DYNAMIC TRADING IN A LIMIT ORDER BOOK: CO-INTEGRATION, OPTION HEDGING AND QUEUEING DYNAMICS

by

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Graduate Department of Statistical Sciences
University of Toronto

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Abstract

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We show how an agent dynamically trades in a limit order book, accounting for asset co-movement, exposure to a contingent claim and queue position of limit orders. Inspired by real-world trading problems, we propose models that capture relevant market dynamics and formulate stochastic control problems for the agent. We derive the associated dynamic programming equations and prove existence and uniqueness of the solutions under mild conditions. We provide numerical schemes to solve the equations and address convergence issues, when appropriate. We calibrate the models to real data and demonstrate the optimal strategies by numerical examples. Our work is complete in that it bridges the gap between abstract mathematical theory and practical implementation.

For an agent executing a basket of assets, we show how she can improve her strategy by employing information from co-movements of multiples assets, even if some assets are outside of the basket. We derive the agent’s trading speed in closed-form and use simulations to demonstrate the performance of the optimal strategy. Furthermore, for an agent who takes a short position in a contingent claim, we show how she maximizes her expected utility of wealth by trading the underlying asset. She employs market orders to keep the inventory on target to replicate the payoff of the claim and uses limit orders to build the inventory at a favorable price and boost expected terminal wealth by completing round-trip trades that earn the spread. Finally, we show how an agent incorporates queue
position of her limit order in the decision of whether to cancel the order or let it rest. The extra information on queue position enables the agent to better predict the execution time of the order and the time that the limit order book switches regime. A simulation study demonstrates that the optimal strategy significantly outperforms a benchmark that ignores the effect of queue position.
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Chapter 1

Introduction

1.1 Background

In the old days, financial markets were quote-driven, where designated market makers quote the buy and sell prices and other market participants trade with them. Nowadays, most major exchanges including NYSE, Nasdaq, CME and LSE have switched to electronic trading systems as the primary trading mechanism. These trading systems are predominantly organized through the limit order book (LOB) in which any market participant can submit limit orders (LOs) and acts as a market maker.

The fundamental change in trading systems has lead to several consequences. One of them is the surge in algorithmic trading (AT). AT is the process of employing computer programs to perform trades that were traditionally carried out by human traders. Today, a large proportion of trades are initiated by computer algorithms. Gerig (2015) estimated that over the period between 2002 and 2012, high-frequency trading (a specific type of algorithmic trading that relies on high speed) accounted for approximately 55% of trading volume in US equity market, 40% in European equity markets, and is quickly growing in Asia and other markets such as fixed income, foreign exchange and commodities. There are various types of AT. Based on their functionalities, they roughly fall into the three
categories below. Note that we sometimes refer to these algorithms as agents. It is to be understood that the agents are the algorithms themselves, not actual human traders.

**Order execution.** These are algorithms deployed by an agency broker who executes a large order on behalf of a client. The execution horizons are typically short, ranging from a few hours to a few trading days. However, the size of the entire order is often much larger than the available liquidity on the market. Therefore it must be broken into smaller child orders to reduce its market impact. Determining the timing to send these child orders is critical, and there are two conflicting goals: if the orders are executed too fast, they incur substantial transaction costs; on the other hand, if the orders are executed too slowly, the price might move away. The key in designing order execution algorithms lies in balancing between transaction cost and uncertainty in execution price.

**Market making.** These are algorithms that take over the role of designated market makers in the LOB. A typical market making agent posts LOs on both sides of the market. The agent makes a profit by completing a “round-trip” trade that consists of two steps. In the first step, the agent’s limit buy order (LBO) matches with another agent’s market sell order (MSO), and shortly after, in the second step, the agent’s limit sell order (LSO) matches with another agent’s market buy order (MBO), or vice versa. The agent has to hold either a long or a short position for a moment, and if the price does not change during this period, she earns a profit that equals to the difference between the best buy and best sell price, i.e., the spread. Of course, the price might move away before the second step of the round-trip trade is fulfilled. Therefore, managing inventory risk is essential for the agent. Furthermore, there is another issue that significantly affects the agent’s profitability - *adverse selection*. Adverse selection happens when the agent’s LO is taken by an informed trader, who possesses better information on future price movement. In this case, it is likely that the price moves in an adverse direction for the agent during the holding period and the agent incurs a loss. To improve her profitability, the agent must try to differentiate between informed and uninformed traders and avoid
Arbitrage. These are algorithms designed to profit from short-term price deviations. There are various ways that price can temporarily depart from its stationary level. For example, one type of these algorithms engages in cross-venue arbitrage, ensuring that prices for the same asset on different exchanges are the same. These algorithms rely on expensive hardware that can send and receive signals in milliseconds. In recent years, the entrants of new trading platforms such as BATS in the US and Chi-X in the UK have created more opportunities for these algorithms to profit from. Another type of the arbitrage algorithms profits from making short-term forecasts on the price, for example, the statistical arbitrage strategies, which attempt to profit from short-term deviations of multiple assets from their stationary relationship.

1.2 The Limit Order Book

Most modern exchanges are organized through the LOB. It is a collection of LOs, which represent interests in buying or selling an asset at a particular price. An LO typically contains the following information: type (buy or sell), price, volume and time of submission. For example, an agent can submit an LO indicating that she is willing to buy 100 shares of INTC\(^1\) for the price 34.51$ per share. Once an LO enters the LOB, it rests on the book until it is matched with an MO or cancelled.\(^2\) For modelling purpose, it is usually convenient to aggregate the total volume of LOs that are of the same type and are on the same price level. By doing so we obtain a sequence of triplets \((\text{type}, \text{price}, \text{volume})\) and we will use that to represent the LOB. Table 1.1 shows a snapshot of an LOB for the ticker INTC at Oct 1, 2014 10 AM EST. On the first level of the buy side of the book, the price is 34.51$ and the volume is 2305. It indicates that there is a total volume of 2305 LOs available at a price 34.51$. Note that the best buy price is always lower than

\(^1\)Intel Corporation on Nasdaq Stock Market

\(^2\)Some exchanges allow participants to use other types of LO, such as iceberg orders.
the best sell price. The difference between them is called the *spread*.

<table>
<thead>
<tr>
<th>Buy</th>
<th>Sell</th>
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<tbody>
<tr>
<td>Price</td>
<td>Volume</td>
</tr>
<tr>
<td>34.51</td>
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<tr>
<td>34.50</td>
<td>2680</td>
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<tr>
<td>34.49</td>
<td>3177</td>
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<td>34.48</td>
<td>3638</td>
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<td>34.47</td>
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Table 1.1: LOB snapshot for the first 5 price levels (INTC at Oct 1 2014 10:00:00 AM EST)

An agent who is patient posts LOs and waits for counterparties to trade with her. Alternatively, when she requires urgency, she executes MOs which immediately match with the resting LOs. However, she has to cross the spread and receives a worse price. Take Table 1.1 as an example. Suppose an agent execute a MBO of volume 2,500. This MBO will first match with the LSOs at price level 34.52$, for a volume of 2,354. After it depletes the first level, the remaining volume 146 will match with the LOs at the next price level, 34.53$, changing the volume of available LSOs from 3,714 to 3,568. The average price is 
\[
\text{average price} = \frac{34.52 \times 2354 + 34.53 \times 146}{2500} = 34.5206\% \text{ per share.}
\]
Note that if the agent chooses to post an LBO instead of executing an MBO, she can post the order at a price 34.51$ or lower. If the LBO eventually matches with an MSO, the execution price will be lower.

When an MO matches with the LOs resting on the LOB, it always match with the LOs at the best price first. There are various different rules governing how LOs of the same price level are selected. Two most popular rules are *time-priority* and *pro-rata*.

**Time-priority.** LOs are ranked based on their time of arrival. When an MO arrives, it matches with LOs starting from the earliest one.

**Pro-rata.** When an MO arrives and does not deplete a certain price level, all LOs at that price level are partially filled with the filled volume proportional to the volume of each LO.
1.3 Literature Review

Research works on algorithmic trading were pioneered by Almgren and Chriss (2001). They considered the problem that an agent liquidated an asset over a finite horizon and derived the optimal liquidation speed analytically. Their work laid down the foundation of optimal execution and is one of the most cited pieces in this field. It was also the first work that explicitly modelled price impact. In their setup, aggressive selling (buying) pushed the asset’s midprice down (up), which was termed permanent price impact. The cost of crossing the spread was captured by that the agent received a price worse than the midprice, and it is termed temporary price impact. Both permanent impact and temporary impact were assumed to be linear in the liquidation speed. Their idea of decomposing price impact into a permanent one and a temporary one has been adopted by many of the later works.

There are various extensions to Almgren and Chriss (2001). Obizhaeva and Wang (2013) proposed the transient price impact, which bridged the gap between permanent and temporary price impact. Their work was later generalized by Alfonsi et al. (2010), who allowed for general shape functions for the LOB. See also Almgren (2012), who incorporated stochastic volatility and liquidity, Gatheral and Schied (2011), for modelling the asset price as a geometric Brownian motion and Jaimungal and Kinzebulatov (2013), who considered the optimal execution problem with a price limiter.

The above studies focus on optimal execution. For market making, Avellaneda and Stoikov (2008) first introduced a framework where an agent posted LOs on both sides of the LOB to maximize an exponential utility. In their work, the agent controlled the distance between her LO and the midprice. The further her LO from the midprice was, the less likely that her LO would be filled when an MO arrived. Their work was also the first in considering optimal trading strategies using LOs. Guéant et al. (2013) considered a similar problem with upper and lower bounds on the agent’s inventory. See also Cartea et al. (2014) for a model that incorporated richer dynamics of MOs arrivals,
adverse selection and a predictable component of the midprice’s drift. While using LOs is essential in market making, it is also possible to consider optimal execution with LOs, see, e.g., Bayraktar and Ludkovski (2014) and Guéant et al. (2012). A related but slightly different problem is to incorporate dark pools. For works in this direction, see e.g., Kratz and Schöneborn (2015).

In recent years, research in algorithmic trading has attracted a lot of attention. Below we select a few topics and briefly discuss recent developments in each of them.

**Limit and market orders.** There have been some efforts in combining LOs and MOs in optimal execution and market making. These problems are usually formulated as combined regular and impulse control problems. For optimal execution, see Cartea and Jaimungal (2015c) and Chevalier et al. (2016), and for market making, see Guilbaud and Pham (2013) and Guilbaud and Pham (2015). Our work in Chapter 3 continues along this direction by considering the problem of option hedging with limit and market orders.

**Order-flow imbalance.** Short-term price movements are often driven by the imbalance between demand and supply of the asset. Cont et al. (2014) proposed a measure called order flow imbalance that could be computed by aggregating total volume of MOs over a moving time window. They showed that their measure possessed predictive power on the direction of next price movement. Bechler and Ludkovski (2015) modelled the imbalance measure using a mean-reverting process and considered the market making problem in the presence of the imbalance measure. Cartea et al. (2015a) proposed another measure that could be directly calculated from the volumes in the LOB and showed that their measure also had predictive power over next price movement. They modelled the imbalance measure using a discrete valued Markov chain and solved the market making problem in the presence of the imbalance measure.

**Statistical arbitrage.** Besides order flow imbalance, co-movement of multiple assets also provide predictive power on short-term price movements. This statistical relationship
is often modelled using co-integration, which refers to some linear combination of the assets being stationary. Tourin and Yan (2013) proposed a continuous-time model for co-integration of two stocks and derived the optimal trading strategies. Their work was generalized to multiple assets by Cartea and Jaimungal (2016b) and Lintilhac and Tourin (2016). Leung and Li (2015) used a different setup. They formulate the problem as a double stopping problem where the agent chose the time to enter and exit a trade. See also Lei and Xu (2015) for a similar setup. Another approach taken by Ngo and Pham (2014) was to formulate the problem as an optimal switching problem. Our work in Chapter 2 contributes to this stream of literature by incorporating statistical arbitrage into optimal execution.

**Event inter-arrival time.** Events in the LOB often exhibits *clustering* behavior, that is, when an event occurs, it is more likely to observe another event occurring within a short time. A popular model for clustering of events is the Hawkes process (c.f. Hawkes (1971)), a self-exciting process in which the intensity of future events jumps up when an event occurs. There is an abundance of literature discussing how to apply Hawkes process in a financial setting, see, for example, Bacry and Muzy (2014). For works that relate Hawkes processes to other ‘macro’ price models, Jaisson et al. (2015) showed that a Hawkes process converges to a Cox-Ingersoll-Ross (CIR) process under suitable scaling. For optimal execution with a Hawkes model for incoming MOs, see Alfonsi and Blanc (2016) and for market making, see Cartea et al. (2014). As an alternative to Hawkes processes, Fodra and Pham (2015b) proposed a semi-Markov model. The same authors applied the model to a market making problem in Fodra and Pham (2015a).

**Queueing dynamics of the LOB.** Many exchanges adopt the time-priority rule as the matching mechanism for the LOB. Under this rule, each price level of the LOB can be viewed as a first-in-first-out queue: an LO always starts from the back of the queue and moves forward when there is a cancellation of LO in front or an MO arrival. This fact has important implications for an agent who uses LOs since the position of her LO
dictates the time that the order will be executed. Therefore, understanding the queueing dynamics of the LOB is an important question in algorithmic trading research and a lot of effort has been devoted. Cont et al. (2010) used a queueing system to model the entire LOB. A simpler and more tractable model can be found in Cont and De Larrard (2013), in which only the best levels of the LOB were modelled. Guo et al. (2015) studied the dynamics of queue position under diffusion limit. For optimal placement problems under fluid (deterministic) queue models, see Maglaras et al. (2014) and Maglaras et al. (2015). Our work in Chapter 4 contributes to this stream of literature by showing an agent chooses the time to place or cancel a limit order in such queues.

1.4 Main Results and Outline

In this thesis, we address three different problems arising in the realm of algorithmic trading. In each problem, we start with a real world trading problem and construct a suitable stochastic model for the market dynamics. We formulate stochastic control problems and use tools such as dynamic programming and numerical partial differential equations to solve them. Verification theorems are provided and convergence of numerical schemes are addressed, when appropriate. We also calibrate the models to real data and illustrate the optimal strategies by numerical examples. In the subsequent subsections, we present a brief description of each problem and discuss the main result. Finally, in Chapter 5, we summarize the main findings in this thesis and pinpoint some directions for future works. Appendix A and B contain acronyms and background material.

1.4.1 Trading Co-Integrated Assets with Price Impact

Chapter 2 is an attempt to unite optimal execution and statistical arbitrage. In the previous literature on optimal execution, interactions between multiple assets are generally overlooked - most works dealt with a single asset; even when multiple assets were indeed
considered, they were often assumed to be correlated only. For example, in a multivariate
version of Almgren and Chriss (2001), it is assumed that the assets’ midprices $P_t \in \mathbb{R}^n$
have the following dynamics

$$dP_t = \nu_t dt + \sigma^\top W_t, \quad (1.1)$$

where $\nu_t \in \mathbb{R}^n$ is the agent’s liquidation speed, $\Sigma = \sigma^\top \sigma$ is the covariance matrix of
asset return and $W_t \in \mathbb{R}^n$ is a standard Brownian motion. The agent seeks the optimal
$\nu_t$ to maximize her expected terminal wealth while penalizing a running penalty. It can
be shown that the optimal $\nu_t$ is a deterministic function of time.

In this chapter, we consider the problem in which the assets are correlated and co-
integrated with the following dynamics

$$dP_t = g(o_t) dt + dS_t, \quad (1.2a)$$
$$dS_t = \kappa (\theta - S_t) dt + \sigma^\top dW_t, \quad (1.2b)$$

where $dS_t$ represents the co-integration component of the midprice dynamics and $g(o_t)$
denotes the effect of order flow from all market participants, including the agent’s trades,
on midprices.

In (1.2a), the term $g(o_t) dt$ generalizes the term $\nu_t dt$ in (1.1), by incorporating order
flows from the other market participant’s trading activity. It allows the agent to ad-
just her trading speed by anticipating the other market participant’s aggregated trading
activity. Moreover, the term $dS_t$ generalizes the term $dW_t$ in (1.1) by incorporating
co-integration dynamics. In econometrics literature, a discrete time version of (1.2b) is
often used to model co-integration, which refers to the statistical relationship that cer-
tain linear combination of asset prices is stationary, though each individual asset price is
nonstationary. As a final point, our setup allows the agent to trade only a subset of the
assets, while still being able to benefit from order flow and the co-integration dynamics.

We solve the associated stochastic control problem and derive the agent’s optimal
trading speed. Our result is similar to that of Almgren and Chriss (2001), but has two additional correction terms: one for co-integration and the other for the order flow and a predefined schedule. Moreover, the optimal trading speed is linear in the midprices and the inventory, where the coefficients solve a matrix Riccati differential equation and a linear matrix PDE. Under mild conditions, we show that both the Riccati equation and the matrix PDE admit bounded solutions, and we use the result to provide a verification theorem for the stochastic control problem.

Finally, to illustrate the performance of the optimal strategy, we calibrate the model parameters using five stocks from the Nasdaq exchange and conduct a simulation study. The result shows that the optimal strategy achieves a relative saving of 4 to 4.5 basis points over the Almgren-Chriss strategy.

1.4.2 Option Hedging with Limit and Market Orders

In Chapter 3, we show how an agent can hedge a short position in an option using LOs and MOs. In the classical option pricing theory, an option can be hedged by holding a portfolio of the underlying asset in which the number of shares held equals to the option delta. The classical theory, however, does not provide a trading algorithm for rebalancing such a portfolio. Moreover, important market microstructural features such as transaction cost of MOs and adverse selection are generally overlooked. Our work in Chapter 3 fills this gap.

In our setup, at time 0, an agent sells an option that expires at a future time $T$. She trades the underlying asset using LOs and MOs between time 0 and time $T$ to maximize her expected utility. LOs are updated by the agent in continuous time and MOs are modeled as impulse controls. We formulate the problem as a combined regular and impulse control problem and solve it using dynamic programming. We establish a comparison principle for the associated dynamic programming equation (DPE) and prove a verification theorem for the value function. To solve the DPE, we provide a numerical
scheme and prove its convergence.

We demonstrate the performance of the optimal strategy by a numerical example in which the agent takes a short position in European options written on E-Mini S&P from CME. The agent’s optimal strategy comprises two parts: hedge and speculate. She employs MOs to ensure that the inventory is on target to replicate the terminal payoff of the option and LOs to build the inventory at favorable prices. In addition, she adjusts the volume of the LOs she posted to execute round-trip trades that earn the spread.

1.4.3 Optimal Decisions in a Time Priority Queue

In Chapter 4, we show how the queue position influences the decisions on whether to cancel an existing LO or let it rest. Our primary goal is to answer the following questions: when is it optimal for an LO to enter the LOB and when is it optimal to cancel an existing LO?

To achieve this goal, our first step is to perform empirical studies on the dynamics of the LOB. We employ data for the ticker INTC from the Nasdaq exchange and analyse various LOB events, including addition and cancellation of LOs, MO arrival, distribution of MO size and distribution of replenished queue size. As far as we know, some of our empirical findings are novel. For example, we find that the intensity of LO cancellation in front of a particular LO depends strongly on the queue position of that LO, but not so much on the queue length. We also find that a large proportion of MOs deplete the entire queue at the best price level, but they rarely go beyond that level.

Based on our empirical studies, we propose a queueing model for a single price level on one side of the LOB. An important feature of our model is the inclusion of a regime, which is a function of volume imbalance and can be interpreted as an abstract trade signal (TS). The gain or loss of a filled LO depends on TS: when TS is in a gainful (adverse) regime, a filled LO is more likely to be a gain (loss). Moreover, the dynamics of the LOB depend on TS as well.
We then apply our queueing model by considering the problem of an agent maximizing her expected utility through placing and cancelling LOs. She keeps track of three state variables: the TS, the queue length, and the queue position of her LO. We present the agent’s optimal strategy after calibrating the model to real data. Our result shows that even at an adverse regime, the agent might still be willing to stay in the queue. This is because she wishes to obtain a good queue position when TS switches to a gainful regime. To illustrate the performance of the optimal strategy, we conduct a simulation study. The result shows that for the same level of standard deviation of terminal wealth, the optimal strategy achieves a mean that is 2.5% higher; or a standard deviation that is 8.8% lower for the same level of mean.
Chapter 2

Trading Co-Integrated Assets with Price Impact

2.1 Introduction

How to optimally execute a large position in an individual stock has been a topic of intense academic and industry research during the last few years. In contrast, there is scant work on the joint execution of large positions in multiple assets. One of the early papers on optimal execution is by Almgren and Chriss (2001) who consider a discrete-time model where the strategy employs market orders (MOs) only. Extensions of their work, where the agent employs MOs and/or limit orders, include Almgren (2012), Kharroubi and Pham (2010), Gueant et al. (2012), Forsyth et al. (2012), Jaimungal and Kinzebulatov (2013), Guilbaud and Pham (2013), and Cartea and Jaimungal (2015c). In the extant literature, if the agent liquidates a portfolio of different assets, these are considered to be correlated, but do not include co-integration, nor do they include the market impact of the order flow from other market participants. This chapter fills this gap. We show how an agent executes a basket of assets employing a framework that models the price impact of order flow, and employs the information provided by the co-integration factors that
drive the joint dynamics of prices – which may include other assets she is not trading in.

In our framework, the agent’s MOs have both temporary and permanent price impact. Temporary impact results from the agent’s MOs walking the limit order book (LOB), and permanent impact results from one-sided trading pressure exerted on prices. In contrast to most of the literature (Cartea and Jaimungal (2016a) and Cartea and Jaimungal (2014) being two notable exceptions), here, MOs of other market participants are treated in the same way as the agent’s order: market buy orders exert upward pressure on prices, and market sell orders downward pressure on prices. Furthermore, order flow in one asset may impact the prices of co-integrated assets. This cross-effect is partly caused by trading algorithms that take positions based on the co-movements of assets. Such strategies induce co-movement in order flow and liquidity displayed in the LBOs of the co-integrated assets.

In our setup, permanent impact of order flow is linear in the speeds of trading of all market participants (including the agent), and temporary impact is also linear in the agent’s speed of trading. We focus on the execution problem where the agent liquidates shares in \( m \) assets and employs information from a collection of \( n \geq m \) co-integrated assets. The agent maximizes the expected terminal wealth and penalizes deviations from an inventory-target schedule. This scenario appears in many applications in practice. For example, agency traders are often faced with liquidating a basket of Eurodollar\(^1\) futures of consecutive maturities. These contracts are highly co-integrated, and not simply correlated, see the discussion in Almgren (2014).

Our setup is related to that of Gārleanu and Pedersen (2013) in which the authors optimize the discounted, and penalized, future expected excess returns in a discrete-time, infinite-time horizon problem. In their model, prices contain an unpredictable martingale component, and an independent stationary predictable component. The penalty is imposed to account for a version of temporary price impact similar to walking the LOB,

\(^1\)Recall that Eurodollar futures are futures contracts on time deposits denominated in USD, but held in a non-US country.
and they include a permanent price impact which reverts to zero if there are no trades. Passerini and Vazquez (2016) numerically study a continuous-time, finite horizon, version of Gârleanu and Pedersen (2013), and account for crossing the spread or posting limit orders. Our approach differs in five main aspects: (i) our setup is in continuous-time, (ii) the execution horizon is finite, (iii) the agent solves an execution problem where prices are co-integrated (rather than having independent predictable components), (iv) the agent’s MOs have permanent and temporary impact, and (v) the MOs of other market participants also have permanent price impact. Moreover, we provide analytic characterizations of the solution to the execution problem.

To illustrate the performance of the strategy we calibrate model parameters to five stocks (INTC, SMH, FARO, NTAP, and ORCL) traded in the Nasdaq exchange and run simulations for variations of the strategy including different levels of urgency and inventory-target schedules, including/excluding a speculative component which allows repurchases of shares. As benchmark we use the multi-asset version of the Almgren-Chriss (AC) strategy where the agent models the correlation between the assets in the basket, but does not model co-integration or employ additional information from other assets. The agent liquidates a basket consisting of 4,600 shares of INTC and 900 shares of SMH which corresponds to 1% and 4% of traded volume over the one hour in which execution occurs.

Additional information from other co-integrated stocks considerably boosts the performance of the strategy. For example, if the level of urgency required by the agent to liquidate the portfolio is high (resp. low) the strategy outperforms AC by 5.5 (resp. 1) basis points. This improvement over AC is due to the quality of the information provided by the co-integrated assets, and due to a speculative component of the strategy which allows the agent to repurchase shares during the liquidation horizon to take advantage of price signals. If the agent is not allowed to speculate, i.e. cannot repurchase shares, the relative savings compared to AC, depending on the level of urgency, are between 0
Finally, we also illustrate how the strategy performs when the agent has access to only one trading day of data, thus parameter estimates are incorrect. We show that the performance of the strategy is broadly the same as that resulting from that when the agent has enough data to obtain correct parameter estimates.

Our model is also related to the literature on pairs trading in that the agent’s strategy benefits from co-integration in asset prices. For example, Mudchanatongsuk et al. (2008) model the log-relationship between a pair of stock prices as an Ornstein-Uhlenbeck process and use this to formulate a trading strategy. More recently, Leung and Li (2015) study the optimal timing strategies for trading a mean-reverting price spread, see also Lei and Xu (2015), and Ngo and Pham (2014). Finally, the work of Tourin and Yan (2013) develops an optimal portfolio strategy for a pair of co-integrated assets. This is generalized to multiple co-integrated assets in Cartea and Jaimungal (2015a), and Lintilhac and Tourin (2016).

The remainder of this chapter is structured as follows. Section 2.2 presents the model for the co-integrated prices and poses the liquidation problem solved by the agent. Section 2.3 presents the dynamic programming equation and shows the optimal liquidation speeds. Section 2.4 discusses the Nasdaq exchange data employed to estimate the co-integrating factor of prices, and illustrates the performance of the strategy under different assumptions. Section 2.6 concludes and proofs are collected in the Appendix.

2.2 Model

The investor must liquidate a portfolio of assets and has a time limit to complete the execution. One simple strategy is to view each stock in the portfolio independently and employ a liquidation algorithm designed for an individual stock, see e.g. Almgren and Chriss (2001), Bayraktar and Ludkovski (2014), Cartea et al. (2015b). Treating
each stock independently is optimal if the assets in the portfolio do not exhibit any co-movements or dependence.

Here we focus on the general case where a collection of traded assets co-move. Modelling the joint dynamics provides the investor with better information to undertake the liquidation strategy. Ideally, the information employed in the execution strategy is not limited to the constituents of the portfolio to be liquidated, it includes other assets that improve the quality of the information employed in the algorithm. See for example, Cartea et al. (2013b) who show how to learn from a collection of assets to trade in a subset of the assets.

The portfolio consists of \( m \) assets which are a subset of the \( n \)-dimensional vector \( P = (P_t)_{0 \leq t \leq T} \) of midprices that the investor employs in the trading algorithm. The midprices are determined by a co-integration factor and the impact of the order flow from all market participants including the investor’s orders. Specifically we assume that the midprices satisfy the multivariate stochastic differential equation (SDE)

\[
dP_t = dS_t + g(o_t) \, dt,
\]

where \( S \) denotes the co-integration component of midprices and satisfies

\[
dS_t = \kappa (\theta - S_t) \, dt + \sigma^\top dW_t.
\]

Here \( \kappa \) is a \( n \times n \) matrix, \( \theta \) is an \( n \)-dimensional vector, and \( \sigma^\top \) is the \( n \times n \) matrix produced by the Cholesky decomposition of the asset prices’ correlation matrix \( \Sigma \) (i.e. \( \Sigma = \sigma^\top \sigma \)), where the operation \( ^\top \) denotes the transpose operator. As usual we work on the filtered probability space \( (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}) \), and \( W = (W_t)_{0 \leq t \leq T} \) is an \( n \)-dimensional Brownian motion with natural filtration \( \mathcal{F}_t \).

Moreover \( g(o_t) \) represents the effect of order flow \( o = (o_t)_{0 \leq t \leq T} \), with \( o_t \in \mathbb{R}^n \), from all market participants (including the investor’s trades) on midprices, and \( g : \mathbb{R}^n \to \mathbb{R}^n \) is
a permanent price impact function. Below we give a more detailed account of the effect of order flow on the midprice dynamics – for more details see Cartea and Jaimungal (2016a) who discuss the effect of market order flow on asset prices.

The investor wishes to liquidate the portfolio of $m$ assets over a time window $[0, T]$ – the setup for the acquisition problem is similar, so we do not discuss it here. Her initial inventory in each asset is given by the vector $Q_0 \in \mathbb{R}^m$ and she must choose the speed at which she liquidates each one of the assets using MOs only.

We denote by $\nu = (\nu_t)_{0 \leq t \leq T}$ the vector of liquidation speeds, and by $Q^\nu = (Q^\nu_t)_{0 \leq t \leq T}$ the vector of (controlled) inventory holding in each asset. The inventory is affected by how fast she trades and satisfies

$$dQ^\nu_t = -\nu_t \, dt .$$

(2.3)

In our model all MOs have price impact. We assume that price impact is linear in the speed of trading (see Cartea and Jaimungal (2016a) for extensive data analysis illustrating this fact) and treat the order flow of the investor and other market participants symmetrically. In particular, we denote other agents’ aggregated net trading speed by $\mu = (\mu_t)_{0 \leq t \leq T}$, which we assume is Markov\(^2\) with infinitesimal generator $L^\mu$, and assume that is independent\(^3\) of the Brownian motion $W$. Thus, the price impact of order flow is:

$$g(o_t) = -b X^\top \nu_t + \bar{b} \mu_t ,$$

(2.4)

where $b$ is the permanent impact $n \times n$ symmetric matrix and $\bar{b}$ is the permanent impact $n \times n$ matrix from other agents trading activity. $X$ is a $m \times n$ matrix with $X_{ij} = 1_{\{i=j\}}$ and maps the first $m$ elements of an $n$-dimensional vector to an $m$-dimensional vector.

Although permanent impact from order flow is treated symmetrically, here we separate

\(^2\)We can easily include other factors that drive order flow, as long as the joint process, consisting of the driving factors and order flow itself, is Markov.

\(^3\)This independence assumption can also be relaxed.
the agent’s impact from that of other participants should we want to focus on either one when analyzing the strategy.

Therefore, after inserting (2.4) in (2.1), the midprice can be expressed as

\[ P^\nu_t = S_t + b X^\top (Q^\nu_t - Q_0) + \bar{b} \mathcal{M}_t, \tag{2.5} \]

where \( \mathcal{M}_t = \int_0^t \mu_u du \) and we use the notation \( P^\nu_t \) to stress that midprices are affected by the investors (controlled) speed of trading.

Our model for price dynamics is related to that used in optimal ‘pairs trading’ where a speculative strategy is designed to profit from the movement of a collection of co-integrated assets, see Tourin and Yan (2013), Leung and Li (2015), and Cartea and Jaimungal (2015a). Our work is different in that the agent’s objective is to execute a basket of co-integrated assets, and more importantly, order flow from all market participants, including the agent’s own trades, is explicitly modelled in the price dynamics, and (as discussed below, we account for temporary price impact).

In addition to permanent price impact, the investor receives worse than quoted mid-prices because her MOs walk the LOBs. This price impact is temporary and only affects the prices the investor receives when selling shares. The execution prices are given by

\[ \tilde{P}^\nu_t = X P^\nu_t - A \nu_t. \tag{2.6} \]

\( A \) is an \( m \times m \) positive definite matrix, so the temporary impact is linear in the speed of trading. Without loss of generality, we assume that the first \( m \) coordinates of \( P^\nu_t \) correspond to the assets the investor trades.

In this setup, the LOBs recover immediately after the execution of the MOs – see Almgren (2003), Alfonsi et al. (2010), Kharroubi and Pham (2010), Gatheral et al. (2012), Schied (2013), Guéant and Lehalle (2013) for further discussions and generalizations.

Finally, the investor’s cash from liquidating shares in the \( m \) assets is denoted by
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\[ X^\nu = (X^\nu)_{0 \leq t \leq T} \] and satisfies the SDE

\[ dX^\nu_t = (X^\nu_t \nu_t - a^\nu_t \nu_t) \nu_t dt. \tag{2.7} \]

### 2.2.1 Performance criteria and value function

The investor aims at liquidating the portfolio by the terminal date \( T \) and maximizes expected terminal wealth while penalizing deviations from a deterministic target inventory \( \mathcal{Q}_t : \mathbb{R}^+ \to \mathbb{R}^m \) satisfying \( \mathcal{Q}_0 = Q_0 \) and \( \mathcal{Q}_T = 0 \).

Her performance criteria is

\[ H^\nu(t, x, p, q, \mu) = \mathbb{E}_{t, x, p, q, \mu} \left[ X^\nu_T + (P^\nu_T)^\top X^\nu_T - (Q^\nu_T)^\top \alpha Q^\nu_T - \phi \int_t^T (Q^\nu_u - \mathcal{Q}_u)^\top \hat{\Sigma} (Q^\nu_u - \mathcal{Q}_u) \, du \right], \tag{2.8} \]

where the expectation operator \( \mathbb{E}_{t, x, p, q, \mu}[\cdot] \) represents expectation conditioned on (with a slight abuse of notation) \( X^\nu_t = x, P^\nu_t = p, Q^\nu_t = q, \) and \( \mu_t = \mu, \) and \( \hat{\Sigma} \) is an \( m \times m \) sub-matrix of the correlation matrix \( \Sigma \) corresponding to the \( m \) assets that are being traded. Her value function is

\[ H(t, x, p, q, \mu) = \sup_{\nu \in \mathcal{A}} H^\nu(t, x, p, q, \mu), \tag{2.9} \]

where \( \mathcal{A} \) is the set of admissible strategies consisting of \( \mathcal{F} \)-predictable processes such that \( \int_0^T |\nu^i_u| \, du < +\infty, \mathbb{P}\text{-a.s.} \), for each asset \( i \) the investor is liquidating. The liquidation speeds are not restricted to remain positive – we return to this point in Section 2.4 when we analyze the empirical performance of the trading strategy.

The first term on the right-hand side of the performance criteria (2.8) is the terminal cash. The second term represents the cash obtained from liquidating all remaining shares at the end of the trading window at the price \( P \). The third term is the market impact.
costs from liquidating final inventory which are encoded in the positive definite matrix $\alpha > 0$.

Furthermore, the term in the second line of (2.8) represents a running inventory-target penalty where $\phi \geq 0$ is a penalty parameter. This inventory penalty does not affect the investor’s revenues, but affects the optimal liquidation rates. When the value of the inventory penalty parameter $\phi$ is high, the strategy is forced to track closely the target $Q_t$. This is similar in spirit to Cartea and Jaimungal (2014) who develop a trading strategy to target VWAP, i.e. volume weighted average price, and similar to Bank et al. (2015) who study how to target general positions (without any price dynamics).

For example, when $Q_t = 0$ over the execution window, $\phi$ may be interpreted as an urgency parameter. High values of $\phi$ correspond to the trader wishing to rid herself of more inventory early on. This particular target is justified in a setting where the investor considers model uncertainty – i.e. she is ambiguity averse. Cartea et al. (2013a) show that including a running penalty that curbs the strategy to draw down inventory holdings to zero is equivalent to the agent considering alternative models with stochastic drifts. In that setting, the higher the value of $\phi$, the less confident is the agent about the trend of the midprice, so the quicker the strategy executes the shares.

### 2.3 Optimal Portfolio Liquidation

In this section we derive the optimal liquidation rates. Our first step is to rewrite the control problem using the fundamental price $S$ as a state variable. Using (2.7) and integration by parts, the investor’s wealth $X_T^\nu$ can be written as

$$X_T^\nu = \int_0^T (\mathbf{X} S_t - a \nu_t)\nu_t dt - \frac{1}{2} (Q_T^\nu - Q_0)^\top \mathbf{X} b \mathbf{X}^\top (Q_T^\nu - Q_0)$$

$$- \mathcal{M}_T^\nu \bar{b}^\top \mathbf{X}^\top Q_T^\nu + \int_0^T \mu_t \bar{b}^\top \mathbf{X}^\top Q_t^\nu dt. \quad (2.10)$$
From (2.5), we have
\[(P^\nu_T)^\top X^\top Q^\nu_T = S^T_T X^\top Q^\nu_T + (Q^\nu_T - Q_0)^\top X b X^\top Q^\nu_T + \mathcal{M}^T_T \bar{b}^\top X^\top Q^\nu_T, \quad (2.11)\]

and using (2.10) and (2.11), the performance criteria can be written as
\[
H^\nu(t, x, s, q, \mu) = \mathbb{E}_{t, x, s, q, \mu} \left[ \int_0^T (\mathcal{X} s_t - a \nu_t)^\top \nu_t dt + \int_0^T \mu_t^\top \bar{b}^\top X^\top Q^\nu_t dt \\
+ (Q^\nu_T)^\top (\frac{1}{2} \mathcal{X} b X^\top - \alpha) Q^\nu_T - \frac{1}{2} (Q_0)^\top \mathcal{X} b X^\top Q_0 \\
- \phi \int_t^T (Q^\nu_u - Q_u)^\top \tilde{\Sigma} (Q^\nu_u - Q_u) du + S^T_T X^\top Q^\nu_T \right].
\]

(2.12)

We further simplify the problem by introducing the transformed processes \(Y^\nu = \{Y^\nu_t\}_{t \geq 0}\) and \(Z = \{Z_t\}_{t \geq 0}\) through the following equalities:
\[
Y^\nu_t = \int_0^t (\mathcal{X} s_u - a \nu_u)^\top \nu_u du + \theta^\top \mathcal{X}^\top (Q^\nu_u - Q_0) + \int_0^t \mu_u^\top \bar{b}^\top X^\top Q^\nu_u du, \quad (2.13)
\]
\[
Z_t = S_t - \theta, \quad (2.14)
\]
in which case \(Z\) and \(Y^\nu\) satisfy the SDEs
\[
dZ_t = -\kappa Z_t dt + \sigma^\top dW_t, \quad (2.15)
\]
\[
dY^\nu_t = \left\{ (\mathcal{X} Z_t - a \nu_t)^\top \nu_t + \mu_t^\top \bar{b}^\top X^\top Q^\nu_t \right\} dt, \quad (2.16)
\]
and the control problem, in the new variables, becomes
\[
H(t, y, z, q, \mu) = \sup_{\nu \in \mathcal{U}} \mathbb{E}_{t, y, z, q, \mu} \left[ Y^\nu_T + Z^\top_T X^\top Q^\nu_T + \theta^\top \mathcal{X}^\top Q_0 + (Q^\nu_T)^\top (\frac{1}{2} \mathcal{X} b X^\top - \alpha) Q^\nu_T \\
- \frac{1}{2} (Q_0)^\top \mathcal{X} b X^\top Q_0 - \phi \int_t^T (Q^\nu_u - Q_u)^\top \tilde{\Sigma} (Q^\nu_u - Q_u) du \right],
\]

(2.17)
where $\mathcal{U}$ is the set of admissible strategies in the new variables.

### 2.3.1 The dynamic programming equation

The dynamic programming principle suggests that the value function (2.17) is the unique viscosity solution to the DPE

$$
\frac{\partial}{\partial t} H + \mathcal{L}^\mu H + \sup_{\nu} \{ \mathcal{L}^\nu H \} - \phi (q - Q_t)^\top \tilde{\Sigma} (q - Q_t) = 0,
$$

subject to the terminal condition

$$
H(T, y, z, q, \mu) = y + z^\top \mathcal{X}^c q + q^\top (\frac{1}{2} \mathcal{X} \mathcal{X}^\top - \alpha) q + \theta^\top \mathcal{X}^c Q_0 - \frac{1}{2} Q_0^\top \mathcal{X} \mathcal{X}^\top Q_0,
$$

and where $\mathcal{L}^\nu$ is the infinitesimal generator of the process $(Y^\nu, Q^\nu, Z)$, which acts on a smooth function $\varphi$ as follows:

$$
\mathcal{L}^\nu \varphi(t, y, z, q, \mu) = \left\{ (\mathcal{X} \nu - a^\nu) \nu + \mu^\top b^\nu \mathcal{X}^\top q \right\} \frac{\partial}{\partial y} \varphi - \nu^\top \frac{\partial}{\partial q} \varphi - z^\top \kappa \frac{\partial}{\partial z} \varphi + \frac{1}{2} \text{Tr} (\Sigma \frac{\partial}{\partial z z} \varphi).
$$

#### Proposition 2.1. Solving the DPE

The DPE (2.18) admits the solution

$$
H(t, y, z, q, \mu) = y + z^\top A(t) z + z^\top B(t, \mu) + q^\top C(t) q + q^\top D(t, \mu) + z^\top E(t) q + F(t, \mu),
$$

if there exists unique matrix-valued functions $A(t)$ $(n \times n)$, $B(t, \mu)$ $(n \times 1)$, $C(t)$ $(m \times m)$, $D(t, \mu)$ $(m \times 1)$, $E(t)$ $(n \times m)$, and function $F(t, \mu)$ that satisfy:

(a) The matrix Riccati equation:

$$
\dot{G} + GM_1 G + GM_2 + M_2^\top G + M_3 = 0,
$$
with terminal condition \( G(T) = \begin{bmatrix} 0^{(n,n)} & 0^{(n,m)} \\ 0^{(m,n)} & \frac{1}{2} \mathcal{X} b \mathcal{X}^T - \alpha \end{bmatrix} \), where \( G = \begin{bmatrix} 2A & E - \mathcal{X}^T \\ E^T - \mathcal{X} & 2C \end{bmatrix} \), \( 0^{(j,k)} \) is a \( j \times k \) matrix of zeros,

\[
M_1 = \frac{1}{2} \begin{bmatrix} 0^{(n,m)} & 0^{(m,m)} \\ 0^{(m,m)} & a^{-1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} -\kappa & 0^{(n,m)} \\ 0^{(m,m)} & 0 \end{bmatrix}, \quad \text{and} \quad M_3 = \begin{bmatrix} 0^{(n,n)} & -\kappa \mathcal{X}^T \\ -\mathcal{X} \mathcal{X}^T & -2\phi \Sigma \end{bmatrix}.
\]

(2.23)

(b) The linear matrix PDEs:

\[
\dot{B} + \mathcal{L} \mu B - \kappa B + \frac{1}{2} (E^T - \mathcal{X}) a^{-1} D = 0^{(n)},
\]

(2.24a)

\[
\dot{D} + \mathcal{L} \mu D + C^T a^{-1} D + 2 \phi \bar{\Sigma} Q_t + \mathcal{X} \bar{b} \mu = 0^{(m)},
\]

(2.24b)

\[
\dot{F} + \mathcal{L} \mu F + \frac{1}{4} D^T a^{-1} D + Tr(\Sigma A) - \phi Q_t^T \bar{\Sigma} Q_t = 0,
\]

(2.24c)

with terminal conditions

\[
B(T, \cdot) = 0^{(n)}, \quad D(T, \cdot) = 0^{(m)}, \quad F(T) = \theta^T \mathcal{X}^T Q_0 - \frac{1}{2} Q_0^T \mathcal{X} b \mathcal{X}^T Q_0;
\]

and \( 0^{(k)} \) denotes a vector of \( k \) zeros.

In the above, the dot notation denotes time derivative.

Proof. See 2.A.1. ☐

The following theorem shows that the solution to (2.22) is bounded on \([0, T]\), as long as we choose the terminal penalty \( \alpha \) to be large enough.

**Theorem 2.2.** If \( \frac{1}{2} \mathcal{X} b \mathcal{X}^T - \alpha \) is negative definite, the matrix Riccati differential equation (2.22) has a bounded solution on \([0, T]\).

Proof. See 2.A.2. ☐

Furthermore, the linear matrix PDEs (2.24) admit a unique probabilistic representation.
Chapter 2. Trading Co-Integrated Assets with Price Impact

**Theorem 2.3.** Suppose the assumption of Theorem 2.2 are enforced, and further \( \mathbb{E}[|\mu_0^\pm|^2] < \infty \) and there exists a constant \( C \) such that and \( \mathbb{E}_{0,\mu} [ |\mu_t^\pm|^2 ] < C(1 + |\mu_t^\pm|^2) \) for all \( t \in [0,T] \). Let \( B(t, \mu), D(t, \mu) \) and \( F(t, \mu) \) be \( C^{1,2}([0,T), \mathbb{R}^m) \) solutions to (2.24), each with quadratic growth in \( \mu \), uniformly in \( t \), then

\[
D(t, \mu) = \int_t^T e^{\int_u^t C^*(s)\,ds} \left\{ 2\phi \sum \mathcal{Q}_u + \mathcal{X} \tilde{b} \mathbb{E}_{t,\mu} [ \mu_u ] \right\} \,du, \\
B(t, \mu) = \frac{1}{2} \int_t^T e^{-\kappa(u-t)} (\mathcal{E}^\top - \mathcal{X}^\top) a^{-1} \mathbb{E} [ D(t, \mu_u) ] \,du, \\
F(t, \mu) = \int_t^T \left\{ \frac{1}{4} \mathbb{E}_{t,\mu} [ D^\top(u, \mu_u) a^{-1} D(u, \mu_u) ] + Tr(\Sigma A(u)) - \phi \mathcal{Q}_u \sum \mathcal{Q}_u \right\} \,du,
\]

where the notation \( \mathbb{E}[\cdot]_u^t \) represents the time-ordered exponential.\(^4\)

**Theorem 2.4. Verification.** Suppose the assumption in Theorem 2.2 and Theorem 2.3 are enforced, then the candidate value function (2.21) is indeed the solution to the control problem. Moreover, the trading rate given by

\[
\nu_t^* = -\frac{1}{2} a^{-1} \left\{ 2 C(t) Q_t^\nu + (\mathcal{E}^\top(t) - \mathcal{X}) (S_t - \theta) + D(t, \mu_t) \right\},
\]

is admissible and optimal.

**Proof.** See 2.A.4. \( \square \)

The optimal trading rate can be interpreted as the liquidation strategy of AC, plus modifications due to: co-integration, order flow impact, and target inventory. In our setup, we obtain the AC strategy to liquidate a portfolio of \( m \) assets by removing co-integration (setting \( \kappa = 0 \) in (2.2)), removing the impact of order flow (setting \( \tilde{b} = 0 \) in (2.5)) and setting the target inventory schedule \( \mathcal{Q}_t = 0 \) for all \( t \in [0, T] \). Note that

\(^4\)Recall that the time-ordered exponential of a time dependent matrix \( \mathcal{A}(t) \) is defined as:
\[
\mathbb{E}(\int_u^t A(s)\,ds) := \lim_{||\Pi||\downarrow 0} \Pi^n_{i=1} e^{\Delta t_i}, \quad \text{where} \quad \Pi := \{u = t_0, t_1, \ldots, t_n = t\} \quad \text{is a partition of} \quad [u, t],
\]
and \( \Delta t_i = (t_i - t_{i-1}). \)
when we trade in all assets, so that $m = n$, $\mathbf{X}$ becomes the identity matrix. With these assumptions $A(t) = 0^{(n,n)}$, $B(t) = 0^{(n)}$, $D(t) = 0^{(m)}$, $E(t) = I_m$ (an $m \times m$ identity matrix), for all $t \in [0,T]$ in (2.22). Therefore, the multi-asset AC strategy is to trade at the speed:

$$\nu_t^{AC} = -a^{-1} C(t) Q_t^{\mu^{AC}}.$$  \hspace{1cm} (2.27)

The difference between the optimal trading strategy (2.26) and the AC strategy consists of two components. The first is $(\mathbf{E}^T - \mathbf{X}) (\mathbf{S}_t - \theta)$, which accounts for co-integration in prices, and allows the trader to take advantage of price deviations from all assets – not just the ones she is trading. This modification vanishes as the strategy approaches the end of the trading horizon because the terminal conditions enforce $\mathbf{E}^T \xrightarrow{t \to T} \mathbf{X}$.

The second component, $D(t, \mu)$, is the adjustment due to the inventory target and order flow. For example, if the agent targets zero inventory throughout the life of the strategy, $Q_t = 0$ for all $t$, and ignores the effect of order flow from other traders, i.e. $\bar{b} = 0$, then $D(t, \mu)$ vanishes. Moreover, the terminal conditions for $D$ imply that the effect of this adjustment in the trading strategy diminishes as the strategy approaches the terminal date.

As a final point, if the order flow of other agents $\mu$ is affine, specifically, if $\mathbb{E}[\mu_u | \mathcal{F}_t] = \alpha(t; u) + \beta(t; u) \mu$, for some deterministic functions $\alpha(t; u)$ ($dim = n \times 1$) and $\beta(t; u)$ ($dim = n \times n$), then the PDE system (2.24) admits the affine ansatz for $B(t, \mu) = B_0(t) + B_1(t) \mu$ and $D(t, \mu) = D_0(t) + D_1(t) \mu$, while $F(t, \mu) = F_0(t) + F_1^\top(t) \mu + \mu^\top F_2(t) \mu$.

Two examples of affine order flow models are (i) the shot-noise processes, where order flow jumps up at Poisson times and mean-reverts back to zero, with idiosyncratic upward and downward jumps, as well as co-jumping order flow; and (ii) the multivariate Hawkes process, where increases in order flow induces excitation in order flow among a subset of assets or all assets. Both of these models have appeared in a number of papers that study the empirical aspects of order flow.
2.3.2 Guaranteed liquidation

To ensure full liquidation by the end of the trading window, i.e. \( Q_T^\nu = 0 \), we make the liquidation penalty arbitrarily expensive, i.e. all the components of \( \alpha \) go to \( \infty \). In this case, the terminal condition for \( C \), see (2.22), becomes arbitrarily large as the entries of the terminal condition \( C(T) \) go to \( -\infty \). Now, let us assume that in this limiting case \( A \), \( C \), and \( E \) have the series representations

\[
A(t) = \sum_{n=0}^{\infty} A_n \tau^n, \quad C(t) = \sum_{n=-1}^{\infty} C_n \tau^n, \quad E(t) = \sum_{n=0}^{\infty} E_n \tau^n, \tag{2.28}
\]

where \( \tau = T - t \) and \( A_n, C_n \) and \( E_n \) are constant matrices with the same dimensions as \( A, C \), and \( E \) respectively.

The terminal conditions imply that the first term in the series representation \( A(t) \) is \( A_0 = 0 \) and in \( E(t) \) is \( E_0 = X^\top \). Moreover, substituting (2.28) into (2.22), and matching terms with the same power in \( \tau \), we obtain the following coefficients for the series of \( A(t) \), \( C(t) \), and \( E(t) \):

\[
A_1 = 0, \quad C_{-1} = \frac{1}{2} X b X^\top - a, \quad C_0 = 0, \quad E_1 = -\frac{1}{2} X^\top . \tag{2.29}
\]

We also show that the term \( D(t, \mu) \) in (2.26) has the asymptotic bound \( O(\tau) \mu + O(\tau) \). See 2.A.5 for further details.

Thus, when \( \alpha \to \infty \), so that \( C(T) \to -\infty \) (recall that \( C(T) = \frac{1}{2} X b X^\top - a \) from (2.22)), we employ the series representations (2.28) (using the first two terms for each series), to write the optimal liquidation speed (2.26) as follows:

\[
\nu_t^* = \frac{Q_t^{\nu^*}}{\tau} + O(\tau) Z_t + O(\tau) \mu + O(\tau) . \tag{2.30}
\]

The result is that near maturity, i.e. \( \tau \to 0 \), the optimal strategy behaves like TWAP (time-weighted-average-price), which is given by the first term in the right-hand side of
Thus, as the strategy approaches the terminal date, the remaining inventory is liquidated at a constant rate.

2.4 Simulations: Portfolio Liquidation

This section shows the performance of the strategy under various assumptions about the set of assets employed by the agent. We first describe how the model parameters are estimated using exchange data from five assets traded in Nasdaq. Then we compare the performance of the strategy to that of Almgren-Chriss when the agent liquidates a portfolio of two assets: using the information of another additional set of three stocks, using only the information of the two assets, and allowing asset repurchases.

2.4.1 Data and model parameters

To focus on the additional value added by the co-integration information, we turn off the permanent impact from order flow of other agents, i.e. we set $b = 0$. For an analysis of how agents can benefit from order flow information see Cartea and Jaimungal (2016a). We also turn off the permanent impact from the agent's own trading activity, i.e., we assume $b = 0$.

We employ high-frequency data from five stocks traded in the Nasdaq exchange: INTC, SMH, FARO, NTAP and ORCL. We use all the messages sent to the exchange in November 3, 2014 to build the LOB at a millisecond frequency. We sample the best quotes and posted volume every 60 seconds during the regular trading hours. Midprices are computed as the weighted average of the best bid and the best ask, with weights equal to the volume posted at the best bid and the best ask respectively – these prices are also referred to as microprice. We remove the first and last half hour to reduce the noise in prices due to the opening and closing auctions. Thus, for the trading day we have a midprice time series of 330 data points per stock.
The five stocks we employ are in the high-tech sector, thus sharing a common trend, so we expect them to be co-integrated. We employ a VAR(1), vector-autoregressive of order 1, model of the joint midprice dynamics which is a discrete-time version of the price process in (2.5), and apply Johansen’s co-integration test to determine the number of co-integrating factors. Table 2.1 reports the p-values of the co-integration test for the number of co-integrating factors, where $r_i$ corresponds to the null hypothesis that there are at most $i$ co-integrating factors. In Figure 2.1 we show the realisation of this factor through the trading day.

<table>
<thead>
<tr>
<th>Model</th>
<th>$r_0$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>p-value</td>
<td>0.001</td>
<td>0.223</td>
<td>0.409</td>
<td>0.4637</td>
<td>0.584</td>
</tr>
</tbody>
</table>

Table 2.1: Johansen’s co-integration test for the number of co-integrating factors, and $r_i$ corresponds to the null hypothesis that there are at most $i$ co-integrating factors. Nasdaq data November 3, 2014.

Here we choose a midprice model with 1 co-integration factor, and show in the first two rows in Table 2.5 the parameter estimates for the mean reverting level $\theta$ and the co-integration factors of the VAR(1) model (data November 3, 2014). The rest of the table is taken from Cartea and Jaimungal (2016a) which employs data for the entire year 2014. Rows 3 and 4 are estimates of temporary and permanent impact with no cross effects. We choose the temporary impact model with no cross effects to keep our model parsimonious. The bottom 2 rows show the average incoming rates of MOs and their average volume: $\lambda^-$ is the average number of sell MO per hour, $E[\eta^-]$ is the average volume of sell MOs. The standard deviation of the estimate is shown in parentheses.
Tables 2.6 and 2.7 show the mean-reverting matrix $\kappa$, the variance-covariance matrix $\Sigma$, respectively. Time is $T = 1$, which corresponds to 6.5 hours (1 trading day).

Below we compare the performance of different trading strategies one of which is Almgren-Chriss. In this particular case we assume that midprices satisfy the SDE

$$dP^\nu_t = -b\nu_t dt + (\sigma^{AC})^T dW^{AC}_t.$$  \hspace{1cm} (2.31)

where $(\sigma^{AC})^T$ is the $n \times n$ matrix produced by the Cholesky decomposition of the asset prices correlation matrix $\Sigma^{AC} = (\sigma^{AC})^T \sigma^{AC}$. That is, for the Almgren-Chriss strategy asset prices are driven by correlated Brownian motions but are not assumed to be co-integrated. Table 2.8 shows the estimated correlation matrix $\Sigma^{AC}$.

### 2.4.2 Liquidation of portfolio with two assets

The investor’s objective is to liquidate 4,600 shares of INTC and 900 shares of SMH over one hour. According to Table 2.5, these two numbers correspond to 1% and 4% of the average number of sell volume, respectively. The investor sends MOs for INTC at 3 second interval and MOs for SMH at 30 second interval. This choice is based on the observation that the arrival rate of orders (buys and sells) for INTC is approximately 1,000 per hour and for SMH is approximately 120 per hour, see Table 2.5. Thus, we assume a pace of trading in agreement with that of both assets. This assumption is also important because the LOB is assumed to be infinitely resilient and the temporary impact parameters in $a$ reflect the individual characteristics of the order book for INTC and SMH.

**Strategies.** To illustrate the performance of the liquidation algorithm we consider the following four strategies:

1. **Unrestricted liquidation (UL):** the strategy $\nu^*_t$ is as in (2.26) with target schedule $Q_t = 0$ for all $t$. Recall that the investor’s set of admissible strategies does not
require the trading speed to remain non-negative. So UL may, if it is optimal, repurchase shares before the end of the trading horizon.

2. **Restricted liquidation (RL):** the strategy is as that of UL, but the liquidation speed is set to zero if it is optimal to repurchase, i.e. \( \max(\nu^*_t, 0) \). This is an ad-hoc adjustment to the optimal strategy to preclude repurchases along the trading window. The max operator \( \max(\cdot, 0) \) is to be interpreted componentwise: when the speed of liquidation of an asset in the vector (2.26) is negative only that component is set to zero. Finally, the strategy stops trading when inventory hits zero. The derivation of the optimal strategy under the constraint \( \nu^*_t \geq 0 \) is beyond the scope of our study.

3. **Unrestricted liquidation with target (ULT):** the strategy is as (2.26) where the target schedule for each asset is the Almgren-Chriss (AC) strategy, i.e., \( Q_t = Q^{AC}_t \) where \( Q^{AC}_{t} \) is the AC liquidation position given by integrating (2.27) with penalty parameter \( \phi^{AC} = 0.1 \).\(^5\) With these parameters, the AC strategy liquidates more than the initial inventory of SMH early on, but repurchases inventory by the trading end.

4. **Almgren-Chriss liquidation (AC):** the strategy is as in (2.27) and the price process is (2.31), so the strategy only uses information from INTC and SMH without a co-integrating factor. This is the benchmark we employ to compare the results of the previous three strategies.

**Scenarios.** We simulate \( 10^6 \) sets of sample price paths and look at the performance of the four strategies when liquidating shares in INTC and SMH for a range of values of the penalty parameter \( \phi \) between \( 10^{-3} \) and \( 10^{-2} \), and the liquidation penalty is \( \alpha = 10^6 \) for both assets, i.e. employ strategies that guarantee full liquidation. We measure the performance by comparing the terminal wealth of UL, RL, ULT, and AC under two

\(^5\)The penalty parameter is embedded in \( C \) which appears explicitly in the liquidation speed (2.27).
Figure 2.2: For trading INTC and SMH. Risk-reward for UL, RL, ULT, and AC. Left (right) panel, strategies employ information from all (only the traded) stocks. Within each panel, the penalty $\phi$ increases moving from the right to the left of the diagrams.

scenarios:

- **Scenario 1.** Liquidate shares in INTC and SMH and employ the additional information provided by three additional assets: FARO, NTAP, ORCL.

- **Scenario 2.** Liquidate shares in INTC and SMH and only employ the information provided by the dynamics of INTC and SMH.

Figure 2.2 shows the mean terminal wealth (aggregate cash from liquidating shares in both INTC and SMH) of the four strategies as a function of its standard deviation. As the penalty parameter $\phi$ increases, the standard deviation and mean of the terminal wealth decrease. To see the intuition behind this relationship let us focus on UL. The agent targets an inventory of zero throughout the life of the strategy, and the value of $\phi$ determines how closely the strategy tracks this target. When the penalty is high, the strategy is less able to trade strategically by either speculating (repurchasing shares) and/or taking advantage of midprice signals that stem from the co-integrating factor. Thus, potential benefits from taking advantage of price movements are outweighed by the requirement that inventory must be drawn to zero very quickly. Conversely, as the penalty becomes smaller, the strategy will have more opportunities to anticipate and
Figure 2.3: Price per share of INTC and SMH for UL, RL, ULT, and AC. Left (right) panels, strategies employ information from all (only the traded) stocks. Within each panel, the penalty $\phi$ increases moving from the right to the left of the diagrams.

take advantage of midprice movements and these will not be curbed by a strict inventory target.

The left-hand panel of the figure shows Scenario 1 where UL, RL, and ULT employ the additional information provided by FARO, NTAP, ORCL. Clearly, UL dominates the other strategies where AC is the worst performer because it does not account for the co-integration of assets. The right-hand panel of Figure 2.2 shows Scenario 2 where only information of the co-integrated pair INTC and SMH is employed. Clearly, not employing the additional information provided by other assets that are co-integrated with those in the liquidating portfolio has a considerable effect on the strategies’ performance.

Figure 2.3 shows the mean price per share for INTC and SMH, respectively, for a range of the value $\phi$. For both shares, the figures in the left-hand panels show that
including information from other co-integrated assets boosts the performance of UL. The right-hand panel shows that UL is more volatile than the other strategies and this is a result of the strategy speculating on repurchases of the assets.

Table 2.2 shows, for different values of the target penalty parameter $\phi$, how often UL repurchases shares and the percentage of times that UL and RL underperform AC. Here UL and RL employ information of the midprice dynamics of the five co-integrated assets. The table shows that UL’s speculative component ranges from 13% to 18% in INTC and 64% to 66% in SMH. UL’s speculative trades are similar to those employed in pairs trading which take advantage of temporary deviations of prices. Moreover, we observe that very seldom do we see UL underperform AC, whereas RL underperforms in around 13% of the runs. Recall, however, that the optimal strategy we derived is the UL strategy, while RL is an ad-hoc sub-optimal adjustment that precludes asset repurchases.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>UL $\phi$</th>
<th>RL $\phi$</th>
<th>$%\nu_{INTC} &lt; 0$</th>
<th>$%\nu_{SMH} &lt; 0$</th>
<th>$%X_T &lt; X_T^{AC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>1E-2 7.3E-3 5E-3</td>
<td>1E-2 7.5E-3 5E-3</td>
<td>18.4 16.0 12.9</td>
<td>65.6 65.4 64.2</td>
<td>0.1 0.3 0.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0 0 0</td>
<td>0 0 0</td>
<td>13.4 13.1 12.2</td>
</tr>
</tbody>
</table>

Table 2.2: Repurchase frequency for UL, and underperformance of UL and RL with respect to AC.

Furthermore, as the value of the parameter $\phi$ decreases, there are fewer instances in which the liquidation speeds for INTC and SMH are negative. At first this might seem counterintuitive, for one expects a more relaxed penalty parameter to allow UL more freedom to speculate. Note however, that a high value of $\phi$ (recall that for UL the inventory-target is $Q_t = 0$ for all $t$) pushes the inventory close to zero early. And once the inventory in both assets is low, the strategy attempts more speculative trades by repurchasing the asset. These speculative trades are small in volume, but frequent.

Figure 2.4 compares the performance of UL and RL with that of AC. The comparison
is in basis points according to the commonly used metric

$$\text{Savings}^j = \frac{X^j_T - X^{AC}_T}{X^{AC}_T} \times 10^4,$$

(2.32)

where $X^j_T$ is the terminal cash\(^6\) obtained from liquidation the two-asset portfolio employing strategy $j \in \{UL, RL\}$. In the left-hand panel the strategy employs information from the price dynamics of the five co-integrated assets. For UL, savings are in the order of 1.5 to 5.5 basis points, and for RL between 0 and 5 basis points. In the right-hand panel, only information provided by the midprice dynamics of the two-asset portfolio is employed, so as expected, the savings are lower. In particular, when the standard deviation is low, RL becomes inferior to AC with a negative saving.

Finally, Table 2.3 shows the quantiles of performance of UL and RL measured using (2.32) for a range of the penalty parameter $\phi$. The strategies use the information of the five co-integrated assets.

---

\(^6\)Recall that we have chosen the terminal penalty very large, so that sample paths end with no inventory, and hence the terminal cash the agent has equals their wealth from liquidating.
Table 2.3: Quantiles of relative savings, measured in basis points using (2.32).

<table>
<thead>
<tr>
<th>Quantile</th>
<th>UL</th>
<th>RL</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>0.56 0</td>
<td>-1.74 -1.79 -1.93</td>
</tr>
<tr>
<td>25%</td>
<td>2.05 2</td>
<td>0.51 0.58 0.65</td>
</tr>
<tr>
<td>50%</td>
<td>3.25 3.31 3.44</td>
<td>2.05 2.22 2.37</td>
</tr>
<tr>
<td>75%</td>
<td>4.67 4.83 5.11</td>
<td>3.75 4.04 4.42</td>
</tr>
<tr>
<td>95%</td>
<td>7.27 7.76 8.45</td>
<td>7.27 7.74 8.51</td>
</tr>
</tbody>
</table>

Investor estimates model parameters with error

Here we assume that the investor does not have access to enough trading data, so the parameter estimates she obtains are incorrect. The investor observes prices for one day for each asset, which she employs to calibrate the model. The prices she observes are simulated using the parameters in Tables 2.6 and 2.7. From the observed data, the investor samples prices every minute to estimate parameters, which are reported in Tables 2.9 and 2.10. Moreover, we use the same set of prices to estimate the coefficients for the benchmark AC strategy – parameter estimates are reported in Table 2.11.

To illustrate how the strategy performs when the model parameters are incorrect, we first proceed as above. Then, we simulate $10^6$ sets of sample price paths (using the original parameters – so that the agent has incorrect parameters in their trading strategy) and look at the performance of the four strategies when liquidating shares in INTC and SMH, and proceed as in Subsection 2.4.2. The results are broadly the same as those obtained in the previous section when the investor’s parameter estimates where the same as those used to simulate price paths. For example, Table 2.4 shows quantiles of relative savings, measured in basis points using (2.32), and parameters are estimated with error. The results in the table are similar to those show above in Table 2.3 when the investor estimated the parameters of the model without error. Finally, we do not present the analogues to the figures shown above because the results are qualitatively the same.
Table 2.4: Quantiles of relative savings, measured in basis points using (2.32), and parameters are estimated with error.

### 2.5 Alternative Model

In this section, we propose an alternative model for the midprice dynamics (2.5) and solve the same stochastic control problem for the agent under this model. The only difference between the two models is that in the main model, the effects from order flows do not affect the co-integration dynamics and hence the effects from order flows and co-integration are decoupled; whereas in the alternative model, these two effects are coupled. Our goal is to provide an alternative way of modelling co-integration under price impact and comparisons of the main and the alternative model are beyond the scope of our study.

We adopt the same technique to reduce the control problem into solving a matrix Ricatti differential equation similar to (2.23) and linear matrix PDEs similar to (2.24). It is worthwhile mentioning that when there is no permanent impact (i.e., \( b = 0 \)), the main model and the alternative model are identical. Hence the solutions are identical too. On the other hand, when \( b \neq 0 \), the alternative model leads to equations that are slightly harder to solve. This is because: first, it is not clear whether the matrix Ricatti differential equation admits a bounded solution like (2.23); second, there is no straightforward way to write down the Feynman-Kac representation of the solution of the linear matrix PDEs like in Theorem 2.3. Nevertheless, we present the result here so that readers have more flexibility in choosing the model.
2.5.1 Midprice Dynamics

Recall that in Section 2.2, we assume that the midprice $P_t$ satisfies the following SDE

$$dP_t = dS_t + g(o_t) dt,$$

where $S_t$ is a multivariate version of Ornstein-Uhlenbeck process and $g(o_t)$ represents the impact from order flow. Under this setup, the price impact from order flow and the co-integration dynamics are decoupled. In other words, price movements due to the co-integration dynamics are not affected by order flow. Alternatively, we may consider the following model where co-integration and order flow are coupled:

$$dP_t = \kappa (\theta - P_t) dt + g(o_t) dt + \sigma^T dW_t. \tag{2.33}$$

In the above formulation, the impact from order flows $g(o_t)$ affects the drift of the midprice dynamics, which in turn affects the co-integration dynamics. We assume that the price impact from order flows follows (2.4), i.e.,

$$g(o_t) = -b X^\intercal \nu_t + b \mu_t.$$

We consider the same control problem for the agent where the midprice $P_t$ follows (2.33). Her performance criterion is given by (2.8), and her value function is given by (2.9).

2.5.2 Solving the Control Problem

To solve the control problem, we employ the following transformation of variables:

$$Y_t^\nu = X_t^\nu + \theta^\intercal \mathcal{X}^\intercal (Q_t^\nu - Q_0), \tag{2.34}$$

$$Z_t^\nu = P_t^\nu - \theta. \tag{2.35}$$
Recall that inventory holdings satisfy $dQ_t^\nu = -\nu_t 
dt$ and write

$$dZ_t^\nu = \left\{ -\kappa Z_t^\nu - bX^\top \nu_t + \bar{b} \mu_t \right\} dt + \sigma^\top dW_t. \tag{2.36}$$

Using (2.7) we write

$$dY_t^\nu = (XZ_t^\nu - a\nu_t)^\top \nu_t dt.$$ 

The control problem, in the new variables, becomes

$$H(t, y, z, q, \mu) = \max_{\nu \in \mathcal{U}} \mathbb{E}_{t, y, z, q} \left[ Y_T^\nu + (Z_T^\nu)^\top X^\top Q_T^\nu - (Q_T^\nu)^\top \alpha Q_T^\nu + \theta^\top X^\top Q_0 \right. 
- \phi \left. \int_t^T (Q_u^\nu - Q_u)^\top \tilde{\Sigma} (Q_u^\nu - Q_u) du \right], \tag{2.37}$$

where $\mathcal{U}$ is the set of admissible strategies in the new variables.

Classical results suggest that the value function satisfies the following DPE

$$\partial_t H + \sup_{\nu} \{ L^\nu H \} - \phi (q - Q_t)^\top \tilde{\Sigma} (q - Q_t) = 0, \tag{2.38}$$

where the infinitesimal generator $L^\nu$ acts on a smooth function $\varphi$ as follows

$$L^\nu \varphi(t, y, z, q, \mu) = (Xz - a\nu)^\top \nu \partial_y \varphi - \nu^\top \partial_y \varphi - z^\top \kappa \partial_z \varphi 
- \nu^\top Xb \partial_z \varphi + \mu^\top \bar{b} \partial_z \varphi + \frac{1}{2} \text{Tr} (\Sigma^u \partial_{zz} \varphi). \tag{2.39}$$

Similar to Proposition 2.1, we can solve the DPE using a quadratic ansatz. The following proposition characterizes the solution to the DPE.
Proposition 2.5. **Solving the DPE.** The DPE (2.38) admits the solution

\[ H(t, y, z, q, \mu) = y + z^\top A(t) z + z^\top B(t, \mu) + q^\top C(t) q + q^\top D(t, \mu) + z^\top E(t) q + F(t, \mu), \]

if there exists unique matrix-valued functions \( A(t) (n \times n), B(t, \mu) (n \times 1), C(t) (m \times m), D(t, \mu) (m \times 1), E(t) (n \times m), \) and function \( F(t, \mu) \) that satisfy:

(a) The matrix Riccati equation:

\[ \dot{G} + GM_1 G + GM_2 + M_2^\top G + M_3 = 0, \]

with terminal condition \( G(T) = \begin{bmatrix} 0^{(n,n)}_\cdot & 0^{(n,m)}_\cdot \\ 0^{(m,n)}_\cdot & \frac{1}{2} \chi b \chi^\top - \alpha \end{bmatrix}, \) where \( G = \begin{bmatrix} 2A & E - \chi^\top \\ E^\top - \chi & 2C \end{bmatrix}, \)

\( 0^{(j,k)} \) is a \( j \times k \) matrix of zeros,

\[
M_1 = \frac{1}{2} \begin{bmatrix} b \chi^\top a^{-1} \chi b & b \chi^\top a^{-1} \\ a^{-1} \chi b & a^{-1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} -\kappa \chi^\top & \frac{1}{2} b \chi^\top a^{-1} \chi b \chi^\top \\ 0^{(m,n)} & \frac{1}{2} a^{-1} \chi b \chi^\top \end{bmatrix},
\]

and \( M_3 = \begin{bmatrix} 0^{(n,n)} & -\kappa \chi^\top \\ -\chi \kappa^\top & -2\phi \tilde{\Sigma} \end{bmatrix}. \]

(b) The linear matrix PDEs:

\[ \dot{B} + \mathcal{L}^\mu B - \kappa B + \frac{1}{2} (E^\top - \chi + 2 \chi b A)^\top a^{-1} (D + \chi b B) = 0^{(n)}, \]

\[ \dot{D} + \mathcal{L}^\mu D + \frac{1}{2} (2C + \chi b E)^\top a^{-1} (D + \chi b B) \]

\[ + 2 \phi \tilde{\Sigma} Q_t + \chi b \mu = 0^{(m)}, \]

\[ \dot{F} + \mathcal{L}^\mu F + \frac{1}{2} (D + \chi b B)^\top a^{-1} (D + \chi b B) \]

\[ + Tr(\Sigma A) - \phi Q_t^\top \tilde{\Sigma} Q_t + B^\top \tilde{b} \mu = 0, \]

with terminal conditions

\( B(T, \cdot) = 0^{(n)}, \quad D(T, \cdot) = 0^{(m)}, \quad F(T) = \theta^\top \chi^\top Q_0, \)
and $0^{(k)}$ denotes a vector of $k$ zeros.

In the above, the dot notation denotes time derivative.

**Proof.** Similar to the proof of Proposition 2.1.

**Theorem 2.6. Verification.** Assume that (2.42) and (2.43) has a classical solution for which the solutions for $A$, $C$, and $E$ remain bounded over $[0, T]$, and assume there exist constants $C_B$, $C_D$ and $C_F$ with $|B(t, \mu)| \leq C_B(1 + |\mu|)$, $|D(t, \mu)| \leq C_D(1 + |\mu|)$ and $|F(t, \mu)| \leq C_F(1 + |\mu|^2)$ for all $t$, then the candidate value function (2.21) is indeed the solution to the control problem. Moreover, the optimal trading rate $\nu^*_t$ can be written as a Markov control:

$$
\nu^*_t = -\frac{1}{2} a^{-1} \left\{ (2 C + X b E) Q^*_{t-1} + (E^\top - X U^{-1} + 2 X b A) Z^*_{t-1} + D + X b B \right\}.
$$

(2.44)

**Proof.** Similar to the proof of Theorem 2.4.

Our main results in the section, Proposition 2.5 and Theorem 2.6 complements the results in Section 2.2. It can be seen that when there is no permanent impact from the agent’s trading, i.e., $b = 0$, the matrix Ricatti equation (2.42) reduces to (2.23) and the matrix PDE (2.43) reduces to (2.24). This is an expected result since the midprice dynamics under the two model are identical in the absence of permanent impact. Finally, it is also interesting to ask the question: which model provides a better fit to the data? We will leave this question for future investigations.

### 2.6 Conclusions

We show how to liquidate a basket of assets whose prices are co-integrated. In our framework, market orders from all participants, including the agent liquidating the basket, have a permanent impact on asset prices. In addition, the agent receives prices worse
than the best quotes because her trades walk the limit order book, i.e. have temporary price impact. We assume that price impact is linear in the speeds of trading and order flow has cross-effects: trade activity in one asset may have a permanent effect on prices of co-integrated assets and a temporary effect on the limit order books that display the liquidity of the co-integrated assets.

The agent maximizes terminal wealth and targets an inventory schedule. The liquidation strategy employs information from \( n \) co-integrated assets and liquidates a basket consisting of a subset of \( m \leq n \) assets. We estimate the model parameters and co-integration factors using trade data from five stocks (INTC, SMH, FARO, NTAP, and ORCL) in the Nasdaq exchange. The agent’s basket consists of 46,00 shares in INTC and 900 shares in SMH. We compare the performance of the strategy, under various assumptions, to that of Almgren-Chriss where the agent models the correlation between the assets in the basket, but does not model co-integration or employ additional information from other assets.

Our simulations of the liquidation program show that additional information from other co-integrated stocks considerably boosts the performance of the strategy. For example, if the level of urgency required by the agent to liquidate the portfolio is high (resp. low) the strategy outperforms Almgren-Chriss by 4 (resp. 4.5) basis points per share. This improvement over Almgren-Chriss is due to the quality of the information provided by the co-integrated assets, and due to a speculative component of the strategy which allows the agent to repurchase shares during the liquidation horizon to take advantage of price signals. If the agent is not allowed to speculate, i.e. cannot repurchase shares, the relative savings compared to Almgren-Chriss, depending on the level of urgency, are between 2.5 to 3.5 basis points per share.
2. A  Proofs

2.A.1  Proof of Proposition 2.1

*Proof.* Substitute the ansatz (2.21) into (2.18), we see that $L^\nu H$ can be simplified to

$$
L^\nu H = \nu^\top a \nu - \nu^\top \left( (E^\top - \mathcal{X}) z + 2 C q + D \right) + \mu^\top b^\top \mathcal{X} q - z^\top (\kappa A + A \kappa^\top) z - z^\top \kappa (E q + B) + \text{Tr} \left( \Sigma^\nu A \right).
$$

(2.45)

The supremum of (2.45) is achieved at

$$
\nu^* = -\frac{1}{2} a^{-1} \left( 2 C q + (E^\top - \mathcal{X}) z + D \right).
$$

(2.46)

Substituting $\nu^*$ into (2.18) we obtain the following equality:

$$
0 = z^\top \dot{A} z + z^\top \left( B + \mathcal{L}^\mu B \right) + q^\top \dot{C} q + q^\top \left( D + \mathcal{L}^\mu D \right) + z^\top \dot{E} q + \dot{F} + \mathcal{L}^\mu F
$$

$$
-\phi (q - Q_t)^\top \tilde{\Sigma} (q - Q_t) - z^\top (\kappa A + A \kappa^\top) z - z^\top \kappa (E q + B) + \text{Tr} \left( \Sigma^\nu A \right)
$$

$$
+ \frac{1}{4} (2 C q + (E^\top - \mathcal{X}) z + D)^\top a^{-1} \left( 2 C q + (E^\top - \mathcal{X}) z + D \right).
$$

Matching the coefficients for $z^\top (\cdot) z$, $(\cdot)^\top q (\cdot) q$, $(\cdot)^\top q$, $z^\top (\cdot) q$ and the constant, and stacking $A$, $C$ and $E$ we obtain the system of matrix Riccati equations (2.22) and the linear PDEs (2.24).

2.A.2  Proof of Theorem 2.2

In this subsection we show that the solution to matrix Riccati equation (2.22) remains bounded on $[0, T]$. To show this, we require two intermediate results.

We first state the following comparison theorem (for a proof, see Theorem 2.2.2 in Kratz (2011)).
Theorem 2.7. Let $L_1(t), L_2(t), M(t), N_1(t), N_2(t) \in \mathbb{R}^{d \times d}$ be piecewise continuous on $\mathbb{R}$. Moreover, suppose $L_1(t), L_2(t), N_1(t), N_2(t)$ ($t \in \mathbb{R}$) and $S_1, S_2 \in \mathbb{R}^{d \times d}$ are symmetric. Let $T > 0$ and

$$S_1 \geq S_2, \quad L_1 \geq L_2 \geq 0, \quad N_1 \geq N_2,$$

on $[0, T]$. Assume that the terminal value problem

$$\dot{H}_1 + H_1 L_1 H_1 + MH_1 + H_1 M + N_1 = 0, \quad H_1(T) = S_1,$$

has a solution $H_1$ on $[0, T]$. Then the terminal value problem

$$\dot{H}_2 + H_2 L_2 H_2 + MH_2 + H_2 M + N_2 = 0, \quad H_2(T) = S_2,$$

has a solution $H_2$ on $[0, T]$ and $H_1(t) \geq H_2(t)$ for all $t \in [0, T]$.

From the theorem above, we can show the existence of solution to (2.22) by bounding it by another matrix Riccati differential equation, for which the solution is bounded. The candidate we consider is

$$\dot{H} + HM_1 H + HM_2 + M_1^2 H + \tilde{M}_3 = 0,$$

with terminal condition

$$H(T) = \begin{bmatrix} 0 & 0 \\ \kappa \lambda & -2\alpha \end{bmatrix},$$

where $M_1$ and $M_2$ are given by (2.23),

$$\tilde{M}_3 = \begin{bmatrix} \gamma_{\text{max}} I_n & 0 \\ 0 & 0 \end{bmatrix},$$

and $\gamma_{\text{max}}$ is the largest eigenvalue of the matrix $\frac{1}{2\alpha} \kappa \lambda \lambda^\top \Sigma^{-1} \kappa \kappa^\top$. 

Chapter 2. Trading Co-Integrated Assets with Price Impact

The following theorem explicitly characterize the solution of (2.47).

**Theorem 2.8.** Suppose $\alpha - \frac{1}{2} X b X^\intercal$ is positive definite, the matrix Riccati differential equation (2.47) admits the solution:

$$H = \begin{bmatrix} H^{11} & 0 \\ 0 & H^{22} \end{bmatrix},$$

where $H^{11}$ is given by

$$H^{11}(t) = \gamma_{\text{max}} \int_t^T e^{\kappa(t-u)} e^{\kappa^\intercal(t-u)} du,$$  \hspace{1cm} (2.48)

and $H^{22}$ is given by

$$H^{22}(t) = -((T-t) a^{-1} + (2\alpha - X b X^\intercal)^{-1})^{-1}.$$  \hspace{1cm} (2.49)

**Proof.** First, write $H$ in block form: $H(t) = \begin{bmatrix} H^{11}(t) & H^{12}(t) \\ H^{21}(t) & H^{22}(t) \end{bmatrix}$. From (2.47), it is clear that $H^{12} = (H^{21})^\intercal$. Moreover, $H^{11}$, $H^{12}$ and $H^{22}$ satisfy

$$\dot{H}^{11} + \frac{1}{2} H^{12} a^{-1} H^{21} - \kappa H^{11} - H^{11} \kappa^\intercal + \gamma_{\text{max}} I_n = 0,$$  \hspace{1cm} (2.50a)

$$\dot{H}^{12} + \frac{1}{2} H^{12} a^{-1} H^{22} = 0,$$  \hspace{1cm} (2.50b)

$$\dot{H}^{22} + \frac{1}{2} H^{22} a^{-1} H^{22} = 0.$$  \hspace{1cm} (2.50c)

It is straightforward to verify that (2.49) is a solution to (2.50c). From (2.50b) and the terminal condition, we have $H^{12}(t) = 0$ for all $t \leq T$. Moreover (2.50a) becomes

$$\dot{H}^{11} - \kappa H^{11} - H^{11} \kappa^\intercal + \gamma_{\text{max}} I_n = 0,$$

whose solution is given by (2.48).

We now state the proof of Theorem 2.2.
Proof. Theorem 2.8 asserts that (2.47) has a bounded solution on \([0, T]\), therefore, by applying Theorem 2.7, it suffices to show that \(\tilde{M}_3 \geq M_3\).

To complete this last step, we decompose \((\tilde{M}_3 - M_3)\) as

\[
\tilde{M}_3 - M_3 = \begin{bmatrix}
\gamma^{\text{max}} I_n & \kappa \mathcal{X}^\top \\
\mathcal{X} & 2\phi \Sigma
\end{bmatrix} = \begin{bmatrix}
\gamma^{\text{max}} I_n - \Gamma & 0 \\
\mathcal{X} & 2\phi \Sigma
\end{bmatrix} + \begin{bmatrix}
\Gamma & \kappa \mathcal{X}^\top \\
\mathcal{X}^\top & 2\phi \Sigma
\end{bmatrix},
\]

where \(\Gamma = \frac{1}{2\phi} \kappa \mathcal{X}^\top \tilde{\Sigma}^{-1} \mathcal{X} \kappa^\top\). Recall that \(\gamma^{\text{max}}\) is the largest eigenvalue of \(\Gamma\), hence \((\mathcal{A})\) is positive semidefinite. It remains to prove that \((\mathcal{B})\) is positive semidefinite as well.

For any \(w \in \mathbb{R}^{n+m}\), write \(w = [w_1^\top, w_2^\top]^\top\) where \(w_1 \in \mathbb{R}^n\) and \(w_2 \in \mathbb{R}^m\), then we have

\[
w^\top \begin{bmatrix}
\Gamma & \kappa \mathcal{X}^\top \\
\mathcal{X} & 2\phi \Sigma
\end{bmatrix} w
= w_1^\top \Gamma w_1 + 2w_2^\top \mathcal{X} \kappa^\top w_1 + 2\phi w_2^\top \Sigma w_2
= w_1^\top \Gamma w_1 + 2 \left(\sqrt{2\phi} \tilde{\sigma} w_2\right)^\top \left(\frac{(\tilde{\sigma}^{-1})^\top}{\sqrt{2\phi}} \mathcal{X} \kappa^\top w_1\right) + \left(\sqrt{2\phi} \tilde{\sigma} w_2\right)^\top \left(\sqrt{2\phi} \tilde{\sigma} w_2\right)
= \left(\frac{(\tilde{\sigma}^{-1})^\top}{\sqrt{2\phi}} \mathcal{X} \kappa^\top w_1 + \sqrt{2\phi} \tilde{\sigma} w_2\right)^\top \left(\frac{(\tilde{\sigma}^{-1})^\top}{\sqrt{2\phi}} \mathcal{X} \kappa^\top w_1 + \sqrt{2\phi} \tilde{\sigma} w_2\right)
\geq 0.
\]

This implies that \((\mathcal{B})\) is positive semidefinite and by the comparison principle of Theorem 2.7, the proof is complete. \(\square\)

2.A.3 Proof of Theorem 2.3

To prove the result, we need to show that (2.25a) is the unique solution to (2.24b). To do this we introduce a sequence of approximating functions that converge to the stated solution.

Let \(\Pi = \{t = t_0, t_1, \ldots, t_{n_\Pi} = T\}\) be a partition of \([0, T]\), let \(|\Pi|\) denote the cardinality of the partition \(\Pi\), and let \(\Delta t_k = (t_k - t_{k-1})\). Next, introduce the following piecewise
(left continuous with right limits) constant approximation of $\tilde{C}(t) \triangleq C^\top(t) a^{-1}$,

$$\tilde{C}^\Pi(t) := \sum_{k=1}^{||\Pi||} C^\top(t_k) a^{-1} 1_{\{t \in [t_{k-1}, t_k]\}}.$$  

The time-ordered exponential of $\tilde{C}^\Pi(t)$ is given by

$$e^{\int_0^u \tilde{C}^\Pi(s) \, ds} := e^{\tilde{C}^\Pi(t_k) (t_k - u)} \left[ \prod_{j=k+1}^{l} e^{\tilde{C}^\Pi(t_j) \Delta t_j} \right] e^{\tilde{C}^\Pi(t_{l+1})(u-t_l)}, \quad (2.51)$$

\forall t \in [t_{k-1}, t_k], \text{ and } u \in [t_l, t_{l+1}], \ l < ||\Pi||. \text{ Note that this is continuous in both } t \text{ and } u \text{ for all } t < u \in [0, T]. \text{ We next define a sequence of functions}

$$D^\Pi(t, \mu) = \mathbb{E}_{t, \mu} \left[ \int_t^T e^{\int_t^u \tilde{C}^\Pi(s) \, ds} : \mathcal{Z}_u \, du \right], \quad (2.52)$$

where we have introduced the process $\mathcal{Z} = (\mathcal{Z}_t)_{t \in [0, T]}$ and

$$\mathcal{Z}_t = \zeta(t, \mu_t) \quad \text{where} \quad \zeta(t, \mu) = 2 \phi \tilde{\Sigma} \mathbf{B} \mu + \mathbf{X} \tilde{b} \mu.$$

We require the following proposition to proceed.

**Proposition 2.9. PDE for approximating functions.** The function $D^\Pi(t, \mu)$ is the unique solution to the vector-valued PDE

$$\dot{D}^\Pi + \mathcal{L}^\mu D^\Pi + \tilde{C}^\Pi D^\Pi + \zeta(t, \mu) = 0^{(m)} \quad (2.53)$$

with terminal condition $D^\Pi(T, \mu) = 0^{(m)}$.

**Proof.** To show this, define the stochastic process $\mathfrak{D}^\Pi = (\mathfrak{D}^\Pi_t)_{t \in [0, T]}$, where

$$\mathfrak{D}^\Pi_t = : e^{\int_0^t \tilde{C}^\Pi(s) \, ds} : D^\Pi(t, \mu_t) + \int_0^t : e^{\int_0^u \tilde{C}^\Pi(s) \, ds} : \mathcal{Z}_u \, du.$$
Due the Markov property of $\mu$, we see that

$$\mathcal{D}_t^\Pi = \mathbb{E} \left[ \int_0^T : e^{\int_0^s \tilde{C}^{\Pi}(s) \, ds} : 3_u \, du \left| \mathcal{F}_t^\mu \right. \right].$$

By the integrability assumptions on the process $\mu$, this is a strict martingale. Moreover, the Markov property implies the existence of a sequence of functions $f_{\Pi} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathcal{D}_t^\Pi = f_{\Pi}(t, \mu_t)$. For any $\mathcal{F}^\mu$-stopping time $\tau \leq T$, by Dynkin’s formula we have

$$0^{(m)} = \mathbb{E} [D_{\tau}^\Pi - D_t^\Pi \mid \mathcal{F}_t^\mu],$$

$$= \mathbb{E} \left[ \int_\tau^T \{ \partial_t f_{\Pi}(u, \mu_u) + L^\mu f_{\Pi}(u, \mu_u) \} \, du \left| \mathcal{F}_t^\mu \right. \right].$$

Taking $\tau = (T - t) \wedge h \wedge \inf\{ s \geq 0 : |\mu_{t+s} - \mu_t| \geq \epsilon \}$, for $h$ small, then

$$0^{(m)} = \mathbb{E} \left[ \frac{1}{h} \int_t^\tau \{ \partial_t f_{\Pi}(u, \mu_u) + L^\mu f_{\Pi}(u, \mu_u) \} \, du \left| \mathcal{F}_t^\mu \right. \right]. \quad (2.54)$$

As $h \downarrow 0$, $\mathbb{P}(\tau \neq h) \downarrow 0$, thus taking the limit as $h \downarrow 0$, and using the fundamental theorem of calculus, we have

$$\partial_t f_{\Pi}(t, \mu_t) + L^\mu f_{\Pi}(t, \mu_t) = 0^{(m)}.$$ \hfill (2.55)

Furthermore, from (2.51),

$$\partial_t f_{\Pi}(t, \mu_t) = : e^{\int_0^t \tilde{C}^{\Pi}(s) \, ds} : \left\{ C^{\Pi}(t) D^{\Pi}(t, \mu_t) + \partial_t D^{\Pi}(t, \mu_t) + \zeta(t, \mu_t) \right\}$$

and $L^\mu f_{\Pi}(t, \mu_t) = : e^{\int_0^t \tilde{C}^{\Pi}(s) \, ds} : L^\mu D^{\Pi}(t, \mu_t)$, hence, since (2.55) holds for all paths of $\mu$, together with these two equalities, (2.55) reduces to (2.53). \hfill $\square$

Now, define the approximation error $e_{\Pi}(t, \mu) \triangleq D^{\Pi}(t, \mu) - D(t, \mu)$. Taking the
difference between (2.24b) and (2.53), we see that $\mathcal{E}^\Pi$ satisfies the linear PDE

$$(\partial_t + \mathcal{L}^\mu) \mathcal{E}^\Pi(t, \mu) + \tilde{C}^\Pi(t) \mathcal{E}^\Pi(t, \mu) + \left(\tilde{C}^\Pi(t) - \tilde{C}(t)\right) D(t, \mu) = 0^{(m)},$$

with terminal condition $\mathcal{E}^\Pi(T, \mu) = 0^{(m)}$. Applying the same argument as above, $\mathcal{E}^\Pi$ admits the representation

$$\mathcal{E}^\Pi(t, \mu) = \mathbb{E}_{t, \mu} \left[ \int_t^T \right. e^{\int_t^s \tilde{C}^\Pi(s) \, ds} \left( \tilde{C}^\Pi(s) - \tilde{C}(s) \right) D(s, \mu_s) \, du \left. \right],$$

which leads to

$$|D(t, \mu)| \leq C_2 (1 + |\mu|^2)$$

for all $t \in [0, T]$. The assumptions on $\mu^\pm$ imply that $\mu$ has a finite $L^2(\Omega \times [0, T])$-norm. Hence $D := \{D(t, \mu_t)\}_{0 \leq t \leq T}$ has a finite $L^1(\Omega \times [0, T])$-norm. The desired result follows from dominated convergence.

2.A.4 Proof of Theorem 2.4

Proof. Under the stated assumptions, the candidate solution is indeed a classical solution of the DPE. Applying standard results (e.g., Øksendal and Sulem (2005)), it suffices to check that: (i) the SDE for $Q^\nu$ has a unique solution for each given initial data; and (ii) $\nu^*_t$ is indeed an admissible control.

To verify (i), substituting the optimal control (2.26) into the dynamics of (2.3), we have the dynamics for $Q^{\nu^*}_t$:

$$dQ^{\nu^*}_t = -\frac{1}{2} a^{-1} \left( 2 C(t) Q^{\nu^*}_t + (E^*(t) - \mathcal{X}) Z_t + D(t, \mu_t) \right) dt.$$
The above equation is an ODE with stochastic source term, and it can be explicitly integrated to find

\[
Q^\nu_t = e^{-\int_0^t a^{-1}C(s)ds} : Q_0 - \int_0^t e^{-\int_0^s a^{-1}C(s)ds} : \left\{ (E^\top(u) - \mathcal{X}) Z_u + D(u, \mu_u) \right\} du. \tag{2.57}
\]

Therefore, \( Q^\nu \) has a unique solution for any initial data.

To verify (ii), it suffices to show that \( \nu_t^* \) has a finite \( L^2(\Omega \times [0, T]) \)-norm. From (2.26), it suffices to show that each of \( Z, Q^\nu \) and \( D := \{ D(t, \mu_t) \}_{0 \leq t \leq T} \) has a finite \( L^2(\Omega \times [0, T]) \)-norm. From the SDE (2.5) we see that \( Z \) satisfies this condition. Moreover, from (2.57) and Theorem 2 which implies that \( C \) and \( E \) are bounded on \([0, T] \), \( Q \) has a finite \( L^2(\Omega \times [0, T]) \)-norm if \( D \) does.

It remains to show that \( D \) has a finite \( L^2(\Omega \times [0, T]) \)-norm. From (2.25a) and the assumptions in Theorem 2, there exists a constant \( C_2 > 0 \) such that

\[
|D(t, \mu)| \leq C_2 (1 + |\mu|),
\]

for all \( 0 \leq t \leq T \) and \( \mu \in \mathbb{R}^n \). Furthermore, since the assumptions imply that \( \mu \) has a finite \( L^2(\Omega \times [0, T]) \)-norm, \( D \) also has a finite \( L^2(\Omega \times [0, T]) \)-norm, and the desired result follows. \( \square \)
2. A. 5 Calculating the coefficients for the limiting case

Substituting the power series representation (2.28) into the matrix differential equations (2.22), we obtain the following equations

\[ 0 = -\sum_{n=0}^{\infty} (n + 1) A_{n+1} \tau^n - \sum_{n=1}^{\infty} [\kappa \mathcal{A}_n + \mathcal{A}_n \kappa^\top] \tau^n + \left[ \frac{1}{2} \sum_{n=1}^{\infty} \mathcal{E}_n \tau^n \right] a^{-1} \left[ \sum_{n=1}^{\infty} \mathcal{E}_n^\top \tau^n \right], \quad (2.58a) \]

\[ 0 = \frac{\mathcal{C}_1}{\tau^2} - \sum_{n=0}^{\infty} (n + 1) \mathcal{E}_{n+1} \tau^n - \phi \Sigma + \left[ \frac{\mathcal{C}_1^\top}{\tau} + \sum_{n=0}^{\infty} \mathcal{E}_n^\top \tau^n \right] a^{-1} \left[ \frac{\mathcal{C}_1}{\tau} + \sum_{n=0}^{\infty} \mathcal{E}_n \tau^n \right], \quad (2.58b) \]

\[ 0 = -\sum_{n=0}^{\infty} (n + 1) \mathcal{E}_{n+1} \tau^n - \sum_{n=0}^{\infty} \kappa \mathcal{E}_n \tau^n + \left[ \sum_{n=1}^{\infty} \mathcal{E}_n \tau^n \right] a^{-1} \left( \frac{\mathcal{C}_1}{\tau} + \sum_{n=0}^{\infty} (\mathcal{E}_n) \tau^n \right). \quad (2.58c) \]

Matching the constant terms in (2.58a), we have \( \mathcal{A}_1 = 0 \). Matching the coefficients for \( \tau^{-2} \) in (2.58b), we have \( \mathcal{C}_1 = -\frac{1}{2} \mathcal{X}^\top \). Matching the coefficients for \( \tau^{-1} \) in (2.58b) yields the following equality

\[ \mathcal{C}_1 a^{-1} \mathcal{C}_0 + \mathcal{C}_0 a^{-1} \mathcal{C}_1 = 0. \]

Therefore \( \mathcal{C}_0 = 0 \).

Finally, by matching the constant terms in (2.58c):

\[ -\mathcal{E}_1 - \mathcal{X}^\top + \mathcal{E}_1 a^{-1} \mathcal{E}_1 = 0. \]

This implies \( \mathcal{E}_1 = -\frac{1}{2} \mathcal{X}^\top \).

It remains to show that \( D(t, \mu) \) admits the asymptotic representation \( O(\tau) \mu + O(\tau) \).

From the assumptions in Theorem 2.3, we have \( E_{0, \mu} [||\mu||] < C (1 + ||\mu||) \) for all \( 0 \leq t \leq T \) and some constant \( C > 0 \). Since we assume that \( \mu \) is Markov, we also have

\[ E_{t, \mu} [||\mu||] < C (1 + ||\mu||), \]
for $0 \leq t \leq u \leq T$. The above bound, together with \((2.25a)\), yields

$$\left| D(t, \mu) \right| \leq \int_t^T C_2 + C_3 |\mu| \, du,$$

for constants $C_2, C_3 > 0$. The desired result follows.

\[\square\]

### 2.B Parameter Estimates

In this appendix, we collect the various parameter estimates from the five Nasdaq traded stocks INTC, SMH, FARO, NTAP and ORCL.

<table>
<thead>
<tr>
<th></th>
<th>INTC</th>
<th>SMH</th>
<th>FARO</th>
<th>NTAP</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}$</td>
<td>34.233</td>
<td>51.720</td>
<td>56.338</td>
<td>43.179</td>
<td>38.885</td>
</tr>
<tr>
<td>Co-int factor</td>
<td>-0.904</td>
<td>0.763</td>
<td>0.048</td>
<td>-0.164</td>
<td>0.931</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>$0.44 \times 10^{-6}$</td>
<td>$0.71 \times 10^{-6}$</td>
<td>$0.32 \times 10^{-3}$</td>
<td>$3.05 \times 10^{-6}$</td>
<td>$1.35 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>$(2.37 \times 10^{-7})$</td>
<td>$(2.58 \times 10^{-7})$</td>
<td>$(1.62 \times 10^{-4})$</td>
<td>$(1.27 \times 10^{-6})$</td>
<td>$(0.56 \times 10^{-6})$</td>
</tr>
<tr>
<td>$\hat{\lambda}^-$</td>
<td>453.91</td>
<td>59.4</td>
<td>21.88</td>
<td>251.87</td>
<td>304.13</td>
</tr>
<tr>
<td></td>
<td>(264.63)</td>
<td>(49.46)</td>
<td>(9.25)</td>
<td>(102.72)</td>
<td>(146.83)</td>
</tr>
<tr>
<td>$E[\eta^-]$</td>
<td>1013.83</td>
<td>380.32</td>
<td>98.58</td>
<td>270.8</td>
<td>505.59</td>
</tr>
<tr>
<td></td>
<td>(306.58)</td>
<td>(121.39)</td>
<td>(15.78)</td>
<td>(55.2)</td>
<td>(100.29)</td>
</tr>
</tbody>
</table>

Table 2.5: First two rows (data November 3, 2014) show mean-reverting level $\theta$ (in dollars) and weights of the co-integrating factor. Rest of table employs data for the entire year 2014. Row 3 shows the estimates of temporary price impact. We assume no cross effects so only provide the diagonal elements of the matrix $a$, and recall we assume no permanent impact. Row 4 shows the standard deviation of the estimates in row 3. The bottom 4 rows show the average incoming rates of MOs and their average volume: $\lambda^-$ is the average number of sell MO per hour over the year 2014, $E[\eta^-]$ is the average volume of MOs. The standard deviation of the estimate is shown in parentheses.
### Chapter 2. Trading Co-Integrated Assets with Price Impact

#### Table 2.6: Estimated mean-reverting matrix $\kappa$ and t statistics.

<table>
<thead>
<tr>
<th></th>
<th>INTC</th>
<th>SMH</th>
<th>FARO</th>
<th>NTAP</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTC</td>
<td>45.66</td>
<td>-38.51</td>
<td>-2.43</td>
<td>8.26</td>
<td>-47.01</td>
</tr>
<tr>
<td></td>
<td>(10.70)</td>
<td>(8.99)</td>
<td>(0.57)</td>
<td>(1.93)</td>
<td>(11.02)</td>
</tr>
<tr>
<td>SMH</td>
<td>-19.83</td>
<td>16.73</td>
<td>1.06</td>
<td>-3.59</td>
<td>20.42</td>
</tr>
<tr>
<td></td>
<td>(13.42)</td>
<td>(11.27)</td>
<td>(0.72)</td>
<td>(2.41)</td>
<td>(13.82)</td>
</tr>
<tr>
<td>FARO</td>
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<td>34.87</td>
<td>2.20</td>
<td>-7.48</td>
<td>42.57</td>
</tr>
<tr>
<td></td>
<td>(51.50)</td>
<td>(43.27)</td>
<td>(2.75)</td>
<td>(9.27)</td>
<td>(53.05)</td>
</tr>
<tr>
<td>NTAP</td>
<td>4.98</td>
<td>-4.20</td>
<td>-0.27</td>
<td>0.90</td>
<td>-5.13</td>
</tr>
<tr>
<td></td>
<td>(12.17)</td>
<td>(10.22)</td>
<td>(0.65)</td>
<td>(2.19)</td>
<td>(12.54)</td>
</tr>
<tr>
<td>ORCL</td>
<td>-6.47</td>
<td>5.45</td>
<td>0.34</td>
<td>-1.17</td>
<td>6.66</td>
</tr>
<tr>
<td></td>
<td>(6.30)</td>
<td>(5.29)</td>
<td>(0.34)</td>
<td>(1.14)</td>
<td>(6.49)</td>
</tr>
</tbody>
</table>

#### Table 2.7: Estimated covariance matrix $\Sigma$.

<table>
<thead>
<tr>
<th></th>
<th>INTC</th>
<th>SMH</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTC</td>
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<td>0.108</td>
</tr>
<tr>
<td>SMH</td>
<td>0.108</td>
<td>0.194</td>
</tr>
</tbody>
</table>

#### Table 2.8: Estimated covariance matrix $\Sigma^{AC}$.

<table>
<thead>
<tr>
<th></th>
<th>INTC</th>
<th>SMH</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTC</td>
<td>0.131</td>
<td>0.105</td>
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<tr>
<td>SMH</td>
<td>0.105</td>
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</table>

#### Table 2.9: Estimated (with error) mean-reverting matrix $\kappa$ and t statistics.

<table>
<thead>
<tr>
<th></th>
<th>INTC</th>
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<th>FARO</th>
<th>NTAP</th>
<th>ORCL</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTC</td>
<td>73.92</td>
<td>-62.79</td>
<td>-3.50</td>
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<tr>
<td></td>
<td>(10.99)</td>
<td>(9.34)</td>
<td>(0.52)</td>
<td>(2.93)</td>
<td>(11.50)</td>
</tr>
<tr>
<td>SMH</td>
<td>8.91</td>
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<td>-0.42</td>
<td>2.36</td>
<td>-9.33</td>
</tr>
<tr>
<td></td>
<td>(14.25)</td>
<td>(12.11)</td>
<td>(0.67)</td>
<td>(3.79)</td>
<td>(14.93)</td>
</tr>
<tr>
<td>FARO</td>
<td>48.73</td>
<td>-41.39</td>
<td>-2.31</td>
<td>12.90</td>
<td>-51.04</td>
</tr>
<tr>
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<td>(50.80)</td>
<td>(43.18)</td>
<td>(2.40)</td>
<td>(13.53)</td>
<td>(53.20)</td>
</tr>
<tr>
<td>NTAP</td>
<td>11.48</td>
<td>-9.75</td>
<td>-0.54</td>
<td>3.04</td>
<td>-12.02</td>
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<tr>
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<td>(10.41)</td>
<td>(0.58)</td>
<td>(3.26)</td>
<td>(12.83)</td>
</tr>
<tr>
<td>ORCL</td>
<td>-15.83</td>
<td>13.45</td>
<td>0.75</td>
<td>-4.19</td>
<td>16.59</td>
</tr>
<tr>
<td></td>
<td>(6.39)</td>
<td>(5.44)</td>
<td>(0.30)</td>
<td>(1.70)</td>
<td>(6.70)</td>
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</table>
Table 2.10: Estimated (with error) covariance matrix $\Sigma$.

<table>
<thead>
<tr>
<th></th>
<th>INTC</th>
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<th>FARO</th>
<th>NTAP</th>
<th>ORCL</th>
</tr>
</thead>
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<tr>
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<td>0.155</td>
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<td>0.032</td>
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<td>0.061</td>
<td>0.084</td>
<td>0.033</td>
</tr>
<tr>
<td>FARO</td>
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<td>3.299</td>
<td>0.144</td>
<td>-0.006</td>
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<tr>
<td>NTAP</td>
<td>0.053</td>
<td>0.084</td>
<td>0.144</td>
<td>0.192</td>
<td>0.021</td>
</tr>
<tr>
<td>ORCL</td>
<td>0.032</td>
<td>0.033</td>
<td>-0.006</td>
<td>0.021</td>
<td>0.053</td>
</tr>
</tbody>
</table>

Table 2.11: Estimated (with error) covariance matrix $\Sigma^{AC}$.

<table>
<thead>
<tr>
<th></th>
<th>INTC</th>
<th>SMH</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTC</td>
<td>0.169</td>
<td>0.160</td>
</tr>
<tr>
<td>SMH</td>
<td>0.160</td>
<td>0.261</td>
</tr>
</tbody>
</table>
Chapter 3

Option Hedging with Limit and Market Orders

3.1 Introduction

In the classical approach to pricing and hedging contingent claims, a widespread approach to is to use Delta hedging. The objective of this hedge-portfolio is to replicate the payoff of the claim, which typically requires dynamically rebalancing the portfolio, and only under ideal conditions is the claim perfectly replicated.

The theory of dynamic replication of payoffs relies on a number of assumptions including the absence of price impact stemming from taking positions in the underlying and other trading costs. Traditional models assume that the buying and selling activity of the hedger does not affect the dynamics of the asset. This assumption contradicts the empirical finding that trades may have an impact on the prices the hedger receives, i.e. temporary impact on execution prices, and may have a permanent impact on the price of the asset.

Furthermore, assuming the hedger can take positions in the underlying asset at no cost other than the price of the asset is also incorrect in the vast majority of cases. A
clear example where this assumption fails is in order-driven equity markets. In these markets trading with aggressive market orders (MOs) is expensive because they incur exchange fees and the assets normally trade with a spread about the midprice of the asset. That is, the price paid for one share is more than the cash received from selling one share when MOs are used.

Earlier models of pricing and replication of claims have been designed without considering how assets are traded in modern electronic markets. In this chapter we show how an agent who shorts a European-style claim replicates its terminal payoff whilst taking speculative positions in the underlying asset to maximize expected utility of terminal wealth.

We consider an agent who solves a combined optimal stopping and control problem where the underlying asset is traded in an order-driven market. The agent employs a combination of limit orders (LOs) and MOs during the life of the strategy, see Guéant et al. (2012), and Cartea and Jaimungal (2015c). LOs are executed at better prices than the quoted midprice: buy LOs are executed at the midprice minus a premium and sell LOs are executed at the midprice plus a premium. The premium earned by the LOs is the depth at which the agent posts sell and buy LOs. Thus, LOs are desirable in the agent’s strategy but their execution is uncertain. If the agent requires certainty in execution, she employs MOs which are more costly because they pay exchange fees and are filled at prices worse than the quoted midprice of the asset, see Cartea and Jaimungal (2015c).

We show how the agent aims to replicate the terminal payoff of the contingent claim and simultaneously speculates in the asset to maximize expected utility of terminal wealth. In the optimal strategy, MOs are used to keep the inventory on target to replicate the payoff, and LOs are employed to: (i) build the inventory at favorable prices and without price impact, and (ii) boost expected terminal wealth by executing roundtrip trades that earn the spread.

Our model incorporates a number of important features such as permanent price
impact and adverse selection costs. Orderflow from MOs typically affect the midprice of the underlying asset, see for instance Cartea and Jaimungal (2016a). When MOs arrive in the exchange, the midprice of the asset jumps by a random amount in the direction of the trade. Under the optimal strategy, the agent accounts for this price impact when she posts LOs to protect her from adverse selection costs. These costs arise when a market participant sends a MO that is filled by the agent’s LO, and immediately after the midprice jumps in the same direction as the MO. We find that the volume of shares the agent is willing to post in the LOB depends on the exposure to adverse selection costs.

Our work is closest to that of Guéant and Pu (2015) who consider the pricing and hedging of European call options in illiquid markets (see also Guéant (2016)). Guéant and Pu assume a Bachelier model for the midprice dynamics of the underlying asset of the options. In their model the hedger only executes MOs, and does so by choosing a continuous trading rate. Illiquidity in the market precludes frictionless hedging of the options. Thus, the authors assume that MOs executed by the hedger have (linear in the speed of trading) permanent impact on the price of the underlying and incur execution costs (also a function of the speed of trading MOs).

We depart from Guéant and Pu in a number of aspects. The main difference is that in our model the hedger employs both LOs and MOs in the trading strategy. LOs are continuously updated in the market, and MOs are modeled as an impulse control. In practice, hedgers draw on both types of order to replicate options at the most favourable prices, and more so in illiquid markets where MOs: affect prices, may walk the limit order book (LOB), and pay exchange fees. On the other hand, LOs are filled at better prices than the midprice of the underlying – LOs earn at least half the bid-ask spread and pay no fees. In our model, MOs from all market participants (not only the hedger’s) have a permanent impact on the price of the underlying asset. Conditional on an MO arriving in the exchange, the underlying price jumps in the direction of the trade. Finally, in our
model the hedger not only targets an amount of shares or cash to settle the options, she also executes speculative trades to maximize expected utility of wealth.

In recent years, advances in optimal execution have inspired a new stream of literature. Rogers and Singh (2010) formulated a stochastic control problem where an agent minimizes the integrated squared error between her inventory process and the option delta. A similar formulation can be found in Naujokat and Westray (2011), and Bank et al. (2015), where the former allows the agent to post passive orders and the latter allows for non-Markovian controls. See also Almgren and Li (2016) for a model with permanent price impact.

Moreover, in the extant literature, several attempts have been made to incorporate transaction costs when trading assets. Leland (1985) proposes a discrete-time hedging strategy as an alternative to the Black-Scholes model when the transaction costs are proportional to the volumes traded in a given time interval. Barles and Soner (1998) use indifference pricing to derive pricing formulae for European call options under proportional transaction costs and also assume that transaction costs are proportional to the volumes traded. Other examples include Boyle and Vorst (1992), Davis et al. (1993), and Cvitanić and Karatzas (1996).

Another stream of literature examines the effects of market impact on option pricing. Cetin et al. (2004) extended the Black-Scholes framework by adding a stochastic supply curve for the underlying asset. Their model was later extended by Roch (2011) to incorporate permanent price impact. A similar set-up was also proposed by Bank and Baum (2004).

The remainder of the chapter is organized as follows. Section 3.2 describes the model components and the optimal control problem solved by the agent, and in Section 3.3 we derive its associated dynamic programming equation (DPE). Section 3.4 presents a numerical example of our strategy’s performance when the agent sells a call option written on the S&P E-Mini Futures. Section 3.5 contains proofs for the convergence of
the numerical scheme we adopt. Section 3.7 concludes and we collect other proofs in the Appendix.

3.2 Model

Fix a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq \hat{T}})\), where \(\mathbb{F}\) is the natural filtration generated by the collection of observable processes to be described below. At time \(t = 0\) the agent sells an option written on a traded asset. The option expires at time \(\hat{T}\) and can only be exercised by the buyer at expiry, i.e. is a European-style claim. The agent is risk-averse and trades the underlying asset of the option over the time window \([0, T]\) to maximize her expected utility at time \(T\), where \(T \leq \hat{T}\).

The asset is traded in an order-driven exchange that matches orders from participants who provide liquidity using LOs with those who take liquidity by executing MOs. The agent employs four types of order to trade the underlying asset: market buy order (MBO), market sell order (MSO), limit buy order (LBO), and limit sell order (LSO).

Below we propose the dynamics of the midprice of the traded asset and describe the dynamics of trading activity, including that of LOs and MOs.

Bid-ask spread and tick size. We denote by \(S = (S_t)_{t \in [0,T]}\) the midprice of the underlying asset. Midprices take on values on a grid determined by the tick size in the exchange. Here we denote the tick size by \(\sigma > 0\) and assume that the asset trades with a bid-ask spread fixed at \(\sigma\).

Agent’s limit orders. The agent’s LOs are always posted at the best bid or best ask, i.e. LSOs are posted at the price level \(S_t + \frac{\sigma}{2}\), and LBOs at the price level \(S_t - \frac{\sigma}{2}\). The volume of the LBOs posted by the agent is denoted by the process \(\ell^+ = (\ell^+_t)_{t \in [0,T]}\) and the volume of the LSOs is denoted by the process \(\ell^- = (\ell^-_t)_{t \in [0,T]}\).

Agent’s market orders. We denote by \(\tau^{0+} = (\tau^{0+}_k)_{k \geq 1}\) the times at which the agent sends MBOs and denote their volume by \(m^+ = (m^+_k)_{k \geq 1}\). Similarly, \(\tau^{0-} = (\tau^{0-}_k)_{k \geq 1}\) are
the times at which the agent sends MSOs and denote their volume by \( m^- = (m_k^-)_{k \geq 1} \).
Here, \( \tau^{0\pm} \) are sequences of increasing stopping times and \( m^+_k \) (resp. \( m^-_k \)) are \( \mathcal{F}_{\tau^+_k} \)-measurable (resp. \( \mathcal{F}_{\tau^-_k} \)-measurable) strictly positive integer-valued random variables. The values that \( m^\pm_k \) takes on are specified in Subsection 3.2.1.

We define the processes \( M^{0+} = (M^{0+}_t)_{t \in [0,T]} \) and \( M^{0-} = (M^{0-}_t)_{t \in [0,T]} \) via
\[
M^{0+}_t = \sum_{k=1}^{\infty} 1_{\tau^+_k \leq t}, \quad \text{and} \quad M^{0-}_t = \sum_{k=1}^{\infty} 1_{\tau^-_k \leq t},
\]
as the total number of MBOs and MSOs that the agent has executed up to time \( t \), respectively. We also define the processes \( M^{0+} = (M^{0+}_t)_{t \in [0,T]} \) and \( M^{0-} = (M^{0-}_t)_{t \in [0,T]} \) via
\[
M^{0+}_t = \sum_{k=1}^{\infty} m^+_k 1_{\tau^+_k \leq t}, \quad \text{and} \quad M^{0-}_t = \sum_{k=1}^{\infty} m^-_k 1_{\tau^-_k \leq t},
\]
as the accumulated volume of MBO and MSO the agent has executed up to time \( t \), respectively.

**Other market participants’ market orders.** We assume that there is enough liquidity at the best prices to fill incoming MOs. Thus, all MSOs are executed at the price \( S_t - \Upsilon \) and all MBOs are executed at \( S_t + \Upsilon \), where \( \Upsilon \) consists of the half-spread \( \frac{\sigma}{2} \) (i.e. half a tick) plus a liquidity taking fee.

MBOs and MSOs sent by other market participants arrive in the exchange at the arrival times of a homogeneous Poisson processes. Buy orders arrive with intensity \( \lambda^+ \) and sell orders with intensity \( \lambda^- \). The counting processes for buy and sell MOs are denoted by \( M^+ = (M^+_t)_{t \in [0,T]} \) and \( M^- = (M^-_t)_{t \in [0,T]} \), respectively, with associated event times denoted by \( \tau^+ = (\tau^+_k)_{k \geq 1} \) and \( \tau^- = (\tau^-_k)_{k \geq 1} \), respectively. We assume that the size of incoming MOs is always larger than the maximum volume the agent posts, so that her LOs are always executed in full when they are executed.

Conditional on the arrival of the \( k^{th} \) MO, the agent’s LO is filled according to a
Bernoulli random variable $\zeta_k$ with probability of success $\rho$. The random variables $\zeta_k$ are independent of all other random variables, and i.i.d.. We denote by $L^+ = (L_t^+)_{t \in [0,T]}$ and $L^- = (L_t^-)_{t \in [0,T]}$ the counting processes for the agent’s filled LBOs and LSOs respectively.

**Midprice dynamics.** Changes in the midprice of the asset reflect the arrival of information that affects the value of the asset. We assume that buying and selling pressure from MOs has an impact on the midprice, and other information and news also affect the midprice of the asset, see for example Cartea and Jaimungal (2015b).

Changes in the midprice are given by

$$dS_t = \sigma (dP_t^+ - dP_t^-), \quad (3.1)$$

where $P_t^+ = (P_t^+)_{t \in [0,T]}$ is the upward pressure on the midprice exerted by market participants as a result of MBOs and reshuffling of limit sell orders. Similarly, $P_t^- = (P_t^-)_{t \in [0,T]}$ represents the downward pressure on the midprice as a result of MSOs and changes in LOs.

We assume that $P_t^\pm$ have the following form:

$$P_t^+ = Z_t^+ + \sum_{k=1}^{M_t^+} \xi_k^+ + \sum_{k=1}^{M_0^+} \xi_k^0 + M_0^+,$$

* and

$$P_t^- = Z_t^- + \sum_{k=1}^{M_t^-} \xi_k^- + \sum_{k=1}^{M_0^-} \xi_k^0 - M_0^-.$$  

(3.2)

We explain the various objects in the three terms on the right-hand side of (3.2).

The first term, represents exogenous changes in the midprice due to reshuffling of limit sell orders as a result of information and news that are impounded in the midprice of the asset. Specifically, $Z^\pm = (Z_t^\pm)_{t \in [0,T]}$ are independent Poisson processes with intensity $\theta$, and are independent of all other processes and random variables.

The second term, $\sum_{k=1}^{M_t^+} \xi_k^+$, is a compound Poisson process that represents the price impact of MBOs sent by all market participants (excluding the agent). Moreover, $\xi^+ = \ldots$
\((\xi_k^\pm)_{k \geq 1}\) is a sequence of i.i.d. Bernoulli random variables with constant success probability \(\alpha\). Recall that \(M_t^+\) is a homogeneous Poisson process with arrival rate \(\lambda^+\), thus, every time an MBO arrives, the midprice jumps by \(\xi^+\).

The third term, \(\sum_{k=1}^{M_t^+} \xi_k^{0+}\), represents the price impact of the agent’s MBOs. Upon the agent sending a buy MO, the midprice jumps up by the random amount \(\xi^{0+}\). These midprice innovations are characterized by \(\xi^{0+} = (\xi_k^{0+})_{k \geq 1}\), which is a sequence of \(\mathcal{F}_{\tau_k^{0+}}\)-measurable Bernoulli random variables. Each \(\xi_k^{0+}\) has probability of success \(\beta^+(m_k^+),\) where \(\beta^+: \mathbb{Z}_+ \rightarrow [0, 1]\) is a deterministic function.\(^1\)

The three terms on the right-hand side of (3.3) have a similar interpretation. The first term represents changes in the midprice due to reshuffling of LOs in the sell side of the book. The second term represents the price impact of market participants’ MSOs, where \(\xi^- = (\xi_k^\pm)_{k \geq 1}\) is a sequence of i.i.d. Bernoulli random variables with constant probability of success \(\alpha\). And the third term is the price impact of the agent’s MSOs, where \((\xi_k^0)^{0-})_{k \geq 1}\) is a sequence of \(\mathcal{F}_{\tau_k^{0-}}\)-measurable Bernoulli random variables. Each \(\xi_k^{0-}\) has probability of success \(\beta^-(m_k^-)\).

### 3.2.1 The agent’s optimization problem

The agent’s objective is to maximize expected utility of wealth at time \(T\). Recall that the agent sells a contingent claim expiring at \(\hat{T} \geq T\) and trades the underlying asset over the time window \([0, T]\). We denote by \(Q = (Q_t)_{t \in [0, T]}\) the agent’s inventory, which takes on integer values and we restrict the strategies so that \(q \leq Q_t \leq \bar{q}\), where \(\bar{q}\) and \(q\) are the maximum and minimum inventory level the agent is willing to hold. As a result of the agent’s LO and MO activity, the inventory \(Q_t\) process satisfies the stochastic differential equation (SDE)

\[
dQ_t = \ell_{t^-} dL_t^- - \ell_{t+} dL_t^+ + dM_t^{0+} - dM_t^{0-},
\]  
(3.4)

\(^1\)We could assume that the price impact of other market participants’ MOs is a function of the volume of the order.
and the volume of the agent’s LOs and MOs are such that

\[
\ell^+ t \in \{0, 1, \ldots, q - Q_t\}, \quad (3.5a)
\]
\[
\ell^- t \in \{0, 1, \ldots, Q_t - q\}, \quad (3.5b)
\]
\[
m^+ k \in \{0, 1, \ldots, q - Q_{t_k}^{-}\}, \quad (3.5c)
\]
\[
m^- k \in \{0, 1, \ldots, Q_{t_k}^{-} - q\}, \quad (3.5d)
\]

for \(0 \leq t \leq T\) and \(k \geq 1\), so the strategy obeys the inventory constraints.

The agent’s wealth process is represented by \(X = (X_t)_{t \in [0, T]}\) and satisfies the SDE

\[
dX_t = -\left(S_t - \frac{\sigma}{2}\right) \ell^- t dL_t^- + \left(S_t + \frac{\sigma}{2}\right) \ell^+ t dL_t^+ + (S_t - \Upsilon) dM_t^0 - (S_t + \Upsilon) dM_t^0. \quad (3.6)
\]

The first two terms in the right-hand side of (3.6) represent changes in the agent’s cash position due to filled LOs. The last two terms represent changes in cash due to the agent’s executed MOs.

We denote by \(G: (\sigma Z) \mapsto \mathbb{R}\) the value function of the contingent claim. At time \(t = 0\) the agent receives cash for the contingent claim, and at expiry the value of the claim is \(G(S_T)\). We assume the agent’s initial wealth \(X_0\) consists of the premium she obtains from selling the contingent claim. Below, in subsection 3.3.1 we show how the agent employs indifference pricing to calculate the premium charged for the claim. In the remainder of the analysis, for simplicity we assume the expiry of the contingent claim and the agent’s horizon are the same, i.e. \(\hat{T} = T\).

The agent’s preferences are given by an exponential utility function of wealth and her value function is

\[
H(t, x, s, q) = \sup_{(\ell^\pm, \tau^\pm, m^\pm) \in \mathcal{A}} \mathbb{E}_{t,x,s,q} \left[ -e^{-\gamma (X_T + S_T Q_T - G(S_T) - U(Q_T, S_T))} \right], \quad (3.7)
\]

where \(\gamma > 0\) is the agent’s risk-aversion parameter and \(\mathbb{E}_{t,x,s,q} [\cdot]\) denotes the \(\mathbb{P}\) ex-
pectation conditional on the initial condition $X_{t^-} = x$, $Q_{t^-} = q$, and $S_{t^-} = s$. Here, $\mathcal{A}$ denotes the set of admissible strategies in which $\ell^\pm = (\ell^\pm_t)_{0 \leq t \leq T}$ are $\mathcal{F}$–adapted processes, $\tau^{0\pm}_k = (\tau^{0\pm}_k)_{k \geq 1}$ are sequences of $\mathcal{F}$–stopping times and $m^\pm = (m^\pm_k)_{k \geq 1}$ are sequences of $\mathcal{F}_{\tau^{0\pm}_k}$–measurable random variables. Moreover, $\ell^\pm$ and $m^\pm$ satisfy (3.5) pointwise. The function $U : \mathbb{Z} \times (\sigma \mathbb{Z}) \mapsto \mathbb{R}$ represents other costs the agent incurs at the terminal date to account for any additional terminal transaction costs. For example, if the option is physically settled, the agent may have to adjust her terminal inventory position to settle the option or unwind inventory that is not required (see the example in (3.13a) which corresponds to the case of physically settling a call option).

### 3.3 The Dynamic Programming Equations

To solve the combined optimal stopping and control problem, we employ the dynamic programming principle. Standard results imply that the value function (3.7) is the unique viscosity solution to the quasi-variational inequality (QVI)

$$
\begin{align*}
\max & \left\{ (\partial_t + \mathcal{L})H(t, x, s, q) \right. \\
+ & \left. \max_{\ell^+ \in \{0, \ldots, q-q\}} \left\{ \lambda^+ \mathbb{E} \left[ H(t, x - \ell^+ \left(s - \frac{\sigma}{2}\right), s - \sigma \xi, q + \ell^+ \zeta) - H(t, x, s, q) \right] \right\} \right. \\
+ & \left. \max_{\ell^- \in \{0, \ldots, q-q\}} \left\{ \lambda^- \mathbb{E} \left[ H(t, x + \ell^- \left(s + \frac{\sigma}{2}\right), s + \sigma \xi, q - \ell^- \zeta) - H(t, x, s, q) \right] \right\} \right. \\
+ & \left. \max_{m^+ \in \{1, \ldots, q-q\}} \left\{ \mathbb{E} \left[ H(t, x + m^+ \left(s + \Upsilon\right), s + \sigma \xi^0 + q + m^+) - H(t, x, s, q) \right] \right\} \right. \\
+ & \left. \max_{m^- \in \{1, \ldots, q-q\}} \left\{ \mathbb{E} \left[ H(t, x + m^- \left(s - \Upsilon\right), s - \sigma \xi^0, q - m^-) - H(t, x, s, q) \right] \right\} \right. \\
\right. = 0,
\end{align*}
$$

subject to the terminal condition

$$
H(t, x, s, q) = -e^{-\gamma(x + s - q - G(s)) - U(s, q)}.
$$
The infinitesimal generator $\mathcal{L}$ in (3.8) acts on a smooth functions $g(t, x, s, q)$ as follows:

$$\mathcal{L}g(t, x, s, q) = \theta \left( g(t, x, s + \sigma, q) + g(t, x, s - \sigma, q) - 2g(t, x, s, q) \right).$$

The expectation operators in the second and third lines of the QVI are with respect to the random variables $\zeta, \xi$, and the expectation operator in the last two lines of the QVI is with respect to the random variables $\xi^{0\pm}$. Recall that $\zeta, \xi, \xi^{0\pm}$ are independent Bernoulli random variables with probability of success $\rho, \alpha, \beta^{\pm}(m^{\pm})$ respectively. For detailed derivation of the above DPE, see Pham (2009).

The terms in the QVI have the following interpretation. The term on the right-hand side of the first line of (3.8) represents the changes in the value function due to time and exogenous changes in the midprice that result from reshuffling of LOs.

The second line represents the changes in the value function due to other market participants' incoming MSOs, which are filled by the agent’s LBOs with probability $\rho$, and which generate price impact on the midprice (represented by $\xi$).

Similarly, the third line represents the changes in the value function due to other agents’ MBOs. Finally, the fourth and fifth lines account for changes in the value function due to the agent’s MOs.

To simplify (3.8), we adopt the ansatz $H(t, x, s, q) = -e^{-\gamma(x + sq + h(t, s, q))}$ and derive
the QVI satisfied by \( h(t, s, q) \):

\[
0 = \min \left\{ -\gamma \partial_t h(t, s, q) + \theta \left( e^{-\gamma (q \sigma + h(t, s, q)) - h(t, s, q))} + e^{-\gamma (-q \sigma + h(t, s, q)) - h(t, s, q))} \right) - 2 \right. \\
+ \lambda^- \min_{\ell^+ \in \{0, \ldots, \bar{q} - q\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left( \ell^+ \zeta - h(t, s, q)) - h(t, s, q)) \right) - 1} \right] \right\} \\
+ \lambda^+ \min_{\ell^- \in \{0, \ldots, \bar{q} - q\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left( \ell^- \zeta - h(t, s, q)) - h(t, s, q)) \right) - 1} \right] \right\} ; \\
\min_{m^+ \in \{0, \ldots, \bar{q} - q\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left( -m^+ \zeta + h(t, s, q)) - h(t, s, q)) \right) - 1} \right] \right\} ; \\
\min_{m^- \in \{0, \ldots, \bar{q} - q\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left( -m^- \zeta + h(t, s, q)) - h(t, s, q)) \right) - 1} \right] \right\} ,
\]

(3.9)

subject to the terminal condition \( h(T, s, q) = -G(s) - U(s, q) \).

### 3.3.1 Indifference price of contingent claim

The agent determines the premium for the options she sells using indifference pricing.

We define the \textit{indifference price} of the options as the amount \( I \geq 0 \) that makes the agent indifferent between the following two scenarios:

1. The agent does not sell contingent claims. She employs all the aforementioned order types (MBO, MSO, LBO and LSO) to maximize her expected utility at time \( T \), subject to the same inventory constraint \( q \leq Q_t \leq \bar{q} \).

2. At time \( t = 0 \) the agent receives the cash amount \( I \) and commits to delivering the options payoffs at time \( T \). Again employing all aforementioned order types to maximize her expected utility.

In Scenario 1, the agent solves the optimal control problem:

\[
\tilde{H}(t, x, s, q) = \sup_{(\ell^\pm, \tau^\pm, m^\pm) \in \mathcal{A}} \mathbb{E}_{t, x, s, q} \left[ -e^{-\gamma (X_T + S_T Q_T - \tilde{U}(Q_T))} \right] ,
\]

(3.10)
where \( \tilde{U}(Q_T) \) represents other (transaction) costs that arise from unwinding terminal inventory.

Classical results suggest that \( \tilde{H} \) satisfies (3.8), subject to the terminal condition

\[
\tilde{H}(T, x, s, q) = -e^{-\gamma(x + s q - \tilde{U}(q))}.
\]

By adopting the ansatz \( \tilde{H}(t, x, s, q) = -e^{-\gamma(x + s q + \tilde{h}(t, s, q))} \), we can show that \( \tilde{h}(t, s, q) \) satisfies (3.9) subject to the terminal condition \( \tilde{h}(T, s, q) = -\tilde{U}(q) \).

On the other hand, in Scenario 2, the agent solves the problem where the indifference price \( I(t, x, s, q) \) satisfies

\[
H(t, x + I(t, x, s, q), s, q) = \tilde{H}(t, x, s, q).
\]

It is straightforward to see that \( I \) can be expressed in terms of \( h(t, s, q) \) and \( \tilde{h}(t, s, q) \):

\[
I(t, s, q) = \tilde{h}(t, s, q) - h(t, s, q),
\]

where we suppress \( x \) in \( I(t, x, s, q) \) because there is no dependency on wealth.

**The value of limit orders**

The agent maximizes expected utility of terminal wealth by: (i) trading in the underlying asset to ensure that the payoff of the options are settled, and (ii) taking speculative positions in the underlying asset to benefit from roundtrip trades. A key driver of revenues in the agent’s strategy is the use of LOs, in both (i) and (ii), because they are priced at the midprice plus or minus half-spread when selling or buying the asset, respectively.

Analogous to finding the value of the option using indifference pricing, we may also use indifference pricing to determine the value of employing LOs in the agent’s strategy.
To this end, consider the following problem

$$
\hat{H}(t,x,s,q) = \sup_{(\tau^0, m^\pm) \in \hat{A}} \mathbb{E}_{t,x,s,q} \left[ -e^{-\gamma (X_T + S_T Q_T - G(S_T) - U(S_T,Q_T))} \right],
$$

(3.11)

where $\hat{A}$ denotes the set of strategies $\{(\ell^\pm, \tau^0, m^\pm) \in A : (\ell^\pm)_{t \in [0,T]} = 0\}$, i.e., the set of admissible strategies where the agent uses MOs only. By adopting the ansatz $\hat{H}(t,x,s,q) = -e^{-\gamma (x+s q + \hat{h}(t,s,q))}$, we can show that $\hat{h}(t,s,q)$ satisfies (3.9) subject to the terminal condition $\hat{h}(T,s,q) = -G(s) - U(s,q)$. The agent’s indifference price of LOs is defined as the function $I^{lo}(t,x,s,q)$ such that

$$
H(t,x,s,q) = \hat{H}(t,x + I^{lo}(t,x,s,q), s, q).
$$

Thus, the value of the LOs is given by

$$
I^{lo}(t,s,q) = h(t,s,q) - \hat{h}(t,s,q),
$$

(3.12)

where $h(t,s,q)$ solves the QVI (3.9), and provides the agent’s value function when she can employ LOs and MOs in the strategy. We suppress $x$ in $I^{lo}(t,x,s,q)$ because there is no dependency on wealth.

### 3.4 Numerical Example

In this section we employ numerical methods to illustrate the performance and features of the agent’s strategy. At time $t = 0$ the agent sells European calls written on the S&P 500 E-Mini futures and trades in the underlying to maximize expected utility of terminal wealth.

**Data** We employ high-frequency data from the Chicago Mercantile Exchange to estimate the model parameters. We focus on the S&P E-Mini contract that matures on July
20, 2014 and use messages sent to the exchange to build the LOB on March 24, 2014. Our
data set contains all messages (FIX format) that traders see to track the liquidity makers
and takers in the exchange. Finally, we employ data between 10:00 and 15:30 Eastern
time to exclude the excessive trading activity which normally occurs around market open
and close.

The E-mini contract is worth $50 \times (S&P 500 \text{ index}) and has a ticksize \( \sigma = \$12.5 \).
In the sequel, we report the price of the contract as the original price divided by 50 to
reflect the value of the index.

Parameter Estimation

- **Arrival rate of MOs.** To estimate the rate of MO arrivals, we divide the total
  number of MOs (buy and sell) in the data set by the total time of 5.5 hours. Next,
  we count the total number of price changes not due to an incoming MO, divide it
  by the total time 5.5 hours, and obtain an estimate of the rate of exogenous price
  changes, which in our model is represented by the parameter \( \theta \). The estimates for
  the rate of MO arrivals and the rate of exogenous price changes are for both sides
  of the LOB.

- **Permanent price impact.** For an estimate of the probability of midprice changes
  after MO arrivals from other traders we use the proportion of MOs that result in a
  midprice change, (i.e., consume at least the first level of the LOB) to estimate the
  parameter \( \alpha \).

- **Fill rate of LOs.** Finally, to estimate the probability of the agent’s LO being
  filled by an incoming MO, i.e. the parameter \( \rho \), we assume that the position of
  the agent’s LO in the queue is uniformly distributed in the best bid or ask level
  when a MO arrives. Therefore, the probability that the agent’s LO is filled by an

\footnote{FIX is the protocol that CME employ to communicate changes in the LOB. See
Table 3.1: Estimated model parameters. Numbers in brackets are standard errors.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>rate of exogenous midprice changes</td>
<td>107.7 (4.4)/hour</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>rate of arrival of other participants MOs</td>
<td>4492.4 (28.6)/hour</td>
</tr>
<tr>
<td>$\rho$</td>
<td>probability of agent’s LO is filled</td>
<td>0.0171 (0.008)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>probability of price impact of other participants’ MOs</td>
<td>0.094 (0.001)</td>
</tr>
</tbody>
</table>

incoming MO is $p_i = M_i / L_{i\tau}$, where $M_i$ is the size of the $i$-th MO and $L_{i\tau}$ is the total volume posted in the first level of the LOB when the $i$-th MO arrives. We report the median of $p_i$ as our estimate of $\rho$.

Table 3.1 reports parameter estimates.

Agent’s optimal trading

We assume that the agent sells $N$ call options, and denote by $S$ the price of the underlying S&P E-Mini Futures. All options expire at time $T$ and have the same strike price $K$. The options are physically settled, i.e. if the option expires in-the-money, the agent delivers $N$ shares and collects the cash amount $N \times K$.

The options’ payoffs and other costs incurred by the agent at the terminal date $T$ are given by

\begin{align}
G(S_T) &= N (S_T - K)_+ , \\
U(S_T, Q_T) &= U(Q_T, N) 1_{S_T \geq K} + U(Q_T, 0) 1_{S_T < K} ,
\end{align}

where the operation $(x)_+$ represents $\max(x, 0)$, $U : \mathbb{Z}^2 \mapsto \mathbb{R}$, and $U(q_1, q_2)$ denotes other costs and fees that the agent incurs when changing her inventory position from $q_1$ to $q_2$. For example, if the option expires in-the-money, and the accumulated inventory is $Q_T < N$, the agent must purchase $N - Q_T$ shares to physically settle the options. The cost of these extra shares consists of the sum of: (i) $S_T Q_T - N (S_T - K) = (N - Q_T) S_T$, which is the accumulated inventory at the terminal date $S_T Q_T$ minus the payoff of the
option (3.13a), plus (ii) the cost \( \Upsilon \), which includes the half-spread and liquidity taking fees, in addition to any other fees that may arise from executing the trade – these costs are included in (3.13b).

In the results of the simulations reported below we assume that the agent is short \( N = 10 \) call options. At the time of selling the options the underlying asset is trading at \( S_0 = 1,856.50 \) and we assume that the options are written at-the-money, thus \( K = S_0 \). Transaction fees and the penalty imposed by the agent stemming from adjusting the inventory position at time \( T \) are given by the function

\[
U(q_1, q_2) = \Upsilon \varphi |q_1 - q_2|,
\]

where \( \varphi \geq 1 \) is a terminal penalty parameter. Recall that \( \Upsilon \) consists of the half-spread \( \frac{\sigma}{2} \) plus a liquidity taking fee. Thus, \( \varphi = 1 \) represents the costs of unwinding the terminal inventory position in the exchange. When, \( \varphi > 1 \), \( U \) contains additional inventory penalties which do not affect the agent’s wealth, but do affect the agent’s trading strategy. For example, a high value of the penalty parameter \( \varphi \) forces the strategy to reach the terminal date \( T \) with the correct amount of shares to settle the option. In the simulations we assume \( \varphi = 1.05 \), which makes the strategy reach the terminal date with the required level of inventory, i.e. \( Q_T = N \) if the options ends up in-the-money or \( Q_T = 0 \) if the options expire out-of-the-money.

Table 3.2 presents the remaining parameters required to perform the simulations. In particular, it shows the expiry of the options, the tick size of the exchange, the half-spread, the price impact function of the agent’s MOs, the agent’s risk-aversion parameter, initial inventory, and inventory constraints.
Table 3.2: Other Parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>time to maturity</td>
<td>1 hour</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>bid-ask spread (tick size)</td>
<td>0.25</td>
</tr>
<tr>
<td>$\Upsilon$</td>
<td>half-spread</td>
<td>0.125</td>
</tr>
<tr>
<td></td>
<td>(assuming no MO fee)</td>
<td></td>
</tr>
<tr>
<td>$\beta^\pm(m)$</td>
<td>agent’s price impact</td>
<td>$0.01 \times m$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>agent’s risk-aversion parameter</td>
<td>0.1</td>
</tr>
<tr>
<td>$q_0$</td>
<td>initial inventory</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>maximum inventory</td>
<td>10</td>
</tr>
<tr>
<td>$q$</td>
<td>minimum inventory</td>
<td>0</td>
</tr>
</tbody>
</table>

3.4.1 Features of the optimal strategy

In this section we solve the QVI system (3.9) numerically to illustrate a number of features of the agent’s strategy. Below, in Subsection 3.4.2 we perform simulations to explore the performance of the agent’s strategy, and in Section 3.5 we provide details of our numerical scheme and proof of its convergence.

Here we discuss how the agent employs MOs and LOs depending on the level of inventory, midprice, and remaining time to expiry of the contingent claims to maximize her expected utility of terminal wealth.

Figure 3.1 shows heat-maps of the agent’s optimal strategy when the inventory is fixed at $q = 0$. On the left panel, the colored areas show the regions in which the agent executes MOs of various volumes, and the empty (i.e. white) area shows the region
in which she does not execute any MOs. Similarly, the right panel shows the optimal posting of buy LOs across time and moneyness of the options.

In particular, the left panel of Figure 3.1 shows that as the option moves deeper in-the-money, the agent executes larger volumes of MBOs in anticipation of the option expiring in-the-money. The left panel also shows that everything else equal, when the option is in-the-money, as the option expiry approaches, the agent executes larger volumes of MBOs.

On the other hand, the right panel of Figure 3.1 shows that when the option is out-of-the-money and the stock price is trading not ‘too far’ from the strike, the agent posts LBOs with volume of 5 shares. As the option moves further out-of-the-money, the agent posts smaller LBOs. Moreover, at a fixed asset price, as maturity approaches, the posted volume decreases.

Figure 3.2 shows the optimal strategy when the inventory is fixed at \( q = 5 \), recall that the agent is short \( N = 10 \) call options. When the option is close to at-the-money, the agent does not execute MOs, but rather posts LOs on both sides of the LOB to maximize revenues by earning the spread from speculative roundtrip trades. As time approaches maturity, if the option is in-the-money, the agent increases the volume of the LBOs and eventually executes a MBO, but still posts sell LOs to maximize revenues from speculative trades. Similarly, if the option is out-of-the-money, the agent increases the volume of the LSOs and, nearer expiry, the strategy eventually executes an MSO, but still posts buy LOs to maximize revenues from speculative roundtrip trades. Moreover, there are regions where the agent posts only LBOs (when the option is moderately in-the-money) or only LSOs (when the option is moderately out-of-the-money), but if the option moves significantly in- or out-of-the-money, she executes MBOs or MSOs.

There are other notable features here. Figure 3.2(a) contains two inner boundaries separating the plane into three different regions: the upper colored region, the middle white region, and the lower colored region. Inside the middle region, the agent does not
execute MOs, and as soon as a boundary is reached, the agent immediately executes a MBO (upper region) or MSO (lower region) and the inventory jumps to a different level. As a result, any point \((t, S)\) in the interior of the colored regions is not attainable unless the agent starts with this particular inventory level at time \(t\) and \(S_t = S\). Figure 3.3 shows the strategy’s inner-most boundaries for different levels of inventory. The left (right) panel shows the boundaries for a given inventory level that the agent would execute a MBO (MSO) when the asset price lies above (below) that boundary.

Next we focus on a sample path to illustrate how the strategy behaves dynamically. To this end, Figure 3.4(a) shows a sample path of the midprice and the strike price is
Figure 3.3: Inner-most boundaries of the region where agent immediately executes an MO. At each point in the left (right) panel, the color represents the minimum inventory level such that agent immediately executes a MBO (MSO) when the midprice lies below(above) the boundary for that inventory level.

depicted by a horizontal line. For this particular realization of the midprice, the option stays in-the-money early on, then switches moneyness a few times before maturity and eventually expires out-of-the-money.

Panel 3.4(b) shows the strategy’s acquired inventory, where solid circles denote the execution of MOs. For this particular price path, most of the changes in the strategy’s inventory are due to filled LOs. Moreover, early on, the strategy sends an MO (indicated by the red dot) taking the inventory to $q = 2$, and near expiry, when the midprice is dropping, and in anticipation of the options expiring out-of-the-money, the strategy executes a series of sell MOs (indicated by the green dots), so at expiry $q_T = 0$.

To glean additional insight into what the strategy is doing, panel 3.4(c) shows the inventory path which the strategy targets using LOs. In particular, the red and blue lines represent the levels at which the inventory would jump if the posted LOs were filled, i.e. $Q_t + \ell^+_t$ if the LBO is filled and $Q_t - \ell^-_t$ if the LSO is filled. For this simulation, the agent posts both LBOs and LSOs simultaneously most of the time. In doing so, the agent is aiming to profit from the bid-ask spread while simultaneously hedging the option payoff.

For illustrative purposes, panel 3.4(c) also shows the inventory path that would result
if the agent employs the classical delta strategy to hedge the options using MOs. Without market impact and adverse selection, the high-frequency limit of our model results in a Brownian motion mid-price dynamics. Hence, the Delta-hedge strategy shown here is that resulting from the Bachelier model. To calculate the delta-hedge positions we assume that the midprice follows the arithmetic Brownian motion

\[ S_t = \sigma^B W_t, \quad (3.15) \]

where \( W = (W_t)_{t \in [0,T]} \) is a standard Brownian motion, and we choose the volatility
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Figure 3.5: Example of an instance where the optimal inventory diverges from the Bachelier Delta, $\gamma = 1$. At time $t = 0.65$, the Bachelier delta is outside the band.

Parameter $\sigma^B$ so that it matches the volatility of (3.1) in the absence of the agent’s MOs. Furthermore, we assume that the Brownian motion is independent of all other random variables in the model. Thus, one can show that

\[
(\sigma^B)^2 = \mathbb{V} \left[ Z_t^+ + \sum_{k=1}^{M_t^+} \xi_k^+ - Z_t^- - \sum_{k=1}^{M_t^-} \xi_k^- \right] = \sigma^2 (2 \theta + 2 \lambda \alpha),
\]

where $\mathbb{V}[\cdot]$ denotes the variance operator.
Interestingly, the LOs in the strategy targets inventory levels which contain the Delta-hedge position, but this is not always the case. Figure 3.5 is similar to Figure 3.4, but has $\gamma = 1$ and panel (c) shows the Bachelier delta is not always enclosed by the targeted optimal level of inventory. For example, at around $t = 0.65$, the Bachelier delta requires an inventory position larger than that sought by the agent who targets an inventory level with LOs.

**Standard deviation of the optimal inventory.** The expression for the dynamics of the inventory (3.4) can be used to derive the drift of the agent’s inventory and the standard deviation of the inventory when it is not optimal to execute MOs. Thus, in the region where the agent does not execute MOs, the drift of the inventory is given by

$$
\mu(t,s,q) = \lim_{\delta \to 0} \frac{E_{t,x,s,q} [Q_{t+\delta}] - q}{\delta} = \rho \left( \lambda^+ \ell_-(t,s,q) - \lambda^- \ell^+(t,s,q) \right),
$$

(3.16)

for $t \in [0,T]$, $q \in \{q,...,\bar{q}\}$ and $s \in \sigma Z$. Recall that $\ell^\pm$ represents the number of shares the agent posts on the buy and sell side of the book. We obtain values of $\mu(t,s,q)$ for any $q \in \mathbb{R}$ by linear interpolation and we define $q^*(t,s)$ as the unique point satisfying $\mu(t,s,q^*(t,s)) = 0$, so that $q^*(t,s)$ represents the optimal inventory to hold at time $t$ given $S_t = s$.

Moreover, we define the variance of inventory at the optimal inventory level to hold as

$$
v(t,s) = \lim_{\delta \to 0} \frac{V_{t,x,s,q^*(t,s)} [Q_{t+\delta} - q^*(t,s)]}{\delta} \\
= \rho \left[ \lambda^+ \left( \ell_-(t,s,q^*(t,s)) \right)^2 + \lambda^- \left( \ell^+(t,s,q^*(t,s)) \right)^2 \right].
$$

(3.17)

We define the standard deviation of the optimal inventory as $\sqrt{v(t,s)}$. The quantity $\sqrt{v(t,s)}$ measures the variability of the agent’s inventory when she holds the optimal inventory level.

Panels (a) and (b) of Figure 3.6 show the optimal level of inventory and the contour
plot of the standard deviation of the optimal inventory $v(t, s)$. The standard deviation is largest when the option is at-the-money.

Figure 3.7 shows how the agent values the use of LOs in the trading strategy. The vertical axis shows the indifference price of LOs, i.e., $I^{lo}(0, s, q)$ computed using equation (3.12), where we assume the inventory is $q = 0$ and $s = $1856.5, and the horizontal axis represents different values of the risk-aversion parameter $\gamma$. Observe that as the agent becomes more risk-averse, the value she attaches to the use of LOs diminishes. When the agent’s degree of risk-aversion is extremely high, the optimal trading strategy relies much less on LOs and mainly employs MOs to guarantee that at the terminal date the agent has the correct amount of the underlying asset to settle the option. The use of LOs increases the gains from speculative roundtrip trades and incur no exchange fees; however, LOs introduce additional tracking error, and makes the agent’s terminal wealth more volatile.

Finally, we discuss how the optimal strategy accounts for adverse selection costs. Recall that the agent’s and other market participants’ MOs have permanent impact on the midprice of the underlying asset. This price impact is captured by the random variables $\xi^\pm$, $\xi^0\pm$. As this impact becomes larger (on average), the agent’s strategy relies less on LOs. That is, everything else equal, the agent will execute more MOs, and LOs are
adjusted in two ways: (i) post less LOs, and (ii) volume of LOs is adjusted downwards. In this way the agent strikes the optimal balance between how often and how many shares she posts at the best bid and best ask of the LOB, and the number of MOs she executes. (In the interest of space we do not include figures to show the effect of adverse selection on the agent’s LOs and MOs.)

### 3.4.2 Simulations: financial performance of the strategy

To illustrate the financial performance of the strategy we simulate $10^5$ sample paths of the midprice. We show how the strategy maximizes the agent’s expected utility of wealth whilst replicating the payoff of the $N$ call options. Figure 3.8 shows the value of the payoff of the 10 call options (solid line) and the strategy’s terminal wealth (blue circles), assuming low risk-aversion ($\gamma = 0.1$) in the left panel, and high risk-aversion ($\gamma = 0.3$) in the right panel. In both cases, for all simulations shown here, the agent super-replicates the call option. Part of what drives the “super-replication” is due to the indifference price the agent charges the client. This price accounts for jump-risk, adverse selection costs, bid-ask spreads, and exchange fees, all of which increase the indifference price. On the other hand, the potential profits from round trip trades using LOs, lower the price. Nonetheless, we see the overall effect is that the strategy lies above the payoff...
in all scenarios.

Clearly, as the agent becomes more risk-averse, the strategy becomes more conservative, so it relies more on MOs and less on LOs. By executing more MOs the agent increases the certainty with which she achieves the desired level of inventory, but pays higher costs due to crossing the spread, in addition to MO fees, and forgoes opportunities to earn the spread with LOs. On the other hand, as the agent become less risk-averse, the strategy relies more on LOs and less on MOs. In this case the agent earns the spread when LOs are filled but bears more volatility in the wealth because there is no guarantee that the LOs will be filled. These effects can be appreciated in Figure 3.8, and are also seen in Figure 3.10, where we show the histogram of total number of executed MOs and matched LOs for the agent’s strategy. Clearly, when $\gamma = 0.3$, the agent uses more MOs and less LOs. Thus, she pays higher transaction costs on average and has a lower terminal portfolio value. These costs are, however, transferred to the client through the indifference price.

Panel (a) Figure 3.9 shows the cost of replicating the options in the Bachelier model. Here, the premium charged by the agent for the options is calculated using the Bachelier model and the delta-hedge strategy is implemented as follows. At every point in time the agent calculates and rounds the Bachelier delta to the nearest integer. If the strategy’s
inventory is different from this integer, the agent executes an MO so the inventory remains on target. For every simulation shown in the figure we see the strategy is a sub-replication one because MOs cross the spread and pay a fee. As a comparison, in Panel (b) of Figure 3.9, the agent receives the same premium calculated from the Bachelier model, but she adopts the optimal strategy with $\gamma = 0.3$. She performs much better than in Panel (a). In most scenarios, she is able to replicate the option payoff with a small error.

Figure 3.11 shows the agent’s mean terminal wealth as a function of the standard deviation of terminal wealth when $\gamma$ varies from 0.1 to 0.3. As $\gamma$ increases, both the
mean and the standard deviation of terminal wealth decrease. As the agent becomes more risk-averse the strategy employs more MOs and less LOs. This mix of market and limit orders reduces the variance of terminal wealth, but as a tradeoff also reduces the mean of terminal wealth because the strategy does not reap the benefits from employing LOs.

### 3.5 Convergence Results

In this section, we describe the numerical algorithm we employ to solve for the value function and obtain the optimal trading strategy. We restrict the asset price to take values on a finite grid \( \bar{S} := \{k\sigma : k \in \mathbb{Z}, k_{\min} \leq k \leq k_{\max}\} \) for some \( k_{\min} \leq k_{\max} \in \mathbb{Z} \).

In this way, the system of QVI shown in (3.9) becomes a coupled system of ordinary differential equations indexed by \((s,q)\) on \( \bar{S} \times \mathbb{Q} \), where \( \mathbb{Q} = \{q, ..., \bar{q}\} \). We write \( \mathbb{S} := \bar{S}/\{k_{\min} \sigma, k_{\max} \sigma\} \) (i.e., excluding the end points) and denote by \( n \) (resp. \( m \)) the number of elements in the set \( \mathbb{S} \) (resp. \( \mathbb{Q} \)). We seek a bounded viscosity solution \( u : [0,T] \to \mathbb{R}^{n \times m} \) to the following system of QVIs:

\[
\begin{aligned}
F_{s,q}(t,u,\partial_t u_{s,q}) &= 0, \\
&0 \leq t < T, \\
u_{s,q}(T) &= -G(s) - U(s,q),
\end{aligned}
\]  

(3.18)
for all \((s, q) \in \bar{S} \times Q\). The mapping \(F\) is defined as

\[
F_{s,q}(t, u, p) = \min\{L_{s,q}(t, u, p), M^+_{s,q}(t, u), M^-_{s,q}(t, u)\},
\]

where

\[
L_{s,q}(t, u, p) = -\gamma p + \kappa u_{s,q} + \theta J_{s,q}(t, u) + \lambda^- K^+_{s,q}(t, u) + \lambda^+ K^-_{s,q}(t, u),
\]

\[
M^+_{s,q}(t, u) = \min_{m \in \{0, 1, \ldots, \bar{q} - q\}} \left\{ \mathbb{E}\left[e^{-\gamma(-m - \gamma + u_{s,q} - (1 + \kappa) u_{s,q})} - 1\right]\right\},
\]

\[
M^-_{s,q}(t, u) = \min_{m \in \{0, 1, \ldots, \bar{q} - q\}} \left\{ \mathbb{E}\left[e^{-\gamma(-m - \gamma + u_{s,q} - (1 + \kappa) u_{s,q})} - 1\right]\right\},
\]

\[
J_{s,q}(t, u) = \begin{cases} e^{-\gamma(q \sigma + u_{s,q} - u_{s,q})} + e^{-\gamma(-q \sigma + u_{s,q} - u_{s,q})} - 2, & s \in S, \\ 2e^{-\gamma(q \sigma + u_{s,q} - u_{s,q})} - 2, & s = k_{\min} \sigma, \\ 2e^{-\gamma(-q \sigma + u_{s,q} - u_{s,q})} - 2, & s = k_{\max} \sigma, \end{cases}
\]

\[
K^+_{s,q}(t, u) = \min_{\ell^+ \in \{0, 1, \ldots, q - q\}} \left\{ \mathbb{E}\left[e^{-\gamma(\ell^+ \zeta - (q + \ell^+ \zeta) \sigma \xi + u_{s,q} - \ell^+ \zeta - u_{s,q})} - 1\right]\right\},
\]

\[
K^-_{s,q}(t, u) = \min_{\ell^- \in \{0, 1, \ldots, q - q\}} \left\{ \mathbb{E}\left[e^{-\gamma(\ell^- \zeta + (q - \ell^- \zeta) \sigma \xi + u_{s,q} - \ell^- \zeta - u_{s,q})} - 1\right]\right\},
\]

where \(\kappa \downarrow 0\) is a robustness parameter. In all the above equations, we define \(\min\{\emptyset\} = +\infty\).

Our first result is a comparison principle for the QVI (3.18).

**Proposition 3.1.** The system (3.18) admits a comparison principle, i.e., if \(u\) (resp. \(v\)) is an upper (resp. lower) semicontinuous subsolution (resp. supersolution), then \(u \leq v\).

**Proof.** See Appendix 3.A. \(\square\)

We introduce the finite difference scheme \(F^* = (F^*_{s,q}) : [0, T] \times \mathbb{R}^{m \times n} \times C([0, T]) \rightarrow \)
$F^\varepsilon_{s,q}(t,u,\phi(\cdot)) = \min\{L^\varepsilon_{s,q}(t,u,\phi(\cdot)), M^+_s(t,u), M^-_{s,q}(t,u)\},$

where

$$L^\varepsilon_{s,q}(t,r,\phi(\cdot)) = -\gamma \frac{\phi(t+\epsilon) - r_{s,q}}{\epsilon} + \theta J_{s,q}(t,r) + \lambda^- K^+_{s,q}(t,r) + \lambda^+ K^-_{s,q}(t,r).$$

Let

$$T_\varepsilon := \{t_j := \epsilon j : 0 \leq N\},$$

where $\epsilon > 0$ is chosen so that $N := T_\varepsilon \in \mathbb{Z}_+$. We show that the solution $u^\varepsilon = (u^\varepsilon_{s,q})$, $u^\varepsilon_{s,q} : T_\varepsilon \to \mathbb{R}$ of the discrete problem

$$\begin{cases}
F^\varepsilon_{s,q}(t_j, u^\varepsilon(t_j), u^\varepsilon_{s,q}) = 0, & 0 \leq j \leq N-1, \\
u^\varepsilon_{s,q}(T) = -G(s) - U(s,q),
\end{cases}$$

(3.21)

can be used to approximate the solution of (3.18).

**Proposition 3.2.** The following are true:

(C1) For all bounded functions $u = (u_{s,q})$, $v = (v_{s,q})$ with $u_{s,q}, v_{s,q} \in C[0,T] \times \mathbb{R}$ and $u_{s,q} \leq v_{s,q}$ on $[0,T] \times \mathbb{R}$, suppose $r, r' \in \mathbb{R}^{m \times n}$ satisfies $r_{s,q} - r'_{s,q} = \max_{s,q} \{r_{s,q} - r'_{s,q}\} = \delta \geq 0$ and $M_{s,q}^*(t,r) \geq 0$ we have

$$F^\varepsilon_{s,q}^*(t,r, u_{s,q}^\varepsilon + \delta) - F^\varepsilon_{s,q}^*(t,r', v_{s,q}^\varepsilon) \geq \min\{(M_{s,q}^*(t,r') + 1)\gamma, 1\} \kappa \delta.$$

(C2) For any $\epsilon > 0$ and any bounded function $\phi \in C([0,T])$, fix $t \in [0,T]$ and $r = (r_{s,q}) \in \mathbb{R}^{m \times n}$, the functions

$$r \mapsto F^\varepsilon_{s,q}(t,r,\phi),$$
are uniformly continuous for \( r \) in a bounded set, uniformly in \( t \in [0, T] \).

\((C3)\) For every \( \psi = (\psi_{s,q}), \psi_{s,q} \in C([0, T]) \cap C^1((0, T)), t \in [0, T] \) and \( \delta > 0 \), there exists \( \epsilon > 0 \) such that

\[
|F^\epsilon_{s,q}(t, \psi(t), \psi_{s,q}) - F_{s,q}(t, \psi, \partial_t \psi_{s,q})| \leq \delta, \quad q \leq q \leq \bar{q}.
\]

\((C4)\) For any \( \epsilon > 0 \), there exists a bounded solution \( u^\epsilon = (u^\epsilon_{s,q}), u^\epsilon_{s,q} : T^\epsilon \to \mathbb{R} \) of the discrete problem (3.21).

With the above results, we define the candidate super- and sub-solution of (3.18) as

\[
\underline{u}_{s,q}(t) = \liminf_{\epsilon \to 0, t' \to t} u^\epsilon_{s,q}(t'), \quad \text{and} \quad (3.22)
\]

\[
\bar{u}_{s,q}(t) = \limsup_{\epsilon \to 0, t' \to t} u^\epsilon_{s,q}(t'), \quad (3.23)
\]

for all \( t \in [0, T], q \leq q \leq \bar{q} \).

Proposition 3.2 allows us to apply the arguments of Proposition 3.3 of Briani et al. (2012) to show that \( u = (u_{s,q}) \) and \( \bar{u} = (\bar{u}_{s,q}) \) are supersolution and subsolution of (3.18). The opposite direction is true by definition and we have that \( u = u = \bar{u} \) is the viscosity solution of (3.18).

### 3.6 Stochastic Intensity

In this section, we extend the model in Section 3.2 to incorporate stochastic intensity in the dynamics of the midprice changes and MO arrivals.

Recall that in Section 3.2, we assume that the midprice dynamics are given by

\[
dS_t = \sigma (dP^+_t - dP^-_t),
\]

where \( P^+_t \) (resp. \( P^-_t \)) is interpreted as the upward (resp. downward) jumps in the
midprice. In the absence of the market impact, $P^\pm_t$ are Poisson processes with constant intensity $\theta$. It is also assumed the intensity of other agent’s MBO (resp. MSO) arrivals equal to a constant $\lambda^+$ (resp. $\lambda^-$).

The constant intensity assumption may be relevant under some circumstances. However, in many other cases, midprice changes exhibit *clustering* behavior, that is, when a change in the midprice occurs, the likelihood that another change in the midprice occurs shortly after increases. This phenomenon contradicts the assumption that $\theta$ is a constant. It suggests that $\theta$ is stochastic and $\theta$ increases when there is a change in the midprice.

The same phenomenon is observed for MO arrivals, hence it is also desirable to make $\lambda^\pm$ stochastic and jumps upwards when an MO arrives.

Based on the above observations, we propose replacing $\theta$ with a stochastic process $(\theta_t)_{0 \leq t \leq T}$ and replacing $\lambda^\pm$ with $a^\pm \theta_t$ for some constants $a^\pm > 0$. The dynamics of $\theta_t$ are given by

$$d\theta_t = -\kappa(\theta_t - \mu)dt + \eta^+ dZ^+_t + \eta^- dZ^-_t + \eta^{\text{MO}+} \xi^{+}_{M^+_t} + \eta^{\text{MO}-} \xi^{-}_{M^-_{t+1}}dM^+_t - \eta^{\text{MO}+} \xi^{-}_{M^-_{t+1}}dM^-_t,$$ (3.24)

for $\kappa, \mu, \eta^+, \eta^-, \eta^{\text{MO}+}, \eta^{\text{MO}-} \geq 0$. We require $\kappa > \eta^+ + a^+ \alpha \eta^{\text{MO}+} + a^- \alpha \eta^{\text{MO}-}$ so that $\theta_t$ is stationary.

Let us interpret the terms in (3.24). In the absence of midprice changes and MO arrivals, only the first term $-\kappa(\theta_t - \mu)dt$ is in effect. This term makes $\theta$ mean-reverting to the long-run level $\mu$. It represents the “calming down” of the process when there is no event. The second and the third term represent the feedback effects from the midprice changes to $\theta_t$. For example, when there is an upward jump in the midprice ($dZ^+_t = 1$), $\theta_t$ increases by $\eta^+$ instantaneously and it becomes more likely for another midprice change to occur in a short time. Note that not only the likelihood of midprices changes increases, it is also true that the rate of MO arrivals increases, since it depends on $\theta_t$.

Our proposed dynamics (3.24) are inspired by the Hawkes process, first introduced
by Hawkes (1971) to model clustering in seismic events. In fact, it can be viewed as a special case of the multivariate Hawkes process, see Bacry et al. (2015). In principle, we can let the rate of MO arrivals be another stochastic process that has a similar dynamics as in (3.24), allowing the ratio between the rate of midprice changes and the rate of MO arrivals to be stochastic. The downside of this approach is that the rate of MO arrivals will become another state variable. This increases the dimension of the DPE that we will derive, making it prohibitive to be numerically solved.

Note that the volatility of the midprice changes is directly controlled by $\theta_t$: the higher $\theta_t$ is, the more volatile the midprice is. In this way, our model is also similar to stochastic volatility models such as Heston (1993). There are two stylized facts that the model in Heston (1993) captured: volatility clustering and leverage effect. The former refers to the fact that volatility tends to remain on the same level over short horizons. The latter refers to the observation that changes in the midprice and changes in the volatility are usually negatively correlated. Our proposed dynamics (3.24) is also capable of capturing both stylized facts. For the volatility clustering, note that when there is a change in the midprice or an MO arrival that causes an upward jump in $\theta_t$, it stays on a high level for a short time period until it mean-reverts back to the long-run level. For the leverage effect, suppose $\eta^- > \eta^+$ in (3.24), a downward jump in the midprice increases $\theta_t$ more than that for an upward jump. As a result, the changes in $\theta_t$ (or equivalently changes in the volatility) is negatively correlated with the changes in the midprice.

As a final point, it is worthwhile mentioning that Jaisson et al. (2015) showed that under suitable scaling, (3.24) converges to a Cox-Ingersoll-Ross (CIR) process used in Heston (1993).

### 3.6.1 The Dynamic Programming Equation

The agent solves a similar optimization problem as in Section 3.2. The only things changed are that the rate of the midprice changes becomes $\theta_t$ and that the arrival rate of
MOs becomes $a^\pm \theta_t$ for constants $a^\pm > 0$, where the dynamics of $\theta_t$ is given by (3.24). The rest of the dynamics, including the inventory (3.4) and the wealth (3.6) are unchanged.

The agent’s value function is given by

$$H(t, x, s, q, \theta) = \sup_{(\ell^\pm, \tau^0, m^\pm) \in A} \mathbb{E}_{t,x,s,q,\theta} \left[ -e^{-\gamma (X_T - S_T - G(Q_T, S_T) - U(Q_T, S_T))} \right].$$ \hspace{1cm} (3.25)

Compared to (3.7), the value function $H$ has an extra dependency on $\theta$, the current value of the rate of the midprice changes. Again, classical results in the dynamic programming principle (see, e.g., Pham (2009)) suggest that $H$ satisfies the following DPE

$$\max \left\{ \left( \partial_t + \mathcal{L}^{s,\theta} \right) H \right\}$$

$$+ \max_{\ell^+ \in \{0, 1, \ldots, q-1\}} \left\{ a^- \theta \mathbb{E} \left[ H \left( t, x - \ell^+ \left( s - \frac{\sigma}{2} \right), q + \ell^+ \zeta, \theta + \eta^+ \xi \right) - H \right] \right\}$$

$$+ \max_{\ell^- \in \{0, 1, \ldots, q-1\}} \left\{ a^+ \theta \mathbb{E} \left[ H \left( t, x + \ell^- \left( s + \frac{\sigma}{2} \right), q - \ell^- \zeta, \theta + \eta^- \xi \right) - H \right] \right\} ;$$

$$\max_{m^+ \in \{1, \ldots, q-1\}} \left\{ \mathbb{E} \left[ H \left( t, x - m^+ \left( s + \Upsilon \right), q + m^+, \theta + \eta^+ \zeta^0 \right) - H \right] \right\} ;$$

$$\max_{m^- \in \{1, \ldots, q-1\}} \left\{ \mathbb{E} \left[ H \left( t, x + m^- \left( s - \Upsilon \right), q - m^-, \theta + \eta^- \zeta^0 \right) - H \right] \right\} \right\} = 0 ,$$ \hspace{1cm} (3.26)

where $\mathcal{L}^{s,\theta}$ is the infinitesimal generator of the joint process $(S_t, \theta_t)$. $\mathcal{L}^{s,\theta}$ acts on a smooth function $g(t, x, s, q, \theta)$ as follows:

$$\mathcal{L}^{s,\theta} g(t, x, s, q, \theta) = -\kappa(\theta - \mu) \partial_\theta g + \tilde{\theta} \left[ g(t, x, q, s + \Delta, \theta + \eta^+) \right. \left. - 2 g \right] ,$$

where $\tilde{\theta} = (1 + a(1 - \rho)\alpha) \theta$. $H$ also satisfies the terminal condition

$$H(T, x, s, q, \theta) = -e^{-\gamma (x + q s - G(s) - U(s, q))}.$$
To simplify (3.26), we adopt the ansatz \( H(t, x, s, q, \theta) = -e^{-\gamma(x + s + h(t, s, q, \theta))} \) and derive the QVI satisfied by \( h(t, s, q) \):

\[
0 = \min \left\{ -\gamma \partial_t h + \gamma \kappa (\theta - \mu) \partial_\theta h 
+ \theta \left( e^{-\gamma \left( q \sigma + h(t, s + \sigma, q, \theta + \eta^+ \xi) \right) - h} + e^{-\gamma \left( q \sigma + h(t, s - \sigma, q, \theta + \eta^- \xi) \right) - h} - 2 \right) 
+ \rho a^- - \theta \min_{\ell^+ \in \{0, 1, \ldots, q-1\}} \left\{ E \left[ e^{-\gamma \left( \ell^+ \xi + (q - \ell^+ \xi) \sigma \xi + h(t, s + \sigma, q, \theta + \eta^+ \xi - \xi) - h - 1 \right) - 1 \right] \right\} 
+ \rho a^+ - \theta \min_{\ell^- \in \{0, 1, \ldots, q-1\}} \left\{ E \left[ e^{-\gamma \left( \ell^- \xi + (q - \ell^- \xi) \sigma \xi + h(t, s + \sigma, q, \theta + \eta^- \xi - \xi) - h - 1 \right) - 1 \right] \right\} ; 
\min_{m^+ \in \{0, 1, \ldots, q-1\}} \left\{ E \left[ e^{-\gamma \left( -m^+ \xi + h(t, s + \sigma, \theta + \eta^+ \xi^0 + \xi) - h \right) - 1 \right] \right\} ; 
\min_{m^- \in \{0, 1, \ldots, q-1\}} \left\{ E \left[ e^{-\gamma \left( -m^- \xi + h(t, s + \sigma, \theta + \eta^- \xi^0 - \xi) - h \right) - 1 \right] \right\} , \quad (3.27)
\]

subject to the terminal condition \( h(T, s, q, \theta) = -G(s) - U(s, q) \) for all \( s, q \) and \( \theta \).

### 3.6.2 Maximum Likelihood Estimation

Let us describe how we estimate parameters in (3.24) using maximum likelihood method.

An advantage of the model we proposed in (3.24) is its easiness for parameter estimation: the likelihood function can be explicitly written down without involving some intractable integrations of hidden variables. Note that the variables \( Z_t^\pm, M_t^\pm \) and \( \xi_{M_t^\pm + 1}^\pm \) are all directly observable. For a given set of parameters \( (\kappa, \mu, \eta^\pm, \eta^\text{MO}^\pm, a^\pm, \alpha) \), the path \( \theta_t(0 \leq t \leq T) \) can be explicitly calculated according to (3.24) and the observed values of \( Z_t^\pm, M_t^\pm \) and \( \xi_{M_t^\pm + 1}^\pm \). Let us denote by \( \{\tau_i\}_{0 \leq i \leq n} \) with \( \tau_0 = 0 \) the sequence of event times.
when there is a jump in $\theta_t$. The likelihood function is given by

$$L(\kappa, \mu, \eta^\pm, \eta^{MO \pm}, a^\pm, \alpha; Z_t^\pm, M_t^\pm, \xi_{M_t^\pm+1}^\pm) = \prod_{i=1}^n e^{-\int_{t_{i-1}}^{t_i} (2+a^+ + a^-) \theta_s ds} \left[ \theta_{t_i} 1_{dZ_{t_i}^\neq 0} + a^+ 1_{dM_{t_i}^+ \neq 0} (\alpha \theta_{t_i} 1_{\xi_{M_{t_i-1}^+} \neq 0} + (1-a^+) \alpha 1_{\xi_{M_{t_i-1}^+} = 0}) ight.$$ 

$$+ a^- 1_{dM_{t_i}^- \neq 0} (\alpha \theta_{t_i} 1_{\xi_{M_{t_i-1}^-} \neq 0} + (1-a^-) \alpha 1_{\xi_{M_{t_i-1}^-} = 0}) \right].$$

(3.28)

The log-likelihood function can be written as

$$\ell(\kappa, \mu, \eta^\pm, \eta^{MO \pm}, a^\pm, \alpha) = -(2+a^+ + a^-) \int_0^T \theta_s ds + \sum_{i:dZ_{t_i}^\neq 0} \log(\theta_{t_i})$$

$$+ \sum_{i:dM_{t_i}^+ \neq 0} \log(a^+ \theta_{t_i}) + \sum_{i:dM_{t_i}^- \neq 0} \log(a^- \theta_{t_i})$$

$$+ \sum_{i:dM_{t_i}^+ \neq 0, \xi_{M_{t_i-1}^+} \neq 0} \log(\alpha) + \sum_{i:dM_{t_i}^- \neq 0, \xi_{M_{t_i-1}^-} = 0} \log(1-\alpha).$$

It can be easily seen that the estimation of $\alpha$ can be done independently of the other parameters and the maximum likelihood estimator is

$$\hat{\alpha} = \frac{\#\{i : dM_{t_i} \neq 0, \xi_{M_{t_i-1}^\pm} \neq 0\}}{\#\{i : dM_{t_i} \neq 0\}},$$

where $\#$ is the operator that counts the number of elements in a given set.

The remaining parameters can be estimated by numerically maximizing

$$\tilde{\ell}(\kappa, \mu, \eta^\pm, \eta^{MO \pm}, a^\pm) = -(2+a^+ + a^-) \int_0^T \theta_s ds + \sum_{i:dZ_{t_i}^\neq 0} \log(\theta_{t_i})$$

$$+ \sum_{i:dM_{t_i}^+ \neq 0} \log(a^+ \theta_{t_i}) + \sum_{i:dM_{t_i}^- \neq 0} \log(a^- \theta_{t_i}).$$

Below we report the parameters estimated from real data. We employ the same
CME data as in Section 3.4. For each day between March 24, 2014 and March 29, 2014, we extract the data between 11:00-12:00 Eastern time and re-build the LOB. We discard events for which the bid-ask spreads are larger than 1 tick (about 0.1% of the total number of events). We conduct the above MLE procedure to obtain the estimated parameters on each day and average parameters over the 5 trading days. The results are summarized in Table 3.3 where the numbers in brackets are the standard errors. The standard errors are obtained through the following bootstrapping procedure. At each time, we re-simulate data using the calibrated parameters and re-estimate the parameters based on simulated data. After repeating for 1,000 times, we obtain a sample of 1,000 estimated parameters. We then compute the mean squared errors for each parameter and use the square root of them as the standard errors.

<table>
<thead>
<tr>
<th>κ</th>
<th>$3.19 \times 10^5$ (7.16 $\times 10^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ</td>
<td>100 (6.3)</td>
</tr>
<tr>
<td>$\eta^+$</td>
<td>$2.01 \times 10^3$ (2.99 $\times 10^2$)</td>
</tr>
<tr>
<td>$\eta^-$</td>
<td>$2.02 \times 10^3$ (3 $\times 10^2$)</td>
</tr>
<tr>
<td>$\eta^{MO+}$</td>
<td>$7.30 \times 10^3$ (4.96 $\times 10^2$)</td>
</tr>
<tr>
<td>$\eta^{MO-}$</td>
<td>$7.20 \times 10^3$ (5.04 $\times 10^2$)</td>
</tr>
<tr>
<td>$a^\pm$</td>
<td>45.3 (2.9)</td>
</tr>
<tr>
<td>α</td>
<td>0.108 (0.003)</td>
</tr>
</tbody>
</table>

### 3.6.3 Numerical Example

Here we present another numerical example. We use the estimated parameters from Table 3.3 and choose $N = 10$ call options with maturity $T = 0.05$. Other model parameters are the same as in Section 3.4.

Figure 3.12 shows the heat-maps of the number of MOs the agent executes when the inventory level is 5. The left panel shows the case when the intensity is low ($\theta_0 = 18,273$). The colored areas show the regions in which the agent executes MOs of various volumes, and the empty (white) area is the region in which she does not execute any MO. Similarly, the right panel shows the case when the intensity is high ($\theta_0 = 139,424$).
Qualitatively, the left and the right panel are quite similar. Let us focus on the left panel and discuss its properties. Clearly, when the asset is close to at-the-money (ATM), the agent does not execute any MOs. When the asset becomes deeper in-the-money (ITM), the agent starts to execute MBOs. The volume of MBOs she executes increases as the asset becomes deeper and deeper ITM. Conversely, the agent executes larger and larger volumes of MSOs when the asset becomes deeper and deeper out-of-the-money (OTM).

The above observations are similar to the case of constant intensity, presented in Section 3.4. To see the effect of stochastic intensity, we shall compare the left and the right panel of Figure 3.12. Fix the time $t = 0$, we can see that the white area in the right panel is wider and for the same price level when the asset is deep ITM or OTM, the agent executes MOs of smaller volume. The intuition behind is that because of the high intensity in the right panel, the asset has a high chance of switching moneyness prior to maturity. To account for that, the agent must be flexible in adjusting the inventory, i.e., avoid accumulating a large inventory. Therefore, in the right panel, she tends to keep her inventory close to $q = 5$ and is less inclined to execute MOs.

As a complement to Figure 3.12, Figure 3.13 shows heat-maps for the number of LBOs the agent posts when the inventory is fixed at $q = 5$, for different levels of intensity. Let
us focus on the left panel, in which $\theta = 18,273$. The agent posts LBOs only when she does not execute any MOs. For a fixed time, the agent posts more and more LBOs when the asset is deeper and deeper ITM, and eventually executes MBOs. For a fixed ITM price level, the agent post less and less LBOs as time approaches maturity, and eventually executes MBOs.

Comparing the left and the right panel, we see that the agent posts more LBOs in the right panel. This is because for a higher level of intensity, the rate of MO arrivals is higher and the agent’s LBOs have higher chances of being executed.

Figure 3.14 shows the indifference price for the option. In the left panel, the blue curve is the indifference price for different spot prices of the asset, and the red curve is the option price (Bachelier price) calculated by assuming the midprice follows (3.15),
i.e., an arithmetic Brownian motion. We can see that the indifference price is lower than
the Bachelier price when the option is ATM and is slightly higher than the Bachelier
price when the option is ITM or OTM. The right panel of Figure 3.14 shows the implied
volatility of the indifference price, which is the volatility in (3.15) that reproduces the
same option price as the stochastic intensity model, for the same spot price. It shows the
pattern of “volatility smile”, i.e., the implied volatility is higher for the ITM and OTM
options.

3.7 Conclusions

We show how an agent maximizes expected utility of wealth when she takes a position
in a contingent claim and employs limit and market orders to trade in the underlying
of the claim. The agent solves a combined optimal stopping and stochastic control
problem, and we characterize the solution in terms of a Hamilton-Jacobi-Bellman quasi-
variational inequality (QVI). We employ a numerical scheme to solve the QVI and prove
a convergence result for the scheme.

In our model, market orders from all traders (including the agent) have price impact
and pay other exchange related fees. The price impact is permanent and creates adverse
selection costs when the agent’s limit orders are filled. That is, the midprice of the
underlying asset jumps in the direction of the market order.

We discuss a particular case where the agent takes a short position in European
options written on the E-mini that tracks the S&P500 index. The agent employs limit
and market orders to hedge the exposure to the contingent claim, and also engages in
speculative trades to maximize expected utility of wealth. Market orders are expensive
because they cross the spread and pay fees to the exchange, but guarantee execution, so
the agent employs these to keep the inventory on target.

Limit orders, on the other hand, do not incur exchange fees, but there is no guarantee
that they will be filled in time. We show that the agent’s strategy relies on limit orders not only to achieve the desired inventory target, which depends on time to expiry and moneyness of the contingent claim, but are also employed in speculative trades to earn the spread from roundtrip trades.

Finally, the agent’s strategy accounts for adverse selection costs by: (i) controlling how often she posts LOs on the bid and ask of the limit order book, and (ii) the volume posted in the limit order. If adverse selection costs increase (decrease), the strategy relies more (less) on market orders and less (more) on limit orders.

3.A Proofs

The following proposition is later used in the prove of the comparison principle.

Proposition 3.3. Let \( u = (u_{s,q}) \), \( v = (v_{s,q}) \) \( \in \mathbb{R}^{n \times m} \). Suppose that \( u_{s^*,q^*} - v_{s^*,q^*} = \max_{s,q} \{ u_{s,q} - v_{s,q} \} \geq 0 \) and \( M_{s^*,q^*}^\pm (t,v) \geq 0 \), then

\[
F_{s^*,q^*}(t,u,p) - F_{s^*,q^*}(t,v,p) \geq \min(1, \gamma) \kappa (u_{s^*,q^*} - v_{s^*,q^*}).
\]

Proposition 3.3. It suffices to show the following:

\[
L_{s^*,q^*}(t,u,p) - L_{s^*,q^*}(t,v,p) \geq \kappa (u_{s^*,q^*} - v_{s^*,q^*}),
\]

\[
M_{s^*,q^*}^+(t,u) - M_{s^*,q^*}^+(t,v) \geq \gamma \kappa (u_{s^*,q^*} - v_{s^*,q^*}).
\]

\[
M_{s^*,q^*}^-(t,u) - M_{s^*,q^*}^-(t,v) \geq \gamma \kappa (u_{s^*,q^*} - v_{s^*,q^*}).
\]

To show (3.29), note that \( u_{s^*,q^*} - v_{s^*,q^*} = \max_{s,q} \{ u_{s,q} - v_{s,q} \} \) implies \( u_{s,q} - u_{s^*,q^*} \leq v_{s,q} - v_{s^*,q^*} \) for all \( s \) and \( q \). This further implies \( J_{s^*,q^*}(t,u) \geq J_{s^*,q^*}(t,v) \), \( K_{s^*,q^*}^+(t,u) \geq K_{s^*,q^*}^+(t,v) \) and \( K_{s^*,q^*}^-(t,u) \geq K_{s^*,q^*}^-(t,v) \). The rest are straightforward calculations.

It remains to show (3.30). Showing (3.31) is similar. Let us write \((s,q) = (s^* +
Note that the left-hand side of (3.30) is greater or equal to

\[
\sigma \xi_{0^+}^{+}, q^* + m^+ \) \]

\[
\min_{m^+ \in \{0, 1, \ldots, q^* - 1\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left(-m^+ T + u_{s,q} - (1+\kappa) u_{s^*,q^*}\right)} - e^{-\gamma \left(-m^+ T + v_{s,q} - (1+\kappa) v_{s^*,q^*}\right)} \right] \right\}
\]

\[
\geq \gamma \kappa (u_{s^*,q^*} - v_{s^*,q^*}) \left( M_{s^*,q^*}^+ (t, v) + 1 \right)
\]

\[
\geq \gamma \kappa (u_{s^*,q^*} - v_{s^*,q^*}).
\]

\[
\]

**Proposition 3.1.** We prove the proposition by contradiction.

Suppose \( u_{s^*,q^*}(\bar{t}) - v_{s,q}(\bar{t}) = \max_{s,q} \sup_{[0,T]} \{u_{s,q} - v_{s,q}\} = \eta > 0 \). Without loss of generality, we may assume \( 0 < \bar{t} < T \). Consider the family of upper semicontinuous functions

\[
\Phi_\epsilon(s, q, t, t') = u_{s,q}(t) - v_{s,q}(t') - \phi_\epsilon(t, t'),
\]

where \( \phi_\epsilon(t, t') = \frac{1}{\epsilon} |t - t'|^2 \) and \( \epsilon > 0 \), and the bounded sequence \((q_\epsilon, s_\epsilon, t_\epsilon, t'_\epsilon)\) attains the maximum of \( \Phi_\epsilon \). By standard arguments, we have

\[
\eta_\epsilon = \max \Phi_\epsilon = \Phi_\epsilon(q_\epsilon, s_\epsilon, t_\epsilon, t'_\epsilon) \to \eta, \frac{1}{\epsilon} |t_\epsilon - t'_\epsilon|^2 \to 0,
\]

as \( \epsilon \to 0 \).

Note that \( u_{s,q}(\cdot) - \phi_\epsilon(\cdot, t'_\epsilon) \) attains maximum at \( t_\epsilon \). By the fact that \( u \) is a subsolution, we have

\[
F_{s_\epsilon,q_\epsilon}(t_\epsilon, u(t_\epsilon), \partial_t \phi_\epsilon) \leq 0.
\]

Similarly we have

\[
F_{s_\epsilon,q_\epsilon}(t'_\epsilon, v(t'_\epsilon), -\partial_{t'} \phi_\epsilon) \geq 0.
\]
The above two inequalities yield

\[
0 \geq F_{s^*,q^*}(t, u_{s^*,q^*}(t), \partial_t \phi) - F_{s^*,q^*}(t', v(t'), -\partial_t \phi)
\]

\[
= F_{s^*,q^*}(t, u_{s^*,q^*}(t), \partial_t \phi) - F_{s^*,q^*}(t, v(t'), \partial_t \phi)
\]

\[
+ F_{s^*,q^*}(t, v(t'), \partial_t \phi) - F_{s^*,q^*}(t', v(t'), -\partial_t \phi)
\]

\[
\geq \min(1, \gamma) \kappa (u_{s^*,q^*}(t) - v_{s^*,q^*}(t')) + F_{s^*,q^*}(t, v(t'), \partial_t \phi) - F_{s^*,q^*}(t', v(t'), -\partial_t \phi)
\]

\[
= \min(1, \gamma) \kappa (u_{s^*,q^*}(t) - v_{s^*,q^*}(t')) ,
\]

where the second inequality follows from Proposition 3.3. We have a contradiction when \( \epsilon \to 0 \).

Below is the proof for Proposition 3.2.

**Proposition 3.2.** The verification of conditions (C2), (C3) and (C4) is immediate. For (C1), it suffices to show the following:

\[
L^\epsilon_{s^*,q^*}(t, r, u_{s^*,q^*} + \delta) - L^\epsilon_{s^*,q^*}(t, r', v_{s^*,q^*}) \geq \kappa \delta , 
\]

\[
M^+_{s^*,q^*}(t, r) - M^+_{s^*,q^*}(t, r') \geq (M^+_{s^*,q^*}(t, r') + 1) \gamma \kappa \delta . 
\]

\[
M^-_{s^*,q^*}(t, r) - M^-_{s^*,q^*}(t, r') \geq (M^-_{s^*,q^*}(t, r') + 1) \gamma \kappa \delta . 
\]

To prove (3.32), note that \( r_{s^*,q^*} - r'_{s^*,q^*} = \max\{r_{s,q} - r'_{s,q}\} \) implies \( r_{s,q} - r_{s^*,q^*} \leq r'_{s,q} - r'_{s^*,q^*} \) for all \( s,q \), which further implies \( e^{-\gamma (r_{s,q} - r_{s^*,q^*})} \geq e^{-\gamma (r'_{s,q} - r'_{s^*,q^*})} \) for all \( s,q \). The rest are straightforward calculations.

It remains to prove (3.33). Proving (3.34) is similar. We write \((s,q) = (s^* + \sigma \xi^0 +, q^* +\)
and proceed as in the proof of Proposition 3.3:

\[
M_{s^*, q^*}(t, r) - M_{s^*, q^*}(t, r') \\
\geq \min_{m^+ \in \{0, 1, \ldots, \bar{q} - q^*\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left( -m^+ \bar{\Upsilon} + r_{s,q}^* - (1+\kappa) r_{s^*, q^*}^* \right)} - e^{-\gamma \left( -m^+ \bar{\Upsilon} + r_{s,q}^* - (1+\kappa) r_{s^*, q^*}^* \right)} \right] \right\} \\
\geq \min_{m^+ \in \{0, 1, \ldots, \bar{q} - q^*\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left( -m^+ \bar{\Upsilon} + r_{s,q}^* - (1+\kappa) r_{s^*, q^*}^* \right)} \left( e^{-\gamma \left( r_{s,q}^* - (1+\kappa) r_{s^*, q^*}^* - r_{s,q}^* + (1+\kappa) r_{s^*, q^*}^* \right)} - 1 \right) \right] \right\} \\
\geq \min_{m^+ \in \{0, 1, \ldots, \bar{q} - q^*\}} \left\{ \mathbb{E} \left[ e^{-\gamma \left( -m^+ \bar{\Upsilon} + r_{s,q}^* - (1+\kappa) r_{s^*, q^*}^* \right)} \left( e^{\gamma \kappa (r_{s^*, q^*} - r_{s, q^*}) - 1} \right) \right] \right\} \\
\geq \gamma \kappa (r_{s^*, q^*} - r_{s^*, q^*}) \left( M_{s^*, q^*}^+ (t, r') + 1 \right) \\
\geq \gamma \kappa \delta \]

\[
\square
\]
Chapter 4

Optimal Decisions in a Time Priority Queue

4.1 Introduction

Most major modern stock exchanges have switched to electronic limit order books (LOBs) as the primary matching mechanism of trades. Along with this evolution comes a surge in computerized trading algorithms. These algorithms are developed to perform a variety of tasks (for example, executing a large order or providing liquidity to the market) by quickly digesting information from the market and sending orders to the exchange. The success of a trading algorithm relies on two fundamental questions: what type of orders should be sent and what is the optimal time to do so?

Many LOBs adopt a price-time priority rule, that is, the priorities of limit orders (LOs) facing execution by a market order (MO) are based first on their price, and then on their time of submission. Within a queue at a fixed price, the first LO which is matched to an incoming MO is the one which was placed at the earliest time. When an LO is placed, it resides at the back of the queue and only moves forward if other LOs in front of it are cancelled or if an incoming MO lifts other orders from the queue.
If an agent has an LO active in the LOB at the best price, there are a number of factors to consider when trying to determine the best course of action with respect to cancelling the order or letting it rest. One factor is the probability that an incoming MO would transact with the agent’s LO. This will depend on the position of the LO in the queue, since if it is farther from the front then it will have a smaller chance of being filled by a MO. Another factor is the likelihood that the agent may want to place the order back in the queue a short time after cancelling it. This will be affected by both the position of the LO as well as the total queue length. If the queue is very long and the agent’s LO is near the front, then it will take a long time for a new order to reach the current position should it be cancelled and replaced later.

In this chapter, we show how to form the optimal decisions to place and cancel LOs in a single queue based on the queue length and position of the LO. To achieve this goal, our first step is to perform an empirical study on LOB dynamics. Using Nasdaq data, we study various LOB features including the rate of addition and cancellation of LOs, rate of MO arrival, distribution of MO size, and distribution of replenished queue length. In particular, we wish to investigate how these events depend on the length of the best price queue. We also attempt to gauge the profitability of an LO execution based on an observable trade indicator. We propose a statistical model to describe the dynamics of these features. To the best of our knowledge, some of these empirical findings are novel, in particular the dependence of these events on the length of the queue. For example, we find that the intensity of cancellation of a limit order at a particular position in the queue has significant dependence on the position of the order, but little dependence on the total length of the queue. We also find that though market orders seldom go beyond the best price level, a large proportion of them deplete the entire queue at the best price level; the size of market orders is well described as a random variable right-censored by the volume at the best price level.

Based on our empirical study, we construct a queuing model for a single price level
on one side of the LOB. In our set-up, the dynamics of this queue are partially dictated by a stochastic time dependent regime which can be interpreted as an abstract trade signal (TS). The gain or loss of a filled LO depends on the regime of the TS: during a gainful regime, a filled LO is likely to be profitable, but the rate of MO arrivals is low so it takes longer for a LO to be executed; during an adverse regime, a filled LO is likely to constitute a loss, and the rate of MO arrivals is high so it takes less time for a LO to be executed. This behaviour is consistent with what we observe in the data.

An important feature of our model is that the agent’s impulse control may have a random effect on the state of the queue. Impulse control problems generally feature the ability for the agent to dictate the precise state of the system after their impulse. Mathematically this is proposed by imposing that the state of the system due to the action be a random variable which is measurable at the time of the impulse. The model proposed here allows for the resulting state to be random from the perspective of the agent without introducing issues related to measurable random variables.

We then consider the problem of an agent that maximizes her expected utility by optimally choosing the timing of placing and cancelling her LOs. For simplicity, we restrict the agent to have at most one LO active in the queue at a single time, and the agent incurs a small fixed cost when she places or cancels the LO. In order to form her optimal decision, she keeps track of three state variables: the TS, the queue length, and the queue position of her LO. When she has an LO active in the queue, she stays in the queue as long as the TS is in a gainful regime. However, it is worthwhile mentioning that even in an adverse regime the agent may keep an active LO in the queue, or place a new LO if she does not have an active one. This is because in an adverse regime, the agent may be willing to wait for her active LO to move forward in the queue so that it is in a good position by the time the TS becomes gainful. In an adverse regime, the agent optimally balances between the loss from an executed order and the gain from a good queue position when the TS changes to a favourable one.
4.1.1 Literature Review

Our work is related to much of the literature on optimal execution. In the seminal work of Almgren and Chriss (2001), the authors consider the problem of executing a large portfolio using MOs. Their work has been generalized in a number of ways, see for example, Alfonsi et al. (2010) and Gatheral et al. (2012). An alternative approach using LOs is taken by Avellaneda and Stoikov (2008), and has been extended by Bayraktar and Ludkovski (2011), Guéant et al. (2012), Guilbaud and Pham (2013), and Cartea and Jaimungal (2015b). In all the above studies, the actual queuing dynamics of the LOB are abstracted away. For example, in Almgren and Chriss (2001), Alfonsi et al. (2010), and Gatheral et al. (2012), the agent does not interact directly with the LOB; instead, her trading activities generate price impacts, which are assumed to be deterministic functions of the trading rate. In Avellaneda and Stoikov (2008) and subsequent studies, it is assumed that the agent’s filled LOs follow a counting process with rate independent of the state of the LOB. In both streams of literature, the state of the LOB becomes irrelevant.

More recent developments incorporate order flow information as a state variable for the LOB. In Bechler and Ludkovski (2015), order flow imbalance is assumed to be an Ornstein-Uhlenbeck process, whereas Cartea et al. (2015a) uses volume imbalance as a measure for order flow imbalance and model it as a discrete state Markov chain. In quantifying the profitability of filled LOs, for the purposes of our work we choose the TS to be volume imbalance. In general, the TS can be chosen to be any observable quantity which the agent believes will offer information about the profitability of trades and the dynamics of the LOB. Our contribution is to quantify when it is beneficial to cancel or place an LO, based on its queue position and the TS. The behavior of the agent adopting the optimal strategy resembles that in Yang and Zhu (2016), in which a high-frequency trader learns from the order flow from fundamental investors and trades according to that information.

Our work is also related to the literature on queuing models for LOBs. This stream of
the literature started with Cont et al. (2010). A simplified version which models only the best bid and ask can be found in Cont and De Larrard (2013), where its diffusive limit is also derived. Diffusive limits of queuing model for LOBs are also studied in Lakner et al. (2013), Jaisson et al. (2015), and Guo et al. (2015). In terms of empirical features, our model is closest to Huang et al. (2015). One distinction between our work and previous works is that we do not assume MOs have a constant size. Instead, we model MO size as a censored distribution. As suggested by our empirical study, a large proportion of MOs deplete the queue at the best price. Therefore, if an agent models MO size as a constant, then she will significantly underestimate the probability of her LO being executed if its position is far from the front of the queue.

It is also worthwhile mentioning Maglaras et al. (2014) and Maglaras et al. (2015), who consider order placement problems under fluid (deterministic) queuing models. Our work is different in that we have a stochastic queuing model where the queue length is discrete-valued. Moreover, we incorporate adverse selection risk, so the agent’s executed LOs are not always considered a gain by earning the spread. Rather, the asset price might move in the direction such that the agent’s executed LO becomes an instantaneous loss.

The rest of the chapter is organized as follows. Section 4.2 presents some empirical findings on LOB dynamics. In Section 4.3 we describe a queuing model which reflects many of our empirical findings and the agent’s stochastic control problem. In Section 4.4 we derive and simplify the dynamic programming equation for the agent’s control problem. Section 4.5 contains a numerical example using model parameters calibrated from real data. Section 4.7 concludes.

### 4.2 Empirical Analysis

In this section, we present empirical findings that motivate our queuing model in Section 4.3, including rate of addition and cancellation of other LOs, rate of MO arrivals,
distribution of MO size, and distribution of replenished queue length. We also choose to use the value of volume order imbalance as the TS and investigate the magnitude of gain or loss of a filled LO as a function of imbalance. In general, other TSs can be used if a dependence structure between the signal and other queue dynamics can be specified. We present the parametric models that we use for modelling these events and describe the estimation procedures. For the purposes of this analysis, we primarily consider only events on the sell side of the LOB, and as such we are concerned only with market buy orders (MBOs).

### 4.2.1 Data

The data that we analyze is the Nasdaq Historical Total View (ITCH) for the ticker INTC on Oct 1, 2014, with the first and last half hours of trading removed. We divide all volumes and order sizes by 100, round them to the nearest integer and discard results that are equal to zero. In other words, we assume that all buy and sell volumes are multiples of 100. Our choice is based on the observation that a large amount of LOs and MOs have a “round-lot” size, i.e, size of multiples of 100. For other equities this typical order size may be different and should be modified accordingly. See, for example, Cartea et al. (2015b) for more details. Henceforth it should be understood that any volume quantity is to be interpreted as representing a multiple of this “round-lot” size. We denote by \( L^b_t \) \( (L^a_t) \in \{1, 2, \ldots\} \) the observed volumes at time \( t \) of LOs posted at the best buy (sell) price. For most of this section, only the sell side of the LOB is examined. We have removed all events when \( L^a_t > 200 \) (0.7% of all events).

### 4.2.2 Volume Imbalance

As many researchers have pointed out, the imbalance between buy and sell order flows exhibits a dependence structure with future price changes. There are various ways of measuring order flow imbalance: Cont et al. (2014) constructs a measure based on net
aggregated volumes of buy and sell orders over a time interval; Cartea et al. (2015a) uses another measure, the volume imbalance, computed from volumes of LOs posted at the best buy and sell price. Both works show that their measures of order flow imbalance were significant in predicting price movement, and so we construct our TS regime from observations of the imbalance process. In this chapter, we consider the volume imbalance as in Cartea et al. (2015a), defined as:

$$\rho_t = \frac{L_t^b - L_t^a}{L_t^b + L_t^a} \in [-1, 1].$$ (4.1)

We construct a three-state regime process $Z_t$ which we will interpret as our TS by dividing the imbalance measure interval $[-1, 1]$ into three subintervals and assigning different values as follows:

1. sell-heavy: $\rho_t \in [-1, 0.2] \Rightarrow Z_t = 1$;
2. neutral: $\rho_t \in (0.2, 0.6] \Rightarrow Z_t = 2$;
3. buy-heavy: $\rho_t \in (0.6, 1] \Rightarrow Z_t = 3$.

As will be shown later, $Z_t = 1$ is considered by the agent a gainful regime, where a filled order is likely to be profitable; $Z_t = 3$ is considered by the agent an adverse regime where a filled order is likely to be considered a loss. In interpreting $Z_t$ as a TS we shall henceforth refer to these regimes as gainful, neutral, and adverse.

From the definition in (4.1) we expect dependence between the distribution of queue length $L_t^a$ and the regime of the TS $Z_t$. To capture this pattern, we propose letting the rate of transition between regimes dependent on the queue length. Let $\lambda^{\bar{z}, \bar{z}, \ell}$ be the rate of transition from regime $z$ to regime $\bar{z}$ ($z \neq \bar{z}$), when the queue length is $\ell > 0$. We estimate $\lambda^{\bar{z}, \bar{z}, \ell}$ through the following procedure. We denote by $N^Z(z, \bar{z}, \ell)$ the total number of transitions from $z$ to $\bar{z}$ when $L_t^a = \ell$ and by $T(z, \ell)$ the occupation time for which $L_t^a = \ell$ and $Z_t = z$. We run a Poisson regression $N^Z(z, \bar{z}, \ell) \sim Poisson(\lambda^{\bar{z}, \bar{z}, \ell}T(z, \ell))$ with the
canonical link function
\[
\lambda^{z,\bar{z},\ell} = \exp(\beta_{0,z,\bar{z}}^{Z} + \beta_{1,z,\bar{z}}^{Z} \ell),
\]
for constants $\beta_{0,z,\bar{z}}^{Z}$ and $\beta_{1,z,\bar{z}}^{Z}$. The coefficients set equal to their maximum likelihood estimates and the estimated functions $\lambda^{z,\bar{z},\ell}$ are truncated for the highest 1% and the lowest 1% values of $\ell$ weighted by occupations time\(^1\) and replaced by linear extrapolation. Estimated coefficients are reported in Table 4.5 and 4.6.

Figures 4.1(a) 4.1(d) show the rates of transition from the gainful to the neutral and adverse regimes. Both rates increase as the queue becomes shorter, an indication that there is an increasing buy pressure, thus it is more and more likely that we transition to an increasingly adverse regime. Also note that the transition rate from gainful to neutral is higher than that from gainful to adverse (the jump which can be interpreted as being of smaller size is more likely to occur). Figures 4.1(b) and 4.1(e) show the transition rates to gainful and adverse regimes from the neutral regime. When the queue is short, the regime is more likely to become adverse, whereas when the queue becomes long, it is more likely to become gainful. Similar patterns can be found in Figures 4.1(c) and 4.1(f), where the rates of transition out of the adverse regime are presented.

### 4.2.3 Gain/Loss of a Filled Limit Order

The agent’s LO faces different levels of adverse selection risk in different regimes. In our study, we use $\theta = \Upsilon - \Delta_{dt}$ as a proxy for the agent’s wealth change when her order is filled, where $\Upsilon$ is equal to the size of a half-tick plus the rebate, multiplied by the size of the agent’s order, and $\Delta_{dt}$ is the change in the best sell price $dt$ seconds after an MBO arrival, multiplied by the size of the agent’s order. $\Upsilon$ can be interpreted as the agent’s gain from an executed limit order if the fundamental asset price does not change upon

\(^{1}\)For each $\ell$, we compute $f_{z,\ell} = \frac{T_{z,\ell}}{\sum_{\ell } T_{z,\ell}}$, the fraction of time that the queue length is equal to $\ell$ when the LOB is in regime $z$. The upper and lower truncation thresholds are defined as: $\ell_{\bar{z}} = \min\{\ell : \sum_{m<\ell} f_{z,m} \geq 0.01\}$ and $\ell_{\bar{z}} = \max\{\ell : \sum_{m<\ell} f_{z,m} \leq 0.99\}$. 
Figure 4.1: Estimated rate of transition. Points represent the empirically observed transition rate in each regime for each value of queue length. Curves represent the estimated fit corresponding to equation (4.2).

Figure 4.2: Histogram of best sell price change 100ms after MO arrival.

execution. In our numerical example in Section 4.5, we will use the empirical distribution of $\Delta dt$ for $dt = 100$ milliseconds.

Figure 4.2 shows the histogram of best sell price changes 100 milliseconds after the arrival of an MBO. It is clear that when the TS is in the adverse regime, the best sell price is more likely to move up after an MBO arrival and the agent suffers more from adverse selection.
Figure 4.3: The rate of cancellation for different queue length and queue position. For each point \((\ell, y)\) on the grid, we calculate the total number of cancellation divided by the total time that the best sell queue length is equal to \(\ell\). We then apply Gaussian smoothing to obtain this figure.

4.2.4 Rate of Addition and Cancellation

We assume that all cancellations are of a unit size. We denote by \(\tilde{\lambda}^{c,z,\ell,y}\) the rate of cancellation of an order at position \(y\) (\(0 < y \leq \ell\)) when the queue length is \(\ell\) and the TS regime is \(z\) (the queue position \(y = 1\) corresponds to the front of the queue, and \(y = \ell\) represents the back). Figure 4.3 shows the empirical estimates of \(\tilde{\lambda}^{c,z,\ell,y}\). Each panel corresponds to a different regime \(z\), and the colour at each point \((\ell, y)\) represents the corresponding rate of cancellation \(\tilde{\lambda}^{c,z,\ell,y}\) (lighter colours indicating a higher rate of cancellation). For each regime, we see that the rate of cancellations depends much more heavily on the position in the queue, \(y\), than on the total length of the queue, \(\ell\). Therefore, a reasonable model for LO cancellation is to assume that \(\tilde{\lambda}^{c,z,\ell,y}\) does not depend on \(\ell\) and henceforth we use the notation \(\tilde{\lambda}^{c,z,y}\).

The agent is not concerned with the rate of cancellation of any particular order, but she is concerned with the total rate of cancellations in front of her order and behind her order. Therefore, we denote by \(\lambda^{c,z,y}\) the total rate of cancellation at or before position \(y\), i.e., \(\lambda^{c,z,y} = \sum_{k=1}^{y} \tilde{\lambda}^{c,z,k}\). For instance, when the LOB is in regime \(z\), the rate of cancellations happening within the interval \((a, b]\) is \(\lambda^{c,z,b} - \lambda^{c,z,a}\). We do not attempt to estimate \(\tilde{\lambda}^{c,z,y}\), but below we describe how we estimate the function \(\lambda^{c,z,y}\).

We begin by taking all events corresponding to LO cancellations in the best sell queue,
marked as \((Q_{z,i}^c, L_{z,i}^c)_{i=1,2,...}\), where \(Q_{z,i}^c\) is the volume of the \(i\)-th cancellation that occurs during regime \(z\) and \(L_{z,i}^c\) is the total volume at the best sell price immediately prior to this cancellation. For each level \(L_{z,i}^c = \ell\) \((\ell = 1, 2, ...\) and each regime we denote by \(N^c(z, \ell)\) the aggregate volume of cancelled LOs, so that we have

\[
N^c(z, \ell) = \sum_{i: L_{z,i}^c = \ell} Q_{z,i}^c.
\]

Recall that \(T(z, \ell)\) denotes the occupation time of the state \((Z_t, L_t^a) = (z, \ell)\). We then conduct a Poisson regression \(N^c(z, \ell) \sim \text{Poisson}(\lambda^c(z, \ell)T(z, \ell))\) with the following link function

\[
\lambda^c(z, \ell) = \beta_0^{c,z}(\exp(\beta_1^{c,z} \ell) - 1), \tag{4.3}
\]

where \(\beta_0^{c,z}\) and \(\beta_1^{c,z}\) are constants. The above form gives \(\lambda^c(z, \ell = 0) = 0\) which will be an important restriction when we write our full queue model to ensure that when the agent’s order is at the front of the queue, the rate of cancellations in front of her order is zero. The values of \(\beta_0^{c,z}\) and \(\beta_1^{c,z}\) are obtained via maximum likelihood estimation.

The rate of addition \(\lambda^{a,z,\ell}\) is estimated through a similar procedure, except that we use the canonical link function

\[
\lambda^{a,z,\ell} = \exp(\beta_0^{a,z} + \beta_1^{a,z} \ell), \tag{4.4}
\]

for constants \(\beta_0^{a,z}\) and \(\beta_1^{a,z}\).

Figure 4.4 shows the estimated rate of addition and cancellation. Coefficients are reported in Table 4.7.

### 4.2.5 Rate of Market Order Arrival

We denote by \(\lambda^{m,z,\ell}\) the rate of MO arrival when the TS is in regime \(z\) and the length of the queue is \(\ell\). In principle, \(\lambda^{m,z,\ell}\) can depend on both \(z\) and \(\ell\), as in the case for the
rate of addition and cancellation. However, given the fact that the number of MO events is significantly smaller than the number of LO additions and cancellations, it is difficult to estimate the dependence of \( \lambda_{m,z,\ell} \) on \( \ell \). For example, in the data set we consider here there are only 2,911 MBOs. In particular, there are only 556 MBOs when the TS is in the gainful regime. Therefore, to keep the model parsimonious, we assume that \( \lambda_{m,z,\ell} \) depends on the regime \( z \) only and henceforth we denote the rate of MOs by \( \lambda_{m,z} \). The quantity \( \lambda_{m,z} \) can be estimated as

\[
\lambda_{m,z} = \frac{M(z)}{T(z)},
\]

where \( M(z) \) is the total number of MBOs when the TS is in regime \( z \) and \( T(z) \) is the occupation time of regime \( z \). Estimated results are present in Table 4.8.

4.2.6 Distribution of Market Order Size

As documented in a number of studies, it is rare for an MO to walk through the first level of the LOB (see for example Cartea et al. (2015a)). Here, however, we present another empirical finding: a non-negligible fraction of MOs deplete the entire queue at the best price. Table 4.1 shows some summary statistics on MO size. The column “Average MO size” contains the average size of MBOs when the queue length is at different intervals.
We can see that as the queue length increases, the average MO size increases as well. The column “% of depletion” contains the proportion of MOs that deplete the entire queue at the best sell price. Surprisingly, these proportions are very high: for example, even when the queue length is between [40, 60), which is about 5 times the average MO size, the probability that an MO depletes the whole queue is 12.1%. Therefore, a realistic model for MO size should incorporate the probability of an LO being executed even it is far away from the front of the queue.

Another observation from Table 4.1 is that the percentage of MOs depleting the entire queue decreases as the queue length increases. This suggests that we model the MO size as a random variable right-censored by the queue length. To validate this hypothesis, we conduct the following bootstrap exercise. First we estimate the empirical distribution of MO size using the non-parametric Kaplan-Meier estimator (Kaplan and Meier (1958)), assuming that MO size is indeed right-censored by the queue length. We then draw independent samples \((\bar{Q}_i^b)_{i=1,2,...}\) from the estimated empirical distribution of the MO size and \((L_i^b)_{i=1,2,...}\) from the empirical distribution of queue length. Next we compute the censored MO size \(Q_i^b = \min\{\bar{Q}_i^b, L_i^b\}\) and the summary statistics based on the bootstrapped dataset \((Q_i^b, L_i^b)\). The results are shown in Table 4.2. It can be seen that the summary statistics in Table 4.2 are very close to those in Table 4.1.

<table>
<thead>
<tr>
<th>Queue Length</th>
<th>Average MO size</th>
<th>% of depletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 20)</td>
<td>4.5</td>
<td>39.8%</td>
</tr>
<tr>
<td>[20, 40)</td>
<td>9.2</td>
<td>14.3%</td>
</tr>
<tr>
<td>[40, 60)</td>
<td>10.0</td>
<td>12.1%</td>
</tr>
<tr>
<td>[60, 80)</td>
<td>13.6</td>
<td>5.8%</td>
</tr>
<tr>
<td>[80, 100)</td>
<td>16.1</td>
<td>4.1%</td>
</tr>
</tbody>
</table>

Table 4.1: Size of market sell orders (empirical).

In later sections, we will model MO size using a negative binomial distribution which depends on the regime of the TS. In particular, for a fixed regime \(z\) and for each pair of observed MO size and queue length \((Q_{m,i}^z, L_{m,i}^z)_{i=1,2,...}\), we assume that \(Q_{m,i}^z = \min\{\bar{Q}_{m,i}^z, L_{m,i}^z\}\), where \((\bar{Q}_{m,i}^z)_{i=1,2,...}\) are i.i.d. negative binomial random variables. The parameters are es-
<table>
<thead>
<tr>
<th>Queue Length</th>
<th>Average MO size</th>
<th>% of depletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 20)</td>
<td>4.3</td>
<td>40.6%</td>
</tr>
<tr>
<td>(20, 40)</td>
<td>8.7</td>
<td>14.8%</td>
</tr>
<tr>
<td>(40, 60)</td>
<td>11.2</td>
<td>9.5%</td>
</tr>
<tr>
<td>(60, 80)</td>
<td>12.8</td>
<td>6.2%</td>
</tr>
<tr>
<td>(80, 100)</td>
<td>13.6</td>
<td>3.7%</td>
</tr>
</tbody>
</table>

Table 4.2: Size of market sell orders (bootstrapped).

(a) gainful  
(b) neutral  
(c) adverse  

Figure 4.5: Estimated and empirical distribution of MO size. The empirical distributions are estimated using Kaplan-Meier estimator and the curves are the result of fitting a negative binomial distribution.

4.2.7 Distribution of Replenished Queue Length

When the best queue in our model is depleted, either due to an MO filling the entire queue, or the cancellation of the last LO in the queue, we assume that the queue is immediately replenished. The new queue length is drawn from a distribution \( \mu^{z} \), which depends on the regime \( z \). To estimate \( \mu^{z} \), we fit negative binomial distributions to the length of the best sell queue. The results are shown in Figure 4.6. Estimated parameters can be found in Table 4.10.
4.3 Model

In this section we propose a model for the queue which encompasses all of the possible events which may alter its state. Once the set of possible events has been established, the rate at which these events occur and the distributions of the changes they cause are chosen to reflect the behaviour observed in Section 4.2.

Our construction is non-standard in the sense that our stochastic processes will be indexed by $\mathbb{R} \times \{1, 2\}$ rather than by $\mathbb{R}$ as usual. We introduce the notation $\tilde{t} = (t, k)$ where $t \in \mathbb{R}$ and $k \in \{1, 2\}$. We equip the set $\mathbb{R} \times \{1, 2\}$ with the lexicographical order, i.e. $\tilde{t}_1 < \tilde{t}_2$ if $t_1 < t_2$ or if $t_1 = t_2$ and $k_1 < k_2$. We require this construction because at a single instant in time, there may occur multiple events which must follow a logical sequence. Thus, the first index denoted $t$ is to be interpreted as the usual flow of continuous time, and the second index denoted $k$ will be used to construct the proper sequencing of events which occur simultaneously.

The need for this type of construction stems from two sources. First, we model a replenishing of the queue to a random length when it is depleted by some event, and second, the agent’s action of cancelling the last LO in the queue will cause the queue to be replenished. The cancellation and replenishing occur at the same instant in time, but within this particular time we would still like to interpret the cancellation as occurring first, and we do not want the agent to have foreknowledge of the replenished length when
she cancels her order.

A specific example which illustrates this sequencing of simultaneous events is the following: suppose there are exactly two LOs active in the queue, the order at the back being the one belonging to the agent. At time $t$:

1. The order at the front of the queue is cancelled.
2. Based on the cancellation of the first order, the agent decides to cancel her order.
3. The cancellation of the last order causes the queue to be replenished at a random length.
4. Based on the new length of the queue, the agent decides to place a new LO at the back of the queue.

This example demonstrates how we would like the decisions of the agent to be based on information in the queue. The agent’s decision to cancel her order in step 2 is not allowed to depend on the length of the replenished queue in step 3, but the agent’s decision to replace the order in step 4 is allowed to depend on this quantity. The typical set-up of an impulse control problem gives the agent perfect knowledge of the effect of her control on the state of the system by stating that the controlled state is measurable with respect to the stopping time of the impulse. See for example Bensoussan et al. (1982), Baccarin and Sanfelici (2006), and Ly Vath and Pham (2007). Our extended indices allow us to circumvent this issue.

Fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T})$. In developing the model it may be useful to consider the filtration $\mathcal{G}$ defined by $\mathcal{G}_t = \mathcal{F}_{(t,1)}$.

### 4.3.1 State Space

The state of the system will be denoted by $\mathcal{S} = (X, Z, L, Y)$, where $X$ represents the agent’s wealth, $Z$ is the regime of the TS, $L$ is the total length of the queue, and $Y$ is the
position of the agent’s order (should it exist). We set a maximum possible queue length so that \( L \in \{0, 1, \ldots, N^L\} \). We allow the length of the queue to be zero, but we will specify the dynamics such that if \( L(t, 1) = 0 \), then \( L(t, 2) > 0 \) due to instantaneous replenishing. In addition, we will always have \( Y \in \{1, \ldots, N^L, \Xi\} \) where \( Y = \Xi \) is a placeholder state which indicates that the agent does not have an active order. Otherwise we must enforce the constraint \( Y \leq L \) which indicates that the position of the agent’s order is at most equal to the length of the queue. We denote the set of feasible states by

\[
S = \left\{ S = (X, Z, L, Y) : X \in \mathbb{R}, Z \in \{1, \ldots, N^Z\}, L \in \{0, 1, \ldots, N^L\}, Y \in \{1, \ldots, N^L, \Xi\}, Y = \Xi \text{ or } Y \leq L \right\}.
\]

(4.5)

We interpret all statements of the form \( \Xi > k \), \( \Xi < k \), and \( \Xi = k \) to be false for \( k \in \mathbb{R} \).

### 4.3.2 Uncontrolled Queue Dynamics

Here we indicate the dynamics of the state given that the agent does not place or cancel any orders. To this end, for each value of \( z, \bar{z} \in \{1, \ldots, N^Z\} \) and \( \ell \in \{1, \ldots, N^L\} \), we let \( N_{z,\bar{z},\ell}^{c}(t, 1) \), \( N_{z,\bar{z},\ell}^{a}(t, 1) \), and \( M_{z}^{a}(t, 1) \) be independent Poisson processes adapted to \( G_{t} \). An arrival of the process \( N_{z,\bar{z},\ell}^{c}(t, 1) \) indicates a change in the TS from \( z \) to \( \bar{z} \) at time \( t \). An arrival of \( N_{z,\bar{z},\ell}^{a}(t, 1) \) represents the cancellation of an order at position \( \ell \) while the TS is equal to \( z \). Lastly, an arrival of \( M_{z}^{a}(t, 1) \) indicates an MO in regime \( z \). We extend these processes to be defined at all \( \tilde{t} \) by setting \( N_{z,\bar{z},\ell}^{c}(\tilde{t}, 1) = N_{z,\bar{z},\ell}^{c}(t, 1) \), and similarly for the others. The interpretation is that these processes may jump at an index of the form \((t, 1)\), but not at one of the form \((t, 2)\). Staying consistent with the notation of Section 4.2, the intensities of these processes are denoted by \( \lambda_{z,\bar{z},\ell} \), \( \tilde{\lambda}_{c,z,\ell} \), \( \lambda^{a,z,\ell} \), and \( \lambda^{m,z} \). In addition, as before, we have \( \lambda^{c,z,\ell} = \sum_{k=1}^{\ell} \tilde{\lambda}^{c,z,k} \). The quantity \( \tilde{\lambda}^{c,z,\ell} \) represents the rate of cancellation.
of an order at position \( \ell \) (if it exists), so \( \lambda_{c,z,\ell} \) is the rate of cancellation of any order in front of (and including) the one in position \( \ell \).

We also suppose that for each \( z \) and each \( t \) we have three random variables \( \epsilon^{z}_{(t,1)} \), \( \theta^{z}_{(t,1)} \), and \( \zeta^{z}_{(t,2)} \), all of which are independent from all other variables and with distributions \( \mu^{\epsilon,z} \), \( \mu^{\theta,z} \), and \( \mu^{\zeta,z} \) respectively. The distribution \( \mu^{\epsilon,z} \) is chosen so that \( \epsilon^{z}_{(t,1)} \) is a positive integer, \( \mu^{\zeta,z} \) is chosen so that \( \zeta^{z}_{(t,2)} \) is an integer between 1 and \( N_L \), and \( \mu^{\theta,z} \) is chosen so that \( \mathbb{E}[e^{-\gamma \theta^{z}_{(t,1)}}] < \infty \) (\( \gamma \) is the agent’s risk aversion parameter which will be introduced later). Note that two of these variables are defined at time \( (t,1) \), and the third is defined at time \( (t,2) \). This is to emphasize that \( \epsilon^{z}_{(t,1)} \) and \( \theta^{z}_{(t,1)} \) are \( \mathcal{F}_{(t,1)} \)-measurable, but \( \zeta^{z}_{(t,2)} \) is \( \mathcal{F}_{(t,2)} \)-measurable. The need for this will be clear below when we specify the dynamics of the queue. The interpretation of these quantities is the following: the value of \( \epsilon^{z}_{(t,1)} \) is the size of an MO that arrives at time \( t \) while in regime \( z \). If the agent has an LO which is filled by an incoming MO, then the change in the agent’s wealth is equal to \( \theta^{z}_{(t,1)} \). The value of \( \zeta^{z}_{(t,2)} \) will be used as the size of the replenished queue if there is an event at time \( t \) which depletes it.

Let \( \bar{\tau} = (\tau, \kappa) \) be a stopping time in the filtration \( \mathcal{F}_t \) and let \( \varphi \) be an \( \mathcal{F}_{\bar{\tau}} \)-measurable random variable taking values in \( \mathcal{S} \) with the restriction that if \( L(\varphi) = 0 \), then \( \kappa = 1 \). We define a sequence of stopping times \( \bar{\rho}_j = (\rho_j, k_j) \) and states \( \varphi_j \) as follows: let \( \bar{\rho}_0 = \bar{\tau} \) and \( \varphi_0 = \varphi \), and set

\[
X_{\bar{\rho}_0, \varphi}^u = X(\varphi) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quan
Now recursively define stopping times by

\[
\bar{\rho}_{j+1} = \begin{cases} 
(\rho_j, 1), & \text{if } L_{\bar{\rho}_j}^u \neq 0, \\
(\rho_j, 2), & \text{if } L_{\bar{\rho}_j}^u = 0,
\end{cases}
\]  

(4.6)

where if \( L_{\bar{\rho}_j}^u \neq 0 \),

\[
\rho_{j+1} = \inf_{\bar{z} \neq \bar{z}_{\bar{\rho}_j}} \left\{ t > \rho_j : \Delta N_t^{Z_{\bar{\rho}_j}^u, \bar{z}, \ell} = 1, \Delta N_t^{c, Z_{\bar{\rho}_j}^u, \ell} = 1, \Delta N_t^{a, Z_{\bar{\rho}_j}^u, L_{\bar{\rho}_j}^u} = 1, \text{ or } \Delta M_t^{Z_{\bar{\rho}_j}^u} = 1 \right\},
\]  

(4.7)

and define the state transition at these times by

\[
\varphi_{j+1} = \begin{cases}
\left( X_{\bar{\rho}_j}^u, Z_{\bar{\rho}_j}^u, L_{\bar{\rho}_j}^u, Y_{\bar{\rho}_j}^u \right), & \text{if } \Delta N_{\bar{\rho}_{j+1}}^{Z_{\bar{\rho}_j}^u, \bar{z}, \ell} = 1, L_{\bar{\rho}_j}^u \neq 0, \\
\left( X_{\bar{\rho}_j}^u, Z_{\bar{\rho}_j}^u, L_{\bar{\rho}_j}^u, 1 + Y_{\bar{\rho}_j}^u \right), & \text{if } \Delta N_{\bar{\rho}_{j+1}}^{a, Z_{\bar{\rho}_j}^u, L_{\bar{\rho}_j}^u} = 1, L_{\bar{\rho}_j}^u \neq 0, \\
\left( X_{\bar{\rho}_j}^u, Z_{\bar{\rho}_j}^u, L_{\bar{\rho}_j}^u - 1, Y_{\bar{\rho}_j}^u - 1_{\ell < Y_{\bar{\rho}_j}^u} \right), & \text{if } \Delta N_{\bar{\rho}_{j+1}}^{c, Z_{\bar{\rho}_j}^u, L_{\bar{\rho}_j}^u} = 1, L_{\bar{\rho}_j}^u \neq 0, \\
\left( X_{\bar{\rho}_j}^u + \theta 1_{\ell \leq Y_{\bar{\rho}_j}^u}, Z_{\bar{\rho}_j}^u, (L_{\bar{\rho}_j}^u - \epsilon) 1_{\ell \leq L_{\bar{\rho}_j}^u} \right), & \text{if } \Delta M_{\bar{\rho}_{j+1}}^{Z_{\bar{\rho}_j}^u} = 1, L_{\bar{\rho}_j}^u \neq 0, \\
\left( X_{\bar{\rho}_j}^u, Z_{\bar{\rho}_j}^u, \zeta, \Xi \right), & \text{if } L_{\bar{\rho}_j}^u = 0
\end{cases}
\]  

(4.8)

where for the sake of brevity in (4.8) we have replaced \( \theta_{\bar{\rho}_{j+1}}^{Z_{\bar{\rho}_j}^u}, \epsilon_{\bar{\rho}_{j+1}}^{Z_{\bar{\rho}_j}^u} \), and \( \zeta_{\bar{\rho}_{j+1}}^{Z_{\bar{\rho}_j}^u} \) by \( \theta, \epsilon, \) and \( \zeta \) respectively. We now set:

\[
X_{\bar{\rho}_{j+1}}^u = X(\varphi_{j+1}), \quad Z_{\bar{\rho}_{j+1}}^u = Z(\varphi_{j+1})
\]  

\[
L_{\bar{\rho}_{j+1}}^u = L(\varphi_{j+1}), \quad Y_{\bar{\rho}_{j+1}}^u = Y(\varphi_{j+1})
\]
This construction requires some explanation. The definition given by (4.6) and (4.7) corresponds to the arrival index of the next feasible event. If there is an event that sends \( L \) to zero (which must happen at an index of the form \((t, 1)\) by construction), then the next stopping time is immediate but occurs at an index of the form \((t, 2)\). If the most recent previous event does not result in \( L \) being sent to zero, then the next event is due to a regime change, MO arrival, LO arrival, or LO cancellation, and must occur at an index of the form \((t, 1)\). Note that the infimum is taken only over values of \( \ell \) corresponding to order positions which are within the queue and not equal to the agent’s order should it exist, hence the agent’s order can never be exogenously cancelled. Also, by independence of the Poisson processes, with probability 1 the increments in (4.7) never coincide, so the definition in (4.8) is well defined.

The first two types of events in (4.8) are rather straightforward. The first type corresponds to a change in the TS from its previous value of \( Z_{\rho_k} \) to the new value of \( \bar{z} \) (by convention we set the intensity \( \lambda^{z,z,\ell} = 0 \) for all \( z \)). The second type corresponds to the addition of a single LO (we set the intensity of \( \lambda^{a,z,N_L} = 0 \) to enforce \( L_t^u \leq N^L \)).

The third type of event corresponds to a cancellation of the order in position \( \ell \). This will reduce the length of the queue by 1 (note that if this results in the queue being depleted, then by (4.6) the next stopping time will be \( \bar{\rho}_{k+2} = (\rho_{k+1}, 2) \) to allow the queue to be randomly replenished immediately). If the agent does not have an order in the queue, then this status is unchanged due to the cancellation. If the agent does have an active order, then its position decreases by one if the cancellation occurs in front of the agent’s order. Finally, note that if there is exactly one order in the queue and the agent has an active order, then an exogenous cancellation cannot occur due to the definition in (4.7).

The fourth event type is the arrival of an MO. If the size of the MO (denoted here by \( \epsilon \)) is large enough to fill the agent’s order, then the wealth changes by the random value \( \theta \). The length of the queue either decreases by \( \epsilon \), or if the MO is large enough to deplete
the queue then it sends $L$ to zero (and it will be immediately replenished as described in the third type of event). Finally, if the MO is not large enough to fill the agent’s LO, then the agent’s position decreases by the size of the MO. Otherwise, the status of the agent’s order changes to $\Xi$ as the order is filled and removed from the queue.

The state of the uncontrolled system starting at index $\bar{\tau}$ in state $\varphi$ is then defined as

$$S_{\bar{t},\bar{\tau},\varphi} = (X_{\bar{t},\bar{\tau},\varphi}, Z_{\bar{t},\bar{\tau},\varphi}, L_{\bar{t},\bar{\tau},\varphi}, Y_{\bar{t},\bar{\tau},\varphi}) = \varphi_j, \quad \text{for } \rho_j \leq \bar{t} < \rho_{j+1} \quad (4.9)$$

### 4.3.3 Controlled Queue Dynamics

The previous subsection outlined how the state of the queue changes if the agent does not intervene. The agent’s control will be defined by a sequence of stopping times whose effect on the state of the queue will be outlined here. The full controlled process will then consist of stopping the uncontrolled queue at times corresponding to the agent’s strategy, then restarting the uncontrolled queue in a new state defined appropriately by the effect of the agent’s action. We will also describe the restrictions put on the stopping times to ensure that the agent’s action is feasible.

Let $\{\bar{\tau}_j\}_{j \geq 1} = \{(\tau_j, \kappa_j)\}_{j \geq 1}$ be a sequence of non-decreasing stopping times which represent the times at which the agent sends a signal to place or cancel an order. We denote the controlled state by $S_{\bar{t}}^c$ and define it iteratively by setting $\tau_0 = (0, 1)$ and $\phi_0 = S_{(0,1)}^c = (x, z, \ell, y) \in S$, and for $\bar{\tau}_j \leq \bar{t} < \bar{\tau}_{j+1}$ we set

$$S_{\bar{t}}^c = (X_{\bar{t}}, Z_{\bar{t}}, L_{\bar{t}}, Y_{\bar{t}}^c) = S_{\bar{t},\bar{\tau}_j,\phi_j}^u. \quad (4.10)$$

It remains to define the states $\phi_j$ which represent the states at which we start the uncontrolled queue immediately after the agent exercises her control. Define the operator
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A : \mathbb{S} \rightarrow \mathbb{S} by

\[
A(x, z, \ell, y) = \begin{cases} 
(x - \eta, z, \ell + 1, \ell + 1), & y = \Xi \\
(x - \eta, z, \ell - 1, \Xi), & y \neq \Xi
\end{cases}
\] (4.11)

and then set

\[
\phi_{j+1} = A(S_{\bar{\tau}_j, \bar{\phi}_j})
\] (4.12)

The value of \(A(x, z, \ell, y)\) represents the state of the queue once the signal is sent given that the state is \((x, z, \ell, y)\) immediately before the signal. Note that the operators incorporate the cost \(\eta > 0\) incurred by the agent for sending a signal.

Finally, we impose the following restrictions on \(\{\bar{\tau}_j\}_{j \geq 1} = \{(\tau_j, \kappa_j)\}_{j \geq 1} : \)

\[
\lim_{j \to \infty} \tau_j > T,
\] (4.13)

\[
Y_{\bar{\tau}_j, \bar{\phi}_{j-1}}^{u, \tau_j-1, \bar{\phi}_{j-1}} = \Xi \Rightarrow L_{\bar{\tau}_j, \bar{\phi}_{j-1}}^{u, \tau_j-1, \bar{\phi}_{j-1}} < N^L
\] (4.14)

\[
\kappa_j = \begin{cases} 
1 & \text{if } L_{\bar{\tau}_j, \bar{\phi}_{j-1}}^{u, \tau_j-1, \bar{\phi}_{j-1}} = 1 \text{ and } Y_{\bar{\tau}_j, \bar{\phi}_{j-1}}^{u, \tau_j-1, \bar{\phi}_{j-1}} = 1, \\
2 & \text{otherwise}
\end{cases}
\] (4.15)

We denote the set of sequences which satisfy these conditions by \(\mathcal{A}\). Condition (4.13) ensures that the number of signals sent by the agent over the trading period is finite almost surely. This simplifying assumption will be satisfied automatically by the optimal strategy anyway since the cost satisfies \(\eta > 0\). Condition (4.14) simply states that if the agent does not have an order in the queue, then she may only place one if the length of the queue is strictly less than the maximum allowable length. This simplifying assumption could be relaxed in a more generalized framework and is also convenient for numerical purposes.

Condition (4.15) is related to the information that the agent has at the time of


executing her strategy. If $L_i^c = Y_i^c = 1$, then the agent’s order is the only one in the queue, and cancelling it would send the length of the queue to zero to be replenished immediately after. We do not wish for the agent’s strategy to consist of a cancellation when she has foreknowledge of the length of the replenished queue. Thus, when cancelling at an index of the form $\bar{t} = (t, 1)$, the queue is replenished at $(t, 2)$ to the level $\zeta_{Z_c(\bar{t}, 1)}^{(t, 1)}$, which is an $\mathcal{F}_{(t, 2)}$ measurable random variable. The agent is then allowed to replace the order at index $(t, 2)$, depending on the value of $\zeta_{(t, 2)}$.

### 4.3.4 Performance Criteria and Optimization

The agent attempts to choose a sequence of stopping times satisfying conditions (4.13) to (4.15), denoted by $\{\bar{\tau}_j\}_{j \geq 1} \in \mathcal{A}$, which maximizes the utility of her terminal wealth. We define the value function as

$$H(\bar{t}, x, z, \ell, y) = \sup_{\{\tau_j\}_{j \geq 1} \in \mathcal{A}} \mathbb{E}\left[ -e^{-\gamma X^c_{\bar{\tau}}(\bar{t}, 2)} \middle| X_i^c = x, Z_i^c = z, L_i^c = \ell, Y_i^c = y \right],$$

(4.16)

where $\gamma$ is a risk aversion parameter. We suppose that the dynamic programming principle holds for any stopping time $\bar{\tau} \preceq (T, 2)$:

$$H(\bar{t}, x, z, \ell, y) = \sup_{\{\tau_j\}_{j \geq 1} \in \mathcal{A}} \mathbb{E}\left[ H(\bar{\tau}, X^c_{\bar{\tau}}, Z^c_{\bar{\tau}}, L^c_{\bar{\tau}}, Y^c_{\bar{\tau}}) \middle| X_i^c = x, Z_i^c = z, L_i^c = \ell, Y_i^c = y \right].$$

(4.17)

Our formulation of the problem (4.17) differs from classical stochastic control problems in that it allows the agent to exercise multiple impulse controls at the same time $t$. Therefore, it is not clear how to apply the dynamic programming principle to this problem. At this point we would like to eliminate the use of the time index notation of the form $\bar{t} = (t, k)$ in order to write a more easily readable dynamic programming equation. To this end, suppose $\bar{t} = (t, 1)$ on the left hand side of (4.17) and $\bar{\tau} = (t, 2)$ on
the right hand side. If $\ell > 1$ or $y = \Xi$ then

$$H((t, 1), x, z, \ell, y) = H((t, 2), x, z, \ell, y),$$

(4.18)

due to (4.15) (the agent is not allowed to take action at this particular time index, so
the uncontrolled dynamics proceed). On the other hand, if $\ell = y = 1$, then

$$H((t, 1), x, z, \ell, y) = \max \left\{ \mathbb{E}\left[ H((t, 2), x - \eta, z, \zeta^z_{(t,2)}, \Xi) \right], H((t, 2), x, z, \ell, y) \right\}$$

(4.19)

where the expectation is taken over the random variable $\zeta^z_{(t,2)}$. Equation (4.19) carries
the intuition that if the agent is the only one with an order in the queue, then the optimal
choice is based on the relative value of proceeding to the next instant in time with no
action versus the average value to the agent if they were to cancel and generate a random
queue length. The form of the first term within the maximum of (4.19) is due to the last
lines of (4.6) and (4.8). By considering both (4.18) and (4.19) we are able to eliminate
any use of time index of the form $(t, 1)$ and write the value function in terms of only
$(t, 2)$. With a slight abuse of notation, from this point forward we now denote the value
function by $H(t, x, z, \ell, y)$ where we are implicitly referring only to the value function at
an expanded time index of the form $(t, 2)$.

### 4.4 Solving for the Optimal Strategy

#### 4.4.1 Dynamic Programming Equation

In this section we write the dynamic programming equation corresponding to the agent’s
optimization problem as described above. Due to the drastic difference in the dynamics
depending on whether $y = \Xi$ or $y \neq \Xi$ as well as the restriction $\ell \leq N^L$, we will separate
the writing of this equation into three cases. We first write the equation for $y = \Xi$ and
$\ell < N^L$. Second, we consider $y = \Xi$ and $\ell = N^L$. Finally, we have the case $y \neq \Xi$. 
Recall that we have introduced the notation $\tilde{\lambda}_{c,z,\ell} = \sum_{k=1}^{\ell} \tilde{\lambda}_{c,z,k}$. This notation will be convenient because any cancellation of an order in front of the agent’s order, regardless of its position, has the same effect on the agent’s value function. Similarly for cancellations behind the agent’s order. For the case $y = \Xi$ and $\ell < N^L$ the dynamic programming equation is

$$\max \left\{ \partial_t H + \sum_{\bar{z} \neq z} \lambda_{z,\bar{z},\ell} \left( H(t, x, \bar{z}, \ell, \Xi) - H \right) \right. \left. + \lambda_{m,z} \mathbb{E} \left[ 1_{\epsilon \leq \ell} H(t, x, \epsilon, \Xi) + 1_{\epsilon > \ell} H(t, x, \epsilon, \Xi) - H \right] \right.$$

$$+ \lambda_{c,z,\ell} \left( 1_{\ell > 1} H(t, x, \ell - 1, \Xi) + 1_{\ell = 1} \mathbb{E} \left[ H(t, x, \ell, \Xi) - H \right] \right) \left. \right. \right.$$

$$(4.20)$$

$$+ \lambda_{a,z,\ell} \left( H(t, x, \ell + 1, \Xi) - H \right) ;$$

$$H(t, x - \eta, z, \ell + 1, \ell + 1) = 0 .$$

For the case $y = \Xi$ and $\ell = N^L$, the equation takes the form

$$\partial_t H + \sum_{\bar{z} \neq z} \lambda_{z,\bar{z},N^L} \left[ H(t, x, \bar{z}, N^L, \Xi) - H \right]$$

$$+ \lambda_{m,z} \mathbb{E} \left[ 1_{\epsilon < N^L} H(t, x, \epsilon, N^L) - \epsilon, \Xi) + 1_{\epsilon \geq N^L} H(t, x, \epsilon, \Xi) - H \right] \right. \left. \right. \right. \right.$$ 

$$(4.21)$$

$$+ \lambda_{c,z,N^L} \left( H(t, x, z, N^L - 1, \Xi) - H \right) = 0 .$$
Finally, when $y \neq \Xi$ the equation is
\[
\max \left\{ \partial_t H + \sum_{\bar{z} \neq z} \lambda^{z,\bar{z},\ell} \left( H(t, x, \bar{z}, \ell, y) - H \right) + \lambda^{m,z,\ell} \mathbb{E} \left[ 1_{\epsilon < y} H(t, x, z, \ell - \epsilon, y - \epsilon) + 1_{y \leq \epsilon < \ell} H(t, x + \theta, z, \ell - \epsilon, \Xi) + 1_{\epsilon \geq \ell} H(t, x + \theta, z, \zeta, \Xi) - H \right] + \lambda^{c,z,y-1} \left( H(t, x, z, \ell - 1, y - 1) - H \right) + (\lambda^{c,z,\ell} - \lambda^{c,z,y}) \left( H(t, x, z, \ell - 1, y) - H \right) + \lambda^{a,z,\ell} \left( H(t, x, z, \ell + 1, y) - H \right) ; \right. \\
1_{\ell=1} \mathbb{E} \left[ H(t, x - \eta, z, \zeta, \Xi) \right] + 1_{\ell>1} H(t, x - \eta, z, \ell - 1, \Xi) - H \right\} = 0 .
\]

The terminal condition does not depend on the restrictions on the state variables and is given by
\[
H(T, x, z, \ell, y) = -e^{-\gamma x} .
\]

The expectations in (4.20) to (4.22) are taken with respect to the independent random variables $\epsilon, \theta,$ and $\zeta$ with marginal distributions $\mu^{\epsilon,z}, \mu^{\theta,z},$ and $\mu^{\zeta,z}$ respectively. Equation (4.21) does not take the form of a quasi-variational inequality because the agent is restricted from adding an order to the queue when $\ell = N^L$.

Each term in the dynamic programming equation has a rather straightforward explanation. We will focus on giving this for equation (4.22), the others follow similarly. The max between two terms represents the optimal decision by the agent whether to cancel her order based on the current value of the state variables, or to take no action and let the uncontrolled dynamics of the state variables proceed.

Within the first quantity being compared via the max operator, the summation term
represents the average rate of change of the value function due to regime changes of the TS. The next term represents the expected rate of change due to the arrival of an MO. Note that there are three possible outcomes: the MO does not fill the agent’s order (in which case the order moves forward by the MO size and the queue length decreases), the MO fills the agent’s order (wealth changes by random quantity and the agent’s order status becomes Ξ), or the MO depletes the entire queue (the agent’s order is filled and the queue replenished at the random size ζ).

The next two terms represent the expected rate of change due to exogenous LO cancellations. These may happen either ahead of the agent’s order (with total rate $\lambda_{c,z,y} - 1$) or behind the agent’s order (with total rate $\lambda_{c,z,\ell} - \lambda_{c,z,y}$). The difference between the two changes in value is that in one case the agent’s order moves closer to the front of the queue and in the other case the position stays the same but total queue length still decreases. The final rate term is similar but corresponds to the addition of a single order to the back of the queue.

The second quantity within the max comparison represents the change in the value function given that the agent cancels her order. Note that there are again two cases depending on the state of the queue. If the agent’s order is the only one present ($\ell = 1$) then a cancellation will cause the queue to be replenished at the random length $\zeta$. Otherwise if $\ell > 1$ the queue simply decreases by size 1.

The form of equations (4.20) to (4.22) along with the terminal conditions (4.23) allow us to make the ansatz $H(t, x, z, \ell, y) = -e^{-\gamma(x + h(t, z, \ell, y))}$. Making this substitution in the
The above equations result in the following: for \( y = \Xi \) and \( \ell < N^L \)

\[
\begin{align*}
\min \left\{ & -\gamma \partial_t h + \sum_{\bar{z} \neq z} \lambda_{\bar{z},z,\ell} \left( e^{-\gamma (h(t,\bar{z},\ell,\Xi) - h)} - 1 \right) \\
& + \lambda_{m,z}^\ell \mathbb{E} \left[ 1_{\epsilon < \ell} e^{-\gamma (h(t,z,\ell - \epsilon,\Xi) - h)} + 1_{\epsilon \geq \ell} e^{-\gamma (h(t,z,\Xi - \epsilon) - h)} - 1 \right] \\
& + \lambda_{c,z}^\ell \left( 1_{\ell > 1} e^{-\gamma (h(t,z,\ell - 1,\Xi) - h)} + 1_{\ell = 1} \mathbb{E} \left[ e^{-\gamma (h(t,z,\Xi) - h)} - 1 \right] \right) \\
& + \lambda_{a,z}^\ell \left( e^{-\gamma (h(t,z,\ell + 1,\Xi) - h)} - 1 \right) ; \\
& e^{-\gamma (h(t,z,\ell + 1,\ell + 1) - h - \eta)} - 1 \right\} = 0,
\end{align*}
\]

for \( y = \Xi \) and \( \ell = N^L \) we have

\[
\begin{align*}
-\gamma \partial_t h + \sum_{\bar{z} \neq z} \lambda_{\bar{z},z,N^L} \left( e^{-\gamma (h(t,\bar{z},N^L,\Xi) - h)} - 1 \right) \\
& + \lambda_{m,z}^{N^L} \mathbb{E} \left[ 1_{\epsilon < N^L} e^{-\gamma (h(t,z,N^L - \epsilon,\Xi) - h)} + 1_{\epsilon \geq N^L} e^{-\gamma (h(t,z,\Xi) - h)} - 1 \right] \\
& + \lambda_{c,z}^{N^L} \left( e^{-\gamma (h(t,z,N^L - 1,\Xi) - h)} - 1 \right) ,
\end{align*}
\]
and finally for \( y \neq \Xi \) we have

\[
\min \left\{ -\gamma \partial_t h + \sum_{\bar{z} \neq z} \lambda^{z,\bar{z},\ell} \left( e^{-\gamma(h(t,\bar{z},\ell,y)-h)} - 1 \right) + \lambda^{m,z} \mathbb{E} \left[ \mathbb{1}_{\ell < y} e^{-\gamma(h(t,z,\ell-1,y-\epsilon-h))} + \mathbb{1}_{y \leq \epsilon < \ell} e^{-\gamma(h(t,z,\ell-\epsilon,\Xi)-h)} \right] + \mathbb{1}_{\ell \geq \epsilon} e^{-\gamma(h(t,z,\ell,\Xi)-h)} - 1 \right. \\
+ \left. \lambda^{c,z,\ell} \left( e^{-\gamma(h(t,z,\ell-1,y-1)-h)} - 1 \right) + (\lambda^{c,z,\ell} - \lambda^{c,z,y}) \left( e^{-\gamma(h(t,z,\ell-1,y)-h)} - 1 \right) + \lambda^{a,z,\ell} \left( e^{-\gamma(h(t,z,\ell+1,y)-h)} - 1 \right) \right. \\
\left. + \mathbb{1}_{\ell=1} \mathbb{E} \left[ e^{-\gamma(h(t,z,\ell,\Xi)-\eta)} + 1_{\ell \geq 1} e^{-\gamma(h(t,z,\ell-1,\Xi)-h-\eta)} - 1 \right] \right\} = 0.
\]

These equations are subject to terminal conditions

\[
h(T, z, \ell, y) = 0. \tag{4.27}
\]

The form of equations (4.24) and (4.26) due to the ansatz allow us to conclude that the agent’s decision to place and cancel orders does not depend on how much wealth she has accumulated over time (the equations are independent of \( x \)). The decision is only based on the current state of the queue and market parameters.

### 4.4.2 Optimal Strategy

Construction of the optimal strategy follows along the same lines as in other optimal impulse control problems. We simply need to make some simple modifications due to the form of the indexing set of our stochastic processes. The system of QVI’s (4.24) to
(4.26) can be written in the form

\[ \min \left\{ F_i(t, h(t), \partial_t h_i(t)) ; G_i(t, h(t)) \right\} = 0, \]  

(4.28)

where \( i = (z, \ell, y) \) indexes the system of equations (the exact form of \( F \) and \( G \) will be given later in Section 4.6). We then use the system to define a continuation region and an impulse region:

\[ C = \left\{ (t, z, \ell, y) : t \leq T \text{ and } G_i(t, h(t)) > 0 \right\}, \]  

(4.29)

\[ I = \left\{ (t, z, \ell, y) : t \leq T \text{ and } G_i(t, h(t)) = 0 \right\}. \]  

(4.30)

The optimal strategy consists of a sequence of stopping times \( \{\bar{\tau}_j\}_{j=1}^\infty \) which will be constructed recursively. By convention we set \( \bar{\tau}_0 = (0, 1) \), then given \( \bar{\tau}_{j-1} \) we first define \( \tau_j \) as:

\[ \tau_j = \inf \left\{ t \geq \tau_{j-1} : (t, Z^{u,\bar{\tau}_{j-1},\phi_{j-1}}, L^{u,\bar{\tau}_{j-1},\phi_{j-1}}, Y^{u,\bar{\tau}_{j-1},\phi_{j-1}}) \in I \right\}. \]  

(4.31)

Recall that \( Z^{u,\bar{\tau}_{j-1},\phi_{j-1}}, L^{u,\bar{\tau}_{j-1},\phi_{j-1}}, \) and \( Y^{u,\bar{\tau}_{j-1},\phi_{j-1}} \) are the uncontrolled processes starting from index \( \bar{\tau}_{j-1} \) in state \( \phi_{j-1} \). Once \( \tau_j \) is defined, \( \kappa_j \) is chosen based on the state of the queue at time \( \tau_j \) as

\[ \kappa_j = \begin{cases} 
1 & \text{if } L^{u,\bar{\tau}_{j-1},\phi_{j-1}}(\tau_j) = 1 \text{ and } Y^{u,\bar{\tau}_{j-1},\phi_{j-1}}(\tau_j) = 1, \\
2 & \text{otherwise} 
\end{cases} \]  

(4.32)

This strategy is easily seen to be admissible: \( \{\bar{\tau}_j\}_{j=1}^\infty \) satisfies (4.13) since with probability 1 there are a finite number of jumps of each of the Poisson processes \( N^z, N^c, N^a, M^z \), so \( \tau_j = \infty \) for all sufficiently large \( j \); it satisfies (4.14) because \( (t, z, N^L, \Xi) \notin I \) by the definition \( G_z, N^L, \Xi = \infty \) (see Section 4.6); and it satisfies (4.15) by virtue of (4.32).
In the next section we graphically illustrate the continuation and impulse regions.

4.5 Numerical Examples

In this section, we present a numerical example that demonstrates the agent’s optimal strategy and interpret the queue as being on the sell side of the LOB.

4.5.1 Parameters

We use volume imbalance to construct the trade signal $Z_t$ (see Subsection 4.2.2). Most parameters, including the rate of addition and cancellation ($\lambda^{a,z,\ell}$ and $\lambda^{c,z,\ell}$), the rate of transition between regimes ($\lambda^{z,\hat{z},\ell}$), the rate of MO arrival ($\lambda^{m,z}$), the distribution of MO size ($\mu^{\ell,z}$), the distribution of gain/loss of a filled LO ($\mu^{\theta,z}$), and the distribution of replenished queue length ($\mu^{\zeta,z}$) are described in Section 4.2. Other constants are $\eta = 10^{-4}$, $\gamma = 0.4$, $\Upsilon = 0.5$ and $T = 50$ seconds.

4.5.2 The Optimal Strategy

Figure 4.7 shows the optimal switching regions when the agent has an active LO ($y \neq \Xi$). The left panel shows both the switching region and the continuation region when the TS is in the neutral regime ($Z_t = 2$) and the right panel shows these regions when the TS is in the adverse regime ($Z_t = 3$). The switching region (represented by yellow colouring in both panels) indicates when it is optimal to cancel an existing LO rather than let it remain in the queue. When the TS is in the gainful regime ($Z_t = 1$), the agent will never cancel an existing LO, so the figure is omitted.

There are several features of these figures which are worthy of discussion. First, for both panels, we can see that if we fix any queue position $Y_t$, as the queue length $L_t$ decreases, the agent eventually changes her strategy from keeping the order to cancelling once a threshold for $L_t$ is crossed. There are two intuitive reasons for this. First, as the
length of the queue decreases, the rate of transition to the gainful regime decreases as well, making the choice to remain in the queue more risky (see Figure 4.1). Second, the agent has an incentive to cancel the order to avoid adverse selection when not in the gainful regime, but if she does and the TS becomes favourable shortly after, then she would like to put the order back in the queue and she has needlessly sacrificed her queue position. However, if the total queue length is shorter, than this sacrifice becomes less significant to the point where it is worth giving up the position to avoid adverse selection, but not difficult to replace the order in the case that the TS becomes gainful again after cancelling.

Second, we see that the switching regions in Figure 4.7 are not monotone in the queue position $Y_t$. For example, on the right panel, when the queue length $L_t$ is around 40, the agent will cancel her LO when $Y_t$ is close to 0 and when $Y_t$ is close to $L_t$; in between, she stays in the queue. The reasoning for this is similar to the second line of intuition above. When the agent’s order is at the back of the queue and there is a small level of risk to being filled by an MO, the agent may simply cancel the order since there is essentially no sacrifice to queue position, and replace the order if the TS becomes favourable shortly after. As the order moves slightly closer to the front, the sacrifice of queue position

Figure 4.7: Optimal strategy when $t = 0$ and $Y_t \neq \Xi$. The parameters are $\eta = 10^{-4}$, $\gamma = 0.4$, $T = 0.5$ and $T = 50$ seconds. The yellow region indicates where the agent’s optimal policy is to cancel her order. The blue region indicates where she takes no action. The red curve represents the position in the queue as a function of its total length which maximizes the agent’s value function.
eventually outweighs the risk of adverse selection, so the agent retains her order in the queue. The value of this queue position will increase if the TS changes to a favourable state. With a further move towards the front of the queue, the risk of adverse selection becomes imminent as most MOs are certain to fill her LO.

Although the switching regions in the two panels of Figure 4.7 look similar, the actual frequency of switching can be quite different. This is because the distribution of queue length is different for the different regimes. From Figure 4.8, we can see that when the regime is neutral, the distribution of queue length peaks at around 40; whereas when the regime is adverse, the distribution peaks at somewhere between 10 and 20. The implication of this is that in the adverse regime, the agent almost always cancels an LO and in the neutral regime, there is some proportion of time that the agent stays in the queue.

Lastly, we also plot the position in the queue in terms of its total length which maximizes the agent’s value function. Precisely stated, the red curve corresponds to:

$$y^*(t, Z_t, L_t) = \arg \max_{y \leq L_t} h(t, Z_t, L_t, y).$$

We only plot the curve within the continuation region, since every position within the cancellation region would have the same value to the agent. In the gainful regime, the most valuable position to the agent is the front of the queue, so this is also not displayed. Of particular significance is the fact that there are certain states in which the front of the queue is not considered the most valuable. This is another illustration of the trade-off between the adverse selection suffered by being filled by an MO and the desire to be near the front of the queue in preparation for when the regime becomes gainful. When the queue length becomes sufficiently long the front position is always the most valuable. This stems from the fact that a longer queue indicates a higher rate of regime change into a gainful regime to the point where this is more likely to occur before the next MO
Figure 4.8: Distribution of queue length in different regimes. The solid lines are the simulation results from our queuing model. The dashed lines are the empirical distributions.

Figure 4.9: Optimal strategy when $Y_t = \Xi$ and $t = 0$. The parameters are $\eta = 10^{-4}$, $\gamma = 0.4$, $\Upsilon = 0.5$ and $T = 50$ seconds.

(see Figure 4.1).

Figure 4.9 shows the optimal strategy as a function of $L_t$ when the agent does not have an active LO ($Y_t = \Xi$). When the LOB is in the gainful regime, the agent always places a new LO immediately. When the LOB is in the neutral or adverse regime, the agent will place an order when the length of the queue is above a certain threshold. The agent does not enter the queue when it is short because she wants to avoid the risk of adverse selection, but when the queue is long this risk is reduced and it is worth placing the order to ensure a good position before the TS changes to a favourable regime. As expected, and the threshold for placing the order in the adverse regime is higher than that in the neutral regime.
4.5.3 Increasing Transaction Cost

Figures 4.10 and 4.11 show the optimal strategy when the transaction cost $\eta$ is increased to $10^{-3}$. Figure 4.10 shows the switching (yellow) region when the agent has an active LO. We can see that compared to Figure 4.7, the yellow regions are smaller and the blue regions are larger. This is expected as the agent has less incentive to cancel her existing LO. Similarly, Figure 4.11 shows the agent’s action when she does not have an active LO. When compared to Figure 4.9 in all regimes, particularly the gainful regime, we see that the agent is less likely to place a new LO. Of particular interest is the fact that the agent will not post an order in the gainful regime when the queue is short. This may seem counter-intuitive at first, but it is due to the fact that the rate of transition to the adverse and neutral regimes increase as the queue length decreases. If the agent places an LO and the queue switches to one of the other two regimes, she will likely be in the cancellation region (yellow region in Figure 4.10). When she cancels, she incurs another transaction cost. To avoid this situation, her optimal strategy is not to enter the queue when it is very short in the gainful regime. As the queue becomes longer she becomes willing to place her order because even in a longer queue there is a significant change that an MO fills her order, but less chance that the regime changes before this occurs. In summary, when the transaction cost $\eta$ increases, the agent is less likely to either cancel or place an LO.

4.5.4 When the Agent Is More Risk-averse

Figure 4.12 and Figure 4.13 demonstrate the optimal strategy when the agent is more risk averse ($\gamma = 1.2$ increased from $\gamma = 0.4$ in the previous examples). Comparing Figure 4.12 to Figure 4.7, the yellow region is much larger because the agent is more inclined to cancel her LO to avoid the risk associated with adverse selection. Similarly, comparing Figure 4.13 to Figure 4.9, we see that the thresholds to place a new LO have increased to avoid the risk of having LOs active during the presence of an undesirable TS.
Chapter 4. Optimal Decisions in a Time Priority Queue

4.5.5 When the Agent Is Less Risk-averse

Figure 4.10: Optimal strategy when $t = 0$ and $Y_t \neq \Xi$. The parameters are $\eta = 10^{-3}$, $\gamma = 0.4$, $\Upsilon = 0.5$ and $T = 50 \text{ seconds}$. The difference from Figure 4.7 is that $\eta$ is 10 times larger.

Figure 4.11: Optimal strategy when $t = 0$ and $Y_t = \Xi$. The parameters are $\eta = 10^{-3}$, $\gamma = 0.4$, $\Upsilon = 0.5$ and $T = 50 \text{ seconds}$. The difference from Figure 4.9 is that $\eta$ is 10 times larger.

Figure 4.12: Optimal strategy and most valuable queue position when $t = 0$ and $Y_t \neq \Xi$. The parameters are $\eta = 10^{-4}$, $\gamma = 1.2$, $\Upsilon = 0.5$ and $T = 50 \text{ seconds}$. The difference from Figure 4.7 is that $\gamma$ is larger.

4.5.5 When the Agent Is Less Risk-averse

Figure 4.14 and Figure 4.15 show the optimal strategy for the agent when she is less risk-averse ($\gamma = 0.1$). Comparing Figures 4.14 and 4.15 to Figures 4.7 and 4.9 shows
Figure 4.13: Optimal strategy when $t = 0$ and $Y_t = \Xi$. The parameters are $\eta = 10^{-4}$, $\gamma = 1.2$, $\Upsilon = 0.5$ and $T = 50$ seconds. The difference from Figure 4.9 is that $\gamma$ is larger.

Figure 4.14: Optimal strategy and most valuable queue position when $t = 0$ and $Y_t \neq \Xi$. The parameters are $\eta = 10^{-4}$, $\gamma = 0.1$, $\Upsilon = 0.5$ and $T = 50$ seconds. The difference from Figure 4.7 is that $\gamma$ is smaller.

Figure 4.15: Optimal strategy when $t = 0$ and $Y_t = \Xi$. The parameters are $\eta = 10^{-4}$, $\gamma = 0.1$, $\Upsilon = 0.5$ and $T = 50$ seconds. The difference from Figure 4.9 is that $\gamma$ is smaller.

typical expected behaviour through the fact that the agent is in all cases more willing to have active LOs, hence the strategy is more aggressive. Note again the presence of non-monotonicity of the strategy with respect to queue position $Y_t$ when the TS is adverse.
4.5.6 Simulation

In this section we attempt to quantify how the incorporation of queue length and order position into the optimal decision policy benefits the agent when compared to the performance of various strategies which only consider the value of the TS. We do this by simulating the dynamics of the queue according to the model outlined in Section 4.3. We simulate $10^6$ sample paths using the parameters $\eta = 10^{-4}$, $\Upsilon = 0.5$ and $T = 50$ seconds. The initial state of the regime $Z_0$ is randomly drawn from the empirical distribution of $Z_t$, and the initial queue length $L_0$ is drawn from the empirical distribution $\mu_{c,z}$, conditioning on $z = Z_0$. The agent starts with zero wealth and no active LO, i.e., $X_0 = 0$ and $Y_0 = \Xi$. The four different order placement strategies which we test in these simulations are outlined below.

- **Optimal**: the optimal strategy (as defined by cancel regions, continue regions, and new order thresholds) with risk aversion parameter $\gamma$ varying from 0.1 to 1.27.

- **Passive Benchmark**: the agent places a new order whenever the TS becomes gainful, and cancels the order whenever the TS becomes neutral or adverse.

- **Aggressive Benchmark**: the agent places a new order whenever the TS becomes gainful or neutral, and cancels the order whenever the TS becomes adverse.

- **Always-post**: the agent never cancels her order and always replaces it when it is filled by an MO.

Figure 4.16 shows the risk-reward plot for all the strategies. The horizontal axis is the standard deviation of the agent’s terminal wealth and the vertical axis is her mean terminal wealth. The blue curve represents the performance of the optimal strategy for various values of the risk-aversion parameter. From the figure, we see the always-post strategy clearly shows poor performance compared to the other strategies. Even though
Figure 4.16: Risk-reward plot. Along the curve indicating performance of the optimal strategy, the right-most point represents $\gamma = 0.1$. Increasing $\gamma$ lowers both the mean and standard deviation of terminal wealth.

The always-post action is likely to occupy order positions which are at the front of the queue for a greater proportion of time than the other strategies, previous analysis and reasoning indicates that such a position can have negative value to the agent if there is a high chance of being filled by an MO and suffering adverse selection.

Table 4.3 compares some of the optimal strategies for different risk aversion parameters with aggressive benchmark. The optimal strategy when $\gamma = 0.1$ and aggressive benchmark have very similar standard deviations for the terminal wealth; but the optimal strategy has a mean terminal wealth of 2.23, which is 2.5% higher than aggressive benchmark. When $\gamma = 0.4$, the optimal strategy and aggressive benchmark have very similar means, but the standard deviation of the optimal strategy is 1.825, which is 8.8% lower.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 0.1$</td>
<td>2.230</td>
<td>1.976</td>
</tr>
<tr>
<td>$\gamma = 0.4$</td>
<td>2.182</td>
<td>1.825</td>
</tr>
<tr>
<td>Aggressive Benchmark</td>
<td>2.175</td>
<td>1.979</td>
</tr>
</tbody>
</table>

Table 4.3: Mean and standard deviation of terminal wealth.

To better understand the differences between these strategies, let us have a look at Figure 4.17, which shows the histograms of terminal wealth for all strategies. Panel (d) shows the case for the naive always-post strategy. Its terminal wealth has a lot of variation, with a large number of samples being negative. Panel (c) shows the case for
aggressive benchmark. Compared to the always-post strategy, aggressive benchmark is more conservative: the variance is smaller and there are fewer samples ending up with negative wealth. Panel (b) shows the case for passive benchmark, which seems to be much more conservative than the rest. Panel (a) shows the case for the optimal strategy ($\gamma = 0.4$). We can see that its variance lies in between passive benchmark and aggressive benchmark. Table 4.4 shows the proportions of LOs executed as three regimes for the four strategies. The always-post strategy has a large fraction of LOs (21.1%) executed at the adverse regime. Passive benchmark goes the other extreme, with all LOs being executed at the gainful regime. For the optimal strategy, we see that there is only a tiny fraction of LOs (0.2%) executed at the adverse regime. Compared to aggressive benchmark, there are relatively more LOs executed at the gainful regime.


<table>
<thead>
<tr>
<th>Strategy</th>
<th>$z = 1$</th>
<th>$z = 2$</th>
<th>$z = 3$</th>
<th>number of filled orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>optimal ($\gamma = 0.4$)</td>
<td>76.5</td>
<td>23.3</td>
<td>0.2</td>
<td>9.85</td>
</tr>
<tr>
<td>passive benchmark</td>
<td>100.0</td>
<td>0.0</td>
<td>0.0</td>
<td>2.43</td>
</tr>
<tr>
<td>aggressive benchmark</td>
<td>64.0</td>
<td>36.0</td>
<td>0.0</td>
<td>11.91</td>
</tr>
<tr>
<td>always post</td>
<td>45.1</td>
<td>33.7</td>
<td>21.1</td>
<td>17.64</td>
</tr>
</tbody>
</table>

Table 4.4: Percentages of orders executed in each regime.

### 4.6 Convergence Results

In this section, we show that the system of QVIs (4.24) to (4.26) admits a unique solution and provide a numerical scheme that converges to this solution.

We denote by $\hat{S}$ the reduced state space, equivalent to that of (4.5) without the $X$ component:

$$\hat{S} = \left\{ \bar{i} = (Z, L, Y) : Z \in \{1, \ldots, N^Z\}, L \in \{0, 1, \ldots, N^L\}, Y \in \{1, \ldots, N^L, \Xi\}, Y = \Xi \text{ or } Y \leq L \right\}.$$  \hspace{1cm} (4.33)

This is the set of state variables of concern with respect to equations (4.24) to (4.26). We also write $\mathbb{R}^{\hat{S}} = \{ r_i : r_i \in \mathbb{R}, i \in \hat{S} \}$. With the above notations, the original system of QVIs can be restated as

$$J_i(t, u, \partial_t u_i) = \min \{ F_i(t, u, \partial_t u_i); G_i(t, u) \} = 0, \hspace{1cm} (4.34)$$

$$u_i(T) = 0,$$

for $i \in \hat{S}$ and $u : [0, T] \to \mathbb{R}^{\hat{S}}$. In other words, we are viewing the original system of QVIs
as a system indexed by $i \in \bar{S}$. The mappings $F_i$ and $G_i$ in (4.34) are defined as

$$F_{z,\ell,\Xi}(t, r, p) = -\gamma p + \sum_{\bar{z} \neq z} \lambda_{z,\bar{z},\ell} \left( e^{-\gamma(r_{z,\ell,\Xi} - r_{z,\ell,\Xi})} - 1 \right)$$

$$+ \lambda_{m,z,\ell} \left( E\left[ 1_{\epsilon < \ell} e^{-\gamma(r_{z,\ell-1,\Xi} - r_{z,\ell-1,\Xi})} + 1_{\epsilon \geq \ell} e^{-\gamma(r_{z,\ell,\Xi} - r_{z,\ell,\Xi})} \right] - 1 \right)$$

$$+ \lambda_{c,z,\ell} \left( 1_{\ell > 1} e^{-\gamma(r_{z,\ell-1,\Xi} - r_{z,\ell-1,\Xi})} + 1_{\ell = 1} E\left[ e^{-\gamma(r_{z,\ell,\Xi} - r_{z,\ell,\Xi})} \right] - 1 \right)$$

$$+ \lambda_{a,z,\ell} \left( e^{-\gamma(r_{z,\ell+1,\Xi} - r_{z,\ell+1,\Xi})} - 1 \right),$$

$$G_{z,\ell,\Xi}(t, r) = e^{-\gamma(r_{z,\ell+1,\Xi} - r_{z,\ell+1,\Xi})} - 1,$$

for $i = (z, \ell, \Xi)$ with $\ell < N^L$, as

$$F_{z,N^L,\Xi}(t, r, p) = -\gamma p + \sum_{\bar{z} \neq z} \lambda_{z,\bar{z},N^L} \left( e^{-\gamma(r_{z,N^L,\Xi} - r_{z,N^L,\Xi})} - 1 \right)$$

$$+ \lambda_{m,z,N^L} \left( E\left[ 1_{\epsilon < N^L} e^{-\gamma(r_{z,N^L-1,\Xi} - r_{z,N^L-1,\Xi})} + 1_{\epsilon \geq N^L} e^{-\gamma(r_{z,\Xi} - r_{z,N^L,\Xi})} \right] \right)$$

$$- 1$$

$$+ \lambda_{c,z,N^L} \left( e^{-\gamma(r_{z,N^L-1,\Xi} - r_{z,N^L-1,\Xi})} - 1 \right),$$

$$G_{z,N^L,\Xi}(t, r) = \infty.$$
for $i = (z, N^L, \Xi)$, and as

$$F_{z, \ell, y}(t, r, p) = -\gamma p + \sum_{\bar{z} \neq z} \lambda^{z, \bar{z}, \ell} \left( e^{-\gamma(r_{z, \ell, y} - r_{z, \ell, y})} - 1 \right) + \lambda^{m, z, \ell} \left( e^{-\gamma(r_{z, \ell-1, y} - r_{z, \ell, y})} + e^{-\gamma(\theta + r_{z, \ell-1, y} - y)} ight) - 1 \right) + \lambda^{c, z, y-1} \left( e^{-\gamma(r_{z, \ell-1, y} - r_{z, \ell, y})} - 1 \right) + \lambda^{a, z, \ell} \left( e^{-\gamma(r_{z, \ell+1, y} - r_{z, \ell, y})} - 1 \right),$$

$$G_{z, \ell, y}(t, r) = 1_{\ell = 1} e^{-\gamma(r_{z, \ell, y} - y)} + 1_{\ell > 1} e^{-\gamma(r_{z, \ell-1, y} - r_{z, \ell, y} - \eta)} - 1.$$

for $i = (z, \ell, y)$ with $y \neq \Xi$. We define $G_{z, N^L, \Xi}(t, r) = \infty$ so that the QVI (4.34) reduces to $F_{z, N^L, \Xi}(t, u, \partial_t u_i) = 0$.

Our first result is the following comparison principle for (4.34), which ensures the uniqueness of the viscosity solution.

**Theorem 4.1.** Let $u$ be a bounded upper-semi-continuous viscosity subsolution and $v$ a bounded lower-semi-continuous viscosity supersolution of (4.34), then $u \leq v$.

**Proof.** See 4.A.1.

For $(z, \ell, \Xi) \in \bar{S}$ with $\ell < N^L$ and $\phi \in C^1([0, T])$ we provide the following finite
difference scheme:

\[
F_{z,\ell,\Xi}^\Delta(t, r, \phi) = \frac{\phi(t) - r_{z,\ell,\Xi}}{\Delta} + \sum_{\bar{z} \neq z} \lambda_{z,\bar{z},\ell} \left( e^{-\gamma(r_{z,\ell,\Xi} - r_{z,\ell,\Xi})} - 1 \right) \\
+ \lambda_{m,z,\ell} \left( \mathbb{E} \left[ \mathbb{1}_{\ell < t} e^{-\gamma(r_{z,\ell-1,\Xi} - r_{z,\ell,\Xi})} + \mathbb{1}_{\ell \geq t} e^{-\gamma(r_{z,\ell,\Xi} - r_{z,\ell,\Xi})} \right] - 1 \right) \\
+ \lambda_{c,z,\ell} \left( e^{-\gamma(r_{z,\ell+1,\Xi} - r_{z,\ell,\Xi})} - 1 \right),
\]

\[
G_{z,\ell,\Xi}(t, r, \phi) = e^{-\gamma(r_{z,\ell+1,\Xi} - r_{z,\ell,\Xi} - \phi(t) - \eta)} - 1.
\]

The obvious analogous expressions are used for all cases in \( \mathfrak{S} \) that remain. Here \( \Delta = T/N \) is the size of time steps where \( N \) is a positive integer. We denote by \( T^\Delta = \{ t_j := \Delta j : 0 \leq j \leq N \} \) the set of grid points for the finite difference scheme. Let us define

\[
J_i^\Delta(t_j, u^\Delta(t_{j+1}), u_i^\Delta) = \min \left\{ F_i^\Delta(t_j, u^\Delta(t_{j+1}), u_i^\Delta) ; G_i(t_j, u^\Delta(t_{j+1}), u_i^\Delta) \right\},
\]

and consider the following discrete problem:

\[
J_i^\Delta(t_j, u^\Delta(t_{j+1}), u_i^\Delta) = 0, \quad \text{(4.35a)} \\
u_i^\Delta(T) = 0, \quad \text{(4.35b)}
\]

for \( i \in \mathfrak{S}, 1 \leq j \leq N - 1 \) and \( u^\Delta : [0, T] \to \mathbb{R}^\mathfrak{S} \). We will show that the solution to (4.35) can be used to approximate the viscosity solution to (4.34). To proceed we first need the following results.

**Proposition 4.2. (Stability)**

Let \( \Delta \) be sufficiently small and \( u^\Delta \) be the solution to (4.35). For any \( i = (z, \ell, y) \in \mathfrak{S}, \)
we have

\[ u_i(t_j) \leq A(T - t_j), \quad (4.36) \]
\[ u_i(t_j) \geq 0, \quad y = \Xi, \quad (4.37) \]
\[ u_i(t_j) \geq -\eta, \quad y \neq \Xi, \quad (4.38) \]

where \( A > 0 \) is a constant.

Proof. See 4.A.2. \( \square \)

**Proposition 4.3. (Monotonicity)**

Consider functions \( u : [0, T] \rightarrow \mathbb{R}^S \) and \( v : [0, T] \rightarrow \mathbb{R}^S \), satisfying \( C_{\min} \leq u \leq v \leq C_{\max} \), for some constants \( C_{\min} \) and \( C_{\max} \). Suppose for some \( i \in \bar{S} \), we have \( u_i \leq v_i \) and \( u_j = v_j \) for \( j \neq i \); and for some \( t \in [0, T] \), we have \( u_i(t) = v_i(t) \). Then

\[ J_i^\Delta(t, u(t'), u_i) - J_i^\Delta(t, v(t'), v_i) \geq 0 \quad (4.39) \]

for all \( t' \in [0, T] \) and \( \Delta \) sufficiently small.

Proof. See 4.A.3 \( \square \)

**Proposition 4.4. (Consistency)**

Fix \( i \in \bar{S} \). For \( \delta > 0 \), \( t \in [0, T] \) and any family of continuous functions \( \Phi \in C^0([0, T], \mathbb{R}) \) with continuously differentiable \( i \)-th component \( \Phi_i \), we have

\[ |J_i^\Delta(t, \Phi(t + \Delta), \Phi_i) - J_i(t, \Phi(t), \Phi_i(t))| \leq o(\Delta), \quad (4.40) \]

where \( o(\Delta) \rightarrow 0 \) as \( \Delta \rightarrow 0 \).

Proof. Immediately follows from the fact that \( \Phi_i \) is continuous and

\[ \frac{\Phi_i(t) - \Phi(t + \Delta)}{\Delta} \rightarrow \partial_t \Phi_i(t), \]
as $\Delta \to 0$. 

With the above results, we are now ready to prove the convergence of our numerical scheme.

**Proposition 4.5.** Let $u^\Delta$ be the solution to (4.35) and let $u$ be the viscosity solution to (4.34). Then $u^\Delta \to u$ as $\Delta \to 0$.

**Proof.** We define the following limits.

$$
\underline{u}(t) = \liminf_{\Delta \to 0, t' \to t} u^\Delta(t'), \quad \bar{u}(t) = \limsup_{\Delta \to 0, t' \to t} u^\Delta(t').
$$

Proposition 4.2 ensures the above limits are well-defined. Following Barles and Souganidis (1991) and Briani et al. (2012), it suffices to show that $\underline{u}$ is a viscosity supersolution and $\bar{u}$ is a viscosity subsolution to (4.34). Then by comparison principle (Theorem 4.1), $\bar{u} \leq \underline{u}$. The opposite inequality is obviously true and we have $\bar{u} = \underline{u} = \lim_{\Delta \to 0} u^\Delta$.

We will prove that $\bar{u}$ is a viscosity subsolution (the proof for $\underline{u}$ being a viscosity supersolution is similar). Let $\phi$ be a smooth function and $i \in \bar{S}$ such that $\bar{u}_i - \phi$ has a strict global maximum at $t_0$ and $\bar{u}_i(t_0) = \phi(t_0)$. Standard arguments imply that there exists sequences $\Delta_n \to 0$ and $t_n \to 0$ such that $u^\Delta_i - \phi$ has a local maximum at $t_n$ and $u^\Delta_n(t_n) \to \bar{u}_i(t_0)$ as $n \to \infty$.

Set $\delta_n := (u^\Delta_i - \phi)(t_n)$ and define the function $\Phi^n(t) = \{\Phi^n_j(t)\}_{j \in \bar{S}}$

$$
\Phi^n_j(t) = \begin{cases} 
\phi(t) + \delta_n & j = i, \\
u^\Delta_j(t) & j \neq i.
\end{cases}
$$
We have

\[
0 = J_i^\Delta_n(t_n, u^\Delta_n(t_n + \Delta_n), u^\Delta_n) \\
\geq J_i^\Delta_n(t_n, \Phi^\Delta_n(t_n + \Delta_n), \Phi^\Delta_n) \\
= J_i(t_n, \phi(t_n + \Delta_n) + \delta_n, \partial_t \phi(t_n)) + o(\Delta_n),
\]

where the second line follows from Proposition 4.3 and the third line follows from Proposition 4.4. Sending \( n \to \infty \), we have the required result.

\[\Box\]

### 4.7 Conclusion

Using tick level data from the Nasdaq exchange, we conduct an empirical study on various limit order book events, including dynamics of volume imbalance, gain/loss of a filled limit order, rate of addition and cancellation, rate of market order arrival, distribution of market order size and distribution of replenished queue length. We propose novel ways of modeling cancellation position of limit orders and distribution of market order size. Based on our empirical findings, we develop a queuing model that captures stylized facts on limit order book data. One important feature of our model is that the dynamics of the limit order book depends on the regime, which is a function of volume imbalance. Moreover, an executed limit order is more likely to be considered a gain (loss) by the agent when the limit order book is at a gainful (adverse) regime.

We show how to apply our proposed queuing model in the context of algorithmic trading. We consider the problem of an agent maximizes her expected utility by placing and canceling limit orders in such queuing model. In our set up, the agent tries to let more of her limit orders be executed at a gainful regime and less of her limit orders be executed at an adverse regime. We demonstrate our result using a numerical example, in which parameters are calibrated from real data. Our result shows that even at an
adverse regime, the agent might still be willing to stay in the queue if she already has an active limit order, or place a new limit order if she does not have one. This is because the agent tries to obtain a good queue position when the regime switches to a gainful one. It also implies that strategies that only look at regimes are sub-optimal. Simulation study shows that the optimal strategy achieves a 2.4% higher mean terminal wealth than the benchmark strategy that only looks at regimes, given the same level of standard deviation of terminal wealth; or 11.4% lower standard deviation given the same level of mean terminal wealth.

4.A Proofs

4.A.1 Proof of Theorem 4.1

Proof. Consider \( \tilde{u} = (1 + \delta)u - \delta A(T - t) \), where \( 0 < \delta < 1 \) and \( A > 0 \) is a constant to be fixed later. We first show that \( \tilde{u} \) is a viscosity subsolution to

\[
J_i(t, \tilde{u}, \partial_t \tilde{u}) \leq -\delta C \tag{4.41}
\]

for \( i \in \mathcal{S} \) and some constant \( C > 0 \), independent of the choice of \( \delta \).

We will show (4.41) for the case \( i = (z, \ell, \Xi) \). Showing the case \( i = (z, \ell, y) \) with \( y \neq \Xi \) is similar. Let \( \phi(t) \) be a smooth test function and \( t_0 < T \) be such that \( \tilde{u}_i(t_0) - \phi(t) \) attains maximum at \( t_0 \). The subsolution property of \( u \) implies either of the following

\[
F_i(t_0, u(t_0), \frac{\partial_t \phi(t_0) - A\delta}{1 + \delta}) \leq 0, \tag{4.42}
\]

\[
G_i(t_0, u(t_0)) \leq 0. \tag{4.43}
\]

Suppose (4.42) holds. Without loss of generality, we assume that \( \lambda^{z,\xi,\ell} = \lambda^{m,z,\ell} = \lambda^{c,z,\ell} = 0 \). The case where these are strictly positive can be treated similarly. Multiplying
(4.42) by $1 + \delta$ and re-arranging the terms, we have

$$
-\gamma \partial_t \phi(t_0) + \lambda^{a,z,\ell} \left( e^{-\gamma(u_{z,\ell+1,\Xi}(t_0)-u_{z,\ell,\Xi}(t_0) - 1) - 1} \right) \leq -\lambda^{a,z,\ell} \left( e^{-\gamma(u_{z,\ell+1,\Xi}(t_0)-u_{z,\ell,\Xi}(t_0)) - 1} \right) \delta - \gamma A \delta.
$$

Using the fact that $u$ is bounded, we can conclude that there exists a constant $C_1$ such that

$$
-\gamma \partial_t \phi(t_0) + \lambda^{a,z,\ell} \left( e^{-\gamma(u_{z,\ell+1,\Xi}(t_0)-u_{z,\ell,\Xi}(t_0)) - 1} \right) \leq (C_1 - \gamma A) \delta.
$$

From (4.44) we have

$$
F_{z,\ell,\Xi}(t_0, \tilde{u}(t_0), \partial_t \phi(t_0))
\leq (C_1 - \gamma A) \delta + \lambda^{a,z,\ell} e^{-\gamma(u_{z,\ell+1,\Xi}(t_0)-u_{z,\ell,\Xi}(t_0))} \left( 1 - e^{\gamma \delta(u_{z,\ell+1,\Xi}(t_0)-u_{z,\ell,\Xi}(t_0))} \right)
\leq (C_1 - \gamma A) \delta - \lambda^{a,z,\ell} e^{-\gamma(1+\delta)(u_{z,\ell+1,\Xi}(t_0)-u_{z,\ell,\Xi}(t_0))} (u_{z,\ell+1,\Xi}(t_0) - u_{z,\ell,\Xi}(t_0)) \gamma \delta,
\leq (C_2 - \gamma A) \delta,
$$

for some constant $C_2$. In the second inequality we use the fact $e^x \geq x + 1$ for all $x$. In the third inequality we use the fact that $u$ and $\delta$ are bounded. Choosing $A > \frac{C_2}{\gamma}$, we have

$$
F_{z,\ell,\Xi}(t_0, \tilde{u}(t_0), \partial_t \phi(t_0)) \leq -C_3 \delta,
$$

for some constant $C_3 > 0$.

For the case where $\lambda^{z,\ell,\ell}, \lambda^{m,z,\ell}$ and $\lambda^{c,z,\ell}$ are nonzero, the same argument applies: on the right-hand-side of inequalities (4.44) and (4.45), we would have different constants $C_1$ and $C_2$. The rest of the proof is the same.
If (4.43) holds we have \( u_{z,\ell+1,\ell+1}(t_0) - u_{z,\ell,\Xi}(t_0) \geq \eta \) and

\[
G_{z,\ell,\Xi}(t_0, \bar{u}(t_0)) = e^{-\gamma(u_{z,\ell+1,\ell+1}(t_0) - u_{z,\ell,\Xi}(t_0) - \eta)}e^{-\gamma \delta(u_{z,\ell+1,\ell+1}(t_0) - u_{z,\ell,\Xi}(t_0))} - 1
\]

\[
\leq e^{-\gamma \delta(u_{z,\ell+1,\ell+1}(t_0) - u_{z,\ell,\Xi}(t_0))} - 1.
\]

Since either (4.46) or (4.47) holds, by choosing \( C = \min\{C_3, \gamma \eta e^{-\gamma \eta}\} \), we have shown that \( \bar{u} \) is a viscosity subsolution to (4.41).

In order to prove \( u \leq v \), it suffices to show

\[
\max_{i \in \mathcal{S}} \sup \{ \bar{u}_i - v_i \} \leq 0,
\]

(4.48)

for all \( \delta > 0 \). The required result can be obtained by sending \( \delta \) to 0.

Let us show (4.48) by contradiction. Suppose there exists some \( \delta > 0 \) and \( i^* \in \mathcal{S} \) such that

\[
\mathcal{M} := \max_{i \in \mathcal{S}} \sup_t (\bar{u}_i - v^0_i) = \sup_t (\bar{u}_{i^*}(t) - v_{i^*}(t)) > 0.
\]

Clearly, \( \bar{u}_{i^*} - v_{i^*} \) attains its maximum \( \mathcal{M} \) at some \( \bar{t} \in [0, T) \), and we can assume without loss of generality that \( \bar{t} > 0 \). Consider the following class of test functions

\[
\Phi_\alpha(t, s) = \bar{u}_{i^*}(t) - v_{i^*}(s) - \alpha |t - s|^2,
\]

with \( \alpha > 0 \). By standard argument, we have

\[
t_{\alpha}, s_{\alpha} \rightarrow \bar{t},
\]

\[
\mathcal{M}_\alpha = \sup \Phi_\alpha = \Phi_\alpha(t_{\alpha}, s_{\alpha}) \rightarrow \mathcal{M},
\]
as $\alpha \to \infty$, where $(t_\alpha, s_\alpha)$ is the maximizer of $\Phi_\alpha$.

Without loss of generality, we assume that $i^* = (z^*, \ell^*, \Xi)$. The case for $i^* = (z^*, \ell^*, y^*)$ with $y^* \neq \Xi$ can be shown using a similar argument. By the viscosity solution property of $\bar{u}$ and $v$, we have

$$\min \{ F_{z^*, \ell^*, \Xi}(t_\alpha, \bar{u}(t_\alpha)), -2\alpha(t_\alpha - s_\alpha) ; G_{z^*, \ell^*, \Xi}(t_\alpha, \bar{u}(t_\alpha)) \} \leq -C\delta,$$

$$\min \{ F_{z^*, \ell^*, \Xi}(s_\alpha, v(s_\alpha)), -2\alpha(t_\alpha - s_\alpha) ; G_{z^*, \ell^*, \Xi}(s_\alpha, v(s_\alpha)) \} \geq 0.$$

The above two inequalities imply either of the following

$$F_{z^*, \ell^*, \Xi}(t_\alpha, \bar{u}(t_\alpha), -2\alpha(t_\alpha - s_\alpha)) - F_{z^*, \ell^*, \Xi}(s_\alpha, v(s_\alpha), -2\alpha(t_\alpha - s_\alpha)) \leq -\delta C, \quad (4.49)$$

$$G_{z^*, \ell^*, \Xi}(t_\alpha, \bar{u}(t_\alpha)) - G_{z^*, \ell^*, \Xi}(s_\alpha, v(s_\alpha)) \leq -\delta C. \quad (4.50)$$

Note that for any $(z, \ell, \Xi)$, we have

$$\liminf_{\alpha \to \infty} e^{-\gamma(\bar{u}_{z, \ell, \Xi}(t_\alpha) - \bar{u}_{z^*, \ell^*, \Xi}(t_\alpha))} - e^{-\gamma(v_{z, \ell, \Xi}(s_\alpha) - v_{z^*, \ell^*, \Xi}(s_\alpha))}$$

$$\geq \liminf_{\alpha \to \infty} \left( e^{-\gamma(\bar{u}_{z, \ell, \Xi}(t_\alpha) - v_{z, \ell, \Xi}(s_\alpha))} - e^{-\gamma(\bar{u}_{z^*, \ell^*, \Xi}(t_\alpha) - v_{z^*, \ell^*, \Xi}(s_\alpha))} \right) C_4$$

$$\geq \left( e^{-\gamma(\bar{u}_{z, \ell, \Xi}(t) - v_{z, \ell, \Xi}(t))} - e^{-\gamma(\bar{u}_{z^*, \ell^*, \Xi}(t) - v_{z^*, \ell^*, \Xi}(t))} \right) C_4$$

$$\geq 0,$$

where $C_4 > 0$ is a constant. (A similar result hold for $(z, \ell, y)$ with $y \neq \Xi$.) Using the above inequality, we can conclude that as $\alpha \to \infty$, the $\liminf$ of the l.h.s of (4.49) and the $\liminf$ of the l.h.s. of (4.50) are both nonnegative, a contradiction. \qed
4. A. 2 Proof of Proposition 4.2

Proof. We prove by induction backwards in $t_j$. Without loss of generality, we assume that $\lambda^{z,\bar{z},\ell} = \lambda^{m,\bar{z},\ell} = \lambda^{c,\bar{z},\ell} = 0$. Equation (4.35a) can be re-written as

$$u_i(t_j) = \max \left\{ u_i(t_{j+1}) - \frac{\Delta \lambda^a}{\gamma} \left( e^{-\gamma(u_k(t_{j+1}) - u_i(t_{j+1}))} - 1 \right) ; \quad u_m(t_{j+1}) - \eta \right\}$$

for $i, k, m \in \bar{S}$ and $\lambda^a = \lambda^{a,\bar{z},\ell}$ for some $z$ and $\ell$. Clearly (4.36), (4.37) and (4.38) holds when $t_j = T$. Suppose they also hold for $t_{j+1}$ where $j \leq N - 1$. Using the fact that $e^y > 0$ for all $y$ and $u(t_{j+1}) \leq A(T - t_{j+1})$, we have

$$u_i(t_j) \leq A(T - t_j - \Delta) + \frac{\Delta \lambda^a}{\gamma} \leq A(T - t_j),$$

for $A > \lambda^a/\gamma$ and we have shown (4.36).

On the other hand, for $i = (z, \ell, \Xi)$, (4.51) implies

$$u_i(t_j) \geq u_i(t_{j+1}) - \frac{\Delta \lambda^a}{\gamma} \left( e^{-\gamma(u_k(t_{j+1}) - u_i(t_{j+1}))} - 1 \right)$$

$$= u_i(t_{j+1}) + \frac{\Delta \lambda^a}{\gamma} e^{-\gamma(u_k(t_{j+1}) - u_i(t_{j+1}))} \left( e^{\gamma(u_k(t_{j+1}) - u_i(t_{j+1}))} - 1 \right)$$

$$\geq u_i(t_{j+1}) + \Delta \lambda^a e^{-\gamma(u_k(t_{j+1}) - u_i(t_{j+1}))} (u_k(t_{j+1}) - u_i(t_{j+1}))$$

$$\geq (1 - \Delta \lambda^a e^{-\gamma AT}) u_i(t_{j+1})$$

$$\geq 0,$$

for $\Delta < e^{-\gamma AT}/\lambda^a$. In the forth line (third inequality) we use the fact that $0 \leq u_i(t_{j+1}) \leq A(T - t_{j+1}) \leq AT$ and $u_k(t_{j+1}) \geq 0$. For $i = (z, \ell, y)$ with $y \neq \Xi$, (4.51) implies

$$u_{z,\ell,y}(t_j) \geq u_{z,\ell-1,\Xi}(t_{j+1}) - \eta$$

$$\geq -\eta.$$
4.A.3 Proof of Proposition 4.3

Proof. It suffices to show
\[
F^\Delta_i(t, u(t'), u_i) - F^\Delta_i(t, v(t'), v_i) \geq 0, \quad (4.52)
\]
\[
G^\Delta_i(t, u(t'), u_i) - G^\Delta_i(t, v(t'), v_i) \geq 0. \quad (4.53)
\]

The inequality in (4.53) immediately follows from \( u_i \leq v_i \). To show (4.52), we assume without loss of generality that \( i = (z, \ell, \Xi) \) and \( \lambda^z,\bar{z},\ell = \lambda^{m,z,\ell} = \lambda^{c,z,\ell} = 0 \). Let \( w = u - v \) and \( i^* = (z, \ell + 1, \Xi) \). Note that \( w_i(t) = 0, w_i(t') \leq 0 \) and \( u_i^* = v_i^* \). The left-hand-side of (4.52) is
\[
\gamma \frac{w_i(t)}{\Delta} - \gamma \frac{w_i(t')}{\Delta} + \lambda^{a,z,\ell} e^{-\gamma(u_i^*(t') - v_i^*(t'))} \left( e^{\gamma w_i(t')} - 1 \right)
\geq - \gamma \frac{w_i(t')}{\Delta} + \lambda^{a,z,\ell} e^{-\gamma(u_i^*(t') - v_i^*(t'))} \gamma w_i(t')
\geq \left( -\frac{1}{\Delta} + \lambda^{a,z,\ell} e^{\gamma(C_{\min} - C_{\max})} \right) \gamma w_i(t')
\geq 0,
\]
for \( \Delta \) sufficiently small. \( \square \)

4.B Parameter Estimates

<table>
<thead>
<tr>
<th>( \bar{z} = 1 )</th>
<th>( \bar{z} = 2 )</th>
<th>( \bar{z} = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = 1 )</td>
<td>7.769 (0.145)</td>
<td>17.962 (0.891)</td>
</tr>
<tr>
<td>( z = 2 )</td>
<td>-14.79 (0.193)</td>
<td>7.939 (0.14)</td>
</tr>
<tr>
<td>( z = 3 )</td>
<td>-19.614 (1.095)</td>
<td>-8.6 (0.15)</td>
</tr>
</tbody>
</table>

Table 4.5: Estimated \( \beta^Z_{0,z,\bar{z}} \) in (4.2). Numbers in brackets represent standard errors.
### Table 4.6: Estimated $\beta^Z_{1,z,\bar{z}}$ in (4.2).

<table>
<thead>
<tr>
<th>$\bar{z}$ = 1</th>
<th>$\bar{z}$ = 2</th>
<th>$\bar{z}$ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$ = 1</td>
<td>-2.024 (0.037)</td>
<td>-6.621 (0.336)</td>
</tr>
<tr>
<td>$z$ = 2</td>
<td>3.932 (0.049)</td>
<td>-2.718 (0.046)</td>
</tr>
<tr>
<td>$z$ = 3</td>
<td>5.033 (0.292)</td>
<td>2.931 (0.048)</td>
</tr>
</tbody>
</table>

### Table 4.7: Estimated rate of addition and cancellation.

<table>
<thead>
<tr>
<th>$z$ = 1</th>
<th>$z$ = 2</th>
<th>$z$ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^c_{0,z}$</td>
<td>$\beta^c_{1,z}$</td>
<td>$\beta^c_{-1,z}$</td>
</tr>
<tr>
<td>1.64 (0.003)</td>
<td>0.302 $\times 10^{-4}$ (0.083 $\times 10^{-5}$)</td>
<td>0.695 (0.005)</td>
</tr>
<tr>
<td>3.45 (0.018)</td>
<td>0.327 $\times 10^{-4}$ (0.184 $\times 10^{-5}$)</td>
<td>1.945 (0.014)</td>
</tr>
<tr>
<td>16.381 (0.189)</td>
<td>0.11 $\times 10^{-3}$ (0.128 $\times 10^{-5}$)</td>
<td>2.932 (0.018)</td>
</tr>
</tbody>
</table>

### Table 4.8: Estimated rate of MO arrival.

<table>
<thead>
<tr>
<th>$z$ = 1</th>
<th>$z$ = 2</th>
<th>$z$ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^a_{0,z}$</td>
<td>$\beta^a_{1,z}$</td>
<td>$\beta^a_{-1,z}$</td>
</tr>
<tr>
<td>0.044 (3.92 $\times 10^{-4}$)</td>
<td>0.175 (0.003)</td>
<td>0.596 (0.012)</td>
</tr>
</tbody>
</table>

### Table 4.9: Estimated parameter of MO size distribution (negative binomial) with probability mass function $p(k) = \binom{k+r-1}{k}(1-p)^r p^k$.

<table>
<thead>
<tr>
<th>$z$ = 1</th>
<th>$z$ = 2</th>
<th>$z$ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$ = 0.477 (0.002)</td>
<td>0.630 (0.002)</td>
<td>0.866 (0.003)</td>
</tr>
<tr>
<td>p $3.339 \times 10^{-4}$ (1.961 $\times 10^{-6}$)</td>
<td>5.408 $\times 10^{-4}$ (2.968 $\times 10^{-6}$)</td>
<td>9.097 $\times 10^{-4}$ (4.697 $\times 10^{-6}$)</td>
</tr>
</tbody>
</table>

### Table 4.10: Estimated parameter of replenished queue length distribution (negative binomial) with probability mass function $p(k) = \binom{k+r-1}{k}(1-p)^r p^k$.

<table>
<thead>
<tr>
<th>$z$ = 1</th>
<th>$z$ = 2</th>
<th>$z$ = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$ = 6.031 (0.294)</td>
<td>7.530 (0.513)</td>
<td>3.618 (0.452)</td>
</tr>
<tr>
<td>p $8.441 \times 10^{-4}$ (4.285 $\times 10^{-5}$)</td>
<td>1.9 $\times 10^{-3}$ (1.345 $\times 10^{-4}$)</td>
<td>2.3 $\times 10^{-3}$ (3.041 $\times 10^{-4}$)</td>
</tr>
</tbody>
</table>
Chapter 5

Conclusions

5.1 Summary of Contributions

In this thesis, we address three real world problems in the realm of algorithmic trading. For each problem, we construct an appropriate model that describes relevant market dynamics and formulate it as a stochastic control problem. We derive the associated dynamic programming equations and provide verification theorems for the value functions. We propose methods to simplify the resulting dynamic programming equations and solve the reduced equations via numerical schemes. Convergence of the numerical schemes is examined, when appropriate. We calibrate our models to real data and investigate the properties of the optimal strategies. Below we describe our contributions to the current academic literature in details.

In Chapter 2, we show how an agent liquidates a portfolio of assets whose midprices are co-integrated. The agent maximizes terminal wealth and targets an inventory schedule while accounting for the price impact of MOs from all market participants. The liquidation strategy employs $n$ co-integrated assets and liquidates a basket consisting of $m \leq n$ assets. We derive the agent’s optimal trading speed in closed-form. To illustrate the performance of the optimal strategy, we calibrate our model to five stocks from the
Chapter 5. Conclusions

Nasdaq exchange and conduct a simulation study in which the agent liquidates 4,600 shares of INTC and 9,00 shares of SMH. Our result shows that the agent outperforms the Almgren-Chriss (AC) strategy by 4 to 4.5 basis points when she adopts the optimal strategy. Even if the agent is not allowed to speculate, i.e., cannot repurchase shares, the relative savings compared to AC is between 2.5 to 3.5 basis points.

In Chapter 3, we show how an agent hedge a short position a contingent claim by trading the underlying asset using LOs and MOs. The agent solves a combined regular and impulse control problem to maximize her expected utility of terminal wealth. Our work is different from the extant literature in: (i) the agent employs LOs, and (ii) the agent’s MOs are modelled as impulse controls. We also incorporate some important features such as price impact and adverse selection costs. We demonstrate the agent’s optimal strategy for a particular case where the agent takes a short position in European call options written on the E-Mini S&P500 futures. In the optimal strategy, the agent employs MOs to ensure the inventory is on target to replicate the payoff of the claim and employs LOs to build the inventory at a favorable price and earn the spread by completing round-trip trades.

In Chapter 4, we show how an agent makes decisions on whether to place or cancel an LO based on its queue position in the LOB. Using tick data from the Nasdaq exchange, we perform empirical studies on various LOB events and propose novel ways to model cancellation position of LOs and distribution of MO size. Based on our empirical findings, we develop a queueing model that captures the stylized facts on LOB data and apply the model in the context of algorithmic trading. We consider the problem of an agent maximizing her expected utility by placing and cancelling LOs in such queueing model. In our framework, the agent’s goal is to let more of her LOs executed at a gainful regime, and less of her LOs executed at an adverse regime. To demonstrate our result, we calibrate our model to real data and provide a numerical example. Our simulation study shows that the optimal strategy achieve a 2.4% higher mean terminal wealth than the
benchmark that only looks at the regimes; or an 11.4% lower standard deviation given the same level of mean terminal wealth.

5.2 Future Works

In this section, we point out a few directions in which works in this thesis can be extended. The choices are based on our experience and reflect what we think that can best contribute to the current literature.

Model for co-integration with permanent impact. In Chapter 2, we derive our main result assuming that the midprice dynamics follows (2.5), in which price impact from order flow and co-integration dynamics do not affect each other. In Section 2.5, we propose an alternative model for the midprice dynamics (2.33) and derive a similar result. In the alternative model, price impact and co-integration dynamics are indeed coupled. It is natural to ask the question: which model is better? Moreover, a better model might be different from the two. A more general question to ask is: what is a good model for price impact under co-integration? Another possible direction to extend our model in Chapter 2 is to incorporate stochastic volatility, for example, using the model proposed by Andresen et al. (2014).

Cross-impact between multiple assets. Besides price impact models under co-integration, it is interesting to simply study price impact models for multiple assets, in particular, the cross-impact: how does trading asset A affect the price of asset B or vice versa? What is the condition for non-arbitrage? There are numerous problems in this direction that are worth investigating.

Model for stochastic intensity. In Section 3.6, we model the inter-event time for midprice changes and MO arrivals using Hawkes process. Our model implicitly assumes an exponential kernel (see Bacry et al. (2015)) for the intensity decay. Looking at Table 3.1 more closely, we see that the rate of mean-reversion $\kappa$ is quite high, in the order of
It suggests that events in the LOB are highly clustered: when an event occurs, the intensity surges to a very high level and mean-reverts to the long-run level quickly. The exceedingly high $\kappa$ is an indication that exponential decay kernel may not be sufficient in capturing the dynamics of the intensity. There are a few potential approaches to deal with this problem. First, we can replace the exponential kernel with other decay kernels such as the power kernel. The downside of this approach is that the dynamics of the system are no longer Markov. Hence dynamic programming may not be suitable any more. Another possible approach is to use semi-Markov model proposed by Fodra and Pham (2015b).

**Option pricing.** In Chapter 3, we show how the agent employs indifference pricing to calculate the premium charged for taking a short position in a contingent claim. This method can be used for option pricing, and it would be interesting to assess how it performs in the real world. For example, does it capture stylized facts in real data? Is it more appropriate for short maturity options? To answer these questions, one could start by comparing option prices on the market with the indifference price predicted by our model.

**Option hedging by tracking a hedging process.** Our set up in Chapter 3 does not require specifying a hedging process before hand. An alternative approach is to consider the problem where an agent tracks a hedging process (e.g., Black-Scholes delta) using limit and market orders. The advantage of this approach is that the hedging process can be non-Markovian. Hence the optimal strategy can be non-Markovian too.

**Rigorous theory for impulse control with a random outcome.** In Chapter 3 and Chapter 4, we encounter a new type of impulse control problem that has not been rigorously treated in the current literature. In the classical impulse control problem, before the agent exercises an impulse, the state of the system right after the impulse is known. The agent, based on her knowledge on the post-impulse state, makes a decision on whether to exercise the impulse or not. For the models in Chapter 3 and Chapter 4
however, the state of the system is random. In Chapter 3, the midprice has a probability of jumping after the agent sends an MO, and in Chapter 4, the queue is instantaneously replenished to a random length when the agent cancels the last LO in the queue. Intuitively, the agent should average over all possible states after the impulse and make decisions based on that. In Chapter 3 and Chapter 4, we derive the dynamic programming equations based on this intuition. However, there are a few subtle issues that need to be addressed in full mathematical rigor. In Chapter 4, we construct the filtrations in a non-standard way to avoid the measurability issue and provide an argument for deriving the dynamic programming equations. For future works, it is worthwhile considering a general version of our construction of filtrations to allow for multiple events occurring at the same time index, yet still possess orders between them. It would also be interesting to prove the dynamic programming principle under this more general setting.

**Applications of queueing models on optimal execution.** In Chapter 4, we apply our queueing model in the context in which an agent maximizes profit. Alternatively, we can apply the queueing model in optimal execution. For example, we can change the agent’s target to liquidating a certain number of shares over a fixed time horizon. We can even generalize the setting by allowing the agent to use market orders. The advantage of incorporating queue position is the ability to better forecast execution time of limit orders.

**The market microstructural foundation of the queueing dynamics.** Our empirical analysis in Chapter 4 reveals certain patterns in the queueing dynamics of the limit order book. Given the fact that the limit order book is the place where different market participants interact, it is interesting to understand the market microstructural foundation that leads to the observed patterns in the queueing dynamics. For example, what are good models for the behavior of different types of traders that produce our observed dynamics?

**Applications of queueing models on market microstructure.** We can also
apply the queueing model in the analysis of market microstructure. For example, it is interesting to understand how market makers and informed traders interact in such queues. This direction extends the approach of Kyle (1985) by accounting for queueing dynamics of the limit order book. For the purpose of an equilibrium analysis, our model might be too complicated. Hence a first step could be simplifying the queueing model.
Appendix A

Acronyms

AT Algorithmic trading
ATM At-the-money
CIR Cox-Ingersoll-Ross
CME Chicago Mercantile Exchange
DPE Dynamic programming equation
FARO FARO Technologies, Inc.
HF High frequency
HFT High frequency trading
HJB Hamilton-Jacobian-Bellman
INCT Intel Inc.
LO Limit order
LBO Limit buy order
LSE London Stock Exchange
LSO Limit sell order
MO Market order
MBO Market buy order
MLE Maximum likelihood
<table>
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<tr>
<th>Acronym</th>
<th>Definition</th>
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<tr>
<td>MSO</td>
<td>Market sell order</td>
</tr>
<tr>
<td>NTAP</td>
<td>NetApp Inc.</td>
</tr>
<tr>
<td>NYSE</td>
<td>New York Stock Exchange</td>
</tr>
<tr>
<td>ORCL</td>
<td>Oracle Corporation</td>
</tr>
<tr>
<td>OTM</td>
<td>Out-of-the-money</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial differential equation</td>
</tr>
<tr>
<td>QVI</td>
<td>Quasi-variational inequality</td>
</tr>
<tr>
<td>RCLL</td>
<td>Right continuous with left limit</td>
</tr>
<tr>
<td>SDE</td>
<td>Stochastic differential equation</td>
</tr>
<tr>
<td>SMH</td>
<td>VanEck Vectors Semiconductor ETF</td>
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<tr>
<td>TS</td>
<td>Trade signal</td>
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Appendix B

Background Materials

B.1 Dynamic Programming Principle

In this section, we collect some results in using dynamic programming principle to solve stochastic control problems.

Fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F})_{0 \leq t \leq T})$. Let $U$ be a given closed and convex set, $W_t$ be a $m$-dimensional Brownian motion with independent components, and $\tilde{N}(dz, dt) = N(dz, dt) - 1_{|z| \leq 1} \nu(dz) dt$ be a compensated Poisson random measure. We assume that the controlled process $X_t \in \mathbb{R}^k$ satisfies the following SDE

$$
dX_t = b(X_t, u_t) \, dt + \sigma(X_t, u_t) \, dW_t + \int \gamma(X_t\cdot, u_t\cdot, y) \tilde{N}(dt, dy), \quad (B.1)$$

where $u = (u_t)_{0 \leq t \leq T}$ is the agent’s control process that is RCLL and $u_t \in U$ for all $t$. The coefficients $b : \mathbb{R}^k \times U \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times U \to \mathbb{R}^k \times \mathbb{R}^m$ and $\gamma : \mathbb{R}^k \times U \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n$
satisfy the following conditions\footnote{It is not difficult to generalize to the case where \( b, \sigma \) and \( \gamma \) are time dependent.},

\[
|b(x, u) - b(y, u)| \leq C|x - y|,
\]
\[
|\sigma(x, u) - \sigma(y, u)| \leq C|x - y|,
\]
\[
\int |\gamma(x, u, z) - \gamma(y, u, z)|^2 \nu(dz) \leq C|x - y|^2,
\]

for a constant \( C > 0 \) and for any \( x, y \in \mathbb{R}^k, u \in U \). The above conditions ensure that the SDE (B.1) admits a unique solution.

Besides controlling the dynamics of \( X_t \) through \( u_t \), the agent also chooses a sequence of stopping times \( \tau = (\tau_k)_{k \geq 1} \) and a sequence of actions (impulses) \( \zeta = (\zeta_k)_{k \geq 1} \). Let \( Z \) be the set where the impulse \( \zeta_k \) takes value in. Naturally, we require that \( \zeta_k \) is measurable w.r.t. \( \mathcal{F}_{\tau_k} \) for all \( k \). Each impulse \( (\tau_k, \zeta_k) \) directly change the states as follows

\[
X_{\tau_k} = \Gamma(X_{\tau_k^{-}}, \zeta_k),
\]

for a measurable function \( \Gamma \).

Let us write \( \alpha = (u, \tau, \zeta) \). The agent’s performance criteria is given by

\[
J^\alpha(t, x) = \mathbb{E}_{t,x} \left[ \int_t^T f(X^\alpha_s, u_s) ds + g(X^\alpha_T) + \sum_{\tau_k \leq T} K(X^\alpha_{\tau_k}, \zeta_k) \right],
\]

where \( \mathbb{E}_{t,x}[\cdot] \) is the short-hand notation for \( \mathbb{E}[\cdot|X_t = x] \). Note that we use the notation \( X^\alpha_t \) to stress the fact that its dynamics are affected by \( \alpha \). The functions \( f \) and \( g \) represent the continuous and the terminal reward for the agent; The function \( K \) represents the cost or reward each time the agent exercises an impulse control. The agent’s goal is to maximize \( J \) over the set of admissible controls:

\[
v(t, x) = \sup_{\alpha \in \mathcal{A}} J(t, x),
\]
where $v$ stands for the agent’s value function.

Let $\mathcal{L}^\alpha$ be the infinitesimal generator of the process $X_t^\alpha$, which acts on a smooth function $\phi = \phi(t, x)$ as follows

$$
\mathcal{L}^\alpha \phi = b(x, u_t)^T \partial_x \phi + \frac{1}{2} \text{Tr}(\sigma(x, u_t) \partial_{xx} \phi)
+ \int \phi(t, x + \gamma(x, u_t)) - \phi(t, x) - \gamma(x, u_t)^T \partial_x \phi 1_{|z| \leq 1} \nu(dz).
$$

Through a dynamic programming argument, we can show that the value function $v$ satisfies the following HJB quasi-variational inequality (QVI):

$$
\max \left\{ \sup_{u \in U} \left\{ \partial_t v + \mathcal{L}^\alpha v + f(x, u) \right\} ; \sup_{z \in Z} \left\{ v(t, \Gamma(x, z)) - v + K(x, z) \right\} \right\} = 0,
$$

with the terminal condition $v(T, x) = g(x)$.

For the detailed derivations of the above results, see Øksendal and Sulem (2005).

### B.2 Viscosity Solution

The HJBQVI (B.3) is easy to work with when its solution is sufficiently smooth. For example, one can show that if (B.3) admits a solution $v \in C^{1,2}([0, T] \times \mathbb{R}^k)$, it coincide with the value function of the control problem. However, it is not always true, or at least easy to deduce, that the solution to (B.3) has sufficient smoothness. In this case, the partial derivative term $\partial_{xx} v$ may not be well-defined.

One remedy for this problem is to defined a weak solution for (B.3). The suitable definition for this type of equations turn out to be the *viscosity solution*, proposed by Crandall and Lions (1983). For a thorough review of this subject, see Crandall et al. (1992).
Let us start by writing (B.3) in the following form:

\[ F(x, v(x), Dv(x), D^2v(x)) = 0, \]  

(B.4)

for \( x \in \mathcal{O} = (0, T) \times \mathbb{R}^k \), where \( F \) is a continuous mapping from \( \mathcal{O} \times \mathbb{R} \times \mathbb{R}^k \times \mathcal{S}_k \) and \( \mathcal{S}_k \) is the set of \( k \times k \) symmetric matrices with real entries. We also require the following *degenerate elliptic* condition to hold:

\[ F(x, r, p, A) \leq F(x, r, p, B) \quad \text{if} \quad A \geq B, \]

for all \( (x, r, p) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^k \).

Let us recall the definition of semi-continuous envelopes of a function.

**Definition 1** (semi-continuous envelope). *For any function \( v \), we denote by \( v^* \) (resp. \( v_* \)) as its upper (resp. lower) semi-continuous envelope, where*

\[
\begin{align*}
v^*(x) &= \limsup_{x_n \to x} v(x_n), \\
v_*(x) &= \liminf_{x_n \to x} v(x_n).
\end{align*}
\]

Now we are ready to state the definition of viscosity solutions.

**Definition 2** (viscosity supersolution). *A function \( v \) is a (viscosity) supersolution of (B.4) if for all \( x_0 \in \mathcal{O}, \phi \in C^2(\mathcal{O}) \) and that \( x_0 \) is a global minimizer of \( u_* - \phi \), we have*

\[ F(x_0, u_*(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0. \]

**Definition 3** (viscosity subsolution). *A function \( v \) is a (viscosity) subsolution of (B.4) if for all \( x_0 \in \mathcal{O}, \phi \in C^2(\mathcal{O}) \) and that \( x_0 \) is a global maximizer of \( u^* - \phi \), we have*

\[ F(x_0, u^*(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0. \]
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Definition 4 (viscosity solution). A function $v$ is a viscosity solution if it is a viscosity supersolution and a viscosity subsolution.

Note that in the above definitions of subsolutions and supersolutions, we replace the differentials $Dv$ and $D^2v$ using a smooth test function $D\phi$ and $D^2\phi$. Defining through a test function circumvents the technical problem that $v$ may not be sufficiently smooth. Moreover, it is easy to deduce that if $v \in C^2$ is a classical solution of (B.4), $v$ is also a viscosity solution.

The most important property for (B.3) to hold is called the strong comparison principle, which states that an upper semi-continuous subsolution is no greater than a lower semi-continuous supersolution. The strong comparison ensures uniqueness of the viscosity solution. To see this, note that for two solutions $u$ and $v$, we have $u^* \leq v^* \leq v^* \leq u^*$. Thus we must have $u = v$ and they are continuous. Strong comparison also allows us to prove that the solution can be approximated by a numerical scheme, see Barles and Souganidis (1991). There are some general results on the conditions for the strong comparison to hold, see Crandall et al. (1992) and the references therein. In many real-world problems however, it has to be treated on a case-by-case basis.

B.3 Feymann-Kac Formula

Fix a finite time horizon $[0, T]$, We consider the following partial-integro differential equation (PIDE):

\[
\partial_t v(t, x) + \mathcal{L} v(t, x) + f(t, x) = r(t, x) v(t, x), \quad (B.5a)
\]

\[
v(T, x) = g(x), \quad (B.5b)
\]

for all $t \in (0, T), x \in \mathbb{R}^n$. Here $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are continuous functions.
Let $(X_t)_{0 \leq t \leq T}$ be a Markov process with infinitesimal generator $\mathcal{L}$. We denote by $X^{t,x}$ the process with the same dynamics as $X$ and started from $x$ at time $t$. The following Feynman-Kac representation theorem states that the solution to (B.5) admits a probabilistic representation.

**Theorem B.1 (Feynman-Kac Representation).** Let $v(t, x) \in C^{1,2}([0, T] \times \mathbb{R}^n)$ be a function with bounded derivative in $x$ and a solution to (B.5). Then $v$ admits the following representation

$$v(t, x) = \mathbb{E} \left[ \int_t^T e^{-\int_s^t r(u, X^t,x_u)du} f(s, X^t,x_s)ds + e^{-\int_t^T r(u, X^t,x_u)du} g(X^t,x_T) \right],$$

(B.6)

for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

For a proof, see Pham (2009). In Appendix 2.A, we prove a generalized version of the Feynman-Kac theorem, where $v$ is a vector-valued function.
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