanged [203]. The magnitude of many of these effects has been estimated and are expected to be reduced as the initial transverse magnetization is decreased [204].

Given the considerations listed above, all of the measurements in this chapter must be understood as an average over an inhomogeneous density distribution in the trap (discussed in more detail below) as well as a dynamical average over a range of temperature and polarization.

Due to the dynamical nature of the experiment a convention for the temperature and interaction strength must be chosen. The temperature reported is the initial temperature of the spin-polarized gas. In two and three dimensions the initial polarized $k_F = k_{Fi}$ is used. However, the logarithmic dependence of the interaction strength on $k_F$ implies that the distinction between the initial and final $k_F$ results in a simple translation by $-\ln(2)/2$ in two dimensions. Conversely, in three dimensions the transformation is to scale the interaction strength by a factor of $2^{-1/3}$.

5.4 Results

My study of spin transport in degenerate Fermi gases has extended over many years. In 2013, the lab first measured the transverse spin diffusivity at unitarity in a three-dimensional gas [28]. In 2013/2014 the
methods used were then improved to measure the Leggett-Rice effect and the transverse spin diffusivity in a three-dimensional gas [109]. Lastly, in 2016 the experiment was extended to an ensemble of two-dimensional Fermi gases [115]. I summarize these results and present a comparison of the data between two and three dimensions.

5.4.1 Spin Transport in Three Dimensions

Figure 5.6 shows measurements of the Leggett-Rice parameter as a function of temperature at unitarity and as a function of scattering length at low temperature. Figure 5.6a shows the Leggett-Rice parameter as a function of reduced temperature at unitarity. At high temperatures $\gamma$ tends to zero in contrast to the temperature insensitivity of spin-wave behaviour in a weakly interacting Fermi gas [107, 108]. The behaviour of $\gamma$ as a function of temperature can be understood from the scattering amplitude in vacuum. At high temperatures the scattering amplitude is purely imaginary at unitarity. At low temperature, however, the degenerate Fermi sea restores a non-zero real part of the scattering amplitude and hence a non-zero value for $\gamma$.

The solid points in Fig. 5.6 and Fig. 5.7 are measured using the analysis method described above. The open points in Fig 5.6a and Fig 5.7a are obtained from a reanalysis of the data presented in [28]. For each of these data sets only $A(t)$ was measured. The amplitude is fit using Eq. (5.13) to measure both $\gamma$ and $D_{\perp}$ simultaneously. As discussed above, using this method to determine the spin transport parameters is not sensitive to the sign of $\gamma$ and is more sensitive to slight nonlinearities in the amplitude.

Figure 5.6b shows measurements of the Leggett-Rice parameter as a function of interaction strength parametrized by $-(k_Fa)^{-1}$. There are two implications of this data. First, the Leggett-Rice effect is non-zero and will therefore cause precession of the transverse spin currents to slow diffusive demagnetization. Further, the value of $\gamma$ appears to saturate for weaker interactions to a value of $\pm 1$. This will be discussed further below. Second, $\gamma$ changes sign in the range $-1.3 < -(k_Fa)^{-1} < 0$. The sign change of $\gamma$ indicates that the effective interaction between fermions changes sign as one tunes the system across the Feshbach resonance.
upper-branch of the energy spectrum [205–207], with repulsive interactions, to the lower-branch, with attractive interactions.

The sign change of $\gamma$ has been discussed in the context of an upper-branch instability [120, 124] in which atoms decay to form bound pairs in the lower-branch [122, 125, 162, 208, 209]. The onset of the instability is predicted to occur near $-(k_F a)^{-1} \sim -1$ [109, 120] which is in the range $\gamma$ is observed to change sign. The pairing instability is discussed further in Chapter 3.

Having measured the Leggett-Rice effect, the bare transverse spin diffusivity is measured at unitarity as a function of temperature and across the BCS-BEC crossover at low temperature. Figure 5.7a shows the temperature dependence of $D_\perp^0$ at unitarity. As temperature decreases, the transverse spin diffusivity decreases and reaches an apparent plateau at low temperature of $D_\perp^0 = 2.3(4)h/m$. The saturation of $D_\perp^0$ can be understood from dimensional arguments. The diffusivity is given by $D \sim \ell_{\text{mfp}} v$ where $\ell_{\text{mfp}}$ is the mean free path and $v$ is the characteristic velocity of particles promoting transport. The mean free path is inversely proportional to the scattering cross-section multiplied by the density. At unitarity $\ell_{\text{mfp}}$ is proportional to $1/k_F$. A typical velocity for a degenerate Fermi gas is the Fermi velocity $v_F = \hbar k_F/m$. Combining these two relations gives $D \sim h/m$. A lower-bound on the spin diffusivity has been discussed extensively in the context of cold atoms [46, 47, 49, 190–194] and has been shown to be respected for three-dimensional unitary Fermi gases [27, 28, 109].

Figure 5.7b shows the dependence of $D_\perp^0$ on interaction strength at low temperature. A minimum is observed near $-(k_F a)^{-1} = 0$ consistent with the understanding that spin diffusivity is smallest for maximal scattering. As with $\gamma$, the value of $D_\perp^0$ for large $|k_F a|^{-1}$ appears to saturate.

The data in Fig. 5.6 and Fig. 5.7 are compared to a kinetic theory [49]. The calculation was performed by T. Enss and the methodology is summarized here. The calculation used the Boltzmann equation to find the non-equilibrium time evolution of the spin distribution function in response to the applied magnetic field gradient. Collisions between fermions of unlike spin use the many-body T-matrix computed in the medium of surrounding fermions [49]. The calculation is done in the limit of large imbalance, which corresponds to the initial conditions in the experiment and supresses superfluidity in the calculation.
Figure 5.7: The bare transverse spin diffusivity in a three-dimensional gas (a) at unitarity as a function of reduced temperature and (b) at \((T/T_F)_1 = 0.18(4)\) as a function of interaction strength. The solid lines show a kinetic theory calculation. Open points are from data taken without measuring the phase, as discussed in the text. The grey area represents the region excluded by the conjectured quantum bound, \(D_\perp^\perp \geq \hbar/m\).

From the transverse scattering time \(\tau_\perp\) [49] one can calculate

\[
D_0^\perp = \frac{\tau_\perp}{2M} \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_i v_{ki} v_{kj} (f_\vec{k} \uparrow - f_\vec{k} \downarrow) \tag{5.14}
\]

and the Leggett-Rice parameter \(\gamma\). The latter is a weighted momentum average of the many-body T matrix \(T(\vec{q},\omega)\):

\[
\gamma = -\frac{n \tau^2}{4\hbar D_0^\perp M^2} \int \frac{d^3\vec{k}_1}{(2\pi)^3} \frac{d^3\vec{k}_2}{(2\pi)^3} v_{1j} (v_{1j} - v_{2j}) (f_{1\uparrow} - f_{1\downarrow}) \times \text{Re} T(\vec{k}_1 + \vec{k}_2, \xi_{1\uparrow} + \xi_{2\downarrow}) \tag{5.15}
\]

Here \(1, 2\) are shorthand for \(\vec{k}_1, \vec{k}_2, f_{i\sigma} \equiv \exp(\beta \xi_{i\sigma})+1\) is the Fermi distribution for \(\xi_{i\sigma} = (\hbar \vec{k})^2/2m - \mu_\sigma\), and \(v_{k_j}\) is the \(j\)th Cartesian component of the velocity. The numerical results for the homogeneous system are related to the measurements in the trapping potential using the local-density approximation.

The solid lines in Figs. 5.6 and 5.7 are calculated within this kinetic theory using the local reduced temperature at peak density. The homogenous calculation captures the trends in the data well at unitarity. The agreement is likely due to the dynamics being hydrodynamic and essentially local. In other words, the mean free path is less than the spin spiral pitch. In Figs 5.6b and 5.7b the homogeneous calculation fails to capture the behaviour of the data for increasing \(|k_Fa|^{-1}\). The discrepancy between data and theory heralds the breakdown of the hydrodynamic model. The mean free path at peak density changes from \(\ell_{\text{mfp}} \approx 300 \text{nm at } (k_F a)^{-1} = 0\) to \(3 \mu\text{m at } |k_F a|^{-1} = 2\), approaching both the pitch of the spin spiral and the Thomas-Fermi radius of the cloud. Thus, the analysis using the homogeneous solution of Eq. (5.9) is expected to be most accurate in the strongly interacting regime.

More recently, a complete analysis of the transverse spin transport dynamics has been undertaken in an inhomogeneous trapping potential [210]. Figure 5.8 shows a comparison of the data with theory for the homogeneous and inhomogeneous solutions of Eq. (5.9). Including effects of inhomogeneity improves the agreement between theory and experiment at larger \(|k_F a|^{-1}\).
Figure 5.8: A comparison of the solutions of the kinetic theory calculation carried out for a homogeneous and inhomogeneous trapping potential. The full solution for the trapped gas shows similar saturation effects to the data. This image is reproduced from [210].

5.4.2 Spin Transport in Two Dimensions

Measurements of $D_0^\perp$ and $\gamma$ are repeated in an ensemble of two-dimensional Fermi gases. The two-dimensional potential is engineered using a one-dimensional optical lattice which is populated with $\sim 200$ individual two-dimensional gases. The confinement strength is roughly 10 times the Fermi energy implying that the experiment is well within the two-dimensional regime. For more discussion on the population of the ensemble of two-dimensional gases see Appendix A.

Figure 5.9 shows measurements of the Leggett-Rice parameter at low temperature as a function of interaction strength and as a function of temperature at $\ln (k_F a_{2D}) \approx 0$. As in Fig. 5.6, this data shows qualitatively similar features. First, Fig. 5.9a shows the temperature dependence of $\gamma$ at $\ln (k_F a_{2D}) = -0.1(2)$. As discussed previously, the temperature behaviour can be understood from the perspective of the scattering amplitude which is purely imaginary when $k_F = a_{2D}$ at high temperatures. At low temperature the degenerate Fermi sea restores a non-zero real part of the scattering amplitude and therefore a non-zero $\gamma$.

Figure 5.9b shows measurements of $\gamma$ as a function of $\ln (k_F a_{2D})$ at $T/T_F = 0.21(3)$ (open squares) and $(T/T_F)_i = 0.31(2)$ (filled circles). The measurement is interpreted as revealing the effective interaction between the spin current and the local magnetization [195, 196, 211, 212]. Repulsive interactions are observed when $\gamma < 0$ for $\ln (k_F a_{2D}) \lesssim -1.5$, whereas when $\gamma > 0$ as observed for $\ln (k_F a_{2D}) \gtrsim -1.5$ interactions are attractive. The sign change of $\gamma$ is associated with a pairing instability in two dimensions [136] and discussed in more detail in Chapter 3.

Figure 5.10 shows measurements of the bare spin diffusivity $D_0^\perp$ at low temperature as a function of interaction strength and as a function of temperature at $\ln (k_F a_{2D}) \approx 0$. Figure 5.10a shows the
temperature dependence of $D^\perp_0$ at $\ln (k_F a_{2D}) = -0.1(2)$. The spin diffusivity seems to decrease monotonically with decreasing temperature. However, a clear plateau at low temperatures (as suggested in the three-dimensional data) is not observed. Figure 5.10b shows $D^\perp_0$ at $(T/T_F)_i = 0.21(3)$ (open squares) and $(T/T_F)_i = 0.31(2)$ (closed circles) versus $\ln (k_F a_{2D})$. The data is compared to a kinetic theory calculation with (solid) and without (dashed) medium scattering. The grey area represents the region excluded by the conjectured quantum bound, $D^\perp_0 \gtrsim \hbar/m$.

Effects related to inhomogeneities are investigated by varying the pitch of the spin spiral created relative to the size of the cloud. Varying the magnetic field gradient applied allows for the tuning of the spin spiral pitch. However, due to the $\sim 1.5$ ms maximum time during which the phase measurement is reproducible the measurement procedure must be modified. For a three-dimensional unitary Fermi
Figure 5.11: The timescale of demagnetization \((D_0^\perp \alpha^2)^{1/3}\) plotted as a function of the magnetic field gradient applied. The shaded region shows corresponds to times excluded by the bound \(D_0^\perp \leq \hbar/m\). A single diffusivity describes the data shown by the solid line, \(D_0^\perp = 1.1(1)\hbar/m\).

gas this was done by carrying out a simple spin-echo sequence initialized with a \(\theta = \pi/2\) pulse and using an estimate of the amplitude of the transverse magnetization [28]. In this section I employ time-resolved spectroscopy of the contact (see Chapter 3) to measure the dependence of the timescale of demagnetization on the applied magnetic field gradient. The sample is initialized with a \(\theta = \pi/2\) pulse and the spin-echo rf pulse is removed as measurements of the contact do not rely on successful rephasing of the spins, which also provides a reassuring check on the fidelity of the spin echo pulse in previous measurements. The connection between the contact and the magnetization comes from the Pauli exclusion principle: pairs of fermions must have a singlet wave function to interact via a contact interaction. The singlet fraction can be no larger than \(1 - |M|\), and would be \((1 - |M|^2)/4\) for uncorrelated spins [28, 80, 137]. A direct comparison of the contact and magnetization shows a relationship that falls in between these two limits: \(1 - |M|^{1.4(2)}\) (see Chapter 3). The method used does not allow for a careful characterization of the Leggett-Rice effect and so to determine the bare spin diffusivity a value of \(\gamma = 0.71\) based off of the temperature and interaction strength) is chosen in the analysis.

Figure 5.11 shows the dependence of the measured timescale of demagnetization as a function of the applied magnetic field gradient at \(\ln (k_F a_{2D}) = -0.1(2)\). For all the data presented, the timescale scales with \(\alpha^{2/3}\) (solid line) and can be explained by a single diffusivity \(D_0^\perp = 1.1(1)\hbar/m\). The data show that the microscopic \(D_0^\perp\) is independent of \(B'\) across the accessible range of gradients, and is thus independent of the pitch of the spin spiral.

### 5.5 Discussion

The data presented show qualitatively similar behaviour in two and three dimensions. In both cases \(D_0^\perp\) is minimum where the scattering cross-section is maximum. Irrespective of dimensionality the quantum bound \(D \gtrsim \hbar/m\) is obeyed which is in stark contrast to a previous measurement of the effective transverse spin diffusivity in two dimensions roughly 100 times smaller than this bound [45]. In both two and three dimensions the Leggett-Rice is observed to slow diffusive demagnetization.

The measurements presented in this chapter provide a complete characterization of the demagnetization dynamics of strongly interacting Fermi gases in two and three dimensions. Future work can help
systematically explore the effects of inhomogeneities on the measurement of the spin transport parameters. While calculations have shown that an inhomogeneous density profile can impact the measured values of the spin transport parameters [210], the geometry of the trap in this experiment can be manipulated to directly observe the effects of the trap size and inhomogeneities. Another systematic effect that should be explored is the impact finite rf pulse length has on the short time magnetization. A direct observation of the effects of finite pulse length can be measured by utilizing short Raman transitions for spin manipulation instead of rf. By improving the measurement techniques further, a precise determination of the spin transport parameters can be made to test various theories of transport in strongly interacting Fermi systems.

Another avenue for future work is to explore the crossover from strongly interacting to weakly interacting systems [45, 107, 108, 214]. Observing spin dynamics in weakly interacting systems can easily be accomplished by tuning the magnetic field to vary the scattering length. In the weakly interacting regime phenomenon like spin waves [107, 108], spin self-rephasing [215, 216], and entanglement can be studied [214, 217].
Chapter 6

Conclusion

This thesis summarizes recent progress in exploring the non-equilibrium dynamics of degenerate Fermi gases in the presence of strong interactions. The essential ingredients for these experiments are Feshbach resonances that allow for the tuning of the two-body interaction and optical potentials that allow for varying the dimensionality. By controlling a few simple parameters, the emergent many-body physics is remarkably rich. The experiments apply a quantum quench to initiate non-equilibrium dynamics that can be measured as the system relaxes to a quasi-steady-state. Combined with an understanding of rf spectroscopy these experiments have shed light on many open problems in the field of ultracold atomic Fermi gases.

The results presented in Chapter 3 highlight contributions towards better understanding rf spectroscopy of strongly interacting Fermi gases in two and three dimensions. In three dimensions the absence of weakly bound dimers in rf spectra is used to infer a regime of metastability of the upper-branch where free scattering atoms interact via an effective repulsive interaction. Exploring the upper-branch prompted preliminary measurements that attempt to observe the “repulsive contact”. This work also involved measuring the contact for a strongly interacting two dimensional Fermi gas. The measurements suggest that a contribution from dimers was not observed and perhaps the contact was measured in an attractive upper-branch. Finally, the dynamical methods previously established in the lab were extended to present the first observation of contact dynamics for a two-dimensional Fermi gas.

There are several new questions motivated by the work presented in this thesis that should be investigated. I am curious to explore the repulsive contact and understand how it fits in to the formalism of universal relations. Ideally, the contact can serve as a proxy for measuring the energetics of a repulsive Fermi gas near a Feshbach resonance if the adiabatic theorem holds. This presents a conceptually simple measurement to determine the energy in the upper-branch which can be used to test various theoretical treatments [38, 161, 218]. Understanding the energetics of a repulsive Fermi gas near a Feshbach resonance is important in answering the question of whether Stoner ferromagnetism can be realized in an ultracold Fermi gas near a Feshbach resonance [120, 123, 125, 162, 206]. The work presented in Chapter 3 helps clarify the region over which the upper-branch is unstable with respect to decay to the lower-branch. The dynamical method used presents an opportunity to delineate between a timescale associated with two-body contact interactions and three-body inelastic decay. More precisely, when the Fermi sea is no longer able to take up the energy to form a dimer, three-body recombination is the primary dimer formation mechanism. Perhaps there is a regime of interactions where the three-body
timescale can be observed to be different from the two-body timescale associated with the two-body contact dynamics. Of general interest is to observe the timescale on which dimers form in two- or three-body processes. The formation of dimers could be observed by quenching and probing the system on a faster timescale using Raman transitions. Using these tools it would be interesting to see if this dynamical method can be used to probe the timescale on which the contact is established.

While spontaneous Stoner ferromagnetism has been largely ruled out due to the instability of the upper-branch [120, 125], theoretical proposals have suggested that preparing fixed domains could prepare a metastable ferromagnetic state [124]. In fact, a single domain wall prepared in a strongly repulsive Fermi gas has shown a dynamical domain of immiscibility which may point towards the existence of a ferromagnetic instability [27, 178, 219]. Another route to observe ferromagnetism may be by utilizing an optical lattice [206] or by increasing the temperature [218]. This experiment may be well suited to look for evidence of a ferromagnetic instability at high temperature using these dynamical methods.

The measurements of the two-dimensional contact have also left me with a number of questions. Why are dissociated dimers absent in the rf spectra? Is this because the dynamical scheme used prepares the gas in the upper-branch? Or is this related to a technical issue with the optical lattice? If this measurement was indeed probing an upper-branch contact it would be exciting to understand the implications for the energetics of the gas. Perhaps there is even a regime of interactions where ferromagnetism could be searched for.

Chapter 4 presented the experimental discovery of the $p$-wave contacts. One immediate extension of this work is to directly measure the energy of the gas to verify the adiabatic theorem as has been done for $s$-wave interactions [17]. The measurements in this chapter have also opened a new avenue of research related to the effect of the effective range on the universality of gases with strong short-ranged interactions [88, 94, 220]. In particular, more study of $C_R$ is warranted to understand its origin and physical significance to gases with $p$-wave interactions. The dynamical method used could also be used to search for the onset of superfluidity on a timescale where a quasi-steady-state has been established.

Another avenue of research is to study $p$-wave interactions in two dimensions. In two dimensions the super-Efimov effect is predicted to exist for identical fermions with $p$-wave interactions [166–168]. This three-body effect could be observed spectroscopically [170] or through measurements of atom loss. It may be interesting to search for a three-body contact in this context as well. Even if evidence for trimer formation cannot be observed, studying the $p$-wave contacts in two dimensions may be interesting in its own right. Tightly confining one of the three trapped directions could be useful to tune the anisotropy of the interaction [145, 146]. The confinement can also be used to tune the magnitude of the effective range which may be helpful in understanding $C_R$.

Importantly, this work has opened up a new avenue of research into the contact and its associated universal relations. Already theoretical work has extended the contact relations to higher order partial wave interactions [96, 97]. Hopefully, the contact will continue to be a hot topic in cold atoms research with connections to other strongly interacting systems [86].

Finally, Chapter 5 presented measurements of the bare transverse spin diffusivity in strongly interacting two- and three-dimensional Fermi gases showing that a general quantum bound $D_\perp \gtrsim \hbar/m$ is respected for all interactions and for the lowest temperatures probed. The Leggett-Rice effect is also measured, which describes a precession of the spin current about the local magnetization. The Leggett-Rice effect reveals the sign of the effective interaction near a Feshbach resonance. Further, the observations in this chapter show that dimensionality seems to only modify the dynamics a small amount
which is consistent with observations of collective dynamics in two-dimensional Fermi gases. Overall, these observations suggest that quantum bounds on transport may be respected generally.

While this work has provided a very complete characterization of the demagnetization dynamics, there remain a few open technical questions this experiment is well suited to study. First, the experiment manipulates the spin using rf pulses with finite duration. To fully investigate the effects of the duration of the rf pulses, Raman transitions should be used to manipulate the spin on a shorter timescale. Second, the experiment takes place in an inhomogeneous trapping potential and with an inhomogeneous density profile. These inhomogeneities likely impact the measured dynamics for weaker interactions [210]. The homogeneous transport parameters could be measured by applying a local probe in the measurement procedure [68]. Lastly, the finite size of the trap could impact the measurements. This effect has been partially explored by varying the linear gradient but a further exploration could be to vary the trapping frequencies or aspect ratio of the trap.

Finally, the demagnetization dynamics and contact dynamics were shown to be connected and studied here for the first time in two dimensions. It has been shown that the singlet fraction can be used as an entanglement witness [137] and could lead to studies of entanglement in strongly interacting Fermi systems.
Appendix A

Apparatus

The “chip experiment” is a versatile apparatus which can produce quantum degenerate gases of both $^{87}$Rb, a boson, and $^{40}$K, a fermion. The experiments described in this thesis are all carried out with $^{40}$K, while $^{87}$Rb is used merely to sympathetically cool the fermionic atom. This apparatus produces spin-polarized fermions that can be used for dynamical measurements. The apparatus is described in great detail in previous theses [116, 221, 222] outlining the general recipe and components used to produce a quantum degenerate gas. In this appendix I provide an overview of the experimental sequence and detail the changes I have made to the apparatus. Particular attention is paid to the construction and implementation of a one-dimensional optical lattice.
A.1 Overview

In general, cold atom experiments follow a similar recipe; prepare the sample, perform an experiment, and measure relevant observables. The chip experiment utilizes a single vacuum chamber and micro-fabricated atom chip trap to quickly produce a degenerate Fermi gas. The proximity of optically trapped atoms to the chip trap allows for fast and homogeneous manipulation of the atoms in dynamic measurements. I describe the typical experimental sequence below.

A.1.1 Experimental Sequence

The experimental sequence takes about 35 seconds. All equipment is controlled by an ADwin sequencer with a minimum time step of 10 µs.

MOT: Atoms are initially collected in a dual species magneto-optical trap (MOT). Three counter-propagating pairs of laser beams intersect at the centre of a quadrupole magnetic field generated by a current running through a pair of “MOT coils” in anti-Helmholtz configuration. An additional coil with symmetry axis perpendicular to the MOT coils (the “transfer coil”) displaces the field minimum 2.5 cm below the geometric centre of the MOT coils. The $^{87}$Rb trapping light is $\sim -26$ MHz detuned from the $|F = 2\rangle \rightarrow |F' = 3\rangle$ transition. The $^{87}$Rb repump light is tuned to the $|F = 1\rangle \rightarrow |F' = 2\rangle$ transition and combined with the trapping light before amplification. The $^{40}$K trapping light is detuned by $\sim -30$ MHz from the $|F = 9/2\rangle \rightarrow |F' = 11/2\rangle$ transition. It is amplified separately from the $^{40}$K repump light which is tuned to the $|F = 7/2\rangle \rightarrow |F' = 9/2\rangle$ transition. The $^{40}$K trapping light is combined with the $^{87}$Rb light and split into six. The $^{40}$K repump light is added to only four of the six beams and has a 1 cm diameter dark spot in the middle. The dark spot creates a volume with no repump light in which cold $^{40}$K accumulates and no longer scatters photons which minimizes excited state collisions with $^{87}$Rb.

The MOT is loaded in two stages: first $^{40}$K is briefly loaded by itself ($\sim 2$ s), then both $^{87}$Rb and $^{40}$K are loaded together ($\sim 10$ s). During the loading stages, the current in dispensers for both $^{87}$Rb and $^{40}$K is pulsed on. In only the $^{87}$Rb loading stage are LEDs at $\sim 405$ nm turned on to release atoms from the glass cell via light-induced atomic desorption. Both of these techniques temporarily increase the vapour pressure during the MOT stage of the experiment. In the final 100 ms of the MOT stage, the power in the $^{40}$K trap beams is increased and the detuning is decreased to compress the $^{40}$K MOT.

Optical Molasses: Following the MOT stage of the experiment a brief stage of optical molasses is used to cool $^{87}$Rb to reach sub-Doppler temperatures. The magnetic field and $^{40}$K trapping beams are extinguished and the $^{87}$Rb trapping and repump light is used to form optical molasses.

Optical Pumping: Next, both species are optically pumped for 200 µs into the stretched magnetic hyperfine states $|F = 2, m_F = 2\rangle$ for $^{87}$Rb and $|F = 9/2, m_F = 9/2\rangle$ for $^{40}$K. A small bias field is applied using the X-bias shim coil during this stage and the optical pumping beams are composed of $\sigma^+$ light.

Quadrupole Magnetic Trapping and Magnetic Transfer: The atoms are then loaded into a quadrupole magnetic trap formed by the MOT and transfer coils. The location of the magnetic field minimum can be controlled using the X-, Y-, and Z-bias shim coils. This position is chosen to optimize the final atom number and temperature in the chip trap. After the atoms are loaded, the quadrupole magnetic trap is compressed in 400 ms. The atoms are then transferred 5 cm vertically by varying the current in the transfer coil.

Loading the Chip Trap: When the quadrupole magnetic trap nears the atom chip, shim fields are used to finely adjust the position of the magnetic field minimum relative to the magnetic field minimum
generated by the Z-wire of the atom chip (see Fig. A.1). Subsequently, the Z-bias and X-bias fields are ramped to values such that they generate an Ioffe-Pritchard type trap in combination with the Z-wire. The quadrupole trap is then quickly turned off by ramping the currents in the MOT and transfer coils to zero. Due to the differential gravitational sag for $^{87}$Rb and $^{40}$K the bias fields can be used to adjust the relative efficiency of loading either species into the chip trap. The local magnetic field minimum generated by the Z-wire is located roughly 200 $\mu$m from the atom chip.

**Forced Rf Evaporation:** The U-wire (see Fig. A.1) is used to deliver a rf magnetic field which selectively removes high momentum $^{87}$Rb atoms from the trap. The remaining $^{87}$Rb atoms rethermalize through collisions with a lower average energy. Collisions between $^{87}$Rb and $^{40}$K atoms sympathetically cool the fermionic atom which is not evaporated directly. During the final stages of evaporation the chip trap is decompressed to minimize three-body loss processes as the density of $^{87}$Rb increases. One advantage of the chip trap is that high magnetic field gradients are easily achievable which leads to rapid rethermalization due to high collision rates leading to a relatively fast ($\sim 6$ s) evaporation stage.

**Loading the Optical Dipole Trap:** The optical dipole trap (ODT) is formed at the focus of two intersecting red-detuned laser beams. The weak “crossing beam” is slowly ramped on after the chip trap is decompressed. This aids in mode-matching between the chip trap and the ODT. The final power in the crossing beam determines the relative efficiency of loading $^{87}$Rb and $^{40}$K into the dipole trap. After evaporation in the chip trap is complete the “trapping beam” is ramped on and the chip trap is quickly turned off. After this point, the atoms are completely optically trapped.

**State Manipulation:** The remaining $^{87}$Rb atoms are transferred into their absolute ground state $|F = 1, m_F = 1\rangle$ with one step of microwave adiabatic rapid passage (ARP) using the microwave antenna (see Fig. A.1). The $^{40}$K atoms are also transferred into their ground state $|F = 9/2, m_F = -9/2\rangle$ with rf ARP. With both species in their ground state, spin-changing collisions are energetically prohibited greatly improving the lifetime of this mixture.

**Evaporative Cooling:** The final stage of cooling occurs in the ODT. The most energetic atoms are removed by weakening the optical trap until they are no longer trapped. This stage preferentially removes $^{87}$Rb atoms as they are heavier. The $^{87}$Rb atoms sympathetically cool the $^{40}$K atoms. The final temperature of the $^{40}$K atoms is determined by the final trap depth of the ODT. After cooling, residual $^{87}$Rb atoms are removed with a resonant optical pulse leaving a spin-polarized gas of fermionic $^{40}$K.

**Experiment:** At this point a variety of experiments are possible: the interaction strength can be varied using a magnetic Feshbach resonance, the dimensionality can be changed by loading the gas into an optical lattice, or the spin-composition can be varied using rf state manipulation. These experimental techniques are discussed in more detail throughout this thesis.

**Imaging:** Lastly, the results of an experiment are measured using absorption imaging after time-of-flight (TOF). In this stage, either the momentum distribution of the trapped atoms or the relative population of a spin-mixture can be measured to construct various observables of the system.

### A.1.2 The Chip

The fabrication and specifications of the atom chip are described in previous theses [221, 223]. The layout of the wires is shown in Fig. A.1 while the uses of each are described in Tab. A.1. During an experimental cycle each of the wires is used at least once.

The close proximity of the atoms to the chip allows for fast rf state manipulation and for control over the magnetic field gradient. These capabilities have allowed the experiment to investigate non-
Figure A.1: (a) Top view of the chip. The grey shading indicates one interconnected area that is grounded. The uses of the brightly coloured wires are summarized in Tab. A.1. (b) A zoomed view of the central portion of the chip. This image was originally produced in [221].

<table>
<thead>
<tr>
<th>Chip wire</th>
<th>Colour</th>
<th>Use(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z-wire</td>
<td>blue</td>
<td>magnetic trapping, Stern-Gerlach pulse</td>
</tr>
<tr>
<td>Rf-antenna</td>
<td>light green</td>
<td>$^{87}\text{Rb}$ rf evaporation, $^{40}\text{K}$ state preparation</td>
</tr>
<tr>
<td>Microwave antenna</td>
<td>dark purple</td>
<td>$^{87}\text{Rb}$ state preparation</td>
</tr>
<tr>
<td>U-wires</td>
<td>red</td>
<td>magnetic field gradient control</td>
</tr>
<tr>
<td>Bar wires</td>
<td>dark green</td>
<td>coherent state manipulation</td>
</tr>
<tr>
<td>Dimple wires</td>
<td>yellow</td>
<td>heating between cycles</td>
</tr>
</tbody>
</table>

Table A.1: A summary of chip wires.

equilibrium dynamics on short timescales. Unfortunately, the experiment has moved away from arbitrary magnetic potentials [221, 222]. Using the chip wires to perform rf tomography of the trapped gas was explored. However, the reduced size of the cloud in the ODT required too large of a magnetic field gradient for this to be viable. The chip has remained a tool to reduce the experimental cycle time and an effective rf antenna.

A.2 Optical Dipole Trap

The final stages of the experimental sequence take place in a crossed beam ODT. This trap is generated by intersecting two perpendicular red-detuned beams. The basis of the optical trap is the AC stark shift where the energy states of an atom are shifted by the presence of far-detuned light [224]. Alternatively, this result can be thought of to arise from the dispersive interaction of the induced electric dipole of the atom with the intensity of light. The potential energy of the atom is proportional to the intensity and the real part of the polarizability. In the Lorentz model of the atom-light interaction for large detuning
and negligible saturation the dipole potential is given by [224]

\[
U_{\text{dip}}(r) = -\frac{3\pi c^2}{2\omega_0^3} \left( \frac{\Gamma}{\omega_0 - \omega} + \frac{\Gamma}{\omega_0 + \omega} \right) I(r)
\]  
\[\text{(A.1)}\]

where \(c\) is the speed of light, \(\Gamma\) is the linewidth of the transition, \(\omega_0\) is the resonant frequency, \(\omega\) is the frequency of light, and \(I\) is the intensity. For a red-detuned trap, the potential energy of the atom is lowered and therefore potential wells form in the intensity maxima. In the rotating wave approximation one arrives at

\[
U_{\text{dip}}(r) = \frac{3\pi c^2}{2\omega_0^3} \frac{\Gamma}{\Delta} I(r)
\]  
\[\text{(A.2)}\]

where \(\Delta \equiv \omega - \omega_0\). This result is reproduced when considering the interaction of an atom with far-detuned light in time-independent second-order perturbation theory [224].

For a single focused Gaussian laser beam propagating along the \(x\)-direction the spatial intensity distribution is [224]

\[
I(r, x) = \frac{2P}{\pi w_0^2} \exp\left( -2 \frac{r^2}{w_0^2} \right)
\]  
\[\text{(A.3)}\]

where \(r\) denotes the radial direction, and \(P\) is the power. The \(1/e^2\) radius \(w(x)\) depends on the axial coordinate \(x\) via

\[
w(x) = w_0 \sqrt{1 + \left( \frac{x}{x_R} \right)^2}
\]  
\[\text{(A.4)}\]

where \(x_R = \pi w_0^2/\lambda\) is the Rayleigh range and \(w_0\) is the beam waist. In the harmonic approximation, the trapping frequency along the radial direction is

\[
\omega_r = \sqrt{\frac{4U_{\text{dip}}(0)}{mw_0^2}}.
\]  
\[\text{(A.5)}\]

The trapping frequency along the direction of propagation is much weaker

\[
\omega_x = \sqrt{\frac{2U_{\text{dip}}(0)}{mx_R^2}}
\]  
\[\text{(A.6)}\]

which necessitates the use of a crossing beam in this experimental setup to achieve a low aspect-ratio trap.

The trap is made from two perpendicular beams of 1064 nm light both focused at the location of the magnetic chip trap. “ODT1” propagates along the \(x\)-direction with Gaussian waist \(\sim 20\,\mu\text{m}\). ODT1 is considered the trapping beam and contributes mostly to the trap depth. “ODT2” propagates along the \(z\)-direction with a Gaussian waist \(\sim 100\,\mu\text{m}\). The weak crossing beam, ODT2, contributes mostly to the trapping frequency. Both ODT1 and ODT2 are generated from the same 10 W ytterbium fiber laser. The total power is divided between two paths for ODT1 and ODT2. They are frequency shifted 80 MHz with respect to one another after passing through AOMs. The optical path nearest to the chamber is shown in Fig. A.2.

The apparatus now includes a second ytterbium fiber laser, which is used to generate an alternate trapping beam “ODT3”. ODT3 co-propagates with ODT1 along the \(x\)-direction (see Fig. A.2). It was designed to have an adjustable waist ranging from \(\sim 30\,\mu\text{m}\) to \(\sim 100\,\mu\text{m}\) using a telescopic beam-
Figure A.2: A simplified sketch of the relevant optical paths near the cell. The lenses have focal lengths $f_1 = 400$ mm, $f_2 = 150$ mm, $f_3 = 300$ mm, $f_4 = 75$ mm, $f_5 = 100$ mm, and $f_6 = 50$ mm.
Appendix A. Apparatus

Figure A.3: The measured trapping frequencies of ODT3 versus optical power. From a fit assuming $\omega \propto \sqrt{P}$ (solid line) the waist of ODT3 was determined to be 45 $\mu$m which is much larger than ODT1 ($\sim 20$ $\mu$m).

expander. The purpose of the larger beam is to provide access to deeper traps with similar trapping frequencies to previous iterations of the experiment.

In practice ODT3 is used to replace ODT1 after evaporation in the dipole trap. To load ODT3 from ODT1, ODT3 is ramped on in 20 ms. This ramp is slow enough to avoid inducing oscillations in the cloud while fast enough to minimize the time spent with both ODT1 and ODT3 on which seems to heat the atoms. Subsequently, ODT1 is ramped off in 20 ms. Currently, an unknown heating mechanism makes using ODT3 impossible for long term experiments but the waist was determined to be 45 $\mu$m from measurements of the trapping frequency as shown in Fig. A.3.

A.3 One-Dimensional Optical Lattice

Another optical potential utilized in this experiment is an optical lattice [9, 225]. An optical lattice is generally produced by interfering two beams of light to produce a standing wave. Atoms are periodically trapped in the intensity maxima or minima of the standing wave. In its simplest realization, an optical lattice is formed by a retro-reflected beam to generate a standing wave. Other realizations of optical lattice potentials use two phase-coherent beams intersecting at a shallow angle to generate a standing wave pattern whose periodicity depends on the angle of intersection. Such “accordion” lattices have recently been employed to isolate single lattice wells [226, 227].

In this experiment the lattice beam propagates along the $x$-direction and is retro-reflected back onto itself to generate a standing wave

$$V(x) = V_0 \cos^2 (kx).$$  \hspace{1cm} (A.7)

Here $k = 2\pi/\lambda$ is the wave vector of the lattice where $\lambda \approx 760$ nm. The lattice depth is typically expressed in units of the recoil energy $E_R = \hbar^2 / (2m\lambda)^2$. 
A.3.1 Lattice Potentials

The motion of a particle in a periodic structure is governed by the Schrödinger equation

\[ H \Phi_n^q(r) = E_n^q \Phi_n^q(r) \]  \hspace{1cm} (A.8)

where \( n \) is the band index and \( q \) is the quasi-momentum of the particle. Here \( H = p^2/(2m) + V(r) \), where \( p \) is the momentum of the particle and \( V(r) \) is the potential it is travelling in. The solutions to this equation are Bloch wave functions

\[ \Phi_n^q(r) = e^{iqr/\hbar} u_n^q(r) \]  \hspace{1cm} (A.9)

where \( u_n^q(r) \) satisfy a slightly modified Schrödinger equation

\[ \tilde{H} u_n^q(r) = E_n^q u_n^q(r) \]  \hspace{1cm} (A.10)

with \( \tilde{H} = (p + q)^2/(2m) + V(r) \). \( V \) and \( u \) can be expanded in Fourier series

\[ V(x) = \sum_r V_r e^{i2rkx}, \quad u_n^q(x) = \sum_\ell c_{\ell}^{n,q} e^{i2k\ell x} \]  \hspace{1cm} (A.11)

such that Eq. (A.10) becomes

\[ \sum_\ell \left( \frac{2\hbar k \ell + q}{2m} \right)^2 c_{\ell}^{n,q} e^{i2k\ell x} + \sum_r \sum_\ell V_r e^{i2(r+\ell)kx} c_{\ell}^{n,q} = E_n^q \sum_\ell c_{\ell}^{n,q} e^{i2k\ell x}. \]  \hspace{1cm} (A.12)

Expanding the potential, Eq. (A.7), in terms of exponentials gives

\[ V(x) = \frac{V_0}{4} \left( e^{2ikx} + e^{-2ikx} + 2 \right) \]  \hspace{1cm} (A.13)

which implies that only the \( r = 0, \pm 1 \) terms in Eq. (A.12) are non-vanishing. This reduces the problem to

\[ \sum_\ell H_{\ell m} c_{\ell}^{n,q} = E_n^q c_{\ell}^{n,q} \]  \hspace{1cm} (A.14)

where

\[ H_{\ell m} = \begin{cases} \left( \frac{2\ell + q}{\hbar k} \right)^2 E_R + \frac{V_0}{2}, & \ell = m \\ \frac{V_0}{4}, & |\ell - m| = 1 \\ 0, & \text{otherwise.} \end{cases} \]  \hspace{1cm} (A.15)

To determine the band-structure one simply needs to solve this eigenvalue problem. The eigenvalues \( E_n^q \) are the energies of the \( n \)-th Bloch band at a given quasi-momentum \( q \). The quasi-momentum is restricted to the first Brillouin zone (i.e., \( q \in [-\hbar k, \hbar k] \)). A numerical diagonalization of Eq. (A.15) is used to find the energies of the \( n \) lowest Bloch bands shown in Fig. A.4. To solve this problem numerically, I truncate Eq. (A.15) at values \( |s| > n \) for which \( c_{s}^{n,q} \) is sufficiently small.

At low lattice depth, there is almost no gap at the edge of the Brillouin zone and the dispersion relation resembles that of a free particle. However, as the lattice depth is increased the Bloch bands
begin to flatten out and are increasingly separated from each other in energy. The tunnelling between neighbouring sites is approximately given by the width of the lowest Bloch band

$$J = \frac{\max(E_0^q) - \min(E_0^q)}{4}. \quad (A.16)$$

For deep lattices ($V_0 = 50E_R$) the tunnelling time is on the order of seconds and so the lattice sites can effectively be considered decoupled along the lattice direction. At lower lattice depth ($\sim 5E_R$) the tunnelling time is on the order of milliseconds and tunnelling on the timescale of the experiments in this thesis is appreciable.

### A.3.2 Experimental Implementation

The optical lattice is generated by a single retro-reflected beam at 760 nm. The optical path is shown in Fig. A.2. The optical lattice propagates along the weak trapping direction of the ODT. As the lattice light is blue-detuned from resonance the atoms are trapped at the intensity minima and the lattice beam provides a small anti-confining potential to the optical trap. The waist of the forward-going beam is approximately 70 $\mu$m while the retro-reflected beam is slightly focused ($\sim 60 \mu$m) to compensate for losses at the cell walls of the chamber.

The optical lattice is loaded while atoms are confined in the ODT in a two-stage process. First, the lattice depth is increased to $\sim 5E_R$ in 100 ms. At this lattice depth, tunnelling between adjacent lattice sites is slowed to a rate of roughly 1 kHz. The lattice depth is then ramped up to $\sim 50E_R$ in 5 ms where the majority of the two-dimensional experiments take place. The fast ramp is necessitated for
two reasons. One, the light source for the optical lattice has frequency noise at $\sim 80$ kHz and $\sim 160$ kHz. The lattice depth must therefore be ramped quickly through low lattice depths where this noise can cause resonant excitations to excited bands of the lattice and subsequent heating. The frequency noise is characterized by observing depletion in the ground band of the optical lattice as a function of the lattice depth. A typical spectrum is shown in Fig. A.5. Loss due to frequency noise is observed at lattice depths of $\sim 15E_R$ and $\sim 35E_R$ which correspond roughly to the frequencies identified above. Secondly, the fast ramp of the lattice depth to a regime where tunnelling is frozen out allows for better matching with simulations of the lattice loading.

The final lattice depth is measured by amplitude modulation spectroscopy. By modulating the amplitude of the optical lattice, energy can be transferred in a deterministic way to atoms in the lattice to resonantly excite them to higher bands. In contrast to frequency modulation, amplitude modulation changes the band index $n \rightarrow n + 2$. Because of the curvature of different bands, only certain modulation frequencies will excite particle-hole pairs at specific quasi-momenta. In this way the band structure can be mapped out by varying the frequency of the modulation as seen in Fig. A.6. At large lattice depths, where the Bloch bands are largely flat, a single lattice depth can be determined by observing the driving frequency at which atoms are excited out of the ground band.

The population in the $n$-th band can be measured by a band-mapping technique where quasi-momentum is mapped onto real momentum in the $n$-th Brillouin zone. Here the lattice depth is ramped down to zero in $\sim 400 \mu s$ before releasing the atoms from the ODT which generates the distribution shown in Fig. A.6. Amplitude modulation spectroscopy combined with band-mapping allows one to determine the lattice depth by observing the energy required to drive these $n \rightarrow n + 2$ excitations and comparing to the calculated band structure.

Band-mapping is used to verify that only the ground Bloch band is occupied when beginning an experiment. The criteria that only the ground band is occupied constrains the ODT parameters from which the optical lattice is loaded. A dipole trap with large transverse confinement will cause loading of higher bands of the lattice during the adiabatic ramp on. Therefore, the ODT is decompressed prior to the ramp on of the optical lattice. The typical time sequence used to load the optical lattice is shown in Fig. A.7. Concurrent compression of the dipole trap while the lattice depth is being increased was.
Figure A.6: Amplitude modulation spectroscopy and band mapping are used together to map out the band structure in a $10E_R$ lattice.
Appendix A. Apparatus

Figure A.7: The typical sequence used to load the optical lattice. After evaporation the ODT is compressed and then relaxed to values such that loading the optical lattice doesn’t cause excitations out of the ground band. The lattice is ramped on to $5E_R$ in 100 ms and then to $50E_R$ in 5 ms for the majority of the experiments presented in this thesis.

explored but did not increase the transverse confinement above which atoms were loaded into excited bands.

After atoms are loaded in the optical lattice, the heating rates are characterized [228]. One might expect that the blue-detuned lattice used has a large scattering rate as the detuning is not large compared to the detuning of the ODT. However, the atoms are trapped at the intensity minima in the blue-detuned lattice as opposed to the red-detuned ODT and so the intensity of light they scatter is much lower. One can also compare the rate at which energy increases in the ground band of the optical lattice. It has been shown that the heating rate is suppressed much quicker in a blue-detuned lattice as compared to a red-detuned lattice as the confinement is increased [228]. This can be explained qualitatively as red-detuned lattices give rise to spontaneous emission events that return atoms to the ground band whereas in blue-detuned lattices these events return atoms to excited bands.

The heating rate is observed via the rate of atom loss. The experiments described in this thesis take place over the course of a few hundred milliseconds and so the heating rates must be small over those timescales. In Fig. A.8 the atom number is measured as a function of time in the optical lattice at the final lattice depth typically used ($50E_R$). No appreciable loss is observed over 200 ms which is sufficient for the dynamical measurements presented in this thesis.

A.4 Thermometry

To determine the temperature of an atomic gas one requires access to the trapped density distribution. The standard procedure to infer the density distribution is to use TOF imaging: the trapping potential
Appendix A. Apparatus

Figure A.8: The atom number as a function of time for a lattice depth of \(12E_R\) (open points) and \(50E_R\) (closed points) is used to determine if there is appreciable heating on the timescale of the experiments.

The gas is suddenly turned off and the gas is allowed to expand ballistically. The resulting distribution is imaged with resonant laser light which creates a column-integrated two-dimensional image of the expanded atom cloud. The shape of this distribution can be related to the temperature of the trapped gas.

A.4.1 Trapped Fermi Gases

For a non-interacting ideal Fermi gas at temperature \(T\), the average occupation of states \(i\) with energies \(E_i\) is given by the Fermi-Dirac distribution [30]

\[
\langle n_i \rangle = \frac{1}{e^{(E_i - \mu)/k_B T} + 1}.
\]  

(A.17)

The chemical potential \(\mu\) for a fixed number of particles \(N\) is chosen such that \(N = \langle N \rangle = \sum_i \langle n_i \rangle\). The total energy is \(U = \sum_i E_i \langle n_i \rangle\).

The atoms are confined to a harmonic trap \(V(r) = \frac{1}{2}m\omega_j^2r^2\) in two- or three-dimensions where \(\omega_j\) is the trapping frequency in the \(j\)-direction. In the Thomas-Fermi approximation the thermal energy \(k_B T \equiv 1/\beta\) is assumed to be much larger than the quantum mechanical level spacings \(\hbar\omega_i\). The discrete level structure can therefore be neglected and instead the density of particles in phase space is considered. The occupation of a phase space cell defined by position \(r\) and momentum \(p\) is then [30]

\[
f(r,p) = \frac{1}{e^{(p^2/2m + V(r) - \mu)/\beta} + 1}.
\]  

(A.18)

The density of a trapped ideal Fermi gas is determined by

\[
n(r) = \int \frac{d^D p}{(2\pi\hbar)^D} f(r,p)
\]  

(A.19)

where \(D = \{2, 3\}\) is the dimension. The Fermi-Dirac integral is evaluated in terms of the special functions

\[
Li_{\nu+1}(-e^x) = -\frac{1}{\Gamma(\nu + 1)} \int_0^\infty \frac{t^{\nu}}{e^{t-x} + 1} dt.
\]  

(A.20)
where $\Gamma$ is the gamma function and $Li_v(y)$ is the polylogarithm defined by

$$Li_v(y) = \sum_{k=1}^{\infty} \frac{y^k}{k^v}. \quad (A.21)$$

Evaluating Eq. (A.19) the density distribution in D-dimensions is

$$n(r) = -\frac{1}{\lambda_T^D} Li_{D/2} \left( -e^{(-V(r)+\mu)\beta} \right) \quad (A.22)$$

where $\lambda_T = \sqrt{2\pi\hbar^2\beta/m}$ is the deBroglie wavelength.

The total number of trapped atoms can be determined by integrating the density distribution or by using knowledge of the density of states $\rho(E)$ as a function of energy. The total number is

$$N = \int_0^\infty \rho(E) \frac{1}{e(E-\mu)\beta + 1} dE, \quad (A.23)$$

the total energy is

$$U = \int_0^\infty E\rho(E) \frac{1}{e(E-\mu)\beta + 1} dE, \quad (A.24)$$

and the entropy is

$$S = k_B \beta(U - \mu N) + \int_0^\infty \rho(E) \log \left( 1 + e^{(\mu-E)\beta} \right). \quad (A.25)$$

In this discussion it is useful to define the fugacity $Z \equiv e^{\beta\mu}$. Quantities are often discussed relative to the Fermi energy, $E_F$, which can be determined by integrating the density of states. The reduced temperature, which quantifies the degeneracy of the gas, $T/T_F = Tk_B/E_F$, is also related to the fugacity.

I summarize useful thermodynamic properties of harmonically trapped Fermi gases in both two- and three-dimensions in Tab. A.2.

### A.4.2 Determining the Temperature

To determine the temperature the trapped density distribution is mapped to the density distribution after ballistic expansion in TOF. The expansion of the trapped gas is considered “ballistic” when the mean free path between collisions is longer than the size of the trapped cloud and therefore the collisions...
can be neglected [30]. For ballistic expansion a particle in the trap at position $r_0$ and with momentum $p_0$ is mapped to position $r = r_0 + p_0 t/m$ after an expansion time $t$. In terms of the phase space density in the trap the density distribution after ballistic expansion is [30]

$$n(r, t) = \int d^3 r_0 \int \frac{d^3 p_0}{(2\pi\hbar)^3} f(r_0, p_0) \delta(r - r_0 - \frac{p_0}{m} t)$$

where the delta-function takes into account all particles at positions and momenta that will contribute at position $r$ after time $t$.

The expansion results in simple scalings of the trapped density distribution. Each spatial dimension $j = x, y, z$ is scaled by the factor $b_j(t) = \sqrt{1 + \omega_j^2 t^2}$. The net effect on the density distribution is characterized by (in 3D)

$$n(r, t) = \frac{1}{b_x(t) b_y(t) b_z(t)} n\left(\frac{x}{b_x(t)}, \frac{y}{b_y(t)}, \frac{z}{b_z(t)}\right).$$

Absorption imaging measures the column-integrated density distribution $n(x, y, t) = \int dz n(r, t)$. This integration gives a density distribution (in 3D)

$$n(x, y, t) = n_0 \frac{Li_2\left(-Z \exp\left[-\frac{m\beta}{2} \left(\frac{\omega_x^2 y^2}{b_y^2(t)} + \frac{\omega_y^2 y^2}{b_y^2(t)}\right)\right]\right)}{Li_2(-Z)}$$

where $n_0$ is a factor that encapsulates the imaging efficiency. The form of the density distribution motivates the use of the fitting function [30]

$$n(x, y) = \frac{Li_2\left(-Z \exp\left[-\frac{Z}{R_x^2} + \frac{Z^2}{R_y^2} f(Z)\right]\right)}{Li_2(-Z)}$$

where the widths

$$R_j^2 = \frac{2\beta^2(t)}{m\omega_j^2} f(Z)$$

with

$$f(x) = \frac{Li_1(-\alpha)}{Li_0(-\alpha)} = \frac{1 + \alpha}{\alpha} \ln(1 + \alpha).$$

### A.4.3 Loading the One-Dimensional Lattice

The atoms are initially confined in a three-dimensional harmonic potential. The introduction of the optical lattice along the $x$-direction modifies the density of states in that direction to be [229, 230]

$$\rho_x(E) = \sqrt{\frac{8}{m\omega_x^2 \lambda^2 E}}.$$

In the other two directions, the harmonic potential remains unchanged and the density of states is simply $\rho_j(E) = 1/(\hbar\omega_j)$. Using these respective density of states, I integrate up to energy $E$ to determine the total number of states. Taking the derivative of the number of states with respect to energy then gives
Figure A.9: The (a) number and (b) reduced temperature on each lattice site after loading the optical lattice starting with $4 \times 10^4$ atoms and a reduced temperature $T/T_F = 0.25$ in the three-dimensional harmonic trap.

From this density of states one can calculate the number, energy, and entropy in the combined harmonic and optical lattice potential.

To calculate the number distribution and temperature while loading the lattice one simply imposes number and entropy conservation. Given the entropy in the three-dimensional harmonic trap, one matches this with the entropy in the combined harmonic and optical lattice potential. The matching criteria is calculated for a $5 E_R$ lattice as motivated by the experimental loading procedure.

Having calculated the entropy in the combined system, the final temperature and number distribution of the ensemble of isolated two-dimensional gases can be determined. The rapid ramp of the lattice depth to $50 E_R$ is considered to only change the confinement of the isolated two-dimensional gases and not the number distribution or temperature. With this assumption, the fugacity on each lattice site $\ell$ is

$$Z_\ell = Z \exp \left[ -\frac{m \omega^2 \lambda^2 \ell^2}{8k_B T} \right] \tag{A.34}$$

where $Z$ is the fugacity in the combined system. A priori it is unknown how many sites will have a population with one or more atom and so for computational ease I calculate $Z_\ell$ for $\ell \in [-200, 200]$ (many more sites than will be occupied) and then post-select on sites with a number greater than one. Having calculated $Z_\ell$ one can determine the properties of the ensemble of two-dimensional Fermi gases as in Tab. A.2.

Loading the optical lattice increases the reduced temperature due to the modified density of states. A typical number and temperature distribution is shown in Fig. A.9 as a function of the lattice site after loading the lattice starting with $4 \times 10^4$ atoms and a reduced temperature $T/T_F = 0.25$ in the three-dimensional harmonic trap. Typically, 200 sites are populated.

One complication involved in utilizing an optical lattice to generate strong confinement is that it populates an ensemble of two-dimensional gases, each with their own Fermi energy. Any measurement must be understood as an average over this ensemble. This complicates measurements used to determine
the temperature of the ensemble after loading the optical lattice. In particular, in ballistic expansion during TOF the large confinement in the $x$-direction results in fast expansion of the density profile in that direction as the scaling factor is proportional to the trapping frequency. This results in the individual two-dimensional Fermi gases overlapping so that the in situ number distribution cannot be determined. As such, the adiabaticity and efficacy of the loading procedure is difficult to determine.

To benchmark the calculation of the loading procedure the ensemble-averaged temperature of the two-dimensional gases is measured after ballistic TOF. The total density distribution is the sum over each two-dimensional gas

$$n(y, z, t) = \sum_\ell n_\ell(y, z, t) = -\sum_\ell \frac{1}{X_\ell b_\ell(t) b_\ell(t)} Li_1(-Z e^{\omega_y^2 y_y^2/b_y^2(t)+\omega_z^2 z_z^2/b_z^2(t)\beta m/2}).$$ \hspace{1cm} (A.35)

Absorption imaging measures the column integrated density distribution $n(y, t) = \int dz n(y, z, t)$. In the experiment, the optical density decreases and the atoms rapidly expand along the tightly confined $x$-direction. Integrating along this direction in the analysis effectively performs the sum in Eq. (A.35) leaving a one-dimensional density distribution that can be fit using

$$n(y) = \tilde{n}_0 \frac{Li_{3/2}(-Z \exp\left[-\frac{y^2}{\xi^2} f(Z)\right])}{Li_{3/2}(-Z)}$$ \hspace{1cm} (A.36)

where $\tilde{n}_0$ encapsulates the effects of imaging efficiency and ballistic expansion.

Figure A.10 shows a comparison of the fitted temperatures and a calculation of the expected measured temperature after TOF and ensemble averaging. They agree within their respective statistical uncertainties providing evidence that the loading procedure of the optical lattice is indeed adiabatic and well understood. It is important to note that after loading the optical lattice the individual two-dimensional gases are not rigorously in thermal equilibrium as they are spin-polarized and thus atoms are unable to scatter. However, the data in Fig. A.10 provides evidence that this effective temperature is close to the true temperature of the two-dimensional gases.
Figure A.11: Measurements of the (a) number and (b) reduced temperature as a function of the final trap depth in the ODT for various initial numbers and temperatures. The potassium number loaded into the ODT is greater than the rubidium number for points in blue. Points in black show data when the rubidium number is greater than the potassium number. Red points show a balanced amount of rubidium and potassium loaded into the ODT.

### A.4.4 Controlling the Temperature

The temperature of the degenerate Fermi gas is primarily determined by the final stage of evaporation performed in the ODT. The relative number of $^{40}$K and $^{87}$Rb also helps to determine the final number and temperature at the end of evaporation. $^{87}$Rb is directly evaporated by weakening the optical potential and allowing the hotter atoms to escape. The remaining atoms collide with $^{40}$K atoms to cool them. For a sufficiently weak optical trap the two species separate due to their differential gravitational sag and sympathetic cooling ceases. A tight trap (as is provided by the small beam waist in this experiment) is required to provide a large parameter space where sympathetic cooling is possible. This has been explored in more detail for the sympathetic cooling of $^{39}$K with $^{87}$Rb [231].

Empirically, the optimal number of $^{40}$K to load into the chip trap to reach the desired reduced temperature can be determined for a particular experiment. This is accomplished by varying the amount of trapping light for $^{40}$K during the MOT and by varying the relative loading efficiency of both species from the chip trap to the ODT. Figure A.11 shows measurements of the number and reduced temperature as a function of final evaporation depth in the ODT for various initial numbers and temperatures. For large initial number of $^{40}$K the amount of $^{87}$Rb is typically insufficient to sympathetically cool until the bottom of the trap is reached. This is indicated by the sharp drop in atom number at an ODT power of 0.12 where it appears that $^{40}$K begins to be directly evaporated. This is confirmed by the increasing reduced temperature at low ODT power: too much $^{40}$K results in inefficient sympathetic cooling. For lower atom numbers the amount of $^{87}$Rb is sufficient to cool the $^{40}$K until the two species separate at an ODT power of approximately 0.08. Typically, the experiment is run in an intermediate regime with a good balance of $^{87}$Rb and $^{40}$K where temperatures as low as $T/T_F \sim 0.15$ and numbers around $4 \times 10^4$ can be achieved.

### A.5 Radio-Frequency State Manipulation

Rf radiation is primarily used to manipulate the internal spin degree of freedom of the atoms. Typical rf frequencies are in the 1 kHz - 100 GHz range. Thus, the wavelength is very long and the momentum
transferred to the atom is negligible. Therefore rf radiation can be thought to just change the internal state of the atom while leaving the kinetic energy and momentum of the system unchanged.

Due to the long lifetime of the internal states one can assume a two-level system: the initial state $|g\rangle$ and the excited state $|e\rangle$, separated by an energy difference $E_e - E_g = \hbar \omega_0$, where $\omega_0$ is the resonant frequency required to drive the transition. The coupling rf field is assumed to be monochromatic with frequency $\omega_L$. Treating this problem semi-classically starting with all atoms in state $|g\rangle$ at time $t = 0$, the probability to find an atom in state $|e\rangle$ is

$$P_e(t) = \frac{\Omega}{\delta^2 + \Omega^2} \sin^2 \left( \frac{\sqrt{\delta^2 + \Omega^2}}{2} t \right)$$  \hspace{1cm} (A.37)

where $\delta = \omega_L - \omega_0$ is the detuning of the rf radiation and $\Omega$ is the Rabi frequency. To fully transfer all of the atoms into state $|e\rangle$ one needs to apply resonant (i.e., $\delta = 0$) rf radiation for $t = \pi/\Omega$. This is referred to as a $\pi$ pulse.

In the experiments described in this thesis the pulse area $\Omega t$ and the frequency of the rf radiation are manipulated. For a non-interacting gas one can vary the frequency of the rf radiation to measure the resonant transition frequency at a given magnetic field. The energy of this transition can be exactly calculated using the Breit-Rabi formula and so a measurement of the transition frequency allows one to calibrate the magnetic field for a given current through the magnetic field coils. The rf line shape has a linewidth that is limited by the inverse of the pulse duration. Ultimately, the linewidth is limited by magnetic field noise and linewidths on the order of 1 kHz are measured which is consistent with the magnetic field stability on the order of 10 mG. A narrow linewidth spectrum is shown in Fig. A.12a.

Varying the pulse area can be accomplished by changing the duration of the rf pulse or by changing the Rabi frequency. By changing the pulse area one can drive Rabi-flopping between the two states as in Fig. A.12b. Typically the pulse area is varied with $\delta = 0$ to calibrate a $\pi$ pulse at a given magnetic field and then the duration of the pulse is varied while keeping the Rabi frequency constant to create spin-mixtures with arbitrary polarization.
Appendix A. Apparatus

A.6 Magnetic Field Control

The MOT coils double as the coils that generate the “Feshbach” field which is used to tune the scattering length between the various spin states. After evaporation in the ODT, the MOT coils are placed in Helmholtz configuration and the current is stabilized to generate a homogeneous magnetic field near 200 G. The magnetic field can be varied from 177 G to 213 G with a stability of approximately one part in $10^5$ [116, 232]. However, due to the location of the ODT relative to the MOT coils there is a residual magnetic field gradient of $\sim 10$ G/cm in the $y$-direction.

To control the magnetic field gradient, small currents are applied through two parallel chip wires (see Tab. A.1) [116]. By varying the sum and difference of the currents the magnetic field gradient can be changed in the $y$- and $z$-directions. The gradient in the $x$-direction also varies slightly as a function of the current through the wires although it always remains small (i.e., $\sim 1$ G/cm). The magnetic field gradient is measured by moving the atoms using piezo-actuated controllers on mirror mounts for the ODT beams and measuring the change in the transition frequency between internal spin states as a function of position using rf spectroscopy.

The chip wires can also change the magnitude of the magnetic field by a few Gauss depending on the current applied. This is advantageous for quick manipulation (approximately 100 $\mu$s) of the magnetic field as the wires have low inductance. Conversely, the MOT coils have a large inductance and so rapid changes of the magnetic field by varying the current through them is difficult and typically cannot be accomplished on a timescale faster than 5 ms. To rapidly change the magnetic field by 5-10 Gauss a “jump circuit” is employed [116, 232]. A capacitor is discharged through the $Z$-bias shim coils to quickly introduce a bias field that effectively changes the magnetic field experienced by the atoms. This magnetic field jump occurs over $\sim 20$ $\mu$s and is useful to quickly change the interaction strength, for instance to make the scattering length small during ballistic expansion in TOF.

A.7 Imaging

In the experiments described in this thesis, optical absorption imaging of $^{40}$K on the $|F = 9/2, m_F = -9/2\rangle$ to $|F' = 11/2, m_F' = -11/2\rangle$ cycling transition is performed at a magnetic field of approximately 209 G where the scattering length is small so the expansion in TOF is ballistic. The path the imaging light takes is shown in Fig. A.2. There are two orthogonal imaging systems that propagate in the same direction as the trapping beams of the ODT. The “radial” imaging system is along the $x$-direction and creates an image of the cloud in the $y-z$ plane. This direction is used primarily for calibration and alignment of optical elements. The “axial” imaging system is along the $z$-direction and creates an image of the cloud in the $x-y$ plane. This is the direction that is used for the data presented throughout this thesis.

When multiple spin states are present they are mapped to the $|F = 9/2, m_F = -9/2\rangle$ state prior to imaging. This is accomplished either with rf state manipulation before turning off the ODT or using a combination of a Stern-Gerlach pulse to separate spin states and subsequent rf state manipulation in a residual magnetic field gradient in TOF. The second method allows for the simultaneous imaging of atoms in multiple spin states [116].
Appendix B

The $p$-Wave Contacts: Some Theoretical Considerations

This appendix summarizes some useful theoretical derivations, identities, and relations relevant to the understanding of the results presented in Chapter 4. These results have been worked out in a number of sources and are reproduced here to collect useful and related concepts. As such, I include results that I have re-derived with appropriate references to the source material. In some cases, Shizhong Zhang or Zhenhua Yu performed the initial derivation which I reproduce here.
B.1 Defining the Contacts

I follow the derivation of the adiabatic theorem for the $p$-wave contacts as presented in [94]. The idea of the derivation is to utilize the ultra-dilute nature of the gas to find a regime in which the two-body and many-body wave functions have the same form, although the normalization of the many-body wave function will be determined by many-body physics. The range of the interaction $r_0$ is much less than the interparticle spacing $k_F^{-1}$, such that when two particles come together the form of the wave function is determined by the two-body physics.

I begin by studying two-body low-energy scattering near a $p$-wave Feshbach resonance. Two particles of mass $m$ interact via a potential $U(r)$ with range $r_0$. The relative wave function in the $p$-wave channel can be written as $\psi_k(r) \equiv \chi_k(r)Y_{1\ell}(\hat{r})/r$, where $Y_{1\ell}$ are the spherical harmonics, $\nu$ labels the projection of angular momentum along the $z$-direction and $k$ is the relative wave vector. For low-energy scattering the radial wave function $\chi_k(r)$ can be expanded in powers of $k^2$, $\chi_k(r) = \chi^{(0)}(r) + k^2\chi^{(1)}(r) + \ldots$. In the asymptotic regime where $k^{-1} \gg r_0$, I fix the normalization such that

$$\chi_k(r) = \left(\frac{1}{r} - \frac{r^2}{3v}\right) + k^2\left(\frac{r}{2} - \frac{r^2}{3R} + \frac{r^4}{30v}\right) + \ldots \tag{B.1}\$$

where $v$ is the scattering volume and $R$ is the effective range introduced in Chapter 4. The short range form of $\chi$ is completely determined by the two-body physics independent of the asymptotic wave vector $k$.

I now consider a system of $N$ spinless fermions confined in a volume $V$ with density $n = N/V = k_F^3/(6\pi^2)$ where $k_F$ is the Fermi wave vector. The two-body density matrix

$$\rho_2(r_1, r_2) = \langle \psi^\dagger(r_1)\psi^\dagger(r_2)\psi(r_2)\psi(r_1) \rangle,$$

where $\psi(r)$ creates a fermion at position $r$, is Hermitian and can be diagonalized

$$\rho_2(r_1, r_2) = \sum_{\alpha} n_\alpha \phi_\alpha^*(r_1, r_2)\phi_\alpha(r_1, r_2). \tag{B.2}$$

The eigenvalues $n_\alpha$ satisfy the condition $\sum_\alpha n_\alpha = N(N-1)$, and the associated pair wave functions $\{\phi_\alpha(r_1, r_2)\}$ form an orthonormal set. In a rotationally invariant system, they can be further written as

$$\phi_\alpha(r_1, r_2) = \frac{1}{\sqrt{Vr}} \exp(i\mathbf{P} \cdot \mathbf{R})\varphi_{j\ell\nu}(r)Y_{1\nu}(\hat{r}), \tag{B.3}$$

where $\mathbf{R} = (r_1 + r_2)/2$ is the centre-of-mass, $\mathbf{r} = r_1 - r_2$ is the relative coordinate, and $r = |\mathbf{r}|$. $\mathbf{P}$ is the centre-of-mass momentum of a pair and $j, \ell, \nu$ label the quantum numbers of the relative radial direction, the angular momentum, and its $z$-projection respectively. The shorthand $\alpha = \{\mathbf{P}, j, \ell, \nu\}$ is used for all the quantum numbers that label the pair wave function. I will limit the discussion to the $p$-wave ($\ell = 1$) component.

The pair wave function $\phi_\alpha(r_1, r_2)$, and by extension $\varphi_{j\ell\nu}(r)$, is not an eigenfunction of the two-body Schrödinger equation, but can be expanded in terms of the $p$-wave functions

$$\varphi_j = \int_0^\infty dk b_{jk}\chi_j(r) + b_{j\kappa}\chi_\kappa(r). \tag{B.4}$$

Here $\{b_{jk}, b_{j\kappa}\}$ are the real expansion coefficients, the integration is over all scattering states, and I have allowed for the possibility of a shallow bound state with radial wave function $\chi_\kappa(r)$ and binding energy.
\( E_d = \hbar^2 k^2 / m = \hbar^2 R / (mv) \), when \( v > 0 \) and \( R > 0 \). The consequence of this expansion is that the form of \( \phi_\alpha \) in the asymptotic regime \( r_0 \ll r \ll k_F^{-1} \) is completely determined by the two-body physics via \( \chi^{(0,1)}(r) \). In fact, I can write the two-body density matrix in terms of the two-body radial wave functions as

\[
\rho_2(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3) = \frac{1}{V} \sum_\alpha n_\alpha e^{iP \cdot (\mathbf{R} - \mathbf{R}')} Y_{1\nu}^* (\hat{\mathbf{r}}) Y_{1\nu} (\hat{\mathbf{r}}') \int_0^\infty dk \int_0^\infty dk' b_{jk} \chi_k (r)b_{jk'} \chi_{k'} (r'),
\]

where \( \mathbf{R}' = (\mathbf{r}_2 + \mathbf{r}_3) / 2 \) and \( r' = |\mathbf{r}_2 - \mathbf{r}_3| \).

The interaction energy of the many-body system can be written in terms of \( \rho_2 \) as \( \langle \mathcal{U} \rangle = \frac{1}{2} \int U(|\mathbf{r}_1 - \mathbf{r}_2|) \rho_2(\mathbf{r}_1, \mathbf{r}_2) d^3r_1 d^3r_2 \), where the factor of \( 1/2 \) arises due to the interaction being between particles in the same spin state. Decomposing \( \chi_k \) in powers of \( k \) gives

\[
\langle \mathcal{U} \rangle = \frac{1}{2} \sum_\nu \left[ C_{v,\nu} \int dr U |\chi^{(0)}|^2 + C_{R,\nu} \int dr U \chi^{(0)} \chi^{(1)} \right],
\]

where I have defined the \( p \)-wave contacts as in Chapter 4

\[
C_{v,\nu} = \sum_{\mathbf{P},j} n_{\mathbf{P},j,\nu} \int dk \int dk' b_{jk} b_{jk'},
\]

\[
C_{R,\nu} = \frac{1}{2} \sum_{\mathbf{P},j} n_{\mathbf{P},j,\nu} \int dk \int dk' b_{jk} b_{jk'} (k^2 + k'^2).
\]

### B.2 Deriving the Momentum Distribution

The derivation of the momentum distribution for \( p \)-wave interactions has had a bit of an untidy history. The original derivation [94] neglected so-called "centre-of-mass" terms [89, 91, 92, 155] that appear at order \( k^{-4} \) which compete with the observed values of \( C_R \) which also enter as \( k^{-4} \). The magnitude of this correction is estimated in Chapter 4. In this section I highlight the details of the derivation of the momentum distribution starting from the two-body density matrix.

Starting from the two-body density matrix one can define the one-body density matrix

\[
\rho_1(\mathbf{r}_1, \mathbf{r}_2) = \int d^3r_3 \rho_2(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3).
\]

The Fourier transform of \( \rho_1 \) then gives the high momentum tails. I follow the supplementary material of [94] and identify quantities under the integral as the inverse Fourier transform. The one-body density matrix can be written as

\[
\rho_1(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{V} \int d^3r_3 \sum_\alpha n_\alpha e^{iP \cdot (\mathbf{R} - \mathbf{R}')} Y_{1\nu}^* (\hat{\mathbf{r}}) Y_{1\nu} (\hat{\mathbf{r}}') \int dk \int dk' b_{jk} b_{jk'}
\]

\[
\times \left[ \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|^2} + \frac{k^2}{2|\mathbf{r}_1 - \mathbf{r}_2|^2} + \frac{1}{|\mathbf{r}_2 - \mathbf{r}_3|^2} + \frac{k'^2}{2|\mathbf{r}_2 - \mathbf{r}_3|^2} \right].
\]
This can be rewritten in terms of the contacts defined above

\[
\rho_1(r_1, r_2) = \frac{1}{V} \int d^3r_3 \sum_\alpha n_\alpha e^{iP.(R^*-R)} Y_{1\nu}(\hat{r}) Y_{1\nu}(\hat{r}')
\]

(B.11)

where

\[
C_{v,\nu} \left[ \frac{C_{R,\nu}}{|r_1 - r_2|^2} + \frac{1}{|r_2 - r_3|^2} + \int dk \int dk'b_{jk}b_{jk'} \left( -\frac{k'^2}{2} \frac{1}{|r_1 - r_2|^2} + \frac{k'^2}{2} \frac{1}{|r_2 - r_3|^2} \right) \right].
\]

The first two terms have been treated previously in the supplementary material of [94], however the last two terms do not cancel in the integral. I will treat them below but first I need a few identities for the Fourier transform of the spherical harmonics.

I start with an expansion of a plane wave in terms of spherical harmonics

\[ e^{ikr} = 4\pi \sum_{\ell,\nu} (i)^{\ell} j_{\ell}(kr) Y_{\ell,\nu}(\hat{k}) Y^*_{\ell,\nu}(\hat{r}) \] (B.12)

where \( j_{\ell}(x) \) is the spherical Bessel function of the first kind. The Fourier transform of \( Y_{\ell,\nu} \)

\[ \text{FT}\{Y_{\ell',\nu'}(\hat{r})\}(\hat{k}) = \int d^3r Y_{\ell',\nu'}(\hat{r}) e^{ikr} \] (B.13)

can be expanded using Eq. (B.12)

\[ \text{FT}\{Y_{\ell',\nu'}(\hat{r})\}(\hat{k}) = \int d^3r Y_{\ell',\nu'}(\hat{r}) 4\pi \sum_{\ell,\nu} (i)^{\ell} j_{\ell}(kr) Y_{\ell,\nu}(\hat{k}) Y^*_{\ell,\nu}(\hat{r}). \] (B.14)

Performing the angular integral enforces \( \nu = \nu' \) and \( \ell = \ell' \) which gives

\[ \text{FT}\{Y_{\ell,\nu}(\hat{r})\}(\hat{k}) = 4\pi (i)^{\ell} Y_{\ell,\nu}(\hat{k}) \int dr r^2 j_{\ell}(kr). \] (B.15)

Specializing to the case of \( \ell' = 1 \) this simplifies to

\[ \text{FT}\{Y_{1,\nu}(\hat{r})\}(\hat{k}) = -8\pi i Y_{1,\nu}(\hat{k}) \frac{1}{k^3}. \] (B.16)

Similarly, one can derive the Fourier transform of \( Y_{1,\nu}/r^2 \),

\[ \text{FT}\{Y_{1,\nu}/r^2\}(\hat{k}) = -4\pi i Y_{1,\nu}(\hat{k}) \frac{1}{k^2}. \] (B.17)

With these identities in mind, the procedure to determine the momentum distribution should look straightforward. One simply tries to write each term as the inverse Fourier transform of the spherical harmonics and then proceeds with the integral over \( k \). The first term in \( \rho_1 \) gives a term proportional to \( k^{-2} \) while the remaining terms give a term proportional to \( k^{-4} \). The details are shown explicitly for the first two terms in the supplementary material of [94]. This finally allows one to write the momentum distribution as it is given in Chapter 4

\[ n_k \to \sum_\nu |Y_{1\nu}(\hat{k})|^2 \left[ \frac{16\pi^2 C_{v,\nu}}{Vk^2} + \frac{32\pi^2 C_{R,\nu}}{Vk^4} - \frac{8\pi^2 C_{cm,\nu}}{3 Vk^4} \right] + \frac{\pi C_{cm,\nu}}{Vk^4} \] (B.18)

where \( C_{cm,\nu} \) is related to the centre-of-mass momentum of the pair.
B.3 Connection Between Closed-Channel Dimers and the Contacts

This derivation is presented in the supplementary material of [95]. I highlight the essential details arriving at simple expressions for the contacts \( C_v \) and \( C_R \) in terms of the number of closed-channel dimers. This derivation is also used to help illustrate the physical significance of \( C_R \).

The connection between the contacts and closed-channel molecules is most easily seen in a two channel model. In momentum space, the Hamiltonian describing the conversion of scattering fermions to molecules can be written as [233–235]

\[
H_{\text{int}} = \bar{g} \sqrt{V} \sum_{k,p} k Y_{1\nu}(\hat{k}) b_{p/2+k}^\dagger a_{p/2-k} + \text{H.c.}
\]  

where \( b_{p}^\dagger \) (\( a_p^\dagger \)) creates a molecule (fermion) with momentum \( p \), and \( \bar{g} \) is the coupling constant. The free Hamiltonian can be written as

\[
H_0 = \sum_k \epsilon(k) a_k^\dagger a_k + \sum_p E(p) b_p^\dagger b_p.
\]

where \( \epsilon(k) = k^2/(2m) \), \( E(p) = p^2/(4m) + \beta_{\nu}(B) \) where \( \beta_{\nu}(B) = \delta \mu_{\nu}(B - B_{0,\nu}) \) is the detuning of the closed-channel molecule and \( \delta \mu_{\nu} \) is the magnetic moment difference between the open- and closed-channel. The goal is to express \( \bar{g} \) and \( \beta_{\nu} \) in terms of the scattering parameters \( v_{\nu} \) and \( R_{\nu} \). The result is

\[
\frac{1}{v_{\nu}} = -\beta_{\nu} \frac{8\pi^2 \hbar^2}{mg^2} + 2\Lambda^3 \frac{3\pi}{2} \quad \text{(B.21)}
\]

\[
\frac{1}{R_{\nu}} = \frac{8\pi^2 \hbar^4}{m^2 g^2} + 2\Lambda \frac{\pi}{\pi} \quad \text{(B.22)}
\]

where \( \Lambda \) is an ultra-violet cutoff in momentum.

To obtain the number of closed-channel molecules it is simplest to use the Hellmann-Feynman theorem, by taking the derivative with respect to \( \beta_{\nu} \). That is

\[
N_{d,\nu} = \frac{\partial E}{\partial \beta_{\nu}} = \frac{\partial E}{\partial v_{\nu}} \frac{\partial v_{\nu}^{-1}}{\partial \beta_{\nu}}.
\]  

The first partial derivative on the right-hand side of Eq. (B.23) is immediately recognized at the adiabatic theorem for \( p \)-waves. The second partial derivative can be evaluated by expressing \( v_{\nu}^{-1} \) in terms of \( \beta_{\nu} \) which when close to the Feshbach resonance gives,

\[
\frac{\partial v_{\nu}^{-1}}{\partial \beta_{\nu}} \approx -\frac{1}{\delta \mu_{\nu} \Delta v_{\nu} b_{\nu}}.
\]  

Therefore, Eq. (B.23) becomes

\[
N_{d,\nu} = \frac{\hbar^2}{2m} C_{\nu,\nu} \frac{1}{\delta \mu_{\nu} \Delta v_{\nu} b_{\nu}}.
\]  

For \( \delta \mu_{\nu} = \partial E_{d,\nu}/\partial B \) the differential magnetic moment can be expressed in terms of scattering parameters.
as \( \delta \mu = \hbar^2 R/\left(m \Delta \nu v^\text{bg} \right) \). This yields a simple expression for \( C_{v, \nu} \) in terms of the number of dimers

\[ C_{v, \nu} \sim 2R \nu N_{d, \nu}. \] (B.26)

Similarly, one can investigate \( C_{R, \nu} \) via the adiabatic theorem. This gives

\[ -\frac{\hbar^2}{2m} C_{R, \nu} = \frac{\partial E}{\partial R} \frac{1}{\nu} \nu \frac{\partial E}{\partial \bar{g}^2} \frac{1}{\partial \bar{R}^2} \] (B.27)

\[ = -\frac{m^2 \bar{g}^2}{8\pi^2 \hbar^4} \left( \beta \nu N_{d, \nu} + \frac{1}{2} \langle H\text{int} \rangle \right). \] (B.28)

This helps provide some understanding of \( C_{R, \nu} \) as the first term on the right-hand side represents something proportional to the number of dimers while the last term involves the coherences between atoms and dimers \( \langle \hat{b}^\dagger p \hat{a} p \rangle \). Ignoring the second term, one can find an approximate form for \( C_{R, \nu} \) near the Feshbach resonance. Noticing that \( m^2 \bar{g}^2/(8\pi^2 \hbar^4) \approx R \nu \nu \delta \mu \nu \nu \Delta \nu v^\text{bg} \nu \nu N_{d, \nu} \) allows one to write

\[ -\frac{\hbar^2}{2m} C_{R, \nu} \approx -\frac{R \nu \nu}{\nu} \nu \delta \mu \nu \nu \Delta \nu v^\text{bg} \nu \nu N_{d, \nu} \] (B.30)

which can be simplified using the expression for \( \delta \mu \nu \nu \) to give

\[ C_{R, \nu} \approx -\frac{2R^2 \nu \nu}{\nu} N_{d, \nu} \approx -\frac{R \nu \nu}{\nu} C_{v, \nu}. \] (B.31)

### B.4 Rate Equations Near a p-Wave Feshbach Resonance

Given the two channel model presented above, the dynamics of \( C_{\nu} \) can be understood by considering how the number of dimers changes as a function of time as is done in the supplementary material of [95]. Let the momentum distribution of the dimers be given by \( n_{\nu}(p) \) and that the conversion rate between dimers and scattering fermions occurs at the same rates as that given in vacuum \( \gamma \nu \nu \). This is a good approximation when very close to the p-wave resonance \( (E_{d, \nu} \gtrsim 0) \). If the centre-of-mass momentum of the dimer is \( p \), then by energy and momentum conservation the incoming scattering fermions must have momentum \( p/2 + k \) and \( p/2 - k \), with \( |k| = k_0 \) and \( \hbar^2 k^2_0/(2m) = -E_{d, \nu} \). The magnitude of \( k \) is fixed by the dimer energy while the direction can be arbitrary. In the vacuum case, the angular average gives rise to the decay rate of the p-wave dimer, \( \gamma \nu \nu \). Thus, one can write down phenomenological rate equations for the populations of dimers and fermions,

\[ \frac{dn_{\nu}(p)}{dt} = \gamma \nu \sum_{|k|=k_0} \left[ f_{p/2+k} f_{p/2-k} (1 + n_{\nu}(p)) - (1 - f_{p/2+k} f_{p/2-k}) n_{\nu}(p) \right], \] (B.32)

where \( f \) is the Fermi distribution function and \( \overline{f f} \) means the angular average over \( k \). The first term describes the conversion of two fermions into a dimer while the second term describes the opposite process in which a dimer disassociates into two fermions, corrected with Fermi and Bose statistics.

At the beginning of the dynamics there is a Fermi sea of Fermi momentum \( k_F \). When \( E_{d, \nu} \) is only
slightly above threshold (i.e., $k_0 \ll k_F$) one can write

$$\frac{dn_{\nu}(p)}{dt} = \gamma_{\nu} \left[ f_{p/2}^2 (1 + n_{\nu}(p)) - (1 - f_{p/2}^2)^2 n_{\nu}(p) \right]$$

$$= \gamma_{\nu} \left[ f_{p/2}^2 - (1 - 2f_{p/2})n_{\nu}(p) \right].$$

(B.33)

Summing over $p$ can be done to determine rate equations for the number observed in experiments

$$\frac{dN_{d,\nu}}{dt} = \gamma_{\nu} \sum_p \left[ f_{p/2}^2 - (1 - 2f_{p/2})n_{\nu}(p) \right]$$

$$= -\gamma_{\nu} N_{d,\nu} + \gamma_{\nu} \sum_p f_{p/2}^2 + 2\gamma_{\nu} \sum_p f_{p/2}n_{\nu}(p).$$

(B.34)

The first term describes vacuum decay of a dimer, proportional to $N_{d,\nu}$. The second term describes the conversion of two fermions into a dimer, which when summed over $p$ gives

$$\sum_p f_{p/2} = \frac{V4\pi}{(2\pi)^3} \int_0^{2k_F} dpp^2$$

which gives $8\gamma_{\nu}N_f$ at zero temperature where $N_f = Vk_F^3/(6\pi^2)$. The last term arises from the Pauli exclusion principle which inhibits the decay of dimers into fermions when the final states are already occupied and therefore reduces the apparent dimer decay rate. This term is more complicated to handle and is combined with the second term to give an empirical loss rate $A_{\nu}$. The same steps can be repeated for the density of scattering fermions and after including fermion-dimer or dimer-dimer loss one arrives at the coupled differential equations for the number of fermions $N_f$ and the number of dimers $N_{d,\nu}$ presented in Chapter 4.
In this appendix I collect some results detailing simulations I carried out of the magnetization dynamics. This work supplements the robust theoretical treatment provided by Tilman Enss. The calculations presented in this appendix are easily implemented and help highlight the effects of some technical issues associated with the measurements, like finite pulses, finite size effects, and inhomogeneities. In this appendix I also provide an overview of Fermi-Liquid theory and interpret the spin transport measurements from Chapter 5 within this framework.
C.1 Interpretation in Fermi-Liquid Theory

At low temperatures, Landau Fermi Liquid theory \[236\] provides a microscopic interpretation of the spin transport parameters. Fermi Liquid theory postulates that there exists a mapping between a strongly interacting gas of atoms and a weakly interacting gas of quasiparticles. These quasiparticles are elementary excitations that represent the motion of many particles of the system simultaneously. Fermi Liquid theory has been successful at predicting the low temperature thermodynamic and transport properties of some strongly interacting systems of fermions.

The extent to which Fermi Liquid theory is appropriate to describe a strongly interacting ultracold gas of fermions in the normal state is still a topic of active research \[69\]. In this section I will report the results of Fermi Liquid theory to provide a comparison to the data presented in Chapter 5 where applicable.

C.1.1 Fermi-Liquid Theory in Two Dimensions

For a dilute, weakly repulsive Fermi gas in two dimensions, the ground state is a stable Fermi liquid \[237\] and the Landau interaction \( f \) function is given to second order in the coupling \( g = -1/\ln (k_F a_{2D}) \) by

\[
F^s(\theta) = g + g^2 \left( 2 + \ln |\cos \frac{\theta}{2}| \right) \\
F^a(\theta) = -g - g^2 \ln |\cos \frac{\theta}{2}|. 
\] (C.1)

The first angular Fourier coefficients defined by \( F^{s,a}(\theta) = \sum_m F^{s,a}_m e^{im\theta} \) are

\[
F^s_0 = g + g^2 (2 - \ln 2), \quad F^a_0 = -g - g^2 \ln 2, \quad F^s_1 = g^2/2, \quad F^a_1 = -g^2/2. 
\] (C.2)

When describing forward scattering between two particles on the Fermi surface, \( \theta \) measures the angle between the incoming particles and thereby parametrizes the angular dependence of the effective interaction. The effective mass \( m^*/m = 1 + F^a_0 = 1 + g^2/2 \) and the spin susceptibility \( \chi/\chi_0 = (1 + F^s_0)/(1 + F^a_0) = 1 + g + g^2(3/2 - \ln 2) \). The effective mass has been measured for a weakly interacting two-dimensional Fermi gas \[81\] but the spin susceptibility has not. For a low temperature gas, the transverse spin diffusivity in two dimensions is given by

\[
D^\perp_0 = \frac{\chi_0 \tau^\perp \epsilon_F}{m^*} = (1 + F^a_0) \frac{\tau^\perp \epsilon_F}{m} 
\] (C.3)

where \( \epsilon_F \) is the Fermi energy and \( \tau^\perp \) is the transport lifetime. For \( (1 + F^a_0) \) of order unity and assuming \( D^\perp_0 \sim \hbar/m \) saturates the quantum bound, \( \tau^\perp \sim \hbar/\epsilon_F \). Since Fermi-Liquid theory requires long lived quasiparticles promoting transport this transport lifetime is at the lower self-consistent bound for a quasiparticle treatment. The Leggett-Rice parameter can also be written in terms of the transport lifetime and a mean-field interaction \( W \) between unlike spins \[49, 210\]

\[
\gamma = \frac{\tau^\perp W n}{\hbar} 
\] (C.4)
where \( n \) is the total number density.

The ratio \( \gamma / D_0^\perp \) eliminates the scattering time and allows one to define the dimensionless effective interaction \( \lambda \) in terms of the quantities \( \gamma \) and \( D_0^\perp \),

\[
\lambda = -\frac{\hbar \gamma}{2m^* D_0^\perp},
\]

Using the definition of the mean-field interaction \( W \) [210] one can express \( \lambda \) in terms of Fermi liquid parameters

\[
\lambda = 1 + F_0^a - \frac{1}{1 + F_1^a}.
\]

To second order in the perturbative weak-coupling expansion \( \lambda \) can be written

\[
\lambda = g - g^2(\ln 2 - 1/2) + \mathcal{O}(g^3)
\]

\[
= -\frac{1}{\ln \left( k_F a_{2D} \right) - (\ln 2 - 1/2)} + \mathcal{O}(|\ln (k_F a_{2D})|^{-3}).
\]

The divergence of \( \lambda \) near \( \ln (k_F a_{2D}) \approx 0.2 \) signals the breakdown of the perturbative expansion. The true solution for \( \lambda \) should exhibit a zero crossing as the interacting gas crosses over from effective repulsive interactions to effective attractive interactions.

### C.1.2 Fermi-Liquid Theory in Three Dimensions

For a weakly interacting Fermi gas in three dimensions parametrized by the coupling \( g_3 = (k_F a) \), the interaction function gives [238]

\[
F^a(\theta) = \frac{2g_3}{\pi} \left[ 1 + \frac{2g_3}{\pi} \left( 1 + \frac{\cos \theta}{2 \sin (\theta/2)} \ln \frac{1 + \sin \left( \theta/2 \right)}{1 - \sin \left( \theta/2 \right)} \right) \right],
\]

\[
F^a(\theta) = -\frac{2g_3}{\pi} \left[ 1 + \frac{2g_3}{\pi} \left( 1 - \frac{\sin \theta}{2} \ln \frac{1 + \sin (\theta/2)}{1 - \sin (\theta/2)} \right) \right].
\]

One can define the first angular Fourier coefficients as above to arrive at

\[
F_0^a = \frac{2g_3}{\pi} + \frac{8(\ln 2 + 2)}{3\pi^2} (g_3)^2, \quad F_1^a = \frac{8(7\ln 2 - 1)}{5\pi^2} (g_3)^2,
\]

\[
F_0^a = -\frac{2g_3}{\pi} + \frac{8(\ln 2 - 1)}{3\pi^2} (g_3)^2, \quad F_1^a = \frac{8(2\ln 2 + 1)}{5\pi^2} (g_3)^2.
\]

In terms of Landau parameters the effective mass \( m^*/m = 1 + F_0^a / 3 \) and the spin susceptibility \( \chi/\chi_0 = (1 + F_1^a / 3)/(1 + F_0^a) \). For a three-dimensional unitary Fermi gas these have been determined to be \( m^*/m = 1.13 \) and \( \chi/\chi_0 = 0.73 \) from a thermodynamic measurement [239, 240]. The three-dimensional transverse spin diffusivity is given by

\[
D_0^\perp = \frac{\chi_0}{\chi} \frac{2\tau_{\perp} \epsilon_F}{3m^*} = \frac{2\tau_{\perp} \epsilon_F}{3m} (1 + F_0^a).
\]
Utilizing the definitions of $\gamma$ and $\lambda$ as described above, the dimensionless effective interaction $\lambda$ can be written in terms of Landau parameters

$$\lambda = \frac{1}{1 + F_0} - \frac{1}{1 + F_1/3}.$$  \hspace{1cm} (C.11)

To first order in the coupling $g_3$ this gives

$$\lambda = \frac{1}{(2g_3/\pi)^{-1} - 1}.$$  \hspace{1cm} (C.12)

which diverges at $g_3 = \pi/2$.

### C.1.3 Comparison of Two- and Three-Dimensional Spin Transport

I plot the dimensionless effective interaction $\lambda_0 \equiv \lambda m^*/m$ versus the coupling $g$ ($g_3$) in two (three) dimensions. The conceptual simplicity of $\lambda_0$ is that it eliminates $\tau_\perp$. Further, $\lambda_0$ has two contributions: $F_0^a$, corresponding to the effective magnetic field produced by the local magnetization and $F_1^a$, corresponding to a spin vector potential created by a local spin current. The latter has no analogue for weakly interacting fermions and could be constrained by these experiments if all other Landau parameters are known.

Figure C.1a shows measurements of $\lambda_0$ plotted versus the coupling $- (g_3)^{-1}$ for a three-dimensional gas. The solid lines show a kinetic theory calculation while the dashed lines show the Fermi liquid theory prediction. $\lambda_0 \approx -0.2$ at unitarity, smaller in magnitude than $2.1 \leq \lambda \leq 2.7$ in liquid $^3$He [241]. Combined with $F_0^a = 1.1(1)$ from thermodynamic measurements [239, 240], this implies that $F_1^a \approx 0.5$ for a unitary Fermi gas. This is near the upper limit to be consistent with $F_1^a = 0.4(1)$ determined from $m^*$ in a balanced gas [239, 240], since Fermi Liquid theory requires $F_1^a < F_1^a$ [242].

Figure C.1b shows measurements of $\lambda_0$ plotted versus the coupling $-g^{-1}$ for a two-dimensional gas. Solid circles show data at $(T/T_F)_i = 0.31(2)$, open squares show data at $(T/T_F)_i = 0.21(3)$, solid lines show the kinetic theory prediction, and dashed lines show the Fermi liquid theory calculation. As compared to the data in three dimensions, $\lambda_0$ is roughly two times greater at peak.

### C.2 Finite Pulse Length

An important difference between the spin-echo experiment described in Chapter 5 and many theoretical treatments of spin-echo sequences is that the $\pi$ pulse that causes the spin spiral to untwist is assumed to happen instantaneously whereas in the experiment it has a finite duration. The spin echo pulses in the experiment can be carried out in $<10 \mu s$, but such high intensities cause transient field noise which impacts the measured magnetization. Therefore, the $\pi$ pulses used are currently limited to $40 \mu s$ in duration.

Working towards quantifying the effect of the pulse length in a spin-echo sequence I perform a simulation of the precession and manipulation of a set of spins pinned to a lattice. I model the precession of a single spin in a magnetic field using the Bloch equations. The spin’s magnetization is given by the vector in spin-space $\mathbf{M} = \langle M_x, M_y, M_z \rangle$ and it evolves in time according to

$$\partial_t \mathbf{M} = \gamma_0 \mathbf{M} \times \mathbf{B} + \text{(diffusive terms)}.$$  \hspace{1cm} (C.13)
Figure C.1: The dimensionless effective interaction $\lambda_0$ in (a) three dimensions versus $-(k_F a)^{-1}$ at $(T/T_F) = 0.18(4)$ and (b) two dimensions versus $\ln(k_F a_{2D})$ at $(T/T_F)_i = 0.21(3)$ (open squares) and $(T/T_F)_i = 0.31(2)$ (closed circles). The solid lines show a kinetic theory calculation. The dashed lines show the perturbative calculation from Fermi Liquid theory.

where $B = (B + x_3 \Delta B) \hat{x}_3$ is the magnetic field, $\Delta B$ is the magnitude of the magnetic field gradient, $\gamma_0$ is the gyromagnetic ratio, and $x_j$ denotes spatial direction with $j \in \{1, 2, 3\}$. In this section I ignore any diffusive terms (see below for their inclusion).

The propagation of an individual spin in spin-space can be described using a series of rotation matrices. In general, the spin after a time interval $\Delta t$ is given by

$$ M(t + \Delta t) = R_z(\theta) R_{rf} M(t) $$

(C.14)

where $R_z$ and $R_{rf}$ are rotation matrices encapsulating the effects of precession in a magnetic field and rotation from an applied rf pulse, respectively. Here $R_z(\theta)$ is given by

$$ R_z(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

(C.15)

and describes a rotation around the $z$-axis in spin-space by an amount $\theta = \Delta \mu x_3 \Delta B \Delta t$ where $\Delta \mu$ is the difference in magnetic moment of the two spin states.

The effect of an rf pulse is described in terms of a series of rotations. For a pulse of duration $\Delta t$ with Rabi frequency $\Omega$, $R_{rf}$ is given by

$$ R_{rf} = R_z(\phi) R_y(\beta) R_z(\Omega_g \Delta t) R_y(-\beta) R_z(-\phi) $$

(C.16)

where $\Omega_g = \sqrt{\Delta^2 + \Omega^2}$ is the effective Rabi frequency, $\beta = \tan^{-1}(\Delta/\Omega)$, and $\phi$ is the phase angle.
of the pulse. Due to the presence of the magnetic field gradient the pulse is detuned by an amount
\[ \Delta = \Delta \mu x \Delta B. \] Each \( R \) is a rotation matrix about the \( x \), \( y \), or \( z \) axis in spin-space.

I implement this simulation by breaking the spin-echo sequence into segments of precession and
segments of rotation during an rf pulse. As I am ignoring diffusive terms, the processes of precession or
rotation occur independent of any other effect and so a single time step is required for each. However, the
appeal of expressing the dynamics of spin in this form is that it allows for an easy discretization of time
which will be included in the next section. Nonetheless, this simulation is sufficient to demonstrate the
effect of the magnitude of the spin spiral pitch \( \ell_{\text{spiral}} = \frac{4\pi}{\hbar} \left( \Delta \mu \Delta B \right) \) as discussed in Chapter 5. The
effect of the magnetic field gradient and the precession time on the measured magnetization is shown in
Fig. C.2. Here, a purely transverse magnetization is initialized, and allowed to follow the typical spin
echo protocol. In Fig. C.2a the transverse magnetization is seen to have spread in spin-space causing an
apparent decrease in the measured magnetization. Figure C.2b shows that as the gradient is increased
(decreasing spin helix pitch) the amplitude of the magnetization at the echo time decreases. However,
the effect is quite small. For typical experimental parameters where the gradient is 20 G/cm and the
maximal time is 2.0 ms, \( \ell_{\text{spiral}} \sim 3.2 \mu m. \)

C.3 Finite Difference Methods

It is straightforward to combine the rotation matrices describing the rf pulses with a finite difference
method to solve the full differential equation describing the demagnetization dynamics including the
Leggett-Rice effect. This method does not determine the spin transport parameters but can be used to
simulate data that includes finite size and finite pulse length effects to see if trends in the data are
reproduced. The exploration of the numerical methods presented here were inspired by [243, 244] where
trap geometry, system size, and boundary conditions were considered.

First, I rescale Eq. (5.9) to be dimensionless
\[ \partial_t \mathbf{m} = \xi \mathbf{s}_3 \times \mathbf{m} + \frac{1}{1 + \alpha^2 m^2} \left[ \nabla^2 \mathbf{m} + \alpha (\mathbf{m} \times \nabla^2 \mathbf{m}) \right] \]  \hspace{1cm} (C.17)
where \( \tau = t((\Delta B)^2 D^z_0)^{1/3} \) is the dimensionless time, \( \xi = x_3((\Delta B/D^z_0)^{1/3} \) is the dimensionless spatial coordinate, \( m = M/M_0 \) is the normalized magnetization, and \( \alpha = \gamma M_0 \). I decompose Eq. (C.17) into components in spin space and numerically solve by discretizing the derivatives on a grid. I use a second-order central difference for the spatial derivative and a Runge-Kutta method for the time derivative.

The computational scheme is to first choose a spatial grid size (301 points). I fix the boundaries to have zero magnetization. I initialize the magnetization with \( M = (0, \sin \theta, \cos \theta) \) where \( \theta \) is the tipping angle of the initialization pulse in the spin-echo sequence. I then step forward the numerical method until the time of the \( \pi \) pulse. Then I perform a finite length \( \pi \) pulse. Next I repeat the precession time by stepping forward the numerical method. Lastly, I record the individual components of the magnetization and can calculate the transverse magnetization or perform a 20 \( \mu s \) \( \pi/2 \) pulse to generate a Ramsey fringe from which I can apply the same analysis techniques as discussed in Chapter 5 to determine the transverse magnetization. Note that during the finite pulses the total rotation angle is broken into segments based on the time step of the numerical method. I calculate the magnetization after each rotation and then use the new magnetization for an iteration of the numerical method and repeat for each time step during the rf pulse.

Some results exploring the duration of the rf pulses and the size of the cloud are shown in Fig. C.3. I first test the numerical method with parameters that match the experimental setup and infinitely fast rf pulses. I choose a diffusivity \( D^z_0 = 2h/m \), a Leggett-Rice parameter \( \gamma = 1.0 \), a magnetic field gradient 20 G/cm, and a system size of 10 \( \mu \)m which matches the tight direction of the trap. The results are shown in Fig. C.3a. Using the analysis procedure outlined in Chapter 5 I find \( D^z_0 = 1.4h/m \) and \( \gamma = 0.7 \). Figure C.3b-d show results with increasing rf pulse duration. For a 40 \( \mu s \) \( \pi \) pulse (Fig. C.3b) the initial magnetization for a \( \pi/4 \) and 3\( \pi/4 \) pulse begin to differ from each other and the theoretically expected value of \( \sim 0.7 \). The difference in the initial magnetization increases as the duration of the pulse is made longer shown in Fig. C.3c (80 \( \mu s \)) and Fig. C.3d (120 \( \mu s \)). However, the best-fit spin transport parameters remain largely unchanged when increasing the duration of the rf pulses. A difference in the initial magnetization for \( \theta = \pi/4, 3\pi/4 \) is also seen in the data presented in Chapter 5. The simulation suggests that this could result partially from the duration of the rf pulses. To test the hypothesis that finite rf pulses impact the initial magnetization one could experimentally increase the duration of the rf pulses and observe if the discrepancy in initial magnetization becomes more pronounced.

The finite difference simulation can be used to explore the effects of finite size and inhomogeneous density distributions. The simulation can also be extended to include longitudinal dynamics. While a substantial amount of work would be required to determine the truncation errors and global errors as well as large computational time to run the simulation with small step sizes, the initial exploration I performed showed that this simulation would be useful to capture some of the experimental details of the spin transport problem.
Figure C.3: Finite difference simulations with finite rf pulses. Data is simulated with various initial values of $M_z$: $M_z = 0$ (black), $M_z = -0.707$ (red) and $M_z = 0.707$ (blue). (a) For instantaneous rf pulses the correct values of $M_z$ are reconstructed. For (b) 40 $\mu$s, (c) 80 $\mu$s, and (d) 120 $\mu$s pulses the values of $M_z$ when $M_z \neq 0$ begin to differ from the expected values.
Bibliography


BIBLIOGRAPHY


