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1 Effect of Soft Outer Lining in Pentafurcated Duct

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4 **Abstract**

5 This paper describes the diffraction of the lowest plane wave prop-
6 agating out of the opening of a semi-infinite hard duct which is sym-
7 metrically placed inside an infinite soft duct. The whole system forms
8 a pentafurcated waveguide whose solution is given by eigenfunction ex-
9 pansion method. Same method has been used to validate the results
10 of a trifurcated duct problem previously tackled by using the standard
11 Wiener-Hopf technique. Some graphical results showing the influence of
12 waveguide spacing on the reflection coefficient are presented. We com-
13 pared our results with the related existing work. We observed that the
14 accumulative value of reflection for current pentafurcated duct is 1375.1
15 which is greater among the related trifurcated and existing pentafurcated
16 ducts. Hence the soft lining on outer plates gives better results to atten-
17 uate the unwanted noise. These problems have application in acoustics.

18 Keywords: Exhaust system; Eigenfunction expansion; Diffraction; Tri and
19 penta furcated ducts; Soft and rigid plates.

20 **1 Introduction**

21 Noise is a serious current issue and a hazard to environmental health [16].
22 Machines running with high sound intensity produce noise through air ducts
23 which act as waveguides for sound propagation. Sound absorptive duct linings
24 and proper geometrical duct patterns can be beneficial to reduce the inten-
25 sity of this unwanted sound. Mathematical determination of the wave prop-
26 agation has application in many areas of physical significance. Bowmen et.
27 al. [1] presented ample results by considering different geometries for electro-
28 magnetic and sound waves while McIver [2] has determined the scattering of
29 water waves for two dimensional problem. Ammari et al ([18],[19]) have dealt
30 with the electromagnetic wave scattering by a thin structure. Many researchers
31 ([3],[4],[7],[22],[26]) considered the acoustic wave scattering satisfying different

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boundary conditions. Jones [8] was first to solve the problem of non-staggered equally spaced half plane which ended in Wiener-Hopf equation matrix. Tayar and Alinur [14] have considered the equidistant plate duct with bounded impedance. Ammari et al [20] used variational approach to analyze the harmonic electromagnetic waves scattering from cavity. Kaplunov et al [27] considered a plane acoustic layer bounded by elastic membranes. Many investigators studied sound attenuating problems by considering various geometries of waveguide ducts with different nature of plates. Some took bifurcated waveguide ([7],[9],[26]) while others considered trifurcated problems ([3],[4]). Many authors ([3, 4, 5, 7, 8, 14]) have used scalar Wiener Hopf method. Wiener Hopf technique is based on the application of integral transforms, analytic continuation of complex valued functions and residue theory. In this method boundary value problem is transformed to Wiener Hopf functional equations/matrix by using integral transforms (formulas for matrix factorization are very limited). Wiener Hopf equation involves two unknown functions of a complex variable which are analytic in two overlapping half planes. This provides sufficient information to apply the theory of analytic continuation. An important step in this method is to decompose the kernel functions into a product and sum of two functions in appropriate half planes. These decomposed/split functions involve infinite products which are very difficult to cope with. An alternate method, known as eigenfunction expansion method employed in many scattering problems ([11],[12],[13],[15],[17],[21],[23],[25],[28],[29],[30],[31],[33]), is comparatively more straightforward and easier to implement. In this method potential solutions satisfying boundary conditions are presented in form of eigen modes which are matched across the boundary of each continuity.

In this paper we solved a five spaced/pentafurcated waveguide problem. Similar duct designed problem has been solved by Hassan et al [24] by taking rigid boundary conditions on all plates while in our work the nature of outer plates are soft. To make the analysis simple and to get first hand approximation of the reflected field in actual system, we took pressure-release condition on outer plates. We split waveguide into six regions for convenience. Potential functions in these regions were found by using Linton and McIver's approach [10]. We used eigenfunction expansion to get six infinite set of unknown coefficients which were determined by MATLAB programming [32]. In appendix we have given solution for a three spaced duct problem by following the same procedure. Hassan and Rawlins [5] have discussed the same trifurcated problem by Wiener-Hopf method. Same results were obtained for the reflection field amplitude as were found in [5] for the given duct's dimensions, which proves the validity of our method. We also determined energy conversation and the impact of change in duct's width on reflection for considered geometries. We have presented the reflection field amplitude comparison among trifurcated (appendix), outer-soft pentafurcated and rigid pentafurcated duct [24] problems. This comparison would be helpful to analyze acoustic radiated energy for considered geometries. It is shown that the soft lining on outer plates has better impact as compared to that of hard boundary in pentafurcated duct, thus resulting in reduction of sound. We have also given comparison of reflection field behavior with existing

1 results of penta-furcated problems ([24],[28], [29]). The results will be helpful for
 2 sound attenuating gadgets (e.g., silencers, band-stop filters etc.).

3 2 Formulation of Outer-Soft Penta-furcated Duct 4 Problem

5 We investigate the acoustic reflection of a lowest wave mode which comes out
 6 from a semi-infinite pipe $|y| < a, x < 0$. Figure 1 shows the schematic geometry
 7 of the problem. The configuration of the penta-furcated problem is such that
 8 a semi infinite hard trifurcated duct is placed symmetrically inside the infinite
 9 soft duct. The sound source field, which propagates modes along $|y| < a$ is
 10 located at (x_0, y_0) , ($x_0 < 0, |y_0| < a$). The acoustic pressure and velocity are
 11 defined by

$$12 \quad p = -\rho_o \frac{\partial \varphi}{\partial t}$$

13 and

$$14 \quad \vec{u} = \text{grad}(\varphi)$$

15 respectively, where $\varphi(x, y, t)$ is the scalar potential function and ρ_o is the equi-
 16 librium state density. The time variant can be removed by writing

$$17 \quad \varphi(x, y, t) = \text{Re}[\phi(x, y)e^{-i\omega t}] \quad (2.1)$$

18 where the angular frequency is $\omega = kc$, (k and c are the wave number and
 19 speed of sound respectively). Then the wave equation

$$20 \quad \nabla^2 \varphi = (1/c^2)\varphi_{tt}$$

21 reduces to two dimensional Helmholtz equation

$$22 \quad \nabla^2 \phi(x, y) + k^2 \phi(x, y) = 0, c_1 \leq y \leq c_2. \quad (2.2)$$

23 We solve (2.2) for $\phi(x, y)$ subject to BCS:

$$24 \quad \phi(x, y) = 0, y = c_1, x \in (-\infty, \infty), \quad (2.3)$$

$$25 \quad \phi_y(x, y) = 0, y = c_2, x \in (-\infty, 0]. \quad (2.4)$$

26 where $c_1 = -d, d$, and $c_2 = -b, -a, a, b$. The following radiation conditions
 27 guarantee the solution to be bounded

$$28 \quad \phi(x, y) - \text{incident} = \text{bounded} + \text{outstanding} \quad \text{as } x \rightarrow \pm\infty \quad (2.5)$$

30

Figure 1: Schematic diagram of penta-furcated duct

3 Determination of Coefficients

3.1 Region 1 $\{-d \leq y \leq -b, x < 0\}$

Here the solution of (2.2) is given by

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n e^{-\bar{\alpha}_n x} \psi_n(y) \quad (3.1)$$

which satisfies (2.3), (2.4) and (2.5). Eigenvalues, associated eigenvalues and eigenfunctions in region 1 are defined by

$$\alpha_n = \frac{(2n-1)\pi}{2(d-b)}, \quad n = 1, 2, \dots,$$

$$\bar{\alpha}_n = \sqrt{k^2 - \alpha_n^2}, \quad n = 1, 2, \dots$$

and

$$\psi_n(y) = \sqrt{\frac{2}{(d-b)}} \cos \alpha_n(y+b), \quad (3.2)$$

respectively. The eigenvalues α_n are the solutions of

$$\cos \alpha_n(d-b) = 0.$$

The eigenfunctions $\psi_n(y)$ satisfy

$$\int_{-d}^{-b} \psi_m(y) \psi_n(y) dy = \delta_{mn} \quad m, n = 1, 2, \dots \quad (3.3)$$

which is the orthonormal relation, where δ is Kronecker delta defined by

$$\delta_{mn} = \begin{cases} 1, & m=n \\ 0, & m \neq n. \end{cases}$$

3.2 Region 2 $\{-b \leq y \leq -a, x < 0\}$

Here the vertical eigenfunctions are given by

$$\xi_n(y) = \begin{cases} \sqrt{\frac{2}{b-a}} \cos \beta_n(y+b), & n \neq 1, \\ \sqrt{\frac{1}{b-a}}, & n = 1. \end{cases} \quad (3.4)$$

The eigenvalues $\beta_n = \frac{(n-1)\pi}{b-a}$ are the solutions of

$$\sin \beta_n(b-a) = 0, \quad n = 1, 2, 3, \dots$$

The eigenfunctions $\xi_n(y)$ satisfy

$$\int_{-b}^{-a} \xi_m(y) \xi_n(y) dy = \delta_{mn}. \quad (3.5)$$

The general potential is given by

$$\phi(x, y) = \sum_{n=1}^{\infty} B_n e^{-\bar{\beta}_n x} \xi_n(y) \quad (3.6)$$

which satisfies (2.2), (2.4) and (2.5) where

$$\bar{\beta}_n = \sqrt{k^2 - \beta_n^2}. \quad (3.7)$$

3.3 Region 3 $\{|y| \leq a, x < 0\}$

Here general potential is

$$\phi(x, y) = \sum_{n=1}^{\infty} C_n e^{-\bar{\gamma}_n x} \eta_n(y) + e^{\bar{\gamma}_1 x} \eta_1(y) \quad (3.8)$$

which satisfies (2.2), (2.4) and (2.5). The associated eigenvalues are

$$\bar{\gamma}_n = \sqrt{k^2 - \gamma_n^2}, \quad n = 1, 2, \dots \quad (3.9)$$

and incident wave $e^{\bar{\gamma}_1 x} \eta_1(y)$ is excited in the lowest mode propagating from $x = -\infty$. The eigenvalues $\gamma_n = \frac{(n-1)\pi}{2a}$ satisfy

$$\sin 2\gamma_n a = 0.$$

The eigenfunctions η_n are given by

$$\eta_n(y) = \begin{cases} \sqrt{\frac{1}{a}} \cos \gamma_n(y+a), & n \neq 1 \\ \sqrt{\frac{1}{2a}}, & n = 1 \end{cases} \quad (3.10)$$

which satisfy

$$\int_{-a}^a \eta_m(y) \eta_n(y) dy = \delta_{mn}, \quad m, n = 1, 2, 3, \dots \quad (3.11)$$

In this region C_n are the reflected field amplitudes. Here $|R| = |C_1|$ is the coefficient of the lowest mode $\exp(-i\bar{\gamma}_1 x)$ which will investigate the radiated energy.

3.4 Region 4 $\{a \leq y \leq b, x < 0\}$

Here the general potential satisfying (2.2), (2.4) and (2.5) is given by

$$\phi(x, y) = \sum_{n=1}^{\infty} D_n e^{-\bar{\beta}_n x i} \zeta_n(y).$$

where the eigenvalues β_n are same as given in region 2. The eigenfunctions $\zeta_n(y)$

$$\zeta_n(y) = \begin{cases} \frac{\sqrt{2}}{\sqrt{b-a}} \cos \beta_n(y-b), & n \neq 1, \\ \frac{1}{\sqrt{b-a}}, & n = 1. \end{cases} \quad (3.12)$$

satisfy the orthonormal relation

$$\int_a^b \zeta_m(y) \zeta_n(y) dy = \delta_{mn}, \quad m, n = 1, 2, 3, \dots \quad (3.13)$$

3.5 Region 5 $\{b \leq y \leq d, x < 0\}$

The general potential which satisfies (2.2), (2.3), (2.4) and (2.5) is

$$\phi(x, y) = \sum_{n=1}^{\infty} E_n e^{-\bar{\alpha}_n x i} \sigma_n(y). \quad (3.14)$$

where the eigenvalues α_n are same as given in region 1.

The eigenfunctions σ_n in region 5 are defined by

$$\sigma_n(y) = \sqrt{\frac{2}{d-b}} \cos \alpha_n(y-b) \quad (3.15)$$

which satisfy

$$\int_b^d \sigma_n(y) \sigma_m(y) dy = \delta_{mn}, \quad m, n = 1, 2, \dots \quad (3.16)$$

3.6 Region 6 $\{-d \leq y \leq d, x > 0\}$

The general potential which satisfies (2.2), (2.3) and (2.5) are given by

$$\phi(x, y) = \sum_{n=1}^{\infty} F_n e^{\bar{\omega}_n x i} \epsilon_n(y), \quad (3.17)$$

where

$$\epsilon_n(y) = \sqrt{\frac{1}{d}} \sin \omega_n(y+d),$$

$$\bar{\omega}_n = \sqrt{k^2 - \omega_n^2} \quad n = 1, 2, 3, \dots$$

and

$$\omega_n = \frac{n\pi}{2d}$$

which are the roots of $\sin 2\omega_n d = 0$.

4 Formulation of the Infinite Set of Equations

4.1 Continuity of Pressure:

The continuity of the pressure across regions 6 and 1 at $x = 0$ gives

$$\sum_{n=1}^{\infty} A_n \psi_n = \sum_{n=1}^{\infty} F_n \epsilon_n.$$

Multiplying by ψ_m , integrating over $[-d, -b]$ and using (3.3), the above equation becomes

$$A_m = \sum_{n=1}^{\infty} F_n L_{mn}, \quad m = 1, 2, \dots \quad (4.1)$$

where

$$L_{mn} = \int_{-d}^{-b} \psi_m(y) \epsilon_n(y) dy$$

$$= \begin{cases} \frac{-\sqrt{2}\omega_n \{\cos \omega_n(d-b) - \cos \alpha_m(d-b)\}}{\sqrt{d(d-b)(\omega_n^2 - \alpha_m^2)}} & \omega_n \neq \alpha_m \\ \sqrt{\frac{d-b}{2d}} \sin \alpha_m(d-b), & \omega_n = \alpha_m. \end{cases}$$

The continuity of the pressure across region 6 and 2 at $x = 0$ gives

$$\sum_{n=1}^{\infty} B_n \xi_n = \sum_{n=1}^{\infty} F_n \epsilon_n.$$

Integrating over $-b \leq y \leq -a$ after multiplying by ξ_m and using (3.5), the above equation becomes

$$B_m = \sum_{n=1}^{\infty} F_n Q_{mn}, \quad m = 1, 2, \dots \quad (4.2)$$

where

$$Q_{mn} = \int_{-b}^{-a} \epsilon_n(y) \xi_m(y) dy$$

$$= \begin{cases} \frac{1}{\sqrt{d(b-a)}} \left[\frac{\cos \omega_n(d-b) - \cos \omega_n(d-a)}{\omega_n} \right], & m = 1 \\ -\sqrt{\frac{2}{d(b-a)}} \omega_n \left[\frac{\cos \beta_m(b-a) \cos \omega_n(d-a) - \cos \omega_n(d-b)}{\omega_n^2 - \beta_m^2} \right], & \omega_n \neq \beta_m \\ \frac{1}{\sqrt{2d(b-a)}} \left[\frac{-\cos \beta_m(b+d-2a) + \cos \beta_m(d-b)}{2\beta_m} + (b-a) \sin \beta_m(d-b) \right], & \omega_n = \beta_m. \end{cases}$$

The continuity of the pressure across region 6 and 3 at $x = 0$ gives

$$\sum_{n=1}^{\infty} C_n \eta_n + \eta_1 = \sum_{n=1}^{\infty} F_n \epsilon_n.$$

Integrating over $|y| \leq a$ after multiplying by η_m and using (3.11), the above equation becomes

$$\delta_{1m} + C_m = \sum_{n=1}^{\infty} F_n T_{mn}, \quad m = 1, 2, \dots \quad (4.3)$$

where

$$T_{mn} = \int_{-a}^a \epsilon_n(y) \eta_m(y) dy$$

$$= \begin{cases} \sqrt{\frac{2}{ad}} \frac{\sin \omega_n a \sin \omega_n d}{\omega_n}, & m = 1 \\ \frac{-\omega_n}{\sqrt{ad}} \left[\frac{-\cos \omega_n (d-a) + \cos \omega_n (a+d) \cos 2\gamma_m a}{\omega_n^2 - \gamma_m^2} \right], & \omega_n \neq \gamma_m \\ \frac{1}{2\sqrt{ad}} \left[\frac{-\cos \gamma_m (3a+d) + \cos \gamma_m (d-a)}{2\gamma_m} - 2a \sin \gamma_m (a-d) \right], & \omega_n = \gamma_m. \end{cases}$$

The continuity of the pressure across regions 6 and 4 at $x = 0$ gives

$$\sum_{n=1}^{\infty} D_n \zeta_n = \sum_{n=1}^{\infty} F_n \epsilon_n.$$

Integrating over $a \leq y \leq b$ after multiplying by ζ_m and using (3.13), we get

$$D_m = \sum_{n=1}^{\infty} F_n R_{mn}, \quad m = 1, 2, \dots \quad (4.4)$$

where

$$R_{mn} = \int_a^b \epsilon_n(y) \zeta_m(y) dy$$

$$= \begin{cases} \sqrt{\frac{2}{d(b-a)}} \left[\frac{\cos \omega_n (b+d) - \cos \omega_n (a+d)}{\omega_n} \right], & m = 1 \\ -\sqrt{\frac{1}{d(b-a)}} \omega_n \left[\frac{\cos \omega_n (b+d) - \cos \omega_n (a+d) \cos \beta_m (b-a)}{\omega_n^2 - \beta_m^2} \right], & \omega_n \neq \beta_m \\ \sqrt{\frac{1}{2d(b-a)}} \left[(b-a) \sin \omega_n (b+d) - \frac{\cos \omega_n (b+d) - \cos \omega_n (2a+d-b)}{2\omega_n} \right], & \omega_n = \beta_m. \end{cases}$$

The continuity of the pressure across region 6 and 5 at $x = 0$ gives

$$\sum_{n=1}^{\infty} E_n \sigma_n = \sum_{n=1}^{\infty} F_n \epsilon_n.$$

Integrating over $b \leq y \leq d$ after multiplying by σ_m and using (3.16), we obtain

$$E_m = \sum_{n=1}^{\infty} F_n S_{mn}, \quad m = 1, 2, \dots \quad (4.5)$$

where

$$S_{mn} = \int_b^d \xi_n(y) \sigma_m(y) dy$$

$$= \begin{cases} \sqrt{\frac{2}{d(d-b)}} \omega_n \left[\frac{\cos \omega_n (d+b)}{\omega_n^2 - \alpha_m^2} \right], & \omega_n \neq \alpha_m \\ \sqrt{\frac{1}{2d(d-b)}} \left[(d-b) \sin \omega_n (b+d) + \frac{\cos \omega_n (b+d) - \cos \omega_n (3d-b)}{2\omega_n} \right], & \omega_n = \alpha_m. \end{cases}$$

4.2 Continuity of Velocity:

Following the above procedure, the continuities of the velocities across different regions at $x = 0$ give the following set of equations

$$-\bar{\alpha}_m A_m = \sum_{n=1}^{\infty} \bar{\omega}_n F_n L_{mn}, \quad m = 1, 2, \dots, \quad (4.6)$$

$$-\bar{\beta}_m B_m = \sum_{n=1}^{\infty} \bar{\omega}_n F_n Q_{mn}, \quad m = 1, 2, \dots, \quad (4.7)$$

$$\bar{\gamma}_1 \delta_{1m} - \bar{\gamma}_m C_m = \sum_{n=1}^{\infty} \bar{\omega}_n F_n T_{mn}, \quad m = 1, 2, \dots, \quad (4.8)$$

$$-D_m \bar{\beta}_m = \sum_{n=1}^{\infty} \bar{\omega}_n F_n R_{mn}, \quad m = 1, 2, \dots \quad (4.9)$$

and

$$-E_m \bar{\alpha}_m = \sum_{n=1}^{\infty} \bar{\omega}_n F_n S_{mn}, \quad m = 1, 2, \dots \quad (4.10)$$

Now we find unknown coefficients in region 1 to region 6 by solving the infinite system of equations (4.1) to (4.10). For this we limit ourselves to $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, 5M$. In figure 2, subfigures 1, 3 and 5 are plotted to observe the absolute error value on log scale against M by using Richardson extrapolation formula while subfigures 2, 4 and 6 present the convergence in reflection as a function of truncation number M . Here the duct dimensions are such that $a = 1$, $b = 2$, $d = 3$ for $k = \pi/4, \pi/3$ and $3\pi/2$. We can see that for $M \geq 110$, the reflection becomes insensitive which indicates that the error lies within the line width.

Figure 2: Absolute error and reflection amplitude versus number of equations for $a = 1$, $b = 2$ and $d = 3$

5 Numerical Outcomes

In this section we plot $|R| = |C_1|$ against k (wave number). This would be of concern to determine the quantity of acoustic energy which is dispersed among the different regions. The energy $(1 - |R|^2)$ coming out from the region $\{|y| < a, x < 0\}$ is distributed among the other waveguide regions. We draw some graphs for $|R|$ for different dimensions of the duct spacing which would be helpful to predict the acoustic radiated power. Before discussing the reflection of penta-furcated problem, we first plot $|R|$ for trifurcated problem given in appendix by solving infinite system of equations (7.10) to (7.15). The graphs 3 and 4 are plotted to have the comparison between eigenfunction expansion (dotted line) and Wiener-Hopf methods (solid line) for $d = \frac{3}{2}$ and $d = 3$ respectively where $a = 1$ and Mach number $M = 0$ which is given by

$$|R| = \frac{1}{4ak^2 G_+^2(k)}. \quad (5.1)$$

where G_+ involves complicated infinite product [5]. This shows that eigenfunction expansion method has same results as found in [5]. We present comparison of $|R|$ for outer-soft penta-furcated (dotted line), rigid penta-furcated (dashed line) and trifurcated (solid line) problems in cases 1 and 2.

Figure 3: $|R|$ versus k for $a = 1$ and $d = 1.5$

Figure 4: $|R|$ versus k for $a = 1$ and $d = 3$

1 **5.1 Case 1:**

2 Here the duct spacing is such that $a = 1$, $b = \frac{5}{4}$, and $d = \frac{3}{2}$ in the frequency
3 range $0 < k < \pi$. Figure 5 depicts the situation graphically. There is no wave
4 before $k = \frac{\pi}{3}$ which is onset of wave propagating in forward direction for both
5 tri and current penta furcated ducts while for rigid penta furcated duct there is
6 no mode before $k = \frac{2\pi}{3}$.

Figure 5: Absolute reflection versus wave number for $a = 1$, $b = 1.25$ and $d = 1.5$

Figure 6: Absolute reflection versus wave number for $a = 1$, $b = 2$ and $d = 3$

Figure 7: Absolute reflection versus wave number for $a = 1, b = 2$ and $d = 3$

Figure 8: Absolute reflection versus wave number for $a = 1, b = 2$ and $d = 3$

1 **5.2 Case 2:**

2 Figure 6 is plotted for duct spacing $a = 1, b = 2, d = 3$ and the frequency
3 range is $0 < k < \pi$. Here the cut on (off) frequencies are at $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$

1 and $\frac{5\pi}{6}$. In case 3, $|R|$ versus k is plotted for current outer-soft pentafurcated
 2 waveguide to observe the effect of varying duct spacing on reflection. Figure 7
 3 is just for frequency range $0 < k < 2\pi$. Similarly figure 8 is plotted to have
 4 comparison with pentafurcated problems having different boundary conditions;
 5 current, rigid [24], outer soft inner hard [28] and soft hard [29].

6 5.3 Case 3:

7 Figures 9, 10 and 11 are plotted for reflection field analysis against wave number
 8 for pentafurcated duct with frequency range $0 < k < \pi$. In Figure 9 the duct
 9 spacing is such that *i*) $a = 0.5$ (solid line) *ii*) $a = 1$ (dotted line) for fixed width
 10 $b = 2$ and $d = 3$. Figure 10 is plotted for *i*) $b = 2$ *ii*) $b = 2.5$ (solid line) for
 11 fixed $a = 1$ and $d = 3$ while figure 11 is drawn for *i*) $d = 3$ *ii*) $d = 4$ (solid line)
 12 for set values of $a = 1$ and $b = 2$.

Figure 9: Absolute reflection versus wave number for fixed $b = 2$ and $d = 3$

Figure 10: Absolute reflection versus wave number for fixed $a = 1$ and $d = 3$

Figure 11: Absolute reflection versus wave number for fixed $a = 1$ and $b = 2$

5.4 Algorithm

Here is the basic algorithm which is the essential part for finding the numerical solutions of unknown coefficients in different regions.

- **Input:** a, b, d, M and k where $a < b < d$.
- Calculate eigenvalues $\alpha_n, \beta_n, \gamma_n$ and associated eigenvalues $\bar{\alpha}_n, \bar{\beta}_n, \bar{\gamma}_n$.
- For $m = 1, 2, 3, \dots, M$
and for $n = 1, 2, 3, \dots, N$, we calculate inner products $L_{mn}, Q_{mn}, T_{mn}, R_{mn}$, and S_{mn} ,
- Write equations (4.1 to 4.10) in matrix form.
- Calculate unknown coefficients $A_n, B_n, \dots, F_n$.

6 Energy Conservation

We can derive an energy conservation equation by using the Green's theorem [6] for one propagating mode in each of the regions 1 – 5 and five modes in region 6. We start with

$$\int_{\Omega} (\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^*) dv = \int_{\partial\Omega} \left(\phi^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial \phi^*}{\partial n} \right) ds \quad (6.1)$$

where complex conjugate is denoted by * and Ω is the region $|y| \leq d, |x| < \infty$ with discontinuity for the semi-infinite duct. This may not be difficult to obtain the equation:

$$\bar{\alpha}_1 |A_1|^2 + \bar{\beta}_1 |B_n|^2 + \bar{\gamma}_1 |C_1|^2 + \bar{\beta}_1 |D_1|^2 + \bar{\alpha}_1 |E_1|^2 + \sum_{n=1}^5 \bar{\omega}_n |F_n|^2 = \bar{\gamma}_1. \quad (6.2)$$

7 Conclusion

We have worked out the problems of tri and penta furcated exhaust models for static air by using the eigenfunction expansion method, assuming that only lowest mode propagates in region $y \leq |a|, x < 0$. The same trifurcated problem has been earlier studied in [5] by the Wiener-Hopf technique with fluid flow which requires the splitting of the kernel of the complex functions while the method used here is straight forward and easy to implement. We have obtained same results for the absolute reflection as were obtained in [5] (see figures 3 and 4) which is the confirmation of the precision of ourwork. Figure 5 compares the reflection field behaviour among trifurcated, current and rigid penta furcated ducts [24] when no mode propagates in coaxial regions. We can observe that behaviour of reflection for both tri and outer-soft penta furcated ducts is exactly the same for $0 \leq k \leq \pi$. For $0 \leq k \leq 1.2$ reflection coefficient for rigid penta furcated duct is less as compared to other two ducts while afterward this mechanism reverses. Hence acoustic energy $(1 - |R^2|)$ coming out from outer-soft penta furcated duct is smaller than that of rigid duct in region $0 \leq k \leq 1.2$. Thus the outer-soft surface has a noise reducing effect on noise transmitted down the duct. Figure 6 shows the reflection field behaviour when two modes propagate in backward direction and three modes travel in forward direction for tri and outer-soft penta furcated ducts while for rigid penta furcated duct there are two forward and one backward mode. Figures 9-11 show the effect on reflection for outer-soft penta furcated duct when one of the width dimension changes while other two are kept fixed. From figures 9 and 10, we notice that as the width of one of the regions decreases, value of $|R|$ increases.

We have observed that when there is no wave in the coaxial regions (figure 5), the accumulative value of reflection for outer-soft penta furcated duct (1307.8) is greater than accumulated value of reflection for hard penta furcated duct (803.3268). While for extended frequency range we can find that accumulated reflection for tri, hard penta and outer-soft penta ducts are 1227.2,

1144.8 and 1375.1 respectively (see figure 7). Thus the reflection for outer soft penta-furcated duct is greatest among all considered ducts which results in lesser transmitted acoustic energy. We have also noticed that the value 1375.1 is greater among the other penta-furcated problems ([28] and [29]). Hence the insertion of soft outer lining in penta-furcated duct can provide more active noise attenuation as compared to related outer soft trifurcated [5] and existing penta-furcated ducts ([24], [28] and [29]) thus improves the performance of duct.

This penta-furcated problem can be solved by using Wiener Hopf method. One can also extend this model to fluid flow. The presented models will be valuable in designing practically efficient exhaust systems.

Appendix A: Eigenfunction Expansion for Trifurcated Duct

We give a brief description of trifurcated problem previously solved in [5] using Wiener Hopf technique. We will use the same procedure as we used for outer-soft penta-furcated problem. This trifurcated problem is given by (2.2), (2.3), (2.4) and (2.5) while plate b is removed. The potential in region 1 $\{-d \leq y \leq -a, x < 0\}$ is given by

$$\phi(x, y) = \sum_{n=1}^{\infty} a_n e^{-\bar{\alpha}_n x} \hat{\psi}_n(y), \quad (7.1)$$

where

$$\bar{\alpha}_n = \sqrt{k^2 - \hat{\alpha}_n}, \quad (7.2)$$

$$\hat{\alpha}_n = \frac{(2n-1)\pi}{2(d-a)} \quad (7.3)$$

which are roots of $\cos \hat{\alpha}_n(d-a) = 0$ and the orthonormal function $\hat{\psi}_n(y)$ are given by

$$\hat{\psi}_n(y) = \sqrt{\frac{2}{d-a}} \sin \hat{\alpha}_n(y+d).$$

The eigenfunction expansion in region 2 $\{|y| < a, x < 0\}$ is

$$\phi(x, y) = e^{i\bar{\beta}_1 x} \eta_1(y) + \sum_{n=1}^{\infty} b_n e^{-\bar{\beta}_n x} \hat{\eta}_n(y) \quad (7.4)$$

where

$$\bar{\beta}_n = \sqrt{k^2 - \hat{\beta}_n}, \quad n = 1, 2, \dots$$

$$\hat{\beta}_n = \frac{(n-1)\pi}{2a}$$

which are roots of $\sin 2\hat{\beta}_n a = 0$ and orthonormal function $\hat{\eta}_n(y)$ are given by

$$\hat{\eta}_n(y) = \begin{cases} \sqrt{\frac{1}{a}} \cos \hat{\beta}_n(y-a), & n \neq 1, \\ \sqrt{\frac{1}{2a}}, & n = 1. \end{cases}$$

1 The incident wave $e^{\bar{\beta}_1 x i} \eta_1(y)$ is traveling from left to right. To have one fun-
 2 damental incident and one reflected wave mode, we restrict $0 < ka < \frac{\pi}{2}$ and
 3 $\bar{\beta}_n = i\sqrt{\frac{(n-1)^2 \pi^2}{4a^2} - k^2}$ so that $\text{Re}\bar{\beta}_n = 0$ and $\text{Im}\bar{\beta}_n > 0$ for $n > 2$.
 4 The eigenfunction in region 3 $\{a < y < d, x < 0\}$ is given by

$$5 \quad \phi(x, y) = \sum_{n=1}^{\infty} c_n e^{-\bar{\alpha}_n x i} \hat{\sigma}_n(y). \quad (7.5)$$

6 where eigenfunctions $\hat{\sigma}_n(y)$ are

$$7 \quad \hat{\sigma}_n(y) = \sqrt{\frac{2}{d-a}} \sin \hat{\alpha}_n(y-d).$$

8 The eigenfunction in region 4 $\{|y| < d, -\infty < x < \infty\}$ is given by

$$9 \quad \phi(x, y) = \sum_{n=1}^{\infty} d_n e^{\bar{\omega}_n x i} \epsilon_n(y) \quad (7.6)$$

10 Appendix B: Orthogonality Products

11 We define the inner product between region 1 and region 4 as

$$12 \quad \hat{L}_{mn} = \int_{-d}^{-a} \hat{\psi}_m(y) \epsilon_n(y) dy \quad (7.7)$$

$$13 \quad = \begin{cases} \sqrt{\frac{2}{d(d-a)}} \left[\frac{\omega_n \sin \hat{\alpha}_n(d-a) \cos \omega_n(d+a)}{\hat{\alpha}_m^2 - \omega_m^2} \right], & \omega_n \neq \hat{\alpha}_m \\ \sqrt{\frac{2(d-a)}{d}}, & \omega_n = \hat{\alpha}_m. \end{cases}$$

15 The inner product between region 2 and 4 is given by

$$16 \quad \hat{T}_{mn} = \int_{-a}^a \hat{\eta}_m(y) \epsilon_n(y) dy \quad (7.8)$$

$$17 \quad = \begin{cases} \frac{-1}{\sqrt{2ad}} \left[\frac{\cos \omega_n(a-d) - \cos \omega_n(a+d)}{\omega_n} \right], & m = 1 \\ -\frac{1}{\sqrt{ad}} \omega_n \left[\frac{\cos \omega_n(a-d) - \cos 2a\beta_m \cos \omega_n(a+d)}{\omega_n^2 - \hat{\beta}_m^2} \right], & \omega_n \neq \hat{\beta}_m \\ \frac{1}{2\sqrt{ad}} \left[\frac{-\cos \hat{\beta}_m(a-d) + \cos \hat{\beta}_m(3a+d)}{2\hat{\beta}_m} + 2a \sin \hat{\beta}_m(a-d) \right], & \omega_n = \hat{\beta}_m. \end{cases}$$

19 and the inner product between region 3 and 4 is given by

$$20 \quad \hat{S}_{mn} = \int_a^d \hat{\sigma}_m(y) \epsilon_n(y) dy \quad (7.9)$$

$$21 \quad = \begin{cases} \sqrt{\frac{2}{d(d-a)}} \left[\frac{\omega_n \sin \hat{\alpha}_m(d-a) \cos \omega_n(d-a)}{\hat{\alpha}_m^2 - \omega_n^2} \right], & \hat{\alpha}_m \neq \omega_n \\ \sqrt{\frac{d-a}{2d}}, & \hat{\alpha}_m = \omega_n. \end{cases}$$

23 Appendix C: System of equations

We equate the potentials and their velocities across $x = 0$. Taking the inner products by multiplying with appropriate eigenfunctions and integrating over respective dimensions, we get the following system of equations.

$$a_m = \sum_{n=1}^{\infty} d_n \hat{L}_{mn} \quad (7.10)$$

$$\delta_{m1} + b_m = \sum_{n=1}^{\infty} d_n \hat{T}_{mn} \quad (7.11)$$

$$c_m = \sum_{n=1}^{\infty} d_n \hat{S}_{mn} \quad (7.12)$$

$$-\bar{\alpha}_m a_m = \sum_{n=1}^{\infty} d_n \bar{\omega}_n \hat{L}_{mn} \quad (7.13)$$

$$\delta_{m1} \bar{\beta}_1 - \bar{\beta}_m b_m = \sum_{n=1}^{\infty} d_n \bar{\omega}_n \hat{T}_{mn} \quad (7.14)$$

$$-\bar{\alpha}_m c_m = \sum_{n=1}^{\infty} d_n \bar{\omega}_n \hat{S}_{mn}. \quad (7.15)$$

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