Squeeze Film Dampers in High-Speed Turbomachinery:
Fluid Inertia Effects, Rotordynamics, and
Thermohydrodynamics

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Department of Mechanical and Industrial Engineering
University of Toronto

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Abstract

Unbalance induced vibrations are the main source of structural vibrations in high-speed turbomachinery. This mass unbalance is associated with the limitations and imperfections in manufacturing rotor systems and leads to a synchronous load cycle in the rotor. Squeeze film dampers (SFDs) are essential components in high-speed turbomachinery, including aircraft jet engines, high-performance compressors, gas turbines, and automotive turbochargers, which are incorporated to attenuate or completely suppress the steady-state unbalance induced vibration amplitudes at the resonance frequencies, reduce the forces transmitted to the supports, and to ensure the stable operation of the system. Conventionally, the dynamics of SFDs are characterized by using Reynolds equation, where it is assumed that the effect of fluid film inertia is negligible relative to the viscous effects. However, at large velocities, the inertia forces are significantly large and the effect of fluid inertia can no longer be neglected. Additionally, the design and analysis of SFDs are generally based on isothermal conditions; however, the thermophysical properties of the bearing lubricant strongly depend on the local state of temperature. At high operating speeds, the bearings may experience significant temperature rise, since the viscous dissipation that is associated with the shear motion, as well as the heat transfer
with the bearing surfaces, can generate significant temperature and viscosity variations within the lubricant film, which ultimately influences the static and dynamic performance of the bearing considerably.

The objective of this thesis is to develop a comprehensive model for SFDs in high-speed turbomachinery. The proposed model incorporates the effect of fluid film inertia and thermal effects to provide a precise prediction of the SFD behavior at high velocities. In order to achieve this objective, the effect of lubricant inertia at different SFD operating conditions, including small and large amplitude motions of the journal center, are formulated by using momentum approximation and perturbation method, and for different damper geometries, including finite-length and short-length dampers. The results of the analysis demonstrate the significant influence of fluid inertia on the shape, phase, and magnitude of the SFD pressure distribution as well as the magnitude and direction of the fluid film reaction forces. Subsequently, a finite element based rotodynamic model is developed for multi-mass flexible rotors and the proposed SFD models are incorporated into the rotodynamic model to study the effect of SFD fluid film inertia on the unbalance induced steady-state vibration amplitudes and transient orbits of high-speed rotors. The results of the rotodynamic analysis confirmed the considerable influence of SFD fluid inertia on the attenuation of the unbalance induced vibrations at resonance zones. Finally, a detailed thermohydrodynamic (THD) model incorporating the full term Navier-Stokes equations, the energy equation for the lubricant, the Laplace heat conduction equation for the surrounding solids, including bushing and shaft, and realistic thermal boundary conditions is developed to investigate the effect of temperature variation on the dynamics of SFDs.
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## Nomenclature

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<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_e$</td>
<td>Shaft element cross section area ($m^2$)</td>
</tr>
<tr>
<td>$A_r$</td>
<td>Radial acceleration of the journal center ($m/s^2$)</td>
</tr>
<tr>
<td>$c_{ij}, k_{ij}$</td>
<td>Rotor Support and SFD retaining spring stiffness and damping components ($N.s/m, N/m$)</td>
</tr>
<tr>
<td>$c$</td>
<td>SFD radial clearance ($m$)</td>
</tr>
<tr>
<td>$C, G, K, M$</td>
<td>Assembled rotor system inertia, support damping, gyroscopic effect, and stiffness matrices</td>
</tr>
<tr>
<td>$C_p$</td>
<td>Lubricant specific heat ($J/kg, ^\circ C$)</td>
</tr>
<tr>
<td>$C_f$</td>
<td>Force conversion coefficient</td>
</tr>
<tr>
<td>$C_{ij}, M_{ij}$</td>
<td>SFD damping and mass coefficients ($N.s/m, kg$)</td>
</tr>
<tr>
<td>$D_{out}, D_{in}$</td>
<td>Shaft inner and outer diameter ($m$)</td>
</tr>
<tr>
<td>$D_{d, out}, D_{d, in}$</td>
<td>Disk element inner and outer diameter ($m$)</td>
</tr>
<tr>
<td>$D, R$</td>
<td>SFD journal diameter/radius ($m$)</td>
</tr>
<tr>
<td>$e$</td>
<td>SFD journal eccentricity ($m$)</td>
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<tr>
<td>$E_e$</td>
<td>Shaft element Young’s modulus ($pa$)</td>
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<tr>
<td>$f_x, f_y$</td>
<td>Support (i.e. Ball bearings and squirrel cage) forces ($N$)</td>
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<tr>
<td>$F(t)$</td>
<td>Rotor External force ($N$)</td>
</tr>
<tr>
<td>$F_0$</td>
<td>Rotor mass unbalance magnitude ($kg-m$)</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$F_r, F_t$</td>
<td>SFD fluid film reaction force radial and tangential components (N)</td>
</tr>
<tr>
<td>$F_{unb}$</td>
<td>Rotor mass unbalance force (N)</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational constant ($m/s^2$)</td>
</tr>
<tr>
<td>$G_{shear,e}$</td>
<td>Shaft element shear modulus (pa)</td>
</tr>
<tr>
<td>$G_{e,M,e,K,e}$</td>
<td>Shaft element inertia, stiffness, and gyroscopic effect matrices</td>
</tr>
<tr>
<td>$G_{d,e,M_{d,e}}$</td>
<td>Disk element gyroscopic effect and inertia matrices</td>
</tr>
<tr>
<td>$h$</td>
<td>SFD fluid film thickness (m)</td>
</tr>
<tr>
<td>$h_b$</td>
<td>Bush convective heat transfer coefficient ($W/m^2 \cdot ^\circ C$)</td>
</tr>
<tr>
<td>$h_s$</td>
<td>Shaft convective heat transfer coefficient ($W/m^2 \cdot ^\circ C$)</td>
</tr>
<tr>
<td>$H$</td>
<td>Dimensionless SFD fluid film thickness</td>
</tr>
<tr>
<td>$I_d$</td>
<td>Disk element diametral moment of inertia (kg.m$^2$)</td>
</tr>
<tr>
<td>$I_e$</td>
<td>Shaft element second moment of inertia (m$^4$)</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Disk element polar moment of inertia (kg.m$^2$)</td>
</tr>
<tr>
<td>$k_a$</td>
<td>Air thermal conductivity ($W/m \cdot ^\circ C$)</td>
</tr>
<tr>
<td>$k_b$</td>
<td>Bush thermal conductivity ($W/m \cdot ^\circ C$)</td>
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<tr>
<td>$k_s$</td>
<td>Shaft thermal conductivity ($W/m \cdot ^\circ C$)</td>
</tr>
<tr>
<td>$k_f$</td>
<td>Lubricant thermal conductivity ($W/m \cdot ^\circ C$)</td>
</tr>
<tr>
<td>$K_r, M_r$</td>
<td>Modal stiffness and modal mass matrices ($N/m, kg$)</td>
</tr>
<tr>
<td>$L$</td>
<td>SFD journal length (m)</td>
</tr>
</tbody>
</table>
\( L_{e} \) Length of shaft element (\( m \))

\( L_{shaft} \) Shaft length (\( m \))

\( m_{d} \) Disk element mass (\( kg \))

\( m_{unb} \) Rotor unbalance mass (\( kg \))

\( M_{e_{i}, M_{e_{i}}} \) Shaft inertia element matrix including rotary inertia and shear effects (\( kg \))

\( n_{e} \) Number of shaft elements

\( Ne \) Eckert number

\( p \) Vector of modal coordinates

\( P \) Fluid pressure (\( pa \))

\( P_{cav} \) Cavitation pressure (\( pa \))

\( Pe \) Peclet number

\( q \) Vector of independent rotor system coordinates (\( m \))

\( r_{b} \) Bush radial component (\( m \))

\( r_{s} \) Shaft radial component (\( m \))

\( r_{u}, r_{v} \) Rotor displacement magnitudes (\( m \))

\( R_{bi}, R_{bo} \) Bush inner and outer radius (\( m \))

\( R_{out}, R_{in} \) Shaft inner and outer radius (\( m \))

\( Re \) Squeeze Reynolds number

\( s \) Laplace operator
$t$  
Time ($s$)

$T$  
Lubricant temperature ($^\circ C$)

$T_{amb}$  
Ambient temperature ($^\circ C$)

$T_b$  
Bush temperature ($^\circ C$)

$T_s$  
Shaft temperature ($^\circ C$)

$T_0$  
Supply lubricant temperature ($^\circ C$)

$u,v,w$  
SFD fluid film velocity components ($m/s$)

$V$  
SFD fluid velocity vector ($m/s$)

$V_t$  
Tangential velocity component of the SFD journal center ($m/s$)

$W_d$  
Disk element width ($m$)

$x,y,z$  
The components of the fixed SFD coordinate system

$X,Y$  
The components of the fixed SFD inertial coordinate system

$\overline{F}$  
Dimensionless SFD fluid film reaction forces

$\overline{F_r}, \overline{F_t}$  
Dimensionless SFD fluid film reaction force radial and tangential components

$\overline{F_x}, \overline{F_y}$  
Dimensionless SFD fluid film reaction force horizontal and vertical components

$\overline{I_{i,j}}$  
Dimensionless SFD momentum flux integrals

$\overline{I_{0,j}}$  
Dimensionless zeroth-order SFD momentum flux integrals

$\overline{P}$  
Dimensionless SFD fluid pressure
\( \varepsilon \)  
SFD journal eccentricity ratio

\( \eta \)  
Dimensionless SFD radial component

\( \eta_u, \eta_v \)  
Rotor displacement angles (rad)

\( \theta \)  
Dimensionless SFD circumferential component (rad)

\( \lambda \)  
Relaxation parameter

\( \mu \)  
Fluid dynamic viscosity (Pa s)

\( \mu_m \)  
Averaged radial lubricant viscosity (Pa s)

\( \mu_0 \)  
Supply lubricant viscosity (Pa s)

\( \xi \)  
Dimensionless SFD axial component

\( \rho \)  
Fluid density (kg/m\(^3\))

\( \rho_e \)  
Shaft element density (kg/m\(^3\))

\( \rho_d \)  
Disk element density (kg/m\(^3\))

\( \sigma, \psi \)  
Rotor mass unbalance force and moment eccentricity (m)

\( \nu_e \)  
Shaft element Poisson ratio

\( \phi \)  
SFD attitude angle (rad)

\( \omega_i \)  
\( i^{th} \) natural frequency of the rotor system (Hz)

\( \Omega \)  
Rotor angular velocity (rad/sec)

\( \Delta t \)  
Time step (s)

\( \mu \)  
Dimensionless fluid viscosity
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{\mu}_m$</td>
<td>Dimensionless averaged radial lubricant viscosity</td>
</tr>
<tr>
<td>$\bar{\phi}$</td>
<td>Mode shapes</td>
</tr>
<tr>
<td>$\theta'$</td>
<td>Dimensionless SFD fixed circumferential component ((rad))</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Lubricant viscosity-temperature coefficient</td>
</tr>
<tr>
<td>$\hat{\phi}$</td>
<td>Shaft element shear coefficient</td>
</tr>
<tr>
<td>$\tilde{\phi}$</td>
<td>Viscous dissipation function</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>------------------------------</td>
</tr>
<tr>
<td>CCO</td>
<td>Circular-Centered Orbit</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree of Freedom</td>
</tr>
<tr>
<td>EOM</td>
<td>Equation of Motion</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Method</td>
</tr>
<tr>
<td>LBA</td>
<td>Long Bearing Approximation</td>
</tr>
<tr>
<td>MDOF</td>
<td>Multiple Degrees of Freedom</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>SAM</td>
<td>Small Amplitude Motion</td>
</tr>
<tr>
<td>SBA</td>
<td>Short Bearing Approximation</td>
</tr>
<tr>
<td>SFD</td>
<td>Squeeze Film Damper</td>
</tr>
<tr>
<td>SOR</td>
<td>Successive Over Relaxation</td>
</tr>
<tr>
<td>THD</td>
<td>Thermohydrodynamic</td>
</tr>
<tr>
<td>WA</td>
<td>Warner Approximation</td>
</tr>
</tbody>
</table>
Chapter 1
Introduction

1.1 Problem Statement and Background

Unbalance induced vibrations are the main source of structural vibrations in high-speed turbomachinery. This mass unbalance is associated with the limitations and imperfections in manufacturing rotor systems and leads to a synchronous load cycle in the rotor. Squeeze film dampers are essential components in high-speed turbomachinery, including aircraft jet engines, high performance compressors, gas turbines, and automotive turbochargers, that are incorporated to attenuate or completely suppress the steady-state unbalance induced vibration amplitudes at the resonance frequencies, reduce the forces transmitted to the supports, and to ensure the stable operation of the system. Figure 1.1 demonstrates the geometry of a conventional SFD. The dimensions of the SFD components in the figure are exaggerated to provide a clearer representation of the SFD. A typical SFD consists of a stationary outer bearing (i.e. the bush) and an inner journal with approximately identical diameters. The journal is assembled on the outer surface of a rolling element and is prevented from rotation by using an anti-rotation mechanism. The annular region between the journal and the housing is generally smaller than 0.25 mm in thickness and is filled with a lubricant. The precession motion of the journal is induced by residual unbalance of the rotor and generates a hydrodynamic squeeze film pressure distribution that applies reaction forces over the journal, providing the damping force to attenuate the transmitted forces and in turn reducing the rotor vibration. The dynamic force response of SFDs is determined by the damper geometry, operating speed, and lubricant properties. Furthermore, since SFDs do not produce direct stiffness; retaining springs are introduced in parallel to the squeeze film (i.e. squirrel cage) [1] or a pair of elastomer O-rings in radial disposition with the purpose of introducing stiffness to the SFD system and centering the journal within the bearing and preventing the rotation of the journal, while still allowing the orbital whirls due to the flexibility of the retaining spring.
1.1.1 Effect of Lubricant Inertia

The pioneering work by Cooper [2] demonstrated the benefits associated with the rotor operation from coupling of a damping element, represented by an oil squeeze film, to an elastic element placed within the rotor support. Ever since, research works have focused on providing greater insight into different features of SFDs, including the effect of lubricant inertia. According to the classic lubrication theory, the pressure distribution in the annular region of a hydrodynamic journal bearing is determined by using the Reynolds equation, where it is assumed that the inertial forces are negligible relative to viscous forces (i.e. $Re \approx 0$) [3]. In modern turbomachinery, increasing velocity and size of rotors as well as application of low-viscosity lubricants requires the fluid inertia effect to be included in design and analysis of journal bearings. Smith [4] used simplified journal bearing geometries, including the short and long bearing models, to determine the effect of fluid inertia on the dynamic characteristics of journal bearings. He concluded that the effect of fluid inertia introduces an added mass to the rotor system, which significantly influences the dynamic characteristics of short stiff rotors.

The effect of fluid inertia for thin films and hydrodynamic journal bearings has been considerably studied in the literature. Typically, the proposed fluid inertia models assume circular-centered orbits (CCOs) of the journal center. Circular-centered orbits of the journal center is a very common type of journal motion in industrial applications of squeeze film dampers, including vertical rotors mounted on SFDs, horizontal rotors mounted on SFDs with centralizing springs, rotors operating close to critical velocities, and for rotor response to large unbalance forces [5]. Additionally, several investigations [1] have assumed small amplitude motions of the journal center in their corresponding investigations. According to order of
magnitude calculations, for small amplitude motions of the journal center, convective inertia terms in the flow equations are negligible relative to unsteady (temporal) inertia terms [6]. However, for large amplitude motions of the journal center, including the displacement at critical speeds, the effect of convective inertia is no longer negligible and should be necessarily incorporated into the calculations.

In general, the existing SFD models apply reducing assumptions to the lubricant flow equations, namely the fluid continuity equation and the momentum transport equations, to develop partial differential expressions for the lubricant pressure. Subsequently, geometric approximations (i.e. short bearing approximation and long bearing approximation) are applied to the pressure expression to directly calculate the SFD force coefficients (i.e. direct and cross-coupled damping and inertia terms). Alternatively, numerical and closed-form analytical solutions are proposed for the analysis of the flow equations [1].

The force coefficients are the most common representation for the SFD forces in the presence of fluid inertia. In this technique, firstly, thin film equations are integrated into an expression for the lubricant pressure distribution by adopting either one of the momentum approximation method (aka method of averaged inertia) [7]–[9], the perturbation method [10], or the energy approximation method. Han and Rogers [11], [12] have provided a detailed comparison of three approximation methods, namely momentum approximation method, iterative method, and energy method, on the force coefficients for short, long, and finite-length cylindrical SFDs. Subsequently, geometric approximations (i.e. short bearing approximation and long bearing approximation) are applied to the pressure expression to provide an approximate closed-form representation of the lubricant pressure distribution. The closed-form pressure expression is integrated over the journal surface to obtain the fluid film reaction forces (i.e. the tangential and radial components). These forces are nonlinear functions of the velocity and the acceleration of the journal center. The Jacobian matrices of the journal forces with respect to the velocities and the accelerations of the journal center are computed to develop equivalent inertia and damping coefficients respectively.

According to the momentum approximation method [13], [14], firstly, the flow equations are integrated along the film thickness to represent the flow dynamics in terms of the mean flows,
averaged inertia, and wall shear stress differences. Subsequently, the terms in the equations are approximated by assuming that the shape of the velocity profiles is not strongly influenced by the inertia forces [15]. San Andres and Vance [16] have implemented the momentum approximation technique to determine the force coefficients for finite-length SFDs executing CCOs. They have included convective fluid inertia as well as end seal effects in the calculations. Furthermore, they have suggested that the proposed model is strictly valid for small Reynolds numbers (i.e. \( \text{Re} \leq 1 \)). San Andres and Vance [17] have analytically investigated the effect of fluid inertia and turbulence on the force coefficients of short and long SFDs. They have included both temporal and convective inertia in their analysis. They have suggested that at small eccentricity ratios, where the effect of temporal inertia is dominant, an added mass is produced, which corresponds to the radial direct inertia coefficient, however, at large eccentricity ratios, where the effect of convective inertia is superior, this effect is completely reversed. Dousti et al. [18] have developed an extended short bearing Reynolds equation that is applicable to both laminar and turbulent flow regimes. Furthermore, the force coefficients were determined based on the proposed pressure expression. Hashimoto [15] has investigated the effect of various boundary conditions on the analysis of long journal bearings with turbulent flow regimes including fluid inertia. Additionally, the force coefficients were developed by using the momentum approximation technique.

Alternatively, according to the perturbation technique, a small first-order perturbation by means of the expressions for the fluid film velocity components and the lubricant pressure distribution that are typically expanded in power series of the squeeze film Reynolds number is applied to the flow equations. This perturbation technique separates the flow equations into a set of inertialess equations that are characterized by using the classic Reynolds equation, and a set of first-order inertial correction equations. According to [16], [17], [19] it is suggested that the first-order perturbation is applicable for SFD flow regimes with \( \text{Re} \leq 25 \). Reinhardt and Lund [20] introduced a first-order perturbation technique to determine the force coefficients for hydrodynamic journal bearings at small Reynolds numbers. They suggested that the inertial corrections to the damping and stiffness coefficients are small, while a large inertia coefficient is introduced that significantly affects the bearing dynamics. San Andres and Vance [21] have combined the momentum approximation technique and the first-order perturbation technique to
develop force coefficients for short and long SFDs executing small amplitude CCOs for both cavitated and uncavitated lubricant conditions. Furthermore, they have introduced a leakage factor to extend the application of the proposed force coefficients to finite-length dampers. Nataraj et al. [22] have adopted the perturbation technique to provide a rigorous derivation of the effect of fluid inertia on the corrected stiffness, damping, and inertia coefficients for short journal bearings.

Finally, the energy approximation firstly develops the expressions for the fluid kinetic energy. Subsequently, the Lagrange’s equations along with the Reynolds transport theorem are applied to the energy expressions to obtain the inertia forces. Similar to the momentum approximation technique, in order to develop the fluid expressions, it is assumed that the shape of the velocity profiles is not strongly influenced by the fluid inertia effects. El-Shafei and Crandall [23] have used an energy approximation method by applying Lagrange’s equation to determine the force coefficients for short and long SFDs, for both cavitated and uncavitated lubricants. They have suggested that the energy approximation provides an accurate estimation of the inertia forces in SFDs. Moreover, they have applied the energy approximation method to determine the force coefficients for long and short SFDs.

The force coefficient approach is particularly valuable for rotordynamic analysis, since it directly provides an accelerated estimation of the fluid film reaction forces. However, in order to model the effect of supplementary dynamic lubricant phenomenon on the damping characteristics of SFDs, including lubricant cavitation and lubricant temperature variation, it is additionally required to develop expressions for the lubricant hydrodynamic pressure distribution and the fluid film velocity components. Furthermore, the force coefficients are typically developed for limiting bearing geometries, which makes their predictions inaccurate for arbitrary bearing geometries.

Alternatively, numerical models provide very accurate predictions for the SFD behaviour. San Andres [24] has proposed a finite difference numerical procedure to determine the effect of fluid inertia on the fluid film reaction forces for long SFDs executing CCOs. The results demonstrated the significant effect of fluid inertia on the dynamic force response of SFDs operating at large Reynolds numbers. Hamzehlouia and Behdinan [25] have used the momentum approximation
technique to develop a modified Reynolds equation to represent the effect of fluid inertia on the
dynamics of finite-length cavitated (i.e. unpressurized) SFDs executing small amplitude CCOs. They have developed a finite difference numerical procedure to determine the SFD fluid film reaction forces. In [26] the perturbation technique was adopted to develop a pressure distribution expression for SFDs executing small amplitude CCOs, including fluid inertia effect. The pressure distribution and the fluid film reaction forces were numerically determined by using finite difference approximation. The bulk flow model [27]–[29] is an alternative numerical technique that is used to study the SFD dynamics. In this technique, bulk flow variables are introduced by calculating the average lubricant velocities across the film thickness and are substituted into the flow equations. Subsequently, the bulk-flow model system of equations, including the continuity equation and the momentum transport equation, is solved for the hydrodynamic pressure and the velocity profiles by using finite volume method. The bulk flow modeling along with the finite volume numerical provide superior accuracy for the prediction of the SFD parameters; however, the process is generally computationally very expensive, especially for integration of the SFD model into rotordynamic systems, where the SFD parameters are calculated over a considerable number of iterations.

Additional analytical techniques have been proposed for calculating the closed-form pressure distribution in SFDs [19], [30], however, either the results are strictly valid for specific SFD geometries and configurations, which makes them inapplicable to arbitrary bearing geometries, or they are extremely complicated for integration into rotordynamic models. Tichy [19] has developed a modified Reynolds equation, including convective inertia terms, for SFDs executing CCOs by using the momentum approximation method. Furthermore, the proposed nonhomogeneous partial differential expression is solved by using the classical separation of variable approach. The results demonstrated the effects of fluid inertia and damper end leakage on the amplitude and the phase shift of the pressure profile. San Andres and Vance [5] have developed a closed-form exact expression for the pressure distribution for finite-length SFDs executing small amplitude CCOs. Additionally, they have determined the exact linearized damping and inertia force coefficients. Similar methodology has been adopted in [23], [31] to determine closed-form expressions for the pressure distribution and fluid film reaction forces for finite-length SFDs executing small amplitude CCOs. Tichy [32] has investigated the effect of
fluid inertia and viscoelasticity on fluid film reaction forces for short pressurized (i.e. uncavitated) SFDs executing small amplitude CCOs. The closed-form pressure distribution and the force coefficients are represented. Qingchang et al. [33] have developed closed-form analytical expressions for the fluid film reaction forces for short SFDs, including inertia effects. They have suggested that the proposed model is superior to the traditional theoretical short bearing models. El-Shafei [34] has proposed a third-order perturbation in the eccentricity ratio to provide an analytical expression for the pressure distribution and the velocity components in the absence of fluid inertia.

1.1.2 Rotordynamic Models Incorporating SFDs

The design and application of SFDs in turbomachinery is expansively represented in [35], [36]. Typically, the rotordynamic models incorporating SFDs are categorized as follows: (1) Rigid Rotor Model, (2) Simple Flexible Rotor Model, and (3) Complex Flexible Rotor Model. The rigid rotor model assumes that the flexibility of the rotor shaft is neglected and represents the equation of motion of the rotor based on the relative position, velocity, and acceleration of the center of the SFD bearing, the center of the SFD journal, and the mass center of the rotor. The dynamics of a rigid rotor incorporating SFDs and fluid-film bearings has been investigated in several studies [37]–[40]. White [41] studied the dynamics of a rigid rotor supported by squeeze film bearings, both experimentally and theoretically, by calculating the SFD forces based on the Reynolds equation. He concluded that three steady-state orbit whirls may exist at the same frequency, confirming the existence of the bi-stable condition and the jump phenomenon. Mohan and Hahn [42] obtained the steady-state response for a horizontal centralized rigid rotor supported by short SFDs executing circular-centered orbits (CCOs) by assuming a π-film model for the lubricant cavitation. They studied the effect of the SFD design parameters on the critical unbalance and the transmissibility (i.e. the forces transmitted to the supports). Hahn [43] investigated the critical unbalance for rigid rotors with SFDs by assuming both short bearing approximation (SBA) and Warner Approximation (WA). Cookson and Kossa [44] have investigated the effect of the design and operating parameters on the effectiveness of SFDs without centering springs, supporting rigid rotors, with possible extension to flexible rotor models and determined the range of the operating parameters for the effective design of SFDs.
The simple flexible rotor model assumes that the total mass of the rotor is either concentrated in the center or in the middle plane of the rotor, or alternatively it is distributed between the rotor center of mass and the center of the bearings. The steady-state orbit whirls of a simple flexible rotor supported by unpressurized (i.e. π-film model) short SFDs executing CCOs was investigated in [45], [46] and the stability of the response was studied by using Routh-Hurwitz criterion. Cunningham et al. [47] theoretically studied the model of a simple flexible rotor incorporating SFDs with centering elements, with masses distributed at the mid-span and supports. Zhao et al. [48] have investigated the bifurcation of the unbalance response of a flexible rotor supported with short and cavitated (i.e. π-film model) SFDs by using the trigonometric collation method and verified the stability of the solutions by applying the Floquet transition matrix method for both eccentric and concentric damper operations. Zhu et al. [49] have studied the behavior of the multiple-solution response in simple flexible rotors supported with short and cavitated (i.e. π-film model) SFDs equipped with centralizing springs by using different methods, including: numerical integration method and slow acceleration method. The comparison between the computational results showed that the numerical integration method is computationally inefficient, while the slow acceleration method can detect multiple-solution speed regions as well as non-synchronous response regions. Inayat-Hussain et al. [50]–[52] studied the effect of the design and the operating parameters on the onset speed of bifurcation and the extent of non-synchronous behavior in the steady-state response of a simple flexible rotor supported by SFDs with and without centering springs by using direct numerical integration.

In many practical examples, the number of degrees of freedom (DOFs) represented by the simple flexible rotor model is insufficient to provide an accurate approximation of the system dynamics. In this case, the continuous rotor model is discretized and a finite number of DOFs are introduced. The dynamics of multi-mass and multi-degrees of freedom (MDOF) rotors supported with SFDs is theoretically studied in [53], [54]. Greenhill and Nelson [55] investigated the steady-state orbit whirls of a multi-mass flexible rotor, by using an iterative technique based on the secant method for solving the nonlinear rotor system equations. Bonello et al. [56] have proposed a receptance harmonic balance method to investigate the steady-state periodic response of complex rotors supported with SFDs and the stability analysis of the periodic solution was verified by applying Floquet Theory to the modal equations of the system. Furthermore, they
have applied the proposed technique to study the interaction between an eccentric SFD and an unbalanced flexible rotor [57]. Chouksey et al. [58] have investigated the influence of internal rotor material damping and the fluid film forces by using the modal analysis of a complex rotor system.

The precedent studies assumed that the effect of lubricant inertia on the fluid film reaction forces in SFDs is negligible and either used the complete or the approximate (i.e. long bearing and short bearing approximations) Reynolds equation to represent the SFD dynamics, where it is assumed that the inertial SFD forces are negligible relative to the viscous forces (i.e. $Re \approx 0$) [3]. In modern turbomachinery, increasing velocity and size of turbomachinery and application of low-viscosity lubricants requires the fluid inertia effect to be included in the design and analysis of hydrodynamic bearings. The effect of lubricant inertia in SFDs is represented by the squeeze Reynolds number (i.e. Re). For large propulsion turbines and aero-engines the operating SFD squeeze Reynolds number is moderately large, typically in the order of one to twenty [59]. In general, the fluid inertia effect improves the damping capacity of SFDs, increasing the tangential component and reducing the radial component of the fluid film reaction forces. The effect of fluid inertia on the damping characteristics of SFDs has been theoretically and experimentally investigated in several studies [4], [1], [18]. The theoretical studies typically focus on developing expressions for the SFD pressure distribution and the fluid film reaction forces, including: closed-form analytical expressions [5], [19], [23], numerical procedures [25], [29], [60], force coefficients to represent the direct and cross-coupled inertia and damping [20], [21], [61], [62], and limiting damper geometries [15], [33]. San Andres and Vance [59] have theoretically investigated the effect of lubricant inertia on the synchronous steady-state response of a centrally preloaded single mass flexible rotor supported by short-length open-ended SFDs and represented the rotor excursion amplitudes and unbalance transmissibility for both unpressurized cavitated (i.e. $\pi$-film model) and pressurized (i.e. $2\pi$-film model) conditions. They have verified that at moderately large lubricant inertia effects, the response amplitude of the rotor and the transmitted forces to the supports are significantly reduced. Furthermore, for unpressurized (i.e. cavitated) dampers the possibility of bi-stable operation and jump phenomenon considerably declines and practically diminishes at sufficiently large inertia effects. El-Shafei [63], [64] has theoretically studied the steady-state unbalance response of a Jeffcott rotor incorporating both short and long
SFDs executing CCOs in the presence of lubricant inertia effects, by using a fast algorithm. It is demonstrated that the lubricant inertia results in the excitation of a second mode for the Jeffcott rotor, declines the likelihood of bi-stable operation and jump phenomenon, and reduces the transmitted forces to the supports. He et al. [65] have proposed a predictor-corrector procedure harmonic balance method to study the forced response of a flexible multi-mass rotor supported on a short SFD including inertia effect.

1.1.3 Thermohydrodynamic Modeling

Conventionally, the design and analysis of journal bearings is based on isothermal conditions; however, the thermophysical properties of the bearing lubricant strongly depend on the local state of temperature. At high operating speeds, the bearings may experience significant temperature rise, since viscous dissipation that is associated with the shear motion as well as the heat transfer with the bearing surfaces can generate significant temperature and viscosity variations within the lubricant film, which ultimately influences the static and dynamic performance of the bearing considerably. Consequently, the isothermal and iso-viscous assumptions are becoming increasingly invalid in the analysis of hydrodynamic journal bearings. The incorporation of lubricant thermal behavior in theoretical studies is required to accurately model and predict the operating characteristics of journal bearings. Thermal effects are essential in the analysis and design of journal bearings. Furthermore, constant monitoring of the lubricant temperature is necessary to prevent seizure (i.e. a thermally induced complete loss of bearing clearance) and orbital instability of the bearing. Additionally, at high operating temperatures, conventional oil-lubricated bearings are subjected to oil cooking and degradation. Therefore, an important objective of the bearing design is to limit the maximum lubricant temperature.

A detailed model for a hydrodynamic bearing that integrates the heat generation and dissipation to the surrounding is referred to as a thermohydrodynamic model. The thermohydrodynamic lubrication model incorporates the generation of frictional heat in the fluid film as well as the heat removal by convection in the oil films and the heat conduction though the surrounding solid walls. A complete analysis of thermal effects in journal bearings requires the study of the heat transfer to the whirling journal and the stationary bush. The study of the heat transfer in the solids results in a coupling between Laplace conduction equation in the solids with energy...
equation in the lubricant, which requires a trial-and-error numerical solution at the oil-solid interface boundary temperatures.

In general, a comprehensive thermohydrodynamic analysis of a bearing incorporates a realistic solution of the lubricant flow equations in which the viscosity field is predicted based on the computation of temperature obtained from the conservation of energy by including the energy equation in the lubricant and the Laplace heat conduction equation in the surrounding solids (i.e. journal and bushing). Khonsari [66], [67] has provided a comprehensive review of the earlier thermohydrodynamic research. The numerical solutions for the thermal analysis of journal bearings can be classified into three general categories: (1) finite difference, (2) finite element, and (3) finite volume. In their pioneering work, McCallion et al. [68] compared the results of a numerical thermohydrodynamic model of a finite-length journal bearing against isothermal and adiabatic models under different operating conditions. Ezzat and Rohde [69] have represented a three-dimensional thermohydrodynamic model to study the effect of bearing geometry, lubricant type, and inlet temperature on the performance of steadily loaded finite inclined slider bearings. The model numerically solved the fluid film momentum, continuity, and energy equations along with the heat conduction equations for the bearing solids by using finite difference approximation. Majumdar [70] studied the effect of temperature on lubricant viscosity and density by numerically solving the two-dimensional energy equation and Reynolds equation simultaneously. Suganami and Szeri [71] proposed a thermohydrodynamic model for journal bearings that is valid in both laminar and super laminar flow regimes. The model equations were solved by using finite difference method. Crosby [72] developed a thermohydrodynamic model, including heat conduction equation for the solids, to investigate the effect of temperature variation across the fluid film thickness on the performance of finite length journal bearings. Smith and Tichy [73] presented an analytical solution for the two-dimensional thermal model for a journal bearing under iso-viscous conditions. The effect of the convection and dissipation in the film were incorporated into the Peclet number. Ferron et al. [74] studied the theoretical and experimental thermohydrodynamic analysis of finite length plain journal bearings, including the heat transfer between the fluid film and both the shaft and the bush, the lubricant cavitation, and lubricant recirculation. Khonsari and Beaman [75] have investigated the thermohydrodynamics of journal bearings operating under steady-state loading conditions. An analytical model
including a finite difference model for Reynolds equation, energy equation, and Laplace heat conduction equations, including correction factors for the lubricant cavitation effects and the mixing of the supply oil and the recirculating oil was developed. Conservative solutions for the film temperature were obtained by assuming an isothermal shaft temperature and adiabatic boundary conditions on the bush inner surface, which resulted in significant computational efficiency. Boncompain et al. [76] proposed a general thermohydrodynamic model, including cavitation in the film, the lubricant recirculation, and the reversed flow at the oil inlet, which solved the generalized Reynolds equation, the energy equation in the fluid film, and the heat transfer in the bush and the shaft. Khonsari and Esfahanian [77] have extended the application of thermohydrodynamic theory to incorporate the effect of solid particles in hydrodynamically lubricated journal bearings. Gero and Ettles [78] proposed a finite element procedure to solve the thermohydrodynamic lubrication problem. The proposed method allowed for iterative solutions of the system of equations for the temperature field, by formulating the discrete equations either by using the parabolic form of energy equation along with appropriate backward differencing, or by using upwinding with the elliptic form of the energy equation. Colynuck and Medley [79] compared finite difference methods, including control volume and Taylor series for numerical analysis of thermohydrodynamic lubrication of a plane inclined surface bearing with infinite length. The results of the analysis demonstrated the similar accuracy and computational efficiency of the two numerical techniques. Han and Paranjpe [80] represented a rigorous thermohydrodynamic model for finite length journal bearings by using a fully conservative finite volume formulation to numerically solve Reynolds equation and energy equation simultaneously. The lubricant cavitation, reverse flow, and recirculating flow were included in the analysis. Khonsari and Wang [81] developed an extended thermoelastohydrodynamic model that included provisions for the shaft thermal dilation as well as the bush thermoelastic deformation and proposed a numerical solution combining the finite difference and finite element methods to solve the model equations. Furthermore, they [82] presented the transient thermohydrodynamic analysis of the lubricating film in a journal bearing, where a simplified energy equation was developed by integrating an assumed form of the temperature distribution along the film thickness. The proposed equation provided the mean film temperature subjected to an adiabatic boundary condition at the oil-bush interface and a boundary condition that imposed a uniform circumferential temperature at the oil-shaft interface. Paranjpe and Han [83] developed
a detailed thermohydrodynamic model for steadily loaded journal bearings, including Elrod cavitation algorithm, three-dimensional energy equation in the oil film, heat conduction equations in the journal and bushing, and mixing of supply and recirculating lubricants. A coupled approach was developed to calculate the temperature distribution in the film and bushing, eliminating the need for iterative solutions between the two domains. Ju and Wang [84] proposed a thermohydrodynamic model for a finite-length hydrodynamic journal bearing with a non-Newtonian lubricant, obeying the power-law model. They incorporated Elrod cavitation algorithm into Reynolds equation. Furthermore, a perturbation technique was incorporated to facilitate the solution for the equation of motion and the integro-differential energy equation. Paranjpe and Han [85] developed a comprehensive transient thermohydrodynamic model for dynamically loaded journal bearings and studied the effect of different time scales for the oil film and moving grids in the oil film on the thermal calculations. Elrod mass conserving cavitation algorithm along with the energy equation in the lubricant film and the heat conduction equation in the bushing were considered in the study. Furthermore, the journal was treated as a lumped thermal element. Khonsari et al. [86] have developed design charts for journal bearings based on the results of extensive amounts of thermohydrodynamic simulations. The generalized results provided reliable predictions of the maximum bearing temperatures as well as a rapid procedure that allows convenient investigation of bearing performance with realistic thermal considerations. Paranjpe [87] provided a comprehensive transient thermohydrodynamic analysis for dynamically loaded journal bearings, including Elrod lubricant cavitation algorithm and Laplace heat conduction in the bush. The journal was treated as a lumped thermal element at uniform temperature. Furthermore, a simplified thermal analysis which used a single effective oil-film temperature was developed. The results of the analysis demonstrated the considerable variation of lubricant temperature over time and space. Fillon and Khonsari [88] represented a generalized thermohydrodynamic analysis that incorporated two temperature-rise parameters that directly provide the maximum and average dimensionless temperature of tilting pad journal bearings based on the ISOADI (i.e. adiabatic condition at the film/pad interface) boundary conditions. The results of the simulations were incorporated into design charts which allowed the prediction of the maximum temperature and a realistic effective temperature of five-show tilting pad bearings. Monmousseau et al. [89] have developed a two-dimensional transient thermohydrodynamic model to study the effect of thermal expansions on the bearing deformation
with application to bearing seizure. The study considered the energy equation in the fluid film as well as the heat conduction equations in the shaft and the bushing. Wilson [90] applied the method of lumped variables to explore the thermohydrodynamic lubrication for a simple slider bearing, which led to an enhanced understanding of the interplay between the different modes of heat transfer and the influence of the corresponding temperature elevation on the bearing performance. Sehgal et al. [91] provided a comparative theoretical thermohydrodynamic investigation of circular, axial groove, and off-set halves journal bearings. The proposed models considered heat conduction in the bush, however, the heat conduction in the shaft and the recirculation of the oil were neglected. Kucinschi et al. [92] proposed and advanced bi-dimensional model, assuming that the temperature variation is negligible in the axial direction, to calculate the transient temperature field in a bearing. The proposed model provided the pointwise variation of viscosity and the thermo-elastic deformations of both the journal and the bush as well as the detailed study of the ruptured zone of the fluid film. Furthermore, they used finite element method with upwind technique to solve the equations. Pierre and Fillon [93] have developed a three-dimensional thermohydrodynamic analysis that included lubricant rupture and re-formation phenomena by conserving the mass flowrate, to study the influence of geometric and operating parameters, including bearing length, diameter, radial clearance, rotational speed, applied load, and lubricant properties on the behavior of plain journal bearings. Wang and Damodaran [94] described a transient two-dimensional thermohydrodynamic model, which investigated a dynamically loaded journal bearing. Furthermore, a single domain pseudospectral method combining Fourier expansions and Chebyshev polynomials for spatial discretization were introduced and integrated with the corresponding time marching technique to solve the unsteady flow equations. Chang et al. [95] proposed a thermoelastohydrodynamic lubrication analysis to study the static performance of tilting pad journal bearings. The numerical solution incorporated the Newton-Raphson method to calculate the hydrodynamic pressure and a sequential seeping method to solve the three-dimensional energy equation for the fluid under each pad and the three-dimensional heat conduction for the pads. Furthermore, the elastic deformation and the thermal expansion of each pad was determined by employing isoparametric finite element method. Costa et al. [96] developed a two-dimensional thermohydrodynamic model to study the influence of oil supply conditions, including oil supply temperature, supply pressure, groove length, and groove location on the performance of journal bearings. Elrod
cavitation algorithm was incorporated into Reynolds equation and finite difference approximation was adopted to numerically solve the energy equation and the heat conduction equation in the bush. Bouyer and Fillon [97] have represented a three-dimensional thermoelastohydrodynamic model to investigate the influence of global and local thermal effects as well as mechanical and thermal deformations on the performance of plain journal bearings. The generalized Reynolds equation, the energy equation, and the Laplace equation in the bushing was solved iteratively by using finite difference method. Pierre et al. [98] have represented a three-dimensional numerical thermohydrodynamic model, including lubricant film rupture and reformation phenomenon by conserving mass flow rate, to predict the performance of plain journal bearings under steady-state conditions. Chun [99] has developed a two-dimensional thermohydrodynamic model to study the effects of variable density and variable specific heat on maximum pressure, maximum temperature, bearing load, frictional loss, and side leakage in high-speed journal bearings. Peng and Khonsari [100] have developed a three-dimensional thermohydrodynamic model to compute the temperature field in an air-lubricated compliant foil journal bearing. Zengeya et al. [101] presented a three-dimensional thermohydrodynamic model for slider bearings, which coupled Reynolds equation and energy equation in finite-element program Sepran, by using streamline upwind Petrov-Galerkin method. Sharma and Pandey [102] described two different temperature approximation profiles, namely Legendre polynomial and parabolic polynomial across the film thickness in energy equation. The validity of the temperature profile approximation was examined for thermohydrodynamic analysis of infinitely wide slider bearings. Zengeya et al. [103] have adopted streamline upwind Petrov-Galerkin finite element method to develop a three-dimensional thermohydrodynamic model for plain journal bearings. Furthermore, they have provided the detailed formulation for the energy equation in a dimensionless form that is applicable to a journal bearing that is mapped by unwrapping. Singh et al. [104] represented a theoretical steady-state thermohydrodynamic analysis of an axial groove journal bearing by simultaneously solving Reynolds equation, energy equation, and the heat conduction equation in the bush and the shaft by employing an iterative numerical technique. Roy [105] has compared the thermohydrodynamic analysis of an axial grooved oil journal bearing at five different feeding locations. The Reynolds equation was simultaneously solved with the lubricant energy equation and the heat conduction equations in the shaft and bushing to determine the pointwise temperature field. Boubendir et al. [106] have developed a
numerical model for the thermohydrodynamic analysis of porous circular finite-length journal bearings with end seals. Zhang et al. [107] described the influence of the thermal boundary conditions on the thermohydrodynamic analysis of plain journal bearings. The proposed THD model incorporated the three-dimensional energy equation and the heat conduction equation in the solids as well as a mass-conserving Elrod cavitation model. Daniel and Cavalca [108] have developed a thermohydrodynamic model to study the thermal effects on the performance of tilting pad bearings. The generalized Reynolds equation and the energy equation were simultaneously solved by employing finite volume method. Chauhan et al. [109] proposed a thermohydrodynamic model incorporating parabolic temperature profile approximation to investigate the temperature profile in the fluid film for circular and offset-halves journal bearing for three different oil grades. Laukiavich et al. [110] studied the effect of thermal expansion on the clearance of a hydrodynamic bearing up to seizure. The three-dimensional Navier-Stokes equations and a fully termed energy equation along with the three-dimensional heat conduction and elasticity equations for the bushing and journal were considered. Furthermore, the effect of viscosity variation and lubricant cavitation were integrated into the model.

The preceding studies assume that the effect of lubricant inertia on the fluid film reaction forces in SFDs is negligible and either use the complete or the approximate (i.e. long bearing and short bearing approximations) Reynolds equation to represent the SFD dynamics, where it is assumed that the inertial SFD forces are negligible relative to the viscous forces (i.e. \( Re \approx 0 \)) [3]. In general, the fluid inertia effect improves the damping capacity of SFDs, increasing the tangential component and reducing the radial component of the fluid film reaction forces [25], [26], [31], [111], [112]. In his early studies, Safar [113] developed a semi-analytical investigation of the influence of fluid inertia and the effects of convection and dissipation on the performance of an infinitely long bearing. The effect of fluid inertia was incorporated into the calculations by assuming that the shape of the velocity profiles was not strongly influenced by fluid inertia. Subsequently, momentum approximation was applied to the equations. Yang et al. [114], [115] have proposed a bulk-flow thermohydrodynamic model for prediction of static and dynamic performance characteristics of turbulent-flow, process-liquid, for hydrostatic journal bearings. The effect of lubricant fluid inertia was included in the analysis. Furthermore, flow turbulence was incorporated through turbulence shear parameters based on friction factors derived from
Moody’s equations. Additionally, a finite difference scheme was implemented to solve the bulk-flow equations. Gandjalikhan Nassab and Moayeri [116] presented a thermohydrodynamic analysis for a finite length axially grooved fluid film journal bearing. A numerical model was developed to solve the full three-dimensional Navier-Stokes equation, including fluid inertia, coupled with the energy equation for the lubricant and the heat conduction equations for the bushing and the shaft. The governing equations were transformed in the computational domain by generating an orthogonal grid by applying conformal mapping. Furthermore, the transformed equations were discretized by using the control volume method and solved by employing the semi-implicit SAMPLE algorithm. Shyu et al. [117] proposed a thermohydrodynamic model incorporating the Legendre collocation method, the bulk-flow model, and the Elrod cavitation algorithm. The Legendre collocation method was implemented to solve the momentum equations, including fluid inertia effects, and the energy equation to determine the velocity components and the temperature distribution, while the pressure distribution was solved by using bulk-flow model and by adopting the SIMPLER scheme. The proposed model is applicable to both laminar and turbulent flow regimes. Arab Solghar and Gandjalikhan Nassab [118] proposed a thermohydrodynamic solution for a finite-width journal bearing with single axial groove on the crown, including turbulence and fluid inertia effects.

1.2 Research Objectives

The objectives of this thesis can be classified into three categories:

1. This work studies the effect of lubricant film inertia on the dynamic performance of squeeze film dampers. Firstly, the fluid film equation of motion, including the continuity equation and the Navier-Stokes momentum transport equations are developed and reduced by applying thin film assumptions. Subsequently both momentum approximation and perturbation methods are implemented to develop analytical expressions for the fluid velocity components, hydrodynamic pressure distribution, and fluid film reaction forces in presence of the lubricant inertia effects, for different SFD operating conditions, including small amplitude motions and large amplitude motions. The solution for the proposed expressions is either determined by developing appropriate numerical procedures, or by applying analytical tools to provide closed-form solutions. An important contribution of this work relative to the
conventional force coefficient technique is that the proposed inertial models directly compute the fluid pressure distribution and fluid velocity components, which are essential for investigating the effect of SFD fluid inertia on thermohydrodynamics and cavitation modeling for the fluid film. Additionally, the proposed models are developed directly for finite-length bearings, which significantly enhances the accuracy of the predictions relative to the typical limiting geometry models.

2. This work develops a finite-element based rotordynamic model for multi-mass flexible rotors. The proposed SFD models are incorporated into this rotordynamic model to investigate the effect of SFD fluid inertia on the steady-state unbalance induced vibration amplitudes and transient orbits of high-speed rotors incorporating SFDs. In the past, the effect of fluid inertia has been either completely neglected in the rotordynamic analysis or it is only represented for rigid and simple flexible models. An important advantage of the proposed rotordynamics model is that it provides an enhanced and realistic prediction of the SFD behavior in high-speed rotors.

3. This work represents a comprehensive thermohydrodynamic model in the presence of fluid inertia to investigate the thermal effects on the dynamic performance of SFDs. Relative to the conventional isothermal SFD models, including the thermal effects improves the accuracy of the SFD predictions, especially at high-speed applications, where the viscous dissipation is considerably larger. Furthermore, this work is among the pioneering works, which incorporated both the fluid inertia effects and the thermal effects into the thermohydrodynamic analysis.

1.3 Thesis Organization

Chapter 2 investigates the effect of lubricant inertia on the fluid velocity components, lubricant pressure distribution, and fluid film reaction forces. Several analytical tools are incorporated to solve the thin film equations and to develop expressions for the fluid velocities, pressure distribution, and lubricant forces. Furthermore, numerical procedures and closed form solutions are developed for the proposed expressions. Additionally, the proposed solution techniques are incorporated into simulation models to study the effect of different SFD operating parameters,
including journal eccentricity ratio, slenderness ratio, and Reynolds number (i.e. inertia effects) on the dynamic performance of SFDs.

Chapter 3 studies the effect of SFD fluid inertia on the steady-state and transient unbalance induced vibrations of high-speed rotors. Firstly, a finite element based flexible multi-mass rotordynamic model is developed. Subsequently, the SFD models that were developed in Chapter 2 are incorporated into the rotordynamic model to represent the thin film damping effects. Finally, simulation results are developed to demonstrate the effect of SFD lubricant inertia on the steady-state and transient vibration amplitudes under different SFD operating conditions, including unbalance magnitudes, fluid inertia components, and SFD radial clearance.

Chapter 4 provides a detailed thermohydrodynamic analysis for SFDs. The hydrodynamic pressure equation as well as the energy equation in the lubricant and the heat conduction equations in the bushing and the shaft are incorporated along with the realistic thermal boundary condition to develop a thermohydrodynamic model that represents the effect of lubricant temperature variation on the performance of SFDs. Furthermore, an iterative numerical procedure is developed to solve the system of partial differential equations. Additionally, the effect of fluid inertia is incorporated into the thermal calculations by using a first order inertia correction for the pressure distribution.
Chapter 2
The Effect of Fluid Inertia

The dynamic behavior of a viscous Newtonian fluid within boundaries is generally characterized by using the three-dimensional continuity and Navier-Stokes equations as follows [3]:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0, \tag{2.1}
\]

\[
\rho \left( \frac{\partial V}{\partial t} + V \cdot \nabla V \right) = -\nabla P + \nabla \cdot (\mu \nabla V) + \nabla \left( -\frac{2\mu}{3} \nabla V \right) + \rho g, \tag{2.2}
\]

where Equation (2.1) is the continuity equation corresponding to the conservation of mass within the fluid boundaries; and Equation (2.2) corresponds to the conservation of momentum within the fluid boundaries. The terms in Equations (2.1) and (2.2) are expanded as follows [119]:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0, \tag{2.3}
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] + \frac{2}{3} \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right] + \frac{2}{3} \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \right] + \rho X, \tag{2.4}
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial w}{\partial x} + \frac{\partial v}{\partial z} \right) \right] + \frac{2}{3} \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \right] + \frac{2}{3} \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right) \right] + \rho Y, \tag{2.5}
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right] + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] + \frac{2}{3} \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} \right) \right] + \frac{2}{3} \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial w}{\partial z} - \frac{\partial v}{\partial y} \right) \right] + \rho Z \tag{2.6}
\]
Furthermore, it is assumed that:

1. The body force terms are small compared to the viscous, inertia, and pressure terms.

2. According to an order of magnitude analysis, the velocity gradients $\frac{\partial u}{\partial y}$ and $\frac{\partial w}{\partial y}$ are large compared to all other velocity gradients.

3. The lubricant is Newtonian, incompressible (i.e. density gradient is zero), and iso-viscous (i.e. the viscosity gradient is zero).

Applying the above assumptions to equations (2.3) to (2.6) gives:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$  \hspace{1cm} (2.7)

$$\rho \left\{ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right\} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2},$$  \hspace{1cm} (2.8)

$$\rho \left\{ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right\} = -\frac{\partial P}{\partial y} + \mu \frac{\partial^2 v}{\partial y^2},$$  \hspace{1cm} (2.9)

$$\rho \left\{ \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right\} = -\frac{\partial P}{\partial z} + \mu \frac{\partial^2 w}{\partial y^2}.$$  \hspace{1cm} (2.10)

Additionally, according to the thin film assumption in hydrodynamic lubrication theory, which is characterized by the small ratio of film thickness to the bearing’s other physical dimensions, i.e. $c \ll R$, it is concluded that [3]:

1. The effect of the curvature of the film is negligible; hence a linear coordinate system is used to describe the lubricant dynamics.

2. The variation of the pressure across the film is negligible (i.e. $\partial P/\partial y = 0$).

The SFD configuration in this work is a symmetric damper about its mid-plane with open ends (i.e. no seal). The geometry of the system is represented in Figure 2.2. An orthogonal Cartesian coordinate system $\{x,y,z\}$ is fixed in the plane of the lubricant, where the $z$-axis is perpendicular to the plane of motion. Furthermore, an orthogonal Cartesian system $\{x',y',z'\}$ translating with angular velocity $R\Omega$ is introduced, where the $x'$-axis is perpendicular to the line connecting the
centers of the inner and outer cylinders, and the $y'$-axis is in the direction of the minimum thickness. The angle $\theta'$ starts from the origin of the fixed Cartesian system and the angle $\theta$ is measured at the maximum film thickness in the direction of the whirling motion. Finally, a fixed inertial coordinate system $\{X,Y\}$ is defined at the center of the bearing. Based on the preceding description, for an incompressible and iso-viscous lubricant, the flow equations in the SFD are reduced to:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.11}
\]

\[
\rho \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right\} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}, \tag{2.12}
\]

\[
\frac{\partial P}{\partial y} = 0, \tag{2.13}
\]

\[
\rho \left\{ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right\} = -\frac{\partial P}{\partial z} + \mu \frac{\partial^2 w}{\partial y^2}. \tag{2.14}
\]
Additionally, the dimensionless velocity boundary conditions for the lubricant are defined as follows:

\[
\begin{align*}
&u = 0, \quad v = 0, \quad w = 0 \quad y = 0 \\
&u = 0, \quad v = \frac{\partial h}{\partial t}, \quad w = 0 \quad y = h,
\end{align*}
\]

where:

\[
h = c + e \cos(\theta). \tag{2.16}
\]

Subsequently, in order to demonstrate the dependence of the fluid inertia terms on the squeeze Reynolds number, the flow equations are normalized by introducing dimensionless parameters as follows:

\[
\begin{align*}
\theta &= \Theta - \phi = \frac{x}{R} - \Omega t, \quad \eta = \frac{y}{c}, \quad \xi = \frac{z}{R}, \quad \tau = \Omega t, \\
\overline{u} &= \frac{u}{R \Omega}, \quad \overline{v} = \frac{v}{c \Omega}, \quad \overline{w} = \frac{w}{R \Omega}, \quad \overline{F} = \frac{F c^2}{\mu \Omega R^4}, \\
\overline{P} &= \frac{P c^2}{R^2 \Omega \mu}, \quad H = \frac{h}{c} = 1 + e \cos \theta, \quad \text{Re} = \frac{\rho \Omega c^2}{\mu}.
\end{align*}
\]

The dimensionless parameters in Equation (2.17) are substituted into Equations (2.11), (2.12), and (2.14) as follows:

\[
\begin{align*}
&\frac{\partial \overline{u}}{\partial \theta} + \frac{\partial \overline{v}}{\partial \eta} + \frac{\partial \overline{w}}{\partial \xi} = 0, \tag{2.18} \\
&\text{Re} \left\{ \frac{\partial \overline{u}}{\partial \tau} + u \frac{\partial \overline{u}}{\partial \theta} + v \frac{\partial \overline{u}}{\partial \eta} + w \frac{\partial \overline{u}}{\partial \xi} \right\} = -\frac{\partial \overline{P}}{\partial \theta} + \frac{\partial^2 \overline{u}}{\partial \eta^2}, \tag{2.19} \\
&\text{Re} \left\{ \frac{\partial \overline{w}}{\partial \tau} + u \frac{\partial \overline{w}}{\partial \theta} + v \frac{\partial \overline{w}}{\partial \eta} + w \frac{\partial \overline{w}}{\partial \xi} \right\} = -\frac{\partial \overline{P}}{\partial \xi} + \frac{\partial^2 \overline{w}}{\partial \eta^2}. \tag{2.20}
\end{align*}
\]
Additionally, the dimensionless velocity boundary conditions for the lubricant are defined as follows:

\[
\begin{align*}
\tilde{u} = 0, \quad \tilde{v} = 0, \quad \tilde{w} = 0 & \quad \eta = 0 \\
\quad \tilde{u} = 0, \quad \frac{\partial H}{\partial \tau}, \quad \tilde{w} = 0 & \quad \eta = H.
\end{align*}
\] (2.21)

Moreover, the boundary conditions for the hydrodynamic pressure in an open ended SFD are given by:

1. The pressure is periodic and continuous in the circumferential direction (\(\theta\)), i.e.
   \[\bar{P}(\theta, \xi) = \bar{P}(\theta + 2\pi, \xi)\]
2. The pressure equals atmospheric pressure at the axial ends of the bearing, i.e.
   \[\bar{P}(\theta, L/D) = \bar{P}(\theta, -L/D) = 0\]
3. The hydrostatic pressure must be above the liquid cavitation pressure, i.e.
   \[\bar{P} \geq \bar{P}_{cav} \quad 0 \leq \theta \leq 2\pi, -L/D \leq \xi \leq L/D\]

where \(\bar{P}_{cav}\) is the saturation pressure of the lubricant or the saturation pressure for release of entrapped gases, typically ambient pressure.

### 2.1 Reynolds Equation

Assuming that the effect of fluid inertia is negligible relative to the viscous effects, Equations (2.18) to (2.20) are reduced to:

\[
\frac{\partial \tilde{u}}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \eta} + \frac{\partial \tilde{w}}{\partial \xi} = 0,
\] (2.22)

\[
-\frac{\partial \bar{P}}{\partial \theta} + \frac{\partial^{2} \tilde{u}}{\partial \eta^{2}} = 0,
\] (2.23)

\[
-\frac{\partial \bar{P}}{\partial \xi} + \frac{\partial^{2} \tilde{w}}{\partial \eta^{2}} = 0.
\] (2.24)
Subsequently the momentum transport equations are integrated in the radial direction to determine the axial and circumferential velocity components. Integrating Equation (2.23) twice in the radial direction gives:

\[ u(\theta, \eta, \xi) = \frac{1}{2} \eta^2 \frac{\partial \bar{P}}{\partial \theta} + C_1 \eta + C_2, \]  

(2.25)

where the integration constants are determined by applying the velocity boundary conditions in Equation (2.21):

\[ u(\theta, \eta, \xi) = \frac{1}{2} \frac{\partial \bar{P}}{\partial \theta} \left( \eta^2 - \eta H \right). \]  

(2.26)

Similarly, the velocity in the axial coordinate is computed:

\[ w(\theta, \eta, \xi) = \frac{1}{2} \frac{\partial \bar{P}}{\partial \xi} \left( \eta^2 - \eta H \right). \]  

(2.27)

Substituting Equations (2.26) and (2.27) into Equation (2.22) gives:

\[ \frac{\partial v}{\partial \eta} = - \left( \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \xi} \right) = - \frac{1}{2} \left[ \frac{\partial^2 \bar{P}}{\partial \theta^2} \left( \eta^2 - \eta H \right) - \eta \frac{\partial P}{\partial \theta} \frac{\partial H}{\partial \theta} + \frac{\partial^2 \bar{P}}{\partial \xi^2} \left( \eta^2 - \eta H \right) \right]. \]  

(2.28)

Equation (2.28) is integrated in the radial direction as follows:

\[ v = \frac{-1}{2} \frac{\partial^3 \bar{P}}{\partial \theta^3} \left( \frac{\eta^3}{3} \right) + \frac{1}{2} \frac{\partial \bar{P}}{\partial \xi} \frac{\partial H}{\partial \theta} \left( \frac{\eta^2}{2} \right) - \frac{1}{2} \frac{\partial^2 \bar{P}}{\partial \xi^2} \left( \frac{\eta^3}{3} \right) - \frac{H \eta^2}{2} + C_3. \]  

(2.29)

Subsequently, applying the velocity boundary condition on the bush surface gives:

\[ v = \frac{-1}{2} \frac{\partial^3 \bar{P}}{\partial \theta^3} \left( \frac{\eta^3}{3} \right) - \frac{H \eta^2}{2} + \frac{1}{2} \frac{\partial \bar{P}}{\partial \theta} \frac{\partial H}{\partial \theta} \left( \frac{\eta^2}{2} \right) - \frac{1}{2} \frac{\partial^2 \bar{P}}{\partial \xi^2} \left( \frac{\eta^3}{3} \right) - \frac{H \eta^2}{2}. \]  

(2.30)

Furthermore, the velocity boundary condition on the journal surface gives:
\[ \frac{\partial}{\partial \theta} \left( H^3 \frac{\partial P}{\partial \theta} \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial P}{\partial \xi} \right) = 12 \frac{\partial H}{\partial \tau}. \] 

Equation (2.31) is referred to as Reynolds equation. Reynolds equation characterizes the hydrodynamic lubricant distribution in SFDs in absence of fluid inertia. In order to determine the fluid film reaction forces, the pressure distribution is integrated over the journal surface as follows:

\[ \frac{F_r}{F_t} = \int_{-L/D}^{L/D} \int_{-\theta}^{\theta} P(\theta, \xi) \left[ \cos(\theta) \sin(\theta) \right] d\theta d\xi. \] 

(2.32)

An analytical solution to Reynolds equation does not exist for arbitrary geometry SFD geometries and generally, the solution to the Reynolds equation can be determined by using numerical methods, once appropriate boundary conditions are assigned to the system [120]. However, the analytical solution to the Reynolds equation can be determined for approximated expressions of the Reynolds equation (i.e. limiting geometries of journal bearings), the most well recognized of which are the short bearing approximation (SBA) and the long bearing approximation (LBA) [3].

2.1.1 Short Bearing Approximation (SBA)

The short bearing approximation [3] corresponds to the limiting hydrodynamic bearing geometry condition \( L/D \gg 1 \), where the length of the bearing is considered very short relative to the diameter of the bearing. The short bearing condition is widely adopted for the analysis of bearings with no end seals (i.e. open ends) and \( L/D \leq 0.5 \). Consequently, the circumferential flow is negligible and the pressure profile is constant along the circumferential direction (i.e. \( \partial P/\partial \theta = 0 \)). This limiting configuration reduces Equation (2.32) to:

\[ \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial P}{\partial \xi} \right) = 12 \frac{\partial H}{\partial \tau}. \] 

(2.33)

The SBA is widely used for accelerated estimation of pressure distribution and fluid film reaction forces in journal bearings with slenderness ratio of \( L/D \leq 0.5 \) and for small values of
journal eccentricity ratio. Direct integration of Equation (2.33) in the axial direction gives the pressure distribution:

\[ \overline{P} = \left( \frac{12}{H^3} \frac{\partial H}{\partial \tau} \right) \frac{\xi^2}{2} + C_1\xi + C_2. \]  

(2.34)

The integration constants are determined by applying the pressure boundary conditions at the axial ends of the bearing:

\[ \overline{P} = \left( \frac{6}{H^2} \frac{\partial H}{\partial \tau} \right) \left[ \xi^2 - \left( \frac{L}{D} \right)^2 \right]. \]  

(2.35)

Equation (2.35) provides a closed-form lubricant pressure expression for short SFDs. Assuming that the SFD executes CCOs, the radial velocity and acceleration of the journal center are neglected and the time derivative of the dimensionless fluid film thickness is given as follows:

\[ \frac{\partial H}{\partial \tau} = \varepsilon \sin \theta \]  

(2.36)

Substituting Equation (2.36) into Equation (2.35) gives:

\[ \overline{P} = \frac{6 \varepsilon \sin \theta}{(1 + \varepsilon \cos \theta)^3} \left[ \xi^2 - \left( \frac{L}{D} \right)^2 \right]. \]  

(2.37)

Subsequently, the fluid film reaction forces are determined by integrating the pressure expression in Equation (2.35) over the journal surface as follows:

\[
\begin{align*}
\overline{F}_r &= 6 \varepsilon \int_0^{\theta_1} \sin \theta \left[ \cos(\theta) \right] d\theta \int_{-L/D}^{L/D} \left[ \xi^2 - \left( \frac{L}{D} \right)^2 \right] d\xi \\
&= -8 \varepsilon \left( \frac{L}{D} \right)^3 \int_0^{\theta_1} \sin \theta \left[ \cos(\theta) \right] d\theta.
\end{align*}
\]  

(2.38)

Booker [121] has defined the following bearing integrals:
Substituting Equation (2.39) into Equation (2.38) gives a closed-form expression for the fluid film reaction forces as follows:

$$\begin{bmatrix} F_r \\ F_i \end{bmatrix} = -8E \left( \frac{L}{D} \right)^3 \begin{bmatrix} J_{11}^3 \\ J_{32}^3 \end{bmatrix}. \tag{2.40}$$

### 2.1.2 Long Bearing Approximation (LBA)

Long bearing approximation [3] corresponds to the limiting hydrodynamic bearing geometry condition $L/D \gg 1$, where the length of the bearing is considered very long relative to the diameter of the bearing. The long bearing condition is widely adopted for the analysis of bearings with $L/D \geq 2$ and tightly sealed bearings, even at small slenderness ratios. Consequently, the pressure distribution in the circumferential direction is effectively very small (i.e. $\partial P/\partial \xi = 0$), reducing Equation (2.31) to:

$$\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial P}{\partial \theta} \right) = 12 \frac{\partial H}{\partial \tau}, \tag{2.41}$$

where:

$$\frac{\partial H}{\partial \tau} = \frac{\partial \theta}{\partial \tau} \frac{\partial H}{\partial \theta} = -\frac{\partial H}{\partial \theta}. \tag{2.42}$$

Substituting Equation (2.42) into Equation (2.41) gives:

$$\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial P}{\partial \theta} \right) = -12 \frac{\partial H}{\partial \theta}. \tag{2.43}$$

Integrating Equation (2.41) along the circumferential direction gives:

$$P = -12 \int \frac{d\theta}{H^2} + C_3 \int \frac{d\theta}{H^3}. \tag{2.44}$$
The above equation is solved by applying the Sommerfeld’s transformation \[122\], since a direct integration leads to discontinuity in the solution. According to the Sommerfeld’s transformation:

\[
1 + \epsilon \cos \theta = \frac{1 - \epsilon^2}{1 - \epsilon \cos \varphi},
\]

where the range \( \theta = 0 - 2\pi \) corresponds to the same range \( \varphi = 0 - 2\pi \), which facilitates a straightforward application of the boundary conditions. Hence:

\[
\cos \theta = \frac{\cos \varphi - \epsilon}{1 - \epsilon \cos \varphi},
\]

\[
\sin \theta = \frac{\sin \varphi \sqrt{1 - \epsilon^2}}{1 - \epsilon \cos \varphi},
\]

\[
d\theta = \frac{\sqrt{1 - \epsilon^2}}{1 - \epsilon \cos \varphi} d\varphi.
\]

Consequently:

\[
\int \frac{d\theta}{(1 + \epsilon \cos \theta)^2} = \frac{1}{(1 - \epsilon^2)^{3/2}} (\varphi - \epsilon \sin \varphi), \quad (2.47)
\]

\[
\int \frac{d\theta}{(1 + \epsilon \cos \theta)^3} = \frac{1}{(1 - \epsilon^2)^{5/2}} \left( \varphi - 2\epsilon \sin \varphi + \frac{\epsilon^2 \varphi}{2} + \frac{\epsilon^2}{4} \sin 2\varphi \right). \quad (2.48)
\]

Substituting Equations (2.47) and (2.48) into Equation (2.44) gives:

\[
\overline{P} = -\frac{12}{(1 - \epsilon^2)^{3/2}} (\varphi - \epsilon \sin \varphi) + \frac{C_1}{(1 - \epsilon^2)^{5/2}} \left( \varphi - 2\epsilon \sin \varphi + \frac{\epsilon^2 \varphi}{2} + \frac{\epsilon^2}{4} \sin 2\varphi \right) + C_2. \quad (2.49)
\]

The integration constants are calculated by applying Summerfield boundary conditions:

\[
\overline{P}(\theta = 0) = 0,
\]

\[
\overline{P}(\theta = 2\pi) = 0, \quad (2.50)
\]
hence:

\[
\bar{P}(\theta) = -12 \varepsilon \left(1 + H\right) \sin \theta \left(2 + \varepsilon^2\right) H^2.
\]  
\[\text{(2.51)}\]

Subsequently, the fluid film reaction forces are determined by integrating the pressure expression in Equation (2.51) over the journal surface as follows:

\[
\left[ \frac{F_r}{F_t} \right] = -12 \varepsilon \left(1 + H\right) \sin \theta \left[ \frac{\cos(\theta)}{2 + \varepsilon^2} H^2 \left(2 + \varepsilon^2\right) \right] d\theta.
\]  
\[\text{(2.52)}\]

Substituting the Booker’s integrals in Equation (2.39) into Equation (2.52) gives a closed-form expression for the fluid film reaction forces as follows:

\[
\left[ \frac{F_r}{F_t} \right] = -12 \varepsilon \left(1 + H\right) \sin \theta \left[ J_2^{11} + J_1^{11} \right].
\]  
\[\text{(2.53)}\]

### 2.1.3 Analytical Solution

This section develops an approximate closed-form analytical solution for Reynolds equation by using the classical separation of variables technique. Firstly, it is assumed that the dimensionless fluid film pressure is separated into the sum of a homogenous solution and a particular solution:

\[
\bar{P}(\theta, \xi) = \bar{P}_H(\theta, \xi) + \bar{P}_F(\theta, \xi).
\]  
\[\text{(2.54)}\]

Starting with the particular solution, the pressure variable is separated into the sum of a circumferential pressure and an axial pressure, which are independent of one another:

\[
\bar{P}_F(\theta, \xi) = \bar{P}_1(\theta) + \bar{P}_2(\xi),
\]  
\[\text{(2.55)}\]

Substituting Equation (2.55) into Equation (2.31) gives:

\[
\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial}{\partial \theta} \left[ \bar{P}_1(\theta) + \bar{P}_2(\xi) \right] \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial}{\partial \xi} \left[ \bar{P}_1(\theta) + \bar{P}_2(\xi) \right] \right) = 12 \frac{\partial H}{\partial \tau}.
\]  
\[\text{(2.56)}\]
After some manipulations, the following expression is achieved:

\[
\frac{1}{H^3} \frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \bar{P}_1(\theta)}{\partial \theta} \right) - \frac{12}{H^3} \frac{\partial H}{\partial \tau} = - \frac{\partial^2 \bar{P}_2(\xi)}{\partial \xi^2},
\]  

(2.57)

where the right-hand side of the above equation is strictly a function of \( \xi \) and the left-hand side is strictly a function of \( \theta \). Since \( \xi \) and \( \theta \) are independent of one another, the only possibility is that both sides of the equation are equal to a constant value:

\[
- \frac{\partial^2 \bar{P}_2(\xi)}{\partial \xi^2} = k,
\]  

(2.58)

\[
\frac{1}{H^3} \frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \bar{P}_1(\theta)}{\partial \theta} \right) - \frac{12}{H^3} \frac{\partial H}{\partial \tau} = k.
\]  

(2.59)

Equation (2.58) is integrated twice to obtain the solution:

\[
\bar{P}_2(\xi) = k \frac{\xi^2}{2} + k_1 \xi + k_2.
\]  

(2.60)

Assuming that \( k = 0 \) and applying the pressure boundary conditions at the axial ends of the bearing gives:

\[
k_1 = k_2 = 0,
\]  

(2.61)

and the solution for Equation (2.60) becomes trivial, implying that the particular part of the pressure distribution does not change in the axial direction. Furthermore, Equation (2.59) is equivalent to the LBA, consequently:

\[
\bar{P}_p(\theta) = -12 \frac{\varepsilon (1 + H) \sin \theta}{\left(2 + \varepsilon^2\right) H^2}.
\]  

(2.62)
Subsequently, the homogenous solution is determined by separating the solution into the products of a function of the circumferential coordinate and a function of the axial coordinate as following:

\[ \overline{P}_H(\theta, \xi) = \overline{P}_3(\theta) \overline{P}_4(\xi). \]  

(2.63)

Substituting Equation (2.63) into the homogeneous equation gives:

\[ \overline{P}_4(\xi) \frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \overline{P}_3(\theta)}{\partial \theta} \right) + H^3 \overline{P}_3(\theta) \frac{\partial}{\partial \xi} \left( \frac{\partial \overline{P}_4(\xi)}{\partial \xi} \right) = 0. \]  

(2.64)

It is assumed that the circumferential pressure distribution component is equivalent to the LBA pressure distribution, since \( P_3(\theta) \) is strictly a function of the circumferential coordinate:

\[ \overline{P}_3(\theta) = -12 \frac{\varepsilon (1 + H) \sin \theta}{(2 + \varepsilon^2) H^2}. \]  

(2.65)

Subsequently, Equation (2.64) is expanded and the homogeneous solution is determined:

\[ \overline{P}_4(\xi) \frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \overline{P}_3(\theta)}{\partial \theta} \right) = -H^3 \overline{P}_3(\theta) \frac{\partial^2 \overline{P}_4(\xi)}{\partial \xi^2}, \]  

(2.66)

hence:

\[ \frac{\partial^2 \overline{P}_4(\xi)}{\partial \xi^2} = \frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \overline{P}_3(\theta)}{\partial \theta} \right) \frac{1}{\overline{P}_4(\xi)} = \overline{\lambda}, \]  

(2.67)

and:

\[ \frac{\partial^2 \overline{P}_4(\xi)}{\partial \xi^2} + \overline{\lambda} \overline{P}_4(\xi) = 0. \]  

(2.68)

Assuming that \( \overline{\lambda}_i = -\kappa_i^2 \) gives:
\[ \frac{\partial^2 \bar{P}_4(\xi)}{\partial \xi^2} - \kappa_1^2 \bar{P}_4(\xi) = 0. \]  

(2.69)

Consequently:

\[ \bar{P}_4(\xi) = C_1 e^{\kappa_1 \xi} + C_2 e^{-\kappa_1 \xi}, \]  

(2.70)

and:

\[ \bar{P}_H(\theta, \xi) = -12 \frac{\varepsilon (1 + H) \sin \theta}{(2 + \varepsilon^2) H^2} \left[ C_1 e^{\kappa_1 \xi} + C_2 e^{-\kappa_1 \xi} \right]. \]  

(2.71)

The total solution for the pressure distribution is determined by accumulating the homogenous and particular solutions:

\[ \bar{P}(\theta, \xi) = -12 \frac{\varepsilon (1 + H) \sin \theta}{(2 + \varepsilon^2) H^2} \left[ 1 + C_1 e^{\kappa_1 \xi} + C_2 e^{-\kappa_1 \xi} \right]. \]  

(2.72)

The integration constants are determined by applying the pressure boundary conditions at the axial ends of the bearing:

\[ \bar{P}(\theta, \xi) = -12 \frac{\varepsilon (1 + H) \sin \theta}{(2 + \varepsilon^2) H^2} \left[ 1 - \frac{e^{\kappa_1 \xi} + e^{-\kappa_1 \xi}}{\frac{\kappa_1 L}{D} + e^{\frac{\kappa_1 L}{D}}} \right], \]  

(2.73)

where:

\[ \cosh x = \frac{e^x + e^{-x}}{2}. \]  

(2.74)

Substituting Equation (2.74) into Equation (2.73) gives:

\[ \bar{P}(\theta, \xi) = -12 \frac{\varepsilon (1 + H) \sin \theta}{(2 + \varepsilon^2) H^2} \left[ 1 - \frac{\cosh \left( \kappa_1 \xi \right)}{\cosh \left( \kappa_1 L/D \right)} \right]. \]  

(2.75)
Furthermore, $\kappa$ is determined by solving the boundary value problem based on the homogeneous solution of the pressure distribution. Tolle and Muster [123] have determined the eigenvalue components for purely viscous fluid flow in the thin film region for tangential and normal motions of the journal center:

$$
\kappa_n = \sqrt{1 + \frac{2\varepsilon^2}{\sqrt{1 - \varepsilon^2}}},
$$

(2.76)

$$
\kappa_t = \sqrt{\frac{2(2 + \varepsilon^2)(1 + \sqrt{1 - \varepsilon^2})}{\sqrt{1 - \varepsilon^2}(4 + 4\sqrt{1 - \varepsilon^2} - \varepsilon^2)}}.
$$

(2.77)

### 2.1.4 Numerical Solution

This section represents a numerical solution for solving Reynolds equation. In order to determine the numerical solution, firstly, a solution domain is defined for the problem. Subsequently, the partial differential pressure expression is discretized over the solution domain by using finite difference approximation. Finally, an iterative Gauss-Seidel numerical algorithm with successive over-relaxation (SOR) is developed to calculate the point-wise pressure distribution in the thin film domain. Firstly, the terms in Equation (2.31) are expanded to facilitate the discretization of the partial differential terms:

$$
\left[ 3H^2 \frac{\partial H}{\partial \theta} \frac{\partial \overline{P}}{\partial \theta} + H^3 \frac{\partial^2 \overline{P}}{\partial \theta^2} \right] + H^3 \frac{\partial^2 \overline{P}}{\partial \xi^2} = 12 \frac{\partial H}{\partial \tau}.
$$

(2.78)

The partial derivatives of the fluid film thickness $H$ are given as follows:

$$
\frac{\partial H}{\partial \theta} = -\varepsilon \sin(\theta),
$$

$$
\frac{\partial H}{\partial \tau} = \dot{\varepsilon} \cos(\theta) + \varepsilon \sin(\theta).
$$

(2.79)

Assuming that the SFD executes CCOs, the radial velocity and acceleration of the journal center become zero and:
\[
\frac{\partial H}{\partial \theta} = -\varepsilon \sin(\theta), \quad (2.80)
\]
\[
\frac{\partial H}{\partial \tau} = \varepsilon \sin(\theta). \quad (2.81)
\]

Substituting Equations (2.80) and (2.81) into Equation (2.78) gives:
\[
\left[ -3\varepsilon H^2 \sin(\theta) \frac{\partial P}{\partial \theta} + H^3 \frac{\partial^2 P}{\partial \theta^2} \right] + H^3 \frac{\partial^2 P}{\partial \xi^2} = 12\varepsilon \sin(\theta). \quad (2.82)
\]

Subsequently, the above equation is discretized based on a finite difference approximation (FDA) technique that uses backward difference approximation for the first order derivative terms and central difference approximation for the second order derivative terms:
\[
\frac{\partial \overline{P}}{\partial \theta} = \frac{\overline{P}(\theta, \xi) - \overline{P}(\theta - \Delta \theta, \xi)}{\Delta \theta}, \quad \text{Equation (2.83)}
\]
\[
\frac{\partial \overline{P}}{\partial \xi} = \frac{\overline{P}(\theta, \xi) - \overline{P}(\theta, \xi - \Delta \xi)}{\Delta \xi}, \quad \text{Equation (2.84)}
\]
\[
\frac{\partial^2 \overline{P}}{\partial \theta^2} = \frac{\overline{P}(\theta + \Delta \theta, \xi) - 2\overline{P}(\theta, \xi) + \overline{P}(\theta - \Delta \theta, \xi)}{\Delta \theta^2}, \quad \text{Equation (2.85)}
\]
\[
\frac{\partial^2 \overline{P}}{\partial \xi^2} = \frac{\overline{P}(\theta, \xi + \Delta \xi) - 2\overline{P}(\theta, \xi) + \overline{P}(\theta, \xi - \Delta \xi)}{\Delta \xi^2}, \quad \text{Equation (2.86)}
\]

Substituting Equations (2.83) to (2.86) into Equation (2.82) provides a discretized expression for Reynolds equation:
\[
\left[ -3\varepsilon H^2 \sin(\theta) \frac{\overline{P}_{i,j} - \overline{P}_{i+1,j}}{\Delta \theta} + H^3 \frac{P_{i+1,j} + \overline{P}_{i-1,j} - 2\overline{P}_{i,j}}{\Delta \theta^2} \right] + H^3 \frac{\overline{P}_{i+1,j} - \overline{P}_{i,j+1}}{\Delta \xi^2} = 12\varepsilon \sin(\theta). \quad (2.87)
\]
The above equation is rearranged as following to solve for the point-wise pressure field:

\[
\bar{P}_{i,j} = \frac{A_1 + [A_1 + A_2] \bar{P}_{i-1,j} + [A_2] \bar{P}_{i+1,j} + [A_3] (\bar{P}_{i,j+1} + \bar{P}_{i,j-1})}{[A_1 + 2A_2 + 2A_3]},
\]

(2.88)

where:

\[
A_1 = -\frac{3H_i^2 (\epsilon \sin \theta_j)}{\Delta \theta},
\]

\[
A_2 = -\frac{H_i^3}{\Delta \theta^2},
\]

\[
A_3 = -\frac{H_i^3}{\Delta \epsilon^2},
\]

\[
A_4 = 12 \epsilon \sin \theta_j.
\]

(2.89)

In general, Reynolds equation is classified as an elliptical partial differential equation (PDE). Assuming that the lubricant is incompressible and iso-viscous, and the journal center executes CCO whirls, the following Gauss-Seidel numerical procedure is used to determine the fluid film pressure distribution for a specified SFD eccentricity ratio:

1. The boundary points are initialized to their prescribed values, and the interior points are adjusted to zero.
2. Equation (2.88) is iteratively solved for the interior points.
3. The iteration is only interrupted when the solution error reaches a convergence criterion.
4. Finally, a successive over-relaxation (SOR) technique is used to accelerate the convergence of the solution:

\[
P_{0_{i,j}}^{(k)} = P_{0_{i,j}}^{(k-1)} + \lambda \left( P_{0_{i,j}}^{(k)} - P_{0_{i,j}}^{(k-1)} \right),
\]

(2.90)

where \(k\) denotes the iteration number. Subsequently, the fluid film reaction force components are computed by numerically integrating the pressure distribution over the journal surface as follows:
\[
\begin{bmatrix}
\ddot{F}_r \\
\ddot{F}_i
\end{bmatrix}
= \sum_{i=1}^{N} \sum_{j=1}^{M} P_{i,j} \begin{bmatrix}
\cos(\theta_i) \\
\sin(\theta_i)
\end{bmatrix} \Delta \theta \Delta \xi.
\] (2.91)

### 2.2 Small Amplitude Motions of the Journal Center

This section studies the fluid inertia effects on the performance of SFDs executing small amplitude motions (SAMs). The assumption of small amplitude motions of the journal center is a very common practice in the turbomachinery industry, which further reduces the SFD flow equations by eliminating the nonlinear convective inertia terms. This model reduction facilitates the investigation of unsteady inertia terms. Tichy [32] has studied the effect of small amplitude CCOs for viscoelastic fluids, showing the significant effect of fluid inertia on the fluid film reaction forces, even at moderate Reynolds numbers. San Andres [5], [124] has further emphasized on the significant effect of fluid inertia on dynamic performance of squeeze film dampers executing small amplitude CCOs by representing the effect of fluid inertia on SFD force coefficients for both central and off-centered motions of the journal center. According to order of magnitude calculations, for small amplitude motions of the journal center, the convective inertia terms in Equations (2.19) and (2.20) are negligible relative to the unsteady (temporal) inertia terms [6]. Therefore, the dimensionless flow equations are reduced to:

\[
\frac{\partial \tilde{u}}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \eta} + \frac{\partial \tilde{w}}{\partial \xi} = 0,
\] (2.92)

\[
\text{Re} \frac{\partial \tilde{u}}{\partial \tau} = -\frac{\partial \tilde{P}}{\partial \theta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2},
\] (2.93)

\[
\text{Re} \frac{\partial \tilde{w}}{\partial \tau} = -\frac{\partial \tilde{P}}{\partial \xi} + \frac{\partial^2 \tilde{w}}{\partial \eta^2}.
\] (2.94)

Subsequently the above system of PDEs is solved to determine the fluid velocity components, hydrodynamic pressure distribution, and fluid film reaction forces.
2.2.1 Linearized Fluid Film Forces

This section develops an analytical closed-form solution for Equations (2.92) to (2.94) by incorporating the classical separation of variables technique. For very small amplitude motions of the journal center, the solution for Equations (10) to (12) is assumed to have the following complex form:

\[
\tilde{u}(\theta, \eta, \xi) = -U(\eta, \xi)e^{i\theta}, \quad (2.95)
\]

\[
\tilde{v}(\theta, \eta, \xi) = -V(\eta, \xi)e^{i\theta}, \quad (2.96)
\]

\[
\tilde{w}(\theta, \eta, \xi) = -W(\eta, \xi)e^{i\theta}, \quad (2.97)
\]

\[
\tilde{P}(\theta, \eta, \xi) = -P_i(\xi)e^{i\theta}. \quad (2.98)
\]

Subsequently, Equations (2.95) to (2.98) are substituted into Equations (2.92) to (2.94). Starting with the circumferential momentum transport equation:

\[
\text{Re} \frac{\partial}{\partial \tau}[-U(\eta, \xi)e^{i\theta}] = -\frac{\partial}{\partial \theta}[-P_i(\xi)e^{i\theta}] + \frac{\partial^2}{\partial \eta^2}[-U(\eta, \xi)e^{i\theta}], \quad (2.99)
\]

hence:

\[
i \text{Re}U(\eta, \xi)e^{i\theta} = iP_i(\xi)e^{i\theta} + e^{i\theta} \frac{\partial^2}{\partial \eta^2}[-U(\eta, \xi)], \quad (2.100)
\]

and:

\[
\frac{\partial^2 U(\eta, \xi)}{\partial \eta^2} + i \text{Re} U(\eta, \xi) = iP_i(\xi). \quad (2.101)
\]

Furthermore, the separation of variables technique is applied to the velocity components as follows:

\[
U(\eta, \xi) = U_1(\eta) + U_2(\xi), \quad (2.102)
\]
\( W(\eta, \xi) = W_1(\eta) + W_2(\xi). \) \hspace{1cm} (2.103)

Substituting Equation (2.102) into Equation (2.101) gives:

\[
\frac{\partial^2}{\partial \eta^2} \left[ U_1(\eta) + U_2(\xi) \right] + i \text{Re} \left[ U_1(\eta) + U_2(\xi) \right] = i P_k(\xi),
\]

and:

\[
\frac{\partial^2 U_1(\eta)}{\partial \eta^2} + i \text{Re} U_1(\eta) = i P_k(\xi) - i \text{Re} U_2(\xi).
\]

(2.104)

(2.105)

The right-hand side of the above equation is a function of \( \xi \) and the left-hand side is a function of \( \eta \). Since \( \xi \) and \( \eta \) are independent of one another, the only possibility is that both sides of the equation are equal to a constant value:

\[
\frac{\partial^2 U_1(\eta)}{\partial \eta^2} + i \text{Re} U_1(\eta) = \kappa,
\]

(2.106)

\[
i P_k(\xi) - i \text{Re} U_2(\xi) = \kappa.
\]

(2.107)

Assuming that \( \kappa = 0 \) gives:

\[
\frac{\partial^2 U_1(\eta)}{\partial \eta^2} + i \text{Re} U_1(\eta) = 0,
\]

(2.108)

\[
i P_k(\xi) - i \text{Re} U_2(\xi) = 0,
\]

(2.109)

hence:

\[
U_1(\eta) = C_1 \cos \left( \sqrt{i \text{Re} \eta} \right) + C_2 \sin \left( \sqrt{i \text{Re} \eta} \right),
\]

(2.110)

\[
U_2(\xi) = \frac{P_k(\xi)}{\text{Re}}.
\]

(2.111)
Consequently:

\[
U(\eta, \xi) = C_1 \cos\left(\sqrt{i \text{Re}} \eta\right) + C_2 \sin\left(\sqrt{i \text{Re}} \eta\right) + \frac{P_k(\xi)}{\text{Re}}. 
\] (2.112)

The integration constants are determined by applying the velocity boundary conditions in Equation (2.21):

\[
C_1 = -\frac{P_k(\xi)}{\text{Re}}, 
\] (2.113)

\[
C_2 = \frac{P_k(\xi)}{\text{Re}} \left[ \frac{\cos\left(\sqrt{i \text{Re}}\right) - 1}{\sin\left(\sqrt{i \text{Re}}\right)} \right], 
\] (2.114)

and:

\[
U(\eta, \xi) = \frac{P_k(\xi)}{\text{Re}} \left[ \frac{\cos\left(\sqrt{i \text{Re}}\right) - 1}{\sin\left(\sqrt{i \text{Re}}\right)} \right] \left[ \sin\left(\sqrt{i \text{Re}} \eta\right) - \cos\left(\sqrt{i \text{Re}} \eta\right) + 1 \right]. 
\] (2.115)

After some manipulations, the circumferential velocity component is given as:

\[
U(\eta, \xi) = -\frac{2P_k(\xi)}{\text{Re}} \sec\left(\frac{1}{2} \sqrt{i \text{Re}}\right) \sin\left(\frac{\eta}{2} \sqrt{i \text{Re}}\right) \sin\left(\frac{1}{2} \sqrt{i \text{Re}} \{1 - \eta\}\right). 
\] (2.116)

The axial velocity component is calculated by using a similar procedure as follows:

\[
W(\eta, \xi) = \frac{2i}{\text{Re}} \frac{dP_k(\xi)}{d\xi} \sec\left(\frac{1}{2} \sqrt{i \text{Re}}\right) \sin\left(\frac{\eta}{2} \sqrt{i \text{Re}}\right) \sin\left(\frac{1}{2} \sqrt{i \text{Re}} \{1 - \eta\}\right). 
\] (2.117)

Furthermore, Equations (2.95) to (2.97) are substituted into Equation (2.92) as follows:

\[
\frac{\partial V(\eta, \xi)}{\partial \eta} = -i U(\eta, \xi) - \frac{\partial W(\eta, \xi)}{\partial \xi}, 
\] (2.118)

Substituting the velocity expressions in Equations (2.116) and (2.117) into Equation (2.118) gives:
\[ \frac{\partial V(\eta, \xi)}{\partial \eta} = -\frac{2i}{\text{Re}} \left[ \frac{d^2 P_i(\xi)}{d\xi^2} - P_i(\xi) \right] \sec\left(\frac{1}{2} \sqrt{i \text{Re}}\right) \sin\left(\frac{\eta}{2 \sqrt{i \text{Re}}}\right) \sin\left(\frac{1}{2} \sqrt{i \text{Re}} \{1 - \eta\}\right). \]  

(2.119)

In order to solve for the radial velocity component, Equation (2.119) is firstly integrated in the radial direction:

\[ V(\eta, \xi) = \frac{i}{\text{Re}} \left[ \frac{d^2 P_k(\xi)}{d\xi^2} - P_k(\xi) \right] \left[ \frac{\sec\left(\frac{1}{2} \sqrt{i \text{Re}}\right) \sin\left(\sqrt{i \text{Re}} \left\{ \frac{1}{2} - \eta \right\}\right)}{\sqrt{i \text{Re}}} + \eta \right] + C_3, \]  

(2.120)

Subsequently the following velocity boundary conditions are applied to calculate the integration constant:

\[
\begin{align*}
V(0, \xi) &= 0 \\
V(1, \xi) &= \frac{\partial H}{\partial \tau} = -i \varepsilon.
\end{align*}
\]  

(2.121)

The first boundary condition gives:

\[ C_3 = -\frac{i}{\text{Re}} \left[ \frac{d^2 P_k(\xi)}{d\xi^2} - P_k(\xi) \right] \tan\left(\frac{1}{2} \sqrt{i \text{Re}}\right) \left[ \frac{\sec\left(\frac{1}{2} \sqrt{i \text{Re}}\right) \sin\left(\sqrt{i \text{Re}} \left\{ \frac{1}{2} - \eta \right\}\right)}{\sqrt{i \text{Re}}} + \eta \right]. \]  

(2.122)

hence:

\[ V(\eta, \xi) = \frac{i}{\text{Re}} \left[ \frac{d^2 P_k(\xi)}{d\xi^2} - P_k(\xi) \right] \left[ \frac{\sec\left(\frac{1}{2} \sqrt{i \text{Re}}\right) \sin\left(\sqrt{i \text{Re}} \left\{ \frac{1}{2} - \eta \right\}\right)}{\sqrt{i \text{Re}}} \right] \right. \\
\left. - \frac{\tan\left(\frac{1}{2} \sqrt{i \text{Re}}\right)}{\sqrt{i \text{Re}}} + \eta \right]. \]  

(2.123)
The second boundary condition gives:

\[-i \varepsilon = \frac{i}{\text{Re}} \left[ \frac{d^2 P_k(\xi)}{d\xi^2} - P_k(\xi) \right] \left[ -2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right) \right] + 1.\]  

(2.124)

Consequently:

\[\frac{d^2 P_k(\xi)}{d\xi^2} - P_k(\xi) = \frac{-\varepsilon \text{Re} \sqrt{i \text{Re}}}{\sqrt{i \text{Re}} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}.\]  

(2.125)

Equation (2.125) is a non-homogenous ordinary differential equation that is solved by adding the homogenous solution and the particular solutions together. Starting with the homogeneous solution:

\[\frac{d^2 P_k(\xi)}{d\xi^2} - P_k(\xi) = 0,\]  

(2.126)

hence:

\[P_{k_h}(\xi) = C_4 e^{\xi} + C_5 e^{-\xi}.\]  

(2.127)

Additionally, the particular solution is given by:

\[P_{k_p}(\xi) = \frac{\varepsilon \text{Re} \sqrt{i \text{Re}}}{\sqrt{i \text{Re}} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}.\]  

(2.128)

Consequently:

\[P_k(\xi) = C_4 e^{\xi} + C_5 e^{-\xi} + \frac{\varepsilon \text{Re} \sqrt{i \text{Re}}}{\sqrt{i \text{Re}} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}.\]  

(2.129)
The integration constants are determined by applying the pressure boundary conditions at the axial ends of the bearing:

\[ C_4 = C_5 = -\frac{\varepsilon \Re \sqrt{i \Re}}{\sqrt{i \Re - 2 \tan \left( \frac{1}{2} \sqrt{i \Re} \right)}} \frac{1}{e^{\frac{L}{D}} + e^{-\frac{L}{D}}}, \quad (2.130) \]

and:

\[ P_1 (\xi) = \frac{\varepsilon \Re \sqrt{i \Re}}{\sqrt{i \Re - 2 \tan \left( \frac{1}{2} \sqrt{i \Re} \right)}} \left[ 1 - \frac{e^\xi + e^{-\xi}}{e^{\frac{L}{D}} + e^{-\frac{L}{D}}} \right]. \quad (2.131) \]

Applying Equation (2.74) to Equation (2.131) gives:

\[ P_1 (\xi) = \frac{\varepsilon \Re \sqrt{i \Re}}{\sqrt{i \Re - 2 \tan \left( \frac{1}{2} \sqrt{i \Re} \right)}} \left[ 1 - \frac{\cosh (\xi)}{\cosh \left( \frac{L}{D} \right)} \right]. \quad (2.132) \]

Consequently, the fluid velocity components and pressure distribution are determined by the real part of the following expressions:

\[ \bar{u}(\theta, \eta, \xi) = -\frac{2 P_1 (\xi)}{\Re} \sec \left( \frac{1}{2} \sqrt{i \Re} \right) \sin \left( \frac{\eta}{2} \sqrt{i \Re} \right) \sin \left( \frac{1}{2} \sqrt{i \Re} \left( 1 - \eta \right) \right) e^{\theta}, \quad (2.133) \]

\[ \bar{w}(\theta, \eta, \xi) = \frac{2i}{\Re} \frac{dP_1 (\xi)}{d\xi} \sec \left( \frac{1}{2} \sqrt{i \Re} \right) \sin \left( \frac{\eta}{2} \sqrt{i \Re} \right) \sin \left( \frac{1}{2} \sqrt{i \Re} \left( 1 - \eta \right) \right) e^{\theta}, \quad (2.134) \]
\[ v(\theta, \eta, \xi) = \frac{i}{\text{Re}} \left[ \frac{d^2 P_k(\xi)}{d \xi^2} - P_k(\xi) \right] \left[ \frac{\sec \left( \frac{1}{2} \sqrt{i \text{Re}} \right) \sin \left( \sqrt{i \text{Re}} \left( \frac{1}{2} - \eta \right) \right)}{\sqrt{i \text{Re}}} \right. \]
\[
- \left. \frac{\tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}{\sqrt{i \text{Re}}} + \eta \right] e^{i \theta},
\]

\[ \bar{P}(\theta, \xi) = \frac{\varepsilon \text{Re} \sqrt{i \text{Re}}}{\sqrt{i \text{Re} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}} \left[ 1 - \frac{\cosh(\xi)}{\cosh \left( \frac{L}{D} \right)} \right] e^{i \theta}. \]

Subsequently, the fluid film reaction forces are determined by integrating the pressure distribution over the journal surface. In order to facilitate the direct integration of Equation (2.136), the pressure distribution function is separated into functions of the axial coordinate and the circumferential coordinate:

\[ \bar{P}(\theta, \xi) = \frac{\varepsilon \text{Re} \sqrt{i \text{Re}}}{\sqrt{i \text{Re} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}} \bar{P}_k(\xi) e^{i \theta}, \]

\[ \bar{P}_k = \left[ 1 - \frac{\cosh(\xi)}{\cosh \left( \frac{L}{D} \right)} \right], \]

hence:

\[ \begin{bmatrix} \bar{F}_r \\ \bar{F}_i \end{bmatrix} = \frac{\varepsilon \text{Re} \sqrt{i \text{Re}}}{\sqrt{i \text{Re} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}} \int_{-L/D}^{L/D} \int_{-\theta_i}^{\theta_i} \bar{P}_k(\xi) e^{i \theta} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} d\theta d\xi. \]

The above integral can be rearranged as follows:
\[ \left[ \frac{F_r}{F_t} \right] = \frac{\varepsilon \text{Re} \sqrt{i \text{Re} \left( \frac{L}{D} \right)}}{\sqrt{i \text{Re} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}} \int_{-L/D}^{L/D} P_k(\xi) d\xi \left[ e^{i\theta} \frac{\cos(\theta)}{\sin(\theta)} \right] d\theta. \]  \hspace{1cm} (2.140)

Subsequently, the integrals are computed separately. For the axial part:

\[ \int_{-L/D}^{L/D} P_k(\xi) d\xi = \int_{-L/D}^{L/D} \left[ 1 - \frac{\cosh(\xi)}{\cosh \left( \frac{L}{D} \right)} \right] d\xi = \frac{2L}{D} \left[ 1 - \tanh \left( \frac{L}{D} \right) \right], \]  \hspace{1cm} (2.141)

and for the circumferential part:

\[ \int_{\theta_1}^{\theta_2} e^{i\theta} \cos(\theta) d\theta = \left[ \frac{\theta}{2} - \frac{i}{4} e^{2i\theta} \right]_{\theta_1}^{\theta_2}, \]  \hspace{1cm} (2.142)

and:

\[ \int_{\theta_1}^{\theta_2} e^{i\theta} \cos(\theta) d\theta = \left[ \frac{\theta}{2} - \frac{i}{4} e^{2i\theta} \right]_{\theta_1}^{\theta_2}. \]  \hspace{1cm} (2.143)

Consequently, for the SFD with the 2\pi-film assumption (i.e. pressurized lubricant without cavitation):

\[ \left[ \frac{F_r}{F_t} \right] = \text{Real} \left[ \frac{2\varepsilon \text{Re} \sqrt{i \text{Re} \left( \frac{L}{D} \right)}}{\sqrt{i \text{Re} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}} \left[ 1 - \tanh \left( \frac{L}{D} \right) \right] \left[ \pi \right] \right], \]  \hspace{1cm} (2.144)

and for the SFD with \pi-film assumption (i.e. unpressurized lubricant with cavitation):

\[ \left[ \frac{F_r}{F_t} \right] = \text{Real} \left[ \frac{2\varepsilon \text{Re} \sqrt{i \text{Re} \left( \frac{L}{D} \right)}}{\sqrt{i \text{Re} - 2 \tan \left( \frac{1}{2} \sqrt{i \text{Re}} \right)}} \left[ 1 - \tanh \left( \frac{L}{D} \right) \right] \left[ \frac{\pi}{2} \right] \right]. \]  \hspace{1cm} (2.145)
The linearized fluid force method is a powerful tool that provides direct and accelerated
calculation of the fluid film reaction force components, as well as the velocities and pressure
distribution, however, the proposed expressions are only valid for a very small range of journal
eccentricity ratios.

2.2.2 Momentum Approximation Method

In order to incorporate the momentum approximation method, Equations (2.92) to (2.94) are
integrated along the film thickness, eliminating the radial velocity components, as follows:

$$\frac{\partial}{\partial \theta} \int_0^H ud \eta + \frac{\partial}{\partial \xi} \int_0^H wd \eta + \vec{v}_0\bigg|_0^H = 0,$$  \hspace{1cm} (2.146)

$$\text{Re} \frac{\partial}{\partial \tau} \int_0^H ud \eta = -H \frac{\partial \bar{P}}{\partial \theta} + \Delta \tau_{\theta \eta},$$  \hspace{1cm} (2.147)

$$\text{Re} \frac{\partial}{\partial \tau} \int_0^H wd \eta = -H \frac{\partial \bar{P}}{\partial \xi} + \Delta \tau_{\xi \eta}.\hspace{1cm} (2.148)$$

The dimensionless lubricant flows are defined by:

$$\bar{q}_i = \int_0^H ud \eta.$$  \hspace{1cm} (2.149)

Substituting Equation (2.149) into Equations (2.146) to (2.148) gives:

$$\frac{\partial \bar{q}_\theta}{\partial \theta} + \frac{\partial \bar{q}_\xi}{\partial \xi} + \frac{\partial H}{\partial \tau} = 0,$$  \hspace{1cm} (2.150)

$$\text{Re} \frac{\partial \bar{q}_\theta}{\partial \tau} = -H \frac{\partial \bar{P}}{\partial \theta} + \Delta \tau_{\theta \eta},$$  \hspace{1cm} (2.151)

$$\text{Re} \frac{\partial \bar{q}_\xi}{\partial \tau} = -H \frac{\partial \bar{P}}{\partial \xi} + \Delta \tau_{\xi \eta}.\hspace{1cm} (2.152)$$
Assuming that the shape of the velocity profiles are not strongly influenced by the inertia forces [15], the wall shear stress differences can be expressed by:

\[ \Delta \tau_{\theta\eta} = -\frac{12}{H^2} q_\theta, \]
\[ \Delta \tau_{\xi\eta} = -\frac{12}{H^2} q_\xi. \]  \hspace{1cm} (2.153)

hence:

\[ \frac{\partial \bar{q}_\theta}{\partial \theta} + \frac{\partial \bar{q}_\xi}{\partial \xi} + \frac{\partial \bar{H}}{\partial \tau} = 0, \]  \hspace{1cm} (2.154)

\[ \text{Re} \frac{\partial \bar{q}_\theta}{\partial \tau} = -H \frac{\partial \bar{P}}{\partial \theta} - \frac{12}{H^2} q_\theta, \]  \hspace{1cm} (2.155)

\[ \text{Re} \frac{\partial \bar{q}_\xi}{\partial \tau} = -H \frac{\partial \bar{P}}{\partial \xi} - \frac{12}{H^2} q_\xi. \]  \hspace{1cm} (2.156)

Furthermore, assuming that the temporal inertia terms in the above equations are approximated by using inertialess dimensionless flows [125], the momentum transport equations become:

\[ \text{Re} \frac{\partial \bar{q}_{\theta_0}}{\partial \tau} = -H \frac{\partial \bar{P}_0}{\partial \theta} - \frac{12}{H^2} q_{\theta_0}, \]  \hspace{1cm} (2.157)

\[ \text{Re} \frac{\partial \bar{q}_{\xi_0}}{\partial \tau} = -H \frac{\partial \bar{P}_0}{\partial \xi} - \frac{12}{H^2} q_{\xi_0}. \]  \hspace{1cm} (2.158)

where:

\[ \bar{q}_{\theta_0} = -\frac{H^3 \partial \bar{P}_0}{12 \partial \theta}, \]
\[ \bar{q}_{\xi_0} = -\frac{H^3 \partial \bar{P}_0}{12 \partial \xi}. \]  \hspace{1cm} (2.159)

Substituting Equation (2.159) into Equations (2.157) and (2.158) gives:
\[
\text{Re} \frac{\partial}{\partial \tau} \left(- \frac{H^3 \partial \bar{P}_0}{12 \partial \theta}\right) = -H \frac{\partial \bar{P}}{\partial \theta} - \frac{12}{H^2} q_\theta, \quad (2.160)
\]

\[
\text{Re} \frac{\partial}{\partial \tau} \left(- \frac{H^3 \partial \bar{P}_0}{12 \partial \xi}\right) = -H \frac{\partial \bar{P}}{\partial \xi} - \frac{12}{H^2} q_\xi, \quad (2.161)
\]

Equations (2.160) and (2.161) are rearranged as follows:

\[
\begin{align*}
- q_\theta &= - \frac{H^3}{12} \frac{\partial \bar{P}}{\partial \theta} - \frac{H^2}{12} \text{Re} \frac{\partial}{\partial \tau} \left(- \frac{H^3 \partial \bar{P}_0}{12 \partial \theta}\right), \\
- q_\xi &= - \frac{H^3}{12} \frac{\partial \bar{P}}{\partial \xi} - \frac{H^2}{12} \text{Re} \frac{\partial}{\partial \tau} \left(- \frac{H^3 \partial \bar{P}_0}{12 \partial \xi}\right).
\end{align*} \quad (2.162, 2.163)
\]

Substituting Equations (2.162) and (2.163) into Equation (2.154) gives:

\[
\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \bar{P}}{\partial \theta} \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial \bar{P}}{\partial \xi} \right) = 12 \frac{\partial H}{\partial \tau} + G(\theta, \xi), \quad (2.164)
\]

where:

\[
G(\theta, \xi) = \frac{\partial}{\partial \theta} \left[ H^2 \text{Re} \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial \bar{P}_0}{12 \partial \theta} \right) \right] + \frac{\partial}{\partial \xi} \left[ H^2 \text{Re} \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial \bar{P}_0}{12 \partial \xi} \right) \right]. \quad (2.165)
\]

Equation (2.165) is expanded as follows:

\[
G(\theta, \xi) = \frac{\partial}{\partial \theta} \left( H^2 \text{Re} \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial \bar{P}_0}{12 \partial \theta} \right) \right) + H^2 \text{Re} \frac{\partial}{\partial \tau} \left( \frac{\partial}{\partial \theta} \left( \frac{H^3 \partial \bar{P}_0}{12 \partial \theta} \right) + \frac{\partial}{\partial \xi} \left( \frac{H^3 \partial \bar{P}_0}{12 \partial \xi} \right) \right). \quad (2.166)
\]

Substituting Equation (2.31) into Equation (2.166) gives:

\[
G(\theta, \xi) = \frac{\partial}{\partial \theta} \left( H^2 \text{Re} \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial \bar{P}_0}{12 \partial \theta} \right) \right) + H^2 \text{Re} \frac{\partial^2 H}{\partial \tau^2}, \quad (2.167)
\]

hence:
Consequently, the zeroth-order pressure distribution is calculated by using Reynolds equation and the solution is substituted into Equation (2.168) to determine the total pressure distribution in SFDs. Finally, the fluid film reaction force components are determined by integrating the total pressure distribution over the journal surface.

Subsequently, an iterative numerical solution similar to Section 2.1.4 is incorporated to determine the lubricant pressure distribution. Firstly, Equation (2.168) is expanded to allow the discretization of the differential terms:

\[
H^3 \frac{\partial^2 \overline{P}}{\partial \theta^2} + 3H^2 \frac{\partial H}{\partial \theta} \frac{\partial \overline{P}}{\partial \theta} + H^3 \frac{\partial^2 \overline{P}}{\partial \xi^2} = 12 \frac{\partial H}{\partial \tau} + G(\theta, \xi),
\]

where:

\[
G(\theta, \xi) = H^2 \text{Re} \frac{\partial^2 H}{\partial \xi^2} + \frac{\text{Re} H \frac{\partial H}{\partial \theta}}{6} \left[ H^3 \frac{\partial^2 \overline{P}_0}{\partial \xi \partial \theta} + 3H^2 \frac{\partial H}{\partial \tau} \frac{\partial \overline{P}_0}{\partial \theta} \right],
\]

and:

\[
\frac{\partial^2 \overline{P}_0}{\partial \xi \partial \theta} = \frac{\partial}{\partial \tau} \left( \frac{\partial \overline{P}_0}{\partial \theta} \right) = \frac{\partial}{\partial \theta} \left( \frac{\partial \overline{P}_0}{\partial \theta} \right) \frac{\partial \theta}{\partial \tau} = -\frac{\partial^2 \overline{P}_0}{\partial \theta^2}.
\]

Equations (2.169) and (2.170) are discretized by using Equations (2.83) to (2.86) and the partial derivatives of the fluid film thickness for CCO are substituted into the equations:

\[
H^3 \frac{\overline{P}_{i+1,j} - 2\overline{P}_{i,j} + \overline{P}_{i-1,j}}{\Delta \theta^2} - 3H^2 \varepsilon \sin(\theta) \frac{\overline{P}_{i,j} - \overline{P}_{i-1,j}}{\Delta \theta} + H^3 \frac{\overline{P}_{i,j+1} - 2\overline{P}_{i,j} + \overline{P}_{i,j-1}}{\Delta \xi^2} = 12 \varepsilon \sin(\theta) + G_{i,j},
\]

where:
\[ G_{i,j} = -H^2 \text{Re} \varepsilon \cos(\theta) - \frac{\text{Re} H}{6} \varepsilon \sin(\theta) \left[ -H^2 \frac{P_{0,i,j+1} - 2P_{0,i,j} + P_{0,i,j-1}}{\Delta \xi^2} ight. \]
\[ + 3H^2 \varepsilon \sin(\theta) \frac{P_{0,i,j} - P_{0,i+1,j}}{\Delta \theta} \]
\]

(2.173)

Rearranging Equation (2.172) gives:

\[ \bar{P}_{i,j} = \left[ A_i \right] + \left[ A_1 + A_2 \right] \bar{P}_{i-1,i,j} + \left[ A_2 \right] \bar{P}_{i+1,i,j} + \left[ A_3 \right] \left( \bar{P}_{i,j+1} + \bar{P}_{i,j-1} \right) \]
\[ \left[ A_4 + 2A_2 + 2A_3 \right] \]

(2.174)

where:

\[ A_i = -\frac{3H^2 (\varepsilon \sin \theta)}{\Delta \theta} \]
\[ A_2 = -\frac{H^3}{\Delta \xi^2} \]
\[ A_3 = -\frac{H^3}{\Delta \xi^2} \]
\[ A_4 = 12\varepsilon \sin \theta + G_{i,j} \]

(2.175)

Subsequently, the fluid film reaction forces are calculated by numerically integrating the pressure distribution over the journal surface by using Equation (2.91).

### 2.2.3 Perturbation Method

Perturbation method is a common technique for the analysis of SFDs. A small first-order perturbation by means of the expressions for the fluid film velocity components and the lubricant pressure distribution that are expanded in power series of the squeeze film Reynolds number is introduced as follows:

\[ \bar{u} = u_0 + \text{Re} \bar{u}_1, \]
\[ \bar{v} = v_0 + \text{Re} \bar{v}_1, \]
\[ \bar{w} = w_0 + \text{Re} \bar{w}_1, \]

(2.176) (2.177) (2.178)
\[ \overline{P} = \overline{P}_0 + \text{Re} \overline{P}. \]  

(2.179)

The first-order perturbation separates the variables into a zeroth-order inertialess component and a first-order inertial correction. Substituting Equations (2.176) to (2.179) into Equations (2.92) to (2.94) gives:

\[ \frac{\partial \overline{u}_0}{\partial \theta} + \frac{\partial \overline{v}_0}{\partial \eta} + \frac{\partial \overline{w}_0}{\partial \xi} = -\text{Re} \left[ \frac{\partial \overline{u}_1}{\partial \theta} + \frac{\partial \overline{v}_1}{\partial \eta} + \frac{\partial \overline{w}_1}{\partial \xi} \right], \]  

(2.180)

\[ -\frac{\partial \overline{P}_0}{\partial \theta} + \frac{\partial^2 \overline{u}_0}{\partial \eta^2} = \text{Re} \left[ -\frac{\partial \overline{u}_0}{\partial \tau} - \frac{\partial \overline{P}_1}{\partial \theta} + \frac{\partial^2 \overline{u}_1}{\partial \eta^2} \right], \]  

(2.181)

\[ -\frac{\partial \overline{P}_0}{\partial \xi} + \frac{\partial^2 \overline{w}_0}{\partial \eta^2} = \text{Re} \left[ -\frac{\partial \overline{w}_0}{\partial \tau} - \frac{\partial \overline{P}_1}{\partial \xi} + \frac{\partial^2 \overline{w}_1}{\partial \eta^2} \right]. \]  

(2.182)

The left-hand side of the above equations represents the inertialess lubricant flow equations and is equal to zero. Consequently, two separate systems of zeroth-order and first-order equations are produced. For the zeroth-order system of equations:

\[ \frac{\partial \overline{u}_0}{\partial \theta} + \frac{\partial \overline{v}_0}{\partial \eta} + \frac{\partial \overline{w}_0}{\partial \xi} = 0, \]  

(2.183)

\[ -\frac{\partial \overline{P}_0}{\partial \theta} + \frac{\partial^2 \overline{u}_0}{\partial \eta^2} = 0, \]  

(2.184)

\[ -\frac{\partial \overline{P}_0}{\partial \xi} + \frac{\partial^2 \overline{w}_0}{\partial \eta^2} = 0, \]  

(2.185)

where the velocity boundary conditions are defined as:

\[ \begin{cases} \overline{u}_0 = 0, \overline{v}_0 = 0, \overline{w}_0 = 0 & \eta = 0 \\ \overline{u}_0 = 0, \overline{v}_0 = \frac{\partial H}{\partial \tau}, \overline{w}_0 = 0 & \eta = H. \end{cases} \]  

(2.186)
The analytical solution for Equations (2.183) to (2.185) was described in Section 2.1 in detail. Furthermore, the first-order system of equations is given as:

\[ \frac{\partial \tilde{u}_i}{\partial \theta} + \frac{\partial \tilde{v}_i}{\partial \eta} + \frac{\partial \tilde{w}_i}{\partial \xi} = 0, \quad (2.187) \]

\[ \frac{\partial \tilde{u}_0}{\partial \tau} = -\frac{\partial \tilde{P}_1}{\partial \theta} + \frac{\partial^2 \tilde{u}_i}{\partial \eta^2}, \quad (2.188) \]

\[ \frac{\partial \tilde{w}_0}{\partial \tau} = -\frac{\partial \tilde{P}_1}{\partial \xi} + \frac{\partial^2 \tilde{w}_i}{\partial \eta^2}, \quad (2.189) \]

where the velocity boundary conditions are defined as:

\[ \begin{aligned} 
\tilde{u}_i &= 0, \tilde{v}_i = 0, \tilde{w}_i = 0 \quad \eta = 0 \\
\tilde{u}_i &= 0, \tilde{v}_i = 0, \tilde{w}_i = 0 \quad \eta = H. 
\end{aligned} \quad (2.190) \]

In order to determine the first-order velocities, the expressions for the zeroth-order velocities in Equations (2.26) and (2.27) are substituted into Equations (2.188) and (2.189). Starting with the circumferential momentum transport equation:

\[ \frac{\partial}{\partial \tau} \left[ \frac{1}{2} \frac{\partial \tilde{P}_0}{\partial \theta} \left( \eta^2 - \eta H \right) \right] = -\frac{\partial \tilde{P}_1}{\partial \theta} + \frac{\partial^2 \tilde{u}_i}{\partial \eta^2}. \quad (2.191) \]

Subsequently, Equation (2.191) is integrated twice in the radial direction:

\[ \tilde{u}_i = \frac{\eta^2}{2} \frac{\partial \tilde{P}_0}{\partial \theta} \frac{\partial}{\partial \tau} \left[ \frac{1}{2} \frac{\partial \tilde{P}_0}{\partial \theta} \left( \frac{\eta^4}{12} - \frac{\eta^3 H}{6} \right) \right] + C_1 \eta + C_2. \quad (2.192) \]

The integration constants are determined by applying the velocity boundary conditions in Equation (2.190):

\[ \tilde{u}_i = \frac{1}{2} \frac{\partial \tilde{P}_0}{\partial \theta} \left( \eta^2 - \eta H \right) + \frac{1}{2} \left[ \frac{\partial}{\partial \tau} \left( \frac{\partial \tilde{P}_0}{\partial \theta} \left( \frac{\eta^4}{12} - \frac{\eta^3 H}{6} + \frac{\eta^3 H^2}{12} \right) \right) - \frac{\partial \tilde{P}_0}{\partial \theta} \frac{\partial H}{\partial \tau} \left( \frac{\eta^3}{6} - \frac{\eta^2 H}{6} \right) \right]. \quad (2.193) \]
Similarly, the axial first-order velocity is calculated:

\[
\frac{w_1}{2} = \frac{1}{2} \frac{\partial \bar{P}_1}{\partial \xi} \left( \eta^2 - \eta \eta H \right) + \frac{1}{2} \left[ \frac{\partial}{\partial \tau} \left( \frac{\partial \bar{P}_1}{\partial \xi} \right) \left( \eta^4 - \eta \eta \eta H + \frac{\eta}{12} H^3 \right) - \frac{\partial \bar{P}_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left( \eta^3 - \frac{\eta}{6} H^2 \right) \right].
\]  

(2.194)

Subsequently, Equation (2.187) is integrated along the film thickness by applying the velocity boundary conditions:

\[
\frac{\partial}{\partial \theta} \left( \int_0^H u \, d\eta \right) + \frac{\partial}{\partial \xi} \left( \int_0^H w \, d\eta \right) = 0.
\]  

(2.195)

Substituting Equations (2.193) and (2.194) into Equation (2.195) gives:

\[
\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \bar{P}_1}{\partial \theta} \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial \bar{P}_1}{\partial \xi} \right) = \frac{\partial}{\partial \theta} \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial \bar{P}_0}{\partial \theta} \right) \left( \frac{\eta^4}{10} \right) + \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial H}{\partial \tau} \left( \frac{H^4}{4} \right) \right] + \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} \left( \frac{\partial \bar{P}_0}{\partial \xi} \right) \left( \frac{H^3}{10} \right) + \frac{\partial \bar{P}_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left( \frac{H^4}{4} \right) \right].
\]  

(2.196)

Subsequently, the total pressure is calculated by using Equation (2.179). The fluid film reaction forces are calculated by integrating the pressure distribution over the journal surface.

Additionally, an iterative numerical solution similar to Section 2.1.4 is incorporated to determine the lubricant pressure distribution. The zeroth-order pressure distribution is calculated by using the numerical procedure for Reynolds equation in Section 2.1.4 and the solution is substituted into Equation (2.196) to determine the first-order pressure distribution in SFDs. Equation (2.196) is rearranged as follows:

\[
\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \bar{P}_1}{\partial \theta} \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial \bar{P}_1}{\partial \xi} \right) = \frac{\partial F(\theta, \xi)}{\partial \theta} + \frac{\partial G(\theta, \xi)}{\partial \xi},
\]  

(2.197)

where:

\[
G_1(\theta, \xi) = \frac{\partial}{\partial \tau} \left( \frac{\partial \bar{P}_0}{\partial \theta} \right) \left( \frac{H^3}{10} \right) + \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial H}{\partial \tau} \left( \frac{H^4}{4} \right).
\]  

(2.198)
\[ G_2(\theta, \xi) = \frac{\partial}{\partial \tau} \left( \frac{\partial P_0}{\partial \xi} \right) \left( \frac{H^5}{10} \right) + \frac{\partial P_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left( \frac{H^4}{4} \right). \] (2.199)

According to the chain rule:

\[ \frac{\partial}{\partial \tau} \left( \frac{\partial P_0}{\partial \theta} \right) = \frac{\partial \theta}{\partial \tau} \frac{\partial P_0}{\partial \theta}, \]

\[ = -\frac{\partial^2 P_0}{\partial \theta^2}. \] (2.200)

\[ \frac{\partial}{\partial \tau} \left( \frac{\partial P_0}{\partial \xi} \right) = \frac{\partial \theta}{\partial \tau} \frac{\partial P_0}{\partial \xi}, \]

\[ = -\frac{\partial^2 P_0}{\partial \theta \partial \xi}. \] (2.201)

Consequently:

\[ G_1(\theta, \xi) = -\frac{\partial^2 P_0}{\partial \theta^2} \left( \frac{H^5}{10} \right) + \frac{\partial P_0}{\partial \theta} \frac{\partial H}{\partial \tau} \left( \frac{H^4}{4} \right), \] (2.202)

\[ G_2(\theta, \xi) = -\frac{\partial^2 P_0}{\partial \theta \partial \xi} \left( \frac{H^5}{10} \right) + \frac{\partial P_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left( \frac{H^4}{4} \right). \] (2.203)

Subsequently, Equations (2.197), (2.202), and (2.203) are discretized by using finite difference approximation as follows:

\[ \left[ -3cH^2 \sin(\theta) \frac{P_{i,j} - P_{i,j}}{\Delta \theta} + H^3 \frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{\Delta \theta^2} \right] \]

\[ + H^3 \frac{P_{i,j+1} - 2P_{i,j} + P_{i,j-1}}{\Delta \xi^2} = \frac{G_{i,j} - G_{i+1,j}}{\Delta \theta} + \frac{G_{i+1,j} - G_{i,j-1}}{\Delta \xi}, \] (2.204)

where:

\[ F_{i,j} = -\frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{\Delta \theta^2} \left( \frac{H^5}{10} \right) + c \sin(\theta) \frac{P_{0,i,j} - P_{0,i-1,j}}{\Delta \theta} \left( \frac{H^4}{4} \right), \] (2.205)

\[ G_{i,j} = -\frac{P_{i,j} - P_{i+1,j} - P_{i,j+1} + P_{i,j-1}}{\Delta \theta \Delta \xi} \left( \frac{H^5}{10} \right) + c \sin(\theta) \frac{P_{i-1,j} - P_{i,j-1}}{\Delta \xi} \left( \frac{H^4}{4} \right). \] (2.206)
Moreover, Equation (2.204) is rearranged as follows to solve for the pointwise pressure:

\[
\bar{P}_{i,j} = \frac{[A_1] + [A_1 + A_2] \bar{P}_{i-1,j} + [A_2] \bar{P}_{i+1,j} + [A_3] \left( \bar{P}_{i+1,j+1} + \bar{P}_{i-1,j+1} \right)}{[A_1 + 2A_2 + 2A_3]},
\]

(2.207)

where:

\[
A_1 = -\frac{3H_i^2 (\varepsilon \sin \theta_i)}{\Delta \theta},
\]

\[
A_2 = -\frac{H_i^3}{\Delta \theta^2},
\]

\[
A_3 = -\frac{H_i^3}{\Delta \xi^2},
\]

\[
A_4 = \frac{G_{1,i+1,j} - G_{1,i-1,j} + G_{2,i+1,j} - G_{2,i-1,j}}{\Delta \theta} \frac{\partial^2 \bar{P}}{\partial \eta^2}.
\]

Finally, the total pressure is calculated by using Equation (2.179) and the fluid film reaction forces are determined by numerically integrating the total pressure distribution over the journal surface based on Equation (2.91).

### 2.3 Large Amplitude Motions of the Journal Center

For large amplitude motions of the journal center, including the displacement at critical speeds, the effect of convective inertia is no longer negligible and should be necessarily included in the calculations. This section incorporates the complete inertia effects in the fluid film to calculate the fluid velocity profiles, hydrodynamic pressure distribution, and reaction force components by using the momentum approximation method and the perturbation method. Assuming that the effect of complete inertia is included, the dimensionless fluid film equations are given as follows:

\[
\frac{\partial \bar{u}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \eta} + \frac{\partial \bar{w}}{\partial \xi} = 0,
\]

(2.209)

\[
\text{Re} \left\{ \frac{\partial \bar{u}}{\partial \eta} + u \frac{\partial \bar{u}}{\partial \theta} + v \frac{\partial \bar{u}}{\partial \eta} + w \frac{\partial \bar{u}}{\partial \xi} \right\} = -\frac{\partial \bar{P}}{\partial \theta} + \frac{\partial^2 \bar{u}}{\partial \eta^2},
\]

(2.210)
\[
\text{Re} \left\{ \frac{\partial w}{\partial \tau} + u \frac{\partial w}{\partial \theta} + v \frac{\partial w}{\partial \eta} + w \frac{\partial w}{\partial \xi} \right\} = - \frac{\partial \bar{P}}{\partial \xi} + \frac{\partial^2 w}{\partial \eta^2},
\]  
(2.211)

where the velocity boundary conditions are defined as:

\[
\begin{align*}
\bar{u}_i &= 0, \bar{v}_i = 0, \bar{w}_i = 0 \quad &\eta = 0 \\
\bar{u}_i &= 0, \bar{v}_i = 0, \bar{w}_i = 0 \quad &\eta = H.
\end{align*}
\]  
(2.212)

### 2.3.1 Momentum Approximation Method

In order to apply the momentum approximation method, firstly, Equations (2.209) to (2.211) are integrated along the film thickness, eliminating the radial velocity components, as follows:

\[
\frac{\partial}{\partial \theta} \int_0^H \bar{u} d\eta + \frac{\partial}{\partial \xi} \int_0^H \bar{w} d\eta + \bar{v}_b = 0,
\]  
(2.213)

\[
\text{Re} \left[ \frac{\partial}{\partial \tau} \int_0^H \bar{u} d\eta + \frac{\partial}{\partial \theta} \int_0^H \bar{w} d\eta + \frac{\partial}{\partial \xi} \int_0^H \bar{u} d\eta \right] = -H \frac{\partial \bar{P}}{\partial \theta} + \Delta \bar{\tau}_{\theta\eta},
\]  
(2.214)

\[
\text{Re} \left[ \frac{\partial}{\partial \tau} \int_0^H \bar{u} d\eta + \frac{\partial}{\partial \theta} \int_0^H \bar{w} d\eta + \frac{\partial}{\partial \xi} \int_0^H \bar{w} d\eta \right] = -H \frac{\partial \bar{P}}{\partial \xi} + \Delta \bar{\tau}_{\xi\eta},
\]  
(2.215)

The dimensionless lubricant flows are defined by:

\[
\bar{q}_i = \int_0^H \bar{u}_i d\eta.
\]  
(2.216)

Furthermore, the momentum flux integrals are defined as follows:

\[
\bar{T}_{ij} = \int_0^H \bar{u}_i \bar{u}_j d\eta.
\]  
(2.217)

Substituting Equations (2.216) and (2.217) into Equations (2.213) to (2.214) gives:
\[
\frac{\partial q_0}{\partial \theta} + \frac{\partial q_\xi}{\partial \xi} + \frac{\partial H}{\partial \tau} = 0, \quad (2.218)
\]

\[
\text{Re} \left\{ \frac{\partial q_0}{\partial \tau} + \frac{\partial I_{0\theta}}{\partial \theta} + \frac{\partial I_{0\xi}}{\partial \xi} \right\} = -H \frac{\partial P}{\partial \theta} + \Delta \tau_{0\theta}, \quad (2.219)
\]

\[
\text{Re} \left\{ \frac{\partial q_\xi}{\partial \tau} + \frac{\partial I_{\xi\theta}}{\partial \theta} + \frac{\partial I_{\xi\xi}}{\partial \xi} \right\} = -H \frac{\partial P}{\partial \xi} + \Delta \tau_{\xi\xi}. \quad (2.220)
\]

Assuming that the shape of the velocity profiles are not strongly influenced by the inertia forces [15], the wall shear stress differences can be expressed by:

\[
\Delta \tau_{0\theta} = -\frac{12}{H^2} q_0,
\]

\[
\Delta \tau_{\xi\xi} = -\frac{12}{H^2} q_\xi,
\]

hence:

\[
\frac{\partial q_0}{\partial \theta} + \frac{\partial q_\xi}{\partial \xi} + \frac{\partial H}{\partial \tau} = 0, \quad (2.222)
\]

\[
\text{Re} \left\{ \frac{\partial q_0}{\partial \tau} + \frac{\partial I_{0\theta}}{\partial \theta} + \frac{\partial I_{0\xi}}{\partial \xi} \right\} = -H \frac{\partial P}{\partial \theta} - \frac{12}{H^2} q_0, \quad (2.223)
\]

\[
\text{Re} \left\{ \frac{\partial q_\xi}{\partial \tau} + \frac{\partial I_{\xi\theta}}{\partial \theta} + \frac{\partial I_{\xi\xi}}{\partial \xi} \right\} = -H \frac{\partial P}{\partial \xi} - \frac{12}{H^2} q_\xi. \quad (2.224)
\]

Furthermore, assuming that the temporal inertia terms in the above equations are approximated by using inertialess dimensionless flow rates [125], the above equations become:

\[
\frac{\partial q_0}{\partial \theta} + \frac{\partial q_\xi}{\partial \xi} + \frac{\partial H}{\partial \tau} = 0, \quad (2.225)
\]
Re \left\{ \frac{\partial \bar{q}_{\theta \theta}}{\partial \tau} + \frac{\partial \bar{I}_{\theta \theta}}{\partial \theta} + \frac{\partial \bar{I}_{\xi \theta}}{\partial \xi} \right\} = -H \frac{\partial \bar{P}}{\partial \theta} - \frac{12}{H^2} \frac{\partial \bar{q}_\theta}{\partial \xi}, \tag{2.226}

Re \left\{ \frac{\partial \bar{q}_{\xi \xi}}{\partial \tau} + \frac{\partial \bar{I}_{\theta \theta}}{\partial \theta} + \frac{\partial \bar{I}_{\xi \xi}}{\partial \xi} \right\} = -H \frac{\partial \bar{P}}{\partial \xi} - \frac{12}{H^2} \frac{\partial \bar{q}_\xi}{\partial \xi}, \tag{2.227}

where:

\bar{q}_{\theta \theta} = \int_0^H u_{\theta \theta} d\eta, \hspace{1cm} (2.228)

\bar{I}_{\theta \theta} = \int_0^H u_{\theta \theta} u_{\theta \theta} d\eta.

The detailed calculation of the inertialess velocity components are represented in Section 2.1.

Substituting Equations (2.26) and (2.27) into Equation (2.228) gives:

\bar{q}_{\theta \theta} = \frac{H^3}{12} \frac{\partial \bar{P}}{\partial \theta}, \hspace{1cm} \bar{I}_{\theta \theta} = \frac{\bar{q}_{\theta \theta}^2}{H}, \hspace{1cm} \bar{I}_{\xi \theta} = \frac{\bar{q}_{\theta \theta} \bar{q}_{\xi \theta}}{H}, \hspace{1cm} (2.229)

\bar{q}_{\xi \xi} = -\frac{H^3}{12} \frac{\partial \bar{P}}{\partial \xi}, \hspace{1cm} \bar{I}_{\xi \xi} = \frac{\bar{q}_{\xi \xi}^2}{H},

Furthermore, rearranging Equations (2.26) and (2.27) the equations give:

\bar{q}_\theta = -\frac{H^3}{12} \frac{\partial \bar{P}}{\partial \theta} - \frac{H^2}{12} \text{Re} \left\{ \frac{\partial \bar{q}_{\theta \theta}}{\partial \tau} + \frac{\partial \bar{I}_{\theta \theta}}{\partial \theta} + \frac{\partial \bar{I}_{\xi \theta}}{\partial \xi} \right\}, \hspace{1cm} (2.230)

\bar{q}_\xi = -\frac{H^3}{12} \frac{\partial \bar{P}}{\partial \xi} - \frac{H^2}{12} \text{Re} \left\{ \frac{\partial \bar{q}_{\xi \xi}}{\partial \tau} + \frac{\partial \bar{I}_{\theta \theta}}{\partial \theta} + \frac{\partial \bar{I}_{\xi \xi}}{\partial \xi} \right\}. \hspace{1cm} (2.231)

Subsequently, Equations (2.230) and (2.231) are substituted into Equation (2.225) as follows:

\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \bar{P}}{\partial \theta} \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial \bar{P}}{\partial \xi} \right) = 12 \frac{\partial \bar{H}}{\partial \tau} + \text{Re} \left[ G_1 (\theta, \xi) + G_2 (\theta, \xi) \right]. \hspace{1cm} (2.232)
The first term on the right-hand side of Equation (2.232) represents the effect of viscous forces. The second term represents the temporal (unsteady) inertia forces and is given as follows:

\[ G_1(\theta, \xi) = -\frac{\partial}{\partial \theta} \left( H^2 \frac{\partial q_{\theta_0}}{\partial \tau} \right) + \frac{\partial}{\partial \xi} \left( H^2 \frac{\partial q_{\xi_0}}{\partial \tau} \right). \]  \hspace{1cm} (2.233)

Finally, the third term represents the convective inertia as follows:

\[ G_2(\theta, \xi) = -\frac{\partial}{\partial \theta} \left( H^2 \left[ \frac{\partial}{\partial \theta} \left( \frac{q_{\theta_0}}{H} \right) + \frac{\partial}{\partial \xi} \left( \frac{q_{\xi_0}}{H} \right) \right] \right) \]

\[ -\frac{\partial}{\partial \xi} \left( H^2 \left[ \frac{\partial}{\partial \theta} \left( \frac{q_{\theta_0}}{H} \right) + \frac{\partial}{\partial \xi} \left( \frac{q_{\xi_0}}{H} \right) \right] \right). \] \hspace{1cm} (2.234)

Substituting Equation (2.229) into Equation (2.233) gives:

\[ G_1(\theta, \xi) = \frac{\partial}{\partial \theta} \left( H^2 \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial P_0}{12 \partial \theta} \right) \right) + \frac{\partial}{\partial \xi} \left( H^2 \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial P_0}{12 \partial \xi} \right) \right). \] \hspace{1cm} (2.235)

The above equation is expanded and rearranged as follows:

\[ G_1(\theta, \xi) = \frac{\partial}{\partial \theta} \left( H^2 \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial P_0}{12 \partial \theta} \right) \right) + \frac{\partial}{\partial \xi} \left( H^2 \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial P_0}{12 \partial \xi} \right) \right). \] \hspace{1cm} (2.236)

Substituting Reynolds equation into the second term in Equation (2.236) gives:

\[ G_1(\theta, \xi) = H^2 \frac{\partial^2 H}{\partial \tau^2} + \frac{\partial}{\partial \theta} \left( H^2 \frac{\partial}{\partial \tau} \left( \frac{H^3 \partial P_0}{12 \partial \theta} \right) \right). \] \hspace{1cm} (2.237)

Additionally, substituting Equation (2.229) into Equation (2.234) gives:
\[
G_2(\theta, \xi) = -\frac{\partial}{\partial \theta} \left[ \frac{5H^6}{144} \frac{\partial H}{\partial \theta} \left( \frac{\partial P_0}{\partial \theta} \right)^2 + \frac{H^7}{72} \frac{\partial P_0}{\partial \theta} \frac{\partial^2 P_0}{\partial \theta^2} + \frac{H^7}{144} \frac{\partial P_0}{\partial \theta} \frac{\partial^2 P_0}{\partial \xi \partial \xi} + \frac{H^7}{144} \frac{\partial P_0}{\partial \theta} \frac{\partial^2 P_0}{\partial \xi^2} \right] \]

Consequently, the zeroth-order pressure distribution is calculated by using Reynolds equation and the solution is substituted into Equation (2.232) to determine the total pressure distribution in SFDs. Finally, the fluid film reaction force components are determined by integrating the total pressure distribution over the journal surface.

Subsequently, an iterative numerical solution similar to Section 2.1.4 is incorporated to determine the lubricant pressure distribution. Firstly, Equation (2.232) is expanded to allow for the discretization of the differential terms:

\[
H^3 \frac{\partial^2 P_0}{\partial \theta^2} + 3H^2 \frac{\partial H}{\partial \theta} \frac{\partial P_0}{\partial \theta} + H^3 \frac{\partial^2 P_0}{\partial \xi^2} = 12 \frac{\partial H}{\partial \tau} + \text{Re} \left[ G_1(\theta, \xi) + G_2(\theta, \xi) \right] \tag{2.239}
\]

where:

\[
G_1(\theta, \xi) = \frac{H^2}{6} \frac{\partial^2 H}{\partial \tau^2} + \frac{H}{\partial \theta} \left[ H^3 \frac{\partial^2 P_0}{\partial \tau \partial \theta} + 3H^2 \frac{\partial H}{\partial \tau} \frac{\partial P_0}{\partial \theta} \right] \tag{2.240}
\]

Based on the chain rule:

\[
\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial \tau} = -\frac{\partial}{\partial \theta}, \tag{2.241}
\]

hence:

\[
G_1(\theta, \xi) = H^2 \frac{\partial^2 H}{\partial \theta^2} - \frac{H}{\partial \theta} \left[ H^3 \frac{\partial^2 P_0}{\partial \theta^2} + 3H^2 \frac{\partial H}{\partial \theta} \frac{\partial P_0}{\partial \theta} \right] \tag{2.242}
\]

Furthermore,
The first and second order partial differential discretization by using finite difference approximation are provided by Equations (2.83) to (2.86). The discretization for higher order and mixed derivatives are given as follows:

\[
\begin{align*}
\frac{\partial^3 \bar{P}}{\partial \theta^3} &= \frac{\bar{P}(\theta + 2\Delta\theta, \xi) - 2\bar{P}(\theta + \Delta\theta, \xi) + \bar{P}(\theta - \Delta\theta, \xi) - \bar{P}(\theta - 2\Delta\theta, \xi)}{2\Delta\theta^3}, \\
\frac{\partial^3 \bar{P}}{\partial \xi^3} &= \frac{\bar{P}(\theta, \xi + 2\Delta\xi) - 2\bar{P}(\theta, \xi + \Delta\xi) + \bar{P}(\theta, \xi - \Delta\xi) - \bar{P}(\theta, \xi - 2\Delta\xi)}{2\Delta\xi^3}, \\
\frac{\partial^2 \bar{P}}{\partial \theta \partial \xi} &= \frac{\bar{P}(\theta, \xi) - \bar{P}(\theta - \Delta\theta, \xi) - \bar{P}(\theta, \xi - \Delta\xi) + \bar{P}(\theta - \Delta\theta, \xi - \Delta\xi)}{\Delta\theta \Delta\xi},
\end{align*}
\]
\[
\frac{\partial^2 P}{\partial \theta^2 \Delta \xi} = \frac{\partial^2 \bar{P}(\theta, \xi + \Delta \xi)}{\Delta \theta \Delta \xi} - \frac{\partial^2 \bar{P}(\theta, \xi)}{\Delta \theta \Delta \xi} - \frac{\partial^2 \bar{P}(\theta - \Delta \theta, \xi) + \partial^2 \bar{P}(\theta - \Delta \theta, \xi - \Delta \xi)}{\Delta \theta \Delta \xi} + \frac{\partial^2 \bar{P}(\theta - \Delta \theta, \xi - \Delta \xi)}{\Delta \theta \Delta \xi} - \frac{\partial^2 \bar{P}(\theta - \Delta \theta, \xi - \Delta \xi)}{\Delta \theta \Delta \xi} - \frac{\partial^2 \bar{P}(\theta - \Delta \theta, \xi - \Delta \xi)}{\Delta \theta \Delta \xi} - \frac{\partial^2 \bar{P}(\theta - \Delta \theta, \xi - \Delta \xi)}{\Delta \theta \Delta \xi} \tag{2.247}
\]

\[
\frac{\partial^2 \bar{P}}{\partial \theta^2 \Delta \xi^2} = \frac{\partial^2 \bar{P}(\theta + \Delta \theta, \xi) - \partial^2 \bar{P}(\theta, \xi) + \partial^2 \bar{P}(\theta - \Delta \theta, \xi) - \partial^2 \bar{P}(\theta - \Delta \theta, \xi - \Delta \xi)}{\Delta \theta^2 \Delta \xi} + \frac{\partial^2 \bar{P}(\theta + \Delta \theta, \xi - \Delta \xi) - \partial^2 \bar{P}(\theta, \xi - \Delta \xi) + \partial^2 \bar{P}(\theta - \Delta \theta, \xi - \Delta \xi) - \partial^2 \bar{P}(\theta - \Delta \theta, \xi)}{\Delta \theta^2 \Delta \xi} \tag{2.248}
\]

Hence, Equation (2.239) is discretized as follows:

\[
H^2 \frac{\partial^2 \bar{P}_{i+1,j} - 2\partial^2 \bar{P}_{i,j} + \partial^2 \bar{P}_{i-1,j}}{\Delta \theta^2} - 3H^2 \bar{P}_{i+1,j} \sin(\theta) \frac{\partial^2 \bar{P}_{i,j}}{\Delta \theta} + H^2 \frac{\partial^2 \bar{P}_{i+1,j} - 2\partial^2 \bar{P}_{i,j} + \partial^2 \bar{P}_{i-1,j}}{\Delta \xi^2} = 12\sin(\theta) + G_{i,j} + G_{2i,j} \tag{2.249}
\]

Rearranging Equation (2.249) gives:

\[
P_{i,j} = \left[ A_1 + \text{Re} \left( G_{i,j} + G_{2i,j} \right) \right] + \left[ A_1 + A_2 \right] \bar{P}_{i-1,j} + \left[ A_1 + A_2 \right] \bar{P}_{i,j} + \left[ A_1 + A_2 \right] \left( \bar{P}_{i+1,j} + \bar{P}_{i-1,j} \right) \tag{2.250}
\]

where:

\[
A_1 = -\frac{3H^2 \sin(\theta)}{\Delta \theta}, \tag{2.251}
\]

\[
A_2 = -\frac{H^2}{\Delta \theta}, \tag{2.251}
\]

\[
A_3 = -\frac{H^2}{\Delta \xi}, \tag{2.251}
\]

\[
A_4 = 12\sin(\theta). \tag{2.251}
\]
Subsequently, the fluid film reaction forces are calculated by numerically integrating the pressure distribution over the journal surface by using Equation (2.91).

### 2.3.2 Perturbation Method

A small first-order perturbation by means of the expressions for the fluid film velocity components and the lubricant pressure distribution that are expanded in power series of the squeeze film Reynolds number is introduced as follows:

\[
\bar{u} = u_0 + \text{Re} u_1, \quad (2.252)
\]

\[
\bar{v} = v_0 + \text{Re} v_1, \quad (2.253)
\]

\[
\bar{w} = w_0 + \text{Re} w_1, \quad (2.254)
\]

\[
\bar{P} = P_0 + \text{Re} \bar{P}_1. \quad (2.255)
\]

The above approximation separates the pressure and the velocity components into a zeroth-order inertialess term and a first-order inertial correction component. Substituting Equations (2.252) to (2.255) into Equations (2.209) to (2.211) gives:

\[
\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial \eta} + \frac{\partial w_0}{\partial \xi} = -\text{Re} \left[ \frac{\partial u_1}{\partial \theta} + \frac{\partial v_1}{\partial \eta} + \frac{\partial w_1}{\partial \xi} \right], \quad (2.256)
\]

\[
-\frac{\partial P_0}{\partial \theta} + \frac{\partial^2 u_0}{\partial \eta^2} = \text{Re} \left[ \frac{\partial u_0}{\partial \tau} + u \frac{\partial u_1}{\partial \theta} + v \frac{\partial v_1}{\partial \eta} + w \frac{\partial w_1}{\partial \xi} + \frac{\partial P_1}{\partial \theta} - \frac{\partial^2 u_1}{\partial \eta^2} \right], \quad (2.257)
\]

\[
-\frac{\partial P_0}{\partial \xi} + \frac{\partial^2 w_0}{\partial \eta^2} = \text{Re} \left[ \frac{\partial w_0}{\partial \tau} + u \frac{\partial w_1}{\partial \theta} + v \frac{\partial v_1}{\partial \eta} + w \frac{\partial w_1}{\partial \xi} + \frac{\partial P_1}{\partial \eta} - \frac{\partial^2 w_1}{\partial \eta^2} \right]. \quad (2.258)
\]

The left-hand side of the above equations represents the inertialess lubricant flow equations and is equal to zero. Consequently, two separate systems of zeroth-order and first-order equations are produced. For the zeroth-order system of equations:
\[
\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial \eta} + \frac{\partial w_0}{\partial \xi} = 0, \quad (2.259)
\]

\[
- \frac{\partial P_0}{\partial \theta} + \frac{\partial^2 u_0}{\partial \eta^2} = 0, \quad (2.260)
\]

\[
- \frac{\partial P_0}{\partial \xi} + \frac{\partial^2 w_0}{\partial \eta^2} = 0, \quad (2.261)
\]

where the velocity boundary conditions are defined as:

\[
\begin{align*}
&u_0 = 0, v_0 = 0, w_0 = 0, & \eta = 0 \\
&u_0 = 0, v_0 = \frac{\partial H}{\partial \tau}, w_0 = 0, & \eta = H.
\end{align*} \quad (2.262)
\]

The detailed solution for Equations (2.183) to (2.185) was described in Section 2.1. Furthermore, the first-order system of equations is given as:

\[
\frac{\partial u_1}{\partial \theta} + \frac{\partial v_1}{\partial \eta} + \frac{\partial w_1}{\partial \xi} = 0, \quad (2.263)
\]

\[
\frac{\partial u_0}{\partial \tau} + u \frac{\partial u}{\partial \theta} + v \frac{\partial u}{\partial \eta} + w \frac{\partial u}{\partial \xi} = - \frac{\partial P_1}{\partial \theta} + \frac{\partial^2 u}{\partial \eta^2}, \quad (2.264)
\]

\[
\frac{\partial w_0}{\partial \tau} + u \frac{\partial w}{\partial \theta} + v \frac{\partial w}{\partial \eta} + w \frac{\partial w}{\partial \xi} = - \frac{\partial P_1}{\partial \xi} + \frac{\partial^2 w}{\partial \eta^2}, \quad (2.265)
\]

where the velocity boundary conditions are defined as:

\[
\begin{align*}
&u_1 = 0, v_1 = 0, w_1 = 0, & \eta = 0 \\
&u_1 = 0, v_1 = 0, w_1 = 0, & \eta = H.
\end{align*} \quad (2.266)
\]

In order to determine the first-order velocities, the expressions for the zeroth-order velocities in Equations (2.26), (2.27), and (2.30) are substituted into Equations (2.264) and (2.265). Furthermore, according to Equation (2.31):
\begin{align*}
\frac{\partial^2 \tilde{P}_0}{\partial \theta^2} + \frac{\partial^2 \overline{P}_0}{\partial \xi^2} &= \frac{12}{H^3} \frac{\partial \tilde{H}}{\partial \tau} - \frac{3}{H} \frac{\partial \overline{P}_0}{\partial \theta} \frac{\partial \tilde{H}}{\partial \theta}. 
\end{align*}
(2.267)

Substituting Equation (2.267) into Equation (2.30) reduces the order of the radial velocity expression to:

\begin{align*}
\overline{v}(\theta, \eta, \xi) &= \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \theta} \frac{\partial \tilde{H}}{\partial \theta} \left( \frac{\eta^3}{H} - \eta^2 \right) - \left( \frac{2\eta^3 - 3\eta^2 H}{H^3} \right) \frac{\partial \tilde{H}}{\partial \tau}. 
\end{align*}
(2.268)

Starting with the circumferential momentum transport equation:

\begin{align*}
\frac{\partial}{\partial \tau} \left[ \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \theta} (\eta^2 - \eta H) + \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \theta} (\eta^2 - \eta H) \right]
= & \frac{1}{2} \left[ \left( \frac{\partial^2 \tilde{P}_0}{\partial \theta^2} + \frac{\partial^2 \overline{P}_0}{\partial \xi^2} \right) \left( \frac{\eta^3}{3} - H \frac{\eta^2}{2} \right) - \frac{\partial \overline{P}_0}{\partial \theta} \frac{\partial \tilde{H}}{\partial \theta} \frac{\eta^2}{2} \right] \frac{\partial \tilde{H}}{\partial \eta} \left[ \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \theta} (\eta^2 - \eta H) \right] \\
&+ \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \xi} (\eta^2 - \eta H) \frac{\partial}{\partial \xi} \left[ \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \theta} (\eta^2 - \eta H) \right] = -\frac{\partial \overline{P}_1}{\partial \theta} + \frac{\partial^2 \overline{u}_1}{\partial \eta^2}. 
\end{align*}
(2.269)

Integrating Equation (2.269) and solving for the first-order velocity gives:

\begin{align*}
\overline{u}_1 &= \left[ \frac{\partial^2 \tilde{P}_0}{\partial \theta^2} \left( \frac{\eta^4}{24} - \frac{\eta^3 H}{12} \right) - \frac{\partial \overline{P}_0}{\partial \tau} \frac{\partial \tilde{H}}{\partial \theta} \frac{\eta^3}{12} \right] - \left( \frac{\partial \overline{P}_0}{\partial \theta} \right)^2 \frac{\partial \tilde{H}}{\partial \eta} \left( \frac{\eta^5}{80} - \frac{\eta^4 H}{48} \right) \\
&+ \frac{\partial \overline{P}_0}{\partial \theta} \frac{\partial^2 \tilde{P}_0}{\partial \theta^2} \left( \frac{\eta^4}{120} - \frac{\eta^3 H}{40} + \frac{\eta^4 H^2}{48} \right) - \left( \frac{\partial \overline{P}_0}{\partial \theta} \right)^2 \frac{\partial \tilde{H}}{\partial \tau} \left( \frac{\eta^5}{60H} - \frac{3\eta^5}{80} + \frac{\eta^4 H}{48} \right) \\
&- \frac{\partial \overline{P}_0}{\partial \theta} \frac{\partial \tilde{H}}{\partial \tau} \left( \frac{\eta^6}{15H^3} - \frac{\eta^5 H}{5H^2} + \frac{\eta^4 H^2}{8H} \right) + \frac{\partial^2 \overline{P}_0}{\partial \xi^2} \frac{\partial \tilde{P}_0}{\partial \xi} \left( \frac{\eta^6}{120} - \frac{\eta^5 H}{40} + \frac{\eta^4 H^2}{48} \right) \\
&+ \frac{\eta^2}{2} \frac{\partial \overline{P}_1}{\partial \theta} + C_1 \eta + C_2,
\end{align*}
(2.270)

and the integration constants are determined by applying the velocity boundary conditions in Equation (2.266):
\[ C_1 = \frac{H}{2} \frac{\partial \bar{P}}{\partial \theta} + \left[ \frac{\partial^2 \bar{P}_0}{\partial \phi \partial \theta} \left( \frac{H^3}{24} \right) + \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial H}{\partial \tau} \right] - \left( \frac{\partial \bar{P}_0}{\partial \theta} \right)^2 \frac{\partial H}{\partial \theta} \left( \frac{H^4}{120} \right) \]
\[ + \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial^2 \bar{P}_0}{\partial \theta^2} \left( \frac{H^4}{120} \right) + \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial H}{\partial \theta} \left( \frac{H^2}{120} \right) - \frac{\partial^2 \bar{P}_0}{\partial \phi \partial \xi} \frac{\partial \bar{P}_0}{\partial \xi} \left( \frac{\eta^5}{240} \right). \]
\[ (2.271) \]

\[ C_2 = 0, \]
\[ (2.272) \]

hence:

\[ \bar{u}_1 = \left( \frac{\eta^2 - H \eta}{2} \right) \frac{\partial \bar{P}_1}{\partial \theta} - \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial H}{\partial \tau} \left( \frac{\eta^6}{15H^3} - \frac{\eta^5}{5H^2} + \frac{\eta^4}{8H} + \frac{\eta^3}{12} - \frac{3H^2 \eta}{40} \right) \]
\[ + \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial H}{\partial \theta} \left( \frac{\eta^6}{60H} - \frac{\eta^5}{20} + \frac{\eta^4H}{24} - \frac{H^2 \eta}{120} \right) + \frac{\partial^2 \bar{P}_0}{\partial \phi \partial \theta} \left( \frac{\eta^4}{24} - \frac{\eta^3H}{12} + \frac{H^3 \eta}{24} \right) \]
\[ + \frac{\partial \bar{P}_0}{\partial \theta} \frac{\partial^2 \bar{P}_0}{\partial \theta^2} + \frac{\partial^2 \bar{P}_0}{\partial \phi \partial \xi} \frac{\partial \bar{P}_0}{\partial \xi} \left( \frac{\eta^6}{120} - \frac{\eta^5H}{40} + \frac{\eta^4H^2}{48} - \frac{H^3 \eta}{240} \right). \]
\[ (2.273) \]

Similarly, the axial first-order velocity component is calculated as follows:

\[ \bar{w}_1 = \left( \frac{\eta^2 - H \eta}{2} \right) \frac{\partial \bar{P}_1}{\partial \xi} - \frac{\partial \bar{P}_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left( \frac{\eta^6}{15H^3} - \frac{\eta^5}{5H^2} + \frac{\eta^4}{8H} + \frac{\eta^3}{12} - \frac{3H^2 \eta}{40} \right) \]
\[ + \frac{\partial \bar{P}_0}{\partial \xi} \frac{\partial H}{\partial \theta} \left( \frac{\eta^6}{60H} - \frac{\eta^5}{20} + \frac{\eta^4H}{24} - \frac{H^2 \eta}{120} \right) + \frac{\partial^2 \bar{P}_0}{\partial \phi \partial \xi} \left( \frac{\eta^4}{24} - \frac{\eta^3H}{12} + \frac{H^3 \eta}{24} \right) \]
\[ + \frac{\partial \bar{P}_0}{\partial \xi} \frac{\partial^2 \bar{P}_0}{\partial \xi^2} + \frac{\partial^2 \bar{P}_0}{\partial \phi \partial \phi} \frac{\partial \bar{P}_0}{\partial \phi} \left( \frac{\eta^6}{120} - \frac{\eta^5H}{40} + \frac{\eta^4H^2}{48} - \frac{H^3 \eta}{240} \right). \]
\[ (2.274) \]

Subsequently, in order to eliminate the radial velocity component, Equation (2.263) is integrated along the film thickness and the velocity boundary conditions are applied:

\[ \frac{\partial}{\partial \theta} \left( \int_0^H u_1 d\eta \right) + \frac{\partial}{\partial \xi} \left( \int_0^H w_1 d\eta \right) = 0, \]
\[ (2.275) \]

where, the first-order lubricant flows are defined as [3]:
\[
q_{\eta} = \frac{H}{0} u_{\eta} \, d\eta,
\]

(2.276)

Substituting Equation (2.276) into Equation (2.275) gives:

\[
\frac{\partial q_0}{\partial \theta} + \frac{\partial q_\xi}{\partial \xi} = 0,
\]

(2.277)

Integrating the first-order velocity expressions in Equations (2.273) and (2.274) along the film thickness and substituting into Equation (2.277) gives:

\[
\frac{\partial}{\partial \theta} \left( H^3 \frac{\partial P_1}{\partial \theta} \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial P_1}{\partial \xi} \right) =
\frac{\partial}{\partial \theta} \left\{ -\frac{3H^2}{560} \frac{\partial}{\partial \theta} \left[ \left( \frac{\partial P_0}{\partial \theta} \right)^2 + \left( \frac{\partial P_0}{\partial \xi} \right)^2 \right] - \frac{3H^6}{140} \frac{\partial H}{\partial \theta} \left( \frac{\partial P_0}{\partial \theta} \right)^2 + \frac{13H^4}{70} \frac{\partial H}{\partial \tau} \frac{\partial P_0}{\partial \theta} + \frac{\partial^2 P_0}{\partial \xi^2} \left( \frac{H^5}{10} \right) \right\}
\]

(2.278)

where:

\[
H^3 \frac{\partial^2 P_1}{\partial \theta^2} + 3H^2 \frac{\partial H}{\partial \theta} \frac{\partial P_1}{\partial \theta} + H^3 \frac{\partial^2 P_1}{\partial \xi^2} = 12 \frac{\partial H}{\partial \tau} + G_1(\theta, \xi) + G_2(\theta, \xi),
\]

(2.279)

where:
Based on the chain rule:

\[ \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial \theta} \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial \theta}. \]  

(2.282)

Furthermore, the partial derivatives of the fluid film thickness \( H \) are given as follows:

\[ \frac{\partial H}{\partial \theta} = -\varepsilon \sin(\theta), \]

(2.283)

\[ \frac{\partial H}{\partial \tau} = \dot{\varepsilon} \cos(\theta) + \varepsilon \sin(\theta). \]

Assuming that the SFD executes CCOs, the radial velocity and acceleration of the journal center become zero and:

\[ \frac{\partial H}{\partial \tau} = \varepsilon \sin(\theta), \]

(2.284)

\[ \frac{\partial H}{\partial \theta} = -\varepsilon \sin(\theta). \]

(2.285)
Subsequently, Equations (2.279) to (2.281) are discretized based on the finite difference approximation procedure that was defined in Sections 2.1.4 and 2.3.3 as follows:

\[
H_i^3 \bar{P}_{h,i,j} - 2\bar{P}_{h,i,j} + \bar{P}_{h,i+1,j} - 3H_i^3 \varepsilon \sin(\theta) \frac{\bar{P}_{h,i,j} - \bar{P}_{h,i-1,j}}{\Delta \theta} \\
+ H_i^3 \frac{\bar{P}_{h,i+1,j} - 2\bar{P}_{h,i,j} + \bar{P}_{h,i-1,j}}{\Delta z^2} = 12\varepsilon \sin(\theta) + G_{h,i,j} + G_{z,i,j}.
\]  

(2.286)

The above equation is rearranged as following to solve for the point-wise pressure field:

\[
P_{h,i,j} = \left[ A_4 + G_{h,i,j} + G_{z,i,j} \right] + \left[ A_1 + A_2 \right] \bar{P}_{h,i-1,j} + \left[ A_2 \right] \bar{P}_{h,i+1,j} + \left[ A_3 \right] \left( \bar{P}_{h,i+1,j} + \bar{P}_{h,i-1,j} \right) \\
\left[ A_1 + 2A_2 + 2A_3 \right],
\]  

(2.287)

where:

\[
A_1 = -\frac{3H_i^3 \left( \varepsilon \sin \theta \right)}{\Delta \theta}, \\
A_2 = -\frac{H_i^3}{\Delta \theta^2}, \\
A_3 = -\frac{H_i^3}{\Delta z^2}, \\
A_4 = 12\varepsilon \sin \theta.
\]  

(2.288)

Subsequently, the total pressure is calculated by using Equation (2.255) and the fluid film reaction forces are calculated by numerically integrating the pressure distribution over the journal surface by using Equation (2.91).

2.4 Short Bearing Approximation

This section develops closed-form analytical expressions for the fluid velocity profiles, lubricant pressure distribution, and fluid film reaction forces in the presence of fluid inertia effects for short-length SFDs, by applying short bearing approximation. The proposed model provides fast predictions of the SFD dynamic performance within acceptable accuracy for short-length SFDs.

The thin film equations are further reduced by applying the SBA as follows:
\[
\frac{\partial \bar{v}}{\partial \eta} + \frac{\partial \bar{w}}{\partial \xi} = 0, \quad (2.289)
\]

\[
\text{Re} \left\{ \frac{\partial \bar{w}}{\partial \tau} + v \frac{\partial \bar{w}}{\partial \eta} + w \frac{\partial \bar{w}}{\partial \xi} \right\} = -\frac{\partial \bar{P}}{\partial \xi} + \frac{\partial^2 \bar{w}}{\partial \eta^2}, \quad (2.290)
\]

where:

\[
\begin{aligned}
\bar{v} = 0, & \quad \bar{w} = 0 \quad \eta = 0 \\
\bar{v} = \frac{\partial H}{\partial \tau}, & \quad \bar{w} = 0 \quad \eta = H.
\end{aligned} \quad (2.291)
\]

Subsequently, Equations (2.289) and (2.290) are solved by applying the momentum approximation method and the perturbation method.

### 2.4.1 Momentum Approximation Method

In order to eliminate the radial velocity components, Equations (2.289) and (2.290) are integrated along the film thickness and the velocity boundary conditions in Equation (2.291) are applied to the Equations:

\[
\frac{\partial}{\partial \xi} \int_0^H \bar{w} d\eta + \bar{v}_o = 0, \quad (2.292)
\]

\[
\text{Re} \left\{ \frac{\partial}{\partial \tau} \int_0^H \bar{w} d\eta + \frac{\partial}{\partial \xi} \int_0^H \bar{w}^2 d\eta \right\} = -H \frac{\partial \bar{P}}{\partial \xi} + \Delta \bar{z}_{\nu}. \quad (2.293)
\]

The average lubricant flow components are defined by integrating the velocities along the film thickness as follows:

\[
\bar{q}_i = \int_0^H \bar{u}_i d\eta. \quad (2.294)
\]

Furthermore, the fluid momentum flux integral is represented as follows:
\[ \overline{I}_y = \int_0^H \overline{u}_i \overline{u}_j d\eta. \]  \hfill (2.295)

Substituting Equations (2.294) and (2.295) into Equations (2.292) and (2.293) gives:

\[ \frac{\partial q_{\xi}}{\partial \xi} + \frac{\partial H}{\partial \tau} = 0, \]  \hfill (2.296)

\[ \text{Re} \left\{ \frac{\partial q_{\xi}}{\partial \tau} + \frac{\partial I_{\xi\xi}}{\partial \xi} \right\} = -H \frac{\partial \overline{P}}{\partial \xi} + \Delta \tau_{\xi\eta}. \]  \hfill (2.297)

Assuming that the shape of the velocity profiles are not strongly influenced by the inertia forces [15], the wall shear stress difference can be expressed by:

\[ \Delta \tau_{\xi\eta} = -\frac{12}{H^2} q_{\xi}, \]  \hfill (2.298)

hence:

\[ \frac{\partial q_{\xi}}{\partial \xi} + \frac{\partial H}{\partial \tau} = 0, \]  \hfill (2.299)

\[ \text{Re} \left\{ \frac{\partial q_{\xi}}{\partial \tau} + \frac{\partial I_{\xi\xi}}{\partial \xi} \right\} = -H \frac{\partial \overline{P}}{\partial \xi} - \frac{12}{H^2} q_{\xi}. \]  \hfill (2.300)

According to Equation (2.296), the axial flow is determined as follows:

\[ q_{\xi} = -\int \frac{\partial H}{\partial \tau} d\xi = -\xi \frac{\partial H}{\partial \tau} + C_1. \]  \hfill (2.301)

Moreover, Equation (2.300) is rearranged as follows:

\[ \frac{\partial \overline{P}}{\partial \xi} = -\text{Re} \left\{ \frac{\partial q_{\xi}}{\partial \tau} + \frac{\partial I_{\xi\xi}}{\partial \xi} \right\} - \frac{12}{H} \overline{q}_{\xi}. \]  \hfill (2.302)
Substituting Equation (2.301) into Equation (2.302) gives:

\[
\frac{\partial \bar{P}}{\partial \xi} = - \frac{\text{Re}}{H} \left\{ \frac{\partial}{\partial \tau} \left[ -\xi \frac{\partial H}{\partial \tau} + C_3 \right] + \frac{\partial I_{\bar{q}_z}}{\partial \xi} \right\} - \frac{12}{H^3} \left[ -\xi \frac{\partial H}{\partial \tau} + C_1 \right]. \tag{2.303}
\]

Furthermore, the axial momentum flux integral is approximated as follows [18]:

\[
\overline{I_{\bar{q}_z}} = \alpha \frac{\bar{q}_z}{H}, \tag{2.304}
\]

where for small to moderate squeeze Reynolds numbers a value of \( \alpha = 1.2 \) is assigned for all the calculations. Consequently:

\[
\frac{\partial \bar{P}}{\partial \xi} = - \frac{\text{Re}}{H} \left\{ \frac{\partial}{\partial \tau} \left[ -\xi \frac{\partial H}{\partial \tau} + C_3 \right] + \frac{\partial}{\partial \xi} \left[ \alpha \left( -\xi \frac{\partial H}{\partial \tau} + C_1 \right)^2 \right] \right\} - \frac{12}{H^3} \left[ -\xi \frac{\partial H}{\partial \tau} + C_1 \right]. \tag{2.305}
\]

Equation (2.305) is integrated in the axial direction as follows:

\[
\bar{P} = \frac{\text{Re}}{H} \frac{\xi^2}{2} \frac{\partial^2 H}{\partial \tau^2} - 2 \alpha \frac{\text{Re}}{H^2} \left( \frac{\xi^2}{2} \left( \frac{\partial H}{\partial \tau} \right)^2 - \xi \frac{\partial H}{\partial \tau} C_1 \right) + \frac{12}{H^3} \frac{\xi^2}{2} \frac{\partial H}{\partial \tau} - \frac{12}{H^3} \frac{\xi}{2} C_1 + C_2, \tag{2.306}
\]

and the integration constant are determined by applying the pressure boundary conditions at the axial ends of the bearing:

\[
C_3 = 0,
\]

\[
C_4 = -\frac{12}{H^3} \frac{\partial H}{\partial \tau} \frac{1}{2} \left( \frac{L}{D} \right)^2 - \frac{\text{Re}}{H} \frac{\partial^2 H}{\partial \tau^2} \frac{1}{2} \left( \frac{L}{D} \right)^2 + 2 \alpha \frac{\text{Re}}{H^2} \left( \frac{\partial H}{\partial \tau} \right)^2 \frac{1}{2} \left( \frac{L}{D} \right)^2. \tag{2.307}
\]

Consequently, the analytical expression for the hydrodynamic pressure field is determined as follows:

\[
\bar{P} = \left[ 6 \frac{\partial H}{H^3} \right] + \frac{0.5 \text{Re} \frac{\partial^2 H}{H \partial \tau^2} - \alpha \text{Re} \left( \frac{\partial H}{\partial \tau} \right)^2 \left( \frac{\xi^2}{2} - \left( \frac{L}{D} \right)^2 \right) \right], \tag{2.308}
\]
where, the first term inside the brackets represents the effect of viscous forces, and the second and third terms denote the effect of temporal (unsteady) inertia and convective inertia respectively. Additionally, the definition for the film thickness is substituted into Equation (2.308) as follows:

$$\bar{P} = \left[ \frac{6\varepsilon \sin \theta}{(1 + \varepsilon \cos \theta)^3} - \frac{0.5 \text{Re} \varepsilon \cos \theta}{1 + \varepsilon \cos \theta} - \alpha \frac{\text{Re} \varepsilon^2 \sin^2 \theta}{(1 + \varepsilon \cos \theta)^2} \right] \left[ \xi^2 - \left( \frac{L}{D} \right)^2 \right].$$  \quad (2.309)

In order to determine the fluid film reaction force components, Equation (2.309) is integrated over the journal surface as follows:

$$\left[ \frac{\bar{F}_r}{\bar{F}_i} \right] = \frac{4}{3} \left( \frac{L}{D} \right)^3 \int_\theta \left[ \frac{6\varepsilon \sin \theta}{(1 + \varepsilon \cos \theta)^3} - \frac{0.5 \text{Re} \varepsilon \cos \theta}{1 + \varepsilon \cos \theta} - \frac{1.2 \text{Re} \varepsilon^2 \sin^2 \theta}{(1 + \varepsilon \cos \theta)^2} \right] \frac{\cos(\theta)}{\sin(\theta)} d\theta. \quad (2.310)

The above integration can be represented in term of the Booker’s integrals [121] in Section 2.1.2 as follows:

$$\left[ \frac{\bar{F}_r}{\bar{F}_i} \right] = \frac{4}{3} \left( \frac{L}{D} \right)^3 \left[ 6\varepsilon J^1_3 - 0.5 \text{Re} \varepsilon J^2_1 - \alpha \text{Re} \varepsilon^2 J^1_2 \right]. \quad (2.311)

2.4.2 Perturbation Method

This section develops closed-form analytical expressions for the fluid velocity components, the hydrodynamic pressure distribution, and the fluid film reaction forces by applying a first-order velocity and pressure perturbation to the flow equations. A small first-order perturbation by means of the expressions for the fluid film velocity components and the lubricant pressure distribution that are expanded in power series of the squeeze film Reynolds number is introduced as follows:

$$\bar{v} = \bar{v}_0 + \text{Re} \bar{v}_1, \quad (2.312)$$

$$\bar{w} = \bar{w}_0 + \text{Re} \bar{w}_1, \quad (2.313)$$
\( \overline{P} = \overline{P}_0 + \text{Re} \overline{P}_i. \) (2.314)

Equations (2.312) to (2.314) separate the pressure and the velocities into an inertialess zeroth-order component and a first-order inertial correction. Subsequently, assuming that the inertial components in Equation (2.290) are approximated by using inertialess velocities [20], Equation (2.312) to (2.314) are substituted into Equations (2.289) and (2.290) as follows:

\[
\frac{\partial \overline{v}_0}{\partial \eta} + \frac{\partial \overline{w}_0}{\partial \xi} = -\text{Re} \left[ \frac{\partial \overline{v}_1}{\partial \eta} + \frac{\partial \overline{w}_1}{\partial \xi} \right], \tag{2.315}
\]

\[
-\frac{\partial \overline{P}_0}{\partial \xi} + \frac{\partial^2 \overline{w}_0}{\partial \eta^2} = \text{Re} \left[ \frac{\partial \overline{w}_0}{\partial \tau} + v \frac{\partial \overline{w}}{\partial \eta} + w \frac{\partial \overline{w}}{\partial \xi} + \frac{\partial \overline{P}_1}{\partial \xi} - \frac{\partial^2 \overline{w}_1}{\partial \eta^2} \right]. \tag{2.316}
\]

Based on the definition of the inertialess continuity equation and axial momentum transport equation, the left-hand side of Equations (2.315) and (2.316) equals zero. Consequently, the following sets of zeroth-order equations and first-order equations are produced:

\[
\frac{\partial \overline{v}_0}{\partial \eta} + \frac{\partial \overline{w}_0}{\partial \xi} = 0, \tag{2.317}
\]

\[
-\frac{\partial \overline{P}_0}{\partial \xi} + \frac{\partial^2 \overline{w}_0}{\partial \eta^2} = 0, \tag{2.318}
\]

where:

\[
\begin{cases}
\overline{u}_0 = 0, \overline{v}_0 = 0, \overline{w}_0 = 0 & \eta = 0 \\
\overline{u}_0 = 0, \overline{v}_0 = \frac{\partial H}{\partial \tau}, \overline{w}_0 = 0 & \eta = H^* \tag{2.319}
\end{cases}
\]

and:

\[
\frac{\partial \overline{v}_1}{\partial \eta} + \frac{\partial \overline{w}_1}{\partial \xi} = 0, \tag{2.320}
\]
\[ \frac{\partial \bar{w}_0}{\partial \tau} + v_0 \frac{\partial \bar{w}_0}{\partial \eta} + \bar{w}_0 \frac{\partial \bar{w}_0}{\partial \xi} = - \frac{\partial \bar{P}_0}{\partial \xi} + \frac{\partial^2 \bar{w}_1}{\partial \eta^2}, \]  

(2.321)

where:

\[
\begin{cases}
  u_i = 0, v_i = 0, w_i = 0 & \eta = 0 \\
  u_i = 0, v_i = 0, w_i = 0 & \eta = H.
\end{cases}
\]

(2.322)

According to Equations (2.317) to (2.319), the zeroth-order inertialess velocities and pressure are calculated as follows:

\[
\bar{w}_0(\theta, \eta, \xi) = \frac{1}{2} \frac{\partial \bar{P}_0}{\partial \xi} (\eta^2 - \eta H),
\]

(2.323)

\[
\bar{v}_0(\theta, \eta, \xi) = - \frac{1}{2} \frac{\partial^2 \bar{P}_0}{\partial \xi^2} \left( \frac{\eta^3}{3} - \frac{\eta^2 H}{2} \right),
\]

(2.324)

\[
\frac{\partial}{\partial \xi} \left( H^3 \frac{\partial \bar{P}_0}{\partial \xi} \right) = 12 \frac{\partial H}{\partial \tau}. 
\]

(2.325)

Furthermore, according to Equation (2.325):

\[
\frac{\partial^2 \bar{P}_0}{\partial \xi^2} = \frac{12}{H^3} \frac{\partial H}{\partial \tau}. 
\]

(2.326)

Substituting Equation (2.326) into Equation (2.324) reduces the order of the zeroth-order radial velocity expression as follows:

\[
\bar{v}_0(\theta, \eta, \xi) = - \frac{\partial H}{\partial \tau} \left( \frac{2\eta^3 - 3\eta^2 H}{H^3} \right).
\]

(2.327)

In order to determine the first-order expressions, the zeroth-order velocity profiles are substituted into Equation (2.321):
\[
\frac{\partial}{\partial \tau} \left[ \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \xi} \left( \eta^2 - \eta H \right) \right] - \frac{\partial H}{\partial \tau} \left( \frac{2\eta^3 - 3\eta^2 H}{H^3} \right) \frac{\partial}{\partial \eta} \left[ \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \xi} \left( \eta^2 - \eta H \right) \right] + \\
\frac{1}{2} \frac{\partial \overline{P}_0}{\partial \xi} \left( \eta^2 - \eta H \right) \frac{\partial}{\partial \xi} \left[ \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \xi} \left( \eta^2 - \eta H \right) \right] = -\frac{\overline{P}_1}{\overline{\xi}} + \frac{\partial^2 w_1}{\partial \eta^2}.
\]

(2.328)

Equation (2.328) is twice integrated in the radial direction to calculate the axial first-order velocity as follows:

\[
-\overline{w}_1 = \frac{1}{2} \left[ \frac{\partial^2 \overline{P}_0}{\partial \xi^2} \left( \frac{\eta^4}{12} - \frac{\eta^3 H}{6} - \frac{\overline{P}_0 \partial H \eta^3}{\partial \xi \partial \tau} \right) + \frac{1}{2} \frac{\partial \overline{P}_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left[ \frac{2\eta^5}{5H^2} - \frac{\eta^4}{4H} - \frac{2\eta^6}{15H^3} \right] \right] + \\
\frac{1}{4} \frac{\partial^2 \overline{P}_0}{\partial \xi^2} \frac{\partial \overline{P}_0}{\partial \xi} \left[ \frac{\eta^6}{30} - \frac{\eta^5 H}{10} + \frac{\eta^4 H^2}{12} - \eta^2 H \right].
\]

(2.329)

and the integration constants are calculated by applying the velocity boundary conditions in Equation (2.322). Hence:

\[
-\overline{w}_1 = \frac{1}{2} \left[ \frac{\partial^2 \overline{P}_0}{\partial \xi^2} \left[ \frac{\eta^4}{12} - \frac{\eta^3 H}{6} + \frac{\eta H^3}{12} \right] \right] + \\
\frac{1}{2} \frac{\partial \overline{P}_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left[ \frac{2\eta^5}{5H^2} - \frac{\eta^4}{4H} + \frac{\eta^3 H^2}{20} + \frac{3\eta H^2}{6} \right] + \\
\frac{1}{4} \frac{\partial^2 \overline{P}_0}{\partial \xi^2} \frac{\partial \overline{P}_0}{\partial \xi} \left[ \frac{\eta^6}{30} - \frac{\eta^5 H}{10} + \frac{\eta^4 H^2}{12} - \eta H \right].
\]

(2.330)

Substituting Equation (2.330) into Equation (2.320) and integrating in the radial direction gives:

\[
\overline{v}_1 = -\frac{1}{2} \frac{\partial^3 \overline{P}_0}{\partial \xi^3} \left[ \frac{\eta^3}{3} - \frac{\eta^2 H}{2} \right] - \frac{1}{2} \frac{\partial^3 \overline{P}_0}{\partial \xi^2} \left[ \frac{\eta^5}{24} - \frac{\eta^4 H}{24} + \frac{\eta^2 H^3}{24} \right] + \\
\frac{1}{2} \frac{\partial^2 \overline{P}_0}{\partial \xi^2} \frac{\partial H}{\partial \tau} \left[ \frac{\eta^6}{15H^2} - \frac{\eta^5}{10H} - \frac{2\eta^7}{20H} + \frac{\eta^4 H^2}{24} - \frac{3\eta^2 H^2}{40} \right] + \\
\frac{1}{4} \left[ \frac{\partial^3 \overline{P}_0}{\partial \xi^3} \frac{\partial \overline{P}_0}{\partial \xi} \right] \left[ \frac{\eta^7}{210} - \frac{\eta^6 H}{60} + \frac{\eta^5 H^2}{60} - \frac{\eta^3 H^5}{120} \right] + C_1.
\]

(2.331)

The integration constant is determined by applying the velocity boundary conditions in Equation (2.322). The first boundary condition on the bush surface gives:
\[ v_t = -\frac{1}{2} \frac{\partial^3 \bar{P}_1}{\partial \xi^2} \left[ \frac{\eta^3}{3} - \frac{\eta^5 H}{2} \right] - \frac{1}{2} \frac{\partial^3 \bar{P}_0}{\partial \xi^2} \left[ \frac{\eta^5}{60} - \frac{\eta^5 H}{24} + \eta^5 H^3 \right] \]

\[ - \frac{1}{2} \frac{\partial^2 \bar{P}_0}{\partial \xi^2} \frac{\partial H}{\partial \tau} \left[ \frac{\eta^6}{15H^2} - \frac{\eta^7}{20H} - \frac{2\eta^7}{105H^3} - \frac{\eta^4}{24} + \frac{3\eta^2 H^2}{40} \right] \]

\[ - \frac{1}{4} \left[ \frac{\partial^3 \bar{P}_0}{\partial \xi^3} \frac{\partial \bar{P}_0}{\partial \xi} + \left( \frac{\partial^2 \bar{P}_0}{\partial \xi^2} \right)^2 \right] \left[ \frac{\eta^4}{210} - \frac{\eta^5 H^2}{60} + \frac{\eta^5 H^2}{60} + \eta^5 H^5 \right]. \]

The second velocity boundary condition on the shaft surface gives:

\[ \frac{H^3}{12} \frac{\partial^2 \bar{P}_1}{\partial \xi^2} = \frac{\partial^3 \bar{P}_0}{\partial \xi^2} \frac{\partial H}{\partial \tau} \left( \frac{H^5}{120} \right) + \frac{\partial^2 \bar{P}_0}{\partial \xi^2} \frac{\partial H}{\partial \tau} \left( \frac{13H^4}{840} \right) \]

\[ \left[ \frac{\partial^3 \bar{P}_0}{\partial \xi^2} \frac{\partial \bar{P}_0}{\partial \xi} + \left( \frac{\partial^2 \bar{P}_0}{\partial \xi^2} \right)^2 \right] \left( \frac{H^7}{1120} \right). \]

Integrating the above equation in the axial direction and rearranging the equation gives:

\[ \frac{\partial \bar{P}_1}{\partial \xi} = \frac{\partial \bar{P}_0}{\partial \xi} \left( \frac{H^2}{10} \right) + \frac{\partial \bar{P}_0}{\partial \xi} \frac{\partial H}{\partial \tau} \left( \frac{13H^4}{70} \right) - \frac{\partial^2 \bar{P}_0}{\partial \xi^2} \frac{\partial \bar{P}_0}{\partial \xi} \left( \frac{3H^4}{280} \right) + C_4. \]

Substituting the zeroth-order pressure from Equation (2.325) into Equation (2.334) gives:

\[ \frac{\partial \bar{P}_0}{\partial \xi} = \left( \frac{12 \partial H}{H^3 \partial \xi} \right) \xi, \]

\[ \frac{\partial^2 \bar{P}_0}{\partial \xi^2} = \left( \frac{12 \partial H}{H^3 \partial \xi} \right), \]

\[ \frac{\partial}{\partial \tau} \left( \frac{\partial \bar{P}_0}{\partial \xi} \right) = \left[ \left( \frac{12 \partial^2 H}{H^3 \partial \xi^2} - \frac{36}{H^4} \left( \frac{\partial H}{\partial \xi} \right)^2 \right) \xi, \right. \]

\[ \frac{\partial \bar{P}_1}{\partial \xi} = \left[ \left( \frac{1.2 \partial^2 H}{H \partial \xi^2} - \frac{3.6}{H^2} \left( \frac{\partial H}{\partial \xi} \right)^2 \right) \xi + \left( \frac{78}{35H^2} \left( \frac{\partial H}{\partial \xi} \right)^2 \right) \xi - \left( \frac{54}{35H^2} \left( \frac{\partial H}{\partial \tau} \right)^2 \right) \xi + C_4. \]

Integrating Equation (2.336) in the axial direction and applying the pressure boundary conditions at the axial boundaries of the bearing gives:
\[
\bar{P}_1 = \left[ \frac{0.6 \partial^2 H}{H \partial \tau^2} - \frac{51 \Re}{35H^2} \left( \frac{\partial H}{\partial \tau} \right)^2 \right] \left[ \xi^2 - \left( \frac{L}{D} \right)^2 \right].
\]

The total pressure distribution is calculated by substituting Equations (2.337) and (2.325) into Equation (2.314) as follows:

\[
\bar{P} = \left[ \frac{6 \partial H}{H^2 \partial \tau} + \frac{0.6 \Re \partial^2 H}{H \partial \tau^2} - \frac{51 \Re}{35H^2} \left( \frac{\partial H}{\partial \tau} \right)^2 \right] \left[ \xi^2 - \left( \frac{L}{D} \right)^2 \right],
\]

where, the first term inside the brackets represents the effect of viscous forces, and the second and third terms denote the effect of temporal (unsteady) inertia and convective inertia respectively. Substituting the definition of the film thickness into Equation (2.338) gives:

\[
\bar{P} = \left[ \frac{6 \varepsilon \sin \theta}{(1 + \varepsilon \cos \theta)^3} - \frac{0.6 \Re \varepsilon \cos \theta}{1 + \varepsilon \cos \theta} - \frac{51 \Re \varepsilon^2 \sin^2 \theta}{35(1 + \varepsilon \cos \theta)^2} \right] \left[ \xi^2 - \left( \frac{L}{D} \right)^2 \right].
\]

The fluid film reaction forces are calculated by integrating the pressure expression in Equation (2.339) over the journal surface as follows:

\[
\left[ \frac{\bar{F}_r}{\bar{F}_t} \right] = -\frac{4}{3} \varepsilon \left( \frac{L}{D} \right)^3 \int_0^\theta \left[ \frac{6 \varepsilon \sin \theta}{(1 + \varepsilon \cos \theta)^3} - \frac{0.6 \Re \varepsilon \cos \theta}{1 + \varepsilon \cos \theta} - \frac{51 \Re \varepsilon^2 \sin^2 \theta}{35(1 + \varepsilon \cos \theta)^2} \right] \left[ \cos(\theta) \right] d\theta.
\]

Substituting the Booker’s integrals [121] into Equation (2.340) gives:

\[
\left[ \frac{\bar{F}_r}{\bar{F}_t} \right] = -\frac{4}{3} \left( \frac{L}{D} \right)^3 \left[ 6 \varepsilon J_3^{11} - 0.6 \Re \varepsilon J_1^{20} - 51/35 \Re \varepsilon^2 J_2^{12} \right].
\]

### 2.5 Results and Discussions

This section represents the simulation results for the pressure distribution and the fluid film reaction forces for the proposed SFD models in Sections 2.1 to 2.3. The proposed SFD models are incorporated into a simulation model in Matlab and the results are represented at different SFD operating conditions, including eccentricity ratios, slenderness ratios (i.e. \( L/D \)), and
squeeze Reynolds numbers (i.e. inertia effects). In order to verify the proposed SFD models, the results of the simulation are compared against the force coefficient model developed by Vance [126]. Vance has developed the force coefficients for short-length open ended SFDs by using the π-film assumption (i.e. film cavitation region is developed in half the damper circumference). The force coefficients are represented as follows:

\[
C_n = \frac{\pi \mu D}{4(1 - \varepsilon^2)^{3/2}} \left( \frac{L}{c} \right)^3, \\
C_r = \frac{\mu \varepsilon D}{(1 - \varepsilon^2)^2} \left( \frac{L}{c} \right)^3, \\
M_n = \frac{\pi \rho D}{24} \left( \frac{L}{c} \right)^2 \left[ 1 - 2(1 - \varepsilon^2)^{1/2} \right] \left\{ \frac{(1 - \varepsilon^2)^{1/2} - 1}{\varepsilon^2(1 - \varepsilon^2)^{1/2}} \right\}, \\
M_r = -\frac{27 \rho D}{140 \varepsilon} \left( \frac{L}{c} \right)^2 \left[ 2 + \frac{1}{\varepsilon} \ln \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right) \right].
\]

The fluid film reaction force components are calculated based on the force coefficients in Equations (2.342) to (2.345) as follows:

\[
F_r = -\left( C_r V_t + M_n A_r \right), \\
F_r = -\left( C_n V_t - M_r A_r \right), \\
F_r = -\left( C_n V_t + M_n A_r \right), \\
M_r = -\left( C_r V_t - M_n A_r \right),
\]

where:

\[
V_t = \varepsilon \omega, \\
A_r = -\varepsilon \omega^2,
\]

and:
\[ \bar{F}_r = F_r / C_f , \]
\[ \bar{F}_t = F_t / C_f , \]  

where:
\[ C_f = \mu \Omega R^4 / c^2 . \]

2.5.1 Reynolds Equation

Figure 2.2 represents the effect of journal eccentricity ratio on the dimensionless mid-plane pressure distribution at small slenderness ratios for the numerical model, the analytical model, and the SBA model. In general, it is assumed that the numerical model provides the highest accuracy relative to the other models and is used as a benchmark to investigate the corresponding accuracy of the other models. The results of the LBA model were excluded from the calculations, since at small slenderness ratios the results of the LBA model are at a much larger scale compared to the other three models. At small eccentricity ratios, the results of the three models are at very close agreement. However, at moderate and large eccentricity ratios, the results of the SBA model diverge from the results of the numerical and analytical model, since in general, the SBA model only provides accurate predictions of the SFD behavior at small eccentricity ratios.

\[ \epsilon = 0.1 \text{ and } L/D = 0.5 \]

(a)  
80
Figure 2.2 The effect of eccentricity ratio on the dimensionless mid-plane pressure distribution at small slenderness ratios for inertialess SFD models
Figure 2.3 represents the effect of journal eccentricity ratio on the dimensionless mid-plane pressure distribution at large slenderness ratios for the numerical model, the analytical model, the SBA model, and the LBA model. In general, at large slenderness ratios, the predictions of the SBA model are very inaccurate, which results in a considerable disagreement with the other three models. Furthermore, at small, moderate, and eccentricity ratios the accuracy of the analytical model is significantly higher compared to the LBA model.

(a)
Figure 2.3 The effect of eccentricity ratio on the dimensionless mid-plane pressure distribution at large slenderness ratios for inertialess SFD models
Figure 2.4 shows the effect of journal eccentricity ratio on the radial and tangential dimensional fluid film reaction forces at small slenderness ratios for the numerical model, the analytical model, and the SBA model. For the radial fluid film forces, at small eccentricity ratios, the results of the three models are in close agreement. However, at moderate and large eccentricity ratios, the results of the analytical and SBA models diverge from the results of the numerical model. Similarly, for the tangential force component, the three models are in very close agreement at small eccentricity ratios. At moderate and large eccentricity ratios, the analytical and SBA models maintain their agreement, however, the predictions of the two models is in disagreement with the numerical model.
**Figure 2.4** The dimensionless radial and tangential fluid film reaction forces at small slenderness ratios for inertialess SFD models

Figure 2.5 demonstrates the effect of journal eccentricity ratio on the radial and tangential dimensional fluid film reaction forces at large slenderness ratios for the numerical model, the analytical model, the SBA model, and the LBA model. For both the radial and tangential forces, the results of the numerical model and the analytical model are in very close agreement. The results of the LBA model demonstrate significant disagreement with the numerical results for both force components. Furthermore, the discrepancy between the results of the SBA model and the numerical model are very large, since the SBA model only provides accurate predictions at small slenderness ratios.
Figure 2.5 The dimensionless radial and tangential fluid film reaction forces at large slenderness ratios for inertialess SFD models
2.5.2 Small Amplitude Motions of the Journal Center

Figure 2.6 represents the effect of fluid inertia on the dimensionless mid-plane pressure distribution at very small eccentricity ratios for the numerical Reynolds model, the SAM analytical model, the SAM perturbation model, and the SAM momentum approximation model. In general, the fluid inertia effect causes a significant elevation in the pressure magnitude, a change in the shape of the pressure profile, and a phase shift of the pressure peak in the direction of the journal precision. At small Reynolds number, where the effect of fluid inertia is small, the results of the Reynolds numerical model and the three models including inertia effects is in very close agreement. At moderately small inertia effects, the discrepancy between the results including inertia effects grows noticeably. This discrepancy is very significant at moderate and moderately large inertia effects. At small Reynolds numbers, the influence of the viscous forces makes the pressure profile closer to a sinusoid, however, at moderate and large Reynolds numbers the pressure is in phase with the gap acceleration and transforms into a cosine wave shape. Furthermore, the maximum pressure magnitude considerably grows with the fluid inertia effects.

(a)
Figure 2.6 The effect of fluid inertia on the mid-plane pressure distribution at very small eccentricity ratios for small amplitude motions of the journal center.

Figure 2.7 represents the effect of fluid inertia on the dimensionless mid-plane pressure distribution at small eccentricity ratios for the numerical Reynolds model, the SAM analytical model, the SAM perturbation model, and the SAM momentum approximation model. At moderate to large Reynolds numbers, the effect of fluid inertia completely dominates the viscous effects, which results in a significant variation in the pressure magnitude and profile. Furthermore, the comparison between the three SAM inertia models demonstrates a significant discrepancy between the results of the SAM analytical model and the SAM perturbation and SAM momentum approximation models, since the analytical model is developed by assuming that the eccentricity ratio of the journal center is very small, such that $H \approx 1$, which makes the model inapplicable for predicting SFD performance at larger eccentricity ratios. However, based on the results of Figure 2.6, the analytical model remains a fast and accurate tool for predicting SFD performance at very small eccentricity ratios.
Figure 2.7 The effect of fluid inertia on the mid-plane pressure distribution at small eccentricity ratios for small amplitude motions of the journal center.
Figure 2.8 displays the effect of fluid inertia on the dimensionless mid-plane pressure distribution at moderately small eccentricity ratios for the numerical Reynolds model, the SAM analytical model, the SAM perturbation model, and the SAM momentum approximation model. At moderately small eccentricity ratios the effect of fluid inertia is very significant even at moderately small Reynolds numbers. Furthermore, the comparison between the SAM inertia models shows that the SAM analytical model is completely inapplicable for moderately small eccentricity ratios.
\( \epsilon = 0.25 \) and \( \text{Re} = 5 \)

(b)

\( \epsilon = 0.25 \) and \( \text{Re} = 10 \)

(c)
Figure 2.8 The effect of fluid inertia on the mid-plane pressure distribution at moderately small eccentricity ratios for small amplitude motions of the journal center.

Figure 2.9 represents the effect of fluid inertia on the SFD fluid film reaction forces at small eccentricity ratios for five SFD models, namely the numerical Reynolds model, the SAM analytical model, the SAM perturbation model, the SAM momentum approximation model, and the force coefficient model [126]. The fluid forces are calculated by numerically integrating the lubricant hydrodynamic pressure distribution over the journal surface. For a π-film SFD model at small to moderate inertia effects, the contribution of the inertia forces to the radial force component is a positive value. This positive contribution is added to the negative viscous radial forces and reduces the magnitude of the radial force component relative to the inertialess model, which diminishes the likelihood of bi-stable rotor operation [127]. Furthermore, the magnitude of the inertialess radial forces is negative, meaning that the force is directed towards the center of the bearing (i.e. inwards). However, introducing the effect of fluid inertia initially changes the value of the forces to positive at small eccentricities, demonstrating the outward direction of the forces (i.e. outwards). Subsequently, at larger eccentricity ratios, the value of the radial forces switches back to negative, meaning that the force direction is once again towards the center of
the bearing. Moreover, the contribution of the inertial forces to the tangential fluid film reaction forces is a negative value, which is added to the already negative purely viscous tangential forces, thus increasing the total magnitude of the tangential forces. The comparison between the SAM analytical model and the other three inertia models shows that the SAM analytical model is only applicable to very small eccentricity ratios of the journal center. Furthermore, the comparison between the Vance force coefficient model and the SAM perturbation and momentum approximation models imply that at small and moderately small inertia effects, the three models are in very close agreement. However, at larger Reynolds numbers, where the effect of fluid inertia is more significant, the results of the force coefficient model diverges from the SAM perturbation and momentum approximation models, since the force coefficient model is developed for short-length bearings, and is generally accurate at small fluid inertia effects.

![Graph showing SFD Dimensionless Radial Forces at Re = 1](image)

(a)
Radial Force $F_r$
(f) SFD Dimensionless Tangential Forces at Re = 10

(g) SFD Dimensionless Radial Forces at Re = 15
2.5.3 Large Amplitude Motions of the Journal Center

According to the discussions in the previous sections, at large amplitude motions of the journal center, the effect of convective inertia terms is no longer negligible relative to the temporal inertia effects and the complete inertia terms must be considered in the calculations. This section compares the effect of convective fluid inertia and temporal fluid inertia on the dynamic performance of SFDs. Figure 2.10 shows the effect of fluid inertia on the dimensionless mid-plane pressure distribution at small eccentricity ratios for five models, namely the numerical Reynolds model, the SAM perturbation model, the SAM momentum approximation model, the complete inertia perturbation model, and the complete inertia momentum approximation model. At small Reynolds numbers, the effect of fluid inertia is small and the viscous forces are dominant, hence, the influence of the viscous forces makes the pressure profile closer to a sinusoid and the results of the five models are in very close agreement. At moderately small Reynolds numbers, the effect of fluid inertia is more significant and the inertia forces begin to dominate the viscous forces, hence, the pressure is in phase with the gap acceleration and
transforms into a cosine wave shape. At moderate and large Reynolds numbers, the effect of fluid inertia further grows, which noticeably elevates the magnitude of the maximum pressure. Furthermore, since the eccentricity ratio of the journal center is small, the predictions of the inertia models for small amplitude motions and large amplitude motions are in good agreement, since at small eccentricity ratios, the effect of convective fluid inertia components is negligible relative to the temporal inertia.

(a)
Figure 2.10 The effect of fluid inertia on the mid-plane pressure distribution at small eccentricity ratios for small and large amplitude motions of the journal center.

Figure 2.11 represents the effect of fluid inertia on the dimensionless mid-plane pressure distribution at moderate eccentricity ratios for five models, namely the numerical Reynolds model, the SAM perturbation model, the SAM momentum approximation model, the complete inertia model with perturbation method, and the complete inertia model with momentum approximation method. At moderate eccentricity ratios, the effect of fluid inertia is slightly more considerable on the phase, the shape, and the magnitude of the maximum pressure, since according to Sections 2.2 and 2.3, the fluid inertia components are directly proportional to the eccentricity ratio of the journal. Similarly, the effect of fluid inertia on the pressure profile is much more significant at moderate and large Reynolds numbers. Furthermore, even at moderate eccentricity ratios, the pressure profiles developed by the small amplitude motion models and the large amplitude motion models begin to demonstrate divergence in terms of the shape and the magnitude of the maximum pressure.
Figure 2.11 The effect of fluid inertia on the mid-plane pressure distribution at moderate eccentricity ratios for small and large amplitude motions of the journal center.
Figure 2.12 displays the effect of fluid inertia on the dimensionless mid-plane pressure distribution at moderate eccentricity ratios for five models, namely the numerical Reynolds model, the SAM perturbation model, the SAM momentum approximation model, the complete inertia model with perturbation method, and the complete inertia model with momentum approximation method. At large eccentricity ratios, the discrepancy between the results of the small amplitude motion models and the large amplitude motion models is quite significant even at moderately small Reynolds numbers, since at large amplitude motions, the effect of convective fluid inertia dominates the temporal inertia, which considerably degrades the accuracy of the SAM models and makes them inapplicable for predicting the performance of the SFD performance. At moderate and large Reynolds numbers, the pressure profiles suggested by the small amplitude motion models and the large amplitude motion models are completely different from one another.

![Dimensionless Pressure](image)

$\epsilon = 0.5$ and $\text{Re} = 1$
Figure 2.12 The effect of fluid inertia on the mid-plane pressure distribution at large eccentricity ratios for small and large amplitude motions of the journal center.

Figure 2.13 The effect of fluid inertia on the SFD fluid film reaction forces at small eccentricity ratios for six SFD models, namely the numerical Reynolds model, the SAM perturbation model, the SAM momentum approximation model, the complete inertia perturbation model, the complete inertia momentum approximation model, and the force coefficient model [126]. The fluid forces are calculated by numerically integrating the lubricant hydrodynamic pressure distribution over the journal surface. For the dimensionless radial force components, in general, the results of the inertia models are in very close agreement at small eccentricity ratios (i.e. $\varepsilon \leq 0.25$). However, at moderate and large eccentricity ratios, the magnitude of the radial forces suggested by the small amplitude motion models is noticeably larger relative to the complete inertia model. This discrepancy is much more significant at larger Reynolds numbers, where the effect of fluid inertia is larger. This is justified, since at small eccentricity ratios, where the effect of temporal inertia is dominant, an added mass is produced, which corresponds to the radial direct inertia coefficient, however, at large eccentricity ratios, where the effect of convective inertia is superior, this effect is completely reversed. Similarly, for the tangential fluid film force
components, at moderate and large eccentricity ratios, the contribution of the convective inertia terms is a negative value, which further increases the magnitude of the forces relative to the SAM models.
SFD Dimensionless Radial Forces at Re = 5

(c)

SFD Dimensionless Tangential Forces at Re = 5

(d)
SFD Dimensionless Radial Forces at Re = 10

SFD Dimensionless Tangential Forces at Re = 10
Figure 2.13 The effect of fluid inertia on the dimensionless radial and tangential fluid film force components for both small and large amplitude motions of the journal center.
2.5.4 Short Bearing Approximation

The SBA model including fluid inertia provides an accelerated and accurate prediction of the SFD performance at different conditions. This is especially valuable in rotordynamic models, where the SFD forces are calculated over thousands or even millions of iterations. In this section, the effect of fluid inertia on the SFD pressure distribution is discussed at eccentricity ratios \( \varepsilon = 0.1 \) to \( \varepsilon = 0.5 \) and at Reynolds numbers \( \text{Re} = 1 \) to \( \text{Re} = 15 \). Eccentricity ratios higher that 0.5 are rarely attained in practical applications of SFDs. Furthermore, in most SFD applications, the squeeze Reynolds is between 1 and 15. Figures 2.14 to 2.16 represent the effect of fluid inertia on the dimensionless axial mid-plane (\( \xi = 0 \)) hydrodynamic pressure distribution for SFDs at different eccentricity ratios and Reynolds numbers. The figures compare the results under different SFD operating criteria, including complete inertia effects (momentum approximation and perturbation), small amplitude motions of the journal center (temporal inertia only for momentum approximation and perturbation), and no inertia effects. In general, including fluid inertia effects causes a significant elevation in the maximum pressure amplitude and a phase shift in the direction of the precession motion, which is significantly greater at the small eccentricity ratios. The pressure amplitude increases more significantly at larger eccentricity ratio values. Furthermore, at small Reynolds numbers, the influence of the viscous forces makes the pressure profile closer to a sinusoid, however, at moderate and large Reynolds numbers the pressure is in phase with the gap acceleration and transforms into a cosine wave shape. Moreover, at small Reynolds numbers, the magnitude and the shape of the mid-plane pressure distribution is very similar for both complete inertia models and temporal inertia models, since at small eccentricity ratios the effect of convective inertia is negligible relative to temporal inertia. However, at large eccentricity ratios, the discrepancy between the results of the complete inertia and SAM models is very noticeable due to the dominance of the convective inertia forces.
Re = 1 and $\epsilon = 0.1$

Re = 5 and $\epsilon = 0.1$
Figure 2.14 The effect of fluid inertia on the mid-plane pressure distribution at small eccentricity ratios for small and large amplitude motions of the journal center for short-length SFDs.
Re = 1 and $\epsilon = 0.25$

Re = 5 and $\epsilon = 0.25$
Figure 2.15 The effect of fluid inertia on the mid-plane pressure distribution at moderate eccentricity ratios for small and large journal amplitude motions for short-length SFDs
Re = 1 and $\epsilon = 0.5$

No Inertia SBA
SAM Momentum App. SBA
SAM Perturbation SBA
Complete Inertia Momentum App. SBA
Complete Inertia Perturbation SBA

(a)

Re = 5 and $\epsilon = 0.5$

No Inertia SBA
SAM Momentum App. SBA
SAM Perturbation SBA
Complete Inertia Momentum App. SBA
Complete Inertia Perturbation SBA

(b)
Figure 2.16 The effect of fluid inertia on the mid-plane pressure distribution at small eccentricity ratios for small and large amplitude motions of the journal center for short-length SFDs
Figure 2.17 illustrates the dimensionless radial and tangential fluid film reaction forces at different eccentricity ratios and Reynolds numbers for short-length SFDs. The figure compares the results under different SFD operating criteria, including complete inertia effects (momentum approximation and perturbation), small amplitude motions of the journal center (temporal inertia only for momentum approximation and perturbation), force coefficients [126], and no inertia effects. For a π-film short SFD model at low to moderate inertia effects, the contribution of the inertia forces to the radial force component is a positive value. This positive contribution is added to the negative viscous radial forces and reduces the magnitude of the radial force component relative to the inertialess model, which diminishes the likelihood of bi-stable rotor operation [127]. Furthermore, the magnitude of the inertialess radial forces is negative, meaning that the force is directed towards the center of the bearing (i.e. inwards). However, introducing the effect of fluid inertia initially changes the value of the forces to positive at small eccentricities, demonstrating the outward direction of the forces (i.e. outwards). Subsequently, at larger eccentricity ratios, the value of the radial forces switches back to negative, meaning that the force direction is once again towards the center of the bearing. Moreover, the contribution of the inertial forces to the tangential fluid film reaction forces is a negative value, and is added to the already negative purely viscous tangential forces, thus increasing the total magnitude of the tangential forces. Finally, comparing the radial and tangential forces for the complete inertia model and the temporal inertia model confirms that the magnitude of the forces is very similar at small eccentricity ratios. However, at large eccentricity ratios, the force values demonstrate notable discrepancy, proving that the effect of convective inertia components can no longer be neglected at large eccentricity ratios.
SFD Dimensionless Radial Forces at Re = 1

(a)

SFD Dimensionless Tangential Forces at Re = 1

(b)
SFD Dimensionless Radial Forces at Re = 5

SFD Dimensionless Tangential Forces at Re = 5

Vance [126]
(e)

SFD Dimensionless Radial Forces at Re = 10

(f)

SFD Dimensionless Tangential Forces at Re = 10
Figure 2.17 The effect of fluid inertia on the dimensionless radial and tangential fluid film force for both small and large amplitude motions of the journal center for short-length SFDs.
2.6 Conclusion

This chapter investigated the effect of fluid inertia on the fluid velocity components, lubricant pressure distribution, and fluid film reaction forces for SFDs. Firstly, the thin film equations were developed by applying appropriate assumptions to the general continuity and Navier-Stokes equations. Subsequently, the thin film equations were solved under different SFD operating conditions, including no fluid inertia effects, small amplitude motions of the journal center, large amplitude motions of the journal center, and short-length bearings.

The SFD pressure distribution in the absence of fluid inertia was characterized by Reynolds equation. Furthermore, the approximate geometry solutions for Reynolds equation, including SBA and LBA were represented. Subsequently, a finite difference approximation based numerical procedure incorporating an iterative Gauss-Seidel algorithm with successive over relaxation was developed to numerically determine the pressure distribution and the fluid film reaction forces. Additionally, a closed-form analytical expression for the pressure distribution was developed by adopting the classical separation of variables technique in PDEs.

Furthermore, the effect of fluid inertia was studied for squeeze film dampers executing SAMs. According to an order of magnitude analysis, at small amplitude motions of the journal center, the effect of convective fluid inertia is negligible relative to the unsteady inertia components. Subsequently, the thin film equations are further reduced and the expressions for the fluid velocities, pressure distribution, and fluid film reaction forces was developed by using analytical separation of variables technique, perturbation method, and momentum approximation method.

For large amplitude motions of the journal center, including at the resonance frequency zone, the effect of convective inertia can be no longer neglected relative to the temporal inertia. Consequently, this thesis adopted perturbation technique and momentum approximation method to develop numerical models to study the effect of convective inertia on the SFD dynamics.

Finally, SBA was applied to the thin film equations to develop closed-form analytical models to provide accelerated estimation of the SBD parameters including the complete inertia effect.
Subsequently, the proposed SFD models were incorporated into simulation models and the results were demonstrated at different SFD operating conditions, including eccentricity ratios, slenderness ratios and Reynolds numbers. The comparison between the proposed models and an existing force coefficient model verified the correctness of the simulation results. Furthermore, the simulation results were incorporated to investigate the effect of fluid inertia on the SFD dynamics. The results of the analysis showed that:

1. Including the inertia effects increases the pressure magnitude, shifts the position of the pressure peak in the direction of the journal precision, and changes the shape of the pressure profile. The shape of the pressure is sinusoidal at small Reynolds numbers, while at moderate and large Reynolds numbers the pressure is in phase with the gap acceleration and transforms into a cosine wave shape.

2. At small to moderate Reynolds numbers, the fluid inertia effect reduces the magnitude of the radial force component relative to the inertialess model, which diminishes the likelihood of bi-stable rotor operation. Furthermore, the direction of the radial forces initially changes to outwards at small eccentricity ratios, however, the direction switches back to inwards at moderate to large eccentricity ratios.

3. The fluid inertia effect increases the magnitude of the tangential reaction forces; however, the direction of the forces remains unaffected.

4. The results of the SAM model and complete inertia models are in very close agreement at small eccentricity ratios, however, at large eccentricity ratios there is a noticeable discrepancy between the predictions of the two models. This is justified, since at small eccentricity ratios, where the effect of temporal inertia is dominant, an added mass is produced, which corresponds to the radial direct inertia coefficient, however, at large eccentricity ratios, where the effect of convective inertia is superior, this effect is completely reversed.
Chapter 3
Rotordynamic Analysis

This chapter develops a finite-element based multi-mass flexible rotordynamic model. Furthermore, in order to investigate the effect of the lubricant fluid inertia on the steady-state and transient unbalance induced vibration amplitudes of the rotor, the proposed SFD models in Chapter 2 are incorporated into the rotordynamic model.

3.1 Finite Element Model of a Multi-Mass Flexible Rotor

The main components of a conventional rotor system include the shaft, the disk, the mass unbalance, the rolling elements, the SFDs, and the retaining springs. This work applies finite element method (FEM) [128] to determine the vibration amplitudes in the rotor system. FEM is a sophisticated numerical technique that is implemented for the analysis of stress, vibration, and other dynamic phenomena. In order to apply FEM to the rotor system, firstly, the structure is discretized into subdivisions of simple geometry that are referred to by elements. Subsequently, the elastic, inertia, damping, and external forces and moments on each element (local forces) are expressed in terms of the local coordinates (i.e. translations and rotations). Furthermore, the forces and the moments from every element are assembled together to produce the global forces in terms of the generalized coordinates. Finally, the discretized equation of motion is solved by using numerical techniques. In a rotor system, the shaft is discretized into several beam elements with nodes at both ends of each element. The disks and the bearings are assumed to be attached to the shaft at these nodes. In this work, the rotor vibrations in the lateral and transverse directions are considered. Consequently, each node has four generalized coordinates, including transverse displacements in the X and Y directions as well as rotations about X-axis and Y-axis (i.e. \( \hat{u}, \hat{v}, \hat{\theta} \) and \( \hat{\psi} \)). The element matrices for the discretized system components are determined as follows [129].

3.1.1 The Disk Element

The properties of a rigid disk are determined by using Lagrange’s energy method. It is assumed that the disk strain energy is negligible with respect to the disk kinetic energy. Subsequently, the kinetic energy of the disk due to translation and rotation are calculated, and Lagrange’s energy
method is applied to the energy equation to determine the element mass matrix and element gyroscopic matrix for the disk as follows:

\[
M_{d,e} = \begin{bmatrix}
 m_d & 0 & 0 & 0 \\
 0 & m_d & 0 & 0 \\
 0 & 0 & I_d & 0 \\
 0 & 0 & 0 & I_d
\end{bmatrix},
\]

(3.1)

\[
G_{d,e} = \begin{bmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & I_p & 0 \\
 0 & 0 & -I_p & 0
\end{bmatrix},
\]

(3.2)

where:

\[
m_d = \rho_d \frac{\pi}{4} \left( D_{d,\text{out}}^2 - D_{d,\text{in}}^2 \right) W_d,
\]

(3.3)

\[
I_d = \rho \frac{\pi}{64} \left( D_{d,\text{out}}^4 - D_{d,\text{in}}^4 \right) W_d + \frac{m_d}{12} W_d^2,
\]

(3.4)

\[
I_p = \rho \frac{\pi}{32} \left( D_{d,\text{out}}^4 - D_{d,\text{in}}^4 \right) W_d.
\]

(3.5)

3.1.2 The Shaft Element

The shaft contributes mass, stiffness, and gyroscopic effect to the rotor model. Similarly, the stiffness, inertia, and gyroscopic element matrices for the shaft are determined by using Lagrange’s energy method. The element matrices are derived for either the Bernoulli beam model or the Timoshenko beam model (includes both rotary inertia and shear effects). In order to determine the element matrices, firstly, the displacement along the shaft element is approximated by using shape functions. Subsequently, expressions for the kinetic energy and the strain energy are used to determine the element matrices. The strain energy of the shaft is calculated to approximate the element stiffness matrix. The kinetic energy due to the translation of the shaft gives the inertia element matrix and the kinetic energy due to the rotational motion of the shaft
gives the gyroscopic effect matrix. The stiffness matrix for a beam element with 4 degrees of
freedom at each node is given as:

\[
K_e = \frac{E_e I_e}{(1 + \hat{\phi})L_e^3} \begin{bmatrix}
12 & 0 & 0 & 6L_e & -12 & 0 & 0 & 6L_e \\
0 & 12 & -6L_e & 0 & 0 & -12 & 6L_e & 0 \\
0 & -6L_e & (4 + \hat{\phi})L_e^2 & 0 & 0 & 6L_e & (2 - \hat{\phi})L_e^2 & 0 \\
6L_e & 0 & 0 & (4 + \hat{\phi})L_e^2 & -6L_e & 0 & 0 & (2 - \hat{\phi})L_e^2 \\
-12 & 0 & 0 & -6L_e & 12 & 0 & 0 & -6L_e \\
0 & -12 & 6L_e & 0 & 0 & 12 & 6L_e & 0 \\
0 & -6L_e & (2 - \hat{\phi})L_e^2 & 0 & 0 & 6L_e & (4 + \hat{\phi})L_e^2 & 0 \\
6L_e & 0 & 0 & (2 - \hat{\phi})L_e^2 & -6L_e & 0 & 0 & (4 + \hat{\phi})L_e^2
\end{bmatrix},
\]

(3.6)

where:

\[
I_e = \frac{\pi}{64} \left(D_{out}^4 - D_{in}^4 \right),
\]

(3.7)

and:

\[
\hat{\phi} = \frac{12E_e I_e}{G_{shear,e} \kappa A_e L_e^2},
\]

(3.8)

where [130]:

\[
G_{shear,e} = \frac{E_e}{2(\nu + 1)},
\]

(3.9)

\[
\hat{\kappa} = \frac{6 \left(1 + \left(\frac{R_{in}}{R_{out}}\right)^2\right) \left(1 + \nu\right)}{\left(1 + \left(\frac{R_{in}}{R_{out}}\right)^2\right)^2 \left(7 + 6\nu\right) + \left(\frac{R_{in}}{R_{out}}\right)^2 \left(20 + 12\nu\right)}.
\]

(3.10)

Moreover, the inertia element matrix including the shear effect is given as follows:
\[
M_{e} = \frac{\rho_e A L_e}{840(1 + \phi)^2} \begin{bmatrix}
  m_1 & 0 & 0 & m_2 & m_3 & 0 & 0 & m_4 \\
  0 & m_1 & -m_2 & 0 & 0 & m_3 & -m_4 & 0 \\
  0 & -m_2 & m_5 & 0 & 0 & m_4 & m_6 & 0 \\
  m_2 & 0 & 0 & m_5 & -m_4 & 0 & 0 & m_6 \\
  m_3 & 0 & 0 & -m_4 & m_1 & 0 & 0 & -m_2 \\
  0 & m_3 & m_4 & 0 & 0 & m_1 & m_2 & 0 \\
  0 & -m_4 & m_6 & 0 & 0 & m_2 & m_5 & 0 \\
  m_4 & 0 & 0 & m_6 & -m_2 & 0 & 0 & m_5 \\
\end{bmatrix},
\]

(3.11)

where:

\[
m_1 = 312 + 588\Phi + 280\Phi^2,
\]

\[
m_2 = \left(44 + 77\Phi + 35\Phi^2\right)L_e,
\]

\[
m_3 = 108 + 252\Phi + 140\Phi^2,
\]

\[
m_4 = -\left(26 + 63\Phi + 35\Phi^2\right)L_e,
\]

\[
m_5 = \left(8 + 14\Phi + 7\Phi^2\right)L_e^2,
\]

\[
m_6 = -\left(6 + 14\Phi + 7\Phi^2\right)L_e^2.
\]

Additionally, the effect of rotary inertia element is given by:

\[
M_{e} = \frac{\rho_e I_e}{830L_e(1 + \phi)^2} \begin{bmatrix}
  m_7 & 0 & 0 & m_8 & -m_7 & 0 & 0 & m_8 \\
  0 & m_7 & -m_8 & 0 & 0 & -m_7 & -m_8 & 0 \\
  0 & -m_8 & m_9 & 0 & 0 & m_8 & m_{10} & 0 \\
  m_8 & 0 & 0 & m_9 & -m_8 & 0 & 0 & m_{10} \\
  -m_7 & 0 & 0 & -m_8 & m_7 & 0 & 0 & -m_8 \\
  0 & -m_7 & m_8 & 0 & 0 & m_7 & m_8 & 0 \\
  0 & -m_8 & m_{10} & 0 & 0 & m_8 & m_9 & 0 \\
  m_8 & 0 & 0 & m_{10} & -m_8 & 0 & 0 & m_9 \\
\end{bmatrix},
\]

(3.13)

where:
\[ m_7 = 36, \]
\[ m_8 = \left(3 - 15\hat{\phi}\right) L_e, \]
\[ m_9 = \left(4 + 5\hat{\phi} + 10\hat{\phi}^2\right) L_e^2, \]
\[ m_{10} = \left(-1 - 5\hat{\phi} + 5\hat{\phi}^2\right) L_e^2, \] (3.14)

and the total mass element matrix is given as follows:

\[ M_e = M_{e_1} + M_{e_2}. \] (3.15)

Furthermore, the gyroscopic element matrix is given as follows:

\[
G_e = -\frac{\rho_e I_e}{15L_e\left(1 + \hat{\phi}\right)^2} \begin{bmatrix}
0 & -g_1 & g_2 & 0 & 0 & g_1 & g_2 & 0 \\
g_1 & 0 & 0 & g_2 & -g_1 & 0 & 0 & g_2 \\
-g_2 & 0 & 0 & -g_3 & g_2 & 0 & 0 & -g_4 \\
0 & -g_2 & g_3 & 0 & 0 & g_2 & g_4 & 0 \\
0 & g_1 & -g_2 & 0 & 0 & -g_1 & -g_2 & 0 \\
-g_1 & 0 & 0 & -g_2 & g_1 & 0 & 0 & -g_2 \\
-g_2 & 0 & 0 & -g_4 & g_2 & 0 & 0 & -g_3 \\
0 & -g_2 & g_4 & 0 & 0 & g_2 & g_3 & 0
\end{bmatrix},
\] (3.16)

where:

\[ g_1 = 36, \]
\[ g_2 = \left(3 - 15\hat{\phi}\right) L_e, \]
\[ g_3 = \left(4 + 5\hat{\phi} + 10\hat{\phi}^2\right) L_e^2, \] (3.17)
\[ g_4 = \left(-1 - 5\hat{\phi} + 5\hat{\phi}^2\right) L_e^2. \]

### 3.1.3 The Rolling and Support Elements

The bearing element includes the stiffness and the damping contribution from the rolling elements as well as the stiffness contribution from the retaining springs in the SFDs. The
bearings are flexible and absorb energy. Consequently, the forces acting on the shaft due to the displacements and velocities at the bearing nodes are approximated by [131]:

\[
\begin{align*}
\begin{bmatrix}
    f_x \\
    f_y
\end{bmatrix} &= -\begin{bmatrix}
    k_{uu} & k_{uv} \\
    k_{vu} & k_{vv}
\end{bmatrix} \begin{bmatrix}
    u \\
    v
\end{bmatrix} - \begin{bmatrix}
    c_{uu} & c_{uv} \\
    c_{vu} & c_{vv}
\end{bmatrix} \begin{bmatrix}
    \dot{u} \\
    \dot{v}
\end{bmatrix}.
\end{align*}
\]

(3.18)

### 3.1.4 The Unbalance Force

The main source of vibrations in a rotor system is the unbalance forces that are attributed to the limitations in manufacturing and processing of mechanical structures. These unbalances lead to a synchronous cycle load in the rotor. Assuming that the unbalance mass is attached to the disk, the unbalance force in the rotor is determined as:

\[
F_{\text{unb}}(t) = \Re(\Omega^2 F_0 e^{i\Omega t}),
\]

(3.19)

where \(\Omega^2 F_0\) is the vector of forces and moments acting at the \(i^{\text{th}}\) node due to a disk with an unbalance, where \(F_0\) is given as follows:

\[
F_0 = \begin{bmatrix}
    m_{\text{unb},i} \sigma e^{j\beta} \\
    -j m_{\text{unb},i} \sigma e^{j\beta} \\
    j(I_{d,i} - I_{p,i}) \psi e^{j\gamma} \\
    (I_{d,i} - I_{p,i}) \psi e^{j\gamma}
\end{bmatrix}.
\]

(3.20)

Assuming that no other unbalance forces are applied to the rotor system, the rest of the force vector entities are zero.

### 3.1.5 The Squeeze Film Damper

This work includes the effect of fluid inertia in SFDs. The measure for the lubricant inertia effect on the SFD damping characteristics is the squeeze Reynolds number. The SFDs are assumed to have small slenderness ratios (i.e. \(L/D \leq 0.5\)) such that the open-ended short bearing approximation is also applicable. Furthermore, the lubricant is assumed unpressurized (i.e. supply pressure at atmospheric pressure) such that the \(\pi\)-film cavitation model is valid for the range of the considered flow conditions. According to the \(\pi\)-film cavitation model, the film
extends for $\pi$ radians in the region of positive pressure. Moreover, the lubricant flow is considered laminar for all range of applied Reynolds numbers and the effect of fluid inertia on the fluid film pressure distribution and the fluid film reaction forces is included. Finally, application of retaining elastic elements (i.e. centralizing springs) ensures the circular-centered orbits (CCOs) of the SFD journal center.

Subsequently, the proposed SFD models in Chapter 2 are incorporated into the rotordynamic model. The pressure distribution expressions are integrated over the journal surface to determine the fluid film reaction force components:

\[
\bar{F}_r = \int_{-L/D}^{L/D} \bar{P}(\theta, \xi) \left[ \cos(\theta) \right] d\theta d\xi. 
\]

Additionally, a transformation is applied to calculate the SFD forces in the fixed inertial coordinate (i.e. \{X,Y\}) as follows:

\[
K_{SFD,eq} = F_r / (c_{SFD}), \\
C_{SFD,eq} = F_r / (\Omega c_{SFD}). 
\]

and:

\[
F_X = K_{SFD,eq} X + C_{SFD,eq} \dot{X}, \\
F_Y = K_{SFD,eq} Y + C_{SFD,eq} \dot{Y}. 
\]

The alternative transformation is to use the SFD attitude angle as follows:

\[
\phi = \begin{cases} 
\arctan \left( \frac{Y}{X} \right) + \frac{3\pi}{2} & X \geq 0 \\
\arctan \left( \frac{Y}{X} \right) + \frac{\pi}{2} & X < 0 
\end{cases}, 
\]

and:
\[
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix} = \begin{bmatrix}
\sin \phi & \cos \phi \\
\cos \phi & -\sin \phi
\end{bmatrix}
\begin{bmatrix}
F_r
\end{bmatrix}.
\]

(3.25)

### 3.1.6 The Equation of Motion of The Rotor System

This section describes the equation of motion (EOM) and the geometry of the rotor system. It is assumed that the rotor is symmetric and a disk is located at the axial center of the rotor. Furthermore, the rotor is supported with SFDs at either axial end. The SFDs are assembled on the outer race of two identical rolling elements that the rotor is assembled on. Moreover, the unbalance forces are present only on the centered disk.

The EOM of the rotor system is determined either by using the Newton’s second law of motion or the Lagrange’s energy method. For a conventional rotor system, the EOM is constructed as follows:

\[
[M]\{\ddot{q}\} + ([C] + \Omega[G])\{\dot{q}\} + [K]\{q\} = \{F(t)\},
\]

(3.26)

where \(q\) is a vector that contains the translations and the rotations at the nodes. The element matrices are 8-by-8 for stiffness, gyroscopic effect, and inertia. The effect of bearings, SFDs, and disks are directly incorporated into the corresponding nodes. Furthermore, the global mass, stiffness, and gyroscopic effect matrices are constructed by transforming the local element matrices into global coordinates and assembling the matrices together.

### 3.2 The Free Response of The Rotor Model

Assuming that the rotor equation of motion does not contain non-linear expressions, the free response and the forced response of the rotor system are directly determined. The free response of the rotor system is determined to identify the natural frequencies and the whirl direction of the rotor system at resonance. In order to determine the free response of the rotor system, firstly, the EOM is transformed into state-space form as follows:

\[
[M]\{\dot{q}\} = -([C] + \Omega[G])\{\dot{q}\} - [K]\{q\} = 0,
\]

(3.27)

hence:
\[
\{\dot{q}\} = -[M]^{-1} ([C] + \Omega[G])\{\dot{q}\} - [M]^{-1} [K]\{q\} = 0.
\] (3.28)

Assuming that:
\[
\{x_1\} = \{q\},
\{x_2\} = \{\dot{x}_1\} = \{\dot{q}\},
\{\dot{x}_2\} = \{\dot{q}\},
\] (3.29)
gives:
\[
\{x_2\} = \{\dot{x}_1\},
\{\dot{x}_2\} = -[M]^{-1} ([C] + \Omega[G])\{x_2\} - [M]^{-1} [K]\{x_1\},
\] (3.30)
hence:
\[
\begin{pmatrix}
\{\dot{x}_1\} \\
\{\dot{x}_2\}
\end{pmatrix}
= 
\begin{bmatrix}
[\emptyset] & [I] \\
-\[M]^{-1} [K] & -\[M]^{-1} ([C] + \Omega[G])
\end{bmatrix}
\begin{pmatrix}
\{x_1\} \\
\{x_2\}
\end{pmatrix}.
\] (3.31)

By identity:
\[
\{\dot{X}(t)\} = [A]\{X(t)\},
\] (3.32)
where:
\[
\{\dot{X}(t)\} = 
\begin{pmatrix}
\{x_1\} \\
\{x_2\}
\end{pmatrix},
\] (3.33)
\[
[A] = 
\begin{bmatrix}
[\emptyset] & [I] \\
-\[M]^{-1} [K] & -\[M]^{-1} ([C] + \Omega[G])
\end{bmatrix}.
\] (3.34)

Applying Laplace transform to Equation (3.32) gives:
\[
s[I]\{X(s)\} = [A]\{X(s)\}.
\] (3.35)
The natural frequencies and mode shapes of the rotor system are determined by solving the eigenvalue problem in Equation (3.35). Unlike conventional vibration systems, where the natural frequencies are scalar values, in rotordynamic systems, the eigenvalues are a function of the rotating speed, since the gyroscopic effect matrix is coupled with the rotor speed. The eigenvalues of the rotor system are represented by using the Campbell Diagram. The Campbell diagram represents the rotor system natural frequencies as a function of the speed of rotation. The damped and undamped natural frequencies of the system are calculated based on the eigenvalue problem in this section for the required range of rotor velocity. Subsequently, the direction of the whirl is determined [129]. The whirl direction is either forward, in the direction of the rotor spin, or backward, in the opposite direction. Assuming that the free response of the rotor system in the translational direction at every node is of the form:

\[
\begin{bmatrix}
\hat{u}(t) \\
\hat{v}(t)
\end{bmatrix} = \Re \left\{ \begin{bmatrix}
 r_u e^{i\eta_u} \\
 r_v e^{i\eta_v}
\end{bmatrix} e^{i\omega t} \right\} = T \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix},
\]  

(3.36)

where \( r_u e^{i\eta_u} \) and \( r_v e^{i\eta_v} \) are the elements of the \( i \)th eigenvector corresponding to the DOF of interest, and:

\[
T = \begin{bmatrix}
 r_u \cos \eta_u & -r_u \sin \eta_u \\
 r_v \cos \eta_v & -r_v \sin \eta_v
\end{bmatrix}.
\]  

(3.37)

The orbit \((\hat{u}, \hat{v})\) forms an ellipse where the major and minor axes of the ellipse are obtained from the eigenvalues of:

\[
H = TT^T = \begin{bmatrix}
 r_u^2 & r_u r_v \cos(\eta_u - \eta_v) \\
 r_u r_v \cos(\eta_u - \eta_v) & r_v^2
\end{bmatrix}.
\]  

(3.38)

The length of the semi-major axes and semi-minor axes are given by:

\[
\sqrt{\lambda_2} = \sqrt{\max(eig(H))},
\]

\[
\sqrt{\lambda_1} = \sqrt{\min(eig(H))}.
\]  

(3.39)
The properties of the orbit are encoded into a single parameter $\kappa$, defined as:

$$\kappa = \pm \sqrt{\dot{\lambda}_2 / \dot{\lambda}_1},$$  \hspace{1cm} (3.40)

where $\kappa$ is positive for forward whirl and negative for backward whirl. Alternatively, the phase difference between the responses is given by $\eta_u - \eta_v$. Assuming that $-\pi < \eta_u - \eta_v < \pi$, if $0 < \eta_u - \eta_v < \pi$, the whirl direction is backwards, (i.e. $\kappa < 0$). Similarly, if $-\pi < \eta_u - \eta_v < 0$, the whirl direction is forwards, (i.e. $\kappa > 0$).

### 3.3 The Forced Response of The Rotor Model

The forced response of the rotor system is calculated to determine the effect of mass unbalance on the vibration amplitudes of the rotor. Assuming that there are no non-linear components in the rotor system (i.e. no SFDs) and the unbalance force is represented by Equation (3.19), the EOM of the rotor system is transformed into:

$$[M]\ddot{q} + ([C] + \Omega[G])\dot{q} + [K]q = \Re(\Omega^2 F_0 e^{j\Omega t}).$$  \hspace{1cm} (3.41)

In order to determine the steady-state response of the system, Equation (3.41) must be solved. Assuming that $X(t) = \Re(q_0 e^{j\Omega t})$, and substituting the solution into Equation (3.41) gives:

$$X_0 = \left[([K] - \Omega^2 [M]) + j\Omega([C] + \Omega[G])\right]^{-1}\Omega^2 F_0.$$

Equation (3.42) is solved at different rotor angular velocities to determine the vibration amplitudes of the rotor system due to the mass unbalance.

Assuming that the rotor system incorporates nonlinear components, including SFDs, the force response of the rotor system can no longer be directly calculated. Instead, the transient response of the rotor system is determined to evaluate the effect of time variant non-linear components (i.e. SFDs) on the performance of the rotor system. The transient solution for the rotor system cannot be obtained analytically and alternative solution techniques are required. The most common and inclusive method for computing transient forced vibrations in rotor systems is the
transient numerical integration approach. In the transient integration technique, the rotor system DOFs are marched forward in time through a force balance at every time step. This technique allows any nonlinear elements (i.e. SFD forces) to be included directly as long as the forces can be represented as a function of position and velocity.

3.3.1 Transient Direct Time Integration

In this method, the complete set of DOFs in the physical coordinates is numerically integrated in time. The method is called direct, since no transformation of system equations is required. This approach provides an accurate approximation of the rotor system behavior, however; it is computationally inefficient for rotor systems with large number of degrees of freedom. The Newmark Beta method is the most common numerical technique that is used for direct time integration of dynamic systems. Newmark Beta is an implicit numerical approach, which assumes that the system acceleration varies linearly between two instances of time. In the Newmark method, the expressions for the system velocity and displacement are given by [128]:

\[
\dot{q}_{t+\Delta t} = \dot{q}_t + \left[(1-\alpha)\ddot{q}_t + \alpha\ddot{q}_{t+\Delta t}\right]\Delta t,
\]

\[
q_{t+\Delta t} = q_t + \dot{q}_t\Delta t + \left[\frac{1}{2}-\beta\right]\ddot{q}_t + \beta\ddot{q}_{t+\Delta t},
\]

(3.43)

where \(\alpha\) and \(\beta\) are the solution parameters, which are adjusted to obtain integration accuracy and stability. For \(\alpha=1/6\) and \(\beta=1/2\), Equation (3.43) corresponds to linear acceleration method. Furthermore, for \(\alpha=1/6\) and \(\beta=1/4\), the numerical solution is unconditionally stable, and Equation (3.43) corresponds to constant average acceleration. The Newmark Beta method procedure is summarized as follows:

1. The Mass \([M]\), Damping \([C]\), and Stiffness \([K]\) matrices are formed.
2. The initial conditions \(\{q\}, \{\dot{q}\}, \{\ddot{q}\}\) are determined.
3. The time step \(\Delta t\), and parameters \(\alpha\) and \(\beta\) are selected, and the integration constants are calculated:
\[ a_0 = \frac{1}{\beta \Delta t^2}, a_1 = \frac{\alpha}{\beta \Delta t}, a_2 = \frac{1}{\beta \Delta t}, a_3 = \frac{1}{2 \beta}, \]
\[ a_4 = \frac{\alpha}{\beta} - 1, a_5 = \frac{\Delta t}{2} \left( \frac{\alpha}{\beta} - 2 \right), a_6 = \Delta t (1 - \beta), a_7 = \Delta t \beta. \]

(3.44)

4. The effective stiffness matrix is determined as follows:

\[
\begin{bmatrix} \hat{K} \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} + a_0 \begin{bmatrix} M \end{bmatrix} + a_1 \begin{bmatrix} C \end{bmatrix}.
\]

(3.45)

5. For every time step:

5.1. The effective force vector is calculated at the next time step:

\[
\begin{bmatrix} \hat{F}_{t+\Delta t} \end{bmatrix} = \begin{bmatrix} F_{t+\Delta t} \end{bmatrix} + \begin{bmatrix} M \end{bmatrix} \left( a_0 \begin{bmatrix} q_i \end{bmatrix} + a_2 \begin{bmatrix} \dot{q}_i \end{bmatrix} + a_3 \begin{bmatrix} \ddot{q}_i \end{bmatrix} \right)
+ \begin{bmatrix} C \end{bmatrix} \left( a_1 \begin{bmatrix} q_i \end{bmatrix} + a_4 \begin{bmatrix} \dot{q}_i \end{bmatrix} + a_5 \begin{bmatrix} \ddot{q}_i \end{bmatrix} \right).
\]

(3.46)

5.2. The displacement vector is calculated at the next time step:

\[
\begin{bmatrix} \hat{K} \end{bmatrix} \begin{bmatrix} q_{t+\Delta t} \end{bmatrix} = \begin{bmatrix} \hat{F}_{t+\Delta t} \end{bmatrix}.
\]

(3.47)

5.3. The velocity and acceleration vectors are calculated at the next time step:

\[
\begin{bmatrix} \dot{q}_{t+\Delta t} \end{bmatrix} = a_0 \left( \begin{bmatrix} q_{t+\Delta t} \end{bmatrix} - \begin{bmatrix} X \end{bmatrix} \right) - a_2 \left( \begin{bmatrix} \dot{q}_i \end{bmatrix} \right) - a_3 \left( \begin{bmatrix} \ddot{q}_i \end{bmatrix} \right),
\]
\[
\begin{bmatrix} \ddot{q}_{t+\Delta t} \end{bmatrix} = a_1 \left( \begin{bmatrix} q_{t+\Delta t} \end{bmatrix} - \begin{bmatrix} X \end{bmatrix} \right) - a_4 \left( \begin{bmatrix} \dot{q}_i \end{bmatrix} \right) - a_5 \left( \begin{bmatrix} \dddot{q}_i \end{bmatrix} \right).
\]

(3.48)
Figure 3.1 represents the direct time integration algorithm with the Newmark-Beta solver.

### 3.3.2 Transient Modal Integration

Modal transformation is a powerful technique to determine the transient solution for dynamic systems with large number of DOFs [132], [133]. This technique is appealing since it facilitates the elimination of higher frequency modes with little impact on the system dynamics, which yields to a significantly lower number of DOFs compared to the direct integration method. In this method, the system EOM is firstly transformed into modal coordinates. Moving the damping terms to the right-hand side of the rotor EOM gives:

\[
[M] \ddot{q} + [K] q = \{F(t)\} - ([C] + \Omega [G]) \dot{q}.
\]  

(3.49)

The homogeneous solution for Equation (3.49) is in the form:

\[
\{q(t)\} = \{\phi(x)\} e^{i \omega t}.
\]  

(3.50)
Substituting Equation (3.50) into the homogeneous equation gives:

\[
([K] - \omega^2 [M])\{\phi\} = 0. \tag{3.51}
\]

The trivial solution for the above eigenvalue problem gives the eigenvalues (i.e. square of natural frequencies) and the mode shapes \( \phi \). Furthermore, the modal transformation is given as follows:

\[
\{q(x,t)\} = \begin{bmatrix} \bar{\phi}(x) \end{bmatrix}\{p(t)\}, \tag{3.52}
\]

Substituting the modal transformation in Equation (3.52) into Equation (3.49) gives:

\[
[M]\begin{bmatrix} \bar{\phi} \end{bmatrix}\{\ddot{p}(t)\} + [K]\begin{bmatrix} \bar{\phi} \end{bmatrix}\{p(t)\} = \begin{bmatrix} F(t) \end{bmatrix} - ([C] + \Omega[G])\begin{bmatrix} \bar{\phi} \end{bmatrix}\{p(t)\}. \tag{3.53}
\]

Multiplying Equation (3.53) by the mode shape matrix transpose gives:

\[
\begin{bmatrix} \phi \end{bmatrix}^T [M]\begin{bmatrix} \bar{\phi} \end{bmatrix}\{\ddot{p}(t)\} + \begin{bmatrix} \phi \end{bmatrix}^T [K]\begin{bmatrix} \bar{\phi} \end{bmatrix}\{p(t)\} = \begin{bmatrix} \phi \end{bmatrix}^T \begin{bmatrix} F(t) \end{bmatrix} - \begin{bmatrix} \phi \end{bmatrix}^T ([C] + \Omega[G])\begin{bmatrix} \bar{\phi} \end{bmatrix}\{p(t)\}. \tag{3.54}
\]

Assuming that the modal mass and the modal stiffness matrices are given as follows:

\[
[K_r] = \begin{bmatrix} \bar{\phi} \end{bmatrix}^T [K]\begin{bmatrix} \bar{\phi} \end{bmatrix}, \tag{3.55}
\]

\[
[K_r] = \begin{bmatrix} \bar{\phi} \end{bmatrix}^T [K]\begin{bmatrix} \bar{\phi} \end{bmatrix}, \tag{3.56}
\]

and substituting Equations (3.55) and (3.56) into Equation (3.54) gives:

\[
[M_r]\{\ddot{p}(t)\} + [K_r]\{p(t)\} = \begin{bmatrix} \phi \end{bmatrix}^T \begin{bmatrix} F(t) \end{bmatrix} - \begin{bmatrix} \phi \end{bmatrix}^T ([C] + \Omega[G])\begin{bmatrix} \bar{\phi} \end{bmatrix}\{p(t)\}. \tag{3.57}
\]

Subsequently, dividing Equation (3.57) by the modal mass gives:

\[
\{\ddot{p}(t)\} + [\omega^2]\{p(t)\} = [M_r]^{-1}\begin{bmatrix} \phi \end{bmatrix}^T \begin{bmatrix} F(t) \end{bmatrix} - [M_r]^{-1}\begin{bmatrix} \phi \end{bmatrix}^T ([C] + \Omega[G])\begin{bmatrix} \bar{\phi} \end{bmatrix}\{p(t)\}. \tag{3.58}
\]

Assuming that:
\{ f(t) \} = [M_r]^{-1} \left[ \bar{\phi}^T \right] \{ F(t) \} - [M_r]^{-1} \left[ \bar{\phi}^T \right] \left( \left\{ C + \Omega [G] \right\} \bar{\phi} \right) \{ p(t) \}, \quad (3.59)

gives:

\{ \ddot{p}(t) \} + [\omega^2] \{ p(t) \} = \{ f(t) \}. \quad (3.60)

The expression for the external forces includes the unbalance forces, the damping forces, and the SFD forces. The above equation is numerically solved by using a predictor-corrector technique. In this study, the predictor uses backward finite difference method and the corrector uses Newmark-Beta method. The solution procedure is as follows:

1. The predicted modal displacement vector \( p^* \) and the modal velocity vector \( \dot{p}^* \) are calculated by using central difference equations:

\[ p_{i+1}^* = p_i + \dot{p}_i \Delta t, \quad (3.61) \]

\[ \dot{p}_{i+1}^* = \dot{p}_i + \ddot{p}_i \Delta t. \quad (3.62) \]

2. The predicted modal unbalance forces, gyroscopic forces, and support forces (i.e. SFD forces) are calculated based on the predicted modal displacement and modal velocity vector.

3. The predicted modal acceleration is calculated based on the modal equation of motion as follows:

\[ \dddot{p}_{i+1}^* = f_{i+1}^* - [\omega^2] p_{i+1}^*. \quad (3.63) \]

4. The corrected modal displacement and modal velocity vectors are calculated by using Newmark-Beta method:

\[ p_{i+1} = p_i + \dot{p}_i \Delta t + \left[ \frac{1}{2} - \alpha \right] \ddot{p}_i + \alpha \dddot{p}_{i+1}^* \Delta t^2, \quad (3.64) \]

\[ \dot{p}_{i+1} = \dot{p}_i + \left[ \frac{1}{2} - \beta \right] \ddot{p}_i + \beta \dddot{p}_{i+1}^* \Delta t. \quad (3.65) \]

5. The modal unbalance forces, gyroscopic forces, and support forces (i.e. SFD forces) are calculated based on the modal displacement and modal velocity vectors.
6. The modal acceleration vector is calculated based on the modal displacement and modal velocity vectors:

\[ \ddot{p}_{i+1} = f_{i+1} - \left( \alpha^2 \right) p_{i+1} \]  

(3.66)

7. The physical displacement vector, velocity vector, and acceleration vector are calculated:

\[ \dot{q}_{i+1} = \left[ \phi \right] p_{i+1}, \]
\[ \ddot{q}_{i+1} = \left[ \phi \right] \ddot{p}_{i+1}, \]
\[ \dddot{q}_{i+1} = \left[ \phi \right] \dddot{p}_{i+1}. \]  

(3.67)

Figure 3.2 demonstrates the modal integration algorithm with the predictor-corrector solver.

![Figure 3.2 The modal integration algorithm with the predictor-corrector solver](image-url)
3.4 Results and Discussions

This section represents the simulation results for the steady-state synchronous response and transient response of a high-speed rotor supported by SFDs in presence of lubricant inertia effects. Firstly, the EOM of the rotor system along with the proposed solution techniques are incorporated into Matlab. Subsequently, the steady-state and transient responses of the rotor system is represented under different operating conditions, including fluid inertia effects and SFD radial clearances. In the simulations, it is assumed that the rotor is symmetric and a disk is located at the axial center of the rotor. Furthermore, the rotor is supported with SFDs at either axial end. The SFDs are open-ended and are assembled on the outer race of two identical rolling elements that the rotor is assembled on. Moreover, the unbalance forces are present only on the centered disk. Additionally, the rotor shaft is discretized into 20 elements. The schematic of the rotor model is represented in Figure 3.3. Furthermore, Table 3.1 summarizes the rotor and SFD parameters.

![Rotor System Schematic Diagram](image)

**Figure 3.3** The rotor system schematic in the simulation model

<table>
<thead>
<tr>
<th>Table 3.1 The simulation parameters for the rotordynamic model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parameter</strong></td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>$c_{11}, c_{22}$</td>
</tr>
<tr>
<td>$c_{12}, c_{21}$</td>
</tr>
<tr>
<td>$c/R$</td>
</tr>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>$D_{out}, D_{in}$</td>
</tr>
<tr>
<td>$D_{d,out}, D_{d,in}$</td>
</tr>
<tr>
<td>$E_e$</td>
</tr>
<tr>
<td>$G_{shear,e}$</td>
</tr>
<tr>
<td>$k_{11}, k_{22}$</td>
</tr>
<tr>
<td>$k_{12}, k_{21}$</td>
</tr>
<tr>
<td>$L_{shaft}$</td>
</tr>
<tr>
<td>$L/D$</td>
</tr>
<tr>
<td>$m_{unb\sigma}$</td>
</tr>
<tr>
<td>$W_d$</td>
</tr>
<tr>
<td>$\alpha, \beta$</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
<tr>
<td>$\rho_e$</td>
</tr>
<tr>
<td>$\rho_d$</td>
</tr>
<tr>
<td>$\rho$</td>
</tr>
<tr>
<td>$\nu_e$</td>
</tr>
<tr>
<td>$\phi$</td>
</tr>
<tr>
<td>$\Omega$</td>
</tr>
</tbody>
</table>
The rotor EOM is numerically solved by using the transient modal integration technique for the first 20 resonance modes of the rotor system. Firstly, the effect of fluid inertia on the steady-state vibration amplitudes of the rotor system is represented. The results of the analysis are compared for 3 different cases: (1) no SFDs, (2) SFDs with no inertia effects, and (3) SFDs with inertia effects. According to the parameter quantities in Table 3.1, the squeeze Reynolds number under different rotor system operating conditions is represented by Figure 3.4. The squeeze Reynolds number increases with the rotor speed, the SFD radial clearance, and the lubricant temperature.

\[ \Delta t = 10^{-5} \text{ s} \]

\[
\begin{array}{|c|c|}
\hline
\text{s} & \text{Squeeze Reynolds Number at } 40^\circ\text{C} \\
\hline
\end{array}
\]

- \( C_{\text{SFD}} \frac{R_{\text{SFD}}}{C_2} = 0.004266 \)
- \( C_{\text{SFD}} \frac{R_{\text{SFD}}}{C_2} = 0.0048532 \)
- \( C_{\text{SFD}} \frac{R_{\text{SFD}}}{C_2} = 0.0072798 \)

Angular Velocity (Ω) [RPM]
Figure 3.4 The SFD squeeze Reynolds number under different operating conditions

In order to verify the correctness of the rotordynamic model and the transient modal integration algorithm, an equivalent model, excluding the SFD effects, was developed in ANSYS based on the parameters in Table 3.1 and the results of the simulation were compared at different unbalance parameters. Figure 3.5 represents the comparison between the simulation results of the proposed rotordynamic model and the ANSYS model. The results represent the steady-state unbalance induced vibrations of the disk center at different rotor velocities. The results demonstrate the close agreement between the two models.
Figure 3.5 Steady-state unbalance induced response of the linear rotor system (No SFD), comparison between the rotordynamic model and the ANSYS model

3.4.1 Short-Length SFDs

This section investigates the effect of the short-length SFD models proposed in this work on the steady-state and transient unbalance induced vibration amplitudes in high-speed rotors. Figure 3.6 represents the effect of SFD viscous forces (i.e. neglecting fluid inertia effect) on the steady-state vibration amplitudes of the rotor system at the disk node for the SFD radial clearance of $1.27 \times 10^{-4}$, at the lubricant temperature of 100°C, and for the three different unbalance magnitudes in Table 3.1. Firstly, the steady-state vibration amplitudes increase with the unbalance magnitude. Furthermore, even for the case where the effect of fluid inertia is neglected, the steady-state vibrations of the disk center is significantly attenuated at the first resonance frequency (i.e. the first bending mode) by incorporating the SFDs into the rotor system.
(a) 

(b) 

\[
\frac{C_{SFD}/R_{SFD}}{,\text{ and } Unb = 7.2008e-05}\]

\[
\frac{C_{SFD}/R_{SFD}}{,\text{ and } Unb = 0.00014402}\]
Figure 3.6 The effect of short-length SFD viscous forces on the steady-state vibration amplitudes of the rotor system

Figure 3.7 represents the effect of SFD fluid inertia on the steady-state vibration amplitudes of the rotor system at the disk node for three different cases, including: (1) Viscous short-length SFD model (i.e. no inertia effects), (2) SAM short-length SFD (i.e. the effect of fluid inertia neglected), and (3) complete inertia short-length. The latter two cases are investigated by using both the perturbation technique and the momentum approximation. The results are represented for 3 SFD radial clearance ratios, at lubricant temperature of 100°C, and for the 3 unbalance magnitudes in Table 3.1. For SFD radial clearance ratio of 0.00242 and for the three unbalance magnitudes, the effect of fluid inertia significantly attenuates the steady-state vibration amplitudes at the first resonance frequency and at large rotor frequencies. Furthermore, at small rotor speeds, since the squeeze Reynolds number is small (Figure 3.4), the results based on the viscous model (i.e. no inertia effect) and the inertia models is in close agreement. However, at moderate and large rotor speeds, the effect of fluid inertia (i.e. the squeeze Reynolds number) significantly increases, and the results of the inertia models notably diverge from the viscous model. Moreover, at small SFD radial clearance ratios (Figure 3.7a to 3.7c), the SFD damping
capacity is higher and the steady-state vibration amplitudes at the first resonance frequency zone (i.e. the first bending mode) are relatively small. Consequently, both the SAM model and the complete inertia model provide an accurate prediction of the steady-state vibration behavior of the rotor system. At moderately large SFD radial clearance ratios (Figure 3.7d to 3.7i), the squeeze Reynolds number is relatively large even at small rotor speeds (i.e. before the resonance zone) and the viscous model and the inertia model demonstrate notable disagreement. Furthermore, the damping capacity of the SFD significantly deteriorates at larger SFD radial clearance ratios, and the steady-state vibration amplitudes grow significantly larger at the resonance frequency zone. Consequently, the SAM inertia model is no longer capable of providing accurate predictions of the rotor vibration characteristics and the effect of convective inertia components should not be neglected in the analysis of the rotor system. Finally, in general, the effect of FSD fluid inertia is represented by and added mass, which is supplemented into the overall rotordynamic system mass, which in turn reduces the first resonance frequency of the system.

\[ \left( \frac{C_{SFD}}{R_{SFD}} = 0.0024266 \text{ and } Unb = 7.2008e-05 \right) \]
(b) \( \frac{C_{SFD}}{R_{SFD}} = 0.0024266 \) and \( Unb = 0.00014402 \)

(c) \( \frac{C_{SFD}}{R_{SFD}} = 0.0024266 \) and \( Unb = 0.00036004 \)
\( \frac{C_{SFD}}{R_{SFD}} = 0.0048532 \) and \( Unb = 7.2008 \times 10^{-05} \)

\( \frac{C_{SFD}}{R_{SFD}} = 0.0048532 \) and \( Unb = 0.00014402 \)

(d)

(e)
\( \left( \frac{C_{\text{SFD}}}{R_{\text{SFD}}} = 0.0048532 \text{ and } Un_b = 0.00036004 \right) \)

\[ \times 10^{-4} \]

Displacement Magnitude [m]

Rotor Speed [rpm]

(f)

\( \left( \frac{C_{\text{SFD}}}{R_{\text{SFD}}} = 0.0072798 \text{ and } Un_b = 7.2008e-05 \right) \)

\[ \times 10^{-4} \]

Displacement Magnitude [m]

Rotor Speed [rpm]

(g)
Figure 3.7 The effect of short-length SFD lubricant inertia on the steady-state vibration amplitudes of the rotor system
Subsequently, the effect of the SFD radial clearance on the steady-state vibration amplitudes of the rotor system is investigated for the 3 SFD radial clearances in Table 3.1. In general, the squeeze Reynolds number (i.e. the effect of fluid inertia) increases with the square of the magnitude of the SFD radial clearance. Furthermore, the SFD forces are inversely proportional to the square of the radial clearance (i.e. \( F = \frac{\bar{F} \mu \Omega R_{SFD}^4}{c_{SFD}^2} \)). Consequently, increasing the SFD radial clearance increases the lubricant inertia effects, while reducing the SFD reaction force magnitudes. Figure 3.8 represents the effect of SFD radial clearance on the rotor steady-state vibration amplitude for both the perturbation technique and the momentum approximation technique at the disk center. The steady-state vibration amplitudes significantly increase with the SFD radial clearance for both solution techniques. Furthermore, the first resonance frequency decreases (i.e. shifts to the left) at larger radial clearances (i.e. larger fluid inertia effects), for the 3 unbalance magnitudes. At larger radial clearances, the squeeze Reynolds number and consequently the fluid inertia effects are larger. This increases the magnitude of the resultant added mass to the system, which reduces the resonance frequency.
Figure 3.8 The effect of short-length SFD radial clearance on the steady-state vibration amplitudes of the rotor system

Figure 3.9 represents the transient orbits of the SFD journal center at the 0.00242 radial clearance ratio and at 0.00036004 kg-m unbalance magnitude at different rotor speeds for four different configurations: (1) No SFD, (2) Viscous SFD model (i.e. no inertia effects), (3) Complete inertia model with the perturbation technique, and (4) Complete inertia model with the
momentum approximation technique. At small rotor speeds (1000 rpm and 5000 rpm), the squeeze Reynolds number is relatively small (Figure 3.4) and the three SFD models are in close agreement. Furthermore, including the SFD damping effect significantly accelerates the convergence of the orbits to the steady-state value. At the resonance zone (7310 rpm and 8250 rpm) the squeeze Reynolds number and the orbit radius is relatively larger and the inertia models demonstrate both significantly faster convergence and smaller orbit radius. At larger rotor speeds (10000 rpm and 12500 rpm) the squeeze Reynolds number is moderately large while the orbit radius rapidly declines relative to the resonance zone.
3.4.2 Finite-Length SFDs

This section studies the effect of finite-length SFD models proposed in this work on the steady-state and transient unbalance induced vibration amplitudes in high-speed rotors. Figure 3.10 represents the effect of SFD viscous forces (i.e. neglecting inertia effect) on the steady-state vibration amplitudes of the rotor system at the disk node. The results are represented for the SFD radial clearance of 1.27e-4, at the lubricant temperature of 100°C, and for the three different unbalance magnitudes in Table 3.1. The steady-state vibration amplitudes increase with the unbalance magnitude. Furthermore, even for the case where the effect of fluid inertia is
neglected, the steady-state vibrations of the disk center are significantly attenuated at the first resonance frequency (i.e. the first bending mode) by incorporating the SFDs into the rotor system.

\[
\frac{C_{SFD}}{R_{SFD}} = 0.0024266 \quad \text{and} \quad Unb = 7.2008 \times 10^{-5}
\]

(a)

\[
\frac{C_{SFD}}{R_{SFD}} = 0.0024266 \quad \text{and} \quad Unb = 0.00014402
\]

(b)

160
Figure 3.10 The effect of finite-length SFD viscous forces on the steady-state vibration amplitudes of the rotor system

Figure 3.11 represents the effect of fluid inertia on the steady-state vibration amplitudes of the rotor system at the disk node for three different cases, namely: (1) Viscous SFD model (i.e. no inertia effects) and (2) complete fluid inertia effects with perturbation method, and (3) complete fluid inertia effects with momentum approximation. The results are represented at lubricant temperature of 100°C and for the 3 unbalance magnitudes in Table 3.1. In general, the fluid inertia effects considerably attenuate the steady-state vibration amplitudes at the resonance zone and at large rotor velocities. At small to moderate rotor speeds, the squeeze Reynolds number is relatively small and the effect of fluid inertia is negligible, consequently, the steady-state vibrations for the three models are very similar. At the first resonance zone (i.e. first bending mode), the squeeze Reynolds number is moderately large and the effect of fluid inertia is significant. Additionally, the vibration amplitudes at the resonance frequency zone are noticeably larger, resulting in larger SFD eccentricity ratios. Consequently, at the first resonance zone, the inertial force component is large and the discrepancy between the results of the inertialess and inertial models is noticeable, demonstrating the influence of fluid inertia effects on the
suppression of the unbalance induced vibration amplitudes. At the post resonance zone, the rotor speeds are large, which results in the elevation of the Reynolds number and the inertia effects, hence, the inertial term remains significant and leads to the reduction of the steady-state vibration amplitudes relative to the no inertia model. Finally, the effect of fluid inertia leads to a shift in the first resonance frequency from 8250 rpm to approximately 8000 rpm.
Figure 3.11 The effect of finite-length SFD lubricant inertia on the steady-state vibration amplitudes of the rotor system
Figure 3.12 represents the transient orbits of the SFD journal center at the 0.00242 radial clearance ratio and at 0.00036004 kg-m unbalance magnitude at different rotor speeds for four different configurations: (1) No SFD, (2) Viscous SFD model (i.e. no inertia effects), (3) Complete inertia model with perturbation method, and (4) Complete inertia model with momentum approximation method. At small rotor speeds (1000 rpm and 5000 rpm), the squeeze Reynolds number is relatively small and the three SFD models are in close agreement. Furthermore, including the SFD effect significantly accelerates the convergence of the orbits to the steady-state value. At the resonance zone (7310 rpm and 8250 rpm) the squeeze Reynolds number and the orbit radius are relatively larger and the inertia models demonstrate both significantly faster convergence and smaller orbit radius. At larger rotor speeds (10000 rpm and 12500 rpm) the squeeze Reynolds number is moderately large while the orbit radius rapidly declines relative to the resonance zone.
Figure 3.12 The transient orbit radius of the finite-length SFD journal center at different rotor speeds.
3.5 Conclusion

This chapter represented a finite element based multi-mass flexible rotordynamic model incorporating the SFD models that were developed in Chapter 2. Additionally, the proposed rotordynamic model was integrated into a simulation model to study the effect of SFD lubricant inertia on the steady-state and transient unbalance induced rotor vibration amplitudes. The results of the simulations were represented for different SFD models, namely short-length SFDs and infinite-length SFDs, and under different operating parameters, including inertia effects, unbalance magnitude, damper radial clearance, and operating speeds. According to the results of the analysis:

1. The SFD viscous forces significantly attenuate the steady-state vibration amplitudes of the rotor system at the resonance zone even when the fluid inertia is neglected.

2. Including the SFD inertia effects notably attenuates the steady-state vibration amplitudes of the rotor system relative to the viscous SFD model (i.e. no inertia) at the resonance zone and at the large rotor speed zones. However, at small rotor speeds where the squeeze Reynolds number is small, the two models are in close agreement.

3. The effect of lubricant inertia reduces the frequency of the first resonance of the rotor system. Increasing the fluid inertia effects further shifts the first resonance to lower frequencies.

4. At large SFD radial clearance, the damping capacity of the SFD is significantly reduced and the steady-state vibrations are significantly larger at the resonance zone. Consequently, the SAM SFD model is no longer applicable for predicting the system vibration behavior.

5. Increasing the SFD radial clearance reduces the magnitude of the SFD forces and consequently, increases the steady-state vibration amplitudes at the resonance zone. Furthermore, the first resonance frequency of the system decreases with the SFD radial clearance, since the fluid inertia effects increases with the radial clearance.
This chapter develops a comprehensive thermohydrodynamic model for squeeze film dampers incorporated into high-speed turbomachinery. Firstly, the generalized expression for Reynolds equation, including variable viscosity, is developed. In order to reduce the complexity of the hydrodynamic equations, an average radial viscosity is defined and integrated into the equations. Subsequently, a first-order inertia correction to the pressure is incorporated by using a perturbation technique to represent the effect of lubricant inertia on the hydrodynamic pressure distribution. Subsequently, a thermal model, including the energy equation, the Laplace heat conduction equation in the surrounding solids, including the journal and the bush, and the thermal boundary conditions at the interfaces is constructed. Furthermore, the system of partial differential hydrodynamic and thermal equations is simultaneously solved by using an iterative numerical algorithm. The proposed model is incorporated into a simulation model and the results are represented at different SFD speeds and eccentricity ratios.

4.1 The Generalized Reynolds Equation

The general flow equation for a fluid within the boundaries are represented by Equations (2.3) to (2.6). These equations are further reduced by assuming that:
1. The body force terms and inertia terms are small compared to the viscous and pressure terms.
2. According to an order of magnitude analysis, the velocity gradients $\frac{\partial u}{\partial y}$ and $\frac{\partial w}{\partial y}$ are large compared to all other velocity gradients.
3. The lubricant is Newtonian, incompressible (i.e. density gradient is zero).
4. The film thickness is very small relative to the other film dimensions:
   a. The effect of the curvature of the film is negligible; hence a linear coordinate system is used to describe the lubricant dynamics.
   b. The variation of the pressure across the film is negligible (i.e. $\frac{\partial P}{\partial y} = 0$ )

Applying the above assumptions gives:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4.1) \]

\[ -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = 0, \quad (4.2) \]

\[ -\frac{\partial P}{\partial z} + \frac{\partial}{\partial y} \left( \mu \frac{\partial w}{\partial y} \right) = 0. \quad (4.3) \]

Additionally, the velocity boundary conditions are given by:

\[
\begin{align*}
&u = 0, v = 0, w = 0 \quad y = 0 \\
&u = 0, v = \frac{\partial h}{\partial t}, w = 0 \quad y = h.
\end{align*}
\quad (4.4)
\]

Subsequently, the axial and circumferential velocity components are determined by integrating the momentum transport equations twice in the radial direction as follows:

\[
\begin{align*}
u &= \frac{\partial P}{\partial x} \int_0^y \frac{y}{\mu} dy + B(x, z) \int_0^y \frac{dy}{\mu} + D(x, z), \\
w &= \frac{\partial P}{\partial z} \int_0^y \frac{y}{\mu} dy + C(x, z) \int_0^y \frac{dy}{\mu} + E(x, z).
\end{align*}
\quad (4.5)
\]

The integration constants are determined by applying the velocity boundary equations in Equation (4.4):

\[
\begin{align*}
&D(x, z) = E(x, z) = 0, \\
&B(x, y) = -\frac{\partial P}{\partial x} \int_0^y \frac{dy}{\mu} \int_0^y \frac{dy}{\mu}, \\
&C(x, y) = -\frac{\partial P}{\partial z} \int_0^y \frac{dy}{\mu} \int_0^y \frac{dy}{\mu}.
\end{align*}
\quad (4.6)
\]

Consequently, the velocity components are calculated as follows:
\[ u = \frac{\partial P}{\partial x} \left[ \int_{0}^{y} \frac{y}{\mu} dy - F_{1} \int_{0}^{y} \frac{dy}{\mu} \right], \]  
\[ w = \frac{\partial P}{\partial z} \left[ \int_{0}^{y} \frac{y}{\mu} dy - F_{1} \int_{0}^{y} \frac{dy}{\mu} \right], \]  

where:

\[ F_{1} = \frac{\int_{0}^{h} \frac{v}{\mu} dy}{\int_{0}^{h} \frac{dy}{\mu}}. \]  

Additionally, the radial fluid velocity component is calculated by using Equation (4.1). According to [76], direct computation of the radial velocity by substituting the velocity expressions in Equation (4.7) into Equation (4.1) yields a radial velocity different from zero on the bush surface due to numerical uncertainties, which is physically invalid and leads to difficulty in the convergence of the hydrodynamic solution. In order to prevent this, Equation (4.1) is differentiated with respect to the radial coordinate as follows:

\[ \frac{\partial^{2} v}{\partial y^{2}} = -\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right), \]  

and the radial velocity component is numerically determined by using finite difference analysis.

\[ v_{b}^{h} = -\int_{0}^{h} \frac{\partial u}{\partial x} dy - \int_{0}^{h} \frac{\partial w}{\partial z} dy. \]  

According to the Leibniz integral rule:

\[ \int_{y_{1}}^{y_{2}} \frac{\partial}{\partial x} f(x, y, z) dy = \frac{\partial}{\partial x} \int_{y_{1}}^{y_{2}} f(x, y, z) dy - f(x, y_{2}, z) \frac{\partial y_{2}}{\partial x} + f(x, y_{1}, z) \frac{\partial y_{1}}{\partial x}, \]  

hence:

\[ \int_{0}^{h} \frac{\partial u}{\partial x} dy = \frac{\partial}{\partial x} \int_{0}^{h} u dy - u(x, h, z) \frac{\partial h}{\partial x}. \]
The above Equation can be rearranged as follows:

\[
\int_0^h \frac{\partial u}{\partial x} \, dy = \frac{\partial}{\partial x} \int_0^h u \, dy - \frac{\partial}{\partial x} \left[ u(x, h, z) h \right] + \frac{\partial}{\partial x} \left[ u(x, h, z) \right] h. \tag{4.13}
\]

Furthermore, according to integration by parts [134]:

\[
\left[ u(x, y, z) y \right]_0^h = \int_0^h y \frac{\partial u}{\partial y} \, dy + \int_0^h \frac{\partial v}{\partial y} \, u \, dy, \tag{4.14}
\]

and:

\[
\left[ u(x, y, z) y \right]_0^h = u(x, h, z) h, \tag{4.15}
\]

hence:

\[
\int_0^h \frac{\partial}{\partial x} \left[ u(x, h, z) h \right] = \frac{\partial}{\partial x} \int_0^h u \, dy - \frac{\partial}{\partial x} \int_0^h y \frac{\partial u}{\partial y} \, dy - \frac{\partial}{\partial x} \int_0^h \frac{\partial v}{\partial y} \, u \, dy
\]

\[
= -\frac{\partial}{\partial x} \int_0^h y \frac{\partial u}{\partial y} \, dy. \tag{4.16}
\]

Substituting Equation (4.16) into Equation (4.13) gives:

\[
\int_0^h \frac{\partial u}{\partial x} \, dy = -\frac{\partial}{\partial x} \int_0^h y \frac{\partial u}{\partial y} \, dy + \frac{\partial}{\partial x} \left[ u(x, h, z) \right] h. \tag{4.17}
\]

Since \( u(x, h, z) = 0 \), the above equation becomes:

\[
\int_0^h \frac{\partial u}{\partial x} \, dy = -\frac{\partial}{\partial x} \int_0^h y \frac{\partial u}{\partial y} \, dy. \tag{4.18}
\]

Similarly:

\[
\int_0^h \frac{\partial w}{\partial z} \, dy = -\frac{\partial}{\partial z} \int_0^h y \frac{\partial w}{\partial y} \, dy. \tag{4.19}
\]

Substituting Equations (4.18) and (4.19) into Equation (4.10) gives:
\[ v^b \big|_0^h = \frac{\partial}{\partial x} \int_0^h y \frac{\partial u}{\partial y} dy + \frac{\partial}{\partial z} \int_0^h y \frac{\partial w}{\partial y} dy. \] (4.20)

Finally, substituting the velocity expressions from Equation (4.7) into Equation (4.20) and applying the velocity boundary conditions in Equation (4.4) provides a generalized form of Reynolds equation with variable viscosity as follows:

\[ \frac{\partial}{\partial x} \left[ F_h \frac{\partial P}{\partial x} \right] + \frac{\partial}{\partial z} \left[ F_h \frac{\partial P}{\partial z} \right] = \frac{\partial h}{\partial t}, \] (4.21)

where:

\[ F_h = \int_0^h \frac{y}{\mu} (y - F_1) dy. \] (4.22)

The hydrodynamic equations are normalized by introducing dimensionless parameters as follows:

\[ \begin{align*}
\theta &= \theta' - \phi = \frac{x}{R} - \Omega t, \quad \eta = \frac{y}{h}, \quad \xi = \frac{z}{R}, \quad \tau = \Omega t, \\
\bar{u} &= \frac{u}{R \Omega}, \quad \bar{v} = \frac{v}{c \Omega}, \quad \bar{w} = \frac{w}{R \Omega}, \quad \bar{\mu} = \frac{\mu}{\mu_0}, \\
\bar{P} &= \frac{P c^2}{R^2 \Omega \mu_0}, \quad H = \frac{h}{c} = 1 + \varepsilon \cos \theta.
\end{align*} \] (4.23)

Additionally, the following operator is defined, by applying the chain rule, to incorporate the effect of transforming the film thickness into a rectangular shape [69]:

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial \theta} - \frac{\eta}{H} \left( \frac{\partial H}{\partial \theta} \right) \frac{\partial}{\partial \eta}. \] (4.24)

Substituting Equations (4.23) and (4.24) into Equations (4.7), (4.9), and (4.21) gives:
\[
\bar{u} = H^2 \frac{\partial P}{\partial \theta} \left[ \int_0^\eta \frac{\eta}{\mu} \, d\eta - \int_0^{\eta_1} \frac{1}{\mu} \, d\eta \right]
- \int_0^{\eta_1} \frac{1}{\mu} \, d\eta \left[ \int_0^\eta \frac{\eta}{\mu} \, d\eta \right] \nabla \mu, \tag{4.25}
\]

\[
\bar{w} = H^2 \frac{\partial P}{\partial \xi} \left[ \int_0^\eta \frac{\eta}{\mu} \, d\eta - \int_0^{\eta_1} \frac{1}{\mu} \, d\eta \right]
- \int_0^{\eta_1} \frac{1}{\mu} \, d\eta \left[ \int_0^\eta \frac{\eta}{\mu} \, d\eta \right] \nabla \mu, \tag{4.26}
\]

\[
\frac{\partial^2 \bar{v}}{\partial \eta^2} = -H \frac{\partial}{\partial \eta} \left( \frac{\partial \bar{v}}{\partial \theta} - \eta \frac{\partial H}{\partial \theta} \frac{\partial \bar{u}}{\partial \eta} + \frac{\partial \bar{w}}{\partial \xi} \right), \tag{4.27}
\]

where:

\[
\bar{F}_3 = \int_0^\eta \frac{\eta}{\mu} \left( \eta - \int_0^{\eta_1} \frac{1}{\mu} \, d\eta \right) \, d\eta. \tag{4.28}
\]

In order to reduce the complexity of the numerical solution for the hydrodynamic problem, an average fluid viscosity in the radial direction is defined as follows:

\[
\mu_m = \frac{1}{h} \int_0^h \mu \, dy. \tag{4.29}
\]

Application of the average radial viscosity significantly reduces the expressions for the fluid velocities and pressure distribution. Substituting Equation (4.29) into Equations (4.7) and (4.21) gives:

\[
u = \frac{1}{2 \mu_m} \frac{\partial P}{\partial x} \left[ y^2 - yh \right], \tag{4.30}
\]

\[
w = \frac{1}{2 \mu_m} \frac{\partial P}{\partial z} \left[ y^2 - yh \right],
\]

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\[
\frac{\partial}{\partial x} \left[ \frac{h^3}{\mu_m} \frac{\partial P}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \frac{h^3}{\mu_m} \frac{\partial P}{\partial z} \right] = 12 \frac{\partial h}{\partial t}.
\] (4.31)

Additionally, the dimensionless average radial viscosity is defined as:

\[
\bar{\mu}_m = \frac{\mu_m}{\mu_0}.
\] (4.32)

Substituting Equation (4.32) into Equations (4.25) to (4.27) gives:

\[
\bar{u} = \frac{H^2}{2\mu_m} \frac{\partial P}{\partial \theta} \left[ \eta^2 - \eta \right],
\]

\[
\bar{w} = \frac{H^2}{2\mu_m} \frac{\partial P}{\partial \xi} \left[ \eta^2 - \eta \right],
\] (4.33)

\[
\frac{\partial^2 \bar{v}}{\partial \eta^2} = -H \frac{\partial}{\partial \eta} \left( \frac{\partial \bar{u}}{\partial \theta} - \frac{\eta}{H} \frac{\partial \bar{H}}{\partial \theta} \frac{\partial \bar{u}}{\partial \eta} + \frac{\partial \bar{w}}{\partial \xi} \right),
\] (4.34)

\[
\frac{\partial}{\partial \theta} \left[ \frac{H^3}{\mu_m} \frac{\partial \bar{P}}{\partial \theta} \right] + \frac{\partial}{\partial \xi} \left[ \frac{H^3}{\mu_m} \frac{\partial \bar{P}}{\partial \xi} \right] = 12 \frac{\partial \bar{H}}{\partial \tau}.
\] (4.35)

The dimensionless velocity boundary conditions are as follows:

\[
\begin{align*}
\bar{u} &= 0, \bar{v} = 0, \bar{w} = 0 & \eta &= 0 \\
\bar{u} &= 0, \bar{v} = \frac{\partial \bar{H}}{\partial \tau}, \bar{w} = 0 & \eta &= 1.
\end{align*}
\] (4.36)

Furthermore, for and open-ended SFD, the pressure boundary conditions are given as:

1. The pressure is periodic and continuous in the circumferential direction (\(\theta\)), i.e.
   \(\bar{P}(\theta, \xi) = \bar{P}(\theta + 2\pi, \xi)\).
2. The pressure equals atmospheric pressure at the axial ends of the bearing, i.e.
   \(\bar{P}(\theta, L/D) = \bar{P}(\theta, -L/D) = 0\).
3. The hydrodynamic pressure must be above the liquid cavitation pressure, i.e.
   \(\bar{P} \geq \bar{P}_{cav} \quad 0 \leq \theta \leq 2\pi, -L/D \leq z \leq L/D\).
4.2 The Effect of Fluid Inertia

Section 2.4.2 used a first-order perturbation technique to develop the expression for the inertial correction for the pressure distribution. Firstly, a small first-order perturbation by means of the expression for the lubricant pressure distribution that is expanded in power series of the squeeze film Reynolds number was introduced as follows:

\[ \bar{P} = \bar{P}_0 + \text{Re} \bar{P}_1. \]  

Subsequently, the perturbation equations for the fluid velocity and pressure distribution were substituted into the SFD flow equations, including fluid inertia effects, and a characteristic Equation was developed for the first-order inertia correction for the pressure as follows:

\[ \frac{\partial}{\partial \theta} \left( H^3 \frac{\partial \bar{P}_1}{\partial \theta} \right) + \frac{\partial}{\partial \xi} \left( H^3 \frac{\partial \bar{P}_1}{\partial \xi} \right) = \frac{3H^7}{560} \frac{\partial}{\partial \theta} \left[ \left( \frac{\partial \bar{P}_1}{\partial \theta} \right)^2 + \left( \frac{\partial \bar{P}_0}{\partial \xi} \right)^2 \right] - \frac{3H^6}{140} \frac{\partial \bar{P}_1}{\partial \theta} \left( \frac{\partial \bar{P}_0}{\partial \theta} \right)^2 + \frac{13H^4}{70} \frac{\partial H}{\partial \tau} \frac{\partial \bar{P}_0}{\partial \theta} + \frac{\partial^2 \bar{P}_0}{\partial \tau^2} \left( \frac{H^5}{10} \right) \right\} \]  

where the zeroth-order inertialess pressure expression is calculated based on Equation (4.35).

4.3 The Thermal Model

The thermal model incorporates the energy equation for the lubricant film as well as the Laplace heat conduction equations for the surrounding solids, namely the bush and the shaft. The general expression for the energy equations is given as:

\[ \rho C_p \left( \frac{\partial T}{\partial t} + \nabla V T \right) = \nabla \cdot \left( k \nabla T \right) + \left( \frac{\partial P}{\partial t} + V \cdot \nabla P \right) + \tilde{Q}. \]  

where the term on the left-hand side of the above equation represents convection and the terms on the right-hand side of the equation denote compression work, conduction, and viscous
dissipation. Assuming that the lubricant is incompressible, the compressions work is neglected and the energy equation is reduced as follows:

\[
\rho C_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \phi, \tag{4.40}
\]

where the viscous dissipation is given by [135]:

\[
\tilde{\phi} = \mu \left[ 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial w}{\partial z} \right)^2 - \frac{2}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} \right) + \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 \right. \\
+ \left. \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right]. \tag{4.41}
\]

Furthermore, assuming that:

1. The heat conductivity is constant.
2. An order of magnitude analysis shows that the dissipation function can be reduced to:

\[
\tilde{\phi} = \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 \frac{\partial w}{\partial y} \right]. \tag{4.42}
\]

3. Heat conduction in the axial and the circumferential directions are negligible relative to the heat convection in the radial direction.

Consequently, the energy equation is reduced to:

\[
\rho C_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \frac{\partial^2 T}{\partial y^2} + \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right]. \tag{4.43}
\]

Finally, for steady-state bearing operation, the energy equation is as follows:

\[
\rho C_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \frac{\partial^2 T}{\partial y^2} + \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right]. \tag{4.44}
\]

The general form of Laplace heat conduction equation for the solids is given as follows:
\[ \rho C_v \frac{\partial T}{\partial t} = k_i \nabla^2 T_i. \quad (4.45) \]

In Cartesian coordinates:

\[ \rho C_v \frac{\partial T}{\partial t} = k_i \left( \frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} + \frac{\partial^2 T_i}{\partial z^2} \right), \quad (4.46) \]

and in cylindrical coordinates:

\[ \rho C_v \frac{\partial T}{\partial t} = k_i \left[ \frac{1}{r_i} \frac{\partial}{\partial r_i} \left( r_i \frac{\partial T_i}{\partial r_i} \right) + \frac{1}{r_i^2} \frac{\partial^2 T_i}{\partial \theta^2} + \frac{\partial^2 T_i}{\partial z^2} \right]. \quad (4.47) \]

hence:

\[ \rho C_v \frac{\partial T_i}{\partial t} = k_i \left[ \frac{\partial^2 T_i}{\partial r_i^2} + \frac{1}{r_i} \frac{\partial T_i}{\partial r_i} + \frac{1}{r_i^2} \frac{\partial^2 T_i}{\partial \theta^2} + \frac{\partial^2 T_i}{\partial z^2} \right]. \quad (4.48) \]

For steady-state conditions, the heat conduction equation is reduced to:

\[ \frac{\partial^2 T_i}{\partial r_i^2} + \frac{1}{r_i} \frac{\partial T_i}{\partial r_i} + \frac{1}{r_i^2} \frac{\partial^2 T_i}{\partial \theta^2} + \frac{\partial^2 T_i}{\partial z^2} = 0. \quad (4.49) \]

Consequently, the steady-state heat conduction equations for the bush and the shaft are given as follows respectively:

\[ \frac{\partial^2 T_b}{\partial r_b^2} + \frac{1}{r_b} \frac{\partial T_b}{\partial r_b} + \frac{1}{r_b^2} \frac{\partial^2 T_b}{\partial \theta^2} + \frac{\partial^2 T_b}{\partial z^2} = 0, \quad (4.50) \]

\[ \frac{\partial^2 T_s}{\partial r_s^2} + \frac{1}{r_s} \frac{\partial T_s}{\partial r_s} + \frac{1}{r_s^2} \frac{\partial^2 T_s}{\partial \theta^2} + \frac{\partial^2 T_s}{\partial z^2} = 0. \quad (4.51) \]

Subsequently, the thermal boundary conditions are described:

I- For Energy Equation

a. Matching temperatures at the oil-bush interface:
\[ T|_{y=0} = T_b|_{r=R_b}. \]  

b. Matching temperatures at the oil-shaft interface:

\[ T|_{y=h} = T|_{r=R}. \]  

2- \textit{For Bush}

a. Heat flux continuity at the oil-bush interface:

\[ -k_b \frac{\partial T_b}{\partial r} \bigg|_{r=R_b} = k(\theta) \frac{\partial T}{\partial y} \bigg|_{y=0}, \]  

where [105]:

\[ k(\theta) = \begin{cases} k & \text{full-film} \\ k + k_{a}/2 & \text{cavitation} \end{cases}. \]

b. Free convection at the outer surface of the bush:

\[ -k_b \frac{\partial T_b}{\partial r} \bigg|_{r=R_0} = h_b \left( T_b|_{r=R_0} - T_0 \right). \]

c. Free convection at the axial ends of the bush:

\[ -k_b \frac{\partial T_b}{\partial z} \bigg|_{z=\pm L/2} = h_b \left( T_b|_{z=\pm L/2} - T_0 \right). \]

3- \textit{For Shaft}

a. Heat flux continuity at the oil-shaft interface:

\[ -k_s \frac{\partial T_s}{\partial r} \bigg|_{r=R} = k(\theta) \frac{\partial T}{\partial y} \bigg|_{y=h}. \]

b. Free convection at the axial ends of the shaft:

\[ -k_s \frac{\partial T_s}{\partial z} \bigg|_{z=\pm L/2} = h_s \left( T_s|_{z=\pm L/2} - T_0 \right). \]
Additionally, dimensionless thermal parameters are defined as follows:

\[ \theta = \theta' - \phi = \frac{x}{R} - \Omega t, \quad \eta = \frac{y}{h}, \quad \xi = \frac{z}{R}, \quad \tau = \Omega t, \]

\[ \mu = \frac{\mu_m}{\mu_0}, \quad \rho = \frac{\rho \Omega c^2}{\mu_0}, \quad H = \frac{h}{c} = 1 + \varepsilon \cos \theta, \quad \bar{T} = \frac{T}{T_0}, \]

Substituting the dimensionless parameters in Equation (4.60) and the transformation defined in Equation (4.24) into Equations (4.44), (4.50), (4.51), and the thermal boundaries gives:

\[ -\frac{\partial \bar{T}}{u \partial \theta} + \left( \frac{v - u \eta}{h} \right) \frac{\partial \bar{T}}{\partial \eta} + \frac{w \partial \bar{T}}{\partial \xi} = \frac{1}{H^2} \frac{1}{Pe} \frac{\partial^2 \bar{T}}{\partial \eta^2} + \frac{\mu}{H^2 \text{Re}_0} \left[ \left( \frac{\partial \bar{u}}{\partial \eta} \right)^2 + \left( \frac{\partial \bar{w}}{\partial \eta} \right)^2 \right], \]

\[ \frac{\partial^2 \bar{T}_b}{\partial r_b^2} + \frac{\partial \bar{T}_b}{\partial r_b} + \frac{1}{r_b^2} \frac{\partial^2 \bar{T}_b}{\partial \eta^2} + \frac{\partial^2 \bar{T}_b}{\partial \xi_b^2} = 0, \]

\[ \frac{\partial^2 \bar{T}_s}{\partial r_s^2} + \frac{\partial \bar{T}_s}{\partial r_s} + \frac{1}{r_s^2} \frac{\partial^2 \bar{T}_s}{\partial \eta^2} + \frac{\partial^2 \bar{T}_s}{\partial \xi_s^2} = 0. \]

1. **For Energy Equation**
   a. Matching temperatures at the oil-bush interface:

   \[ \bar{T} \bigg|_{\eta=0} = \bar{T}_b \bigg|_{r_b=R_b/R} \]

   b. Matching temperatures at the oil-shaft interface:

   \[ \bar{T} \bigg|_{\eta=1} = \bar{T}_s \bigg|_{r_s=R_s/R}. \]

2. **For Bush**
   a. Heat flux continuity at the oil-bush interface:
\[-\frac{\partial T_b}{\partial r_b}\bigg|_{r_b=R_{\text{in}}/R} = k(\theta) \frac{R}{k_b} \frac{\partial T}{\partial \eta}igg|_{\eta=0} \]. \hspace{1cm} (4.66)

b. Free convection at the outer surface of the bush:

\[-\frac{\partial T_b}{\partial r_b}\bigg|_{r_b=R_{\text{out}}/R} = B_{ib} \left( T_b\bigg|_{r_b=R_{\text{out}}/R} - 1 \right), \hspace{1cm} (4.67)\]

where:

\[B_{ib} = \frac{R_{h_b}}{k_b}. \hspace{1cm} (4.68)\]

c. Free convection at the axial ends of the bush:

\[-\frac{\partial T_b}{\partial \xi}\bigg|_{\xi=\pm L/D} = B_{ib} \left( T_b\bigg|_{\xi=\pm L/D} - 1 \right). \hspace{1cm} (4.69)\]

3- For Shaft

a. Heat flux continuity at the oil-shaft interface:

\[-\frac{\partial T_s}{\partial r_s}\bigg|_{r_s=1} = k(\theta) \frac{R}{k_s} \frac{\partial T}{\partial \eta}igg|_{\eta=1} \]. \hspace{1cm} (4.70)

b. Free convection at the axial ends of the shaft:

\[-\frac{\partial T_s}{\partial r_s}\bigg|_{\xi=\pm L/D} = B_{is} \left( T_s\bigg|_{\xi=\pm L/D} - 1 \right), \hspace{1cm} (4.71)\]

where:

\[B_{is} = \frac{R_{h_s}}{k_s}. \hspace{1cm} (4.72)\]
4.4 The Viscosity-Temperature Relationship

The thermophysical properties of the lubricant, including fluid viscosity, are strong functions of the lubricant temperature. The relationship between the fluid viscosity and temperature is measured as follows [136]:

\[ \mu = \mu_0 e^{-\beta(T-T_0)}. \] (4.73)

In dimensionless form:

\[ \bar{\mu} = e^{-\beta T_0(T-1)}, \] (4.74)

and:

\[ \bar{\mu}_m = \int_0^1 \bar{\mu} d\eta. \] (4.75)

4.5 The Thermohydrodynamic Numerical Model

This section develops the numerical procedure to solve the thermohydrodynamic model. In general, the hydrodynamic equations and the thermal model are coupled through the temperature-viscosity relationship and the thermal boundary conditions. The numerical solution for this system of PDEs is obtained by using an iterative numerical scheme. Conventionally, the hydrodynamic flow equations are firstly solved under isothermal conditions, subsequently, the results of the Reynolds equation are incorporated to solve the temperature distribution by using the thermal equations. Afterwards, the changes in the fluid viscosity are applied in a new iteration of the Reynolds equation and the thermal model. The iteration is interrupted as soon as the change of the solution in two successive iterations falls below a specified error tolerance. This type of procedure is generally referred to as the thermohydrodynamic analysis.

The axial and circumferential velocity components in Equation (4.33) are discretized as follows:
\[-u_{i,j,k} = \frac{H_i^2}{2\mu_{mi,k}} \frac{\bar{P}_{i,k} - \bar{P}_{i-1,k}}{\Delta \theta} \left[ \eta_j^2 - \eta_j \right], \quad (4.76)\]
\[-w_{i,j,k} = \frac{H_i^2}{2\mu_{mi,k}} \frac{\bar{P}_{i,k} - \bar{P}_{i,k-1}}{\Delta \xi} \left[ \eta_j^2 - \eta_j \right]. \]

For the radial velocity component:

\[-v_{i,j+1,k} = 2\bar{v}_{i,j,k} - \bar{v}_{i,j-1,k} - \Delta \eta^2 H_i \left[ \frac{\bar{u}_{i,j,k} - \bar{u}_{i-1,j,k} - \bar{u}_{i,j-1,k} + \bar{u}_{i-1,j-1,k}}{\Delta \theta \Delta \eta} + \frac{\varepsilon \sin(\theta_i) \bar{u}_{i,j,k} - \bar{u}_{i,j-1,k}}{H_i} \right] \]
\[\quad + \frac{\eta_j \varepsilon \sin(\theta_i) \bar{u}_{i,j,k} - 2\bar{u}_{i,j,k} + \bar{u}_{i,j-1,k}}{\Delta \eta^2} \frac{\bar{w}_{i,j,k} - \bar{w}_{i,j,k-1} - \bar{w}_{i,j-1,k} + \bar{w}_{i,j-1,k-1}}{\Delta \xi \Delta \eta} \right]. \quad (4.77)\]

Subsequently, Equation (4.35) is expanded:

\[3H^2 \frac{\partial H}{\partial \theta} \frac{\partial \bar{P}}{\partial \theta} - H^3 \frac{\partial \mu_m}{\partial \theta} \frac{\partial \bar{P}}{\partial \theta} + H^3 \frac{\partial^2 \bar{P}}{\partial \theta^2} - H^3 \frac{\partial \mu_m}{\partial \xi} \frac{\partial \bar{P}}{\partial \xi} + H^3 \frac{\partial^2 \bar{P}}{\partial \xi^2} = 12 \frac{\partial H}{\partial \tau}. \quad (4.78)\]

The partial derivatives of the fluid film thickness $H$ are given as follows:

\[\frac{\partial H}{\partial \theta} = -\varepsilon \sin(\theta), \quad (4.79)\]
\[\frac{\partial H}{\partial \tau} = \varepsilon \cos(\theta) + \varepsilon \sin(\theta). \]

Assuming that the SFD executes CCOs, the radial velocity and acceleration of the journal center become zero and:

\[\frac{\partial H}{\partial \tau} = \varepsilon \sin(\theta), \quad (4.80)\]
\[\frac{\partial^2 H}{\partial \tau^2} = -\varepsilon \cos(\theta). \quad (4.81)\]

Substituting Equations (4.80) and (4.81) into Equation (4.78) and discretizing the differential terms by using a similar technique to Section 2.1.4, and rearranging the equation gives:
\[
\bar{P}_{i,k} = \left[ A_3 - \left[ A_2 - A_4 \right] P_{i-1,k} - [A_4 - A_3] P_{i,k-1} - \left[ A_4 - A_3 \right] P_{i,k-1} - \left[ A_4 \right] P_{i,k+1} \right] \left[ A_1 - 2A_2 + A_3 - 2A_4 \right], \tag{4.82}
\]

where:

\[
A_1 = -\frac{3H^2 \varepsilon \sin(\theta)}{\mu_{mi,k} \Delta \theta} - \frac{H^3 \mu_{mi,k} - \mu_{mi-1,k}}{\mu_{mi,k} \Delta \theta^2},
\]

\[
A_2 = \frac{H^3 \mu_{mi,k} - \mu_{mi,k-1}}{\mu_{mi,k} \Delta \xi^2},
\]

\[
A_3 = \frac{H^3 \mu_{mi,k} - \mu_{mi,k-1}}{\mu_{mi,k} \Delta \xi^2}, \tag{4.83}
\]

\[
A_4 = \frac{H^3}{\mu_{mi,k} \Delta \xi^2},
\]

\[
A_5 = 12 \varepsilon \sin(\theta).
\]

Additionally, an iterative Gauss-Seidel numerical method with SOR, similar to Section 2.1.4, is used to solve the pressure distribution. Subsequently, the first-order inertial pressure correction is discretized. The first-order pressure distribution in the fluid film is characterized by Equation (4.38). This equation is first expanded to facilitate the discretization of the differential terms:

\[
H^3 \frac{\partial^2 P}{\partial \theta^2} + 3H^2 \frac{\partial H}{\partial \theta} \frac{\partial P}{\partial \theta} + H^3 \frac{\partial^2 P}{\partial \xi^2} = 12 \frac{\partial H}{\partial \tau} + G_1(\theta, \xi) + G_2(\theta, \xi), \tag{4.84}
\]

where:

\[
G_1(\theta, \xi) = -\frac{3H^6}{40} \frac{\partial H}{\partial \theta} \left[ \frac{\partial^2 P}{\partial \theta \partial \xi} + \frac{\partial^2 P}{\partial \xi^2} \right] - \frac{3H^7}{280} \left[ \left( \frac{\partial^2 P}{\partial \theta^2} \right)^2 + \frac{\partial^2 P}{\partial \theta^2} \frac{\partial^2 P}{\partial \theta \xi} + \frac{\partial^2 P}{\partial \theta \xi} \frac{\partial^2 P}{\partial \xi^2} \right] - 9H^5 \frac{\partial H}{\partial \theta} \left[ \frac{\partial^2 P}{\partial \theta^2} \right]^2
\]

\[
- \frac{3H^6}{140} \frac{\partial^2 H}{\partial \theta^2} \left[ \frac{\partial^2 P}{\partial \theta^2} \right]^2 - \frac{3H^6}{70} \frac{\partial^2 H}{\partial \theta^2} \frac{\partial^2 P}{\partial \theta \xi} + \frac{26H^3}{35} \frac{\partial^2 H}{\partial \theta^2} \frac{\partial^2 P}{\partial \theta \xi} + \frac{13H^4}{70} \frac{\partial^2 H}{\partial \theta^2} \frac{\partial^2 P}{\partial \theta \xi} \frac{H^5}{10} + \frac{\partial^2 P}{\partial \theta^2} H^4 \frac{\partial H}{\partial \theta}. \tag{4.85}
\]
\[ G_2(\theta, \xi) = -\frac{3H^7}{280} \left( \frac{\partial^2 P_0}{\partial \theta \partial \xi} \right)^2 + \frac{\partial P_0}{\partial \theta} \frac{\partial^3 P_0}{\partial \theta \partial \xi^2} + \left( \frac{\partial^2 P_0}{\partial \xi^2} \right)^2 + \frac{\partial P_0}{\partial \xi} \frac{\partial^3 P_0}{\partial \xi^3} \right) \]

\[ - \frac{3H^6 \partial H}{140} \frac{\partial^3 P_0}{\partial \theta \partial \xi^2} \frac{\partial \xi}{\partial \xi} - \frac{3H^6 \partial H}{140} \frac{\partial^2 P_0}{\partial \theta \partial \xi} + \frac{13H^4 \partial H}{70} \frac{\partial^2 P_0}{\partial \xi^2} + \frac{\partial^3 P_0}{\partial \xi^3} \left( \frac{H^5}{10} \right). \]  

(4.86)

According to the chain rule:

\[ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial \tau} = -\frac{\partial}{\partial \theta}. \]  

(4.87)

Furthermore, the partial derivatives of the fluid film thickness \( H \) are given as follows:

\[ \frac{\partial H}{\partial \theta} = -\epsilon \sin(\theta), \]

\[ \frac{\partial H}{\partial \tau} = \dot{\epsilon} \cos(\theta) + \epsilon \sin(\theta). \]  

(4.88)

Assuming that the SFD executes CCOs, the radial velocity and acceleration of the journal center become zero and:

\[ \frac{\partial H}{\partial \tau} = \epsilon \sin(\theta), \]  

(4.89)

\[ \frac{\partial H}{\partial \theta} = -\epsilon \sin(\theta). \]  

(4.90)

Subsequently, Equations (4.84) to (4.86) are discretized based on the finite difference approximation procedure that was defined in Sections 2.1.4 and 2.3.3 as follows:

\[ H_i \frac{P_{i,j+1} - 2P_{i,j} + P_{i,j-1}}{\Delta \theta^2} - 3H_i^2 \dot{\epsilon} \sin(\theta_i) \frac{P_{i,j} - P_{i,j-1}}{\Delta \theta} \]

\[ + H_i \frac{P_{i,j+1} - 2P_{i,j} + P_{i,j-1}}{\Delta \xi^2} = 12\epsilon \sin(\theta_i) + G_{v,i} + G_{z,i}. \]  

(4.91)

Equation (4.91) is rearranged to develop the solution for the pointwise temperature:
\begin{equation}
\begin{aligned}
P_{i,j} &= \left[ B_4 + G_{i,j} + G_{2,i,j} \right] + \left[ B_1 + B_2 \right] \bar{P}_{v_{i+1,j}} + \left[ B_2 \right] \bar{P}_{v_{i,j+1}} + \left[ B_3 \right] \left( \bar{P}_{v_{i-1,j}} + \bar{P}_{v_{i,j-1}} \right), \\
&= \frac{\left[ B_1 + B_2 + B_3 \right]}{\left[ B_1 + B_2 + B_3 \right]},
\end{aligned}
\end{equation}

where:

\begin{equation}
\begin{aligned}
B_1 &= -\frac{3H_i^2 (e \sin \theta_i)}{\Delta \theta}, \\
B_2 &= -\frac{H_i^2}{\Delta \theta^2}, \\
B_3 &= -\frac{H_i^2}{\Delta \xi^2}, \\
B_4 &= 12 \varepsilon \sin \theta_i.
\end{aligned}
\end{equation}

Subsequently, Gauss-Seidel technique with SOR is used to solve Equation (4.92). The total pressure distribution is calculated by using Equation (4.37). Similarly, Equations (4.61) to (4.63) of the thermal model are discretized respectively as follows:

\begin{equation}
\begin{aligned}
\bar{T}_{i,j,k} &= C_1 \bar{T}_{i-1,j,k} + (C_2 + C_4) \bar{T}_{i,j-1,k} + C_2 \bar{T}_{i,j,k-1} + C_4 \bar{T}_{i,j+1,k} + C_4, \\
&= \frac{C_1 + C_2 + C_3 + 2C_4}{C_1 + C_2 + C_3 + 2C_4},
\end{aligned}
\end{equation}

where:

\begin{equation}
\begin{aligned}
C_1 &= \bar{u}_{i,j,k}, \\
C_2 &= \frac{1}{\Delta \eta H_i} \left( \bar{v}_{i,j,k} + \bar{u}_{i,j,k} \varepsilon \sin(\theta_i) \right), \\
C_3 &= \bar{w}_{i,j,k}, \\
C_4 &= \frac{1}{H_i^2 \text{Pe} \Delta \eta^2}, \\
C_5 &= \frac{\mu_{i,j,k}}{H_i^2} \frac{\text{Ne}}{\text{Re}_0} \left[ \left( \frac{\bar{u}_{i,j,k} - \bar{u}_{i,j-1,k}}{\Delta \eta} \right)^2 + \left( \frac{\bar{w}_{i,j,k} - \bar{w}_{i,j-1,k}}{\Delta \eta} \right)^2 \right],
\end{aligned}
\end{equation}

and:
\[ T_{bi,j,k} = \frac{D_1T_{bi,j+1,k} + T_{bi,j-1,k}(D_1 - D_2) + D_3(T_{bi+1,j,k} + T_{bi-1,j,k}) + D_4(T_{bi,j,k+1} + T_{bi,j,k-1})}{(2D_1 - D_2 + 2D_3 + 2D_4)}, \] (4.96)

where:

\[ D_1 = \frac{1}{\Delta r_b}, \]
\[ D_2 = \frac{1}{r_b \Delta r_b}, \]
\[ D_3 = \frac{1}{r_b \Delta \theta^2}, \]
\[ D_4 = \frac{1}{\xi^2}. \] (4.97)

and:

\[ \bar{T}_{si,j,k} = \frac{E_1T_{si,j+1,k} + T_{si,j-1,k}(E_1 - E_2) + E_3(T_{si+1,j,k} + T_{si-1,j,k}) + E_4(T_{si,j,k+1} + T_{si,j,k-1})}{(2E_1 - E_2 + 2E_3 + 2E_4)}, \] (4.98)

where:

\[ E_1 = \frac{1}{\Delta r_s}, \]
\[ E_2 = \frac{1}{r_s \Delta r_s}, \]
\[ E_3 = \frac{1}{r_s \Delta \theta^2}, \]
\[ E_4 = \frac{1}{\xi^2}. \] (4.99)

Furthermore, the discretized thermal boundary conditions are given by:

1- **For Energy Equation**

a. Matching temperatures at the oil-bush interface:

\[ \bar{T}_{i,j,k} = \bar{T}_{bi,i,k}. \] (4.100)

b. Matching temperatures at the oil-shaft interface:
\[ T_{i,\text{end},k} = T_{s_i,\text{end},k}. \] (4.101)

2- **For Bush**

a. Heat flux continuity at the oil-bush interface:

\[
\bar{T}_{bi,1,k} = \frac{r_k(r_k(\theta) - 1) R}{k_b} \frac{T_{i,2,k} - T_{i,1,k}}{H_c} + \bar{T}_{bi,2,k}. \tag{4.102}
\]

b. Free convection at the outer surface of the bush:

\[
\bar{T}_{bi,\text{end},k} = \frac{\bar{T}_{bi,\text{end},-1,k} + r_k B_{ib}}{1 + r_k B_{ib}}. \tag{4.103}
\]

c. Free convection at the axial ends of the bush:

\[
\bar{T}_{bi,j,1} = \frac{\bar{T}_{bi,j,2} - \xi B_{ib}}{1 - \xi B_{ib}},
\]
\[
\bar{T}_{bi,j,\text{end}} = \frac{\bar{T}_{bi,j,\text{end},-1} + \xi B_{ib}}{1 + \xi B_{ib}}. \tag{4.104}
\]

3- **For Shaft**

a. Heat flux continuity at the oil-shaft interface:

\[
\bar{T}_{s_i,\text{end},k} = \bar{T}_{bi,\text{end},-1,k} - \frac{r_k(r_k(\theta) - 1) R}{k_s} \frac{T_{i,\text{end},k} - T_{i,\text{end},-1,k}}{H_c} + \bar{T}_{s_i,1,k}. \tag{4.105}
\]

b. Free convection at the axial ends of the shaft:

\[
\bar{T}_{s_i,j,1} = \frac{\bar{T}_{s_i,j,2} - \xi B_{is}}{1 - \xi B_{is}},
\]
\[
\bar{T}_{s_i,j,\text{end}} = \frac{\bar{T}_{s_i,j,\text{end},-1} + \xi B_{is}}{1 + \xi B_{is}}. \tag{4.106}
\]
Figure 4.1 The numerical thermohydrodynamic procedure
Subsequently, the discretized thermal model is solved by using Gauss-Seidel method. The summary of the solution procedure is represented in Figure 4.1. The detailed numerical procedure to solve the thermohydrodynamic model is given as follows:

1. An initial dimensionless temperature distribution in the lubricant and in the solids is assumed (typically the supply oil temperature for the lubricant and the ambient temperature for the solids).
2. The dimensionless fluid viscosity field is calculated by using Equation (4.74).
3. The dimensionless average radial viscosity field is calculated by using Equation (4.75).
4. The dimensionless inertialess pressure distribution is determined by solving Equation (4.82) iteratively by using Gauss-Seidel method with SOR until the convergence criterion is met.
5. The dimensionless velocity fields are calculated by using Equations (4.76) and (4.77).
6. The pressure distribution and the velocity fields from steps (4) and (5) along with the thermal boundaries in Equations (4.100) and (4.101) are incorporated to solve Equation (4.94) iteratively by using Gauss-Seidel method.
7. The thermal boundary conditions in equations (4.102) to (4.104) are incorporated to solve Equation (4.96) for the bush temperature by using Gauss-Seidel method.
8. The thermal boundary conditions in equations (4.105) and (4.106) are incorporated to solve Equation (4.98) for the shaft temperature by using Gauss-Seidel method.
9. The temperature convergence at the oil-bush interface and the oil-shaft interface are evaluated. If the convergence criterion is not met, steps (2) to (8) are repeated.
10. The inertialess pressure distribution is incorporated into Equation (4.92) to determine the first-order pressure correction by using iterative Gauss-Seidel method with SOR.
11. Equation (4.37) is used to determine the total lubricant pressure.

### 4.6 Results and Discussions

The proposed THD model along with the numerical procedure in Section 4.5 are incorporated into a simulation model to study the effect of temperature variation in the presence of fluid inertia on the dynamic performance of SFDs. Table 4.1 summarizes the parameter values in the simulations.
Table 4.1 The simulation parameters for the thermohydrodynamic model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_p$</td>
<td>2000</td>
<td>J/kg °C</td>
</tr>
<tr>
<td>$c/R$</td>
<td>0.00242</td>
<td></td>
</tr>
<tr>
<td>$k_a$</td>
<td>0.025</td>
<td>W/m °C</td>
</tr>
<tr>
<td>$h_b, h_s$</td>
<td>80</td>
<td>W/m² °C</td>
</tr>
<tr>
<td>$k_b, k_s$</td>
<td>50</td>
<td>W/m °C</td>
</tr>
<tr>
<td>$k_f$</td>
<td>0.13</td>
<td>W/m °C</td>
</tr>
<tr>
<td>$L/D$</td>
<td>0.325</td>
<td></td>
</tr>
<tr>
<td>$R_{w_i}/R$</td>
<td>1.0024</td>
<td></td>
</tr>
<tr>
<td>$R_{w_i}/R$</td>
<td>1.3375</td>
<td></td>
</tr>
<tr>
<td>$T_0$</td>
<td>40</td>
<td>°C</td>
</tr>
<tr>
<td>ε</td>
<td>0.1 to 0.5</td>
<td></td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>0.0277 at 40°C</td>
<td>Pa – s</td>
</tr>
<tr>
<td>$\rho$</td>
<td>860</td>
<td>kg/m³</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>1000 to 15000</td>
<td>rpm</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.034</td>
<td></td>
</tr>
</tbody>
</table>
This section represents the results of the THD analysis at four shaft velocities of 1000, 5000, 10000, and 15000 rpm and for eccentricity ratios of 0.1 to 0.5. Figures 4.2 to 4.5 demonstrate the lubricant temperature contours for the four shaft velocities at different eccentricity ratios and at the bearing axial mid-plane. In general, the lubricant temperature variation for SFDs is not very significant even at high shaft velocities, since unlike plain journal bearings, the SFD journal is prevented from rotation, which results in considerably smaller velocities in the circumferential direction. However, even small variations in the lubricant temperature distribution leads to changes in the lubricant viscosity, which ultimately influences the SFD performance. The viscous dissipation in the lubricant is directly proportional to the shaft velocity. Consequently, at small shaft velocities (i.e. 1000 rpm) the variation in the lubricant temperature is very small even at large eccentricities. Furthermore, the lubricant pressure distribution and ultimately the lubricant velocities are proportional to the journal eccentricity ratio. At moderate shaft velocities (i.e. 5000 rpm) and at large eccentricity ratios of 0.4 and 0.5, the temperature variation within the fluid film is much more noticeable. At moderately large and large shaft speeds, the temperature variation is significant even at small eccentricity ratios. Additionally, at small eccentricity ratios, the lubricant temperature is mostly affected at the vicinity of the solid surroundings. Increasing the eccentricity ratios spreads the temperature variation towards the dimensionless radial mid-plane of the bearing. The maximum lubricant temperature is encountered circumferentially either immediately after the end of the rupture zone or at the maximum lubricant pressure angle. Finally, the lubricant pressure distribution remains unchanged at the rupture zone of the film, since according to the π-film cavitation model, the pressure distribution and the fluid film velocity profiles are both zero at the cavitation zone, which results in no viscous dissipation and heat conduction in the rupture zone of the fluid film.
Figure 4.2 The lubricant temperature distribution at 1000 rpm
Figure 4.3 The lubricant temperature distribution at 5000 rpm
Figure 4.4 The lubricant temperature distribution at 10000 rpm
Figure 4.6 represents the maximum film temperature at different shaft speeds and journal eccentricity ratios. At 1000 rpm, the temperature variation in the fluid film is very small and the maximum temperature is approximately equal to the supply conditions. At 5000 rpm and at larger eccentricity ratios, the maximum temperature grows noticeably and reaches 41°C at 0.5 eccentricity ratio. At larger shaft speeds the maximum temperature is noticeably larger even at small eccentricity ratios and ends up at several degrees of centigrade above the supply temperature for large eccentricity ratios.

![Maximum Lubricant Temperature](image)

**Figure 4.6** The maximum lubricant temperature at different shaft speeds and journal eccentricity ratios

Figure 4.7 displays the mean lubricant temperatures at different shaft speeds and journal eccentricity ratios. Similar to the previous discussion, at 1000 rpm, the variation in the fluid temperature is very small and the mean lubricant temperature stays very close to the supply conditions. The results of the simulations at higher shaft speeds show that the fluid film mean temperature increases with both the shaft speed and journal eccentricity ratios. However, the mean temperatures are significantly smaller compared to the maximum film temperatures under similar SFD operating parameters. This is justified, since according to the results and analysis of
the mid-plane lubricant temperature contours, the lubricant temperature stays at the supply condition at the rupture zone, which stands for half of the total volume of the lubricant film.

**Figure 4.7** The mean lubricant temperature at different shaft speeds and journal eccentricity ratios

Figure 4.8 represents the effect of lubricant temperature variation and inertia effects on the radial and tangential fluid film reaction forces for SFDs. In order to emphasize on the thermal and inertia effects, the comparison of the simulation results is represented for three different models, namely (a) isothermal Reynolds equation, (b) thermohydrodynamic model with no inertia effects, and (c) thermohydrodynamic model including fluid inertia effects. The simulation results are compared at different shaft speeds (i.e. Reynolds numbers). In general, the contribution of fluid inertia to the radial reaction force is a positive value, which reduces the magnitude of the radial forces and improves the stability of the SFD. Additionally, the inertia effects contribute a negative value to the tangential forces and increase their magnitude. At small shaft velocities, the squeeze Reynolds number is very small, which translate into negligible fluid inertia effects. Furthermore, according to the previous discussions, the viscous dissipation and heat conduction
are very insignificant at small shaft speeds. Consequently, at small shaft velocities, the results of the three models are in very close agreement. At higher shaft velocities (i.e. 5000 rpm) the squeeze Reynolds number grows, which results in the larger contribution of the inertia effects to the force components. However, the temperature variation is still small even at large eccentricity ratios. Consequently, the discrepancy between the THD model and the other two models become noticeable. At larger shaft speeds (i.e. 10000 rpm and 15000 rpm) the effect of fluid inertia grows significantly larger. Furthermore, the fluid film temperature is much more considerable, especially at larger eccentricity ratios. Consequently, the disagreement between the results of the three models is very noticeable especially at large eccentricity ratios. The results of this section imply that the effect of fluid inertia must be included in the SFD analysis even at moderate shaft velocities. Additionally, the thermal effects significantly contribute to the SFD dynamics when both shaft speeds and journal eccentricity ratios are large.
Tangential Force $[F_t]$

Radial Force $[F_r]$

SFD Dimensionless Tangential Force at Re=0.52439

SFD Dimensionless Radial Force at Re=0.78659
Figure 4.8 The dimensionless radial and tangential fluid film reaction force components at different shaft speeds

4.7 Conclusion

This chapter developed a comprehensive thermohydrodynamic analysis of SFDs. Firstly, the generalized Reynolds equation and inertia-less velocity components were developed for variable viscosity conditions. Subsequently, an averaged radial viscosity matrix was defined to reduce the hydrodynamic equations. Additionally, a first order inertia correction for the lubricant pressure distribution was introduced to account for the effect of fluid inertia on the SFD dynamics. Moreover, the energy equation in the lubricant domain, the Laplace heat conduction equations in the bushing and the shaft, and the thermal boundary conditions were described. The hydrodynamic and thermal equations and the boundary conditions were incorporated into an iterative numerical procedure to determine the temperature distributions in the lubricant and the solids and the lubricant pressure field. The proposed system of equations along with the numerical procedure was integrated into a simulation model in Matlab and the results were represented for different SFD operating parameters. According to the results:
1. In general, the lubricant temperature variation increases with the shaft velocity and journal eccentricity ratio. Furthermore, the fluid temperature remains at supply conditions in the rupture zone.

2. At small eccentricity ratios, the maximum lubricant temperature distribution is closer to the solid surfaces of the bearing, however, at moderate and large eccentricity ratios, the distribution spreads to the radial mid-pane of the bearing.

3. At small shaft velocities, the heat dissipation and heat conduction within the lubricant is very small. Consequently, the temperature variation in the fluid film is negligible even at large eccentricity ratios. Furthermore, the squeeze Reynolds number is very small, which makes the effect of fluid inertia negligible.

4. At moderate shaft velocities, the influence of fluid inertia is more significant and contributes significantly to the magnitude of the fluid film reaction forces. Additionally, at large eccentricity ratios, the thermal effects result in a notable temperature variation in the lubricant which can no longer be neglected.

5. At moderately large and large shaft speeds, the fluid inertia effects significantly influence the SFD dynamics. Furthermore, the lubricant temperature variation is much larger even at small eccentricity ratios, and considerably contributes to the fluid film reaction forces.
Conclusion

Unbalance-induced engine vibrations are the main source of steady-state vibrations in high-speed turbomachinery including jet engines. SFDs are essential components in high-speed turbomachinery that are incorporated to attenuate the steady-state vibrations at resonance frequencies. The critical design consideration in SFDs is the magnitude of the produced damping, since large damping amplitudes force the SFD to act as a rigid constraint to the rotor system with large forces transmitted to the supporting structure, while small damping allows large amplitudes of vibration level. The dynamic force response of SFDs is determined by the damper geometry, operating speed, and lubricant properties. Consequently, in order to design successful SFDs for turbomachinery, it is required to develop accurate models which provide realistic predictions of the SFD behaviour. Development of an accurate and computationally efficient model for squeeze film dampers has been an ongoing challenge due to their nonlinear dynamic behavior. This thesis provided a detailed theoretical analysis of the SFD dynamics in high-speed turbomachinery. Generally, SFD models assume that the influence of lubricant inertia and the thermal effects in the thin film and the surrounding solids is negligible on the SFD. Additionally, the existing SFD fluid inertia models either apply geometry approximations (i.e. SBA and LBA), which deteriorates the accuracy of the predictions for arbitrary damper geometries, or they are computationally inefficient for integration into rotordynamic models, where the SFD parameters are calculated over thousands of iterations. Furthermore, the existing thermohydrodynamic SFD models typically neglect the effect of fluid inertia on the SFD dynamics. This work has implemented a variety of analytical tools to successfully develop several dynamic models for SFDs in the presence of fluid inertia, which accurately and efficiently predict the SFD behavior under different conditions. Subsequently, a comprehensive THD analysis, including fluid inertia effects, was represented.

Firstly, several analytical and numerical models were proposed to describe the SFD behavior in presence of fluid inertia. The results of the analysis demonstrated the significant effect of fluid inertia on the SFD pressure distribution and fluid forces especially at large shaft velocities. Furthermore, the comparison between the results of the proposed models and the force coefficient technique confirmed the superior accuracy of the proposed models especially at larger
journal eccentricity ratios and Reynolds numbers. The proposed SFD models provide substantial tools for enhanced understanding of the effect of lubricant inertia on the damping characteristics of SFDs as well as an accurate estimation for the effect of SFD geometry and lubricant properties on the SFD dynamics.

Subsequently, a finite element based multi-mass flexible rotordynamic model incorporating the proposed SFD models was developed. The existing rotordynamics models incorporating SFDs either completely neglect the lubricant inertia effects and incorporate Reynolds equation to represent the SFD dynamics or implement a simple flexible Jeffcott rotordynamic model to study the effect of SFD fluid inertia on the unbalance-induced stead-state vibrations. The application of the multi-mass flexible rotordynamic model significantly improves the accuracy of the analysis. Furthermore, application of the modal integration algorithm in the numerical model considerably improves the computational efficiency. The results of the rotordynamic analysis demonstrated the significant influence of the SFD fluid inertia on attenuating the steady-state unbalance induced rotor vibration magnitudes at the resonance zone and the high-speed regions. The proposed rotordynamic model is a very powerful tool for the analysis of the steady-state and transient rotor vibrations in the presence of SFD fluid inertia. Furthermore, the model provides a platform to estimate the SFD parameters that are required to achieve a certain rotor operating criterion, including resonance frequencies and maximum allowed vibration amplitudes.

Finally, a comprehensive thermohydrodynamic analysis of SFDs was represented. Typically, the existing THD analysis neglect the effect of SFD lubricant inertia. In this work, a first order inertia correction for the lubricant pressure distribution was introduced to account for the effect of fluid inertia on the SFD dynamics. The results of the analysis demonstrated the significant effect of the SFD thermal interactions at high rotor velocities and large eccentricity ratios.

The results of this work are especially valuable to the high-speed turbomachinery industry (i.e. jet engines and gas turbines) since it provides a method to quickly and accurately review damper designs and provide results as inputs to the system engineering team during the conceptual design phase of an engine/turbine. This could lead to significant development cost reduction through reduced system design iteration. Furthermore, the damper model is effectively integrated
into the rotordynamic model of the complete system, providing a very powerful simulation tool to accurately predict the system vibrations during the development phase.
Contributions

Development of accurate and computationally efficient models to incorporate the effect of fluid inertia in the analysis of SFDs in high-speed turbomachinery:

1. Unlike the conventional force coefficient models, which apply limiting geometry assumption to develop expressions for the inertia and damping force coefficients for SFDs, this work incorporates analytical and numerical techniques to directly develop models to analyze finite-length SFDs. This would significantly expand the applications of the proposed model to arbitrary SFD geometries and will considerably enhance the accuracy of the calculations.

2. An additional advantage of the proposed models is that they provide expressions for the lubricant pressure distribution as well as the fluid velocity components in presence of fluid inertia. The proposed expressions are essential for studying the effect of fluid cavitation as well as developing a thermohydrodynamic model, which includes the lubricant inertia effects.

3. This work proposed closed-form expressions for the velocity components, the lubricant pressure distribution, and the fluid film reaction forces for short-length SFDs in the presence of fluid inertia effects. The proposed analytical expressions provide accelerated calculation of the SFD parameters by reducing the computation time approximately 80 times relative to the numerical models. This is especially valuable for rotordynamics analysis, where the SFD parameters are calculated over thousands or even millions of iterations.

Development of a finite element based multi-mass flexible rotordynamic model to study the effect of SFD fluid inertia on the steady-state and transient unbalance induced vibration amplitude of high-speed rotors

1. The existing rotodynamic SFD models either completely neglect the effect of SFD fluid inertia in the calculations or incorporate simplified rotor models, including rigid models and simple flexible models. This work provides the first application of a complex rotor model for the study of fluid inertia effects on the rotor vibrations.
2. The application of the numerical transient modal integration along with the complex rotor model significantly enhances the accuracy and the computational efficiency of the calculations.

**Development of a comprehensive thermohydrodynamic model to simultaneously investigate the fluid inertia effects and thermal effects on the SFD dynamic performance**

1. The existing rotodynamic THD models typically ignore the fluid inertia effects in the calculations, which makes them inapplicable for analysis of high-speed applications, where the inertia effects are significantly large and can no longer be ignored. Additionally, the temperature variation in the shaft and the axial temperature variation in the lubricant and bushing is conventionally assumed negligible. This work represents a detailed THD model for SFDs by including all the above effects into the model.

2. The focus of the current THD analysis in modern lubrication has been on plain journal bearings, tilting pad bearings, and thrust bearings. This work provides a detailed derivation of the thermohydrodynamic model for SFDs, including the hydrodynamic and thermal model equations and boundary conditions as well as a detailed numerical procedure to solve the system of partial differential equations.
Future Work

Although this study represented a comprehensive investigation of SFDs in high-speed turbomachinery, including significant contributions to this research area, however, additional studies can be performed to further advance the knowledge base in this research area. The following points are a summary of recommendations for future work:

1. Investigation of the effect of fluid cavitation in presence of fluid inertia effects. In the present study, a simplistic π-film model is incorporated to include the effect of lubricant cavitation in the studies. However, in order to even further enhance the accuracy of the SFD calculations, it is required to integrate a more realistic cavitation algorithm, such as Elrod cavitation algorithm, in the SFD calculations.

2. This work studies the dynamics of open-ended (i.e. no end seals) SFDs. The application of the proposed SFD models could be extended to sealed SFDs by incorporating the appropriate end boundary equations into the calculations.

3. The proposed THD model could be further extended to account for the effect of elastic deformations due to the lubricant and solids thermal effects as well as the influence of solid residues in the lubricant on the SFD dynamics.

4. The application of magneto-rheological (MR) fluids is fast immersing in the lubrication industry. The current MR fluid models neglect the effect of fluid inertia and thermal interactions on the dynamics of the corresponding bearings. The proposed models in this thesis could be employed as the guideline to gain further insight into the modeling of these bearings.

5. The results of the analysis in this work were incorporated by the researchers at ARL-MLS Laboratory at the University of Toronto to design and develop a world-class experimental setup to study the effect of SFD lubricant inertia, thermal variations, and operating parameters on the steady-state vibrations of high-speed rotors. The experimental setup will be implemented to validate the analytical and numerical models that are proposed in this work. Furthermore, the effect of SFD geometry, including journal diameter and radial clearance, lubricant properties, and end seals on the SFD dynamics will be evaluated.
References


[15] H. Hashimoto, “Boundary conditions for the calculation of the dynamic characteristics of


